Question 5

a.

Proof: By induction on n.

Let $P(n)$ be the proposition that $n^3 + 2n$ is divisible by 3.

Base case:

When $n = 1$, $n^3 + 2n = 1^3 + 2 \times 1 = 1 + 2 = 3$, 3 is divisible by 3.
Therefore, when $n = 1$, $P(1)$ is true.

Inductive step:

Assume the inductive hypothesis is that: $P(k)$ is true for any positive integer k, that is $k^3 + 2k$ is divisible by 3.

Now we have to show that $P(k + 1)$ is also true, that is $(k + 1)^3 + 2(k + 1)$ is divisible by 3.

$$
\begin{aligned}
(k + 1)^3 + 2(k + 1) &= k^3 + 3k^2 + 3k + 1 + 2k + 2 \\
&= k^3 + 2k + 3k^2 + 3k + 3 \\
&= (k^3 + 2k) + 3(k^2 + k + 1)
\end{aligned}
$$

Since:

$(k^3 + 2k)$ is divisible by 3, by the inductive hypothesis,

$3(k^2 + k + 1)$ is also divisible by 3.

Thus, $(k^3 + 2k) + 3(k^2 + k + 1)$ is divisible by 3.

Therefore, if $P(k)$ is true, we show that $P(k + 1)$ is also true.

b.

Proof: By strong induction on n.
Let $P(n)$ be the proposition that n can be written as a product of primes.

Base case:
When $n = 2$, since 2 is a prime number, it can be written as a product primes, just itself.
Therefore, when $n = 2$, $P(2)$ is true.

Inductive step:
Assume the inductive hypothesis is that: for $k \geq 2$ and for every integer j, where $2 \leq j \leq k$, $P(j)$ can be written as a product of primes.
Now we have to show that $P(k + 1)$ is also true, that is $P(k + 1)$ can be written as a product of primes.
Case 1:

If $k + 1$ is prime, it is already a product of primes, just itself.

Thus, $P(k + 1)$ is true.

Case 2:

If $k + 1$ is not prime, then by definition, $k + 1$ can be expressed as $k + 1 = a \times b$, where a and b are integers such that $2 \leq a \leq k$ and $2 \leq b \leq k$.

- Since $2 \leq a \leq k$, by inductive hypothesis, $P(a)$ is true. Thus, a can be written as a product of primes: $a = p_1 \times p_2 \times ... \times p_r$.
- Since $2 \leq b \leq k$, by inductive hypothesis, $P(b)$ is true. Thus, b can be written as a product of primes: $b = q_1 \times q_2 \times ... \times q_r$.

Therefore, $k + 1 = a \times b = (p_1 \times p_2 \times ... \times p_r) \times (q_1 \times q_2 \times ... \times q_r)$, means that $k + 1$ can be written as a product of primes, so $P(k + 1)$ is true.

Therefore, any positive integer $n \geq 2$ can be written as a product of primes.

Question 6

a.

7.4.1 a

When $n = 3$, the left side of the equation is: $\sum_{j=1}^{3} j^2 = 1^2 + 2^2 + 3^2 = 1 + 4 + 9 = 14$

When $n = 3$, the right side of the equation is: $\frac{3(3+1)(2\times3+1)}{6} = \frac{3\times4\times7}{6} = 14$

Therefore, when $n = 3$, $\sum_{j=1}^{3} j^2 = \frac{3(3+1)(2\times3+1)}{6} = 14$, $P(3)$ is true.

7.4.1 b

$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.

7.4.1 c

$P(k + 1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$.

7.4.1 d

In the base case, we must prove that $P(1)$ is true, which means for $n = 1$, the left side of the equation $\sum_{j=1}^{1} 1^2$ should equal to the right side of the equation $\frac{1(1+1)(2\times1+1)}{6}$.

7.4.1 e

In the inductive step, we must prove that for any positive integer n, if $P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$ is true, then $P(k + 1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}$ is also true.

