03: Types of Matrix Transformations

Chapter Goal: To build a visual catalog of common 2x2 matrices and their corresponding geometric effects on space. This solidifies the idea of a matrix as a transformation.

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1. Basic Transformations

Identity Matrix (`I`)

- **Form:** $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (The columns are the standard basis vectors).
- **Effect:** Does nothing. $\hat{\imath}$ stays at `[1,0]`, $\hat{\jmath}$ stays at `[0,1]`. For any vector $\vec{v}$, $I\vec{v} = \vec{v}$.
- It's called the "Identity" because it preserves the identity of every vector.

Scaling

- **Form:** A diagonal matrix, $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$.
- **Effect:**
  - Stretches or squishes space in the x-direction by a factor of `a`.
  - Stretches or squishes space in the y-direction by a factor of `d`.
- The unit square transforms into a rectangle.
- If `a` or `d` is a fraction (between 0 and 1), space is squished.

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2. "Flipping" Transformations

Axis Flip (Reflection over an axis)

- **Form:** A diagonal matrix with a negative value, e.g., $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$.
- **Effect:** $\hat{\imath}$, which was at `[1,0]`, is now "flipped" to `[-1,0]`.
- **Consequence:** This changes the **orientation** or "handedness" of the coordinate system.

Inversion

- **Form:** $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (which is equal to `-I`).
- **Effect:** Flips both axes. Every vector `[x,y]` becomes `[-x,-y]`. This is equivalent to a **180-degree rotation**.

General Reflections (Mirrors)

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: Reflects space across the line `y=x`. ($\hat{\imath}$ and $\hat{\jmath}$ swap places).
- $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$: Reflects space across the line `y=-x`.
- $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$: A vertical mirror (reflects across the y-axis).
- $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$: A horizontal mirror (reflects across the x-axis).

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3. "Slanting" Transformations

Shear

- **Form:** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
- **Effect:**
  - The $\hat{\imath}$ axis stays in place ($\hat{\imath}_{new} = [1, 0]$).
  - The $\hat{\jmath}$ axis is "pushed" or "sheared" to a new position ($\hat{\jmath}_{new} = [1, 1]$).
- **Visualization:** The unit square transforms into a parallelogram.

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4. "Turning" Transformations

Rotation

- **Example: 90° Counter-Clockwise Rotation**
  - $\hat{\imath}$ moves to the position `[0, 1]`.
  - $\hat{\jmath}$ moves to the position `[-1, 0]`.
  - **Matrix Form:** $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- **General Formula for Rotation by an angle `θ`:**

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

  (Note: There might be slight notational/convention differences in the video, but this is the standard form for counter-clockwise rotation).

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5. Key Message

- All complex transformations can essentially be seen as a **combination** of these basic transformations (stretching, rotating, reflecting, shearing).
- In data science, these transformations are very useful, for example, to normalize facial data so that all faces are oriented in the same direction before processing.
- The next step is to think about what happens when we apply one transformation, and then follow it with another.

Tags: #mml-specialization #linear-algebra #matrices #transformations #scaling #rotation #shear #reflection