Problem 1: Suppose that we flip a loaded coin twice, with probability p of landing heads. Let $X = 1$ if the first flip lands heads, and $X = 0$ if it lands tails. Similarly, let $Y = 1$ if the second flip lands heads, and $Y = 0$ if it lands tails. Compute the joint distribution of (X, Y). Verify that your answer is correct by checking that all probabilities sum to 1.

We may specify the joint distribution by computing the four probabilities

$$P(X=0,Y=0),\ P(X=1,Y=0),\ P(X=0,Y=1),\ P(X=1,Y=1).$$

It will be convenient to display these in a table:

$X \diagdown Y$	0	1
0		
1		

However, since the two events are independent (the first flip and the second), we may compute the probabilities by multiplying the probabilities of X and Y. But since $X, Y \sim \mathcal{B}er(p)$, we have

$X \diagdown Y$	0	1
0	$(1-p)^2$	$p(1-p)$
1	$p(1-p)$	p^2

To verify that our probabilities are correct, we sum them to obtain:

$$(1-p)^2 + 2p(1-p) + p^2 = 1 - 2p + p^2 + 2p - 2p^2 + p^2 = 1.$$

Problem 2: Let (X, Y) be discrete with probability mass function $p(x, y)$ given in the following table:

$x \diagdown y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

(a) Compute $P(X \le 2, Y \le 2)$.

We have

$$\begin{aligned} P(X \le 2, Y \le 2) &= \sum_{x \le 2, y \le 2} p(x,y) \\ &= 0.08 + 0.07 + 0.06 + 0.06 + 0.10 + 0.12 + 0.05 + 0.06 + 0.09 \\ &= 0.69. \end{aligned}$$

(b) Compute $P(X = Y)$.

We have

$$P(X = Y) = \sum_{x=y} p(x,y) = 0.08 + 0.10 + 0.09 + 0.03 = 0.3.$$

(c) Compute $P(X > Y)$.

We have

$$P(X > Y) = \sum_{x>y} p(x, y) = 0.06 + 0.05 + 0.06 + 0.02 + 0.03 + 0.03 = 0.25.$$

Problem 3: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} cx^2y & : x^2 \leq y \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c that makes f a valid density.

We have

$$\iint_{\mathbb{R}^2} f(x, y)\ \mathrm{d}y\mathrm{d}x = \int_{-1}^{1} \int_{x^2}^{1} cx^2y\ \mathrm{d}y\mathrm{d}x = \frac{4}{21}c,$$

so that we must have $c = 21/4$.

(b) Compute $P(X \geq Y)$.

We have

$$P(X \geq Y) = \iint_{\{x \geq y\}} f(x, y)\ \mathrm{d}y\mathrm{d}x = \frac{21}{4} \int_{0}^{1} \int_{x^2}^{x} x^2y\ \mathrm{d}y\mathrm{d}x = \frac{3}{20}.$$

Problem 4: Suppose that the continuous random vector (X, Y) is uniformly distributed over the triangle in $\mathbb{R}^2$ with vertices $(-1, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Compute the density function of (X, Y).

The area of the triangle is $\frac{1}{2} \times 2 \times 1 = 1$. Since the volume under the density surface and above this triangle must be 1, we must have

$$f(x, y) = \begin{cases} 1 & : (x, y) \text{ is in the triangle,} \\ 0 & : \text{otherwise.} \end{cases}$$

(b) Compute $P(X \leq 3/4, Y \leq 3/4)$.

This problem is best solved by drawing a picture. See class notes. We may compute the probability by removing two triangles from the triangle of area 1, of areas 1/16 and 1/32. The answer is therefore 29/32.

Problem 5: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} 30xy^2 & : x - 1 \leq y \leq 1 - x,\ 0 \leq x \leq 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute $F(1/2, 1/2)$.

