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PRISM Case Study - Investor in Market

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1 Introduction

In this report, we implement a case study for the PRISM[4] model checking tool by extending an existing case study showcased on their website [6].

1.1 PRISM

PRISM is a probabilistic model checker, a tool for formal modelling and analysis of systems that exhibit random or probabilistic behaviour. It can be used to analyse systems from many different application domains, such as communication and multimedia protocols, randomised distributed algorithms, security protocols, and biological systems.

1.2 PRISM Usage

For learning how to use PRISM, we invite the reader to their well-documented tutorials and manual. A very useful tool from PRISM, called experiments, allows us, either from the GUI or CLI, to verify/check properties for different model constants/parameters in one go.

1.3 Futures Market Investor

We now discuss the original case study Futures Market Investor [6], found from their case studies list. The case study is based on an investor in a futures market. This example can be considered as a two-player game where one player (the investor) tries to maximize his return while the other player (the market) attempts to minimise the return of the investor. The below summarizes their description of the scenario:

"The Investor can make a single investment in 'futures': a fixed number of shares in a specific company that he can reserve on the first day of any month he chooses. Exactly one month later, the shares will be delivered and will collectively have a market value on that day – he or she can then sell the shares. The investor wants to make a reservation such that the sale has maximum value.

The market value v of the shares is a whole number of euros between €0 and €10 inclusive; it has a probability p of going up by €1 in any month, and 1-p of going down by €1, remaining within bounds. This probability represents short-term volatility.

The probability p itself varies month-by-month in steps of 0.1 between zero and one: p rises with probability 2/3 when v is less than €5, and when v is more than €5 the probability of p falling is 2/3. When v is €5 p rises or falls with equal probability. This represents market trends, where we assume the price of a stock revolves around its "true" value, here €5.

There is a cap c on the value of v, initially €10, which has probability 1/2 of falling by €1 in any month; otherwise it remains where it is. This models a company that is in a slow decline.

If in a given month the investor does not reserve, then at the very next month the market can temporarily bar the investor from reserving. But the market cannot bar the investor in two consecutive months."– *from the Futures Market Investor case study* [6]

Note that even though the case study could be considered a two-player game, with actors investor and market, in PRISM the market's strategy is fixed in advance. Their analysis thus focuses on a single player, the investor.

1.4 Motivations

After observing the model and their description, we have some observations on their use of terminology and description of the scenario.

Firstly, this is not a representation of a Future stock, which is a contract to buy a stock at a pre-determined future time and price. Instead, the investor is simply buying shares, that he receives with some delay. However, even this does not accurately match the model, as there would be a cost related to buying the shares.

Secondly, after the investor receives the stock, there is an imposition that they sell the stock immediately, when more realistically they could wait further before selling them.

A proper scenario description matching their model would be as follows. An investor has a number of shares. They would like to sell the shares, but there is a time delay before the sell is executed. They want to time the sell in order to maximize the return. The market actions and share value variance is as previously defined.

The original scenario has some room for possible extensions. In our implementation, we explore a few differences.

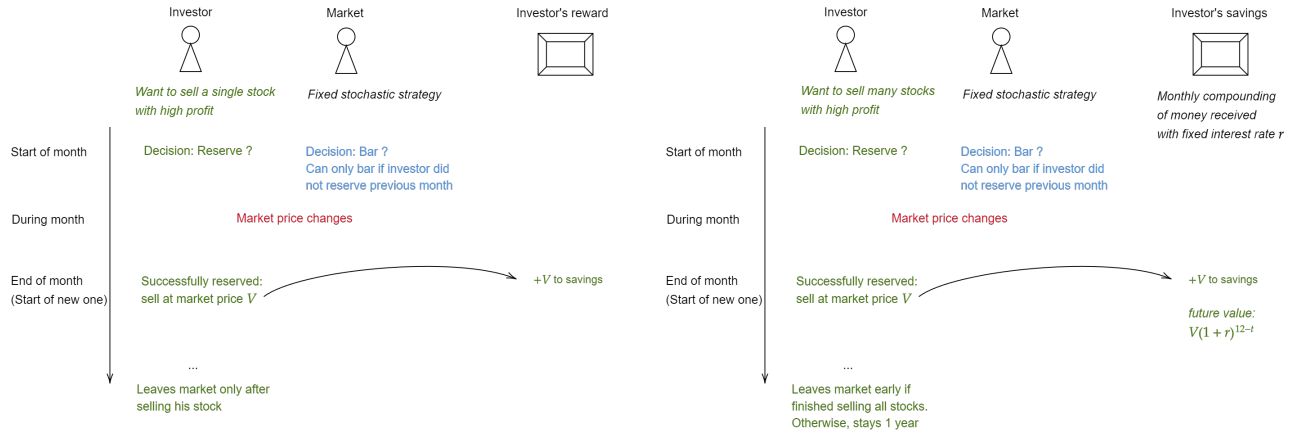
In the original model, the end condition is when the investor performs the single sell of his shares but there's a possibility of never reaching it, as the investor can choose to never reserve or sell. We change the end condition to a time-based one, permitting both the investor to continuously act in the market and the investor to leave the market after some time horizon. This of course would not make sense without any further changes, as the scenario has become such that the investor has access to an infinite amount of shares at his disposal. There needs to thus be a cost/limitation of some sort.

One implementation could be to add a cost (or negative reward) for performing the reservation. At first this seems to just become a lower bound for the stock value at which the investor would be willing to reserve his sell. However, if the cost is still considered even when the market bars the investor's sell that month, this variable would be interesting to observe, as it is then part of the dynamic between the two parties. However, PRISM at the moment strictly does not support costs (negative rewards). This idea is thus reserved for future work, since although for Markov Chains negative costs might be easier to implement, for Markov Decision Processes it is not as easy to handle.

Another way of implementing the limitation, one which we settled on, is simply to have a limit of shares that the investor has at hand.

Our last extension is to add a time-based reward to the sell of the shares. This is done by having the sell value accrue interest, compounded monthly. In other words, it's as if the received money is stored and grows with a fixed interest rate. This is in-line with the standard financial philosophy of "money now is worth more than money later".

We show in Figure 1a and Figure 1b diagrams intuitively explaining both the original case study and our extension.



(a) Original case study

(b) Our case study

Figure 1: Diagrams representing the original and our case study scenarios

2 Background

We detail below some background knowledge that can be helpful; a few probabilistic models, how to specify some properties for these models, and what they mean.

The reader can skip this section if the reader wants to directly dive into our model's case study. However, it's at least recommended to read the part 2.1.5 on our transition graphs and the subsection 2.3 on examples.

2.1 Models

A model is a representation, and usually a simplification of a real system, such that the focus is on the critical aspects. An example of a model is a transition system, which is a combination of states and transitions.

A state can represent, for example, variables of a program taking particular values. This state can change over time.

The transition from a state to another can be randomly triggered or can also be caused by the actions of the users. The transition can also happen every time an instruction is executed in a program.

We might want to check or verify whether our system follows some requirements. This generally requires a property specification language and property verification algorithms.

2.1.1 (Finite) Kripke Structure

A finite Kripke Structure is an extension of a transition system and is defined by $\mathcal{K} = \langle \mathcal{S}, S^0, \rightarrow, AP, L \rangle$ where:

- $\mathscr{S}$ finite set of states
- $S^0 \subseteq \mathscr{S}$ non-empty set of initial states
- $\rightarrow \subseteq \mathscr{S} \times \mathscr{S}$ left-total[1] binary relation on $\mathscr{S}$ representing the transitions
- AP the set of atomic propositions
- $L : \mathscr{S} \rightarrow \mathscr{P}(AP)$ state labelling function that labels each state with a set of atomic propositions that hold on that state

This model is non-deterministic as we don't necessarily know in which state we'll end up next when faced with many successor states. However, given a strategy to choose the next state when faced with many successor states (e.g. user(s) deciding which state to pick), the system becomes deterministic.

Relation to property specification: From a given state of the Finite Kripke Structure, we can form a tree of all the possible trajectories from that state. Each step down the tree corresponds to the one transition step in the Kripke Structure.

A property one could want to verify is whether a set of atomic propositions $\subseteq AP$ stay true on all the states in the future from a given state. We could also verify whether there exists at least one path/run/trajectory/execution in which all the encountered states satisfy a set of atomic propositions $\subseteq AP$.

Specifying such properties can be done with Computation Tree Logic (CTL) formulas (syntax). Verifying a CTL formula (semantics) would give us a set of states satisfying the CTL formula.

2.1.2 Discrete Time Markov Chain, absorbing states, extension with rewards

A DTMC is a probabilistic model where the next states are chosen with some probability. We can also have states (terminal/absorbing) where we are guaranteed not to move away from. Our values are then set in stone.

