RESEARCH STATEMENT

ZENGRUI HAN

Contents

1. Introduction

My research centers around mirror symmetry, a phenomenon originally observed by physicists [CdlOGP91] in the early 1990s, which, roughly speaking, describes a relationship between the algebraic geometry of a space and the symplectic geometry of its mirror space. The first example of mirror symmetry is the observation that one can compute the number of curves on the smooth quintic threefold in $\mathbb{CP}^4$ by looking at an ODE (the Picard-Fuchs equation) coming from a "mirror manifold" of the quintic threefold.

Since then, there have been various mathematical formulations of mirror symmetry. One of the most remarkable formulations is the homological mirror symmetry proposed by Kontsevich in 1994. The statement, roughly speaking, says that the derived category of coherent sheaves on a Calabi-Yau manifold should be equivalent to the Fukaya category of the mirror manifold. Homological mirror symmetry has been the most active area in the study of mirror symmetry.

Apart from attempts to formulate the phenomenon of mirror symmetry in a mathematically rigorous way, another direction in the study of mirror symmetry is to construct explicit examples of mirror pairs. In this direction, the most famous construction is the Batyrev-Borisov mirror construction, which allows one to start with certain combinatorial datum and construct mirror pairs as Calabi-Yau complete intersections in toric varieties. The area of the study of such mirror pairs is called *toric mirror symmetry*.

The first part of my research focuses on the application of homological mirror symmetry to the field of toric mirror symmetry. More precisely, the main object of my study is the so-called *better-behaved GKZ hypergeometric system*, which serves as a generalization of the aforementioned Picard-Fuchs equation for Batyrev-Borisov mirror pairs. One of the major open problems in toric mirror symmetry is to construct an isotrivial family of triangulated categories over the stringy Kähler moduli space that categorifies such systems of PDEs. Jointly with Lev Borisov, I was able to make progress toward this problem. Moreover, I applied my result to the study of Hori-Vafa mirrors (local mirror symmetry), settling a conjecture of Hosono [Hos06], thereby highlighting the importance of GKZ systems.

Since the late 1990s, it has been realized that the phenomenon of mirror symmetry exists for more general spaces, not just Calabi-Yau manifolds (sigma models). A powerful framework to unify different types of models is provided by Witten's *gauged linear sigma model* (GLSM). In the context of mirror symmetry, different phases in the same GLSM share the same mirror family, therefore their mirror-symmetry-theoretic properties should be all equivalent. A pair of spaces with the same mirror family is called *double mirror pair* and has attracted a lot of interest in recent research.

Depending on whether the gauge group is abelian or not, GLSMs can be divided into two classes. The abelian GLSMs are closely related to toric mirror symmetry and are relatively better understood. The nonabelian GLSMs are more difficult to study and new phenomena occur. For example, different geometric phases are not necessarily birationally equivalent to each other.

The second part of my research focuses on a nonabelian GLSM example known as the *Pfaffian double mirrors*. This construction also fits into a conceptual framework developed by Kuznetsov [Kuz07] called *Homological Projective Duality*, which suggests a remarkable relationship between the structures of categorical resolutions on the two sides of the double mirror pair. The Homological Projective Duality for Pfaffian varieties in even-dimensional cases remains an open problem, partially due to the difficulty of finding the correct categorical crepant resolution along with a Lefschetz decomposition. In [Han24b], my results on the stringy Hodge numbers of Pfaffian double mirrors allow me to make conjectural predictions regarding the possible forms of the Lefschetz decompositions for such categorical resolutions.

In the next few sections, I will explain my research in detail.

