07: The Multivariate Taylor Series

Chapter Goal: To "level up" the Taylor Series from the one-dimensional version into its more general and powerful multivariate form, bringing together all the concepts we've learned in this course.

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1. Background: From 1D to Multi-Dimensional Approximations

- **Recap of 1D Taylor Series (using `Δx` notation):**
  We can approximate a function `f` at a point `x + Δx` using information from point `x`:

$$f(x+\Delta x) \approx f(x) + f'(x)\Delta x + \frac{f''(x)}{2!}(\Delta x)^2 + \dots$$

- **The New Problem:** How do we do this for a function with multiple inputs, like $f(x, y)$?
- **Goal:** To approximate the function at a nearby point $(x+\Delta x, y+\Delta y)$.
- **The Visual:**
  - A 1D function is a **curve**. Its Taylor Series approximation is another (polynomial) **curve**.
  - A 2D function is a **surface**. Its Taylor Series approximation will be another (polynomial) **surface**.

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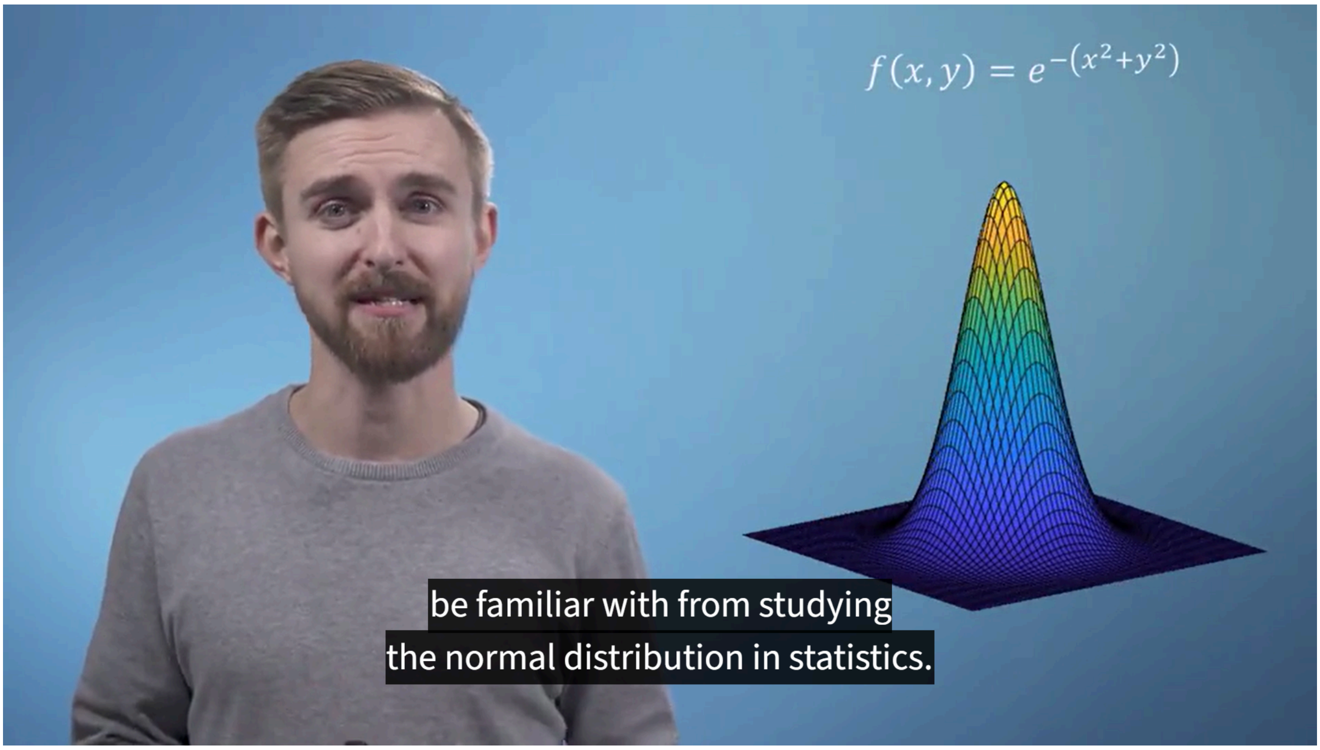
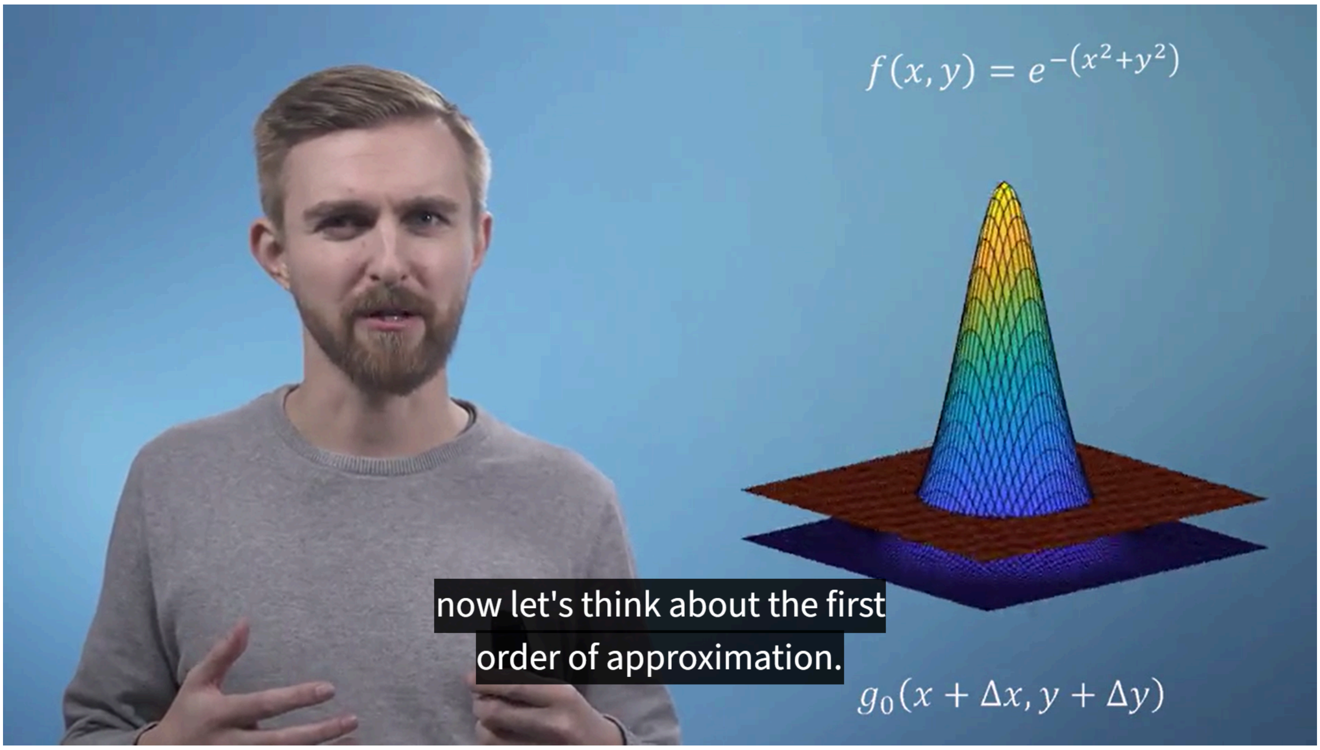
2. The Layers of Multivariate Approximation (Visualized)

Let's look at the 2D Gaussian function, $f(x, y) = e^{-(x^2+y^2)}$, as our example "landscape".