A Discrete Time Markov Chain (DTMC) [1] is formally a memory-less discrete-time finite state space stochastic process, meaning a discrete-time finite state stochastic process (sequence of random variables $S_1, S_2, \ldots$) with states satisfying the Markov property:

- $\mathscr{S}$ finite set of states
- $s_0 \in \mathscr{S}$ the initial state
- $P[S_{t+1} = s' | S_t = s]$ the transition probability such that the states satisfy the Markov property:

$$P[S_{t+1}|S_t] = P[S_{t+1}|S_t, \ldots, S_0]$$

If the DTMC is time-homogeneous, meaning that the probability does not depend on time, we can write the transition probabilities in a state transition matrix.

Absorbing states: We can also introduce the notion of absorbing states as "terminal" states of the system in which there's a probability of 1 to stay in the state and a probability of 0 to exit the state:

$$\sum_{s' \neq s_{\text{absorbing}}} P[S_{t+1} = s' | S_t = s_{\text{absorbing}}] = 0, \quad P[S_{t+1} = s_{\text{absorbing}} | S_t = s_{\text{absorbing}}] = 1$$

This is useful when we want to define a finite time horizon process as in our case study.

Relation to property specification: In addition to our definition, we can define a set of atomic propositions AP and a state labelling function $L : \mathscr{S} \rightarrow \mathscr{P}(AP)$ just like in Finite Kripke structures. We can similarly verify properties based on those atomic propositions.

[1] each state has at least one child

(Finite) Markov Reward Process We can add the notion of rewards to Markov Processes (in our case DTMC). A reward can be attached to a state or to a transition.

So our DTMC with rewards can be defined as:

- $\mathscr{S}$ finite set of states
- $s_0 \in \mathscr{S}$ the initial state
- $P[S_{t+1} = s', R_{t+1} = r' | S_t = s]$[2] the probability, from state s, to end up in s' and receive a particular reward r' after one time step.
- $\gamma \in [0,1]$ discount factor weighing the importance of immediate rewards versus future rewards. We'll keep it at $\gamma = 1$ for the rest of the document meaning that we're only working with the expected sum of future rewards.

We can then, for example, compute how "good" it is to be in a given state. This measure is related to how much reward we can expect to earn from a given state.

More formally, we can compute for each state $s \in \mathscr{S}$, the state value

$$v(s) = \mathbb{E}\left[\sum_{i=1}^{\infty} \gamma^{i-1} R_{t+i} \middle| S_t = s\right] \tag{1}$$

where v is the state-value function and γ is the discount factor. The discount factor γ is usually helpful for obtaining a finite expected discounted sum of rewards for continuous tasks (i.e. when can have infinite trajectories with non-zero probability). But this factor also weighs the importance of rewards received over time.

As mentioned previously, we consider $\gamma = 1$ for the rest of the document. This is for the two following reasons:

- To the best of our knowledge, PRISM only handles discount factors of 1.
- While γ is used to weigh the importance of immediate reward vs future reward, in our case study, this is already being taken into account directly through the reward by considering monthly compounding interest.

Note that while $v(s)$ can theoretically be infinite, in our examples and our case study, we always have finite expected cumulative rewards as there is probability of 1 that we end up in an absorbing state.

Depending on the reward distribution, the state-value ($v(s)$) function can have different meanings. For instance, rewards can be used to measure the expected number of steps from the initial state to an absorbing state. We would then be evaluating the state-value where s is the initial state.

Note that we must assign a reward of 0 for transitions past the absorbing state in order to avoid infinitely positive or negative cumulative rewards. Moreover, although Markov Reward Processes include rewards that permit us to specify even more properties that we would like to verify, there is no notion of actions in this environment; it is purely deterministic.

2.1.3 (Finite) Markov Decision Process

In contrast to DTMCs, Finite Markov Decision Processes (MDP) have transitions with no given probabilities. Some actor/player, must then choose a transition from the set of possible transitions, which are then called actions.

A finite MDP can then be defined with the following:

- $\mathscr{S}$ finite set of states
- $\mathscr{A}$ finite set of actions
- $s_0 \in \mathscr{S}$ the initial state
- $P[S_{t+1} = s', R_{t+1} = r' | S_t = s, A_t = a]$[3] the probability to end up in s' and receive a particular reward r' after taking action a from state s.
- $\gamma \in [0,1]$ discount factor weighing the importance of immediate rewards versus future rewards. As with MRPs, we only consider $\gamma = 1$.

[2] Up to our best knowledge, PRISM only supports special cases with deterministic rewards and not stochastic rewards

[3] PRISM only supports, up to our best knowledge, some sub-case of $P[S_{t+1} = s', R_{t+1} = r' | S_t = s, A_t = a]$, for example with positive deterministic rewards

Assigning probabilities to actions in an MDP would revert it back to an MRP, as they are the defining characteristic separating the two models. A possible assignment is called a policy/strategy $\pi : \mathscr{S} \times \mathscr{A} \to [0,1]$. Once we have an MRP, we can compute the state-value function. Note that this state-value function now depends on the strategy, of which there are many (infinite). Due to this, we use in the context of MDP v_π instead of v, where actions are taken under the fixed policy π.

The goal of an agent is to find a policy/strategy that might maximize expected cumulative future rewards, rewards given by the environment.

More formally, the optimal/best policy π^* is obtained by maximizing the expected cumulative reward starting from the initial state:

$$\pi^* = \text{argmax}_\pi \mathbb{E}_\pi \left[\sum_{i=1}^{\infty} R_i \middle| S_0 = s_0 \right] \tag{2}$$

We can similarly define the worst policy which can be of interest if we want to see what would happen if an user behaves in the worst possible way in our system.

Relation to Futures Market Investor case study: As previously mentioned, the market's strategy is fixed in advance, with a set probability to bar the investor from reserving. Because the market's strategy or policy is given, the actions of the market agent can be thought of as part of the environment dynamics. We therefore have a Markov Decision Process, with the investor being the only actor/agent/player.

Relation to property specification: As before, we can define a set of atomic propositions AP and a state labelling function $L : \mathscr{S} \to \mathscr{P}(AP)$, with which we verify some properties.

2.1.4 Turned-based stochastic game

Turned-based stochastic games are extensions of MDP with multiple agents. They are essentially MDP in which states are attached to a finite number of players. Only one player/agent can act from a given state.

Therefore, a TSG is an MDP with a partition of the state-space $\mathscr{S}$ describing which agent can act from which state.

- $\langle \mathscr{S}, \mathscr{A}, s_0, P[S_{t+1} = s', R_{t+1} = r' | S_t = s, A_t = a] \rangle$ an MDP
- N agents/players and $P = \{\mathscr{S}_i\}_{i=1}^N$ a partition of $\mathscr{S}$

The goal for each agent is to learn a good policy $\pi_i : \mathscr{S}_i \times \mathscr{A} \to [0,1]$ where $\mathscr{S}_i$ represents the set of states in which the agent i can act.

This type of model could be used for future works but it also requires a more general property specification language that deals with TSGs [5] as well as a new tool, the PRISM-games extension to PRISM.

2.1.5 Our transition graphs

We represent in Subsection 3.1 the PRISM models that implement the original case study and our extension. To do so, we utilize many separate graphs instead of a single graph that one would expect from MDPs[4].

We choose to represent our PRISM model in this way for many reasons:

- While the PRISM code itself is fairly brief (see Appendix), it represents a huge number of states that cannot all be displayed.
- It provides a visualization of the PRISM implementations by including PRISM code details such as local variables.
- It bridges the gap between the informal figures (See Figure 1a and Figure 1b) and the formal MDP state transition representation by aggregating states.

Each of the transition graphs corresponds to a single PRISM module, which can informally be described with:

- A set of nodes, each representing one or many states of the MDP.

[4] See subsection 2.3 for an example with the classical graphical representation of an MDP

- An initial node representing a set of states of the MDP that includes[5] the initial state of the MDP. This initial node also describes what values the local variables of the PRISM module take initially.
- Each node contains values of some or all local variables[6] that are satisfied in all the states related to the node.
- A node may have additional information noting how a local variable is updated each time we traverse it.
- Dashed or non-dashed directed edges (transitions) represent one or many transitions in the MDP.
- Different quantities can be attached to directed edges:
  - In red: transition name
  - In gray: brief description explaining the transition
  - In green: conditions necessary to move along the transition. These conditions can involve local variables from other modules[7].
  - In black: information telling how a local variable is updated each time we traverse the transition
  - In blue: probability to transition

  When there's a blue and green description/quantity over a transition, it means that we first need the green quantity to be satisfied before moving according to the blue quantity. If it is not satisfied, we stay in the current state.
- Dashed directed edges instead of solid ones are used mostly for perceptual convenience, but it also serves as a reminder that the transition does not necessarily lead to the same state, a variable might have been updated since.
- Green-colored directed edge represents a directed edge with a green quantity/condition.

Note that all of the modules are linked by synchronization, where transitions that have the same name across modules must satisfy their respective conditionals (in green) to move to their respective states. Recall that the basis for our graphical representation is the MDP model defined in PRISM, meaning that there are nodes from which an agent has to decide which action to take. Without a given policy, the model is non-deterministic.

2.2 Property Specification and Verification

Recall that in general, we have this following scheme:

- Define model
- Define properties we would like to verify
- Verify properties

This subsection mostly deals with the syntax and semantics of the CTL and Probabilistic Computation Tree Logic (PCTL) formulas. This subsection does not cover the algorithms behind property verification.

