

Discretization Methods

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Exam

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Part 1

Consider the advection diffusion equation given as

$$\frac{\partial u(x, t)}{\partial t} + U_0(x) \frac{\partial u(x, t)}{\partial x} = \nu \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (1)$$

where $U_0(x)$ is periodic and bounded and ν is assumed to be constant. Also $u(x, t)$ is assumed to be smooth and periodic as is the initial condition.

- (a) State sufficient conditions on $U_0(x)$ and ν that ensures Eq. 1 to be well-posed.
- (b) Assume that Eq. 1 is approximated using Fourier Collocation method. Is the approximation consistent and what is the expected convergence rate when increasing N , the number of modes used in the approximation.
- (c) Assume now that $U_0(x)$ is constant and Eq. 1 is approximated using a Fourier Collocation method with odd number of modes. Prove that the semi-discrete approximation, i.e. continuous time and approximated space, is stable.

Part 2

Consider now Burger's equation given as

$$\frac{\partial u(x, t)}{\partial t} + u(x, t) \frac{\partial u(x, t)}{\partial x} = \nu \frac{\partial^2 u(x, t)}{\partial x^2}, \quad (2)$$

where $u(x, t)$ is assumed periodic. Eq. 2 has an analytic solution given as

$$u(x, t) = c - 2\nu \frac{\frac{\partial \phi(x-ct, t+1)}{\partial x}}{\phi(x-ct, t+1)}, \quad \phi(a, b) = \sum_{k=-\infty}^{\infty} \exp\left(-\frac{(a - (2k+1)\pi)^2}{4\nu b}\right).$$

This solution represents a sawtooth-like traveling wave which propagates at the velocity c , while slowly decaying due to dissipation. In all subsequent tests use $c = 4.0$ and $\nu = 0.1$ and consider the problem in the standard interval of $x \in [0, 2\pi]$.

- (a) Construct a program that solves Eq. 2 using Fourier Collocation method with an odd number of grid points, i.e. the grid points are given as

$$x_j = \frac{2\pi}{N+1} j = j\Delta x, \quad j \in [0, N],$$

and use the exact solution at $t = 0$ as the initial condition.

To advance the equation in time, use 4th order Runge-Kutta method obtained by first defining the function, $F(u^n)$, as

$$\frac{du^n}{dt} = F(u^n) = -u^n \frac{\partial u^n}{\partial x} + \nu \frac{\partial^2 u^n}{\partial x^2},$$

where u^n represents the solution at $t = n\Delta t$ and Δt is the time step. The 4th order Runge-Kutta method for advancing from u^n to u^{n+1} is then defined as

$$u_1 = u^n + \frac{\Delta t}{2} F(u^n)$$

$$u_2 = u^n + \frac{\Delta t}{2} F(u_1)$$

$$u_3 = u^n + \Delta t F(u_2)$$

$$u^{n+1} = \frac{1}{3} \left(-u^n + u_1 + 2u_2 + u_3 + \frac{\Delta t}{2} F(u_3) \right),$$

where u_1 , u_2 and u_3 are introduced for convenience as help-functions/arrays.

The time step, Δt , is given as

$$\Delta t \leq \text{CFL} \times \left[\max_{x_j} \left(\frac{|u(x_j)|}{\Delta x} + \frac{\nu}{(\Delta x)^2} \right) \right]^{-1}. \quad (3)$$

- (b) Determine by experiment the maximum value of the number CFL in Eq. 3 that results in a stable scheme for $N = 16, 32, 48, 64, 96, 128, 192, 256$. Does the definition of the time-step restriction as given in Eq. 3 seem reasonable (think of the CFL-condition known from Finite Difference Methods)? Use these values of CFL in what remains.
- (c) Measure the L^∞ -error between the computed solution at $t = \pi/4$ and the exact solution for $N = 16, 32, 48, 64, 96, 128, 192, 256$. What is the convergence rate observed and does it correspond to what you would expect?
- (d) Plot the solution for $N = 128$ at $t = 0, \pi/8, \pi/6, \pi/4$.

Part 3

Consider the same problem, parameters, etc as in Part 2.

- (a) Construct a program that solves Eq. 2 using Fourier Galerkin method.

Use the quadrature formula to compute the expansion coefficients of the initial conditions and use the 4th order Runge-Kutta method described in Part 2 to advance the Fourier-Galerkin equations in time.

The time step, Δt , is given as

$$\Delta t \leq \text{CFL} \times \left[\max_{x_j} (|u(x_j)| k_{\max} + \nu (k_{\max})^2) \right]^{-1}, \quad (4)$$

where $k_{\max} = N/2$.

- (b) Determine by experiment the maximum value of the number CFL in Eq. 4 that results in a stable scheme for $N = 16, 32, 48, 64, 96, 128, 192, 256$. Use these values of CFL in what remains.
- (c) Measure the L^∞ -error between the computed solution at $t = \pi/4$ and the exact solution for $N = 16, 32, 48, 64, 96, 128, 192, 256$. What is the convergence rate observed and does it correspond to what you would expect?
- (d) Compare the results with those in Part 2. Do you see any significant difference between the two solutions? Any reason to use the Galerkin formulation?