Question 7

a.

6.1.5 b

The answer is: $\frac{C(13,1)\times C(4,3)\times C(12,2)\times 4^2}{C(52,5)} \approx 0.0211$

$P(three\ of\ a\ kind) = \frac{|three\ of\ a\ kind|}{|5-card\ hand|}$

For $|three\ of\ a\ kind|$:

- Choose the rank for the three of a kind: $C(13, 1)$
- For the rank above, choose 3 suits out of 4: $C(4, 3)$
- Choose other 2 distinct ranks from 12 remaining ranks: $C(12, 2)$
- For the 2 ranks above, choose suits, since every rank have 4 choices: 4^2
- The total number: $C(13, 1) \times C(4, 3) \times C(12, 2) \times 4^2 = 13 \times 4 \times 66 \times 16 = 54912$

For $|5 - card\ hand|$:

- Choose 5 cards from 52 cards: $C(52, 5) = \frac{52!}{47!\times 5!} = 2598960$

Therefore $p(three\ of\ a\ kind) = \frac{|three\ of\ a\ kind|}{|5-card\ hand|} = \frac{C(13,1)\times C(4,3)\times C(12,2)\times 4^2}{C(52,5)} = \frac{54912}{2598960} \approx 0.0211$

6.1.5 c

The answer is: $\frac{C(4,1)\times C(13,5)}{C(52,5)} \approx 0.0020$

$P(5\ same\ suit) = \frac{|5\ same\ suit|}{|5-card\ hand|}$

For $|5\ same\ suit|$:

- Choose 1 suit out of 4: $C(4, 1)$
- Choose 5 cards from 13 cards in that suit: $C(13, 5)$
- The total number: $C(4, 1) \times C(13, 5) = 4 \times 1287 = 5148$

For $|5 - card\ hand|$:

- Choose 5 cards from 52 cards: $C(52, 5) = \frac{52!}{47!\times 5!} = 2598960$

Therefore $p(three\ of\ a\ kind) = \frac{|three\ of\ a\ kind|}{|5-card\ hand|} = \frac{C(4,1)\times C(13,5)}{C(52,5)} = \frac{5148}{2598960} \approx 0.0020$

6.1.5 d

The answer is: $\frac{C(13,1)\times C(4,2)\times C(12,3)\times 4^3}{C(52,5)} \approx 0.4226$

$P(two\ of\ a\ kind) = \frac{|two\ of\ a\ kind|}{|5-card\ hand|}$

For $|two\ of\ a\ kind|$:

- Choose the rank for the two of a kind: $C(13, 1)$
- For the rank above, choose 2 suits out of 4: $C(4, 2)$
- Choose other 3 distinct ranks from 12 remaining ranks: $C(12, 3)$
- For the 3 ranks above, choose suits, since every rank have 4 choices: 4^3
- The total number: $C(13, 1) \times C(4, 2) \times C(12, 3) \times 4^3 = 13 \times 6 \times 220 \times 64 = 1098240$

For $|5 - card\ hand|$:

- Choose 5 cards from 52 cards: $C(52, 5) = \frac{52!}{47!\times 5!} = 2598960$

Therefore $p(three\ of\ a\ kind) = \frac{|three\ of\ a\ kind|}{|5-card\ hand|} = \frac{C(13,1)\times C(4,2)\times C(12,3)\times 4^3}{C(52,5)} = \frac{1098240}{2598960} \approx 0.4226$

b.