This subsection also assumes that the reader is familiar with CTL as we only briefly recall formal descriptions.

Specifying properties for Finite Kripke Structures can be done with CTL formulas. However, CTL does not support probabilities and as such cannot verify properties in probabilistic models such as DTMC. PCTL is an extension of CTL that solves this issue, and can be used for DTMCS. PRISM uses its own property specification language which includes PCTL among others. We only present PCTL, as it is sufficient to understand the PRISM language and how it related to our work.

The reader is also invited to read the PRISM documentation on property specification and reward-based properties in order to specify properties for MDPs as some cannot be specified due to the MDP non-determinism. Some examples of this are listed at the end of this subsection.

[5] A set of states instead of the single initial state due to the so-called synchronization of PRISM modules

[6] The values of the local variables that are not described can be deduced from the values written in the initial node as well as from other nodes or transitions.

[7] e.g for module `barred`, there's no local variable `i` and for module `investor`, there's no local variable `b`

2.2.1 CTL: Computation Tree Logic

Minimal syntax for specifying properties: Let's denote CTL to be the set of CTL formulas.

- $a \in AP \Rightarrow a \in CTL$
- $\Phi_1, \Phi_2 \in CTL \Rightarrow \neg\Phi_1, \Phi_1 \vee \Phi_2, \mathbf{EX}\Phi_1, \mathbf{EG}\Phi_1, \Phi_1\mathbf{EU}\Phi_2 \in CTL$

Semantics of minimal syntax: Let's denote by $[[\Phi]]$ the set of states satisfying $\Phi \in CTL$ for a Kripke structure. Let's also denote by ρ a run/path within the transition system: $\rho : \mathbb{N} \to \mathscr{S}$.

- $[[a]] = v(a)$ if we have $v : AP \to \mathscr{P}(S)$ that gives the set of states satisfying the atomic proposition a. Otherwise, one can use the labeling function L to retrieve it.
- $[[\neg\Phi_1]] = S \setminus [[\Phi_1]]$
- $[[\Phi_1 \vee \Phi_2]] = [[\Phi_1]] \cup [[\Phi_2]]$
- $[[\mathbf{EX}\Phi]] = \{s \in \mathscr{S} | \exists t \in \mathscr{S} \text{ s.t } s \to t \text{ and } t \in [[\Phi]]\}$ (predecessors of states satisfying Φ)
- $[[\mathbf{EG}\Phi]] = \{s \in \mathscr{S} | \exists \text{ run/path } \rho \text{ with } \rho(0) = s \text{ and } \rho(i) \in [[\Phi]], \forall i \geq 0\}$ (greatest fixed point)
- $[[\Phi_1\mathbf{EU}\Phi_2]] = \{s \in \mathscr{S} | \exists \text{ run/path } \rho \text{ with } \rho(0) = s \text{ and } k \geq 0 \text{ s.t } \rho(i) \in [[\Phi_1]], \forall i < k \text{ and } \rho(k) \in [[\Phi_2]]\}$

To compute the two last, we can use fixed point computations in which we either start from $\mathscr{S}$ and reduce the set (called greatest fixed point and it uses ν) or start from $\emptyset$ and increase the set (called least fixed point and it uses μ).

- $[[\mathbf{EG}\Phi]] = \nu Y.[[\Phi]] \cap pre_\exists(Y)$ (greatest fixed point)
- $[[\Phi_1\mathbf{EU}\Phi_2]] = \mu Y.[[\Phi_2]] \cup ([[\Phi_1]] \cap pre_\exists(Y))$

where

$$pre_\exists(Y) \doteq \{s \in \mathscr{S} | \exists t \in \mathscr{S} \text{ s.t } s \to t \text{ and } t \in Y\} \tag{3}$$

is the set of states containing the predecessors of states in Y.

Extended syntax: Let's define $\Phi_1 \equiv \Phi_2 \iff \forall \mathscr{K}, [[\Phi_1]]_\mathscr{K} = [[\Phi_2]]_\mathscr{K}$ (Set of states satisfying Φ_1 are the same as the ones satisfying Φ_2 for any Kripke structure)

- $true \equiv a \vee \neg a$
- $false \equiv \neg true$
- $\Phi_1 \wedge \Phi_2 \equiv \neg(\neg\Phi_1 \vee \neg\Phi_2)$
- $\mathbf{AX}\Phi \equiv \neg\mathbf{EX}\neg\Phi$
- $\mathbf{AG}\Phi \equiv \neg\mathbf{EF}\neg\Phi$
- $\mathbf{EF}\Phi \equiv true\mathbf{EU}\Phi$
- $\mathbf{AF}\Phi \equiv \neg\mathbf{EG}\neg\Phi$
- $\Phi_1\mathbf{AU}\Phi_2 \equiv (\Phi_1\mathbf{AW}\Phi_2) \wedge \mathbf{AF}\Phi_2$ (all (paths/runs/executions) strong until. Φ_2 holds eventually.)
- $\Phi_1\mathbf{AW}\Phi_2 \equiv \neg(\neg\Phi_2\mathbf{EU}\neg(\Phi_1 \vee \Phi_2))$
- $\Phi_1\mathbf{EW}\Phi_2 \equiv (\Phi_1\mathbf{EU}\Phi_2) \vee (\mathbf{EG}\Phi_1)$ (exist (path/run) weak until)

Note that we can extend it even further (e.g $\Longrightarrow$). The semantic is not extended by the extended syntax.

Remarks: To remember the formulas for fixed points, $\mathbf{E}X, \mathbf{E}F, \mathbf{E}G$ use $pre_\exists$ (exist a path) while $\mathbf{A}X, \mathbf{A}F, \mathbf{A}G$ use $pre_\forall$ (see SMV course). Each time we see F (finally), we use an **union** (so least fixed point) and each time we see a G (globally), we use an **intersection** (so greatest fixed point) (because we want something globally true on the whole path).

2.2.2 CTL with state and path formulas

Minimal syntax: Inspired by a PRISM lecture on probabilistic temporal logic, our CTL minimal syntax can also be split into a set of state formulas $\mathscr{F}_S$ and a set of path formulas $\mathscr{F}_P$ where $CTL = \mathscr{F}_S$. This decomposition can help better understand the PCTL syntax as PCTL also uses state or path formulas.

- $a \in AP \Rightarrow a \in \mathscr{F}_S$
- $\Phi_1, \Phi_2 \in \mathscr{F}_S \Rightarrow \neg\Phi_1, \Phi_1 \vee \Phi_2 \in \mathscr{F}_S$
- $\Psi \in \mathscr{F}_P \Rightarrow \mathbf{E}\Psi \in \mathscr{F}_S$
- $\Phi_1, \Phi_2 \in \mathscr{F}_S \Rightarrow \mathbf{X}\Phi_1, \mathbf{G}\Phi_1, \Phi_1\mathbf{U}\Phi_2 \in \mathscr{F}_P$

State formulas are for properties of states while path formulas are for properties of paths.

Semantics of minimal syntax: Splitting the minimal syntax leads to a rewriting of the semantics (but the meaning/semantics of the CTL formulas do not change). Let's recall that $[[\cdot]]$ represents a set of states. We also denote by $[[\Psi]]^P$ the set of paths that satisfy $\Psi \in \mathscr{F}_P$.

Inspired by the PRISM lecture on probabilistic temporal logic, our semantics can also be rewritten as:

- Semantics of state formulas:
  - $[[a]] = \nu(a)$ if we have $\nu : AP \rightarrow \mathscr{P}(S)$ that gives the set of states satisfying the atomic proposition a. Otherwise, one can use the labeling function L to retrieve it.
  - $[[\neg\Phi_1]] = S \setminus [[\Phi_1]]$
  - $[[\Phi_1 \vee \Phi_2]] = [[\Phi_1]] \cup [[\Phi_2]]$[8]
  - $[[\mathbf{E}\Psi]] = \{s \in \mathscr{S} | \exists \text{ run/path } \rho \text{ with } \rho(0) = s \text{ and } \rho \in [[\Psi]]^P\}, \quad \mathbf{E}\Psi \in \mathscr{F}_S$
- Semantics of path formulas:
  - $[[\mathbf{X}\Phi_1]]^P = \{\rho | \rho(1) \in [[\Phi_1]]\}, \quad \mathbf{X}\Phi_1 \in \mathscr{F}_P$
  - $[[\mathbf{G}\Phi_1]]^P = \{\rho | \rho(i) \in [[\Phi_1]], \forall i \geq 0\}, \quad \mathbf{G}\Phi_1 \in \mathscr{F}_P$
  - $[[\Phi_1\mathbf{U}\Phi_2]]^P = \{\rho | \exists k \geq 0 \text{ s.t } \rho(i) \in [[\Phi_1]], \forall i < k \text{ and } \rho(k) \in [[\Phi_2]]\}, \quad \Phi_1\mathbf{U}\Phi_2 \in \mathscr{F}_P$

where $\Phi_1, \Phi_2 \in \mathscr{F}_S, a \in AP, \Psi \in \mathscr{F}_P$

2.2.3 PCTL: Probabilistic Computation Tree Logic

Specifying properties of a DTMC can be done via PCTL. Although CTL can work on the underlying Finite Kripke Structure of a probabilistic model such as DTMCs or MDPs[9], it cannot verify properties that has :

- A time component (something that holds within some period of time)
- Probabilities

Informally, we can say that PCTL is an extension of CTL but with a probability that something is true within some time units.