7.4.1 f

The inductive hypothesis is: assume $P(k)$ is true, means $\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.

7.4.1 g

Proof: By induction on n.

Let $P(n)$ be the proposition that $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$.

Base case:

When $n = 1$, the left side of the equation is: $\sum_{j=1}^{1} j^2 = 1^2 = 1$

When $n = 1$, the right side of the equation is: $\frac{1(1+1)(2\times1+1)}{6} = \frac{1\times2\times3}{6} = 1$

Therefore, when $n = 1$, $\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2\times1+1)}{6}$, $P(1)$ is true.

Inductive step:

Assume the inductive hypothesis is that: $P(k)$ is true for any positive integer k, that is

$$\sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}.$$

Now we have to show that $P(k + 1)$ is also true, that is

$$\sum_{j=1}^{k+1} j^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6} = \frac{(k+1)(k+2)(2k+3)}{6}.$$

The left side of the equation is: $\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k + 1)^2$ [using inductive hypothesis]

$$= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)(k(2k+1)+6(k+1))}{6}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

Therefore, if $P(k)$ is true, we show that $P(k + 1)$ is also true.

b.

7.4.3 c

Proof: By induction on n.

Let $P(n)$ be the proposition that $\sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$.

Base case:

When $n = 1$, the left side of the inequation is: $\sum_{j=1}^{1} \frac{1}{j^2} = 1$

When $n = 1$, the right side of the inequation is: $2 - \frac{1}{1} = 2 - 1 = 1$

Therefore, when $n = 1$, $\sum_{j=1}^{1} \frac{1}{j^2} \leq 2 - \frac{1}{1}$, $P(1)$ is true.

Inductive step:
Assume the inductive hypothesis is that: $P(k)$ is true for $k \geq 1$, that is

$$\sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}.$$

Now we have to show that $P(k + 1)$ is also true, that is

$$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}.$$

The left side of the inequation is: $\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2}$ [using inductive hypothesis]

$$= 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\leq 2 - \frac{1}{k} + \frac{1}{k(k+1)} = 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)}$$

$$= 2 - \frac{k+1-1}{k(k+1)}$$

$$= 2 - \frac{k}{k(k+1)}$$

$$= 2 - \frac{1}{k+1}$$

Thus, $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$.

Therefore, if $P(k)$ is true, we show that $P(k + 1)$ is also true.

c.

7.5.1 a

Proof: By induction on n.

Let $P(n)$ be the proposition that $3^{2n} - 1$ is evenly divisible by 4.

Base case:

When $n = 1$, $3^{2n} - 1 = 3^{2} - 1 = 9 - 1 = 8$.
Since $8 = 4 \times 2$, thus 4 evenly divides 8.
Therefore, when $n = 1$, $P(1)$ is true.

Inductive step:

Assume the inductive hypothesis is that: $P(k)$ is true for any positive integer k, that is $3^{2k} - 1$ is evenly divisible by 4.

Now we have to show that $P(k + 1)$ is also true, that is $3^{2(k+1)} - 1$ is evenly divisible by 4.

$$
\begin{aligned}
3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\
&= 3^{2k} \cdot 3^{2} - 1 \\
&= 3^{2k} \cdot 9 - 1 \\
&= 3^{2k} \cdot 9 - 9 + 8 \\
&= (3^{2k} - 1) \cdot 9 + 8 \\
&= 9 \cdot (3^{2k} - 1) + 8
\end{aligned}
$$

Since:

$(3^{2k} - 1)$ is evenly divisible by 4, by the induction hypothesis,

$9 \cdot (3^{2k} - 1)$ is also evenly divisible by 4.

Since 8 is evenly divisible by 4.

Thus, $9 \cdot (3^{2k} - 1) + 8$ is evenly divisible by 4.

Therefore, if $P(k)$ is true, we show that $P(k + 1)$ is also true.