03: The Cosine Rule & The Dot Product

Chapter Goal: To uncover the **geometric meaning** of the Dot Product and understand its role as a measure of "alignment" between vectors.

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1. Core Idea: Finding the Geometric Meaning

- We already know **how to calculate** the dot product ($r_1 s_1 + \dots$).
- Now we want to know **what it means geometrically**. The answer comes from the **Cosine Rule**.

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2. Deriving the Formula (Logical Flow)

- **Setup:** Create a triangle with vectors `r`, `s`, and `r-s` as its sides. The angle between `r` and `s` is `θ`.
- **The Cosine Rule (Geometric Version):**

> $|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos(\theta)$

- **Rewrite with the Dot Product:**
  - We know from the previous lesson that $|v|^2 = v \cdot v$. Let's apply this to the left side of the equation.
  - $|r-s|^2 = (r-s) \cdot (r-s)$
  - **Expand:** $(r-s)\cdot(r-s) = r\cdot r - r\cdot s - s\cdot r + s\cdot s = |r|^2 - 2(r\cdot s) + |s|^2$
- **Equate the Two Expressions:**
  $|r|^2 - 2(r\cdot s) + |s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos(\theta)$
- **Simplify:**
  - Cancel $|r|^2$ and $|s|^2$ from both sides.
  - $-2(r\cdot s) = -2|r||s|\cos(\theta)$
  - Cancel `-2` from both sides.

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3. The "Aha!" Moment: The Geometric Definition of the Dot Product

- **Final Formula:**

> `r · s = |r| |s| cos(θ)`

- This is the **geometric definition** of the dot product. It tells us the dot product is:

  > "The product of the lengths of the two vectors, multiplied by the cosine of the angle between them."

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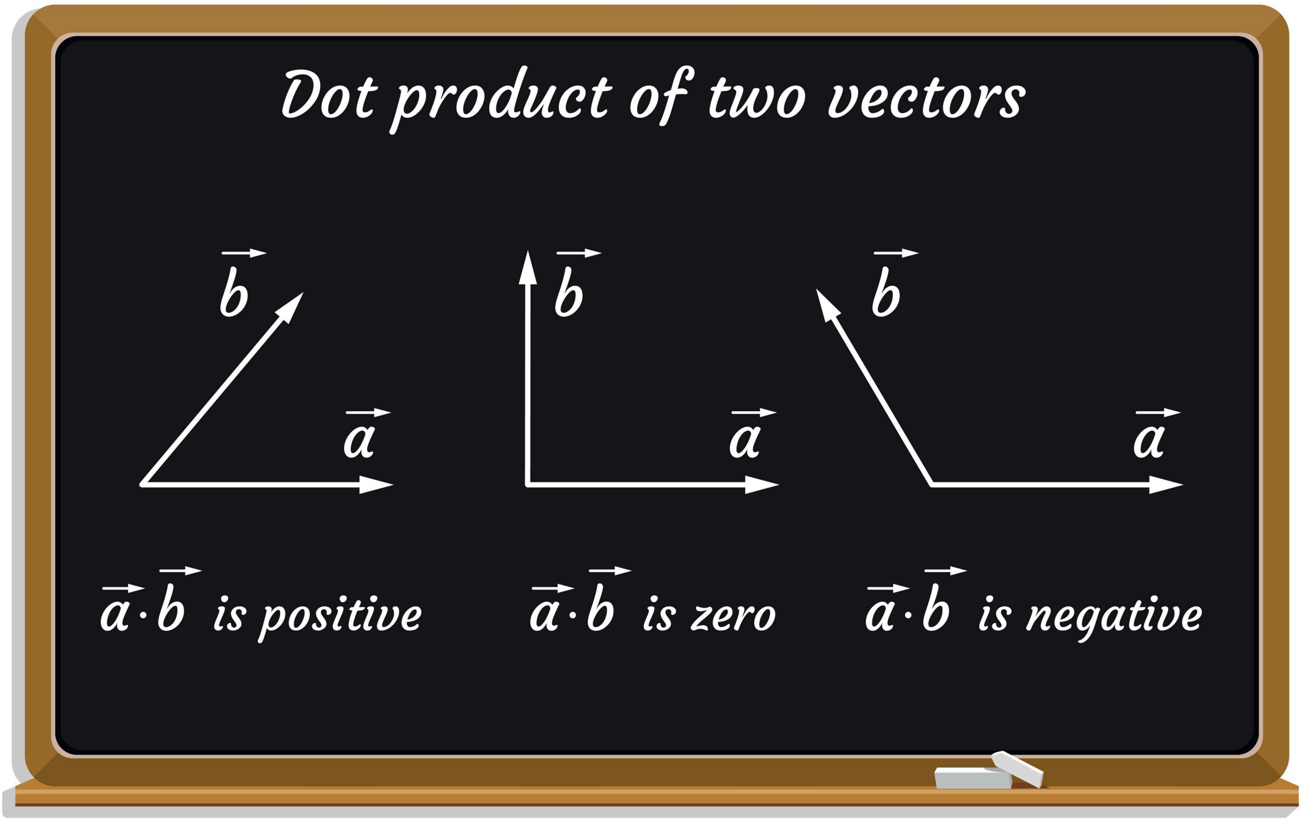
4. Geometric Interpretation (The Meaning of `cos(θ)`)

The dot product is a **"measure of alignment"**. The `cos(θ)` factor tells us about the relative direction of the two vectors.

- **Case 1:** `θ = 0°` **(Same Direction)**
  - `cos(0°) = 1`.
  - `r · s = |r| |s|`. The dot product is at its maximum positive value.
- **Case 2:** `θ = 90°` **(Perpendicular / Orthogonal)**
  - `cos(90°) = 0`.
  - `r · s = 0`.
  - **Key Conclusion:** If the dot product of two non-zero vectors is zero, the vectors are mutually **orthogonal**.
- **Case 3:** `θ = 180°` **(Opposite Directions)**
  - `cos(180°) = -1`.
  - `r · s = -|r| |s|`. The dot product is at its maximum negative value.
- **Conclusion about the Sign:**
  - `r · s > 0`: The vectors are generally pointing in the same direction (angle < 90°).
  - `r · s = 0`: The vectors are perpendicular.
  - `r · s < 0`: The vectors are generally pointing in opposite directions (angle > 90°).



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