PCTL also applies to DTMCs instead of Finite Kripke Structure. PRISM's logics includes extensions of PCTL with rewards and with quantitative properties, properties that return numerical values.

Specifying properties using PCTL formulas: Our formulas are adapted or deduced from the paper of the authors of PCTL [3] in order to look similar to what is used in PRISM. The PCTL syntax definition in that paper [3] doesn't have the "next" operator nor a derivation/equivalence from its minimal syntax. And because a paper from the authors of PRISM [2], as well as from the PRISM lectures, contain the "next" operator in their minimal syntax, we also added the "next" operator to our minimal syntax.

[8] As defined in CTL

[9] In which we ignore all the transition probabilities and the rewards. In PRISM, we can still verify CTL formulas even though the model is a DTMC or MDP.

Minimal syntax: Let's denote $PCTL = \mathscr{F}_S$ to be the set of PCTL formulas where $\mathscr{F}_S$ is the set of state formulas and $\mathscr{F}_P$ is the set of path formulas (see original paper [3])

- $a \in AP \Rightarrow a \in \mathscr{F}_S$
- $\Phi_1, \Phi_2 \in \mathscr{F}_S \Rightarrow \neg\Phi_1, \Phi_1 \vee \Phi_2 \in \mathscr{F}_S$
- $\Phi_1, \Phi_2 \in \mathscr{F}_S$ and $t \in \mathbb{N} \cup \{\infty\} \Rightarrow (\Phi_1 \mathbf{U}^{\leq t} \Phi_2), (\Phi_1 \mathbf{W}^{\leq t} \Phi_2) \in \mathscr{F}_P$[10] where **U** is for strong until and **W** for weak until[11]
- $\Psi \in \mathscr{F}_P$ and $p \in [0,1] \Rightarrow P_{\geq p}[\Psi], P_{>p}[\Psi] \in \mathscr{F}_S$
- $\Phi_1 \in \mathscr{F}_S \Rightarrow \mathbf{X}\Phi_1 \in \mathscr{F}_P$[12]

State formulas are for properties of states while path formulas are for properties of paths. Moreover, recall that we're working with Discrete Time Markov Chains so t represents t discrete time units.

Semantics of minimal syntax: Let's denote by $[[\Phi]]$ the set of states satisfying $\Phi \in \mathscr{F}_S$ and denote by $[[\Psi]]^P$ the set of paths satisfying $\Psi \in \mathscr{F}_P$.

Our semantics are as follows and where in gray, we have parts that are similar to what we had in CTL with state and path formulas:

- Semantics of state formulas:
  - $[[a]] = v(a)$ if we have $v : AP \to \mathscr{P}(S)$ that gives the set of states satisfying the atomic proposition a. Otherwise, one can use the labeling function L to retrieve it.
  - $[[\neg\Phi_1]] = S \setminus [[\Phi_1]]$
  - $[[\Phi_1 \vee \Phi_2]] = [[\Phi_1]] \cup [[\Phi_2]]$[13]
  - $[[P_{\geq p}[\Psi]]] = \{s \in \mathscr{S} | \Pr(\{\text{run/path } \rho \in [[\Psi]]^P | \rho(0) = s\}) \geq p\}, \quad P_{\geq p}[\Psi] \in \mathscr{F}_S$
  - $[[P_{>p}[\Psi]]] = \{s \in \mathscr{S} | \Pr(\{\text{run/path } \rho \in [[\Psi]]^P | \rho(0) = s\}) > p\}, \quad P_{>p}[\Psi] \in \mathscr{F}_S$
- Semantics of path formulas:
  - $[[\Phi_1 \mathbf{U}^{\leq t} \Phi_2]]^P = \{\rho | \exists k \geq 0 \text{ with } k \leq t \text{ s.t } \rho(i) \in [[\Phi_1]], \forall i < k \text{ and } \rho(k) \in [[\Phi_2]]\}, \quad \Phi_1 \mathbf{U}^{\leq t} \Phi_2 \in \mathscr{F}_P$
  - $[[\Phi_1 \mathbf{W}^{\leq t} \Phi_2]]^P = [[\Phi_1 \mathbf{U}^{\leq t} \Phi_2]]^P \cup \{\rho | \rho(i) \in [[\Phi_1]], \forall i \geq 0 \text{ s.t. } i \leq t\}, \quad \Phi_1 \mathbf{W}^{\leq t} \Phi_2 \in \mathscr{F}_P$
  - $[[\mathbf{X}\Phi_1]]^P = \{\rho | \rho(1) \in [[\Phi_1]]\}, \quad \mathbf{X}\Phi_1 \in \mathscr{F}_P$

where $\Phi_1, \Phi_2 \in \mathscr{F}_S, a \in AP, \Psi \in \mathscr{F}_P$. Note that "negation on probabilities flips" the bound: e.g. $[[\neg P_{\geq p}[\Psi]]] = [[P_{<p}[\Psi]]]$

Extended syntax: Let's define $\Phi_1 \equiv \Phi_2 \iff \forall \mathscr{M}, [[\Phi_1]]_{\mathscr{M}} = [[\Phi_2]]_{\mathscr{M}}$ (Set of states satisfying Φ_1 are the same as the ones satisfying Φ_2 for any DTMC.). Similarly, we can also define the equivalence between two path formulas $\Psi_1 \equiv^P \Psi_2$.

- $true \equiv a \vee \neg a$
- $false \equiv \neg true$
- $\Phi_1 \wedge \Phi_2 \equiv \neg(\neg\Phi_1 \vee \neg\Phi_2)$
- $\Phi_1 \implies \Phi_2 \equiv \neg\Phi_1 \vee \Phi_2$
- $\Phi_1 \mathbf{U} \Phi_2 \equiv^P \Phi_1 \mathbf{U}^{\leq\infty} \Phi_2$
- $\Phi_1 \mathbf{W} \Phi_2 \equiv^P \Phi_1 \mathbf{W}^{\leq\infty} \Phi_2$
- $\mathbf{F}\Phi \equiv^P true \mathbf{U}^{\leq\infty} \Phi$
- $\mathbf{F}^{\leq k}\Phi \equiv^P true \mathbf{U}^{\leq k} \Phi$
- $\mathbf{G}\Phi \equiv^P \Phi \mathbf{W}^{\leq\infty} false$

[10] They are the so-called bounded variants of path properties. In PRISM, to get the unbounded version, we have to remove $\leq t$. Otherwise, in our formal description, we can have $t = \infty$

[11] The weak until does not require Φ_2 to hold

[12] As the paper on PCTL [3] doesn't explain how to obtain the next operator and after fruitless attempts to derive it from the 4 previous points, we just added it directly to the minimal syntax

[13] As defined in CTL

Example: A state s satisfying $P_{\geq p}[\Phi_1 \mathbf{U}^{\leq t} \Phi_2]$ means that there's at least a probability of p that Φ_2 will be satisfied within t time units (and that Φ_1 is satisfied in all the states in between).

Relation with EX, EG and EU of CTL: Intuitively, from the brief example, we might already see a relation with CTL by varying that probability and time-bound t, what if $p > 0, t = \infty$? What if $p = 1, t = \infty$? A non-zero probability means "there exists a path" while a probability of 1 is similar but weaker to "for all the paths" as there are infinite paths with 0 probability.

We denote with $\leftrightarrow$ the relation between PCTL and CTL.

- $\mathbf{EX}\Phi \leftrightarrow P_{>0}[\mathbf{X}\Phi]$
- $\mathbf{EG}\Phi \leftrightarrow P_{>0}[\mathbf{G}\Phi]$
- $\Phi_1 \mathbf{EU} \Phi_2 \leftrightarrow P_{>0}[\Phi_1 \mathbf{U} \Phi_2]$

(Optional) Relation with the extended syntax of CTL: We can also write the following relations between the extended syntax of CTL and the extended syntax of PCTL. However, they're not completely equivalent for **A** (all). We also omit "true", "false" and "and".

- $\mathbf{AX}\Phi \leftrightarrow P_{\geq 1}[\mathbf{X}\Phi]$
- $\mathbf{AG}\Phi \leftrightarrow P_{\geq 1}[\mathbf{G}\Phi]$
- $\mathbf{EF}\Phi \leftrightarrow P_{>0}[\mathbf{F}\Phi]$
- $\mathbf{AF}\Phi \leftrightarrow P_{\geq 1}[\mathbf{F}\Phi]$
- $\Phi_1 \mathbf{AU} \Phi_2 \leftrightarrow P_{\geq 1}[\Phi_1 \mathbf{U} \Phi_2]$
- $\Phi_1 \mathbf{AW} \Phi_2 \leftrightarrow P_{\geq 1}[\Phi_1 \mathbf{W} \Phi_2]$
- $\Phi_1 \mathbf{EW} \Phi_2 \leftrightarrow P_{>0}[\Phi_1 \mathbf{W} \Phi_2]$

PCTL is an extension of CTL: The **E** (exists) and a weaker version of **A** (all) of CTL are hidden behind the probabilities. Essentially, to go from **E** (exist) to a weaker version of **A** (all), we just need to change the probability from > 0 to ≥ 1.

