Question 3

a. 8.2.2.b

Proof.
To proof $f(n) = n^3 + 3n^2 + 4$ is $\Theta(n^3)$, we need to show the upper bound $f(n)$ is $O(n^3)$, and the lower bound $f(n)$ is $\Omega(n^3)$.

Select $c_2 = 1$, $c_1 = 8$ and $n_0 = 1$. We will show that for any $n \geq 1$, $n^3 \leq f(n) \leq 8n^3$, implies $n^3 \leq n^3 + 3n^2 + 4 \leq 8n^3$, which means f is $\Theta(n^3)$.

For upper bound, we will show that: take $c_1 = 8$ for all $n \geq 1$, $f(n) \leq 8n^3$.

Since $n \geq 1$, then $n^2 \leq n^3$, $4 \leq 4n^3$,
adding the inequalities, then: $3n^2 + 4 \leq 3n^3 + 4n^3$,
adding both sides with n^3, then: $n^3 + 3n^2 + 4 \leq n^3 + 3n^3 + 4n^3 = 8n^3$,
which means $f(n) = n^3 + 3n^2 + 4 \leq 8n^3$.

We can get that for any $n \geq 1$, $f(n) \leq 8n^3$, which means $f(n) = O(n^3)$

For lower bound, we will show that: take $c_2 = 1$ for all $n \geq 1$, $n^3 \leq f(n)$.

Since $n \geq 1$, then $n^2 \geq 0$, thus: $3n^2 + 4 \geq 0$,
adding both sides with n^3, then: $n^3 + 3n^2 + 4 \geq n^3$,
which means $n^3 \leq n^3 + 3n^2 + 4 = f(n)$.

We can get that for any $n \geq 1$, $n^3 \leq f(n)$, which means $f(n) = \Omega(n^3)$

Therefore, for any $n \geq 1$, $n^3 \leq f(n) \leq 8n^3$, f is $\Theta(n^3)$.
Conclusion, $f(n) = n^3 + 3n^2 + 4$ is $\Theta(n^3)$ with constants $c_2 = 1$, $c_1 = 8$, when $n_0 = 1$.

b.

8.3.5 a

The algorithm partitions the input sequence around the p's value. It rearranges the elements of the input sequence, the elements less than p are moved to the left side, while the elements greater than or equal to p are moved to the right side. This is achieved by increasing i from the start until an element greater than or equal to p is found, decrementing j from the end until an element less than p is found, and swapping these elements if $i < j$. This process continues until i and j meet.

For example, suppose input sequence [3, -1, 5, 0, -2], n = 5, p = 0,
Now we have $a_1 = 3,\ a_2 = -1,\ a_3 = 5,\ a_4 = 0,\ a_5 = -2.$
Starting with the initial value: $i = 1, j = 5,$
Since $i = 1, j = 5, i < j$ is true,

Check the first inner loop: since $a_1 = 3 > p$, so i will not increase,

Check the second inner loop: since $a_5 = -2 < p$, so j will not decrease.

Since $i < j$, swap a_1 and a_5, then the input sequence changed to: -2, -1, 5, 0, 3.

End of the first iteration. Now back to the outer loop: $i = 1, j = 5, i < j$ is true,

Check the first inner loop:

Since $a_1 = -2 < p$, then $i = i + 1$, i increases to 2.

Since $a_2 = -1 < p$, then $i = i + 1$, i increases to 3.

Since $a_3 = 5 > p$, then i stops increasing.

Exit the first inner loop.

Check the second inner loop:

Since $a_5 = 3 > p$, then $j = j - 1,\ j$ decreases to 4.

Since $a_4 = 0 = p$, then $j = j - 1,\ j$ decreases to 3.

Since $a_3 = 5 > p$, then $j = j - 1,\ j$ decreases to 2.

Since $a_2 = -1 < p$, then j stops decreasing.

Exit the second inner loop.

Now $i = 3, j = 2, i < j$ is false, exit the outer loop.

Therefore the final output: -2, -1, 5, 0, 3.

8.3.5 b

The number of times i is incremented or j is decremented is exactly $n - 1$, regardless of the values in the input sequence.

