04: Matrix Multiplication as a Composition of Transformations

Chapter Goal: To understand that multiplying two matrices is the algebraic representation of a geometric action: **composing** or applying two transformations one after the other.

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1. Core Idea: Applying Transformations Sequentially

- **Problem:** What happens if we apply one transformation (A_1), and then apply another transformation (A_2) to the result?
- **Composition:** The action of "applying one transformation then another" is called a **Composition**.
- **Final Result:** The outcome of this entire process (from start to finish) is a **single, new transformation** that encapsulates both actions.

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2. Two Ways of Thinking, One Answer

There are two ways to find the matrix of this combined transformation:

Method #1: Geometric (Following the Basis Vectors)

1. Start with the standard basis vectors $\hat{\imath}$ and $\hat{\jmath}$.
2. Apply the **first transformation (**A_1**)** to $\hat{\imath}$ and $\hat{\jmath}$ to get $\hat{\imath}'$ and $\hat{\jmath}'$.
3. **NOW**, apply the **second transformation (**A_2**)** to the results from step 2 ($\hat{\imath}'$ and $\hat{\jmath}'$) to get the final positions $\hat{\imath}''$ and $\hat{\jmath}''$.
4. The combined matrix ($A_2 A_1$) is the matrix whose columns are the **FINAL** positions of the basis vectors, namely $\hat{\imath}''$ and $\hat{\jmath}''$.

Method #2: Algebraic (Matrix Multiplication)

1. The combined matrix ($A_2 A_1$) can be calculated directly by multiplying matrix A_2 by matrix A_1.
2. **Reading Rule:** Just like function composition `g(f(x))`, we read it from **right to left**. $A_2 A_1$ means "**apply** A_1 **first, then apply** A_2".
3. **Calculation Rule:** "Row times column". To find the entry in row `i`, column `j` of the resulting matrix, you multiply row `i` of the left matrix with column `j` of the right matrix.

The video demonstrates that both of these methods (Geometric and Algebraic) will produce the **exact same final matrix**.

3. Important Properties of Matrix Multiplication

- **NOT Commutative:**

> $$A_1A_2 \neq A_2A_1$$

- **Meaning:** The **order** of transformations matters GREATLY.
- **Visual Intuition:** "Rotating by 90° then reflecting vertically" gives a **DIFFERENT** final result than "Reflecting vertically first, then rotating by 90°". The video proves this visually and with matrix calculations.
- **Associative:**

> $$A_3(A_2A_1) = (A_3A_2)A_1$$

- **Meaning:** The way we **group** the multiplications doesn't matter, as long as the order remains the same.
- **Visual Intuition:** "Apply A_1, then A_2, then A_3" will always be the same, regardless of whether you think of it as `(A₃(A₂A₁))` or `((A₃A₂)A₁)`. The sequence of actions remains `A₁ → A₂ → A₃`.

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4. Key Message

- Perkalian Matriks is not just an arbitrary operation on numbers.
- It is the algebraic representation of a geometric action: **composing transformations in sequence.**
- Understanding this helps us see why its properties (like being non-commutative) make perfect sense.

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5. Worked Example: Proving `FR ≠ RF`

Let's prove that matrix multiplication is not commutative with a concrete example.

Initial Setup

Let's define two simple transformations:

- **Transformation `R` (90° Counter-Clockwise Rotation):**
  - $\hat{i}$ becomes `[0, 1]`
  - $\hat{j}$ becomes `[-1, 0]`
  - Matrix $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- **Transformation `F` (Horizontal Flip / Mirror):**
  - $\hat{\imath}$ becomes `[1, 0]` (stays the same)
  - $\hat{\jmath}$ becomes `[0, -1]` (flips down)
  - Matrix $F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

We will investigate the difference between `F * R` (rotate first, then flip) and `R * F` (flip first, then rotate).

