04: Orthogonal Matrices

Chapter Goal: To introduce a special class of "superhero" matrices in linear algebra and data science: **Orthogonal Matrices**.

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1. A New Tool: The Matrix Transpose (A^T)

- **Definition:** The **transpose** of a matrix `A` (denoted A^T) is a "mirrored" version of `A` where the rows become columns and the columns become rows.
- **How to do it:** "Flip" the elements of the matrix across its main diagonal.
- The entry `(i, j)` in A^T is the entry `(j, i)` from `A`.
- **Example:**

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

(The element `2` in row 1, col 2 moves to row 2, col 1).

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2. Our "Superhero": The Orthogonal Matrix

- **Definition:** A square matrix `A` is called **orthogonal** if all of its column vectors have two special properties:
  1. **Ortho- (Orthogonal):** Every column is perpendicular to every other column.
     - $\vec{a}_i \cdot \vec{a}_j = 0$ if $i \neq j$.
  2. **-normal (Normal):** Every column has a length of 1 (it's a unit vector).
     - $\vec{a}_i \cdot \vec{a}_i = 1$.
- **Combined:** The columns of an orthogonal matrix form an orthonormal basis.
- **Example:** Rotation matrices are perfect examples of orthogonal matrices.

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3. Superpower #1: The Inverse is Incredibly Easy to Find

This is the main "Aha!" Moment of the video.

- Let's calculate $A^T A$ for an orthogonal matrix `A`.
- **"Row-times-Column" Logic:**
  - The `i`-th row of A^T is actually the `i`-th column of `A` ($\vec{a}_i$) "laid down".
  - The `j`-th column of `A` is $\vec{a}_j$.

- Therefore, the `(i, j)` entry of the product $A^T A$ is the Dot Product $\vec{a}_i \cdot \vec{a}_j$.
- **Calculation Result:**
  - If $i \neq j$ (off-diagonal entries), $\vec{a}_i \cdot \vec{a}_j = 0$ (from the orthogonal property).
  - If $i = j$ (on-diagonal entries), $\vec{a}_i \cdot \vec{a}_i = 1$ (from the normal property).
- The resulting matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  This is the **IDENTITY MATRIX (`I`)**!
- **Magical Conclusion:**
  We found that $A^T A = I$. Recalling that the definition of an inverse is $A^{-1} A = I$, it must be that:

  > For an orthogonal matrix, its **Inverse IS its Transpose!**
  >
  > $$A^{-1} = A^T$$

- **Why this is awesome:** Calculating a transpose (just flipping rows/columns) is computationally much faster and easier than calculating an inverse (with Gauss-Jordan Elimination).

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4. Other Superpowers

- **Determinant of ±1:** Because orthogonal matrices only "rotate" space without stretching or squishing it, their area/volume scaling factor (Determinan) must be `1` (if orientation is preserved) or `-1` (if orientation is flipped, e.g., a reflection).
- **Invertible Transformation:** Since $\det(A) \neq 0$, the transformation never "squishes" space. All information is preserved.
- **Change of Basis becomes the Dot Product:** As seen before, if a new basis (the columns of `A`) is orthonormal, the change of basis from the standard world to the new world can be calculated with simple dot products, not a complicated matrix inverse multiplication.

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5. Key Message for Data Science

- Whenever possible, we want to work with **orthonormal bases**.
- If we can transform our data into a basis where the "features" are mutually orthogonal, many calculations become incredibly simple and stable.

- This is one of the reasons why methods like **PCA (Principal Component Analysis)** are so powerful, because they specifically search for a new, optimal orthonormal basis for the data.

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6. Concrete Example: The 90° Rotation Matrix

Let's take our "superhero," the 90° counter-clockwise rotation matrix (`R`), and prove its superpowers.

Matrix `R`:

- $\hat{\imath}$ becomes `[0, 1]`, and $\hat{\jmath}$ becomes `[-1, 0]`.

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Step 1: Verify if `R` is Truly a "Superhero" (Orthogonal)

We must check the two conditions for its columns, $\vec{c}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\vec{c}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

1. Are they Unit Length (Normal)?

- $|\vec{c}_1| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$. (OK ✔)
- $|\vec{c}_2| = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$. (OK ✔)

2. Are they Perpendicular (Ortho)?

- We calculate their dot product: $\vec{c}_1 \cdot \vec{c}_2$.
- $[0, 1] \cdot [-1, 0] = (0 \cdot -1) + (1 \cdot 0) = 0 + 0 = 0$.
- Since the result is 0, they are mutually orthogonal. (OK ✔)

Conclusion: Matrix `R` is proven to be an orthogonal matrix.

Step 2: Find its Inverse the "Hard Way" (Standard Method)

Now, let's find R^{-1} using the standard 2x2 inverse formula:

$$R^{-1} = \frac{1}{\det(R)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

- **Calculate the Determinant of `R`:**
  $\det(R) = (0 \cdot 0) - (-1 \cdot 1) = 0 - (-1) = 1$.
  (As we expected, the determinant is 1!)