8.3.5 c

The total number of times that the swap operation is executed depend on the actual value of the numbers in the sequence, and the range from 0 to $\lfloor \frac{n}{2} \rfloor$.

The maximize number of swaps is $\lfloor \frac{n}{2} \rfloor$, when the left half of the input sequence has all elements greater than or equal to p, and the right half of the input sequence has all elements less than p.

The minimize number of swaps is 0, when the input sequence is already correctly partitioned.

8.3.5 d

The lower bound is $\Omega(n)$. It's not important to consider the worst-case input for determining the asymptotic lower bound.

Because the lower bound represents the minimum work the algorithm must do for any input of size n. Even in the best-case, the algorithm must process each element at least once, i, j start at opposite ends of the input sequence, i need to increase, j need to decrease until $i \geq j$. However the maximize number of swaps is $\lfloor \frac{n}{2} \rfloor$. So it's not important to consider the worst-case input.

8.3.5 e

The upper bound is $O(n)$. Because the algorithm processes each element a constant number of times, leading to a linear time complexity. Thus, the time complexity is $\Theta(n)$.

Question 4

a.

5.1.2 b

The answer is: $40^7 + 40^8 + 40^9$

Total characters = 10 + 26 + 4 = 40
For length 7: 40^7
For length 8: 40^8
For length 9: 40^9
The total number of possible passwords is $40^7 + 40^8 + 40^9$

5.1.2 c

The answer is: $14 \times (40^6 + 40^7 + 40^8)$

Total characters = 10 + 26 + 4 = 40
Sum of digits and special characters = 10 + 4 = 14
For length 7, the first character cannot be a letter: 14×40^6
For length 8, the first character cannot be a letter: 14×40^7
For length 9, the first character cannot be a letter: 14×40^8
The total number of possible passwords is
$14 \times 40^6 + 14 \times 40^7 + 14 \times 40^8 = 14 \times (40^6 + 40^7 + 40^8)$

b.

5.3.2 a

The answer is: 3×2^9

Total characters is 3 from the set $\{a, b, c\}$.
For the first character, there are 3 choices.
For the subsequent characters, since two consecutive characters can not be the same, there are 2 choices.
Thus, the total number of strings is: $3 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 3 \times 2^9$

c.

5.3.3 b

The answer is: $10 \times 26^4 \times 9 \times 8$

For the first digit: there are 10 choices.
For the next four characters: there are $26 \times 26 \times 26 \times 26 = 26^4$ choices.
For the next digit, since no digit appears more than once, and the first digit is taken, there are 9 choices.
For the last digit, it must be different from the other two, so there are 8 choices.
Thus, the total number of possible license plate number is: $10 \times 26^4 \times 9 \times 8$.

5.3.3 c

The answer is: $10 \times 26 \times 25 \times 24 \times 23 \times 9 \times 8$

For the first digit: there are 10 choices.
For the next four characters:
there are 26 choices for the first letter,
Since it must be different from the first, there are 25 choices for the second letter,
24 choices for the third letter,
and 23 choices for the fourth letter.
For the next digit, since no digit appears more than once, and the first digit is taken, there are 9 choices.
For the last digit, it must be different from the other two, so there are 8 choices.
Thus, the total number of possible license plate number is: $10 \times 26 \times 25 \times 24 \times 23 \times 9 \times 8$.

d.

5.2.3 a

Function definition:
For any 9-bit string b in B^9, construct a binary string e in E_{10} by appending a 10th bit to b.

- If the number of 1s in b is even, append a 0 to the end of b.
- If the number of 1s in b is odd, append a 1 to the end of b.

Explanation:

For one-to-one:

Suppose there are different binary strings b_1, b_2 in B^9, which means $b_1 \neq b_2$.

- If the number of b_1, b_2 is even, append a 0 to the end of them, then $e_1 \neq e_2$
- If the number of b_1, b_2 is odd, append a 1 to the end of them, then $e_1 \neq e_2$
- If the number of b_1, b_2 one is odd another is even, append a 0 to one of them, append a 1 to another of them, then $e_1 \neq e_2$.

Therefore, the function is one-to-one.

For onto: Since every binary string e in E_{10} is constructed by appending a 10th bit to b in B^9, so for every string in E_{10} we can find a corresponding string in B^9. Thus, the function is onto.

