Question 3:

a)

4.1.3 b

Answer is: $f(x) = \frac{1}{x^2-4}$ **is not a function from** $\mathbb{R}$ **to** $\mathbb{R}$**.**

For $f(x) = \frac{1}{x^2-4}$ **,** $x^2 - 4 \neq 0$. However, the domain is $\mathbb{R}$, when $x = 2$ or $x = -2$, $x^2 - 4 = 0$, there is no mapped element in the target set. Therefore, $f(x) = \frac{1}{x^2-4}$ is not a function from $\mathbb{R}$ to $\mathbb{R}$.

4.1.3 c

Answer is: $f(x) = \sqrt{x^2}$ **is a function, its range is** $[0, \infty)$**.**

For all real number $x \in \mathbb{R}$, since $x^2 \geq 0$, then $\sqrt{x^2} = |x|$, so $f(x) = \sqrt{x^2}$ is a function from $\mathbb{R}$ to $\mathbb{R}$. Since $|x| \geq 0$, its range is $[0, \infty)$.

b)

4.1.5 b

Answer is: $\{4, 9, 16, 25\}$**.**

For $f(x) = x^2$, its domain is $\{2, 3, 4, 5\}$, then $f(2) = 4$, $f(3) = 9$, $f(4) = 16$, $f(5) = 25$, therefore its range is $\{4, 9, 16, 25\}$.

4.1.5 d

Answer is: $\{0, 1, 2, 3, 4, 5\}$**.**

For $x \in \{0, 1\}^5$, its domain is $\{00000, 00001, \ldots, 11111\}$. Since $f(x)$ is the number of 1s that occur in x, no matter how 0s and 1s place, the numbers of 1s that occur in x can only be 0, 1, 2, 3, 4, 5, so its range is $\{0, 1, 2, 3, 4, 5\}$.

4.1.5 h

Answer is: $\{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$**.**

Since $A = \{1, 2, 3\}$, then $A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$.

Since $f: A \times A \to \mathbb{Z} \times \mathbb{Z}$, then all possible of $f(x, y) = (y, x)$ are:

$f(1, 1) = (1, 1)$
$f(1, 2) = (2, 1)$
$f(1, 3) = (3, 1)$
$f(2, 1) = (1, 2)$
$f(2, 2) = (2, 2)$
$f(2, 3) = (3, 2)$
$f(3, 1) = (1, 3)$
$f(3, 2) = (2, 3)$
$f(3, 3) = (3, 3)$

Therefore its range is $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.

4.1.5 i

Answer is: $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$**.**
Since $A = \{1, 2, 3\}$, then $A \times A = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$.
Since $f: A \times A \to \mathbb{Z} \times \mathbb{Z}$, then all possible of $f(x, y) = (x, y + 1)$ are:

$f(1, 1) = (1, 2)$
$f(1, 2) = (1, 3)$
$f(1, 3) = (1, 4)$
$f(2, 1) = (2, 2)$
$f(2, 2) = (2, 3)$
$f(2, 3) = (2, 4)$
$f(3, 1) = (3, 2)$
$f(3, 2) = (3, 3)$
$f(3, 3) = (3, 4)$

Therefore its range is $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$.

4.1.5 l

Answer is: $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$**.**
Since $A = \{1, 2, 3\}$, then $P(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.
Since $f: P(A) \to P(A)$, $X \subseteq A$, so all possible of $f(X) = X - \{1\}$ are:

$f(\emptyset) = \emptyset$
$f(\{1\}) = \emptyset$
$f(\{2\}) = \{2\}$
$f(\{3\}) = \{3\}$
$f(\{1, 2\}) = \{2\}$
$f(\{1, 3\}) = \{3\}$
$f(\{2, 3\}) = \{2, 3\}$
$f(\{1, 2, 3\}) = \{2, 3\}$

Therefore, its range is $\{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$.

<u>Question 4:</u>

I.a

4.2.2 c

Answer is: one-to-one, but not onto. 2 is not in the range.

For one-to-one: every $x \in \mathbb{Z}$, when $x_1 \neq x_2$, then $x_1^3 \neq x_2^3$, so $f(x_1) \neq f(x_2)$, every element in the domain maps to a unique element in the target, therefore $f(x) = x^3$ is one-to-one.