2. GKZ systems and toric mirror symmetry

2.1. **Background and motivation.** Let Δ be a d-dimensional lattice polytope in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^d$, and let $C = \mathbb{R}_{\geq 0}(\Delta \oplus 1)$ be the $(d+1)$-dimensional cone over Δ. From this combinatorial data we can construct a Gorenstein toric variety $X = \operatorname{Spec} \mathbb{C}[C^\vee \cap N]$. Furthermore, a triangulation Σ of the polytope Δ gives rise to a smooth toric Deligne-Mumford stack $\mathbb{P}_\Sigma$ that is a crepant resolution of X. According to the prediction of string theory, different crepant resolutions should have equivalent physics theories. In mathematical terms, this means that all the toric stacks $\mathbb{P}_\Sigma$ have equivalent derived categories.

A crucial observation is that there is no canonical equivalence between derived categories of different crepant resolutions. This fact suggests that instead of a finite number of equivalences of derived categories, there should exist a *continuous family* of triangulated categories over a certain moduli space, the *stringy Kähler moduli space*, in which there are *large radius limit points* that are in 1-1 correspondence with the crepant resolutions $\mathbb{P}_\Sigma$. The fibers in a neighborhood of the large radius limit point corresponding to $\mathbb{P}_\Sigma$ are given by $D^b(\mathbb{P}_\Sigma)$, and the choice of a derived equivalence between two crepant resolutions is realized by the choice of a path connecting the corresponding large radius limit points.

The construction of such an isotrivial family of triangulated categories is an open problem. However, its decategorification is a genuine local system over the moduli space that can be described as a system of linear PDEs, known as the *GKZ hypergeometric system.*

Definition 2.1 (better-behaved GKZ systems, [BH13] and [Iri11])**.** For each lattice point c in the cone C we attach a holomorphic function $\Phi_c(x_1, \ldots, x_n)$ defined on the stringy Kähler moduli space, and consider a linear system of PDEs:

$$\text{bbGKZ}(C): \begin{cases} \partial_i \Phi_c = \Phi_{c+v_i}, \quad \forall c \in C, i = 1, \cdots, n \\ \sum_{i=1}^n \langle \mu, v_i \rangle x_i \partial_i \Phi_c + \langle \mu, c \rangle \Phi_c = 0, \quad \forall c \in C, \mu \in N^\vee \end{cases}$$

A compactly-supported version bbGKZ(C°) could be defined similarly by considering lattice points in the interior C° only.

The key property of these systems is that their solution spaces are finite-dimensional and can be canonically identified, via certain hypergeometric series, with the Grothendieck groups of the derived categories $D^b(\mathbb{P}_\Sigma)$ and $D^b_c(\mathbb{P}_\Sigma)$ in a neighborhood of the large radius limit point corresponding to Σ.

2.2. **A pair of conjectures of Borisov and Horja.** In [BH15], Borisov and Horja proposed two conjectures on bbGKZ systems: the *duality conjecture* and the *analytic continuation conjecture*. I was able to settle both of them in full generality.

2.2.1. *Duality of GKZ systems.* As explained in the previous subsection, the solution spaces of GKZ systems are naturally identified with $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$, near any large radius limit point. There is a natural Euler pairing between $K_0(\mathbb{P}_\Sigma)$ and $K_0^c(\mathbb{P}_\Sigma)$

$$\chi_\Sigma : K_0(\mathbb{P}_\Sigma) \times K_0^c(\mathbb{P}_\Sigma) \longrightarrow \mathbb{Z}, \quad (\mathcal{F}^\bullet, \mathcal{G}^\bullet) \mapsto \sum_{i\geq 0} (-1)^i \dim \operatorname{Ext}^i(\mathcal{G}^\bullet, \mathcal{F}^\bullet)$$

This pairing is only **locally** defined near the large radius limit points. Motivated by the conjectural existence of the isotrivial family of triangulated categories, Borisov and Horja [BH15] conjectured that there exists a **globally** defined pairing, in terms of the solutions to bbGKZ systems, that recovers the Euler pairing near any large radius limit point. In joint work with Lev Borisov, I was able to find an explicit formula for such a pairing.