5.2.3 b

$\left|E_{10}\right| = 512$

Since B^9 and E_{10} are two finite sets, and there is a bijection from B^9 to E_{10},
thus $\left|E_{10}\right| = \left|B^9\right| = 2^9 = 512$.

Question 5

a.

5.4.2 a

The answer is: 2×10^4

The first three digits are 824 or 825, so there are 2 choices.
Since there are seven digits long, for the next four digits, there are $10 \times 10 \times 10 \times 10$ choices.
Thus, the total number of possible phone number is: 2×10^4.

5.4.2 b

The answer is: $2 \times 10 \times 9 \times 8 \times 7$

The first three digits are 824 or 825, so there are 2 choices.
Since there are seven digits long, for the next four digits, when they are all different:

- For the first digit in four, there are 10 choices.
- For the second digit in four, there are 9 choices.
- For the third digit in four, there are 8 choices.
- For the last digit in four, there are 7 choices.

Thus, the total number of possible choices is: $2 \times 10 \times 9 \times 8 \times 7$.

b.

5.5.3 a

The answer is: 2^{10}

A 10-bit string has 10 positions, each position can be 0 or 1. There are no restrictions, so there are 2 choices in each position. Thus, the total number of possible is: 2^{10}.

5.5.3 b

The answer is: 2^7

Since the string starts with 001, there are 7 positions remaining. Each position can be 0 or 1, there are 2 choices in each position. Thus, the total number of string that start with 001 is: 2^7

5.5.3 c

The answer is: $2^7 + 2^8$

The number of string that start with 001 is 2^7, we did it from #b.

The number of string that start with 10 is 2^8, because there are 8 positions remaining, each position can be 0 or 1, there are 2 choices in each position.
And there is no overlap between 001 and 10.

Thus, the total number of the string starts with 001 or 10 is: $2^7 + 2^8$.

5.5.3 d

The answer is: 2^8

For the first two bits, there are 2 choices for each position, so the number is 2^2.
The last two bits are the same as the first two, so there is no more possible.

For the middle 6 bits, there are no restrictions, so there is 2^6
So the total number of the string that first two bits are the same as the last two bits is $2^2 \times 2^6 = 2^8$.

5.5.3 e

The answer is: $C(10, 6)$

The string has exactly six 0s, which means choose 6 positions out of 10 for 0s.
So the number is: $C(10, 6)$.

5.5.3 f

The answer is $C(9, 6)$

Since the first bit is 1, there are 9 remaining bits.
The string has exactly six 0s, which means choose 6 positions out of 9 for 0s.
So the number is: $C(9, 6)$

5.5.3 g

The answer is $C(5, 1) \times C(5, 3)$

Since the first half of the string has 5 bits, and it contains exactly 1, which means choose 1 position out of 5 for 1s. So the number is $C(5, 1)$.

The second half contains exactly three 1s, which means choose 3 positions out of 5. So the number is $C(5,3)$.
So the total number is: $C(5,1) \times C(5,3)$

c.

5.5.5 a

The answer is: $C(30,10) \times C(35,10)$

Select the boys: select 10 boys from 30 boys is $C(30,10)$.
Select the girls: select 10 girls from 35 girls is $C(35,10)$.
The total number of ways is: $C(30,10) \times C(35,10)$.

d.

5.5.8 c

The answer is: $C(26,5)$

There are 13 hearts and 13 diamonds, then the total number is $13 + 13 = 26$.
Choose 5 cards from these 26 card is: $C(26,5)$.

5.5.8 d

The answer is: $C(13,1) \times C(48,1)$

For four cards of same rank, there are only one way to choose four cards from 4 suits and keep them same rank, so choose the rank from all ranks is: $C(13,1)$.
Choose the fifth cards from the remaining $52 - 4 = 48$ cards is: $C(48,1)$.
Total number of ways is: $C(13,1) \times C(48,1)$.

5.5.8 e

The answer is: $C(13,1) \times C(4,2) \times C(12,1) \times C(4,3)$

For two cards of the same rank, choose the rank is $C(13,1)$, choose the suits is $C(4,2)$, then the number of ways is: $C(13,1) \times C(4,2)$.
For three cards of the same rank, choose the rank (remaining 12 choices) is $C(12,1)$, choose the suits is $C(4,3)$, then the number of ways is: $C(12,1) \times C(4,3)$.
Total number of ways is: $C(13,1) \times C(4,2) \times C(12,1) \times C(4,3)$.

