02: How Matrices Transform Space

Chapter Goal: To reinforce the core intuition from 3Blue1Brown that a matrix is fundamentally a **linear transformation** of space, and to connect the visual properties of this transformation with its formal algebraic rules.

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1. Core Idea: A Matrix is a Transformation Machine

- **Reminder:** The **columns** of a matrix tell us the final destination of the Basis vectors $(\hat{i}, \hat{j})$ after the transformation.
- **Implication:** Since every vector in the space can be written as a Kombinasi Linear of the basis vectors ($\vec{v} = x\hat{i} + y\hat{j}$), then...

> ...the final position of $\vec{v}$ after the transformation ($\vec{v}_{new}$) will be the **same linear combination** of the **new basis vectors** ($\vec{v}_{new} = x \cdot \hat{i}_{new} + y \cdot \hat{j}_{new}$).

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2. Visual Properties of a Matrix Transformation (Linearity)

- **"Aha!" Moment (from 3Blue1Brown):** The rule above has very specific visual consequences for what a matrix transformation can do to space.
  - Grid lines remain **straight** (nothing gets curved).
  - Grid lines remain **parallel**.
  - The **origin stays fixed**.
- **What CAN happen?** Space can be **stretched**, **squished**, **rotated**, or **sheared**.
- **What CANNOT happen?** Space cannot be "bent" or **warped**.

This is the essence of the word **"Linear"** in Linear Algebra.

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3. Algebraic Properties of a Matrix Transformation (Linearity)

The visual properties above can also be described by two formal algebraic rules (this is the universal definition of "linear", as seen in 3b1b Ch. 16).

If `A` is a matrix and $\vec{r}, \vec{s}$ are vectors:

- **Scaling Rule:**

> $A(n\vec{r}) = n(A\vec{r})$

- **Meaning:** "Stretching a vector by `n` times, then transforming it" is the **SAME AS** "Transforming the vector first, then stretching the result by `n` times".
- **Addition Rule (Additivity):**

> $A(\vec{r} + \vec{s}) = A\vec{r} + A\vec{s}$

- **Meaning:** "Adding two vectors first, then transforming the result" is the **SAME AS** "Transforming each vector separately, then adding the results".
- **Conclusion:** Matrix transformations "respect" the basic vector operations (addition and scalar multiplication).

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4. Verification with an Example

The video verifies the ideas above with an example.

- **Problem:** Calculate $A\vec{v}$ where $A = \begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Method #1: Mechanical (The "School" Way)

- Row 1 * Column: $(2 \cdot 3) + (3 \cdot 2) = 6 + 6 = 12$.
- Row 2 * Column: $(10 \cdot 3) + (1 \cdot 2) = 30 + 2 = 32$.
- **Result:** $\begin{bmatrix} 12 \\ 32 \end{bmatrix}$.

Method #2: Intuitive (The 3Blue1Brown / Geometric Way)

1. **Deconstruct** `v` into a linear combination of basis vectors:
   $\vec{v} = 3\hat{\imath} + 2\hat{\jmath}$.
2. **Apply the transformation** `A` to the whole expression:
   $A(3\hat{\imath} + 2\hat{\jmath})$.
3. **Use the properties of linearity:**
   $3(A\hat{\imath}) + 2(A\hat{\jmath})$.
4. We know $A\hat{\imath}$ is the first column ($\begin{bmatrix} 2 \\ 10 \end{bmatrix}$) and $A\hat{\jmath}$ is the second column ($\begin{bmatrix} 3 \\ 1 \end{bmatrix}$).
5. **Substitute and calculate:**
   $3\begin{bmatrix} 2 \\ 10 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 30 \end{bmatrix} + \begin{bmatrix} 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \end{bmatrix}$.

- **The results are the same!**

Instructor's Key Message:
Method #2, while it may seem longer, provides a much deeper understanding. It proves that

we don't have to just memorize the "row-times-column" mechanic. We can always think of matrix-vector multiplication as **"finding the linear combination of the matrix columns (the new basis vectors)"**.

Tags: #mml-specialization #linear-algebra #matrices #transformations #linearity