Problem 1: Suppose that we flip a loaded coin twice, with probability p of landing heads. Let $X = 1$ if the first flip lands heads, and $X = 0$ if it lands tails. Similarly, let $Y = 1$ if the second flip lands heads, and $Y = 0$ if it lands tails. Compute the joint distribution of (X, Y). Verify that your answer is correct by checking that all probabilities sum to 1.

Problem 2: Let (X, Y) be discrete with probability mass function $p(x, y)$ given in the following table:

$x \diagdown y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

(a) Compute $P(X \leq 2, Y \leq 2)$.

(b) Compute $P(X = Y)$.

(c) Compute $P(X > Y)$.

Problem 3: Suppose that (X, Y) is continuous with probability density function

$$f(x,y) = \begin{cases} cx^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c that makes f a valid density.

(b) Compute $P(X \geq Y)$.

Problem 4: Suppose that the continuous random vector (X, Y) is uniformly distributed over the triangle in $\mathbb{R}^2$ with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Compute the density function of (X, Y).

(b) Compute $P(X \leq 3/4, Y \leq 3/4)$.

Problem 5: Suppose that (X, Y) is continuous with probability density function

$$f(x,y) = \begin{cases} 30xy^2 & : x - 1 \leq y \leq 1 - x,\ 0 \leq x \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute $F(1/2, 1/2)$.

Problem 6: Compute the marginal probability mass function distribution $p_X(x)$ of the discrete random vector (X, Y) with probability mass function

$x \diagdown y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Verify that your computations are correct by making sure all marginal probabilities sum to 1. How would you compute the other marginal mass function $p_Y(y)$?

Problem 7: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2 y & : x^2 \le y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

Problem 8: Suppose the joint PMF of two discrete random variables X and Y is given by

$x \diagdown y$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

Determine the conditional mass function $p_{X|Y}(x|1)$.

Problem 9: Suppose that (X, Y) is continuous with probability density function

$$f(x,y) = \begin{cases} \frac{21}{4}x^2 y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the conditional density function $f_{Y|X}(y|x)$.

Problem 10: A soft-drink machine has a random amount Y in supply at the beginning of a given day and dispenses a random amount X during the day (with measurements in gallons). It is not resupplied during the day, hence $X \leq Y$. It has been observed that X and Y have joint density given by

$$f(x,y) = \begin{cases} 1/2, & : 0 \leq x \leq y \leq 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Find the conditional density $f(x|y)$ and evaluate the probability that less than $1/2$ gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

Problem 11: Suppose that a person's score X on a mathematics aptitude test is a number between 0 and 1, and that their score Y on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores X and Y are distributed according to the following joint PDF

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & : 0 \le x, y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

(b) If a student's score on the music test is 0.3, what is the probability that their score on the mathematics test will be greater than 0.8?

(c) If a student's score on the mathematics test is 0.3, what is the probability that their score on the music test will be greater than 0.8?

Problem 12: Let X be the number of heads obtained from a single flip of a coin, so that $X \sim \mathcal{B}er(\theta)$ for some unknown probability θ. Suppose further that θ is an observed value of a $\mathcal{B}eta(2,2)$ random variable. If we flip the coin and obtain $x = 1$, how should we "update" the distribution of θ?

Problem 13: Suppose that three random vectors X, Y, and Z are jointly continuous with density function

$$f(x,y,z) = \begin{cases} c(x+2y+3z) & : 0 \le x,y,z \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c that makes $f(x,y,z)$ a valid density function.

(b) Compute the marginal density $f_{XY}(x,y)$.

(c) Compute the probability $P\left(Z < 1/2 \mid X = 1/4, Y = 3/4\right)$.

Problem 14: Suppose that X, Y, and Z have joint "mixed density" function

$$f(x,y,z) = \begin{cases} cx^{1+y+z}(1-x)^{3-y-z} & : 0 < x < 1,\ \ y,z \in \{0,1\}, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

(b) Compute the marginal "density" $f_{XY}(x,y)$.

(c) Compute the conditional "density" $f_{Z|XY}(z|1/4,1)$.

Problem 15: Suppose that two measurements X and Y are made of the rainfall at a certain location on May 1 of two consecutive years. Supposing that X and Y are independent and that their marginal density functions are each given by

$$f_X(x) = \begin{cases} 2x & : 0 \leq x \leq 1, \\ 0 & : \text{otherwise}, \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & : 0 \leq y \leq 1, \\ 0 & : \text{otherwise}, \end{cases}$$

determine their joint density and compute the probability $P(X + Y \leq 1)$.

Problem 16: Suppose that the joint density function of two continuous random variables X and Y is given by

$$f(x, y) = \begin{cases} kx^2y^2 & : x^2 + y^2 \leq 1, \\ 0 & : \text{otherwise}, \end{cases}$$

for some constant k. Prove that X and Y are dependent.

Problem 17: Suppose that a point (X, Y) is chosen at random from the rectangle R defined as follows:

$$R = \{(x, y) : 0 \leq x \leq 2,\ 1 \leq y \leq 4\}.$$

(a) Determine the joint density of X and Y, the marginal density of X, and the marginal density of Y.

(b) Are X and Y independent?

Problem 18: Let $n \geq 1$ be an integer and suppose $X \sim \Gamma(n+1, 1)$. Suppose that $Y_1, Y_2, \ldots, Y_n$ is an IID random sample such that the conditional distributions of each Y_i given X have densities

$$f(y_i|x) = \begin{cases} \frac{1}{x} & : 0 < y_i < x, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the joint density of the random sample.

(b) Determine the conditional density of X for any given observed values of the random sample.