- **Apply the Inverse Formula:**
  $R^{-1} = \frac{1}{1}\begin{bmatrix} 0 & -(-1) \\ -1 & 0 \end{bmatrix}$

$$R^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

  So, the inverse of our 90° rotation matrix is $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Step 3: Find its Inverse with the "Superpower" (Transpose)

Now, let's try the "superpower" of an orthogonal matrix. The theory says $A^{-1} = A^T$. Let's prove it.

- **Take the Original Matrix `R`:**

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- **Calculate its Transpose (R^T):**
  We "flip" the matrix along its diagonal. Rows become columns, columns become rows.
  - The first row `[0, -1]` becomes the first column.
  - The second row `[1, 0]` becomes the second column.

$$R^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

"Aha!" Moment: Compare the Results!

Look at what we just found:

- **Inverse from the Hard Way (R^{-1}):** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$
- **The Transpose (R^T):** $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

They are EXACTLY the same!

We have proven with a concrete example that for a rotation matrix (which is orthogonal), calculating its inverse is as simple as calculating its transpose.

Additional Geometric Intuition

- `R` is a **90° counter-clockwise** rotation.

- R^{-1} (its inverse) should be the "undo" action, which is a **90° clockwise** rotation.
- What does the matrix $R^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ do?
  - $\hat{i}$ becomes `[0, -1]` (points down).
  - $\hat{j}$ becomes `[1, 0]` (points right).
- This is precisely a 90° clockwise rotation! Everything connects beautifully.

> **Note**
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> Ini contoh bahasa indonesia, yang menurut aku lebih mudah dipahami dan lebih intuitive dengan contoh yang lebih konkret dan pengerjaan yang lebih runut

Kita akan bedah prosesnya pakai **CONTOH NYATA** dengan angka. Kita akan membuat sebuah matriks, mengecek apakah dia Orthogonal, dan membuktikan "keajaibannya".

Mari kita pakai angka dari segitiga siku-siku (3, 4, 5) biar gampang hitungnya. Kita bagi 5 biar panjangnya jadi 1 (0.6 dan 0.8).

Langkah 1: Mari Kita "Ciptakan" Matriks A

Kita buat matriks 2×2. Anggap ini Matriks Transformasi.

Kolom-kolomnya adalah vektor basis kita.

$$A = \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

Mari kita lihat isinya:

- **Kolom 1 (Vektor $\mathbf{a}_1$):** $\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$
- **Kolom 2 (Vektor $\mathbf{a}_2$):** $\begin{bmatrix} -0.8 \\ 0.6 \end{bmatrix}$

Langkah 2: Audisi Masuk Klub (Cek Syarat)

Apakah matriks A ini layak disebut **Orthogonal**? Dia harus lolos 2 tes ketat.

Tes A: Apakah mereka Tegak Lurus? (Dot Product harus 0)

Mari kita dot-kan Kolom 1 dengan Kolom 2.

$$\mathbf{a}_1 \cdot \mathbf{a}_2 = (0.6 \times -0.8) + (0.8 \times 0.6)$$
$$= -0.48 + 0.48 = \mathbf{0}$$

✅ **Lolos!** Mereka tegak lurus (90°).

Tes B: Apakah Panjangnya Satu? (Unit Length)

Mari kita cek panjang Kolom 1 ($\mathbf{a}_1 \cdot \mathbf{a}_1$).

$$|\mathbf{a}_1|^2 = (0.6 \times 0.6) + (0.8 \times 0.8)$$
$$= 0.36 + 0.64 = \mathbf{1}$$

✅ **Lolos!** Panjangnya 1. (Berlaku juga untuk Kolom 2, coba cek: $(-0.8)^2 + 0.6^2 = 1$).

Kesimpulan: Matriks A adalah **Matriks Orthogonal**.

Langkah 3: Operasi Transpose (A^T)

Sekarang kita cari A^T (Transpose).

Caranya: Baris jadi Kolom.

Matriks Awal:

$$A = \begin{bmatrix} \color{blue}{0.6} & \color{blue}{-0.8} \\ \color{red}{0.8} & \color{red}{0.6} \end{bmatrix}$$

(Baris 1 warna biru, Baris 2 warna merah)

Matriks Transpose (Diputar):

$$A^T = \begin{bmatrix} \color{blue}{0.6} & \color{red}{0.8} \\ \color{blue}{-0.8} & \color{red}{0.6} \end{bmatrix}$$

Lihat? Yang tadinya mendatar (biru), sekarang jadi tegak.

