06: Example - Reflection in a Plane

Chapter Goal: To demonstrate the combined power of the Gram-Schmidt Process and Change of Basis by solving a geometrically difficult problem in a very elegant way.

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1. The Problem: Reflection in a Skewed Mirror

- **Goal:** We want to find the result of reflecting a vector $\vec{r}$ across a "mirror" (a plane) that is positioned at a skewed angle in 3D space.
- **The Difficulty:** Solving this with standard trigonometry would be extremely hard because all the angles are "weird".
- **Known Information:**
  - We don't know the equation of the mirror plane, but we know two vectors that lie on it: $\vec{v}_1$ and $\vec{v}_2$.
  - We also know a third vector, $\vec{v}_3$, that is not on the plane.

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2. The Clever Strategy: Change Your Perspective!

- **Core Idea:** Instead of solving the problem in "our world" where the axes (`x,y,z`) are not aligned with the mirror, let's move to a new, simpler "world".
- **The Ideal "Mirror World":**
  - Imagine a new coordinate system $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ where:
    - $\vec{e}_1$ and $\vec{e}_2$ lie perfectly **on the surface** of the mirror.
    - $\vec{e}_3$ points **perpendicularly out** of the mirror (this is called the "normal" vector).
  - These three vectors, $\vec{e}_1, \vec{e}_2, \vec{e}_3$, form an **orthonormal basis**.
- **Why is this World "Easy"?**
  - In this mirror world, the action of "reflection" becomes incredibly simple:
    - The $\vec{e}_1$ component of a vector **does not change**.
    - The $\vec{e}_2$ component of a vector **does not change**.
    - The $\vec{e}_3$ component (the distance from the mirror) simply has its **sign flipped** (`z` becomes `-z`).

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3. The Game Plan (Three Main Steps)

1. Build the Mirror World (Using Gram-Schmidt)

- We start with our "messy" vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.
- Use the Gram-Schmidt Process to transform them into a "nice" orthonormal basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$.
- $\vec{e}_1$ and $\vec{e}_2$ will now define the mirror plane, and $\vec{e}_3$ will be its normal vector.

2. Define the Reflection in the Mirror World

- Inside the basis `E` = $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, the transformation matrix for the reflection (T_E) is very simple:

$$T_E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \begin{matrix} (\vec{e}_1 \text{ is unchanged}) \\ (\vec{e}_2 \text{ is unchanged}) \\ (\vec{e}_3 \text{ is flipped}) \end{matrix}$$

3. Take the Translation Journey (Using `A⁻¹MA` Logic)

- To reflect our original vector $\vec{r}$ (which lives in our world), we take a 3-step journey:
  a. **Translate $\vec{r}$ to the Mirror World:** $\vec{r}_E = E^{-1}\vec{r}$
  b. **Perform the Easy Reflection there:** $\vec{r}'_E = T_E\vec{r}_E$
  c. **Translate the Result Back to Our World:** $\vec{r}' = E\vec{r}'_E$
- **The "Super-Machine" for Reflection in Our World (`T`):**
  Combining all three steps, we get the total reflection matrix in our world:

