06: Finding the Inverse Matrix with Gaussian Elimination

Chapter Goal: To use the same method as Gaussian Elimination, but with a broader goal: not just to solve a single problem, but to find the **Inverse Matrix (A^{-1})** itself.

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1. The Core Idea: Solving Many Problems at Once

- **Problem:** We want to find a matrix `B` (which will be A^{-1}) such that $A \cdot B = I$.
- **Analyzing the Equation:**
  $A \cdot [\text{col}_1(B), \text{col}_2(B), \text{col}_3(B)] = [\text{col}_1(I), \text{col}_2(I), \text{col}_3(I)]$
- **"Aha!" Moment:** This one large matrix equation can actually be broken down into three separate systems of linear equations:

  1. $A \cdot (\text{col}_1(B)) = \text{col}_1(I) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
  2. $A \cdot (\text{col}_2(B)) = \text{col}_2(I) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
  3. $A \cdot (\text{col}_3(B)) = \text{col}_3(I) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

  We could solve each of these systems with Gaussian Elimination to find each column of the inverse matrix `B`.

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2. A Clever Trick: Solving in Parallel

- **Problem:** Performing Gaussian Elimination three times separately is tedious and repetitive, because the row operations performed on matrix `A` will be identical in all three cases.
- **Solution:** Let's solve all three systems **simultaneously** in one process.
- **Setup: The "Augmented Matrix"**

  1. Write matrix `A` on the left side.
  2. Draw a vertical line.
  3. Write the **Identity Matrix** `I` on the right side.

  > $[A|I]$

The right side (`I`) acts as a "container" for our three separate $\vec{v}$ vectors ($\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$) at the same time.

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3. The Gauss-Jordan Elimination Process

- **Goal:** Transform the matrix `A` on the left side into the **Identity Matrix** `I`.
- **Method:** Use elementary row operations (adding/subtracting multiples of rows, multiplying rows by scalars).
  1. **Forward Elimination:** Make all entries **below** the main diagonal zero (as before).
  2. **Backward Elimination:** Make all entries **above** the main diagonal zero.
  3. **Normalization:** Ensure all entries on the main diagonal are `1` (by dividing each row by its diagonal value if needed).
- **The Golden Rule:**

> Every row operation you perform on the **left side** MUST also be applied to the **right side** simultaneously.

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4. The Magical Final Result

After we have successfully transformed the left side into the Identity Matrix `I`, let's see what happened to our equation:

- We started with: $[A|I]$.
- After the operations, we get: $[I|B]$.

This algebraically means we have transformed the equation $A \cdot X = I$ into $I \cdot X = B$, which simplifies to $X = B$.

- **Conclusion:** The matrix that appears on the **right side** at the end of the process **is the Inverse Matrix** A^{-1} that we were looking for!

> $[A|I] \xrightarrow{\text{row operations}} [I|A^{-1}]$

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5. Key Message

- This method (called **Gauss-Jordan Elimination**) is a very systematic and computationally efficient way to find the inverse of a matrix.

- This is the algorithm often used inside a computer when you call a function like `inv(A)`.
- By finding A^{-1}, we now have a "super tool" to solve $A\vec{x} = \vec{v}$ for **any** $\vec{v}$ with a single multiplication: $\vec{x} = A^{-1}\vec{v}$.

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6. Worked Example: Finding the Inverse of Matrix `A`

Matrix `A`:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Goal: Find A^{-1} using the $[A|I] \to [I|A^{-1}]$ method.

Step 1: Setup the Augmented Matrix

We write matrix `A` on the left and the Identity matrix `I` on the right.

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}$$

Phase 1: Forward Elimination (Zeros Below the Diagonal)

Goal: Make the entries at positions (2,1) and (3,1) zero.

Operation 1: `R₂' = R₂ - R₁`

- **Left Side:** `[1, 2, 4] - [1, 1, 3] = [0, 1, 1]`
- **Right Side:** `[0, 1, 0] - [1, 0, 0] = [-1, 1, 0]`

Operation 2: `R₃' = R₃ - R₁`

- **Left Side:** `[1, 1, 2] - [1, 1, 3] = [0, 0, -1]`
- **Right Side:** `[0, 0, 1] - [1, 0, 0] = [-1, 0, 1]`

Result after Steps 1 & 2:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$$

Phase 1 is almost done. We just need to make the diagonal `1`.