Theorem 2.2 ([BH24]). Let Φ and Ψ be solutions to bbGKZ(C) and bbGKZ(C°) respectively. The **GKZ pairing** defined by

$$\langle \Phi, \Psi \rangle_{\mathrm{GKZ}} = \sum_{\substack{c\in C, d\in C^\circ \\ I\subseteq\{1,\cdots,n\}, |I|=\mathrm{rk} N}} \xi_{c,d,I} \operatorname{Vol}_I \left(\prod_{i\in I} x_i \right) \Phi_c \Psi_d$$

where the coefficients $\xi_{c,d,I} = 0, \pm 1$ is determined by the combinatorics of C, and Vol_I denotes the volume of the cone generated by I. Then $\langle \Phi, \Psi \rangle_{\mathrm{GKZ}}$ is a constant for any solutions Φ and Ψ. Furthermore, $\langle -, - \rangle_{\mathrm{GKZ}}$ agrees with the Euler pairing $\chi(-,-)$, in the neighborhood of any large volume limit Σ.

This result can be viewed as the B-model interpretation of Iritani's Gamma integral structure [Iri09], within the context of local mirror symmmetry.

2.2.2. *"Analytic Continuation = Fourier-Mukai".* The second conjecture of Borisov-Horja [BH15] concerns the relationships between solutions near different large radius limit points and their K-theoretic counterparts. Let Σ_1 and Σ_2 be two different triangulations of the cone C, then solutions to bbGKZ near one of them can be analytically continued to the neighborhood of the other. This yields a natural map $\mathrm{Sol}_{\Sigma_1} \to \mathrm{Sol}_{\Sigma_2}$ between solution spaces. Borisov and Horja conjectured that under the identification of the solution spaces Sol_Σ with the K-theory $K_0(\mathbb{P}_\Sigma)$, the corresponding maps between K-theories can be realized as a Fourier-Mukai transform. My second result is as follows.

Theorem 2.3 ([Han23]). For any two adjacent triangulations Σ_1 and Σ_2, the operation of analytic continuation from the neighborhood of Σ_1 to the neighborhood of Σ_2 is realized by a natural Fourier-Mukai transform $K_0(\mathbb{P}_{\Sigma_1}) \to K_0(\mathbb{P}_{\Sigma_2})$. Similar result also holds for the dual system bbGKZ(C°).

2.3. **Hori-Vafa mirrors and Hosono's conjecture.** In this subsection, I will describe my work on the applications of GKZ systems to Hori-Vafa mirrors. The B-models of Hori-Vafa mirrors are toric Calabi-Yau orbifolds $\mathbb{P}_\Sigma$ introduced in previous sections, while the A-models are Landau-Ginzburg models $((\mathbb{C}^*)^d, f)$, where $f : (\mathbb{C}^*)^d \to \mathbb{C}$ is a Laurent polynomial with fixed Newton polytope.

The homological mirror symmetry conjecture for such mirror pairs predicts an equivalence between the derived category of $\mathbb{P}_\Sigma$ and the Fukaya-Seidel category of the LG model $((\mathbb{C}^*)^d, f)$. While the equivalence of these categories remains an open problem, the GKZ systems provide tools to relate the *central charges* on both sides, providing evidence for the homological mirror symmetry of Hori-Vafa mirrors.

The notion of central charge appears in physics literature and has played an important role in Douglas's Π-stability [Dou02] and its mathematical formalization [Bri07] by Bridgeland. In the context of Hori-Vafa mirrors, I generalized the concepts of A-brane and B-brane central charges from 3-dimensional cases considered by Hosono [Hos06] to arbitrary dimensions, interpreting them as period integrals and hypergeometric series respectively. By combining the tropical geometry methods of Abouzaid-Ganatra-Iritani-Sheridan [AGIS20] with the hypergeometric duality [BH24], I was able to prove the equality of the A-brane and B-brane central charges, thereby establishing (a generalization of) Hosono's conjecture [Hos06, Conjecture 2.4].