Zeroth-Order Approximation (g_0)

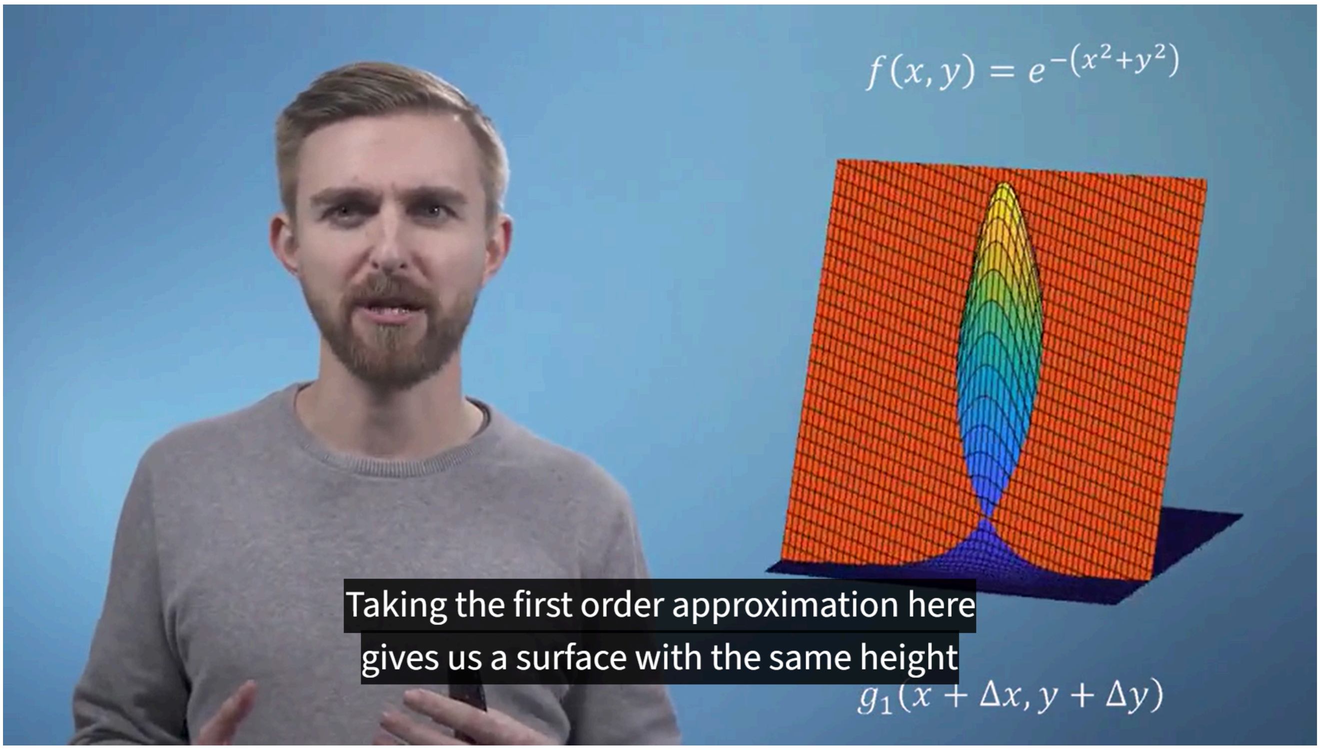
- **Formula:** $g_0(x + \Delta x, y + \Delta y) = f(x, y)$
- **Idea:** Match only the **height** of the function at the point of expansion.
- **Visual:** A **flat, horizontal plane**.
    - *As seen in the screenshot,* if we approximate at the peak, it's a flat plane at the maximum height. If we approximate on the side of the hill, it's a flat plane at that specific height.



