**Problem 1:** Suppose that we flip a loaded coin twice, with probability p of landing heads. Let X = 1 if the first flip lands heads, and X = 0 if it lands tails. Similarly, let Y = 1 if the second flip lands heads, and Y = 0 if it lands tails. Compute the joint distribution of (X, Y). Verify that your answer is correct by checking that all probabilities sum to 1.

**Problem 2:** Let (X,Y) be discrete with probability mass function p(x,y) given in the following table:

$x \setminus y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
	0.06				
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

(a) Compute  $P(X \leq 2, Y \leq 2)$ .

(b) Compute P(X = Y).

(c) Compute P(X > Y).

**Problem 3:** Suppose that (X,Y) is continuous with probability density function

$$f(x,y) = \begin{cases} cx^2y & : x^2 \le y \le 1, \\ 0 & : \text{ otherwise.} \end{cases}$$

(a) Determine the value of c that makes f a valid density.

(b) Compute  $P(X \ge Y)$ .

**Problem 4:** Suppose that the continuous random vector (X,Y) is uniformly distributed over the triangle in  $\mathbb{R}^2$  with vertices (-1,0), (1,0), and (0,1).

(a) Compute the density function of (X, Y).

(b) Compute  $P(X \le 3/4, Y \le 3/4)$ .

**Problem 5:** Suppose that (X,Y) is continuous with probability density function

$$f(x,y) = \begin{cases} 30xy^2 & : x - 1 \le y \le 1 - x, \ 0 \le x \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute F(1/2, 1/2).

**Problem 6:** Compute the marginal probability mass function distribution  $p_X(x)$  of the discrete random vector (X,Y) with probability mass function

$x \setminus y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Verify that your computations are correct by making sure all marginal probabilities sum to 1. How would you compute the other marginal mass function  $p_Y(y)$ ?

**Problem 7:** Suppose that (X,Y) is continuous with probability density function

$$f(x,y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \le y \le 1, \\ 0 & : \text{ otherwise.} \end{cases}$$

Compute the marginal density functions  $f_X(x)$  and  $f_Y(y)$ .

**Problem 8:** Suppose the joint PMF of two discrete random variables X and Y is given by

$x \setminus y$	1	2	3	$\mid 4 \mid$
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

Determine the conditional mass function  $p_{X|Y}(x|1)$ .

**Problem 9:** Suppose that (X,Y) is continuous with probability density function

$$f(x,y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \le y \le 1, \\ 0 & : \text{ otherwise.} \end{cases}$$

Compute the conditional density function  $f_{Y|X}(y|x)$ .

**Problem 10:** A soft-drink machine has a random amount Y in supply at the beginning of a given day and dispenses a random amount X during the day (with measurements in gallons). It is not resupplied during the day, hence  $X \leq Y$ . It has been observed that X and Y have joint density given by

$$f(x,y) = \begin{cases} 1/2, &: 0 \le x \le y \le 2, \\ 0 &: \text{ otherwise.} \end{cases}$$

Find the conditional density f(x|y) and evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

**Problem 11:** Suppose that a person's score X on a mathematics aptitude test is a number between 0 and 1, and that their score Y on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores X and Y are distributed according to the following joint PDF

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & : 0 \le x, y \le 1, \\ 0 & : \text{ otherwise.} \end{cases}$$

(a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

(b) If a student's score on the music test is 0.3, what is the probability that their score on the mathematics test will be greater than 0.8?

(c) If a student's score on the mathematics test is 0.3, what is the probability that their score on the music test will be greater than 0.8?

**Problem 12:** Let X be the number of heads obtained from a single flip of a coin, so that  $X \sim \mathcal{B}er(\theta)$  for some unknown probability  $\theta$ . Suppose further that  $\theta$  is an observed value of a  $\mathcal{B}eta(2,2)$  random variable. If we flip the coin and obtain x=1, how should we "update" the distribution of  $\theta$ ?

**Problem 13:** Suppose that three random vectors X, Y, and Z are jointly continuous with density function

$$f(x, y, z) = \begin{cases} c(x + 2y + 3z) & : 0 \le x, y, z \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

- (a) Determine the value of c that makes f(x, y, z) a valid density function.
- (b) Compute the marginal density  $f_{XY}(x, y)$ .
- (c) Compute the probability P(Z < 1/2 | X = 1/4, Y = 3/4).

**Problem 14:** Suppose that X, Y, and Z have joint "mixed density" function

$$f(x, y, z) = \begin{cases} cx^{1+y+z} (1-x)^{3-y-z} & : 0 < x < 1, \ y, z \in \{0, 1\}, \\ 0 & : \text{ otherwise.} \end{cases}$$

- (a) Determine the value of c.
- (b) Compute the marginal "density"  $f_{XY}(x, y)$ .
- (c) Compute the conditional "density"  $f_{Z|XY}(z|1/4,1)$ .

**Problem 15:** Suppose that two measurements X and Y are made of the rainfall at a certain location on May 1 of two consecutive years. Supposing that X and Y are independent and that their marginal density functions are each given by

$$f_X(x) = \begin{cases} 2x & : 0 \le x \le 1, \\ 0 & : \text{otherwise,} \end{cases} \text{ and } f_Y(y) = \begin{cases} 2y & : 0 \le y \le 1, \\ 0 & : \text{otherwise,} \end{cases}$$

determine their joint density and compute the probability  $P(X + Y \le 1)$ .

**Problem 16:** Suppose that the joint density function of two continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} kx^2y^2 & : x^2 + y^2 \le 1, \\ 0 & : \text{ otherwise,} \end{cases}$$

for some constant k. Prove that X and Y are dependent

**Problem 17:** Suppose that a point (X,Y) is chosen at random from the rectangle R defined as follows:

$$R = \{(x, y) : 0 \le x \le 2, \ 1 \le y \le 4\}.$$

(a) Determine the joint density of X and Y, the marginal density of X, and the marginal density of Y.

(b) Are X and Y independent?

**Problem 18:** Let  $n \ge 1$  be an integer and suppose  $X \sim \Gamma(n+1,1)$ . Suppose that  $Y_1, Y_2, \dots, Y_n$  is an IID random sample such that the conditional distributions of each  $Y_i$  given X have densities

$$f(y_i|x) = \begin{cases} \frac{1}{x} & : 0 < y_i < x, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the joint density of the random sample.

(b) Determine the conditional density of X for any given observed values of the random sample.