

03: The Formal Definition of a Derivative


Chapter Goal: To translate our visual understanding of "gradient/slope" into a formal mathematical notation that we can use for calculations. This is a formal version of 3Blue1Brown's Chapter 2.

1. Background: From Visual to Formal

- **Goal:** To translate our visual understanding of "gradient/slope" into a formal mathematical notation that we can use for calculations.
- **Initial Intuition (Straight Line):** For a straight line, the slope is always constant and easy to calculate using Rise / Run ($\frac{\Delta y}{\Delta x}$) between any two points.

Definition

| | |
|------------------------------|---|
| The Rise Over the Run | Another way of referring to slope: The ratio of the change in vertical distance over the change in horizontal distance. |
|------------------------------|---|


$$\begin{aligned}\text{Slope} &= \frac{\text{Rise}}{\text{Run}} \\ &= \frac{\Delta y}{\Delta x}\end{aligned}$$

2. Handling a Curved Graph

- **Problem:** For a curve, the Rise/Run slope will be different depending on which two points we choose.

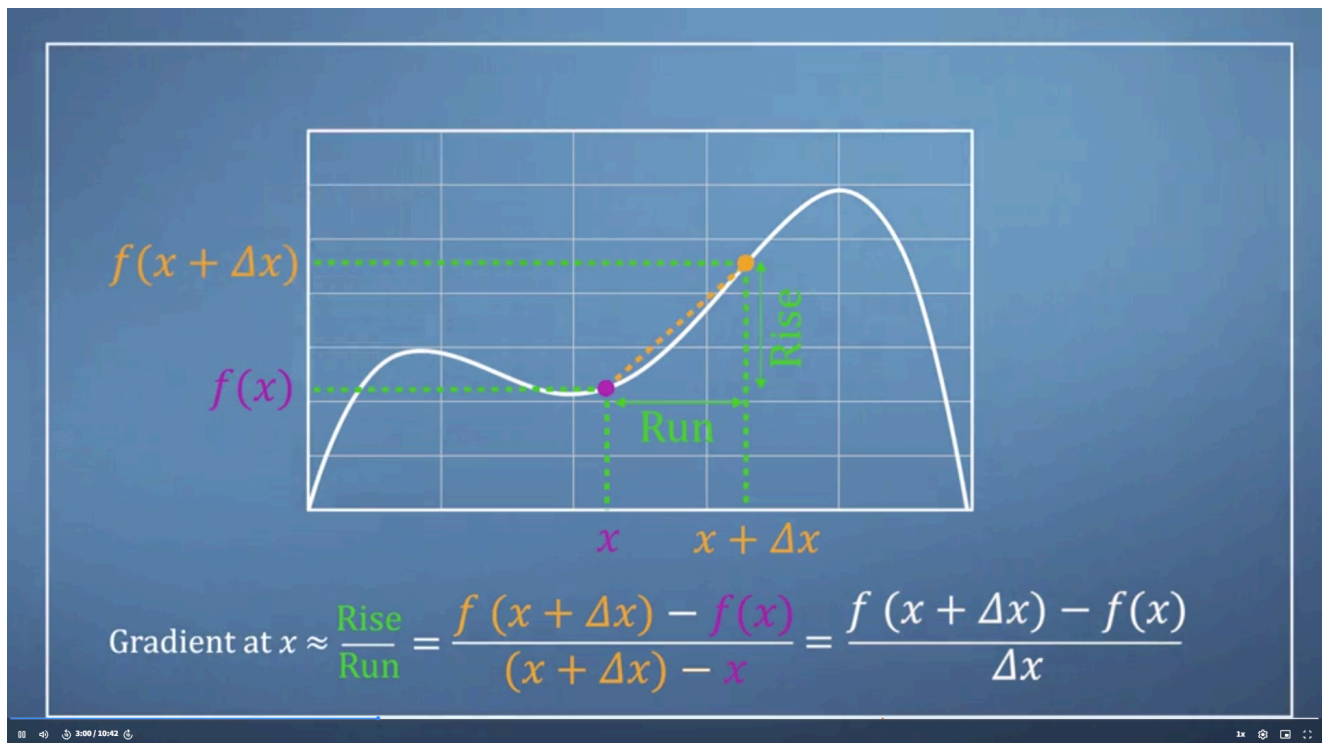
- **Strategy:**

1. Choose a single point x where we want to know the instantaneous slope.
2. Choose a second point that is very close, at $x + \Delta x$. Δx is our "tiny nudge".
3. Calculate the slope of the **secant line** (the connecting line) between these two points.

- **Run:** Δx

- **Rise:** $f(x + \Delta x) - f(x)$

4. **Approximated Slope:** $\frac{f(x + \Delta x) - f(x)}{\Delta x}$

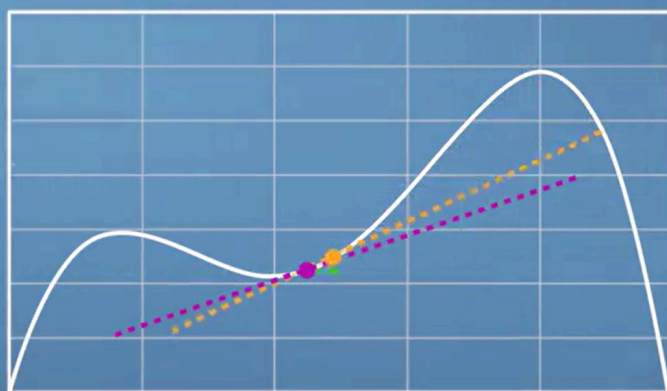


- **The Calculus Step (Limit):**

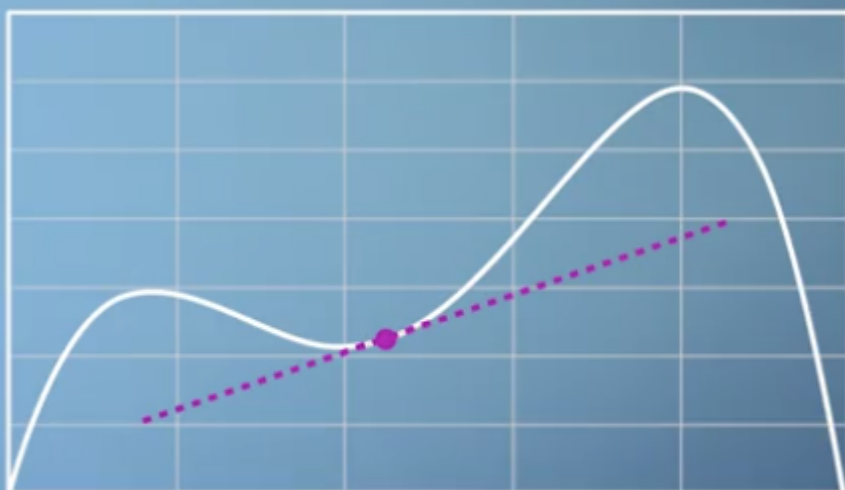
- This approximation gets better and better as Δx gets smaller.
- **Formal Definition of a Derivative:** The derivative $f'(x)$ is the **LIMIT** of this Rise/Run ratio as Δx approaches zero.

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

- The process of using this definition to find a derivative is called **Differentiation**.



