03: Calculating Eigenvectors

Chapter Goal: To translate the geometric intuition of an <u>eigenvector</u> into a formal algebraic procedure for finding both eigenvalues and eigenvectors.

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1. Translating Intuition into Algebra

- **Geometric Intuition:** An eigenvector $\vec{x}$ is a vector that, when transformed by `A`, stays on its own span, and is only scaled by a factor `λ`.
- **Formal Equation:** This idea can be written directly as:

$$A\vec{x} = \lambda\vec{x}$$

- **Anatomy of the Equation:**
    - `A`: The `n x n` transformation matrix.
    - $\vec{x}$: The `n`-dimensional eigenvector we want to find.
    - `λ`: The eigenvalue (a single number/scalar) paired with $\vec{x}$.
- Our task is to find the pairs of $\vec{x}$ (non-zero) and `λ` that satisfy this equation.

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2. The Algebraic "Trick" to Simplify the Problem

- **Problem:** The equation $A\vec{x} = \lambda\vec{x}$ is difficult to solve directly because it involves matrix multiplication on the left and scalar multiplication on the right.
- **Solution:** We rearrange the equation so that all terms are on one side and we can factor out $\vec{x}$.
    1. Rewrite $\lambda\vec{x}$ as matrix multiplication: $\lambda\vec{x} = (\lambda I)\vec{x}$, where `I` is the identity matrix.
    2. Move to the left side: $A\vec{x} - (\lambda I)\vec{x} = \vec{0}$.
    3. Factor out $\vec{x}$:

$$(A - \lambda I)\vec{x} = \vec{0}$$

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3. The "Aha!" Moment: The Connection to the Determinant

- The equation $(A - \lambda I)\vec{x} = \vec{0}$ is a system of linear equations.
- We are looking for a **non-zero** solution for $\vec{x}$ (a non-trivial solution).
- **Key Condition:** A system of equations $M\vec{x} = \vec{0}$ only has a non-trivial solution if and only

if the matrix `M` "squishes" space into a lower dimension.
- **The "Squish" Test:** A matrix squishes space if and only if its **determinant is zero**.
- **Conclusion:** To find a non-zero eigenvector $\vec{x}$, the matrix $(A - \lambda I)$ **must have a determinant of zero**.

$$\det(A - \lambda I) = 0$$

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4. The "Recipe" for Hunting Eigenvalues and Eigenvectors

This gives us a two-step recipe:

Step 1: Hunt for Eigenvalues (λ)

1. Form the matrix $A - \lambda I$.
2. Calculate its determinant: $\det(A - \lambda I)$. The result will be a polynomial in λ (called the **Characteristic Polynomial**).
3. Solve the equation $\det(A - \lambda I) = 0$ to find all possible values of λ .

Step 2: Hunt for Eigenvectors (`x`)

1. Take each λ value you found, one by one.
2. Plug that value of λ back into the equation $(A - \lambda I)\vec{x} = \vec{0}$.
3. Solve this system of linear equations to find the vector $\vec{x}$ (or set of vectors) that is the solution. This is the **Null Space** of the matrix $A - \lambda I$.

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5. Practical Examples

A. Vertical Scaling: $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

- **Step 1 (Find λ):**
$\det\left(\begin{bmatrix} 1 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix}\right) = (1 - \lambda)(2 - \lambda) = 0$.
The solutions are $\lambda_1 = 1$ and $\lambda_2 = 2$.
- **Step 2 (Find `x` for λ=1):**
$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This results in the equation $x_2 = 0$. The eigenvectors are all vectors on the x-axis, of the form `[t, 0]` .
- **Step 2 (Find `x` for λ=2):**
$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This results in the equation $-x_1 = 0$. The eigenvectors are all vectors on the y-axis, of the form `[0, t]` .

B. 90° Rotation: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

- **Step 1 (Find λ):**
  $\det\left(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}\right) = (-\lambda)(-\lambda) - (-1)(1) = \lambda^2 + 1 = 0.$

- The equation $\lambda^2 + 1 = 0$ has **no real number solutions**.

- **Conclusion:** There are no real eigenvalues, which means there are no real eigenvectors. This matches our visual intuition.

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6. Final Message

- Although this process seems long, it is a systematic method to find the "axes of action" for any transformation.

