2.10. Distribution and quantile functions

Definition 2.10

A *distribution function* of a probability measure P on $\mathbb{R}$ is the function $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(s) = P\big((-\infty, s]\big).$$

In particular:

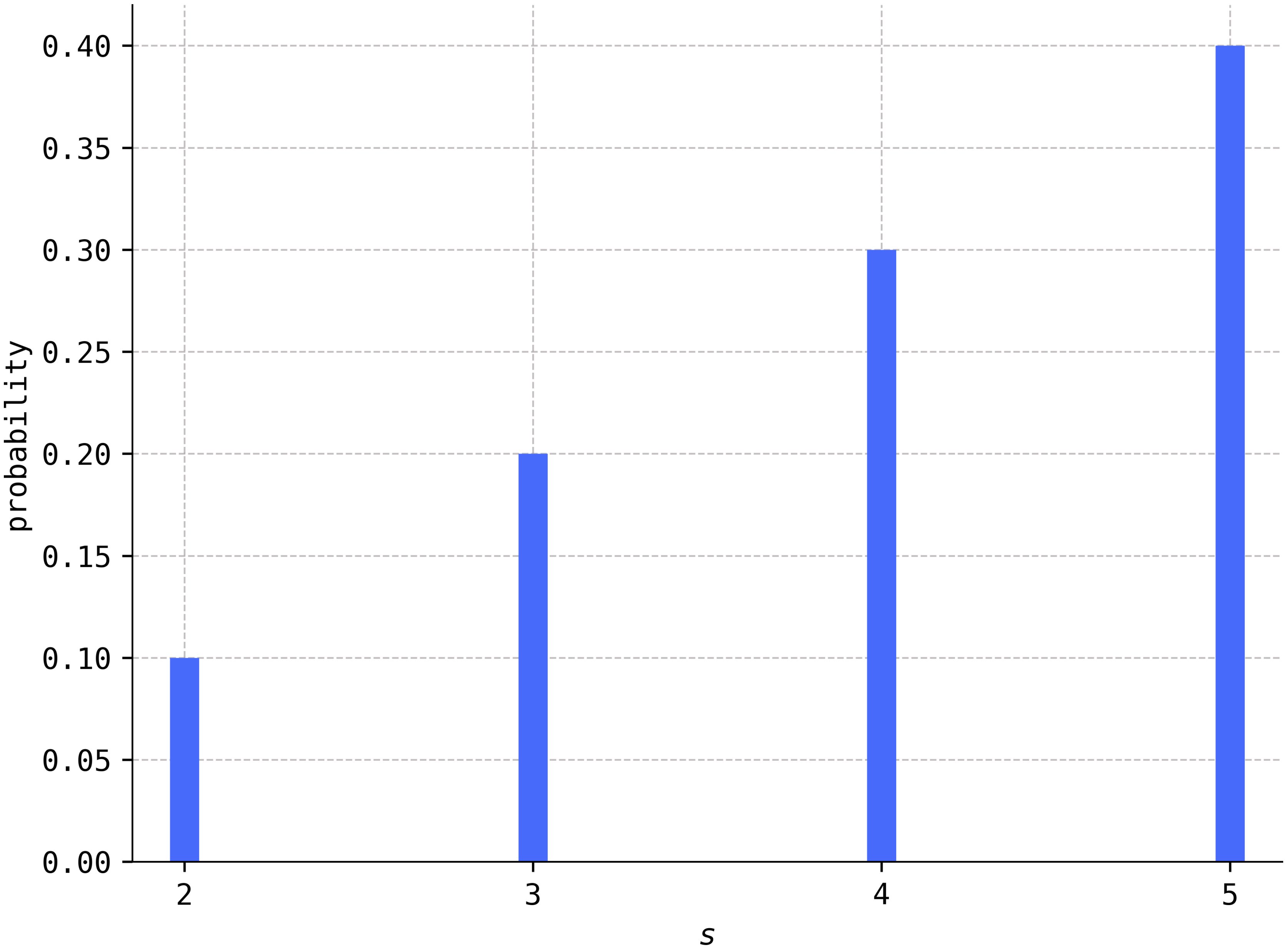
1. If P is discrete with probability mass function $p(s)$, then