5.5.8 f

The answer is: $C(13,5) \times 4^5$

For five-card hands do not have any two cards of the same rank, which means each card has a unique rank. Choose 5 different ranks from 13 choices is: $C(13,5)$.
For each rank choose one suit, there are 4 suits for each rank, so the number of ways is:

$4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

Total number of ways is: $C(13,5) \times 4^5$.

e.

5.6.6 a

The answer is: $C(44,5) \times C(56,5)$

For 10 senate members with the same number of each party, need to select 5 Democrats and 5 Republicans.
Choose 5 Democrats from 44 is: $C(44,5)$.
Choose 5 Republicans from 56 is: $C(56,5)$.
Total number of ways is: $C(44,5) \times C(56,5)$.

5.6.6 b

The answer is: $44 \times 43 \times 56 \times 55$

Choose the speaker and vice speaker for the Democrats: 44×43.
Choose the speaker and vice speaker for the Republicans: 56×55.
Total number of ways is: $44 \times 43 \times 56 \times 55$.

Question 6

a.

5.7.2 a

The answer is: $C(52,5) - C(39,5)$

A standard playing card contains 52 cards. The total number of ways to choose a 5-card hand from 52 cards is: $C(52,5)$.
There are 13 club cards, then non-club cards is $52 - 13 = 39$. the total number of ways to choose 5-card hand with no club is: $C(39,5)$.
Thus the number of ways that at least one club is: $C(52,5) - C(39,5)$.

5.7.2 b

The answer is: $C(52,5) - C(13,5) \times 4^5$

A standard playing card contains 52 cards. The total number of ways to choose a 5-card hand from 52 cards is: $C(52,5)$.
There are 13 ranks, choose different ranks from 13 choices is $C(13,5)$. Every rank has 4 suits choices, so the number of ways to choose the suit for the 5 is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$. Thus, the total number of no cards of the same rank is: $C(13,5) \times 4^5$.
Therefore, the number of ways that at least two cards with the same rank is:
$C(52,5) - C(13,5) \times 4^5$.

b.

5.8.4 a

The answer is: 5^{20}

Since there are 5 choices for each of the 20 comic books, there are no restrictions on how many go to each kid, so for each books there are 5 choices, the number of ways is: 5^{20}

5.8.4 b

The answer is: $C(20,4) \times C(16,4) \times C(12,4) \times C(8,4) \times C(4,4)$

Since the comic book are divided evenly, so each kids get $20 \div 5 = 4$ books.
Choose 4 comic books for the first kid, the number of ways is: $C(20, 4)$,
Choose 4 comic books for the second kid, there are 16 comic books left, the number of ways is: $C(16, 4)$,
Choose 4 comic books for the third kid, the number of ways is: $C(12, 4)$,
Choose 4 comic books for the fourth kid, the number of ways is: $C(8, 4)$,
Choose 4 comic books for the fifth kid, the number of ways is: $C(4, 4)$,
Thus the total number of ways is: $C(20, 4) \times C(16, 4) \times C(12, 4) \times C(8, 4) \times C(4, 4)$.

Question 7

a.

The answer is: 0

Since there are 5 elements in the domain and 4 elements in the codomain. The elements in the domain are larger than the codomain. Thus, it can not be a one-to-one function.

b.

The answer is: $5!$

Since there are both 5 elements in the domain and codomain, the number of one-to-one functions is the number of ways to assign the 5 elements of the domain to the 5 elements of the codomain. So the number is: $5!$.

c.

The answer is: $P(6,5)$

Since there are 5 elements in the domain and 6 elements in the codomain. The number of one-to-one functions is the number of ways to assign the 5 elements of the domain to the 6 elements of the codomain. So the number is: $P(6,5)$.

d.

The answer is: $P(7,5)$

Since there are 5 elements in the domain and 7 elements in the codomain. The number of one-to-one functions is the number of ways to assign the 5 elements of the domain to the 7 elements of the codomain. So the number is: $P(7,5)$.