Gradient at $x \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$



$$\frac{df}{dx} = f'(x) = \lim_{\Delta x \rightarrow 0} \left(\frac{f(x + \Delta x) - f(x)}{\Delta x} \right)$$

3. Practical Examples: Applying the Definition

A. Linear Function: $f(x) = 3x + 2$

1. Plug into the Formula:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{[3(x+\Delta x)+2] - [3x+2]}{\Delta x}$$

2. Expand the Algebra:

$$= \lim_{\Delta x \rightarrow 0} \frac{3x+3\Delta x+2-3x-2}{\Delta x}$$

3. Simplify (many terms cancel):

$$= \lim_{\Delta x \rightarrow 0} \frac{3\Delta x}{\Delta x}$$

4. **Cancel Δx :**

$$= \lim_{\Delta x \rightarrow 0} 3$$

5. **Take the Limit:** Since there is no Δx left, the limit is simply 3.

- **Result:** $f'(x) = 3$. This matches our intuition that the slope of the line $3x+2$ is always a constant 3.

B. Quadratic Function: $f(x) = 5x^2$

1. **Plug into the Formula:**

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{5(x+\Delta x)^2 - 5x^2}{\Delta x}$$

2. **Expand the Algebra:**

$$\begin{aligned} &= \lim_{\Delta x \rightarrow 0} \frac{5(x^2 + 2x\Delta x + (\Delta x)^2) - 5x^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{5x^2 + 10x\Delta x + 5(\Delta x)^2 - 5x^2}{\Delta x} \end{aligned}$$

3. **Simplify:**

$$= \lim_{\Delta x \rightarrow 0} \frac{10x\Delta x + 5(\Delta x)^2}{\Delta x}$$

4. **Cancel Δx (divide each term in the numerator by Δx):**

$$= \lim_{\Delta x \rightarrow 0} (10x + 5\Delta x)$$

5. **Take the Limit (as $\Delta x \rightarrow 0$):**

- The $10x$ term is unaffected.
- The $5\Delta x$ term becomes $5 \cdot 0 = 0$ and vanishes.

- **Result:** $f'(x) = 10x$.

4. The "Shortcuts": Rules of Differentiation

Going through the limit definition from first principles every time is tedious. Therefore, mathematicians discovered "shortcuts" or general rules after seeing patterns from many examples.

- **Power Rule:**

$$\frac{d}{dx}(ax^b) = abx^{b-1}$$

(The power b comes down to multiply a , and then the power is reduced by one).

- **Example:** $\frac{d}{dx}(5x^2) = 5 \cdot 2 \cdot x^{2-1} = 10x^1 = 10x$. It matches!
- **Sum Rule:**

$$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$$

(The derivative of a sum is the sum of the derivatives).

- **Example:** $\frac{d}{dx}(3x + 2) = \frac{d}{dx}(3x) + \frac{d}{dx}(2) = 3 + 0 = 3$. It matches!

- **Key Message:** These rules are simply "shortcuts" that have been proven from the fundamental limit definition of a derivative.

Rumus Turunan fungsi aljabar

1. $y = e \rightarrow y' = 0$
2. $y = x^n \rightarrow y' = nx^{n-1}$
3. $y = ax^n \rightarrow y' = na x^{n-1}$
4. $y = u + v \rightarrow y' = u' + v'$
5. $y = u - v \rightarrow y' = u' - v'$
6. $y = u \cdot v \rightarrow y' = u'v + uv'$
7. $y = \frac{u}{v} \rightarrow y' = \frac{u'v - uv'}{v^2}$

Rumus Turunan Fungsi Eksponen Logaritma

1. $y = \ln x \rightarrow y' = \frac{1}{x}$
2. $y = \ln ax \rightarrow y' = \frac{a}{x}$
3. $y = {}^a \log x \rightarrow y' = \frac{1}{x \ln a}$
4. $y = e^x \rightarrow y' = e^x$
5. $y = e^{ax} \rightarrow y' = ae^{ax}$