For onto: since $f(x) \in \mathbb{Z}$, take $f(x) = 2$, there is no $x \in \mathbb{Z}$ such that $f(x) = x^3 = 2$. This means 2 is not in the range, therefore $f(x) = x^3$ is not onto.

4.2.2 g

Answer is: one-to-one, but not onto. $(1, 1)$ **is not in the range.**

For one-to-one: every $(x, y) \in \mathbb{Z} \times \mathbb{Z}$, if $f(x_1, y_1) = f(x_2, y_2)$, then $(x_1 + 1,\ 2y_1) = (x_2 + 1,\ 2y_2)$, this means $x_1 + 1 = x_2 + 1$ and $2y_1 = 2y_2$, therefore $x_1 = x_2$ and $y_1 = y_2$. So $f(x, y) = (x + 1,\ 2y)$ is one-to-one.

For onto: since $2y \in \mathbb{Z}$, take $2y = 1$, then $y = \frac{1}{2} \notin \mathbb{Z}$, so for the element in the target, there is no element in the domain. This means $(x + 1,\ 1)$ is not in the range, for example $(1,\ 1)$, therefore $f(x, y) = (x + 1,\ 2y)$ is not onto.

4.2.2 k

Answer is: neither one-to-one nor onto. Not one-to-one: $f(1, 3) = f(2, 1) = 5$**. Not onto: 1 is not in the range.**

For one-to-one: take $(x, y) = (1, 3)$, then $f(1, 3) = 2^1 + 3 = 5$,
take $(x, y) = (2, 1)$, then $f(2, 1) = 2^2 + 1 = 5$,
This means, when $(1, 3) \neq (2, 1)$, $f(1, 3) = f(2, 1)$, therefore $f(x, y) = 2^x + y$ is not one-to-one.

For onto: since $2^x + y \in \mathbb{Z}^+$, take $2^x + y = 1$, if $x = 1$, then $2^1 + y = 1$, $y =- 1 \notin \mathbb{Z}^+$, there is no element in the domain. This means 1 is not in the range, therefore $f(x, y) = 2^x + y$ is not onto.

Therefore $f(x, y) = 2^x + y$ is neither one-to-one nor onto.

I.b

4.2.4 b

Answer is: neither one-to-one nor onto. Not one-to-one: $f(001) = f(101) = 101$**. Not onto: 001 is not in the range.**

For one-to-one: since f is obtained by taking the input string and replacing the first bit by 1, then $f(001) = f(101) = 101$, as $001 \neq 101$, therefore f is not one-to-one.

For onto: target is $\{0, 1\}^3$, since f is obtained by taking the input string and replacing the first bit by 1, then any string starting with 0 cannot in the range, such as 001 is not in the range, therefore f is not onto.

4.2.4 c

Answer is: both one-to-one and onto.

Since $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$,
Then: $f(000) = 000$,
$f(001) = 100$,
$f(010) = 010$,
$f(011) = 110$,
$f(100) = 001$,
$f(101) = 101$,
$f(110) = 011$,
$f(111) = 111$,
This means, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, and its range = $\{0, 1\}^3$, therefore f is one-to-one and onto.

4.2.4 d

Answer is: one-to-one, but not onto. 0001 is not in the range.

For one-to-one: since $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$,
Then: $f(000) = 0000$,
$f(001) = 1001$,
$f(010) = 0100$,
$f(011) = 1101$,
$f(100) = 0010$,
$f(101) = 1011$,
$f(110) = 0110$,
$f(111) = 1111$,
This means, when $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$, therefore f is one-to-one.

For onto: $f(x) \in \{0, 1\}^4$, take $f(x) = 0001$, the first bit is 0 and the end bit is 1, so the end bit is not a copy of the first bit. 0001 is not in the range, therefore f is not onto.

4.2.4 g

Answer is: neither one-to-one nor onto. Not one-to-one: $f(\{1, 2\}) = f(\{2\}) = \{2\}$**. Not onto:** $\{1\}$ **is not in the range.**

Since $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then $P(A) = \{\emptyset, \{1\}, \{2\},... \{1, 2\}, \{1, 3\}, \{1, 4\}..., \{1, 2, 3, 4, 5, 6, 7, 8\}\}$.

