06: Changing Basis

Chapter Goal: To understand that coordinate systems are arbitrary and to learn a projection-based method for finding the coordinates of a vector in a **new orthogonal basis**.

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1. Core Idea: Coordinate Systems are Arbitrary

- **Reminder:** A vector $\vec{r}$ is a geometric object (an arrow) that exists independently of any coordinate system.
- **Coordinates** (e.g., `[3, 4]`): Are just a "recipe" or a set of instructions to get to the tip of that arrow.
- `[3, 4]` in the standard basis ($\vec{e}_1, \vec{e}_2$) means: "Walk 3 units in the direction of $\vec{e}_1$, then 4 units in the direction of $\vec{e}_2$."
- **"Aha!" Moment:** The choice of our "rulers" or basis vectors ($\vec{e}_1, \vec{e}_2$) is entirely **arbitrary**. We can choose other "rulers" that are skewed, not perpendicular to each other, or have different lengths.
- The same vector $\vec{r}$ will have a **different "recipe" (coordinates)** if we use a different set of basis vectors (e.g., a basis of $\vec{b}_1, \vec{b}_2$).
- **Conclusion:** The vector itself is fundamental; the numbers (coordinates) we use to describe it are not. Coordinates only have meaning once we know which Basis is being used.

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2. Special Case: Changing to a New Orthogonal Basis

- **Problem:** We have a vector $\vec{r}$ in the standard basis `e` ($\vec{r}_e = [3, 4]$). We have a new basis formed by $\vec{b}_1, \vec{b}_2$. We want to find the coordinates of $\vec{r}$ in the `b` basis ($\vec{r}_b = [?, ?]$).
- **Special Condition:** This method **ONLY WORKS** if the new basis vectors ($\vec{b}_1$ and $\vec{b}_2$) are **mutually orthogonal** (perpendicular).
- **How to Check for Orthogonality:**
  - Calculate their Dot Product: $\vec{b}_1 \cdot \vec{b}_2$.
  - If $\vec{b}_1 \cdot \vec{b}_2 = 0$, then they are orthogonal.

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3. Geometric Intuition: Projection as a Coordinate-Finding Tool

- If the new basis $\vec{b}_1, \vec{b}_2$ is orthogonal, then they form a new set of perpendicular "axes".

- To find "how much of $\vec{r}$ goes in the direction of $\vec{b}_1$," we can simply project $\vec{r}$ onto $\vec{b}_1$.
- **Logic:**
  - The vector projection of $\vec{r}$ onto $\vec{b}_1$ is the component of $\vec{r}$ that is parallel to $\vec{b}_1$.
  - The vector projection of $\vec{r}$ onto $\vec{b}_2$ is the component of $\vec{r}$ that is parallel to $\vec{b}_2$.
- Because $\vec{b}_1$ and $\vec{b}_2$ are orthogonal, $\vec{r}$ can be perfectly reconstructed by summing these two projection vectors:

$$\vec{r} = \text{proj}_{\vec{b}_1}(\vec{r}) + \text{proj}_{\vec{b}_2}(\vec{r})$$

- **"Aha!" Moment:** The scalar coefficients of each basis vector in this sum are our new coordinates!

$$\vec{r} = (c_1 \cdot \vec{b}_1) + (c_2 \cdot \vec{b}_2)$$

  (Where $c_1 \cdot \vec{b}_1 = \text{proj}_{\vec{b}_1}(\vec{r})$)
  Therefore, the coordinates of $\vec{r}$ in basis `b` are $\vec{r}_b = [c_1, c_2]$.

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4. The Calculation Process

- **Goal:** Find the scalars c_1 and c_2.
- Remember the scalar multiple from the Vector Projection formula:

$$\text{proj}_{\vec{s}}(\vec{r}) = \left(\frac{\vec{s} \cdot \vec{r}}{\vec{s} \cdot \vec{s}}\right)\vec{s}$$

  The coefficient `c` we are looking for is that scalar part: $c = \frac{\vec{s} \cdot \vec{r}}{\vec{s} \cdot \vec{s}}$.
- **Calculating c_1 (the first coordinate in basis `b`):**

$$c_1 = \frac{\vec{b}_1 \cdot \vec{r}_e}{\vec{b}_1 \cdot \vec{b}_1} = \frac{\vec{b}_1 \cdot \vec{r}_e}{|\vec{b}_1|^2}$$

- **Calculating c_2 (the second coordinate in basis `b`):**

$$c_2 = \frac{\vec{b}_2 \cdot \vec{r}_e}{\vec{b}_2 \cdot \vec{b}_2} = \frac{\vec{b}_2 \cdot \vec{r}_e}{|\vec{b}_2|^2}$$

- **Example from the Video:**
  - $\vec{r}_e = [3, 4]$
  - $\vec{b}_1 = [2, 1]$, $\vec{b}_2 = [-2, 4]$
  - **Check Orthogonality:** $\vec{b}_1 \cdot \vec{b}_2 = (2)(-2) + (1)(4) = -4 + 4 = 0$. OK.
  - **Calculate c_1:**

$$c_1 = \frac{[2, 1] \cdot [3, 4]}{[2, 1] \cdot [2, 1]} = \frac{6 + 4}{4 + 1} = \frac{10}{5} = 2$$

- **Calculate c_2:**

$$c_2 = \frac{[-2,4]\cdot[3,4]}{[-2,4]\cdot[-2,4]} = \frac{-6+16}{4+16} = \frac{10}{20} = 0.5$$

- **Result:** The new coordinates of $\vec{r}$ in basis `b` are $\vec{r}_b = [2, 0.5]$.
- **Verification:**
  $2\vec{b}_1 + 0.5\vec{b}_2 = 2[2,1] + 0.5[-2,4] = [4,2] + [-1,2] = [3,4]$. Matches the original $\vec{r}_e$.