Case 1: `F * R` (Rotate First, then Flip)

Method #1: Geometric (Following the Basis Vectors)

1. **Start:** $\hat{\imath} = [1, 0]$, $\hat{\jmath} = [0, 1]$.
2. **Step 1 (Apply `R`):**
   - $\hat{\imath}$ is rotated 90° to become $\hat{\imath}' = [0, 1]$.
   - $\hat{\jmath}$ is rotated 90° to become $\hat{\jmath}' = [-1, 0]$.
3. **Step 2 (Apply `F` to the RESULT):**
   - We flip $\hat{\imath}' = [0, 1]$ horizontally. This flips the y-component: $\hat{\imath}'' = [0, -1]$.
   - We flip $\hat{\jmath}' = [-1, 0]$ horizontally. This flips the y-component, but it's 0, so it stays: $\hat{\jmath}'' = [-1, 0]$.
4. **Resulting Matrix:** The columns are $\hat{\imath}''$ and $\hat{\jmath}''$.

$$FR = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Method #2: Algebraic (Matrix Multiplication)

$$FR = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- **Entry (1,1):** (Row 1 of F) · (Col 1 of R) = $[1, 0] \cdot [0, 1] = (1 \cdot 0) + (0 \cdot 1) = 0$.
- **Entry (1,2):** (Row 1 of F) · (Col 2 of R) = $[1, 0] \cdot [-1, 0] = (1 \cdot -1) + (0 \cdot 0) = -1$.
- **Entry (2,1):** (Row 2 of F) · (Col 1 of R) = $[0, -1] \cdot [0, 1] = (0 \cdot 0) + (-1 \cdot 1) = -1$.
- **Entry (2,2):** (Row 2 of F) · (Col 2 of R) = $[0, -1] \cdot [-1, 0] = (0 \cdot -1) + (-1 \cdot 0) = 0$.
- **Resulting Matrix:** $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

Both methods give the same result! ✔️

Case 2: `R * F` (Flip First, then Rotate)

Method #1: Geometric (Following the Basis Vectors)

1. **Start:** $\hat{i} = [1, 0]$, $\hat{j} = [0, 1]$.
2. **Step 1 (Apply `F`):**
   - $\hat{i}$ is flipped to become $\hat{i}' = [1, 0]$.
   - $\hat{j}$ is flipped to become $\hat{j}' = [0, -1]$.
3. **Step 2 (Apply `R` to the RESULT):**
   - We rotate $\hat{i}' = [1, 0]$ by 90°. It moves to the y-axis: $\hat{i}'' = [0, 1]$.
   - We rotate $\hat{j}' = [0, -1]$ by 90°. It moves to the positive x-axis: $\hat{j}'' = [1, 0]$.
4. **Resulting Matrix:** The columns are $\hat{i}''$ and $\hat{j}''$.

$$RF = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Method #2: Algebraic (Matrix Multiplication)

$$RF = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- **Entry (1,1):** $[0, -1] \cdot [1, 0] = 0$.
- **Entry (1,2):** $[0, -1] \cdot [0, -1] = 1$.
- **Entry (2,1):** $[1, 0] \cdot [1, 0] = 1$.
- **Entry (2,2):** $[1, 0] \cdot [0, -1] = 0$.
- **Resulting Matrix:** $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Both methods give the same result! ✔

Conclusion

Compare our two final results:

- `F * R` = $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (This is a reflection across the line `y=-x`).
- `R * F` = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (This is a reflection across the line `y=x`).

The results are completely different. This proves concretely that the order of matrix multiplication matters: `FR ≠ RF`.

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6. Practice Problems

Problem 1: Standard Multiplication

Calculate $A \cdot B$ where:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution:

$$A \cdot B = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

- **(1,1):** $[1, 2] \cdot [5, 7] = (1 \cdot 5) + (2 \cdot 7) = 5 + 14 = 19$
- **(1,2):** $[1, 2] \cdot [6, 8] = (1 \cdot 6) + (2 \cdot 8) = 6 + 16 = 22$
- **(2,1):** $[3, 4] \cdot [5, 7] = (3 \cdot 5) + (4 \cdot 7) = 15 + 28 = 43$
- **(2,2):** $[3, 4] \cdot [6, 8] = (3 \cdot 6) + (4 \cdot 8) = 18 + 32 = 50$

Answer:

$$A \cdot B = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Problem 2: Composition (Scaling then Shear)

Calculate $S \cdot T$ where `S` is a Shear and `T` is a Scaling transformation.

$$S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

Meaning: Apply Scaling first, then apply Shear.