PCTL can work with a weak version of **A** (all) where paths with 0 probabilities are implicitly ignored whereas CTL does take them into account.

However, we can still derive all the CTL operators (syntax without the state/path formulas separation) from PCTL by using:

- $\equiv$-equivalence of (P)CTL extended syntax with (P)CTL minimal syntax
- $\leftrightarrow$-relation between CTL minimal syntax and PCTL minimal or extended syntax.

Key differences between PCTL and CTL

- We can have a "portion" of the paths instead of at least 1 or all paths (Generally more freedom on the quantification over paths).
- We can specify properties that hold during a specify time interval (More freedom on the quantification over time).

2.2.4 Extensions

We informally describe below a few extensions of PCTL with rewards and with quantitative properties, i.e. properties that return numerical values.

Extension with rewards: Instead of verifying properties on probabilities, we can verify properties on rewards. For instance, for a DTMC extended with rewards, we can verify if the expected sum of future rewards from the initial state satisfies some bound.

The DTMC extended with rewards can have many reward structures. Therefore, reward-based properties are conditioned on the reward structure. More information on different reward-based properties is described in their manual.

Extension with quantitative properties/rewards: We might sometimes want to ask what a particular probability or expected reward is instead of asking whether it satisfies some bound. This is done via quantitative properties:

- $P_{=?}[\cdot]$, $P_{min=?}[\cdot]$ and $P_{max=?}[\cdot]$ for quantitative versions of P (See page 27 of [2] and the PRISM manual)
- $R^r_{=?}[\cdot]$, $R^r_{min=?}[\cdot]$ and $R^r_{max=?}[\cdot]$ for quantitative versions of R[14] (See page 30 of [2] and the PRISM manual)

These operators cannot be nested within PCTL formulas. They are the outermost operators.

The two quantitative properties $R^r_{min=?}[\cdot]$ and $R^r_{max=?}[\cdot]$ in which $\cdot$ is replaced by an expression representing the absorbing states, ask what are the worst and best expected cumulative reward obtainable (e.g in a MDP) from the initial state.

$$\min_{\pi} \mathbb{E}_{\pi,r}\left[\sum_{t=1}^{\infty} R_t \middle| S_0 = s_0\right], \quad \max_{\pi} \mathbb{E}_{\pi,r}\left[\sum_{t=1}^{\infty} R_t \middle| S_0 = s_0\right] \tag{4}$$

where π is the policy and r is the reward structure.

Property specification in PRISM: PRISM property specification language syntax is very similar to what we described with PCTL and its extensions with rewards and quantitative properties.

Here's an example of how we can specify that we use the reward structure named "steps":

```
R{"steps"}max=? [F is_done=1]
```

Some properties cannot be used for MDP due to its non-determinism. For instance, we cannot use `R=?` and `P=?`, we need to use `Rmax=?` or `Rmin=?` and `Pmax=?` or `Pmin=?` instead. Intuitively, we understood it as:

- We need to resolve the non-determinism by giving some strategy/policy as the behavior of the whole system depends on them.
- Min and max are related to two strategies, the worst one and the best one that would lead to the worst or best numerical value (depending on what property we want to verify).

Note that `Pmax=?` or `Pmin=?` do not depend on the reward structure.

[14] The lower case r represents the reward structure.

2.3 Examples

We now present examples from the formalism given above. We then show how they can be encoded in PRISM. This should give the reader an intuition on how DTMC's and MDP's, in general, are modeled in PRISM, and how to represent and verify properties in the PRISM tool.

Dice example: The dice example from PRISM tutorial Part 1, which tries to model a throw of a 6-face fair dice using fair coins, can be formally described via a DTMC:

- $\mathscr{S} \subseteq \{0,1,\ldots,7\} \times \{0,1,\ldots,6\}$ the set of reachable states where the first dimension represents `s` and the second dimension represents `d`, the dice number.
- $s_0 = (0,0) \in \mathscr{S}$ the initial state
- $P[S_{t+1} = s' | S_t = s] = T_{s,s'}$ the transition probability matrix (row stochastic matrix) in which we omit the unreachable states:

	(0, 0)	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)	(6, 0)	(7, 1)	(7, 2)	(7, 3)	(7, 4)	(7, 5)	(7, 6)
(0, 0)	0	0.5	0.5	0	0	0	0	0	0	0	0	0	0
(1, 0)	0	0	0	0.5	0.5	0	0	0	0	0	0	0	0
(2, 0)	0	0	0	0	0	0.5	0.5	0	0	0	0	0	0
(3, 0)	0	0.5	0	0	0	0	0	0.5	0	0	0	0	0
(4, 0)	0	0	0	0	0	0	0	0	0.5	0.5	0	0	0
(5, 0)	0	0	0	0	0	0	0	0	0	0	0.5	0.5	0
(6, 0)	0	0	0.5	0	0	0	0	0	0	0	0	0	0.5
(7, 1)	0	0	0	0	0	0	0	1	0	0	0	0	0
(7, 2)	0	0	0	0	0	0	0	0	1	0	0	0	0
(7, 3)	0	0	0	0	0	0	0	0	0	1	0	0	0
(7, 4)	0	0	0	0	0	0	0	0	0	0	1	0	0
(7, 5)	0	0	0	0	0	0	0	0	0	0	0	1	0
(7, 6)	0	0	0	0	0	0	0	0	0	0	0	0	1

  $s = 7$ means that we have decided a dice number, thus reaching an absorbing/terminal state, where there is a probability of 1 (in blue) to loop.

The DTMC graphical representation from the tutorial is shown in Figure 2a. The PRISM encoding or description of the model is:

```
dtmc

module die
    // local state
    s : [0..7] init 0;
    // value of the die
    d : [0..6] init 0;

    [] s=0 -> 0.5 : (s'=1) + 0.5 : (s'=2);
    [] s=1 -> 0.5 : (s'=3) + 0.5 : (s'=4);
    [] s=2 -> 0.5 : (s'=5) + 0.5 : (s'=6);
    [] s=3 -> 0.5 : (s'=1) + 0.5 : (s'=7) & (d'=1);
    [] s=4 -> 0.5 : (s'=7) & (d'=2) + 0.5 : (s'=7) & (d'=3);
    [] s=5 -> 0.5 : (s'=7) & (d'=4) + 0.5 : (s'=7) & (d'=5);
    [] s=6 -> 0.5 : (s'=2) + 0.5 : (s'=7) & (d'=6);
    [] s=7 -> (s'=7);
endmodule
```

A property we can check is the probability to end up in an absorbing state with a particular dice number x by throwing fair coins successively and moving in our DTMC. It should be around $\frac{1}{6}$ because it's a fair dice.

This corresponds to the following quantitative property, which is in the extension of PCTL to quantitative properties:

$$P_{=?}[\mathbf{F}(s = 7) \wedge (d = x)] \tag{5}$$

This formula can also be written in PRISM in a similar manner:

```
P=? [ F s=7 & d=x ]
```

where x is defined as a constant[15]. Running "Verify" would give us, using some iterative algorithm, a value close to $\frac{1}{6}$ for the probability.

If instead of asking for the value, we wish to check whether the value of the probability satisfies some condition, for instance, if the probability is greater than 0.166, we can specify the PCTL formula $P_{>0.166}[\mathbf{F}(s = 7) \wedge (d = x)]$ which corresponds to the following in PRISM:

[15] can be specified in the properties file using `const int x;`

```
P>0.166 [ F s=7 & d=x ]
```

and should return true.

A simple MDP example: Suppose we have an MDP where a player can choose to crouch or jump but has some probability to die depending on the actions chosen.

Let's say that the agent receives a reward of 1 for every time step without dying and a reward of 0 otherwise. Let's also say that crouching leads to a probability of living of 0.8 while jumping leads to a probability of living of 0.3. Clearly, the best action to pick would be to crouch.

- $\mathscr{S} = \{\text{dead}, \text{alive}\}$ finite set of states.
- $\mathscr{A}(\text{dead}) = \{\text{noop}\}, \mathscr{A}(\text{alive}) = \{\text{jump}, \text{crouch}\}$ finite set of actions
- $s_0 = \text{alive} \in \mathscr{S}$ the initial state
- $P[S_{t+1} = s', R_{t+1} = r' | S_t = s, A_t = a]$ the probability to end up in s' and receive a particular reward r' after taking action a from state s:
  - $P[S_{t+1} = \text{alive}, R_{t+1} = 1 | S_t = \text{alive}, A_t = \text{crouch}] = 0.8$
  - $P[S_{t+1} = \text{dead}, R_{t+1} = 0 | S_t = \text{alive}, A_t = \text{crouch}] = 0.2$
  - $P[S_{t+1} = \text{alive}, R_{t+1} = 1 | S_t = \text{alive}, A_t = \text{jump}] = 0.3$
  - $P[S_{t+1} = \text{dead}, R_{t+1} = 0 | S_t = \text{alive}, A_t = \text{jump}] = 0.7$
  - $P[S_{t+1} = \text{dead}, R_{t+1} = 0 | S_t = \text{dead}, A_t = \text{noop}] = 1$

  All the other probabilities are 0. Given a policy for our agent, it can expect to stay alive for a longer period of time or not.

