Discretization Methods
Spring Semester 2024

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Exam

Date of submission: 07.06.2024

Part 1

Consider the advection diffusion equation given as

$$\frac{\partial u(x,t)}{\partial t} + U_0(x)\frac{\partial u(x,t)}{\partial x} = \nu\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{1}$$

where $U_0(x)$ is periodic and bounded and ν is assumed to be constant. Also $u(x,t)$ is assumed to be smooth and periodic as is the initial condition.

(a) State sufficient conditions on $U_0(x)$ and ν that ensures Eq. 1 to be well-posed.

(b) Assume now that $U_0(x)$ is constant and Eq. 1 is approximated using a Fourier Collocation method with odd number of modes. Prove that the semi-discrete approximation, i.e. continuous time and approximated space, is stable.

Part 2

Consider now Burger's equation given as

$$\frac{\partial u(x,t)}{\partial t} + u(x,t)\frac{\partial u(x,t)}{\partial x} = \nu\frac{\partial^2 u(x,t)}{\partial x^2}, \tag{2}$$

where $u(x,t)$ is assumed periodic. Eq. 2 has an analytic solution given as

$$u(x,t) = c - 2\nu\frac{\frac{\partial\phi(x-ct,t+1)}{\partial x}}{\phi(x-ct,t+1)}, \qquad \phi(a,b) = \sum_{k=-\infty}^{\infty}\exp\left(-\frac{(a-(2k+1)\pi)^2}{4\nu b}\right).$$

This solution represents a sawtooth-like traveling wave which propagates at the velocity c, while slowly decaying due to dissipation. In all subsequent tests use $c = 4.0$ and $\nu = 0.1$ and consider the problem in the standard interval of $x \in [0, 2\pi]$.

(a) Construct a program that solves Eq. 2 using Fourier Galerkin method.

Use the quadrature formula to compute the expansion coefficients of the initial conditions and use the 4th order Runge-Kutta method to advance the Fourier-Galerkin equations in time.

The time step, Δt, is given as

$$\Delta t \le \text{CFL} \times \left[\max_{x_j}\left(|u(x_j)|k_{\max} + \nu(k_{\max})^2\right)\right]^{-1}, \tag{3}$$

where $k_{\max} = N/2$.

(b) Determine by experiment the maximum value of the number CFL in Eq. 3 that results in a stable scheme for $N = 16, 32, 48, 64, 96, 128, 192, 256$. Use these values of CFL in what remains.

(c) Measure the L^∞-error between the computed solution at $t = \pi/4$ and the exact solution for $N = 16, 32, 48, 64, 96, 128, 192, 256$ (use t from the last time integration step, since it may not be exactly equal to $\pi/4$). What is the convergence rate observed and does it correspond to what you would expect?

Part 3

Consider the equation given as

$$\frac{\partial u(x,t)}{\partial t} = -x\frac{\partial u(x,t)}{\partial x} \tag{4}$$
$$u(x,0) = f(x) \tag{5}$$

in the interval $-1 \le x \le 1$, with no boundary conditions.

(a) Let $E(t) = \int_{-1}^{1} u^2(x,t)dx$, show that $E(t) \le E(0)e^t$.

(b) Show that the solution of Eq. 4 is given by $u(x,t) = f(xe^{-t})$ and get a better estimate for $E(t)$.

(c) Consider now the Chebyshev collocation method for Eq. 4 based on grid points $x_j = -\cos(\frac{\pi j}{N})$. Denote by u_N the solution of the Chebyshev approximation. Let $E_N(t) = \int_{-1}^{1} u_N^2(x,t)dx$, show that $E_N(t) \le E_N(0)e^t$

(d) Let $f(x) = \cos(5\pi x)$. Apply the Chebyshev collocation method to Eq. 4. Integrate with the 4th order Runge-Kutta scheme to $t = 5$ with $N = 16, 32, 64, 128$. Use time steps Δt such that $N^2\Delta t$ is fixed. If you need boundary conditions, assume that they are given. What is the convergence rate observed? Discuss the quality of the results.