Langkah 4: Keajaiban ($A^T \times A$)

Ini momen "Simsalabim"-nya. Mari kita kalikan Transpose dengan Aslinya.

$$A^T A = \begin{bmatrix} 0.6 & 0.8 \\ -0.8 & 0.6 \end{bmatrix} \times \begin{bmatrix} 0.6 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}$$

Mari kita hitung satu per satu elemennya (Baris Kiri $\times$ Kolom Kanan):

1. Kiri Atas: (Baris 1 A^T $\times$ Kolom 1 A)

   $(0.6)(0.6) + (0.8)(0.8) = 0.36 + 0.64 = \mathbf{1}$

(Ini sebenarnya $\mathbf{a}_1 \cdot \mathbf{a}_1$, makanya hasilnya 1)

2. Kanan Atas: (Baris 1 A^T $\times$ Kolom 2 A)

   $(0.6)(-0.8) + (0.8)(0.6) = -0.48 + 0.48 = \mathbf{0}$

   (Ini sebenarnya $\mathbf{a}_1 \cdot \mathbf{a}_2$, makanya hasilnya 0)

3. Kiri Bawah: (Baris 2 A^T $\times$ Kolom 1 A)

   $(-0.8)(0.6) + (0.6)(0.8) = -0.48 + 0.48 = \mathbf{0}$

4. Kanan Bawah: (Baris 2 A^T $\times$ Kolom 2 A)

   $(-0.8)(-0.8) + (0.6)(0.6) = 0.64 + 0.36 = \mathbf{1}$

Kenapa Matrix Multiplication = Dot Product?

Ini inti kebingunganmu:

> *"Kenapa (Baris Kiri x Kolom Kanan) itu kamu sebut $\mathbf{a}_1 \cdot \mathbf{a}_1$?"*

Mari kita lihat secara visual "Bedah Mayat" matriksnya.

Matriks Asli A (Kolomnya adalah vektor $\mathbf{a}_1$ dan $\mathbf{a}_2$):

$$A = \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix}$$

(Bayangkan $\mathbf{a}_1$ dan $\mathbf{a}_2$ itu tiang berdiri).

Matriks Transpose A^T (Barisnya adalah vektor $\mathbf{a}_1$ dan $\mathbf{a}_2$):

$$A^T = \begin{bmatrix} — & \mathbf{a}_1 & — \\ — & \mathbf{a}_2 & — \end{bmatrix}$$

(Karena di-transpose, tiangnya jadi tidur).

Sekarang Kita Kalikan ($A^T \times A$):

Aturan Matriks kan **"Baris dikali Kolom"**.

1. Elemen Kiri Atas:

   Kita ambil Baris 1 dari A^T dikali Kolom 1 dari A.

   - Baris 1 dari A^T itu siapa? **Vektor $\mathbf{a}_1$**.
   - Kolom 1 dari A itu siapa? **Vektor $\mathbf{a}_1$**.

   Jadi operasinya adalah: $\mathbf{a}_1$ dikali $\mathbf{a}_1$.

Karena cara ngalinya "elemen per elemen lalu dijumlah", itu sama persis dengan definisi Dot Product ($\mathbf{a}_1 \cdot \mathbf{a}_1$).

2. Elemen Kanan Atas:

   Kita ambil Baris 1 dari A^T dikali Kolom 2 dari A.

   - Baris 1: $\mathbf{a}_1$.
   - Kolom 2: $\mathbf{a}_2$.
   - Operasinya: $\mathbf{a}_1 \cdot \mathbf{a}_2$.

Visualisasinya:

$$\begin{bmatrix} —\mathbf{a}_1— \\ —\mathbf{a}_2— \end{bmatrix} \times \begin{bmatrix} | & | \\ \mathbf{a}_1 & \mathbf{a}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} (\mathbf{a}_1 \cdot \mathbf{a}_1) & (\mathbf{a}_1 \cdot \mathbf{a}_2) \\ (\mathbf{a}_2 \cdot \mathbf{a}_1) & (\mathbf{a}_2 \cdot \mathbf{a}_2) \end{bmatrix}$$

Hasilnya (Sesuai sifat Orthogonal):

- $\mathbf{a}_1 \cdot \mathbf{a}_1 = 1$ (Panjangnya 1).
- $\mathbf{a}_1 \cdot \mathbf{a}_2 = 0$ (Tegak lurus).

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Langkah 5: Hasil Akhir

Hasil perkalian tadi adalah:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Ini adalah **Matriks Identitas (I)**!

Apa artinya?

Dalam matematika, jika $X \times Y = I$, maka X adalah Invers dari Y.

Karena kita barusan membuktikan $A^T \times A = I$, maka:

> A^T **adalah Invers dari** A**.**

Kenapa ini bikin hidup indah?

Bayangkan kamu punya matriks raksasa 1000×1000.

- Kalau matriks biasa: Nyari inversnya butuh superkomputer berjam-jam.
- Kalau matriks orthogonal: Cukup "balik" baris jadi kolom (Transpose). Selesai dalam 1 detik.

Gimana? Dengan angka 0.6 dan 0.8 ini, apakah alurnya lebih kelihatan nyata?

Tags: #mml-special-ization #linear-algebra #orthogonal-matrix #transpose #inverse-matrix #pca