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5. Important Point

- This projection method is a quick and elegant way to perform a change of basis, **BUT ONLY if the new basis is orthogonal.**
- If the basis is not orthogonal, we need a more powerful tool: **matrices** (which will be covered later in this specialization and was detailed in the 3Blue1Brown series).

Tags: #mml-specialization #linear-algebra #change-of-basis #basis-vectors #orthogonality #projection

> **Note**
>
> Catatan tambahan agar lebih mudah memahami (lebih ditekankan pemahaman fundamental daripada hanya jago menghitung)

Mode 2: Intuitive Explanation (Currency & Language Analogy)

Okay, take a deep breath. The concept of "Changing Basis" sounds intimidating, but it's actually something very simple that we do in everyday life. Let's use an analogy to make it "click".

1. The Intuition: "Same Thing, Different Label"

Imagine you have a **$100 USD bill** in your hand.

- If you ask an American (USD Basis): "What's the value of this?", they'll say: **"100"**.
- If you ask an Indonesian (Rupiah Basis): "What's the value of this?", they'll say: **"1,600,000"**.

Key Point:

- Did the value of the money change? **No.** The money is still that single piece of paper.
- What changed was the **number** we use to describe it, depending on which "standard" (**basis**) we're using.

In Vectors:

- The vector $\vec{r}$ is the money (the real object in space).
- The Basis ($\vec{e}$ or $\vec{b}$) is the currency (our way of measuring it).

2. The Visualization: The Grid Paper

So far, we've always used the Standard Basis ($\vec{e}$).

- $\vec{e}_1$ (X-axis): 1 step straight to the Right.
- $\vec{e}_2$ (Y-axis): 1 step straight Up.

When we say a vector $\vec{r} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, it's an instruction: "Walk 3 times in the $\vec{e}_1$ direction, then 4 times in the $\vec{e}_2$ direction."

What is Changing Basis?
Imagine an alien comes to visit. This alien has a weird way of walking.

- Their Step 1 ($\vec{b}_1$): Diagonally right and slightly up (e.g., our $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$).
- Their Step 2 ($\vec{b}_2$): Diagonally left and up (e.g., our $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$).

Now, the alien asks:

> "Hey human, that vector $\vec{r}$ of yours... if I use my walking style, how many of my $\vec{b}_1$ steps and how many of my $\vec{b}_2$ steps does it take?"

That is the essence of Changing Basis. We want to change the recipe $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ into a new recipe, alien-style.

3. The Connection to Projection

Now, this is the best part. You've already learned about Projection. In the video, the instructor says there's an easy way to find the "Alien's Recipe" using Projection, **BUT** there's a condition:

- **Mandatory Condition:** The Alien's Step 1 ($\vec{b}_1$) and Step 2 ($\vec{b}_2$) must be **Perpendicular ($90°$).**

Why? Because if they are perpendicular, we can use the "shadow" logic we just learned.

Remember the Vector Projection formula?

$$\text{proj}_{\vec{b}}(\vec{r}) = \left(\frac{\vec{r} \cdot \vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

The number inside the parentheses, $\left(\frac{\vec{r} \cdot \vec{b}}{|\vec{b}|^2}\right)$, is the **NEW COORDINATE!**

It's the answer to the question: "How many $\vec{b}$'s do I need?"

4. Breaking Down the Video's Example

Let's break down the instructor's calculation step-by-step.

- **Given:**
  - Target Vector: $\vec{r} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ (This is our "money").
  - Alien Basis 1: $\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.
  - Alien Basis 2: $\vec{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

Step 1: Check if the alien walks perpendicularly (Dot Product)

$$\vec{b}_1 \cdot \vec{b}_2 = (2 \times -2) + (1 \times 4) = -4 + 4 = 0$$

The result is 0. Great! They are orthogonal. We can use the projection formula.

Step 2: Find the new coordinate for the $\vec{b}_1$ direction
We project $\vec{r}$ onto $\vec{b}_1$.

$$\text{Coordinate 1} = \frac{\vec{r} \cdot \vec{b}_1}{|\vec{b}_1|^2}$$

- $\vec{r} \cdot \vec{b}_1 = (3)(2) + (4)(1) = 6 + 4 = 10.$
- $|\vec{b}_1|^2 = 2^2 + 1^2 = 5.$
- Result: $\frac{10}{5} = 2.$
  This means: We need **2 steps** in the $\vec{b}_1$ direction.

Step 3: Find the new coordinate for the $\vec{b}_2$ direction
We project $\vec{r}$ onto $\vec{b}_2$.

$$\text{Coordinate 2} = \frac{\vec{r} \cdot \vec{b}_2}{|\vec{b}_2|^2}$$

- $\vec{r} \cdot \vec{b}_2 = (3)(-2) + (4)(4) = -6 + 16 = 10.$
- $|\vec{b}_2|^2 = (-2)^2 + 4^2 = 4 + 16 = 20.$
- Result: $\frac{10}{20} = 0.5.$
  This means: We need **0.5 steps** in the $\vec{b}_2$ direction.

CONCLUSION:

- In the standard world, the vector is called $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.
- In the Alien's world ($\vec{b}$), the EXACT SAME vector is called $\begin{bmatrix} 2 \\ 0.5 \end{bmatrix}$.

This means: "Walk 2 times diagonally right ($\vec{b}_1$), then walk half a time diagonally left ($\vec{b}_2$), and you will arrive at the same point as $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$."

The core idea is just changing the "walking recipe" to get to the same destination.