The MDP graphical representation is shown in Figure 2b. The PRISM similar encoding or similar description of the model is:

```
mdp

module mdp_example
    s: [0..1] init 1;  // alive initially

    // non-deterministic choice between jump and crouch from s=1
    [jump] (s=1) -> 0.3: (s'=1) + 0.7: (s'=0);
    [crouch] (s=1) -> 0.8: (s'=1) + 0.2: (s'=0);
    [noop] (s=0) -> (s'=0);  // absorbing states
endmodule

rewards
    (s=1) : 1;  // reward for staying alive
    (s=0) : 0;  // died
endrewards

rewards "steps"
    true : 1;
endrewards
```

However, it's not completely faithful to our formal definition of our model, as the rewards here are based on states instead of transitions. We need to remove 1 from the cumulative reward in order to get the correct expected sum of rewards, as we receive this extra reward from starting at the initial state of alive.

Two quantitative properties we can ask, on this modified MDP, are what are the worst and best expected cumulative reward obtainable:

```
Rmin=? [ F s=0 ]
Rmax=? [ F s=0 ]
```

`Rmin=? [ F s=0 ]` gives us $1 + \frac{0.3}{1-0.3} \approx 1.428571$ and `Rmax=? [ F s=0 ]` gives us[16] $1 + \frac{0.8}{1-0.8} = 5$.

Therefore, the worst and best expected cumulative reward obtainable in our original simple MDP example are $E_{min} = \frac{0.3}{1-0.3} \approx 0.428571$ and $E_{max} = \frac{0.8}{1-0.8} = 4$ respectively.

These expected cumulative rewards correspond to the two cases:

- When the strategy is to always pick `jump` when alive:

$$\begin{aligned} E_{min} &= 0.3 \cdot (E_{min} + 1) \\ (1 - 0.3) E_{min} &= 0.3 \\ E_{min} &= \frac{0.3}{1 - 0.3} \approx 0.428571 \end{aligned} \tag{6}$$

[16] PRISM gave something very close to 5 but the exact value is 5

- When the strategy is to always pick `crouch` when alive:

$$\begin{aligned} E_{max} &= 0.8 \cdot (E_{max} + 1) \\ (1 - 0.8) E_{max} &= 0.8 \\ E_{max} &= \frac{0.8}{1 - 0.8} = 4 \end{aligned} \tag{7}$$

Our MDP example with a fixed strategy: If we fix the stochastic policy or strategy, meaning that $\pi(\text{alive}, \text{crouch}) = p_{\text{crouch}}$ and $\pi(\text{alive}, \text{jump}) = 1 - p_{\text{crouch}}$ are fixed, we fall back to a Markov Reward Process in which we can use PCTL to specify some properties.

The DTMC with rewards (Finite Markov Reward Process) is then defined by

- $\mathscr{S} = \{\text{dead}, \text{alive}\}$ finite set of states.
- $s_0 = \text{alive} \in \mathscr{S}$ the initial state
- $P[S_{t+1} = s', R_{t+1} = r' | S_t = s]$ the probability to end up in s' and receive a particular reward r' after moving from state s:
  - $P[S_{t+1} = \text{alive}, R_{t+1} = 1 | S_t = \text{alive}] = 0.8 \cdot p_{\text{crouch}} + 0.3 \cdot (1 - p_{\text{crouch}})$
  - $P[S_{t+1} = \text{dead}, R_{t+1} = 0 | S_t = \text{alive}] = 0.2 \cdot p_{\text{crouch}} + 0.7 \cdot (1 - p_{\text{crouch}})$
  - $P[S_{t+1} = \text{dead}, R_{t+1} = 0 | S_t = \text{dead}] = 1$

  All the other probabilities are 0.

The MRP graphical representation is left to the imagination of the reader. The PRISM similar encoding or similar description of the model is

```
dtmc

const double p_crouch;  // defines the policy

// MDP example with a given policy -> falls back to MRP
// p_crouch = 0.5 -> random policy
module mdp_example_random_policy
    s: [0..1] init 1;  // alive initially

    // Jump and crouch according to policy from s=1
    [jump_or_crouch] (s=1) -> 0.3*(1-p_crouch): (s'=1)  // jump
                + 0.7*(1-p_crouch): (s'=0)  // jump
                + 0.8*p_crouch: (s'=1)  // crouch
                + 0.2*p_crouch: (s'=0); // crouch
    [noop] (s=0) -> (s'=0);  // absorbing states
endmodule

rewards
    (s=1) : 1;  // reward for staying alive
    (s=0) : 0;  // died
endrewards

rewards "steps"
    true : 1;
endrewards
```

As before, the rewards for the default reward structure don't completely match. We need to subtract 1 from the cumulative reward in order to get the correct expected sum of rewards.

We show below three examples we can specify with CTL, PCTL, PRISM's logics respectively:

- CTL: Exist path where always alive: $\mathbf{EG}\, s = 1$ (In PRISM, it's written as `E [G s=1]`)
- PCTL (with extended syntax): Exist a path where alive for at least 10 steps: $P_{>0}[\mathbf{G}^{\leq 10}\, s = 1]$ (In PRISM, it's written as `P>0[G<=10 s=1]`)
- PRISM can get the value of that probability (quantitative property) $P_{=?}[\mathbf{G}^{\leq 10} s = 1]$ (`P=?[G<=10 s=1]` in PRISM)

and all of them, from the initial state $s_0 = \text{alive}$.

If we set p_{crouch} to 1, we get that:

- CTL: $\mathbf{EG}\, s = 1$ gives true.
- PCTL (with extended syntax): $P_{>0}[\mathbf{G}^{\leq 10}\, s = 1]$ gives true.
- PRISM `P=?[G<=10 s=1]` gives $0.10737418239999985 \approx 0.8^{10} = P[S_{t+1} = \text{alive}, R_{t+1} = 1 | S_t = \text{alive}]^{10}$.

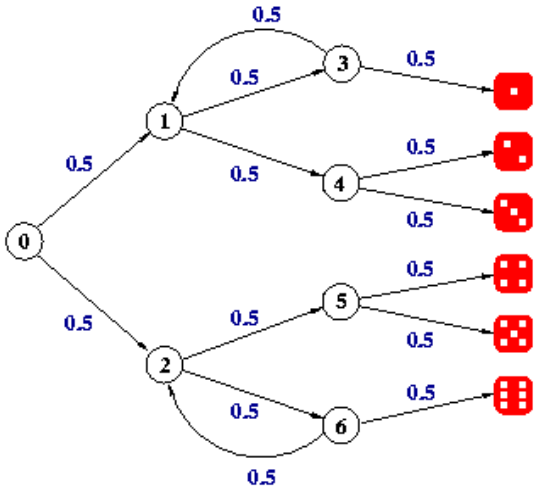
And if we set p_{crouch} to 0, we get that:

- CTL: $\mathbf{EG}\, s = 1$ gives true.
- PCTL (with extended syntax): $P_{>0}[\mathbf{G}^{\leq 10}\, s = 1]$ gives true.
- PRISM `P=?[G<=10 s=1]` gives $5.904900000075486 \cdot 10^{-6} \approx 0.3^{10} = P[S_{t+1} = \text{alive}, R_{t+1} = 1 | S_t = \text{alive}]^{10}$.

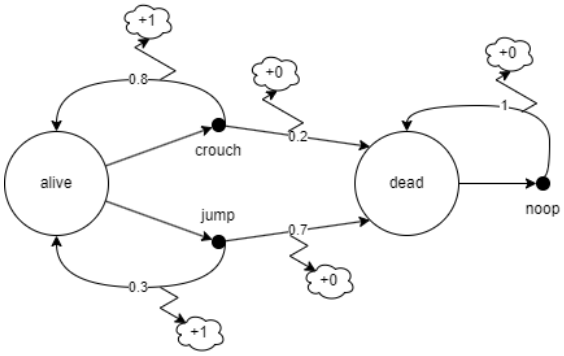
To observe why we said that a probability of 1 doesn't exactly correspond to all the paths, we can compare what PRISM gives us when we check:

- CTL: All paths will lead to the agent dying: $\mathbf{AF}\, s = 0$ (In PRISM, it's written as `A [F s=0]`)
- PCTL (with extended syntax): Agent will die with probability 1: $P_{\geq 1}[\mathbf{F} s = 0]$ (In PRISM, it's written as `P>=1 [ F s=0 ]`)

The CTL formula gives false because there are two paths, from the initial state, from which we can stay alive indefinitely. But these two infinite paths have zero probabilities no matter the strategy, meaning that the PCTL formula gives true, as the probability to die is 1.