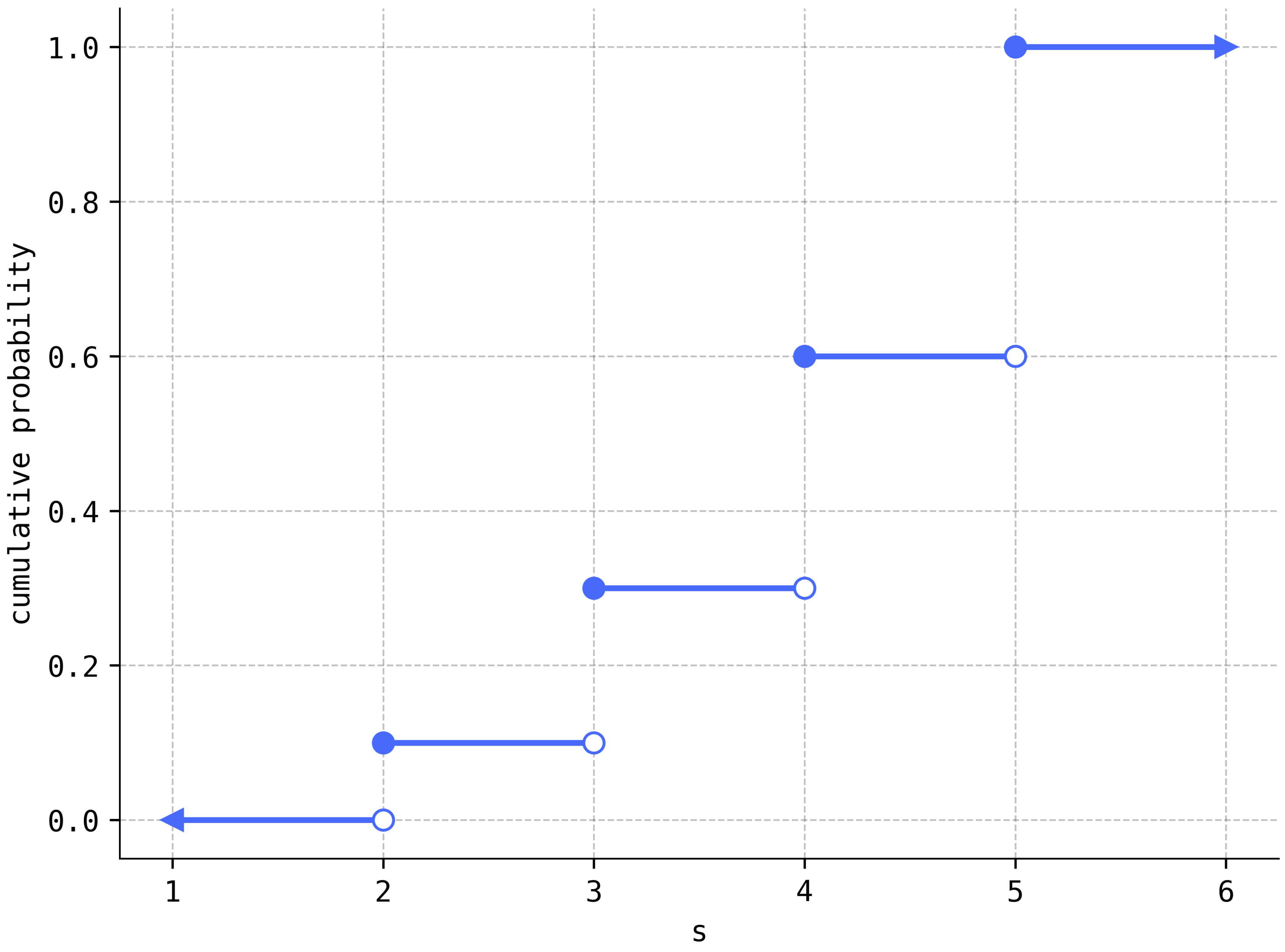
$$F(s) = \sum_{t \leq s} p(t), \tag{2.5}$$