First-Order Approximation (g_1) - Linearisation

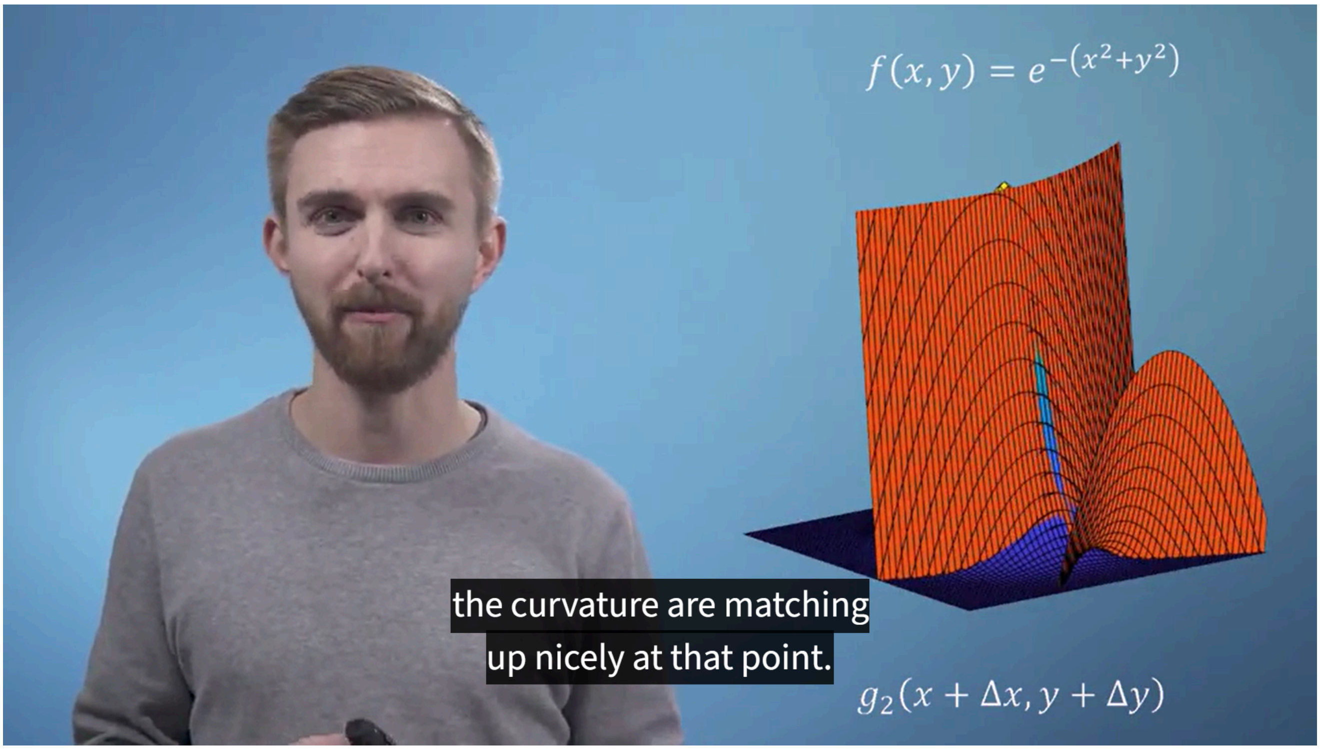
- **Formula:** Incorporates the gradient information.

- **Idea:** Match the **height AND the slope (gradient)** of the function at the point.
- **Visual:** A **slanted, flat plane** that is tangent to the surface at that point.
  - *As seen in the screenshot,* this slanted plane is a much better approximation near the point than the horizontal one.



Second-Order Approximation (g_2)

- **Formula:** Incorporates the Hessian information.
- **Idea:** Match the height, slope, **AND the curvature** of the function at the point.
- **Visual:** A **curved, parabolic surface** that "hugs" the original function.
  - *As seen in the screenshot,* this curved surface (like a saddle or a bowl) is an even better approximation, as it captures the local shape of the landscape much more accurately.



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3. The Formula: Unifying Calculus and Linear Algebra

Let's look at the terms of the Multivariate Taylor Series expansion around a point $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$.

We want to approximate $f(\vec{x} + \Delta\vec{x})$.

The First-Order Term

- **Expanded Form:**

$$\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y$$

- **"Aha!" Moment #1 (Connection to Jacobian/Gradient):**
  This is the Jacobian/Gradient (as a row vector) multiplied by the change vector $\Delta\vec{x}$.

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = J_f \Delta\vec{x}$$

  Or, using Gradient notation: $(\nabla f)^T \Delta\vec{x}$.