For one-to-one: take $X = \{1, 2\}$, then $f(\{1, 2\}) = \{1, 2\} - \{1\} = \{2\}$,
take $X = \{2\}$, then $f(\{2\}) = \{2\} - \{1\} = \{2\}$,
since $\{1, 2\} \neq \{2\}$, $f(\{1, 2\}) = f(\{2\})$, therefore $f(X)$ is not one-to-one.

For onto: $\{1\} \in P(A)$, however $B = \{1\}$, so $\{1\}$ can not be the range of $f(X) = X - B$, therefore $f(X)$ is not onto.

II.a

Answer is: $f: \mathbb{Z} \to \mathbb{Z}^+$, $f(x) = \begin{cases} 2x + 3, \ if\ x \geq 0 \\ -2x, \ if\ x < 0 \end{cases}$

For one-to-one: non-negative integers map to odd number, which $f(x) \geq 3$, and negative integers map to even number, so if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, therefore, $f(x)$ is one-to-one.

For onto: when $f(x) = 1$, then $2x + 3 = 1$, $x = -2$, which is opposite the condition $x \geq 0$, so 1 is not in the range. Therefore $f(x)$ is not onto.
Conclusion, $f(x)$ is one-to-one, but not onto.

II.b

Answer is: $f: \mathbb{Z} \to \mathbb{Z}^{+}$, $f(x) = |x| + 1$

For one-to-one: take $x = -2$, $f(-2) = |-2| + 1 = 3$, take $x = 2$, $f(2) = |2| + 1 = 3$, then $f(-2) = f(2) = 3$, $f(x)$ is not one-to-one.

For onto: for any integer $|x| \geq 0$, then $|x| + 1 \geq 1$, the range is the set of positive integers, therefore $f(x)$ is onto.

Conclusion, $f(x)$ is onto, but not one-to-one.

II.c

Answer is: $f: \mathbb{Z} \to \mathbb{Z}^{+}$, $f(x) = \begin{cases} 2x + 1, \ if\ x \geq 0 \\ -2x, \ if\ x < 0 \end{cases}$

For one-to-one: non-negative integers map to odd number, and negative integers map to even number, so if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, therefore, $f(x)$ is one-to-one.

For onto: when $x \geq 0$, $2x + 1 \geq 1$, so the range is all the odd positive integers. When $x < 0$, $-2x > 0$, so the range is all even positive integers. Therefore the range is all the positive integers.

Conclusion, $f(x)$ is both one-to-one and onto.

II.d

Answer is: $f: \mathbb{Z} \to \mathbb{Z}^{+}$, $f(x) = x^2 + 1$

For one-to-one: take $x = -2$, $f(-2) = (-2)^2 + 1 = 5$, take $x = 2$, $f(2) = 2^2 + 1 = 5$, then $f(-2) = f(2) = 5$, $f(x)$ is not one-to-one.

For onto: $f(x) \in \mathbb{Z}^{+}$, take $f(x) = 3$, then $x^2 + 1 = 3$, $x^2 = 2$. Since there is no integer that let $x^2 = 2$, so 2 is not in the range. Therefore $f(x)$ is not onto.

Conclusion, $f(x)$ is neither one-to-one nor onto.

Question 5:

a)

4.3.2.c

Answer is: $f^{-1}(x) = \frac{x-3}{2}$

$f(x)$ has a well-defined inverse. Because it is both one-to-one and onto.

- For one-to-one: if $f(x_1) = f(x_2)$, then $2x_1 + 3 = 2x_2 + 3$, this means $x_1 = x_2$. Therefore $f(x)$ is one-to-one.
- For onto: $y \in \mathbb{R}$, then $y = 2x + 3 \in \mathbb{R}$, then $x = \frac{y-3}{2} \in \mathbb{R}$, so the range of $f(x)$ is equal to the target. Therefore, $f(x)$ is onto.

Therefore $f^{-1}(x) = \frac{x-3}{2}$.