We have

$$F(1/2, 1/2) = \int_{-\infty}^{1/2} \int_{-\infty}^{1/2} f(x, y)\ \mathrm{d}y\mathrm{d}x = 30 \int_{0}^{1/2} \int_{x-1}^{1/2} xy^2\ \mathrm{d}y\mathrm{d}x = \frac{9}{16}.$$

Problem 6: Compute the marginal probability mass function distribution $p_X(x)$ of the discrete random vector (X, Y) with probability mass function

$x \diagdown y$	0	1	2	3	4
0	0.08	0.07	0.06	0.01	0.01
1	0.06	0.10	0.12	0.05	0.02
2	0.05	0.06	0.09	0.04	0.03
3	0.02	0.03	0.03	0.03	0.04

Verify that your computations are correct by making sure all marginal probabilities sum to 1. How would you compute the other marginal mass function $p_Y(y)$?

To compute $p_X(x)$, we simply sum across all y-values (i.e., rows):

$$p_X(0) = \sum_{y=0}^{4} p(0, y) = 0.08 + 0.07 + 0.06 + 0.01 + 0.01 = 0.23.$$

Similarly:

$$\begin{aligned} p_X(1) &= 0.35, \\ p_X(2) &= 0.27, \\ p_X(3) &= 0.15. \end{aligned}$$

To check our work, we verify that these probabilities sum to 1:

$$p_X(0) + p_X(1) + p_X(2) + p_X(3) = 0.23 + 0.35 + 0.27 + 0.15 = 1.$$

To compute $p_Y(y)$, we would sum instead over all x-values (i.e., columns).

Problem 7: Suppose that (X, Y) is continuous with probability density function

$$f(x, y) = \begin{cases} \frac{21}{4}x^2 y & : x^2 \le y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the marginal density functions $f_X(x)$ and $f_Y(y)$.

By "integrating out" the dependence on y, we compute

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\ \mathrm{d}y = \frac{21}{4} \int_{x^2}^{1} x^2 y\ \mathrm{d}y = \frac{21}{8} x^2 (1 - x^4).$$

Note that this is valid as long as $-1 \le x \le 1$; otherwise, we have $f_X(x) = 0$. Then, by "integrating out" the dependence on x, we compute

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\ \mathrm{d}x = \frac{21}{4} \int_{-\sqrt{y}}^{\sqrt{y}} x^2 y\ \mathrm{d}x = \frac{7}{2} y^{5/2}.$$

Note that this is valid as long as $0 \le y \le 1$; otherwise, we have $f_Y(y) = 0$.

Problem 8: Suppose the joint PMF of two discrete random variables X and Y is given by

$x \diagdown y$	1	2	3	4
1	0.1	0	0.1	0
2	0.3	0	0.1	0.2
3	0	0.2	0	0

Determine the conditional mass function $p_{X|Y}(x|1)$.

We have

$$p_{X|Y}(1|1) = \frac{p_{XY}(1,1)}{p_Y(1)} = \frac{0.1}{0.1+0.3+0} = 0.25$$
$$p_{X|Y}(2|1) = \frac{p_{XY}(2,1)}{p_Y(1)} = \frac{0.3}{0.1+0.3+0} = 0.75$$
$$p_{X|Y}(3|1) = \frac{p_{XY}(3,1)}{p_Y(1)} = \frac{0}{0.1+0.3+0} = 0.$$

Problem 9: Suppose that (X,Y) is continuous with probability density function

$$f(x,y) = \begin{cases} \frac{21}{4}x^2 y & : x^2 \le y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Compute the conditional density function $f_{Y|X}(y|x)$.