6.2.4 a

The answer is: $1 - \frac{C(39,5)}{C(52,5)} \approx 0.7785$

For five-card hands, the total number is: $C(52, 5) = \frac{52!}{47!\times 5!} = 2598960$

When there is no club card, there are $52 - 13 = 39$ cards, then the total number of five-card hands is: $C(39, 5) = \frac{39!}{34!\times 5!} = 575757$

The complement of “at least one club” is “no clubs”, then

$p(at\ least\ one\ club\ card) = 1 - P(no\ club\ card) = 1 - \frac{C(39,5)}{C(52,5)} = 1 - \frac{575757}{2598960} \approx 0.7785$

6.2.4 b

The answer is: $1 - \frac{C(13,5)\times 4^5}{C(52,5)} \approx 0.4929$

For five-card hands, the total number is: $C(52, 5) = \frac{52!}{47!\times 5!} = 2598960$

There are 13 ranks, when there is no same rank in five-card hand, the total number is:

$C(13, 5) \times 4^5 = \frac{13!}{8!\times 5!} \times 4^5 = 1287 \times 1024 = 1317888$

The complement of “at least two cards with the same rank” is “no same rank”, then

$p(at\ least\ two\ cards\ with\ same\ rank) = 1 - P(no\ same\ rank) = 1 - \frac{C(13,5)\times 4^5}{C(52,5)} = 1 - \frac{1317888}{2598960} \approx 0.4929$

6.2.4 c

The answer is: $\frac{2\times C(13,1)\times C(39,4)-C(13,1)\times C(13,1)\times C(26,3)}{C(52,5)} \approx 0.6538$

For five-card hands, the total number is: $C(52,5) = \frac{52!}{47!\times 5!} = 2598960$

For the hand has exactly one club:

- Choose one rank from club: $C(13,1)$
- Choose others from $52 - 13 = 39$ remaining cards: $C(39,4)$
- The total number is: $C(13,1) \times C(39,4)$

For the hand has exactly one spade:

- Same as above, so the total number is: $C(13,1) \times C(39,4)$

For the intersection exactly one club and exactly one spade, the total number is:
$C(13,1) \times C(13,1) \times C(26,3)$
The total number of the hand exactly one club or exactly one spade is:
$2 \times C(13,1) \times C(39,4) - C(13,1) \times C(13,1) \times C(26,3)$
$p(exactly\ one\ club\ or\ exactly\ one\ spade) = \frac{2\times C(13,1)\times C(39,4)-C(13,1)\times C(13,1)\times C(26,3)}{C(52,5)} = \frac{2\times 1069263-439400}{2598960} \approx 0.6538$

6.2.4 d

The answer is: $1 - \frac{C(26,5)}{C(52,5)} \approx 0.9747$

For five-card hands, the total number is: $C(52,5) = \frac{52!}{47!\times 5!} = 2598960$
When there is neither clubs nor spades, there are $52 - 13 - 13 = 26$ cards, the total number is: $C(26,5) = 65780$
$p(at\ least\ one\ club\ or\ at\ least\ one\ spade) = 1 - \frac{C(26,5)}{C(52,5)} = 1 - \frac{65780}{2598960} \approx 0.9747$

Question 8

a.

6.3.2 a

The answer is: $p(A) = \frac{1}{7}$, $p(B) = \frac{1}{2}$, $p(C) = \frac{1}{42}$

Event A:
- b is in the middle position, so there are 6 letters remaining, the number of arrangement is: $6!$
- All the ways of arrangement is: $7!$
- $p(A) = \frac{6!}{7!} = \frac{6\times5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \frac{1}{7}$

Event B:
- Letter c on the right of b, since letter c either left or right, these are equally likely.
- $p(B) = \frac{1}{2}$

Event C:
- Group “def” together and keep the order, there are $4 + 1 = 5$ items, the number of arrangement is: $5!$
- $P(C) = \frac{5!}{7!} = \frac{5\times4\times3\times2\times1}{7\times6\times5\times4\times3\times2\times1} = \frac{1}{42}$

6.3.2 b

The answer is: $p(A|C) = \frac{1}{10}$

Event $A \cap C$:
- Since “b” is in the middle and “def” occurs together and keep the order, there are $2 \times 1 = 2$ ways to arrange them.
- There are 3 remaining letters, the number of ways: $3!$
- The total number of ways is: $2 \times 3!$

Event C:
- From 6.3.2a, the number of ways is $5!$

Therefore, $p(A|C) = \frac{p(A\cap C)}{p(C)} = \frac{|A\cap C|}{|C|} = \frac{2\times3!}{5!} = \frac{1}{10}$

6.3.2 c

The answer is: $p(B|C) = \frac{1}{2}$

Event $B \cap C$:

- Since "def" occurs together and keeps the order, it does not affect the relative order of b and c.
- We want c to be to the right of b, since letter c either left or right, these are equally likely.

