Question 5:

a)

1. 1.12.2 (b):

	Statements	Justifications
1	$\neg q$	Hypothesis 2
2	$\neg q \lor \neg r$	Addition, 1
3	$\neg(q \land r)$	De Morgan's laws, 2
4	$p \rightarrow (q \land r)$	Hypothesis 1
5	$\neg p$	Modus tollens, 3, 4

1.12.2 (e):

	Statements	Justifications
1	$p \lor q$	Hypothesis 1
2	$\neg p \lor r$	Hypothesis 2
3	$q \lor r$	Resolution, 1, 2
4	$\neg q$	Hypothesis 3
5	r	Disjunctive syllogism, 3, 4

2. 1.12.3 (c):

	Statements	Justifications
1	$p \lor q$	Hypothesis 1
2	$\neg\neg p \lor q$	Double negation law, 1
3	$\neg p \rightarrow q$	Conditional identities, 2
4	$\neg p$	Hypothesis 2
5	q	Modus ponens, 3, 4

3. 1.12.5 (c):

Assign variable names to each of the individual propositions:

c: I will buy a new car.
h: I will buy a new house.
j: I get a job.

Replacing English phrases with variable names results in the following argument form:

$(c \land h) \to j$
$\neg j$

$\therefore \neg c$

To prove whether the argument is valid or invalid, assume both hypothesis are true:

	Statements	Justifications
1	$(c \land h) \to j$	Hypothesis 1
2	$\neg j$	Hypothesis 2
3	$\neg(c \land h)$	Modus ponens, 1, 2
4	$\neg c \lor \neg h$	De Morgan's laws, 3

We can get $(\neg c \lor \neg h)$ is true, means $\neg c$ or $\neg h$ is true.
When $\neg c$ is false, $\neg h$ is true, $(c \land h) \to j$ is true.
When $\neg j$ is true, the hypotheses are all true but the conclusion is false.
Therefore, this argument is **invalid**.

1.12.5 (d):

Assign variable names to each of the individual propositions:

c: I will buy a new car.
h: I will buy a new house.
j: I get a job.

Replacing English phrases with variable names results in the following argument form:

$(c \land h) \to j$
$\neg j$
h

$\therefore \neg c$

To prove whether the argument is valid or invalid, assume both hypothesis are true:

	Statements	Justifications
1	$(c \land h) \to j$	Hypothesis 1
2	$\neg j$	Hypothesis 2
3	$\neg(c \land h)$	Modus ponens, 1, 2
4	$\neg c \lor \neg h$	De Morgan's laws, 3
5	$\neg h \lor \neg c$	Commutative laws, 4
6	h	Hypothesis 3
7	$\neg\neg h$	Double negation law, 6
8	$\neg c$	Disjunctive syllogism, 5, 7

Therefore, this argument is **valid**.

b)

1. 1.13.3 (b):

   Suppose the argument is invalid, then both hypotheses should be true but the conclusion should be false.

   - For conclusion $\exists x P(x)$ is false, then every value in P(x) should be false. Thus, P(a) = F, P(b) = F.

   From the truth table:

	P(x)	Q(x)	¬Q(x)
a	F	T	F
	F	F	T
b	F	T	F
	F	F	T

   - For hypothesis 1 $\exists x(P(x) \lor Q(x))$: when x = a, P(a) = F, Q(a) = T, P(a) $\lor$ Q(a) = F $\lor$ T = T. There exists one x, make P(x) $\lor$ Q(x) true. So hypothesis 1 is true.

- For hypothesis 2 $\exists x \neg Q(x)$: when x = b, Q(b) = F, ¬Q(b) = ¬F = T. There exists one x, make ¬Q(x) true. So hypothesis 2 is also true.

Thus, both hypotheses are true but the conclusion is false, the invalid values:
- P(a) = F, P(b) = F
- Q(a) = T, Q(b) = F

Therefore, the argument is invalid.

2. 1.13.5 (d)

Assign variable names to each of the individual propositions:

M(x): x missed class.
D(x): x got a detention.
p represents Penelope.