We have that

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)}.$$

But from a previous problem above we computed

$$f_X(x) = \begin{cases} \frac{21}{8}x^2(1-x^4) & : -1 \le x \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Thus, we have $f_X(x) \neq 0$ only when $-1 < x < 1$ and $x \neq 0$. For these x-values, we have

$$f_{Y|X}(y|x) = \begin{cases} \frac{2y}{1-x^4} & : x^2 \le y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Problem 10: A soft-drink machine has a random amount Y in supply at the beginning of a given day and dispenses a random amount X during the day (with measurements in gallons). It is not resupplied during the day, hence $X \le Y$. It has been observed that X and Y have joint density given by

$$f(x,y) = \begin{cases} 1/2, & : 0 \le x \le y \le 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Find the conditional density $f(x|y)$ and evaluate the probability that less than 1/2 gallon will be sold, given that the machine contains 1.5 gallons at the start of the day.

We first need to compute the marginal density of Y:

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\, \mathrm{d}x = \int_0^y \frac{1}{2}\, \mathrm{d}x = \frac{1}{2}y.$$

Notice that this is only valid if $0 \le y \le 2$; otherwise, we have $f(y) = 0$. In particular, we have $f(y) \neq 0$ only when $0 < y \le 2$. For these y-values, we thus have

$$f(x|y) = \frac{f(x,y)}{f(y)} = \begin{cases} \frac{1}{y} & : 0 \le x \le y \le 2, \\ 0 & : \text{otherwise.} \end{cases}$$

Then:

$$P(X \le 1/2 | Y = 1.5) = \int_{-\infty}^{1/2} f(x|1.5)\ \mathrm{d}x = \frac{2}{3}\int_0^{1/2} \mathrm{d}x = \frac{1}{3}.$$

Problem 11: Suppose that a person's score X on a mathematics aptitude test is a number between 0 and 1, and that their score Y on a music aptitude test is also a number between 0 and 1. Suppose further that in the population of all college students in the United States, the scores X and Y are distributed according to the following joint PDF

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & : 0 \le x, y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) What proportion of college students obtain a score greater than 0.8 on the mathematics test?

It will be convenient to first get the marginal densities for both X and Y:

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = \frac{2}{5}\int_0^1 (2x+3y)\ \mathrm{d}y = \frac{1}{5}(4x+3), \quad 0 \le x \le 1,$$

and

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}x = \frac{2}{5}\int_0^1 (2x+3y)\ \mathrm{d}x = \frac{1}{5}(6y+2), \quad 0 \le y \le 1.$$

Then, to answer this part of the problem, we compute:

$$P(X \ge 0.8) = \int_{0.8}^{\infty} f_X(x)\ \mathrm{d}x = \frac{1}{5}\int_{0.8}^1 (4x+3)\ \mathrm{d}x = 0.264.$$

(b) If a student's score on the music test is 0.3, what is the probability that their score on the mathematics test will be greater than 0.8?

We are asked to compute $P(X \ge 0.8 | Y = 0.3)$. So, we compute:

$$\begin{aligned} P(X \ge 0.8 | Y = 0.3) &= \int_{0.8}^{\infty} f_{X|Y}(x|0.3)\ \mathrm{d}x \\ &= \int_{0.8}^{\infty} \frac{f(x,0.3)}{f_Y(0.3)}\ \mathrm{d}x \\ &= \int_{0.8}^{1} \frac{\frac{2}{5}(2x+3\cdot 0.3)}{\frac{1}{5}(6\cdot 0.3+2)}\ \mathrm{d}x \\ &\approx 0.114. \end{aligned}$$

(c) If a student's score on the mathematics test is 0.3, what is the probability that their score on the music test will be greater than 0.8?

We are asked to compute $P(Y \geq 0.8|X = 0.3)$. So, we compute:

$$\begin{aligned} P(Y \geq 0.8|X = 0.3) &= \int_{0.8}^{\infty} f_{Y|X}(y|0.3)\ \mathrm{d}y \\ &= \int_{0.8}^{\infty} \frac{f(0.3, y)}{f_X(0.3)}\ \mathrm{d}y \\ &= \int_{0.8}^{1} \frac{\frac{2}{5}(2 \cdot 0.3 + 3y)}{\frac{1}{5}(4 \cdot 0.3 + 3)}\ \mathrm{d}y \\ &\approx 0.314. \end{aligned}$$

Problem 12: Let X be the number of heads obtained from a single flip of a coin, so that $X \sim \mathcal{B}er(\theta)$ for some unknown probability θ. Suppose further that θ is an observed value of a $\mathcal{B}eta(2, 2)$ random variable. If we flip the coin and obtain $x = 1$, how should we "update" the distribution of θ?