The Second-Order Term

- **Expanded Form (Messy):**

$$\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y}\Delta x \Delta y + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2\right)$$

- **"Aha!" Moment #2 (Connection to Hessian):**
  This entire messy expression is actually a compact quadratic form involving the **Hessian**

matrix.

$$\frac{1}{2}\begin{bmatrix}\Delta x & \Delta y\end{bmatrix}\begin{bmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2}\end{bmatrix}\begin{bmatrix}\Delta x \\ \Delta y\end{bmatrix} = \frac{1}{2}(\Delta \vec{x})^T H_f \Delta \vec{x}$$

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4. The Final, Compact Formula

By combining these insights, we can write the second-order Multivariate Taylor Series in a beautiful and compact form that brings together everything we've learned:

> $$f(\vec{x} + \Delta \vec{x}) \approx f(\vec{x}) + J_f \Delta \vec{x} + \frac{1}{2}(\Delta \vec{x})^T H_f \Delta \vec{x}$$

- This immediately generalizes from 2D to any number of dimensions. This is the power of using Linear Algebra to express complex Calculus concepts.

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5. A Concrete Worked Example: Step-by-Step

Let's make this tangible with a full example. We will approximate a function using all three layers (0th, 1st, and 2nd order approximations) and see how the error improves at each step.

The Problem Setup

- **Our Function (The "Landscape"):**

$$f(x, y) = x^2 + 3xy + y^2$$

- **The Center Point (where we have information):**
  $\vec{p} = \begin{bmatrix}1 \\ 2\end{bmatrix}$
- **The Target Point (where we want to approximate):**
  $\vec{x} = \begin{bmatrix}1.1 \\ 2.1\end{bmatrix}$
- **The "Small Step" Vector:**
  $\Delta \vec{x} = \vec{x} - \vec{p} = \begin{bmatrix}1.1 - 1 \\ 2.1 - 2\end{bmatrix} = \begin{bmatrix}0.1 \\ 0.1\end{bmatrix}$
- **The "Actual Value" (for comparison):**
  $f(1.1, 2.1) = (1.1)^2 + 3(1.1)(2.1) + (2.1)^2 = 1.21 + 6.93 + 4.41 = \mathbf{12.55}$

Approximation #0: The Flat Plane (0th-Order)

- **Goal:** Match only the height of the function at the center point.
- **Formula:** $g_0(\vec{x}) \approx f(\vec{p})$
- **Ingredient Needed:** The value of `f` at our center point `p=(1,2)`.

$$f(1,2) = 1^2 + 3(1)(2) + 2^2 = 1 + 6 + 4 = 11$$

- **Approximation:** Our guess for $f(1.1, 2.1)$ is simply **11**.
- **Error:** `Actual - Approx = 12.55 - 11 = 1.55`. (Very rough).

Approximation #1: The Slanted Plane (1st-Order / Linearisation)

- **Goal:** Match the height AND the slope (Gradient).
- **Formula:** $g_1(\vec{x}) \approx f(\vec{p}) + J_f(\vec{p}) \cdot \Delta\vec{x}$
- **New Ingredient: The Jacobian/Gradient `J_f` at `p=(1,2)`**
    - First, find the partial derivatives:
        - $\frac{\partial f}{\partial x} = 2x + 3y$
        - $\frac{\partial f}{\partial y} = 3x + 2y$
    - Now, evaluate them at `(1,2)`:
        - $\frac{\partial f}{\partial x}(1,2) = 2(1) + 3(2) = 8$
        - $\frac{\partial f}{\partial y}(1,2) = 3(1) + 2(2) = 7$
    - The Jacobian (as a row vector) is $J_f(1,2) = [8 \quad 7]$.
- **Calculate the 1st-Order Term:**

$$J_f\Delta\vec{x} = [8 \quad 7] \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = (8 \cdot 0.1) + (7 \cdot 0.1) = 0.8 + 0.7 = 1.5$$

- **Build the Full Approximation:**
  $g_1(\vec{x}) \approx f(\vec{p}) + 1.5 = 11 + 1.5 = 12.5$.
- **Approximation:** Our new guess for $f(1.1, 2.1)$ is **12.5**.
- **Error:** `Actual - Approx = 12.55 - 12.5 = 0.05`. (Much better!)