  > `T = E * T_E * E⁻¹`

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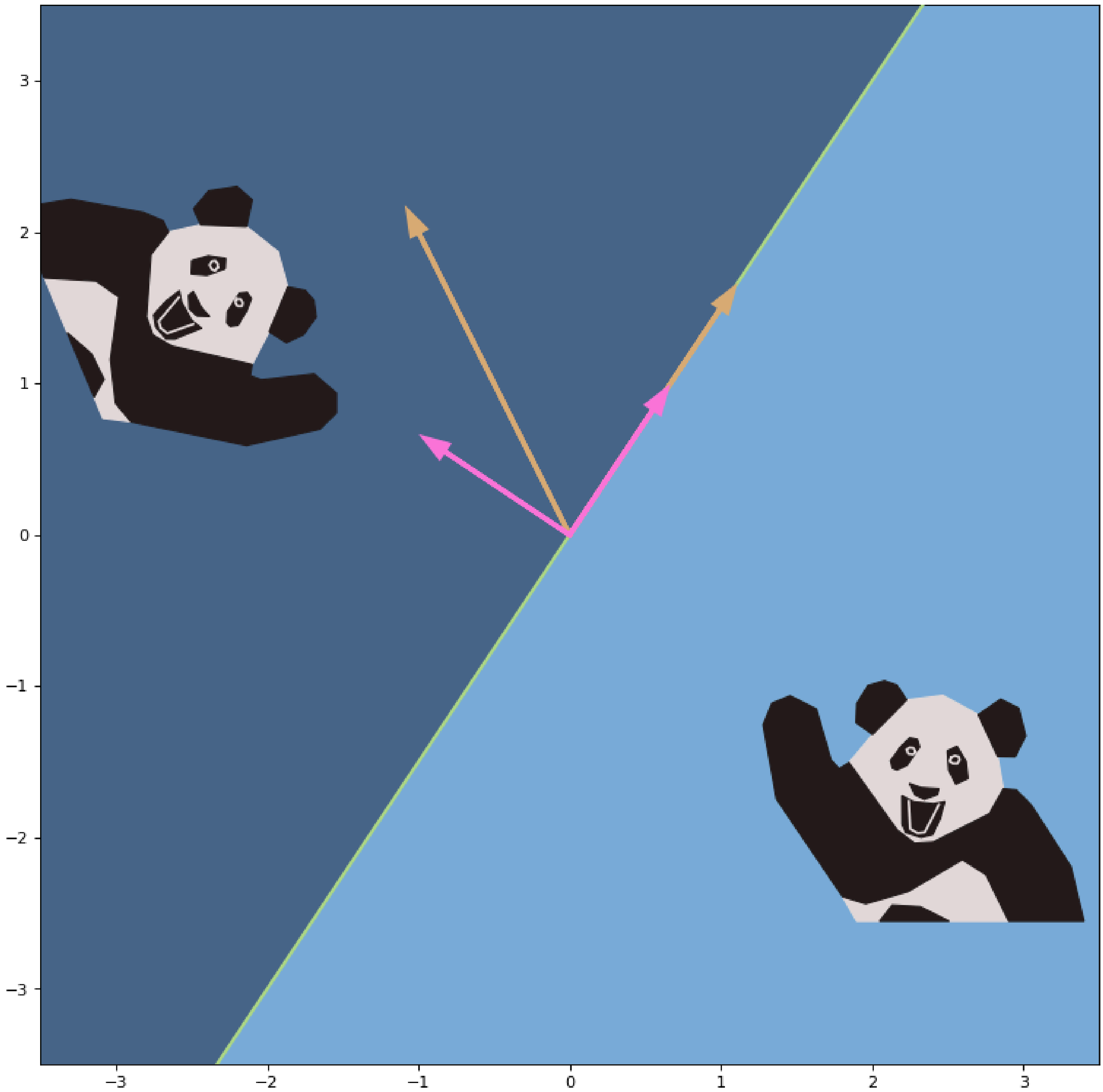
4. An Added Bonus

- **Easy Inverse:** Because we built the basis `E` using Gram-Schmidt, it is guaranteed to be an orthonormal matrix.
- **Superpower of Orthogonal Matrices:** We know that for an orthogonal matrix, $E^{-1} = E^T$ (the Inverse is the Transpose).
- **Simpler Final Formula:**

  > `T = E * T_E * Eᵀ`

- This is much easier to compute than finding the inverse of a general 3x3 matrix.

Final Conclusion:
By combining the **Gram-Schmidt Process** (to create a nice basis) and Change of Basis (to work within that nice basis), we can transform a very difficult geometric problem (reflection in a skewed plane) into a series of matrix multiplications that can be easily solved by a computer. This is a pinnacle of the power of linear algebra.



Tags: #mml-specialization #linear-algebra #gram-schmidt #change-of-basis #orthogonal-matrix #reflection