This is a problem in Bayesian statistics. The idea is that we have a *prior probability distribution* of the probability of heads θ given by $\mathcal{B}eta(2, 2)$. In particular, the expected value of θ at the outset is 0.5. But when we see that we obtain a head on a single flip, this might suggest to us that θ is likely to be greater than 0.5. We want to "update" our prior probability distribution $\mathcal{B}eta(2, 2)$ to reflect the arrival of the new data point $x = 1$. In the Bayesian lingo, this is the process of "updating the prior distribution to obtain the posterior distribution," where the *posterior distribution* is the updated one.

Let's begin. We are told that

$$p(x|\theta) = \theta^x (1 - \theta)^x, \quad x = 0, 1,$$

and

$$f(\theta) \propto \theta(1 - \theta), \quad 0 < \theta < 1.$$

The density function of the posterior distribution is (by definition) the conditional density $f(\theta|x)$. According to Bayes' Theorem, it is obtained via the formula

$$f(\theta|x) = \frac{p(x|\theta) f(\theta)}{p(x)}.$$

But we observed $x = 1$, so in fact we have

$$f(\theta|1) = \frac{p(1|\theta) f(\theta)}{p(1)} \propto \theta^2 (1 - \theta), \quad 0 < \theta < 1.$$

Now, technically we don't know the posterior density $f(\theta|1)$ *exactly*, we only know it up to some proportionality constant. But here's a trick that is often used in Bayesian statistics: The variable part of the density $\theta^2(1-\theta)$ is recognizable as the variable part of a $\mathcal{B}eta(3, 2)$ density. So, whatever the proportionality constant of $f(\theta|1)$ is, it must be the same one as a $\mathcal{B}eta(3, 2)$ density since both densities must integrate to 1 over $[0, 1]$. Thus, the posterior distribution must be a $\mathcal{B}eta(3, 2)$ distribution! In particular, the posterior expected value of θ increases to $3/(3 + 2) = 0.6$.

Problem 13: Suppose that three random vectors X, Y, and Z are jointly continuous with density function

$$f(x,y,z) = \begin{cases} c(x+2y+3z) & : 0 \le x,y,z \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c that makes $f(x,y,z)$ a valid density function.

As usual, we solve

$$\iiint_{\mathbb{R}^3} f(x,y,z)\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 1$$

for c. But the integral on the left-hand side is

$$\int_0^1 \int_0^1 \int_0^1 c(x+2y+3z)\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = 3c,$$

so we must have $c = 1/3$.

(b) Compute the marginal density $f_{XY}(x,y)$.

The marginal density is obtained from the joint one by "integrating out" the variable z. So:

$$f_{XY}(x,y) = \int_{\mathbb{R}} f(x,y,z)\ \mathrm{d}z = \frac{1}{3}\int_0^1 (x+2y+3z)\ \mathrm{d}z = \frac{1}{3}\left(x+2y+3/2\right),$$

for $0 \le x,y \le 1$, and $f_{XY}(x,y) = 0$ otherwise.

(c) Compute the probability $P\left(Z < 1/2 \mid X = 1/4, Y = 3/4\right)$.