Operation 3: $R_3'' = R_3' * (-1)$ (Normalize Row 3)

- **Left Side:** `[0, 0, -1] * (-1) = [0, 0, 1]`
- **Right Side:** `[-1, 0, 1] * (-1) = [1, 0, -1]`

Final Result of Phase 1 (Row Echelon Form):

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Phase 2: Backward Elimination (Zeros Above the Diagonal)

Now we work from the bottom up.

Goal: Make the entries at (2,3) and (1,3) zero, using R_3.

Operation 4: $R_2'' = R_2' - R_3''$

- **Left Side:** `[0, 1, 1] - [0, 0, 1] = [0, 1, 0]`
- **Right Side:** `[-1, 1, 0] - [1, 0, -1] = [-2, 1, 1]`

Operation 5: $R_1' = R_1 - 3 * R_3''$

- **Left Side:** `[1, 1, 3] - 3*[0, 0, 1] = [1, 1, 3] - [0, 0, 3] = [1, 1, 0]`
- **Right Side:** `[1, 0, 0] - 3*[1, 0, -1] = [1, 0, 0] - [3, 0, -3] = [-2, 0, 3]`

Intermediate Result:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Almost there! Just one more entry to zero out.

Goal: Make the entry at position (1,2) zero, using R_2.

Operation 6: $R_1'' = R_1' - R_2''$

- **Left Side:** `[1, 1, 0] - [0, 1, 0] = [1, 0, 0]`
- **Right Side:** `[-2, 0, 3] - [-2, 1, 1] = [0, -1, 2]`

Final Result of the Process

Let's assemble our final matrix:

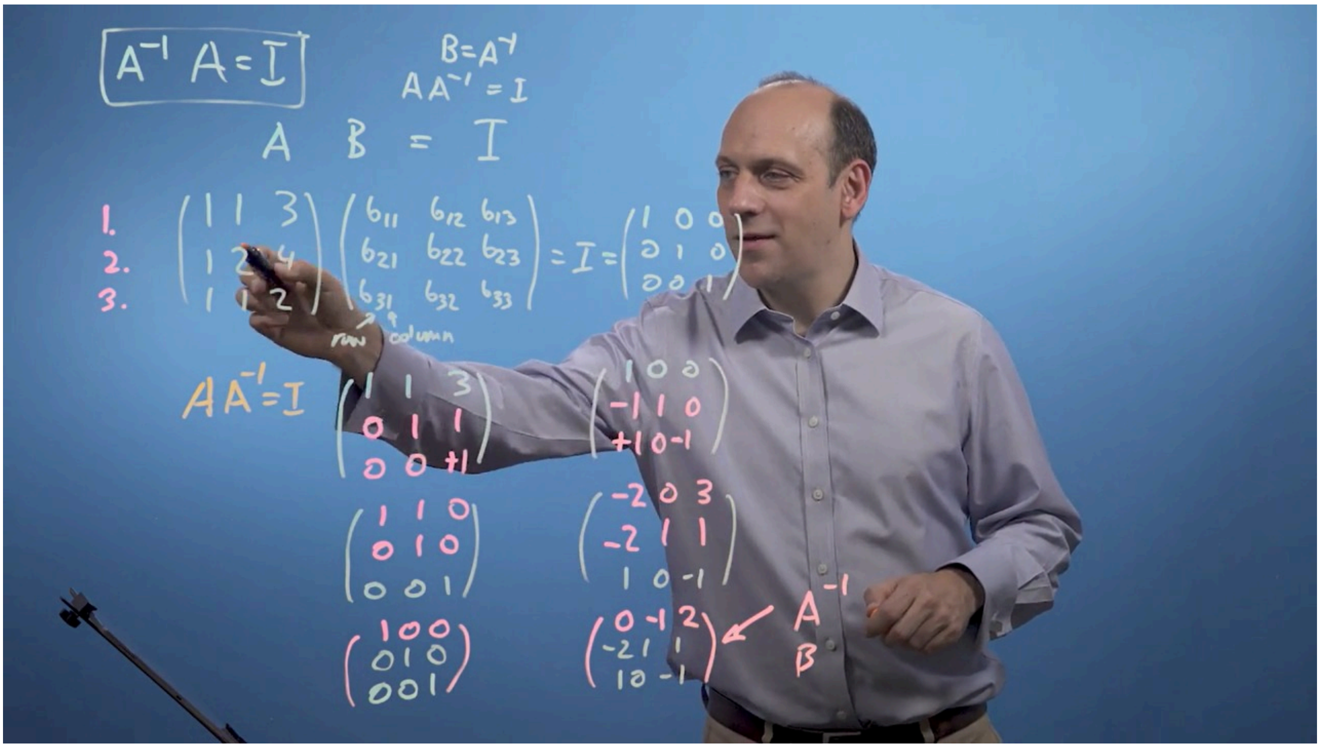
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