4.3.2.d

Answer is: $f(x)$ **doesn't have a well-defined inverse.**

Because $f(x)$ is not one-to-one, multiple subsets have the same size:
For $X \subseteq A$, take $X = \{1\}$, then $f(X) = f(\{1\}) = |\{1\}| = 1$,
take $X = \{2\}$, then $f(X) = f(\{2\}) = |\{2\}| = 1$,
Then $f(\{1\}) = f(\{2\}) = 1$, so $f(x)$ is not one-to-one, means it does have a well-defined inverse.

4.3.2.g

Answer is: $f^{-1}(a_1a_2a_3) = a_3a_2a_1$

$f(x)$ has a well-defined inverse. Because it is both one-to-one and onto.
Since $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$,
Then: $f(000) = 000$,
$f(001) = 100$,
$f(010) = 010$,
$f(011) = 110$,
$f(100) = 001$,
$f(101) = 101$,
$f(110) = 011$,
$f(111) = 111$,

This means, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$, and its range is $\{0, 1\}^3$, therefore f is one-to-one and onto.

$f(x)$ is obtained by taking the input string and reversing the bits. So $f^{-1}(x)$ is also reversing bits, then $f^{-1}(a_1a_2a_3) = a_3a_2a_1$.

4.3.2.i

Answer is: $f^{-1}(a, b) = (a - 5,\ b + 2)$

$f(x)$ has a well-defined inverse. Because it is both one-to-one and onto.

- For one-to-one: different input gives different outputs. Therefore $f(x)$ is one-to-one.
- For onto: for any $f(x, y) \in \mathbb{Z} \times \mathbb{Z}$, there is $(x - 5, y + 2) \in \mathbb{Z} \times \mathbb{Z}$ mapped to, and the range of f is equal to the target. Therefore, $f(x)$ is onto.

Therefore, $f^{-1}(a, b) = (a - 5,\ b + 2)$.

b)

4.4.8.c

Answer is: $(f \circ h)(x) = 2x^2 + 5$

$(f \circ h)(x) = f(h(x)) = f(x^2 + 1) = 2(x^2 + 1) + 3 = 2x^2 + 2 + 3 = 2x^2 + 5$

4.4.8.d

Answer is: $(h \circ f)(x) = 4x^2 + 12x + 10$

$(h \circ f)(x) = h(f(x)) = h(2x + 3) = (2x + 3)^2 + 1 = 4x^2 + 12x + 9 + 1 = 4x^2 + 12x + 10$

c)

4.4.2.b

Answer is: $(f \circ h)(52) = 121$

Since $(f \circ h)(x) = f(h(x)) = f(\lceil \frac{x}{5} \rceil) = (\lceil \frac{x}{5} \rceil)^2$

For $x = 52$, then $(f \circ h)(52) = (\lceil \frac{x}{5} \rceil)^2 = (\lceil \frac{52}{5} \rceil)^2 = 11^2 = 121$

4.4.2.c

Answer is: $(g \circ h \circ f)(4) = 16$

Since $(g \circ h \circ f)(x) = g(h(fx)) = g(h(x^2)) = g(\lceil \frac{x^2}{5} \rceil) = 2^{\lceil \frac{x^2}{5} \rceil}$

For $x = 4$, then $(g \circ h \circ f)(4) = 2^{\lceil \frac{x^2}{5} \rceil} = 2^{\lceil \frac{4^2}{5} \rceil} = 2^4 = 16$

4.4.2.d

Answer is: $(h \circ f)(x) = \lceil \frac{x^2}{5} \rceil$

$(h \circ f)(x) = h(f(x)) = h(x^2) = \lceil \frac{x^2}{5} \rceil$

d)

4.4.6.c

Answer is: $(h \circ f)(010) = 111$

$(h \circ f)(010) = h(f(010))$
Since $f(010) = 110$, by replacing the first bit by 1.
Then $h(f(010)) = h(110) = 111$, by replacing the last bit with a copy with the first bit.
So $(h \circ f)(010) = 111$.