We have:

$$\begin{aligned} P\left(Z < 1/2 \mid X = 1/4, Y = 3/4\right) &= \int_{-\infty}^{1/2} f_{Z|XY}(z \mid 1/4, 3/4)\ \mathrm{d}z \\ &= \int_{-\infty}^{1/2} \frac{f(1/4,3/4,z)}{f_{XY}(1/4,3/4)}\ \mathrm{d}z \\ &= \int_0^{1/2} \frac{\frac{1}{3}(1/4+2(3/4)+3z)}{\frac{1}{3}(1/4+2(3/4)+3/2)}\ \mathrm{d}z \\ &= \frac{5}{13} \\ &\approx 0.385. \end{aligned}$$

Problem 14: Suppose that X, Y, and Z have joint "mixed density" function

$$f(x,y,z) = \begin{cases} cx^{1+y+z}(1-x)^{3-y-z} & : 0 < x < 1,\ \ y,z \in \{0,1\}, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the value of c.

We solve

$$\sum_{y,z\in\{0,1\}} \int_{\mathbb{R}} f(x,y,x)\ \mathrm{d}x = 1$$

for c. But the expression on the left-hand side is

$$\sum_{y,z\in\{0,1\}} \int_0^1 cx^{1+y+z}(1-x)^{3-y-z}\ \mathrm{d}x = \frac{1}{6}c,$$

so we must have $c = 6$.

(b) Compute the marginal "density" $f_{XY}(x, y)$.

We have

$$f_{XY}(x,y) = \sum_{z\in\{0,1\}} f(x,y,z) = 6x^{1+y}(1-x)^{3-y} + 6x^{2+y}(1-x)^{2-y}$$

for $y \in \{0, 1\}$ and $0 < x < 1$, and $f_{XY}(x, y) = 0$ otherwise.

(c) Compute the conditional "density" $f_{Z|XY}(z|1/4, 1)$.

We have

$$f_{Z|XY}(z|1/4,1) = \frac{f(1/4,1,z)}{f_{XY}(1/4,1)} = \frac{3^{3-z}/128}{9/32} = \frac{3^{1-z}}{4},$$

and so

$$f_{Z|XY}(z|1/4,1) = \begin{cases} 3/4 & : z = 0, \\ 1/4 & : z = 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Problem 15: Suppose that two measurements X and Y are made of the rainfall at a certain location on May 1 of two consecutive years. Supposing that X and Y are independent and that their marginal density functions are each given by

$$f_X(x) = \begin{cases} 2x & : 0 \le x \le 1, \\ 0 & : \text{otherwise,} \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2y & : 0 \le y \le 1, \\ 0 & : \text{otherwise,} \end{cases}$$

determine their joint density and compute the probability $P(X + Y \le 1)$.

Since the variables are independent, their joint density is

$$f(x,y) = f_X(x)f_Y(y) = \begin{cases} 4xy & : 0 \le x, y \le 1, \\ 0 & : \text{otherwise.} \end{cases}$$

Then

$$P(X+Y\le 1) = \iint_{\{x+y\le 1\}} f(x,y)\ \mathrm{d}x\mathrm{d}y = 4\int_0^1\int_0^{-x+1} xy\ \mathrm{d}y\mathrm{d}x = \frac{1}{6}.$$

Problem 16: Suppose that the joint density function of two continuous random variables X and Y is given by

$$f(x,y) = \begin{cases} kx^2y^2 & : x^2+y^2 \le 1, \\ 0 & : \text{otherwise,} \end{cases}$$

for some constant k. Prove that X and Y are dependent.

We will show that the joint density *cannot* be factored into the marginal densities. For this, we first compute

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = \frac{2}{3}kx^2\left(1-x^2\right)^{3/2}$$

for $-1 \le x \le 1$, and $f_X(x) = 0$ otherwise. Likewise, we have

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}x = \frac{2}{3}ky^2\left(1-y^2\right)^{3/2}$$

for $-1 \le y \le 1$ and $f_Y(y) = 0$ otherwise. But notice that

$$f(0.9, 0.9) = 0$$

since the point $(0.9, 0.9)$ lies outside the unit circle, while

$$f_X(0.9) \ne 0 \quad \text{and} \quad f_Y(0.9) \ne 0.$$

Thus, we have $f(0.9, 0.9) \ne f_X(0.9)f_Y(0.9)$, which proves the variables are dependent.