The left side has become the Identity matrix `I`. Therefore, the right side is the inverse matrix we were looking for.

Inverse Matrix A^{-1}:

$$A^{-1} = \begin{bmatrix} 0 & -1 & 2 \\ -2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

This is the exact same result obtained in the video.



Note

Menurut aku untuk soal latihan corsera bagian ini, bagus apabila ditaruh di catatan

← Back **Solving linear equations using the inverse matrix**
Practice Assignment • 30 min

1. You go to the shops on Monday and buy 1 apple, 1 banana, and 1 carrot; the whole transaction totals €15. On Tuesday you buy 3 apples, 2 bananas, 1 carrot, all for €28. Then on Wednesday 2 apples, 1 banana, 2 carrots, for €23.

1 point

Construct a matrix and vector for this linear algebra system. That is, for

$$A\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}s_{\text{Mon}}\\s_{\text{Tue}}\\s_{\text{Wed}}\end{bmatrix}$$

Where a, b, c, are the prices of apples, bananas, and carrots. And each s is the total for that day.

Fill in the components of A and $\mathbf{s}$.

```
# Replace A and s with the correct values below:
A = [[1, 1, 1],
     [3, 2, 1],
     [2, 1, 2]]

s = [15, 28, 23]
```

Run

Reset

```
[15, 28, 23]
```

1.

2. Given another system, $B\mathbf{r}=\mathbf{t}$,

1 / 1 point

$$\begin{array}{c}①:\\②:\\③:\end{array}\begin{bmatrix}4&6&2\\3&4&1\\2&8&13\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}9\\7\\2\end{bmatrix}$$

We wish to convert this to echelon form, by using elimination. Starting with the first row, ①, if we divide the whole row by 4, then the top-left element of the matrix becomes 1,

$$\begin{array}{c}①':\\②':\\③':\end{array}\begin{bmatrix}1&3/2&1/2\\3&4&1\\2&8&13\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}9/4\\7\\2\end{bmatrix}$$

Next, we need to fix the second row. This results in the following,

$$\begin{array}{c}①'':\\②'':\\③'':\end{array}\begin{bmatrix}1&3/2&1/2\\0&1&1\\2&8&13\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}9/4\\-1/2\\2\end{bmatrix}$$

What steps did we take?

- (●) The new second row, ②$''$ is the old second row minus three times the old first row, then all multiplied by -2, i.e., $②''=[②'-3①']\times(-2)$.
- ( ) The new second row, ②$''$ is the old second row minus two times the old first row, i.e., $②''=[②'-2①']$.
- ( ) The new second row, ②$''$ is the old second row divided by four minus the old first row, i.e., $②''=②'/4-①'$.
- ( ) The new second row, ②$''$ is the old second row minus three, i.e., $②''=②'-3$.

Correct
We've made the new second row a linear combination of previous rows.

2.

3. From the previous question, our system is almost in echelon form.

1 / 1 point

$$\begin{array}{c} ①'' \\ ②'' \\ ③'' \end{array} \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 2 & 8 & 13 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 2 \end{bmatrix}$$

Fix row 3 to be a linear combination of the other two. What is the echelon form of the system?

- ○ $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ -5/2 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 1/2 \end{bmatrix}$
- ◉ $\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ -1/4 \end{bmatrix}$

Correct
This system is now in echelon form.