Rumus Turunan fungsi Trigonometri

1. $y = \sin x \rightarrow y' = \cos x$
2. $y = \cos x \rightarrow y' = -\sin x$
3. $y = \tan x \rightarrow y' = \sec^2 x$
4. $y = \cotan x \rightarrow y' = -\operatorname{cosec}^2 x$
5. $y = \operatorname{cosec} x \rightarrow y' = -\operatorname{cosec} x \cdot \cotan x$
6. $y = \sec x \rightarrow y' = \sec x \cdot \tan x$
7. $y = \sin ax \rightarrow y' = a \cos ax$
8. $y = \cos ax \rightarrow y' = -a \sin ax$
9. $y = \tan ax \rightarrow y' = a \sec^2 ax$
10. $y = \cotan ax \rightarrow y' = -a \operatorname{cosec}^2 ax$

1. $y = U^n \rightarrow y' = nU^{n-1} \cdot U'$
2. $y = \ln U \rightarrow y' = \frac{1}{U} \cdot U'$
3. $y = a^U \rightarrow y' = a^U \ln a \cdot U'$
4. $y = e^U \rightarrow y' = e^U \cdot U'$
5. $y = \sin U \rightarrow y' = \cos U \cdot U'$
6. $y = \cos U \rightarrow y' = -\sin U \cdot U'$
7. $y = \tan U \rightarrow y' = \sec^2 U \cdot U'$
8. $y = \cotan U \rightarrow y' = -\operatorname{cosec}^2 U \cdot U'$

1.2 Rumus-rumus Integral Tak Tentu

1. $\int k x^n dx = \frac{k}{n+1} x^{n+1} + C$ dengan syarat $n \neq -1$

Contoh: $\int 2x^3 dx = \frac{2}{3+1} x^{3+1} + C = \frac{1}{2} x^4 + C$

2. $\int k dx = kx + C$; suatu konstanta

Contoh: $\int 5 dx = 5x + C$

3. $\int \frac{1}{x} dx = \ln x$, jika $x > 0$

4. $\int e^x dx = e^x + C$

7. $\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C$

8. $\int \cos x dx = \sin x + C$

9. $\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$

Contoh:

$$\int \cos(5x-1) dx = \frac{1}{5} \sin(5x-1) + C$$

10. $\int [f(x)]^n \cdot f'(x) dx = \frac{1}{n+1} [f(x)]^{n+1} + C$

Contoh:

Tentukan $\int 2x\sqrt{x^2-1} dx$

Jawab:

Misalkan $\int f(x) = x^2 - 1$ maka $f'(x) = 2x$

$$\begin{aligned} \int 2x\sqrt{x^2-1} dx &= \int [f(x)]^{1/2} \cdot f'(x) dx \\ &= \int \frac{1}{1+\frac{1}{2}} [f(x)]^{\frac{1}{2}+1} + C \\ &= \frac{2}{3} [x^2-1]^{\frac{3}{2}} + C \end{aligned}$$

11. $\int \tan x dx = \ln \sec x = -\ln \cos u + C$

12. $\int \cotan x dx = \ln \sin u + C$

13. $\int \sec x dx = \tan x + C$

14. $\int \operatorname{cosec} x dx = \ln(\operatorname{cosec} x - \cotan x) + C$

15. $\int \sec^2 x dx = \tan x + C$

16. $\int \operatorname{cosec}^2 x dx = -\cotan x + C$

17. $\int \tan^2 x dx = \tan x - x + C$

18. $\int \cotan^2 x dx = -\cotan x - x + C$

19. $\int \sin^2 x dx = \frac{1}{2}(x - \sin x \cdot \cos x) + C$

20. $\int \cos^2 x dx = \frac{1}{2}(x + \sin x \cdot \cos x) + C$

21. $\int \sec x \cdot \tan x dx = \sec x + C$

22. $\int \operatorname{cosec} x \cdot \cotan x dx = -\operatorname{cosec} x + C$

23. $\int \sin^n x \cos x dx = \frac{1}{n+1} \sin^{n+1} x + C$

Contoh: $\int \sin^5 x \cos x dx = \frac{1}{6} \sin^6 x + C$

Tags: #mml-specialization #multivariate-calculus #derivatives #limits #first-principles
#power-rule