Problem 17: Suppose that a point (X, Y) is chosen at random from the rectangle R defined as follows:

$$R = \{(x,y) : 0 \le x \le 2,\ 1 \le y \le 4\}.$$

(a) Determine the joint density of X and Y, the marginal density of X, and the marginal density of Y.

The joint distribution must be uniform over the rectangle, thus the density surface is a horizontal plane over R. Since the volume underneath it must be 1 and the rectangle R has area 6, we must have $f(x,y) = 1/6$ for all $(x,y) \in R$, and $f(x,y) = 0$ otherwise. Then, we have

$$f_X(x) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}y = \frac{1}{6}\int_1^4 \mathrm{d}y = \frac{1}{2}$$

for $0 \le x \le 2$ and $f_X(x) = 0$ otherwise, as well as

$$f_Y(y) = \int_{\mathbb{R}} f(x,y)\ \mathrm{d}x = \frac{1}{6}\int_0^2 \mathrm{d}x = \frac{1}{3}$$

for $1 \le y \le 4$ and $f_Y(y) = 0$ otherwise.

(b) Are X and Y independent?

Yes. For proof, notice that from part (a) we have

$$f_X(x)f_Y(y) = \begin{cases} \frac{1}{6} & : (x,y) \in R, \\ 0 & : (x,y) \notin R, \end{cases}$$

which is exactly the formula for the joint density $f(x,y)$.

Problem 18: Let $n \ge 1$ be an integer and suppose $X \sim \Gamma(n+1, 1)$. Suppose that $Y_1, Y_2, \ldots, Y_n$ is an IID random sample such that the conditional distributions of each Y_i given X have densities

$$f(y_i|x) = \begin{cases} \frac{1}{x} & : 0 < y_i < x, \\ 0 & : \text{otherwise.} \end{cases}$$

(a) Determine the joint density of the random sample.

Note that

$$f(x) = \begin{cases} \frac{1}{n!} x^n e^{-x} & : x > 0 \\ 0 & : \text{otherwise.} \end{cases}$$

We are asked to compute $f(y_1, \ldots, y_n)$. To do this, we "integrate out" the dependence on x of the joint density $f(x, y_1, \ldots, y_n)$. But, by independence of the random sample, we have

$$f(x, y_1, \ldots, y_n) = f(y_1, \ldots, y_n|x)f(x) = \prod_{i=1}^{n} f(y_i|x)f(x) = \frac{1}{n!}e^{-x}$$

for $0 < y_1, \ldots, y_n < x$ and $x > 0$, and $f(x, y_1, \ldots, y_n) = 0$ otherwise. Thus, for $y_1, \ldots, y_n > 0$ we must have

$$f(y_1, \ldots, y_n) = \int_{\mathbb{R}} f(x, y_1, \ldots, y_n)\ \mathrm{d}x = \frac{1}{n!}\int_M^\infty e^{-x}\ \mathrm{d}x = \frac{1}{n!}e^{-M}$$

where M is the maximum of the y_i's, and $f(y_1, \ldots, y_n) = 0$ otherwise.

(b) Determine the conditional density of X for any given observed values of the random sample.

We have

$$f(x|y_1, \ldots, y_n) = \frac{f(x, y_1, \ldots, y_n)}{f(y_1, \ldots, y_n)}$$

for those y_i's such that the denominator is not zero. But from (a), we have $f(y_1, \ldots, y_n) \neq 0$ if $y_1, \ldots, y_n > 0$, and for these y_i's we have

$$f(x|y_1, \ldots, y_n) = \frac{\frac{1}{n!}e^{-x}}{\frac{1}{n!}e^{-M}} = e^{-x+M}$$

for $x > 0$ and $f(x|y_1, \ldots, y_n) = 0$ otherwise.