3.
Langkah - langkahnya :

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 2 & 8 & 13 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 5 & 12 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ -2{,}5 \end{bmatrix} \quad -5/2 \quad -2{,}5$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix} \quad -3/4$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -1/2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ -1/2 \\ 0 \end{bmatrix}$$

4. Taking your answer from the previous part, use back substitution to solve the system. **1 / 1 point**

What is the value of $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$?

- (selected) $\mathbf{r} = \begin{bmatrix} 3 \\ -1/2 \\ 0 \end{bmatrix}$
- $\mathbf{r} = \begin{bmatrix} 9/4 \\ -1/2 \\ 0 \end{bmatrix}$
- $\mathbf{r} = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$
- $\mathbf{r} = \begin{bmatrix} 9 \\ 7 \\ 2 \end{bmatrix}$

Correct
Well done!

4.

5. Let's return to the apples and bananas from Question 1. **3 / 3 points**

Take your answer to Question 1 and convert the system to echelon form. I.e.,

$$\begin{bmatrix} 1 & A'_{12} & A'_{13} \\ 0 & 1 & A'_{23} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} s'_1 \\ s'_2 \\ s'_3 \end{bmatrix}.$$

Find values for A' and $\mathbf{s}'$.

```
# Replace A and s with the correct values below:
A = [[ 1 , 1, 1],
     [ 0,  1 ,  2],
     [ 0,  0 ,  1  ]]
s = [15, 17, 5]

```

Run

Reset

Correct
Correct! Well done.

6. Following on from the previous question; now let's solve the system using back substitution. **3 / 3 points**

What is the price of apples, bananas, and carrots?

```
# Replace a, b, and c with the correct values below:
s = [3, 7, 5]