Thus, $p(B|C) = \frac{1}{2}$

6.3.2 d

The answer is: $p(A|B) = \frac{1}{7}$

Event $A \cap B$:

- Since b is in the middle, then c can be in 3 of the remaining 6 positions, so the number of ways is: $3 \times 5!$
- $p(A \cap B) = \frac{3\times5!}{7!} = \frac{1}{14}$

Thus, $p(A|B) = \frac{p(A\cap B)}{p(B)} = \frac{1/14}{1/2} = \frac{1}{7}$

6.3.2 e

The answer is: event A and B are independent, event B and C are independent.

Since:

- $p(A|B) = p(A) = \frac{1}{7}$,
- $p(B|C) = p(B) = \frac{1}{2}$,
- $p(A|C) = \frac{1}{10} \neq p(A)$**.**

Thus:

- Event A and B are independent,
- Event B and C are independent,
- Event A and C are not independent.

b.

6.3.6 b

The answer is: $(\frac{1}{3})^5 \times (\frac{2}{3})^5 = \frac{32}{59049}$

Since coin flipped 10 times and independent:

- For the first 5 flips come up heads probability is: $(\frac{1}{3})^5$
- For the last 5 flips come up tails probability is: $(\frac{2}{3})^5$
- Total probability is: $(\frac{1}{3})^5 \times (\frac{2}{3})^5 = \frac{32}{59049}$

6.3.6 c

The answer is: $\frac{1}{3} \times (\frac{2}{3})^9 = \frac{512}{59049}$

Since coin flipped 10 times and independent:

- For the first flips come up heads probability is: $\frac{1}{3}$
- For the rest of the flips come up tails probability is: $(\frac{2}{3})^9$
- Total probability is: $\frac{1}{3} \times (\frac{2}{3})^9 = \frac{512}{59049}$

c.

6.4.2 a

The answer is: $\frac{(\frac{1}{6})^6 \times \frac{1}{2}}{(\frac{1}{6})^6 \times \frac{1}{2} + (0.15)^4 \times (0.25)^2 \times \frac{1}{2}} \approx 0.4038$

Since Ariel choose a die at random, thus:

- $p(Fair) = p(Biased) = \frac{1}{2}$

Ariel rolls six times, getting 4, 3, 6, 6, 5, 5:

- For fair die, each roll has probability $\frac{1}{6}$, then:

  $$p(Six \mid Fair) = (\frac{1}{6})^6$$

- For biased die, probability of 6 is 0.25, others is 0.15, then:

  $$p(Six \mid Biased) = (0.15)^4 \times (0.25)^2$$

Bayes' theorem:

$$p(Fair \mid Six) = \frac{p(Six \mid Fair)p(Fair)}{p(Six \mid Fair)p(Fair)+p(Six \mid Biased)p(Biased)} = \frac{(\frac{1}{6})^6 \times \frac{1}{2}}{(\frac{1}{6})^6 \times \frac{1}{2} + (0.15)^4 \times (0.25)^2 \times \frac{1}{2}} \approx 0.4038$$

Question 9

a.

6.5.2 a

The answer is: $\{0, 1, 2, 3, 4\}$**.**

Since there are 4 aces in the standard deck, the maximum number of aces in a 5-card hand is 4, the minimum number of aces in a 5-card deck is 0. Thus, the range of A is $\{0, 1, 2, 3, 4\}$.