Approximation #2: The Curved Surface (2nd-Order)

- **Goal:** Match the height, slope, AND the curvature.
- **Formula:** $g_2(\vec{x}) \approx g_1(\vec{x}) + \frac{1}{2}(\Delta\vec{x})^T H_f(\vec{p})\Delta\vec{x}$
- **New Ingredient: The Hessian Matrix `H_f` at `p=(1,2)`**
    - Find the second partial derivatives:
        - $\frac{\partial^2 f}{\partial x^2} = 2$
        - $\frac{\partial^2 f}{\partial y^2} = 2$

        - $\frac{\partial^2 f}{\partial x \partial y} = 3$
    - Assemble the Hessian matrix (in this case, it's constant):

$$H_f = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$$

- **Calculate the 2nd-Order Term:**

$$\frac{1}{2}(\Delta\vec{x})^T H_f \Delta\vec{x} = \frac{1}{2}[0.1 \quad 0.1] \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

    1. First, `H * Δx`:
       $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 2(0.1) + 3(0.1) \\ 3(0.1) + 2(0.1) \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$
    2. Next, `(Δx)ᵀ * (result)`:
       $[0.1 \quad 0.1] \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = (0.1 \cdot 0.5) + (0.1 \cdot 0.5) = 0.05 + 0.05 = 0.1$
    3. Finally, multiply by `1/2`:
       $\frac{1}{2} \cdot (0.1) = 0.05$
- **Build the Full Approximation:**
  $g_2(\vec{x}) \approx g_1(\vec{x}) + 0.05 = 12.5 + 0.05 = 12.55.$
- **Approximation:** Our final guess for $f(1.1, 2.1)$ is **12.55**.
- **Error:** `Actual - Approx = 12.55 - 12.55 = 0`.

Summary of Results

Approximation Order	Formula Used	Result	Error
Actual Value	$f(1.1, 2.1)$	**12.55**	-
0th-Order	$f(1, 2)$	**11**	`1.55`
1st-Order	$f(1, 2) + J_f\Delta\vec{x}$	**12.5**	`0.05`
2nd-Order	$f(1, 2) + J_f\Delta\vec{x} + \frac{1}{2}\Delta\vec{x}^T H\Delta\vec{x}$	**12.55**	`0`

(The error is exactly zero because our original function was a quadratic, so a 2nd-order approximation is perfect. For a more complex function, there would still be a very small error).

This example clearly shows how each successive term in the Taylor Series adds another layer of correction, making the approximation increasingly accurate.

✎ Versi kedua

Catatan: Deret Taylor Multivariat (MML Course)

1. Ide Utama: Dari Kurva ke Lanskap

- **Tujuan:** Mengaproksimasi sebuah **"lanskap" fungsi multi-variabel** (`f(x,y)`) yang rumit dengan sebuah **permukaan polinomial** yang lebih sederhana di sekitar satu titik.
- **Notasi:** Kita akan menggunakan notasi "langkah kecil" yang lebih intuitif: `f(x+Δx, y+Δy)` .
    - `(x,y)` : Titik pusat aproksimasi kita.
    - `(Δx, Δy)` : "Langkah kecil" yang kita ambil dari titik pusat.

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2. Proses Aproksimasi Selapis demi Selapis (Visual)

Bayangkan kita ingin meniru sebuah **"lanskap pegunungan" biru (** `f(x,y)` **)** di sekitar satu titik.

Lapisan 1: Aproksimasi Orde ke-0 (Lantai Datar)

- **Rumus:** `f(x+Δx, y+Δy) ≈ f(x,y)`
- **Intuisi:** "Tebakan terbaikku untuk ketinggian di dekat sini adalah sama dengan ketinggian di tempatku berdiri sekarang."
- **Bentuk Visual:** Sebuah **bidang horizontal (lantai datar)** yang diletakkan pada ketinggian `f(x,y)` .
  (Seperti gambar kedua di screenshotmu).