```

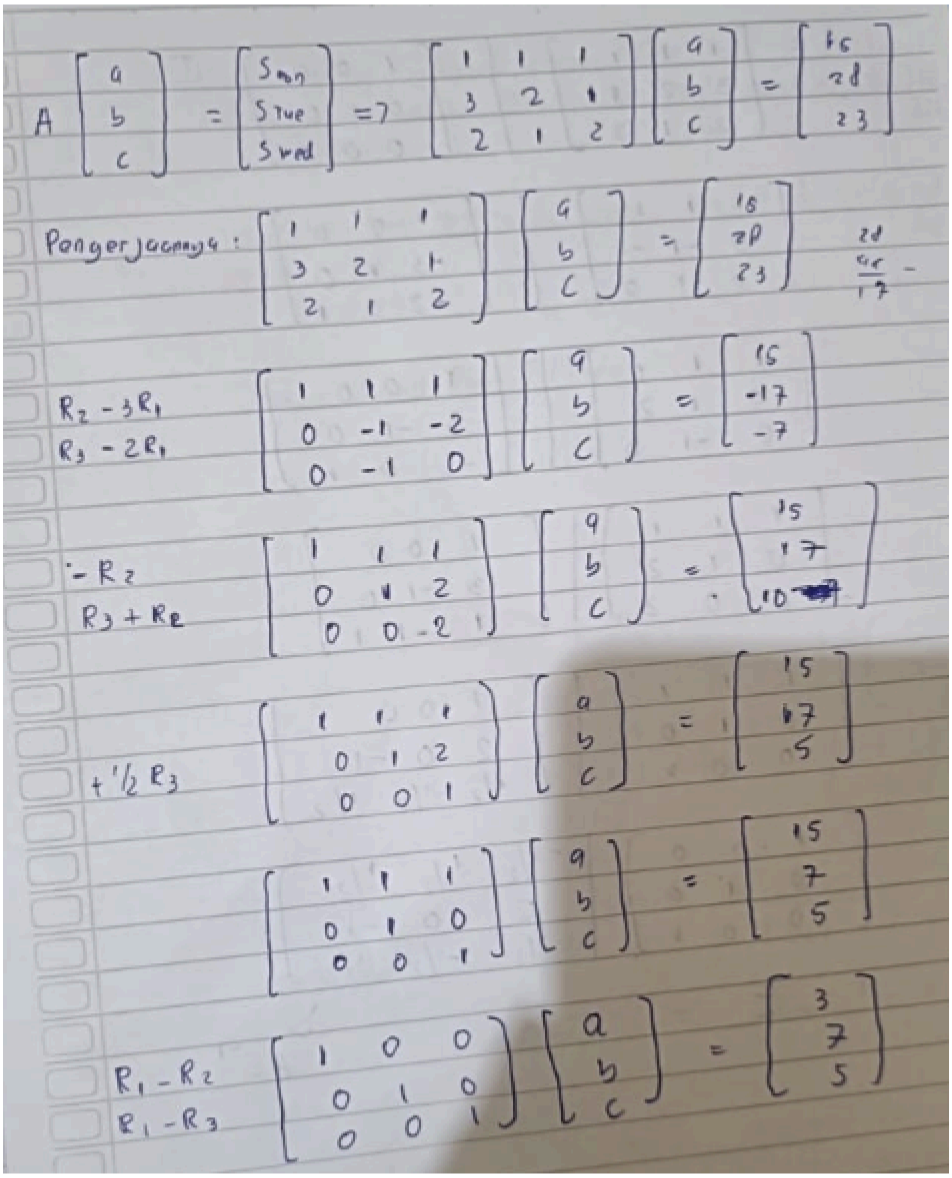
Run

Reset

Correct
Correct! Well done.

5.

langkah-langkah :



$$A\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}S_{mon}\\S_{Tue}\\S_{wed}\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1\\3&2&1\\2&1&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}15\\28\\23\end{bmatrix}$$

Pengerjaannya :

$$\begin{bmatrix}1&1&1\\3&2&1\\2&1&2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}15\\28\\23\end{bmatrix}$$

$R_2 - 3R_1$, $R_3 - 2R_1$

$$\begin{bmatrix}1&1&1\\0&-1&-2\\0&-1&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}15\\-17\\-7\end{bmatrix}$$

$-R_2$, $R_3 + R_2$

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&-2\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}15\\17\\10\end{bmatrix}$$

$+\frac{1}{2}R_3$

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}15\\17\\5\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}15\\7\\5\end{bmatrix}$$

$R_1 - R_2$, $R_1 - R_3$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}3\\7\\5\end{bmatrix}$$

7. If every week, you go to the shops and buy the same amount of apples, bananas, and oranges on Monday, Tuesday, and Wednesday; and every week you get a new list of daily totals - then you should solve the system in general. **3 / 3 points**

That is, find the inverse of the matrix you used in Question 1.

```
# Replace the matrix elements with the correct values below:
Ainv = [[-3/2, 1/2, 1/2],
        [2, 0, -1],
        [1/2, -1/2, 1/2]]

```

Run

Reset

Correct

Correct! Well done.

6.

Langkah-langkahnya :

$$A = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \quad I$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & -3 & 1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & -1 & 0 & -2 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/2 & 1/2 & 1/2 \\ 0 & 1 & 0 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right]$$

$I \qquad\qquad A^{-1}$

8. In practice, for larger systems, one never solves a linear system by hand as there are software packages that can do this for you - such as *numpy* in Python.

1 / 1 point

Use this code block to see *numpy* invert a matrix.

You can try to invert any matrix you like. Try it out on your answers to the previous question.

```
import numpy as np

A = [[1, 1, 1],
     [3, 2, 1],
     [2, 1, 2]]
Ainv = np.linalg.inv(A)

```

Run

Reset

```
[[ -1.50000000e+00   5.00000000e-01   5.00000000e-01]
 [  2.00000000e+00   5.55111512e-17  -1.00000000e+00]
 [  5.00000000e-01  -5.00000000e-01   5.00000000e-01]]
```

In general, one shouldn't calculate the inverse of a matrix unless absolutely necessary. It is more computationally efficient to solve the linear algebra system if that is all you need.

Use this code block to solve the following linear system with *numpy*. $A\mathbf{r} = \mathbf{s}$,

$$\begin{bmatrix} 4 & 6 & 2 \\ 3 & 4 & 1 \\ 2 & 8 & 13 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 9 \\ 7 \\ 2 \end{bmatrix}$$

```
import numpy as np
A = [[4, 6, 2],
     [3, 4, 1],
     [2, 8, 13]]

s = [9, 7, 2]

r = np.linalg.solve(A, s)

```

Run

Reset

Correct
In cases when you don't need the inverse matrix itself,
linear algebra routines are quicker to solve the system for each case.

7.

Tags: #mml-specialization #linear-algebra #inverse-matrix #gauss-jordan-elimination