   where the sum ranges over all $t \in \mathbb{R}$ with $t \leq s$.
2. If P is continuous with probability density function $f(s)$, then

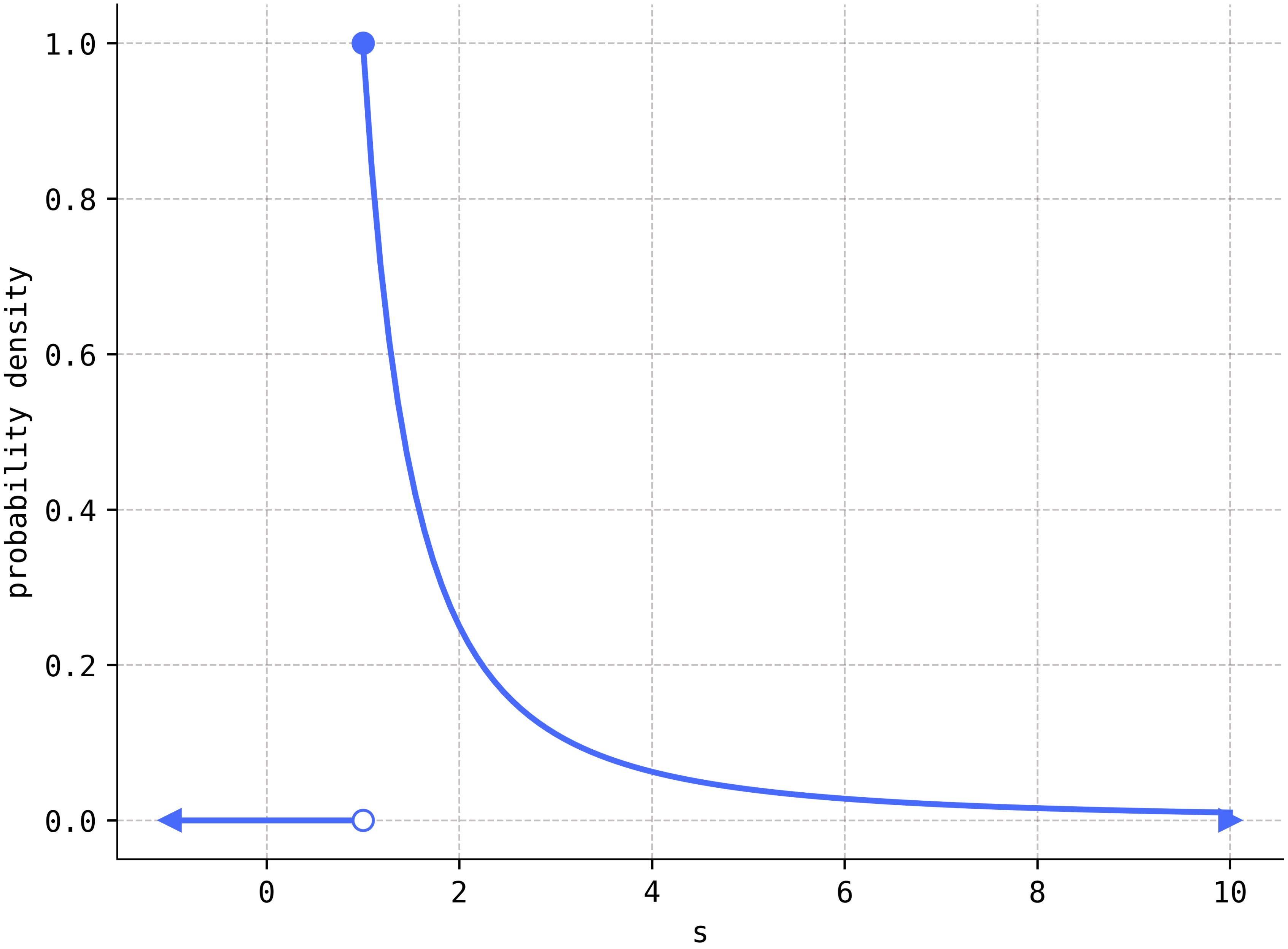
$$F(s) = \int_{-\infty}^{s} f(t)\, \mathrm{d}t.$$