(a) Dice example, image taken from the tutorial



(b) Our simple MDP example

3 Implementation

3.1 Model

In the following, we represent our PRISM model using a graphical representation of the three main modules (month, investor, market) (See 2.1.5 for some explanation), textual descriptions of other modules (cap, value etc.) and textual description of the reward structures.

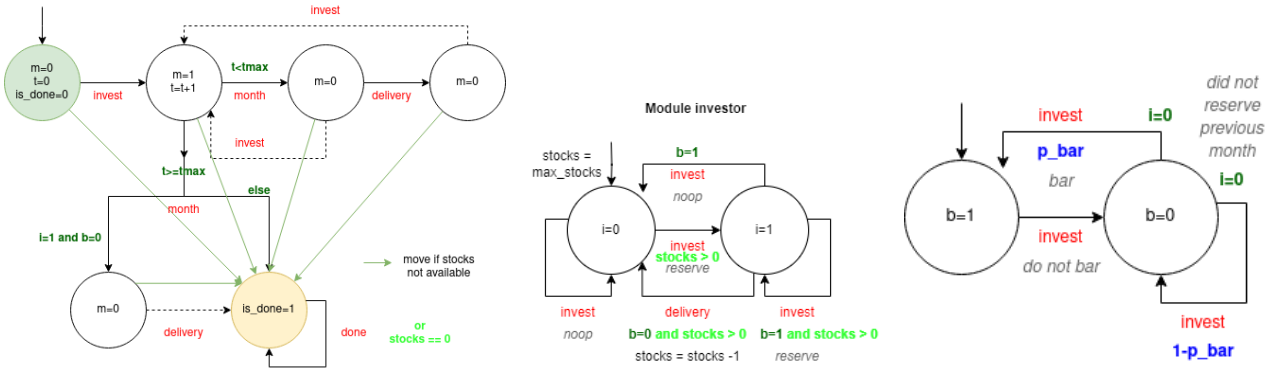
The model is parameterized by:

- `p_bar` : the probability of the market to bar
- `interest` : the fixed monthly interest rate from keeping money in savings
- `v_init` : the initial market price
- `tmax` : the number of months
- `max_stocks` : the number of available stocks to sell

Changing these parameters can lead to different quantitative properties as well as a different number of states in the resulting MDP.

We can describe most of the model's behavior via three main modules, the interactions between them, and the different reward structures.

The month module, shown in Figure 3a, is the backbone module that the investor and barred/market modules will synchronize their actions with. It represents the passing of time with transitions occurring at the start, during, and end of each month. The variable `m` represents this, with `m=1` being the start of the month, `m=0` being the end, so the transition between the two is the "during". We have synchronized actions `[invest]` , `[delivery]` , and `[month]` for the aforementioned time of the month respectively. These will be discussed with their respective modules. The end condition is also monitored in this module, moving to the absorbing state with value `is_done=1` when the investor is out of stocks or when we reach the time limit.



(a) Month Module (b) Investor Module (c) Barred Module (Market module)

The investor module, as shown in Figure 3b, represents the actions the investor takes, which are to reserve or not the sell at the beginning of the month, at synchronized action `[invest]` , and to perform the sell at the end of the month, at synchronized action `[delivery]` . Delivery also decrements the number of shares available to the investor, as they were just sold.

The barred/market module, as shown in Figure 3c, represents the actions the market takes, which is to either bar the investor from successfully reserving the sell or not. This is notably independent of the action the investor decides to take at the same time, since they are synchronized on `[invest]` , but the market can only bar if the investor did not invest the previous month. Note that this model, as with the original case study, is a Markov Decision Process with a single player (investor), since the strategy of the market is still fixed.

The remaining modules, for the value of the share, the probability of the value increasing/decreasing, and for the cap[17], update their values as described in subsection 1.3. The evolution of the value of this stock can be represented with a time series. An example is presented in Figure 4.

[17] ceiling of the value of the share that can decrease over time

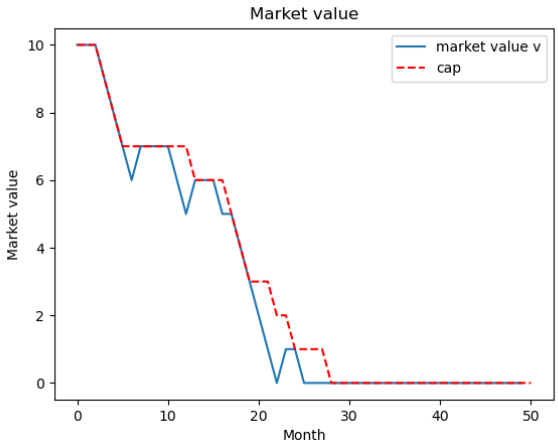


Figure 4: Example Time Series of the Stock Value. Note that the market value is not directly given as reward due to the `interest` parameter

Rewards: The model would be incomplete without reward structures. We define two reward structures:

1. Setting a reward of 1 for every transition except for transitions from absorbing states[18].
2. Setting a reward of $v \cdot (1+\text{interest})^{\text{tmax}-\text{time}}$ every time a stock is sold/delivered (transition `[delivery]`) where v is the price of the stock for which the investor sold.

The first reward structure is used to count the number of steps or transitions until reaching an absorbing state. The second reward structure is used to see how much the investor can earn, taking into account accrued interest with monthly compounding, given some policy.

4 Results

In Figure 5b, we verify 4 quantitative properties within `tmax=12` months with no reinvestment (meaning `interest=0`) and `max_stocks=12` such that the end condition of running out of stocks never happens. We thus only observe the effect of changing our end condition to a time-based one in comparison to the original case study.

- The first property (top left) shows the maximum cumulative reward/money the investor can get. As in the original case study in Figure 5a, the higher the initial value of the stock, the higher the expected reward. We also see however an increasing expected reward the less likely the market bars. This is due to the possibility of selling multiple times. Note that the investor is selling at every month in this case as they have enough stocks and it is thus the optimal strategy.
- The second property (top right) shows the minimum cumulative reward/money the investor can get. It's 0 because the investor can just do `noop` all the time. In other words, the investor never reserves. Note that it's independent of `v_init` and `p_bar`.
- The third property (bottom left) shows the maximum number of steps that the investor can get. We can see that it's independent of `v_init` but decreases as the `p_bar` increases. The maximum number of steps is when the investor always reserves.
  In particular;
  - When the market never bars, the maximum number of steps is $3 \cdot \text{tmax} = 3 \cdot 12 = 36$, for 3 transitions: `[month]`, `[invest]`, `[delivery]`.
  - When the market always bars when it can, the maximum number of steps is $2 \cdot \text{tmax}/2 + 3 \cdot \text{tmax}/2 = 12 + 18 = 30$ because the market bars only at second, fourth, sixth, ..., 12th month. Each time the market bars, it removes the `[delivery]` transition.

[18] Transitions from absorbing states come back to absorbing states

- The fourth property (bottom right) shows the minimum number of steps that the investor can get. We can see that it's independent of both `v_init` and `p_bar`. The minimum number of steps is when the investor never reserves and the value is $2 \cdot \text{tmax} = 2 \cdot 12 = 24$ due to never having the `[delivery]` transition. We see some color variation due to the slight deviation from 24 which might be caused by the iterative algorithm.

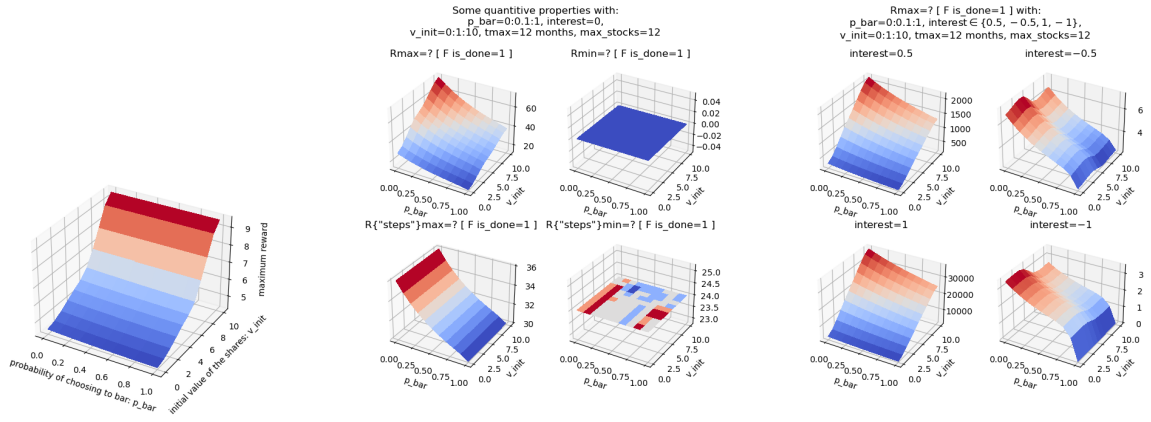
See our graphical representation of the module `month` in Figure 3a for more intuition.