- In the real world, for large matrices, computers use iterative numerical methods, not solving polynomials.

- The most important takeaway is understanding the concept behind the method ($\det(A - \lambda I) = 0$), not becoming an expert in manual calculation.

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7. Worked Example: Hunting for the Eigenvectors

Context: We have already "hunted" for the eigenvalues. For the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, we found two "magic" values: $\lambda_1 = 1$ and $\lambda_2 = 2$.

Current Goal: For each λ, we must find the corresponding eigenvector x. We will be finding the "special line" (the Null Space) of the matrix $A - \lambda I$.

Hunt #1: The Eigenvector for $\lambda = 1$

1. **The "Recipe" to Use:**
   $(A - \lambda I)\vec{x} = \vec{0}$. We will plug in $\lambda = 1$.

2. **Prepare the "Squish" Matrix:**
   $A - 1I = \begin{bmatrix} 1 - 1 & 0 \\ 0 & 2 - 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

3. **Write the System of Equations:**
   We are looking for a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ that satisfies:

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4. **Translate to Regular Algebra (Row-times-Column):**
   - **Row 1:** $(0 \cdot x_1) + (0 \cdot x_2) = 0 \implies 0 = 0$. This equation is true, but gives us no information.
   - **Row 2:** $(0 \cdot x_1) + (1 \cdot x_2) = 0 \implies x_2 = 0$. **This is the key piece of information!**
5. **Interpret the Result ($x_2 = 0$):**
   - This system of equations gives us one condition: "The second component (x_2) of the eigenvector must be zero."
   - What about x_1? The equations say nothing about x_1. This means x_1 is **free** to be any number.
6. **Write the General Solution:**
   - The eigenvector $\vec{x}$ must be of the form $\begin{bmatrix} x_1 \\ 0 \end{bmatrix}$.
   - To show that x_1 can be any number, we replace it with a parameter, typically `t`.

$$\vec{x} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   This is a mathematical description of the entire x-axis.
   - **Examples:** If t=1, $\vec{x} = [1, 0]$. If t=5, $\vec{x} = [5, 0]$. If t=-100, $\vec{x} = [-100, 0]$. All are valid eigenvectors.

- **Conclusion of Hunt #1:**
  The eigenvectors paired with $\lambda = 1$ are all vectors that lie on the x-axis, which can be represented as `[t, 0]`.

Hunt #2: The Eigenvector for `λ = 2`

We repeat the exact same process.

1. **The "Recipe" to Use:**
   $(A - \lambda I)\vec{x} = \vec{0}$ with $\lambda = 2$.
2. **Prepare the "Squish" Matrix:**
   $$A - 2I = \begin{bmatrix} 1 - 2 & 0 \\ 0 & 2 - 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$
3. **Write the System of Equations:**

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

4. **Translate to Regular Algebra:**
   - **Row 1:** $(-1 \cdot x_1) + (0 \cdot x_2) = 0 \implies -x_1 = 0 \implies x_1 = 0$. **This is the key info!**
   - **Row 2:** $(0 \cdot x_1) + (0 \cdot x_2) = 0 \implies 0 = 0$. No new information.

5. **Interpret the Result ($x_1 = 0$):**
   - The condition is: "The first component (x_1) of the eigenvector must be zero."
   - x_2 is free to be any number.
6. **Write the General Solution:**
   - The eigenvector $\vec{x}$ must be of the form $\begin{bmatrix} 0 \\ x_2 \end{bmatrix}$.
   - We replace x_2 with the parameter `t`.

$$\vec{x} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   This is a mathematical description of the entire y-axis.

- **Conclusion of Hunt #2:**
  The eigenvectors paired with $\lambda = 2$ are all vectors that lie on the y-axis, which can be represented as `[0, t]`.

Answering Your Confusion:

- The terms `0*x₂` and `-1*x₁` are simply the results of the "row-times-column" multiplication shown in the video and worked out above.
- The notations `[t, 0]` and `[0, t]` are not single specific vectors. They are a representation of a whole "family" or "line" of vectors. The free parameter `t` tells you that any scalar multiple of the basis vector `[1,0]` (for the first case) or `[0,1]` (for the second case) is also a valid eigenvector.

Tags: #mml-specialization #linear-algebra #eigenvectors #eigenvalues #characteristic-polynomial