Distribution functions are also frequently called *cumulative distribution functions* (CDFs).

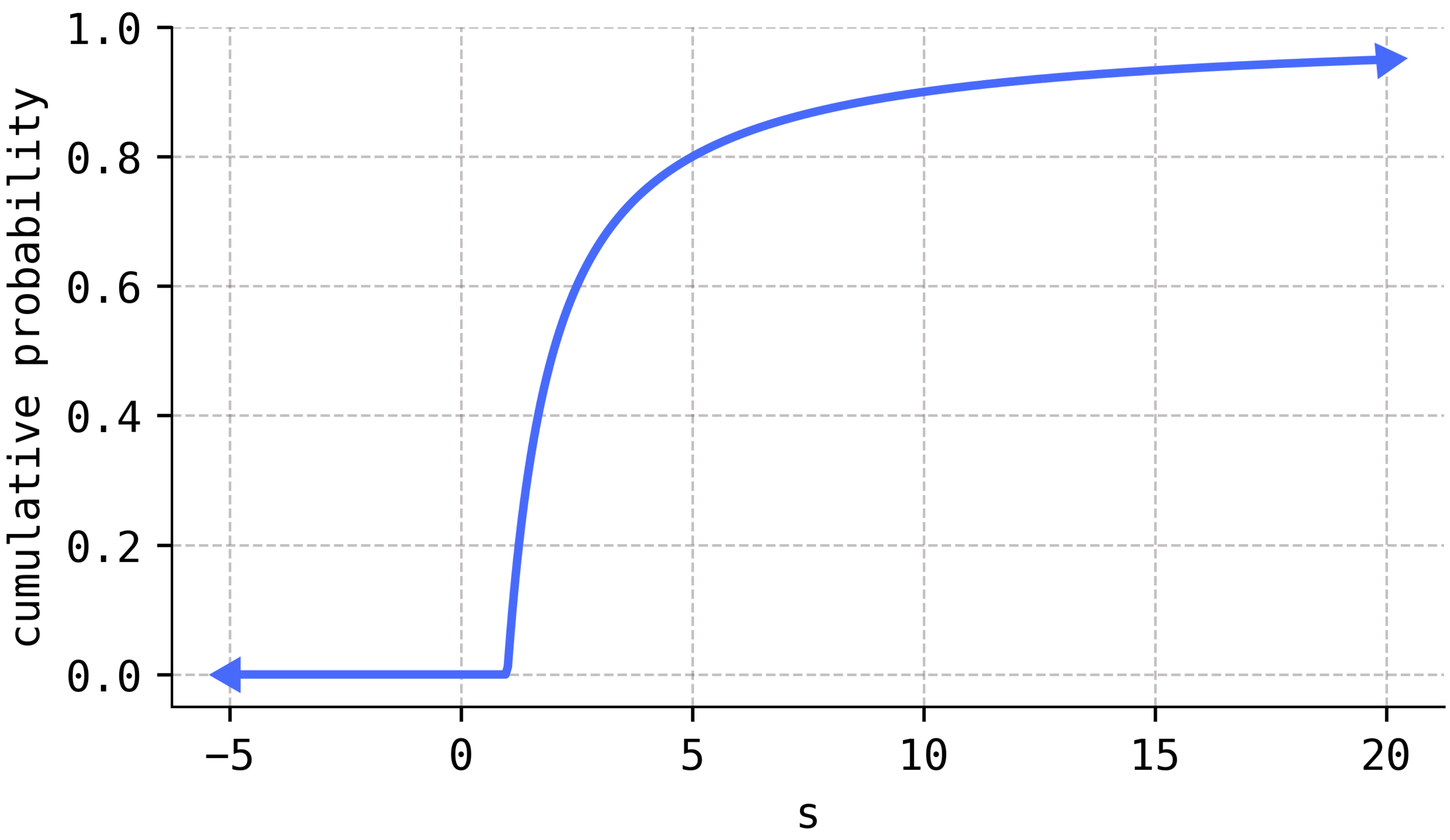

0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
probability
2
3
4
5
s



cumulative probability
1.0
0.8
0.6
0.4
0.2
0.0
1
2
3
4
5
6
s



probability density
1.0
0.8
0.6
0.4
0.2
0.0
0
2
4
6
8
10
s


1.0
0.8
0.6
0.4
0.2
0.0
cumulative probability
−5
0
5
10
15
20
s

Theorem 2.7 (Properties of Distribution Functions)

Let $F(s)$ be the distribution function of a probability measure P on $\mathbb{R}$. Then:

1. We have

$$\lim_{s\to\infty} F(s) = 1 \quad \text{and} \quad \lim_{s\to-\infty} F(s) = 0.$$

2. $F(s)$ is a non-decreasing function, in the sense that

$$s \leq t \quad \Rightarrow \quad F(s) \leq F(t).$$

3. $F(s)$ is *right-continuous* at every $s \in \mathbb{R}$, in the sense that

$$F(s) = \lim_{t\to s^+} F(t).$$

4. The probability measure P is discrete if and only if $F(s)$ is a step function.
5. The probability measure P is continuous if and only if $F(s)$ is continuous.

Problem Prompt

Time for some practice with distribution functions. Do problems 15 and 16 on the worksheet.

Theorem 2.8 (The Fundamental Theorem of Calculus (probability version))

Let $F(s)$ be the distribution function of a probability measure P on $\mathbb{R}$. If $F(s)$ is continuous, then:

1. The measure P is continuous.
2. Wherever the derivative $F'(s)$ exists, we have

$$F'(s) = f(s),$$

where $f(s)$ is the density function of P.

Problem Prompt

Do problem 17 on the worksheet.