Solution:

$$S \cdot T = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

- **(1,1):** $[1, 1] \cdot [2, 0] = (1 \cdot 2) + (1 \cdot 0) = 2$
- **(1,2):** $[1, 1] \cdot [0, 3] = (1 \cdot 0) + (1 \cdot 3) = 3$
- **(2,1):** $[0, 1] \cdot [2, 0] = (0 \cdot 2) + (1 \cdot 0) = 0$
- **(2,2):** $[0, 1] \cdot [0, 3] = (0 \cdot 0) + (1 \cdot 3) = 3$

Answer:

$$S \cdot T = \begin{bmatrix} 2 & 3 \\ 0 & 3 \end{bmatrix}$$

Problem 3 (Challenge): Reverse Order

Calculate $T \cdot S$ from the previous problem. Is the result the same?

$$T \cdot S = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$T \cdot S = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

- **(1,1):** $[2, 0] \cdot [1, 0] = (2 \cdot 1) + (0 \cdot 0) = 2$
- **(1,2):** $[2, 0] \cdot [1, 1] = (2 \cdot 1) + (0 \cdot 1) = 2$
- **(2,1):** $[0, 3] \cdot [1, 0] = (0 \cdot 1) + (3 \cdot 0) = 0$
- **(2,2):** $[0, 3] \cdot [1, 1] = (0 \cdot 1) + (3 \cdot 1) = 3$

Answer:

$$T \cdot S = \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$$

Conclusion: $S \cdot T \neq T \cdot S$. This proves again that order matters.

Problem 4 (Non-Square Matrices)

Calculate $M \cdot N$ where `M` is a 2x3 matrix and `N` is a 3x2 matrix.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad N = \begin{bmatrix} 7 & 8 \\ 9 & 1 \\ 2 & 3 \end{bmatrix}$$

(The result will be a 2x2 matrix)

Solution:

$$M \cdot N = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

- **(1,1):** $[1, 2, 3] \cdot [7, 9, 2] = (1 \cdot 7) + (2 \cdot 9) + (3 \cdot 2) = 7 + 18 + 6 = 31$
- **(1,2):** $[1, 2, 3] \cdot [8, 1, 3] = (1 \cdot 8) + (2 \cdot 1) + (3 \cdot 3) = 8 + 2 + 9 = 19$
- **(2,1):** $[4, 5, 6] \cdot [7, 9, 2] = (4 \cdot 7) + (5 \cdot 9) + (6 \cdot 2) = 28 + 45 + 12 = 85$
- **(2,2):** $[4, 5, 6] \cdot [8, 1, 3] = (4 \cdot 8) + (5 \cdot 1) + (6 \cdot 3) = 32 + 5 + 18 = 55$

Answer:

$$M \cdot N = \begin{bmatrix} 31 & 19 \\ 85 & 55 \end{bmatrix}$$

CONTOH PERKALIAN MATRIKS

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 1 \\ -1 & 2 \end{bmatrix} =$$

$$\begin{bmatrix} 1.1 + 2.3 + 3.\text{-}1 & 1.2 + 2.1 + 3.2 \\ 4.1 + 0.3 + 1.\text{-}1 & 4.2 + 0.1 + 1.2 \end{bmatrix} =$$

$$\begin{bmatrix} 4 & 10 \\ 3 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 5 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} (4 \times 1) + (5 \times 3) & (4 \times 2) + (5 \times 4) \\ (2 \times 1) + (6 \times 3) & (2 \times 2) + (6 \times 4) \end{bmatrix}$$

$$= \begin{bmatrix} 4 + 15 & 8 + 20 \\ 2 + 18 & 4 + 24 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 28 \\ 20 & 28 \end{bmatrix}$$

Tags: #mml-special-ization #linear-algebra #matrix-multiplication #compositions #transformations