Theorem 2.4 ([Han24a]). The A-brane and B-brane central charges are identified under the (conjectural) homological mirror symmetry equivalence $FS((\mathbb{C}^*)^d, f) \xrightarrow{\sim} D^b(\mathbb{P}_\Sigma)$.

3. Pfaffian double mirrors and Homological Projective Duality

Let V be a n-dimensional complex vector space. The Pfaffian variety $\mathrm{Pf}(2k, V)$ is defined as the subvariety of $\mathbb{P}(\wedge^2 V^\vee)$ consisting of skew forms on V whose rank does not exceed $2k$. We consider the pair of projectively dual Pfaffian varieties $\mathrm{Pf}(2k, V^\vee)$ and $\mathrm{Pf}(2\lfloor n/2 \rfloor - 2k, V)$, and define a pair of linear sections X_W and Y_W in these varieties associated to a generic subspace $W \subseteq \wedge^2 V^\vee$ of dimension l as follows.

- X_W is the intersection of $\mathbb{P}W^\perp$ and the Pfaffian variety $\mathrm{Pf}(2k, V^\vee)$ in $\mathbb{P}(\wedge^2 V)$.
- Y_W is the intersection of $\mathbb{P}W$ and the Pfaffian variety $\mathrm{Pf}(2\lfloor n/2 \rfloor - 2k, V)$ in $\mathbb{P}(\wedge^2 V^\vee)$.

This construction has been studied by Borisov-Căldăraru [BC09], Hori-Tong [HT07], Borisov-Libgober [BL19] and many others. As explained in section 1, it fits nicely into a conceptual framework known as the *Homological Projective Duality* (HPD for short) developed by Kuznetsov [Kuz07], which suggests a remarkable relationship between the structures of the categorical crepant resolutions of X_W and Y_W.

Establishing HPD for Pfaffian varieties in even-dimensional cases remains an open problem. One of the main difficulties lies in constructing categorical resolutions of Pfaffians and their Lefschetz decompositions.

My first result in this project establishes a relationship between the *stringy E-functions* of X_W and Y_W, extending previous work of Borisov-Libgober [BL19]. The stringy E-function $E_{\mathrm{st}}(X; u, v)$ of a singular variety X, introduced by Batyrev [Bat98], is a two-variable function that encodes information about the Hodge numbers of the variety. It serves as a replacement of the usual Hodge polynomial in the context of mirror symmetry. In the case of even-dimensional Pfaffian varieties, certain subtleties arise because their stringy E-functions are *not* polynomials. To address this issue, I introduced a modified version of E-function by modifying discrepancies of log resolutions used in the definition of the original stringy E-function.

Theorem 3.1 ([Han24b]). Let X_W and Y_W be as defined above. We have the following relation between their E-functions:

$$q^{2k\lfloor \frac{n}{2} \rfloor} E_{\mathrm{st}}(Y_W) - q^l E_{\mathrm{st}}(X_W) = \frac{q^l - q^{2k\lfloor \frac{n}{2} \rfloor}}{q-1} \binom{\lfloor \frac{n}{2} \rfloor}{k}_{q^2}$$

where $q = uv$, and E_{st} is the stringy E-function of Batyrev when n is odd, and the modified E-function when n is even.

The second part of my research focused on the relationship between the classical aspect (Hodge numbers) and the homological aspect (derived categories) of the study of Pfaffian double mirrors. More precisely, by applying my main result Theorem 3.1, I was able to make numerical predictions regarding the Lefschetz decomposition of the categorical crepant resolutions of the Pfaffian varieties in the even-dimensional cases.