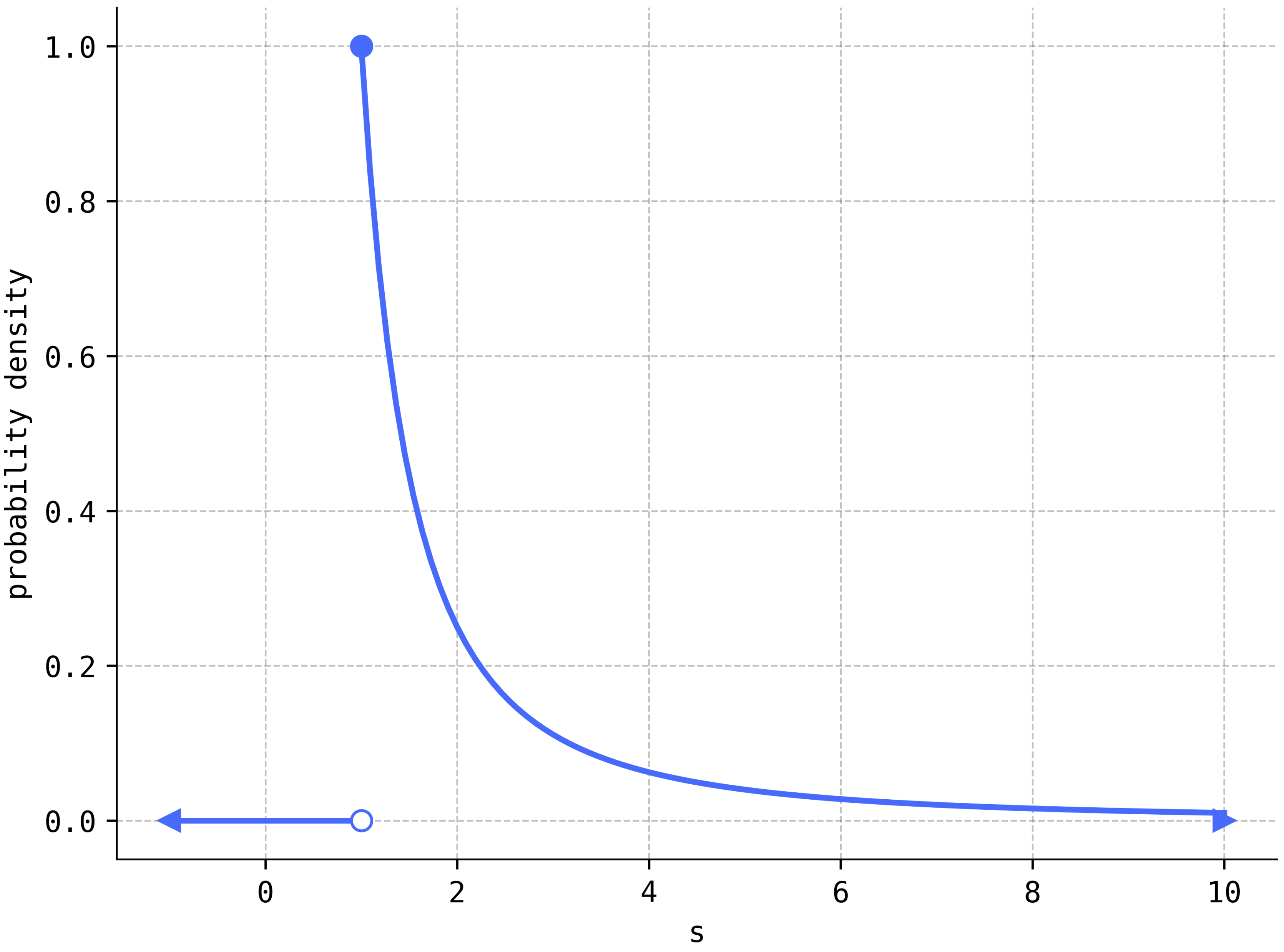
Definition 2.11

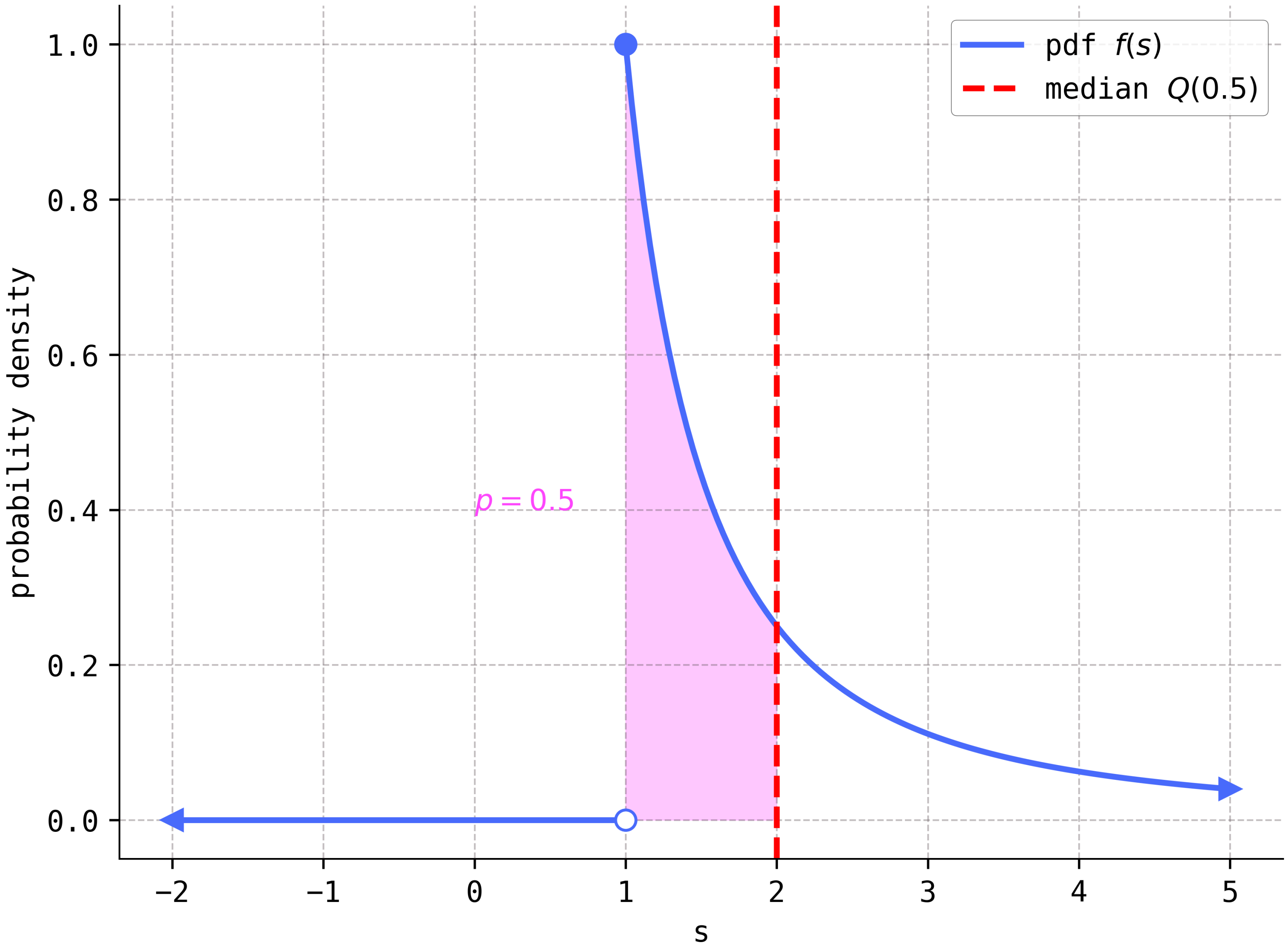
Let P be a probability measure on $\mathbb{R}$ with distribution function $F : \mathbb{R} \to [0,1]$. The *quantile function* $Q : [0,1] \to \mathbb{R}$ is defined so that