4.4.6.d

Answer is: $\{101, 111\}$

$(h \circ f)(x) = h(f(x))$
For $f(x)$, since $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$, and f is obtained by taking the input string and replacing the first bit by 1, then all the results of $f(x)$ are: 100, 101, 110, 111.
For $h(x)$, since $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$, and f is obtained by taking the input string and replacing the last bit with a copy of the first bit, then:

$h(100) = 101,\ h(101) = 101,\ h(110) = 111,\ h(111) = 111$

So the range of $(h \circ f)(x)$ is $\{101, 111\}$.

4.4.6.e

Answer is: $\{001, 011, 101, 111\}$

$(g \circ f)(x) = g(f(x))$

For $f(x)$, since $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$, and f is obtained by taking the input string and replacing the first bit by 1, then all the results of $f(x)$ are: 100, 101, 110, 111.

For $g(x)$, since $f: \{0, 1\}^3 \rightarrow \{0, 1\}^3$, and f is obtained by taking the input string and reversing the bits, then:

$$h(100) = 001,\ h(101) = 101,\ h(110) = 011,\ h(111) = 111$$

So the range of $(h \circ f)(x)$ is $\{001, 011, 101, 111\}$.

e)

4.4.4.c

Answer is: No.

By assumption $f\colon X \to Y$ is not one-to-one, this means:
there exists $x_1, x_2 \in X$, where $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Since $g\colon Y \to Z$ is a well-defined function, an element in the domain can not map to two different elements in the target, so
for any $y_1, y_2 \in Y$, if $y_1 = y_2$, then $g(y_1) = g(y_2)$.

For $(g \circ f)(x) = g(f(x))$, thus $(g \circ f)(x_1) = g(f(x_1))$, $(g \circ f)(x_2) = g(f(x_2))$.

Since $f(x_1) = f(x_2)$, then $g(f(x_1)) = g(f(x_2))$, means $(g \circ f)(x_1) = (g \circ f)(x_2)$.

By assumption $x_1 \neq x_2$, and $(g \circ f)(x_1) = (g \circ f)(x_2)$, thus $(g \circ f)(x)$ is not one-to-one.
Therefore, it it not possible that f is not one-to-one and $g \circ f$ is one-to-one.

4.4.4.d

Answer is: Yes.
Example:

- $f\colon \{1, 2\} \to \{a, b, c, d\}$**:** $f(1) = a$**,** $f(2) = b$**,**
- $g\colon \{a, b, c, d\} \to \{101, 110, 111\}$**:** $g(a) = 101$**,** $g(b) = 110$**,** $g(c) = 111$**,** $g(d) = 111$**.**

By assumption $g\colon Y \to Z$ is not one-to-one, this means:
there exists $y_1, y_2 \in Y$, where $y_1 \neq y_2$, but $g(y_1) = g(y_2)$.

Since $(g \circ f)(x) = g(f(x))$, assume $g \circ f$ is one-to-one, then:
for all $x_1, x_2 \in X$, if $x_1 \neq x_2$, then $g(f(x_1)) \neq g(f(x_2))$.

Therefore when $f(x)$ doesn't include such pair $\{y_1, y_2\}$, where $y_1 \neq y_2$, but $g(y_1) = g(y_2)$, then for all $x_1, x_2 \in X$, if $x_1 \neq x_2$, make $f(x_1) \neq f(x_2)$ and $g(f(x_1)) \neq g(f(x_2))$ available. Thus, the assumption is true.

Example:

Let:

- $X = \{1, 2\}$,
- $Y = \{a, b, c, d\}$,
- $Z = \{101, 110, 111\}$,

Define the functions as:

- f: $X \to Y$: $f(1) = a$, $f(2) = b$,
- g: $Y \to Z$: $g(a) = 101$, $g(b) = 110$, $g(c) = 111$, $g(d) = 111$

For g: since $c, d \in Y$, $c \neq d$, $g(c) = f(d) = 111$, then g is not one-ton-one.

For $g \circ f$: since $1, 2 \in X$,

$(g \circ f)(1) = g(f(1)) = g(a) = 101$,

$(g \circ f)(2) = g(f(2)) = g(b) = 110$.

Thus, $1 \neq 2$, $(g \circ f)(1) \neq (g \circ f)(2)$, then $g \circ f$ is one-to-one.