Lapisan 2: Aproksimasi Orde ke-1 (Lereng Lurus / Linearisasi)

- **Rumus:** `f(x+Δx, y+Δy) ≈ f(x,y) + (∂f/∂x)Δx + (∂f/∂y)Δy`
- **Intuisi:** "Ketinggian baru adalah ketinggian awal, ditambah `(kemiringan arah x * langkah arah x)` ditambah `(kemiringan arah y * langkah arah y)` ."
- **Bentuk Visual:** Sebuah **bidang miring (lereng lurus)** yang memiliki **ketinggian DAN kemiringan yang sama persis** dengan lanskap asli di titik kontak. Ini adalah **"bidang singgung" (tangent plane)**.
  (Seperti gambar ketiga di screenshotmu).

Lapisan 3: Aproksimasi Orde ke-2 (Permukaan Melengkung)

- **Rumus:** `... + ½(∂²f/∂x²)(Δx)² + (∂²f/∂x∂y)ΔxΔy + ½(∂²f/∂y²)(Δy)²`
- **Intuisi:** Sekarang kita tidak hanya mencocokkan ketinggian dan kemiringan, tapi juga **semua kelengkungan** di titik kontak.

- **Bentuk Visual:** Sebuah **permukaan paraboloid** (seperti mangkok, puncak bukit, atau pelana kuda) yang "memeluk" lanskap asli dengan sangat erat.
  (Seperti gambar keempat di screenshotmu, yang menunjukkan bentuk pelana kuda).

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3. "Aha!" Moment: Notasi Matriks yang Elegan

- Lihat kembali rumus aproksimasi orde ke-2 yang panjang itu. Ternyata, ia bisa ditulis dengan sangat ringkas menggunakan alat-alat yang sudah kita kenal!

Rumus Panjang:
`f(x+Δx) ≈ f(x) + [∂f/∂x, ∂f/∂y] · [Δx, Δy] + ½[...]`

Aproksimasi Orde ke-1 (Linearisasi):

- Suku `(∂f/∂x)Δx + (∂f/∂y)Δy` adalah **DOT PRODUCT** antara **Gradien (** `∇f` **)** dengan **vektor langkah (** `Δx` **)**.

> `f(x+Δx) ≈ f(x) + J_f * Δx`
> *(Di mana `J_f` adalah Jacobian/Gradien dan `Δx` adalah vektor `[Δx, Δy]`).*

Aproksimasi Orde ke-2:

- Suku `½(∂²f/∂x²)(Δx)² + ...` yang rumit itu ternyata bisa ditulis sebagai perkalian matriks yang melibatkan **Matriks Hessian (** `H_f` **)**.

> `f(x+Δx) ≈ f(x) + J_f * Δx + ½ * Δxᵀ * H_f * Δx`

(Kamu tidak perlu khawatir tentang detail `Δxᵀ H Δx` sekarang. Intinya adalah untuk melihat betapa rapinya rumus ini menjadi saat ditulis dalam bahasa matriks).

Kesimpulan Akhir:

- Deret Taylor bisa digeneralisasi ke multi-dimensi untuk mengaproksimasi "lanskap".
- **Aproksimasi Orde 0:** Sebuah titik → menjadi sebuah **bidang datar**.
- **Aproksimasi Orde 1 (Linearisasi):** Sebuah garis singgung → menjadi sebuah **bidang singgung (tangent plane)**. Rumusnya menggunakan **Gradien/Jacobian**.
- **Aproksimasi Orde 2:** Sebuah parabola → menjadi sebuah **permukaan paraboloid**. Rumusnya menggunakan **Hessian**.

Ini adalah puncak dari penyatuan Aljabar Linear dan Kalkulus Multivariat. **Gradien** dan **Hessian** adalah "bahan bangunan" fundamental dari Deret Taylor multi-dimensi.

Tags: #mml-specialization #multivariate-calculus #taylor-series #jacobian #hessian #linear-approximation #quadratic-approximation