6.5.2 b

The answer is: $\{(0,\ 0.6588),\ (1,\ 0.2995),\ (2,\ 0.0399),\ (3,\ 0.0017),\ (4,\ 0.000018)\}$**.**

Since the range of A is $\{0, 1, 2, 3, 4\}$, A denotes the number of aces in the hand.
For five-card hands, the number of ways is: $C(52, 5)$.
There are 4 aces in the standard deck, the number of remaining cards is: $52 - 4 = 48$.
For $r = 0$, $p(A = 0) = \frac{C(48,5)}{C(52,5)} \approx 0.6588$
For $r = 1$, $p(A = 1) = \frac{C(4,1)\times C(48,4)}{C(52,5)} \approx 0.2995$
For $r = 2$, $p(A = 2) = \frac{C(4,2)\times C(48,3)}{C(52,5)} \approx 0.0399$
For $r = 3$, $p(A = 3) = \frac{C(4,3)\times C(48,2)}{C(52,5)} \approx 0.0017$
For $r = 4$, $p(A = 4) = \frac{C(4,4)\times C(48,1)}{C(52,5)} \approx 0.000018$
Therefore, the distribution of over random variable A is:
$\{(0,\ 0.6588),\ (1,\ 0.2995),\ (2,\ 0.0399),\ (3,\ 0.0017),\ (4,\ 0.000018)\}$.

b.

6.6.1 a

The answer is: $E[G] = 1.4$

Since there are 7 girls and 3 boys, the total number is 10.
Select 2 of them, the number of ways is: $C(10, 2)$
Probabilities:

- $p(G = 0) = \frac{C(3,2)}{C(10,2)} = \frac{3}{45} = \frac{1}{15}$
- $p(G = 1) = \frac{C(7,1)\times C(3,1)}{C(10,2)} = \frac{7\times 3}{45} = \frac{7}{15}$
- $p(G = 2) = \frac{C(7,2)}{C(10,2)} = \frac{21}{45} = \frac{7}{15}$

$E[G] = 0 \times \frac{1}{15} + 1 \times \frac{7}{15} + 2 \times \frac{7}{15} = \frac{7}{5} = 1.4$

c.

6.6.4 a

The answer is: $E[X] = \frac{91}{6} \approx 15.1667$

Since a fair die is rolled once, each with probability is $\frac{1}{6}$, then:

- For die comes up 1, $X = 1^2 = 1$, $p(X = 1) = \frac{1}{6}$
- For die comes up 2, $X = 2^2 = 4$, $p(X = 4) = \frac{1}{6}$
- For die comes up 3, $X = 3^2 = 9$, $p(X = 9) = \frac{1}{6}$
- For die comes up 4, $X = 4^2 = 16$, $p(X = 16) = \frac{1}{6}$
- For die comes up 5, $X = 5^2 = 25$, $p(X = 25) = \frac{1}{6}$
- For die comes up 6, $X = 6^2 = 36$, $p(X = 36) = \frac{1}{6}$

Thus, $E[X] = 1 \times \frac{1}{6} + 4 \times \frac{1}{6} + 9 \times \frac{1}{6} + 16 \times \frac{1}{6} + 25 \times \frac{1}{6} + 36 \times \frac{1}{6} = \frac{91}{6} \approx 15.1667$

6.6.4 b

The answer is: $E[Y] = 3$

Since a fair coin is tossed three times, then the total number of ways is $2^3 = 8$.

- For 0 heads, $Y = 0$, $p(Y = 0) = \frac{1}{8}$
- For 1 heads, $Y = 1^2 = 1$, $p(Y = 1) = \frac{C(3,1)}{8} = \frac{3}{8}$
- For 2 heads, $Y = 2^2 = 4$, $p(Y = 4) = \frac{C(3,2)}{8} = \frac{3}{8}$
- For 3 heads, $Y = 3^2 = 9$, $p(Y = 9) = \frac{C(3,3)}{8} = \frac{1}{8}$

Thus, $E[Y] = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 4 \times \frac{3}{8} + 9 \times \frac{1}{8} = \frac{24}{8} = 3$

d.