Replacing English phrases with variable names results in the following argument form:

$\forall x(M(x) \rightarrow D(x))$
Penelope is a student
$\neg M(p)$

$\therefore \neg D(p)$

To prove whether the argument is valid or invalid, assume both hypothesis are true:

	Statements	Justifications
1	$\forall x(M(x) \rightarrow D(x))$	Hypothesis 1
2	Penelope is a student	Hypothesis 2
3	$M(p) \rightarrow D(p)$	Universal instantiation, 1, 2
4	$\neg M(p) \lor D(p)$	Conditional identities, 3
5	$\neg M(p)$	Hypothesis 3

Since, ¬M(p) = T, ¬M(p) ∨ D(p) = T ∨ D(p) = T.
When D(p) = T, ¬D(p) = F, will make all hypotheses true but the conclusion is false.
Therefore, this argument is **invalid**.
The invalid values are:
- M(x) = F (Student didn’t miss class)
- D(x) = T (Student got a detention)

1.13.5 (e)

Assign variable names to each of the individual propositions:

M(x): x missed class.
D(x): x got a detention.
A(x): x got an A.
p represents Penelope.

Replacing English phrases with variable names results in the following argument form:

$\forall x((M(x) \lor D(x)) \to \neg A(x))$
Penelope is a student in the class
$A(p)$

$\therefore \neg D(p)$

To prove whether the argument is valid or invalid, assume both hypothesis are true:

	Statements	Justifications
1	$\forall x((M(x) \lor D(x)) \to \neg A(x))$	Hypothesis 1
2	Penelope is a student	Hypothesis 2
3	$(M(p) \lor D(p)) \to \neg A(p)$	Universal instantiation, 1, 2
4	$A(p)$	Hypothesis 3
5	$\neg\neg A(p)$	Double negation laws, 4
6	$\neg(M(p) \lor D(p))$	Modus tollens, 3, 5
7	$\neg M(p) \land \neg D(p)$	De Morgan's laws, 6
8	$\neg D(p) \land \neg M(p)$	Commutative, 7
9	$\neg D(p)$	Simplification, 8

Therefore, this argument is **valid**.

Question 6:

2.4.1 (d)

Proof:
Let x and y be odd integers. We will show that xy is also odd.
Since:

- x is odd, $x = 2m+1$, for some integer m.
- y is odd, $y = 2n+1$, for some integer n.

Plug $x = 2m+1$, $y = 2n+1$ into xy to get:

$$\begin{aligned} xy &= (2m+1)(2n+1) \\ &= 4mn + 2n + 2m + 1 \\ &= 2(2mn + n + m) + 1 \end{aligned}$$

Since m, n are integers, $(2mn + n + m)$ will be also an integer.
Thus, $xy = 2(2mn + n + m) + 1$ can be expressed as $2j + 1$, where $j = 2mn + n + m$ is an integer.
Therefore, the product of two odd integers is odd. ■

2.4.3 (b)

Proof:
Let x be a real number and $x \le 3$, we will prove that $12 - 7x + x^2 \ge 0$.
Since: $12 - 7x + x^2 = x^2 - 7x + 12 = (x - 4)(x - 3)$
Given: $x \le 3$, then:

- $x - 4 \le 3 - 4 = -1$
- $x - 3 \le 3 - 3 = 0$
- Thus $(x - 4)(x - 3) \ge 0$

Therefore: $12 - 7x + x^2 = (x - 4)(x - 3) \ge 0$
So if x is a real number and $x \le 3$, then $12 - 7x + x^2 \ge 0$. ■

Question 7:

2.5.1 (d)

Proof:
Let n be an integer. We assume that if n is even and will prove that $n^2 - 2n + 7$ is odd.
Since n is even, n = 2k, for some integer k.
Plug n = 2k into $n^2 - 2n + 7$ to get:

$$\begin{aligned} n^2 - 2n + 7 &= (2k)^2 - 2(2k) + 7 \\ &= 4k^2 - 4k + 7 \\ &= 2(k^2 - 2k + 3) + 1 \end{aligned}$$

Since k is an integer, then $k^2 - 2k + 3$ will be also an integer.
Therefore, $n^2 - 2n + 7 = 2(k^2 - 2k + 3) + 1$ can be expressed as 2j + 1, where $j = k^2 - 2k + 3$ is an integer. We can conclude that $n^2 - 2n + 7$ is odd. ■

2.5.4 (a)

Proof:
Let x, y be real numbers. We assume that if x > y and will prove that $x^3 + xy^2 > x^2y + y^3$.
Given: x > y, then x - y > 0.