Conjecture 3.2 ([Han24b]). Let V be a even-dimensional complex vector space. The categorical crepant resolution $\widetilde{D}^b(\mathrm{Pf}(2k, V^\vee))$ of the Pffafian variety $\mathrm{Pf}(2k, V^\vee)$ has a non-rectangular Lefschetz decomposition of the form

$$\widetilde{D}^b(\mathrm{Pf}(2k, V^\vee)) = \Big\langle \mathcal{A}_0, \mathcal{A}_1(1), \cdots, \mathcal{A}_{nk-\frac{n}{2}-1}(nk - \frac{n}{2} - 1), \\ \mathcal{A}_{nk-\frac{n}{2}}(nk - \frac{n}{2}), \cdots, \mathcal{A}_{nk-1}(nk-1) \Big\rangle$$

where $\mathcal{A}_0 = \cdots = \mathcal{A}_{nk-\frac{n}{2}-1}$ are $nk - \frac{n}{2}$ blocks of size $\binom{n/2}{k}$, and $\mathcal{A}_{nk-\frac{n}{2}} = \cdots = \mathcal{A}_{nk-1}$ are $n/2$ blocks of size $\binom{n/2-1}{k}$. Similarly, the dual Lefschetz decomposition of the dual $\mathrm{Pf}(n-2k, V)$ has the form

$$\widetilde{D}^b(\mathrm{Pf}(n-2k, V)) = \Big\langle \mathcal{B}_{\frac{n^2}{2}-nk-1}(-\frac{n^2}{2} + nk + 1), \cdots \mathcal{B}_{\frac{n(n-1)}{2}-nk}(-\frac{n(n-1)}{2} + nk), \\ \mathcal{B}_{\frac{n(n-1)}{2}-nk-1}(-\frac{n(n-1)}{2} + nk + 1), \cdots, \mathcal{B}_1(-1), \mathcal{B}_0 \Big\rangle$$

where $\mathcal{B}_0 = \cdots = \mathcal{B}_{\frac{n(n-1)}{2}-nk-1}$ are $\frac{n(n-1)}{2} - nk$ blocks of size $\binom{n/2}{n/2-k}$, and $\mathcal{B}_{\frac{n(n-1)}{2}-nk} = \cdots = \mathcal{B}_{\frac{n^2}{2}-nk-1}$ are $n/2$ blocks of size $\binom{n/2-1}{n/2-k}$.

4. Further directions

In the future I plan to pursue a number of directions, which are directly or indirectly related to my work described above. Below I have outlined some of them.

4.1. **Categorification of GKZ hypergeometric systems.** As explained in section 2, the construction of the isotrivial family of triangulated categories is a major open problem in toric mirror symmetry and has numerous applications across different aspects of this field. For example, solving this problem could lead to the construction of non-commutative crepant resolution of toric Gorenstein singularities in arbitrary dimensions. Jointly with Lev Borisov, I am currently exploring various possible approaches to this problem.

4.2. **Homological projective duality for Pfaffian varieties.** My results on the stringy E-function of Pfaffian varieties and their linear sections indicate a conjectural structure for the Lefschetz decomposition of categorical resolutions of Pfaffian varieties. I am currently exploring the feasibility of constructing the categorical resolution as an admissible subcategory of the derived category of certain log resolution $\widehat{\mathrm{Pf}(2k, V)} \to \mathrm{Pf}(2k, V)$, generated by exceptional collections of the length predicted by the conjecture.

4.3. **Non-commutative spaces in mirror symmetry.** A relatively recent discovery in mirror symmetry is that certain "non-commutative algebraic varieties" can appear as phases of GLSMs. The simplest example is the double mirror pair consisting of a $(2, 2, 2, 2)$-complete intersection in $\mathbb{P}^7$ and a noncommutative variety $(\mathbb{P}^3, \mathcal{B}_0)$, where $\mathcal{B}_0$ is the even part of a Clifford algebra. This double mirror pair fits within the framework of homological projective duality and can be generalized to higher dimensions. In the future, I plan to study various mirror-symmetric aspects of such spaces, including enumerative invariants such as Gromov-Witten invariants.

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Department of Mathematics, Rutgers University, Piscataway, NJ 08854
Email address: `zh223@math.rutgers.edu`