6.7.4 a

The answer is: 1

Since there are 10 students, the teacher hands one coat selected at random to each student. Thus, the probability of each student gets his or her own coat is $\frac{1}{10}$.

For each student i, let:

- $X_i = 1$, if student i gets their own coat,
- $X_i = 0$, if student i does not get their own coat,

The total number of students who get their own coat is: $X = X_1 + X_2 + ... + X_{10}$

Using the linearity of expectation, the expected number of students who get their own coat is:

$$E[X] = E[X_1] + E[X_2] + ... + E[X_{10}] = 10 \times \frac{1}{10} = 1$$

Question 10

a.

6.8.1 a

The answer is: $p(exactly\ 2\ defects) = C(100, 2) \times (0.01)^2 \times (0.99)^{98} \approx 0.1849$

Since there are exactly 2 out of 100 have defects, then:
$p(X = 2) = C(100, 2) \times (0.01)^2 \times (0.99)^{98} \approx 0.1849$

6.8.1 b

The answer is:
$p(at\ least\ 2\ defects) = 1 - (0.99)^{100} - C(100, 1) \times (0.01) \times (0.99)^{99} \approx 0.2642$

There are at least 2 out of 100 have defects, means $X \geq 2$.
Using the complement rule: $p(X \geq 2) = 1 - p(X = 0) - p(X = 1)$
Thus,
$p(X \geq 2) = 1 - (0.99)^{100} - C(100, 1) \times (0.01) \times (0.99)^{99} = 1 - (0.99)^{100} - (0.99)^{99} \approx 0.2642$

6.8.1 c

The answer is: $E[defects] = 1$

$E[defects] = n \times p = 100 \times 0.01 = 1$

6.8.1 d

The answer is:

- $p(at\ least\ 2\ defects) = 1 - (0.99)^{50} \approx 0.3950$
- $E[defects] = 1$
- **Comparison: the expected number is the same, but the probability of at least 2 defects is higher in the batch scenario.**

Since the circuit boards are made in batches of 2, then 100 circuit boards means 50 batches. At least 2 have defects, which means at least 1 of 50 batches defects. Thus:

$$p(at\ least\ 2\ defects) = p(Y \geq 1) = 1 - p(Y = 0) = 1 - (0.99)^{50} \approx 0.3950$$

The expected number of defective batches is: $E[Y] = n \times p = 50 \times 0.01 = 0.5$
Since each defective batch have 2 circuit boards, the expected number of defective boards is:

$$E[defects] = 2 \times E[Y] = 2 \times 0.5 = 1$$

b.

6.8.3 b

The answer is: $\sum_{k=4}^{10} C(10, k) \times (0.3)^k \times (0.7)^{10-k} \approx 0.3504$

Since the observer reaches an incorrect conclusion when the coin is biased, this means that the observer guessed fair when the coin is biased, which means that there are at least 4 heads in 10 flips of a biased coin, and $p = 0.3$.
The probability of getting exactly k heads in 10 trials is:

$$p(X = k) = C(10, k) \times (0.3)^k \times (0.7)^{10-k}$$

Thus:

$$\begin{aligned} p(X \geq 4) &= \sum_{k=4}^{10} C(10, k) \times (0.3)^k \times (0.7)^{10-k} \\ &= 1 - p(X = 0) - p(X = 1) - p(X = 2) - p(X = 3) \\ &= 1 - (10, 0) \times (0.3)^0 \times (0.7)^{10} - (10, 1) \times (0.3)^1 \times (0.7)^9 \\ &\quad - (10, 2) \times (0.3)^2 \times (0.7)^8 - (10, 3) \times (0.3)^3 \times (0.7)^7 \\ &\approx 1 - 0.0282 - 0.1211 - 0.2335 - 0.2668 \\ &\approx 0.3504 \end{aligned}$$