- Multiplying both sides by x^2: $x^2(x - y) = x^3 - x^2y > 0$
- Multiplying both sides by y^2: $y^2(x - y) = xy^2 - y^3 > 0$

Adding the inequalities: $(x^3 - x^2y) + (xy^2 - y^3) > 0$

$$x^3 + xy^2 - x^2y - y^3 > 0$$

Gives that $x^3 + xy^2 > x^2y + y^3$. ■

2.5.4 (b)

Proof:
Let x, y be real numbers. We assume that if $x \le 10$ and $y \le 10$, and will prove that $x + y \le 20$.
Given: $x \le 10$ and $y \le 10$,
Adding the inequalities: $x + y \le 10 + 10 = 20$
Gives that $x + y \le 20$. ■

2.5.5 (c)

Proof:
Let x be a nonzero real number. We assume that if $\frac{1}{x}$ is rational, and will prove that x is also rational.
Since $\frac{1}{x}$ is rational, we can write $\frac{1}{x} = \frac{a}{b}$, where a and b are integers and $b \neq 0$.

Since x is a nonzero real number, then a ≠ 0.

- $\frac{1}{x} = \frac{a}{b}$ can be ax = b, then x = $\frac{b}{a}$.

Since a and b are integers and a ≠ 0, x can be expressed as a ratio of two integers.
Therefore, x is also rational. ■

Question 8:

2.6.6 (c)

Proof:

Proof by contradiction: Assume that the theorem is false, which means that the average of three real numbers is less than all three numbers.

Assume that we have three real numbers a, b, c, then the average is $\frac{a+b+c}{3}$.

By assumption: $\frac{a+b+c}{3} < a$, $\frac{a+b+c}{3} < b$, $\frac{a+b+c}{3} < c$

- Multiplying all three inequalities by 3: a + b + c < 3a, a + b + c < 3b, a + b + c < 3c
- Adding these inequalities: 3(a +b + c) < 3a + 3b + 3c = 3(a + b + c)

This is a contradiction as no number can be less than itself.

Therefore, the assumption that the average of three real numbers is less than all three numbers is false. The average of three real numbers must be greater than or equal to at least one of the numbers. ■

2.6.6 (d)

Proof:

Proof by contradiction: Assume that there exists a smallest integer.

Let n is the smallest integer number.

Since n is integer, then n - 1 is also an integer.

By assumption: n -1 > n

- Subtract n from both side gives: -1 > 0

This is a contradiction as -1 < 0.

Therefore, the assumption that there exists a smallest integer is false. There is no smallest integer. ■

Question 9:

2.7.2 (b)

Proof:
Let x and y be integers with the same parity.

Case 1: x and y are both even.
Since x is even, $x = 2k$ for some integer k.
Since y is even, $y = 2j$ for some integer j.
Plug $x = 2k$, $y = 2j$ into $x + y$ to get: $x + y = 2k + 2j = 2(k + j)$
Since k and j are both integers, $k + j$ is also an integer, and $x + y = 2(k + j)$ is even.
$x + y$ is even, when x and y are both even.

Case 2: x and y are both odd.
Since x is odd, $x = 2k +1$ for some integer k.
Since y is odd, $y = 2j + 1$ for some integer j.
Plug $x = 2k + 1$, $y = 2j + 1$ into $x + y$ to get: $x + y = (2k + 1) + (2j + 1) = 2k + 2j + 2 = 2(k + j + 1)$
Since k and j are both integers, $k + j + 1$ is also an integer and $x + y = 2(k + j + 1)$ is even.
$x + y$ is even, when x and y are both odd.

Therefore, if x and y have the same parity, then $x + y$ is even. ■