

**Problem 1:** Suppose that we flip a loaded coin twice, with probability  $p$  of landing heads. Let  $X = 1$  if the first flip lands heads, and  $X = 0$  if it lands tails. Similarly, let  $Y = 1$  if the second flip lands heads, and  $Y = 0$  if it lands tails. Compute the joint distribution of  $(X, Y)$ . Verify that your answer is correct by checking that all probabilities sum to 1.

**Problem 2:** Let  $(X, Y)$  be discrete with probability mass function  $p(x, y)$  given in the following table:

$x \backslash y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

(a) Compute  $P(X \leq 2, Y \leq 2)$ .

(b) Compute  $P(X = Y)$ .

(c) Compute  $P(X > Y)$ .

**Problem 3:** Suppose that  $(X, Y)$  is continuous with probability density function

$$f(x, y) = \begin{cases} cx^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of  $c$  that makes  $f$  a valid density.

(b) Compute  $P(X \geq Y)$ .

**Problem 4:** Suppose that the continuous random vector  $(X, Y)$  is uniformly distributed over the triangle in  $\mathbb{R}^2$  with vertices  $(-1, 0)$ ,  $(1, 0)$ , and  $(0, 1)$ .

(a) Compute the density function of  $(X, Y)$ .

(b) Compute  $P(X \leq 3/4, Y \leq 3/4)$ .

**Problem 5:** Suppose that  $(X, Y)$  is continuous with probability density function

$$f(x, y) = \begin{cases} 30xy^2 & : x - 1 \leq y \leq 1 - x, \ 0 \leq x \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute  $F(1/2, 1/2)$ .

**Problem 6:** Compute the marginal probability mass function distribution  $p_X(x)$  of the discrete random vector  $(X, Y)$  with probability mass function

$x \backslash y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Verify that your computations are correct by making sure all marginal probabilities sum to 1. How would you compute the other marginal mass function  $p_Y(y)$ ?

**Problem 7:** Suppose that  $(X, Y)$  is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the marginal density functions  $f_X(x)$  and  $f_Y(y)$ .

**Problem 8:** Suppose the joint PMF of two discrete random variables  $X$  and  $Y$  is given by

$x \backslash y$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

Determine the conditional mass function  $p_{X|Y}(x|1)$ .

**Problem 9:** Suppose that  $(X, Y)$  is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the conditional density function  $f_{Y|X}(y|x)$ .

**Problem 10:** A soft-drink machine has a random amount  $Y$  in supply at the beginning of a given day and dispenses a random amount  $X$  during the day (with measurements in gallons). It is not resupplied during the day, hence  $X \leq Y$ . It has been observed that  $X$  and  $Y$  have joint density given by

$$f(x, y) = \begin{cases} 1/2, & : 0 \leq x \leq y \leq 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Find the conditional density  $f(x|y)$  and evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

**Problem 11:** Suppose that a person's score  $X$  on a mathematics aptitude test is a number between 0 and 1, and that their score  $Y$  on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores  $X$  and  $Y$  are distributed according to the following joint PDF

$$f(x, y) = \begin{cases} \frac{2}{5}(2x + 3y) & : 0 \leq x, y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

(b) If a student's score on the music test is 0.3, what is the probability that their score on the mathematics test will be greater than 0.8?

(c) If a student's score on the mathematics test is 0.3, what is the probability that their score on the music test will be greater than 0.8?

**Problem 12:** Let  $X$  be the number of heads obtained from a single flip of a coin, so that  $X \sim \mathcal{Ber}(\theta)$  for some unknown probability  $\theta$ . Suppose further that  $\theta$  is an observed value of a  $\mathcal{Beta}(2, 2)$  random variable. If we flip the coin and obtain  $x = 1$ , how should we “update” the distribution of  $\theta$ ?

**Problem 13:** Suppose that three random vectors  $X$ ,  $Y$ , and  $Z$  are jointly continuous with density function

$$f(x, y, z) = \begin{cases} c(x + 2y + 3z) & : 0 \leq x, y, z \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of  $c$  that makes  $f(x, y, z)$  a valid density function.

(b) Compute the marginal density  $f_{XY}(x, y)$ .

(c) Compute the probability  $P(Z < 1/2 \mid X = 1/4, Y = 3/4)$ .

**Problem 14:** Suppose that  $X$ ,  $Y$ , and  $Z$  have joint “mixed density” function

$$f(x, y, z) = \begin{cases} cx^{1+y+z}(1-x)^{3-y-z} & : 0 < x < 1, \ y, z \in \{0, 1\}, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of  $c$ .

(b) Compute the marginal “density”  $f_{XY}(x, y)$ .

(c) Compute the conditional “density”  $f_{Z|XY}(z|1/4, 1)$ .

**Problem 15:** Suppose that two measurements  $X$  and  $Y$  are made of the rainfall at a certain location on May 1 of two consecutive years. Supposing that  $X$  and  $Y$  are independent and that their marginal density functions are each given by

$$f_X(x) = \begin{cases} 2x & : 0 \leq x \leq 1, \\ 0 & : \text{otherwise,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & : 0 \leq y \leq 1, \\ 0 & : \text{otherwise,} \end{cases}$$

determine their joint density and compute the probability  $P(X + Y \leq 1)$ .

**Problem 16:** Suppose that the joint density function of two continuous random variables  $X$  and  $Y$  is given by

$$f(x, y) = \begin{cases} kx^2y^2 & : x^2 + y^2 \leq 1, \\ 0 & : \text{otherwise,} \end{cases}$$

for some constant  $k$ . Prove that  $X$  and  $Y$  are dependent.

**Problem 17:** Suppose that a point  $(X, Y)$  is chosen at random from the rectangle  $R$  defined as follows:

$$R = \{(x, y) : 0 \leq x \leq 2, 1 \leq y \leq 4\}.$$

(a) Determine the joint density of  $X$  and  $Y$ , the marginal density of  $X$ , and the marginal density of  $Y$ .

(b) Are  $X$  and  $Y$  independent?

**Problem 18:** Let  $n \geq 1$  be an integer and suppose  $X \sim \Gamma(n+1, 1)$ . Suppose that  $Y_1, Y_2, \dots, Y_n$  is an IID random sample such that the conditional distributions of each  $Y_i$  given  $X$  have densities

$$f(y_i|x) = \begin{cases} \frac{1}{x} & : 0 < y_i < x, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the joint density of the random sample.

(b) Determine the conditional density of  $X$  for any given observed values of the random sample.