$$Q(p) = \min\{s \in \mathbb{R} : p \leq F(s)\}.$$

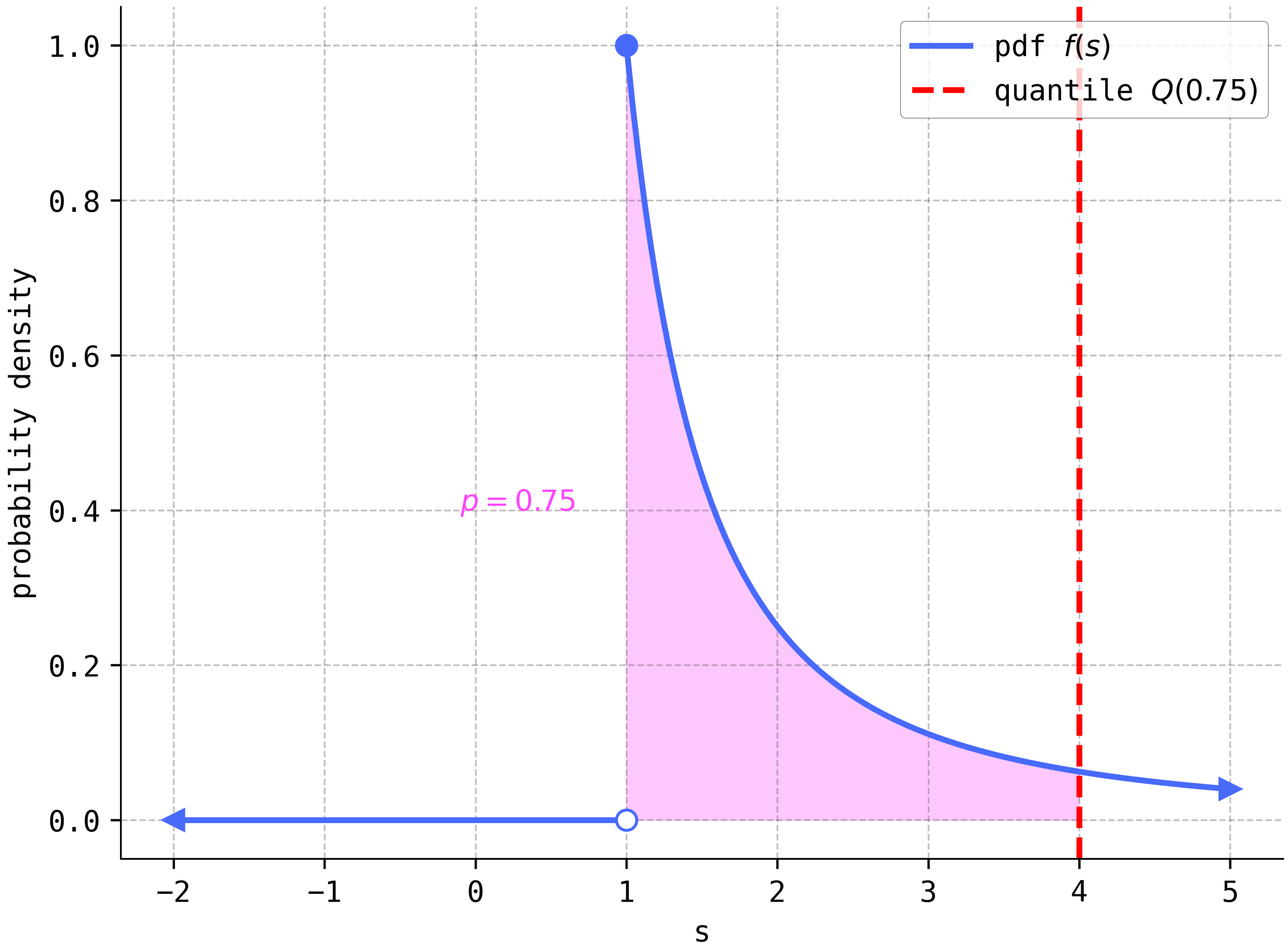
In other words, the value $s = Q(p)$ is the smallest $s \in \mathbb{R}$ such that $p \leq F(s)$.

1. The value $Q(p)$ is called the *p-th quantile*.
2. The quantile $Q(0.5)$ is called the *median* of the probability measure P.


1.0
0.8
0.6
0.4
0.2
0.0
probability density
0
2
4
6
8
10
s



pdf $f(s)$
median $Q(0.5)$
$p = 0.5$
probability density
s
1.0
0.8
0.6
0.4
0.2
0.0
−2
−1
0
1
2
3
4
5



pdf $f(s)$
quantile $Q(0.75)$
$p = 0.75$
probability density
s
1.0
0.8
0.6
0.4
0.2
0.0
−2
−1
0
1
2
3
4
5

Problem Prompt

Do problems 18 and 19 on the worksheet.