In Figure 5c, we verify one property for different values of `interest` to observe its effect. All the other parameters are the same as previously. The property we want to verify is the maximum cumulative reward/money the investor can get, but this time, with a non-zero interest rate. As changing the interest rate doesn't affect the number of steps when the investor always stay for the `tmax` months due to our choice of `max_stocks`, we don't investigate the two properties based on the reward structure `"steps"`.

We can observe that a higher interest rate leads to a higher maximum expected cumulative reward. However, when the interest rate is negative, we see some harder-to-interpret wavy surface plots. This is caused by a combination of different factors:

- High `v_init`:
  - Cap can decrease over time and more likely that `v` initially **decreases** due to the market dynamics.
  - Trade-off between wanting to sell early due to the initial price fall & selling more stocks versus wanting to sell later due to losing money from the negative interest rate.
  - High negative impact of interest rate when we sell early.
- Low `v_init`:
  - While the cap will decrease over time, it is more likely that `v` initially **increases** due to the market dynamics.
  - Trade-off between wanting to sell early due to selling more stocks versus wanting to sell later due to losing money from the negative interest rate & initial price growth.
  - Low negative impact of interest rate when we sell later.

In general, the incentive to sell early is no longer dominant and there's now a trade-off between selling early and selling later.



(a) (b) (c)

Figure 5: (a) Max. expected cumulative reward in the original case study (b) Max./min. expected cumulative reward (first row) and max./min. expected number of steps (second row) in our case study for zero interest rate (c) Max. expected cumulative reward in our case study for non-zero interest rate

5 Future Work

Although we added the `max_stocks` variable, we leave the analysis as future work due to lack of time. Mainly, one could observe the effect of `max_stocks` on the expected reward.

We could also look into the effect of both `interest` and `max_stocks`. When `max_stocks` is less than `t_max`, running out of stocks becomes an end condition. We can then look at when we reach this end condition. Notably, for a low initial value but a high interest rate, will all the stocks be sold early in the year due to the high interest rate, or will the investor wait for the value to rise before investing?

As noted in subsection 1.3, the original case study and our extension can be considered a two-player game, with both the investor and the market being independent actors. However, our project only focused on a single-player game, in which the market is given a fixed strategy.

The proper model to accurately represent the case study is a turned-based stochastic game, which was briefly touched upon in 2.1.4. Future work could look into implementing the original model and our extension in PRISM-games, which is an extension of PRISM with the ability to deal with TSGs. Further properties revolving around the dynamics between the investor and market could then be explored.

For example, an additional investor could be introduced, with the two investors interacting while the market attempts to minimize both their rewards. A further idea could be to implement a Future stock, where the two investors must settle on a future price and date to sell/buy the stock.

6 Conclusion

In conclusion, we delved into the formalism of state transitions, Markov Decision Processes, and property verification. We explored how these models and properties are specified and evaluated in PRISM. Finally, we then take an existing case study, the Futures Market Investor [6], extend it with time-based quantities, and analyze its properties. Future work could import these models as turn-based stochastic games into PRISM-games, and analyze further properties.

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7 Appendix

7.1 PRISM limitations

Up to our best knowledge and understanding, we show below a non-exhaustive list of some limitations that we encountered when using PRISM:

- No for loops, no lists or compact way to specify many similar transitions. See Bluetooth case study:

```
[reply]  receiver=2 & y1=0 -> 1/(maxr+1) : (receiver'=3) & (y1'=0) // reply and make random choice
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*1)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*2)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*3)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*4)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*5)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*6)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*7)
               [...]
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*122)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*123)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*124)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*125)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*126)
               + 1/(maxr+1) : (receiver'=3) & (y1'=2*127);
```

- No way to search/find elements (ctrl + f).
- No negative rewards possible for MDP.
- No probability distribution over rewards. We cannot assign rewards by checking the values of the resulting state.
- No $\gamma \neq 1$ for MRPs or MDPs to have finite expected cumulative rewards in continuous tasks.
- No direct way to specify/check a property on a transition based on a transition label: e.g where we only know the transition label from which we can reach a particular state but we don't know the properties that the state should satisfy. It's the case when we cannot identify the set of states.
- Variable ordering can affect performance due to the symbolic data structures and algorithms used in PRISM (Similar to how ordering matters in SFDDs).
- Inherent to the models, state-space explosion is an issue which can be further explored. PRISM includes symbolic data structures and algorithms (based on BDDs, MTBDDs) to help with state-space explosion just like SFDD can help for encoding Finite Kripke Structures in which the state labelling function is injective. Depending on the model, symbolic data structures doesn't necessarily help. In practice, when models become too big, one might want to work with function approximations (see deep reinforcement learning) even though it's uncertain about how one could verify properties.

  One has to focus on critical aspects of systems when modeling instead of trying to model everything. This helps reduce the number of states.
- Limited graphing tools in PRISM: e.g we cannot have surface plots but we can just extract the results in CSV and use another programming language.
- Although PRISM can implicitly use the best or worst strategy in order to max./min. the expected cumulative reward or evaluate $P_{\min}/P_{\max}$ formulas, these strategies are not available to us in the Simulator. On the contrary, when using the simulator, PRISM resolves the determinism counter-intuitively, using a uniform strategy.

7.2 PRISM Model Code

Below we give the PRISM code for our extension of the case study. Recall that the original case study and its code can be found on their official website.

```
// EXTENDED "INVESTING IN THE FUTURES MARKET" from McIver and Morgan 03
// To finite horizon, multiple investments by Stephane NGUYEN and Tansen RAHMAN

mdp

const double p_bar;
const double interest;
const int v_init;
const int tmax;
const int max_stocks; // number of stocks investor has at the start

// module used to synchronize transitions (time)
module month

    m : [0..1];
    time : [0..tmax] init 0;
    is_done : [0..1] init 0;

    // Increment time, and perform transitions for the start of a new month (invest/bar).
    [invest] (m=0) & (time < tmax) -> (m'=1) & (time'=time+1);

    // Perform transitions that occur during the month (time series).
    [month] (m=1) & (time < tmax) & (is_done=0) -> (m'=0);

    // If last month, go to absorbing state, cashing in the shares if invested previous month
    [month] ((i=0)|((i=1) &(b=1))) & (m=1) & (time >= tmax) & (is_done=0) -> (is_done'=1);
    [month] (i=1) & (b=0) & (m=1) & (time >= tmax) & (is_done=0) -> (m'=0);

    // cash in shares
    [delivery] (m=0) & (time < tmax) & (is_done=0) -> (m'=0);
    [delivery] (m=0) & (time >= tmax) & (is_done=0) -> (is_done'=1);

    // two end conditions
    [done] (is_done=1) -> (is_done'=1);
    [done] (stocks=0) -> (is_done'=1);
endmodule

// the investor
module investor

    stocks: [0..max_stocks] init max_stocks; // number of stocks available to investor
    i : [0..1]; // i=0 no reservation and i=1 made reservation

    [invest] (i=0) -> (i'=0); // do nothing
    [invest] (i=0) & (stocks>0) -> (i'=1); // make reservation
    [invest] (i=1) & (b=1) -> (i'=0); // barred previous month: try again and do nothing
    [invest] (i=1) & (b=1) & (stocks>0) -> (i'=1); // barred previous month: make reservation
    [delivery] (i=1) & (b=0) & (stocks>0) -> (i'=0) & (stocks'=stocks-1); // cash in shares

endmodule

// market barring
module barred

    // b=0 - not barred and b=1 - barred, initially cannot bar
    b : [0..1] init 1;

    // do not bar this month
    [invest] (b=1) -> (b'=0);
    // bar this month (cannot have barred the previous month), only when investor did not invest last month
    [invest] (b=0) & (i=0) -> p_bar: (b'=1) + (1-p_bar): (b'=0);
    // case of b=0 and i=1 never happens because delivery would update i=0

endmodule

// value of the shares
module value

    v : [0..10] init v_init;

    [month] true -> p/10 : (v'=min(v+1,c)) + (1-p/10) : (v'=min(max(v-1,0),c));

endmodule

// probability of shares going up/down
module probability

    p : [0..10] init 5; // probabilitity is p/10 and initially the probability is 1/2

    [month] (v<5) -> 2/3 : (p'=min(p+1,10)) + 1/3 : (p'=max(p-1,0));
    [month] (v=5) -> 1/2 : (p'=min(p+1,10)) + 1/2 : (p'=max(p-1,0));
    [month] (v>5) -> 1/3 : (p'=min(p+1,10)) + 2/3 : (p'=max(p-1,0));

endmodule

// cap on the value of the shares
module cap

    c : [0..10] init 10; // cap on the shares
    // probability 1/2 the cap decreases
    [month] true -> 1/2 : (c'=max(c-1,0)) + 1/2 : (c'=c);

endmodule

rewards
    // reward from transition [delivery], accrued monthly interest
    [delivery] true : v * pow(1 + interest, tmax - time);
endrewards

rewards "steps"
    true : 1;
endrewards
```