Better-behaved GKZ systems and toric mirror symmetry

Zengrui Han
(Joint work with Lev Borisov)

Rutgers, the State University of New Jersey

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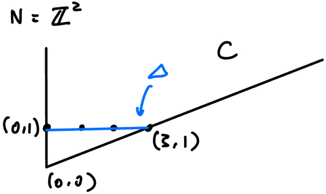
Combinatorial set up

- $N = \mathbb{Z}^d$ is a lattice, C is a polyhedral cone whose ray generators are all of height 1.



- $\mathrm{Spec}(\mathbb{C}[C^\vee \cap N^\vee])$ is a (singular) affine toric Gorenstein variety.
- A regular triangulation Σ of C gives a crepant resolution $\mathbb{P}_\Sigma \to \mathrm{Spec}(\mathbb{C}[C^\vee \cap N^\vee])$, where $\mathbb{P}_\Sigma$ is a (non-compact) Calabi-Yau toric orbifold.

Definition of better-behaved GKZ systems

For any lattice point $c \in C \cap N$ we attach a holomorphic function $\Phi_c(x_1, \cdots, x_n)$, each variable corresponds to a lattice point (of height 1).

Definition (Borisov-Horja, 2010)

The **better-behaved GKZ system** $bbGKZ(C, 0)$ is given by

$$\frac{\partial}{\partial x_i}\Phi_c = \Phi_{c+v_i}, \text{ for any } i, c \in C$$

and

$$\sum_{i=1}^{n} \mu(v_i) x_i \frac{\partial}{\partial x_i}\Phi_c + \mu(c)\Phi_c = 0, \text{ for any } \mu \in N^{\vee}, c \in C$$

Similarly we can define $bbGKZ(C^{\circ}, 0)$ by considering $c \in C^{\circ}$ only.

Gamma series solutions

- We can construct series solutions over certain region of convergence from the K-theory of $\mathbb{P}_\Sigma$. (Some kind of "central charge")

$$K_0(\mathbb{P}_\Sigma) \otimes \mathbb{C} \xrightarrow{\sim} \{\text{Solutions to } bbGKZ(C^\circ)\}$$

$$K_0^c(\mathbb{P}_\Sigma) \otimes \mathbb{C} \xrightarrow{\sim} \{\text{Solutions to } bbGKZ(C)\}$$

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- "A-model pairing":

$$\chi : K_0(\mathbb{P}_\Sigma) \times K_0^c(\mathbb{P}_\Sigma) \to \mathbb{Z},\ (\mathcal{F}, \mathcal{G}) \mapsto \sum_{i=0}^{\infty} (-1)^i \dim \mathrm{Ext}^i(\mathcal{F}, \mathcal{G})$$

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- Question: What is the pairing on the side of solutions ("B-model pairing")?

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- We find an explicit combinatorial formula for the pairing on the B-side.

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- Fix a generic $v \in C^\circ$, define the pairing between solution spaces of $bbGKZ(C,0)$ and $bbGKZ(C^\circ,0)$ by the following formula:

$$\langle \Phi, \Psi \rangle_{GKZ} := \sum_{\substack{c \in C, d \in C^\circ \\ I \subseteq \{1,\cdots,n\}, |I| = \mathrm{rk}\, N}} \xi_{c,d,I} \mathrm{Vol}_I \left(\prod_{i \in I} x_i \right) \Phi_c \Psi_d$$

where

$$\zeta_{c,d,I} = (-1)^{\deg c}, \text{ if } \dim \sigma(I) = \mathrm{rk} N, c + \epsilon v, d - \epsilon v \in \sigma(I)^\circ$$
otherwise 0.

Main theorem

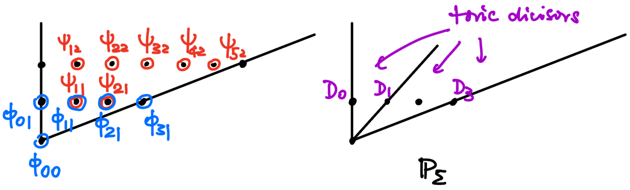
Theorem (Borisov-H., 2023)

- *The B-model pairing* $\langle \Phi, \Psi \rangle_{GKZ}$ *provides a non-degenerate pairing*

$$bbGKZ(C, 0) \times bbGKZ(C^\circ, 0) \to \mathbb{C}.$$

- *For any regular triangulation* Σ*, the B-model pairing* $\langle \Phi, \Psi \rangle_{GKZ}$ *agrees with the A-model pairing* $\chi : K_0(\mathbb{P}_\Sigma) \times K_0^c(\mathbb{P}_\Sigma) \to \mathbb{C}$.

Example: $[\mathbb{C}^2/\mathbb{Z}_3]$



- In this case the formula is

$$\begin{aligned}\langle \Phi, \Psi \rangle_{GKZ} &= \Phi_{00}(x_0 x_1 \Psi_{12} + 2 x_0 x_2 \Psi_{22} + 3 x_0 x_3 \Psi_{32}) \\ &\quad - \Phi_{11}(2 x_0 x_2 \Psi_{11} + 3 x_0 x_3 \Psi_{21}) - \Phi_{21}(3 x_0 x_3 \Psi_{11})\end{aligned}$$

- Consider e.g. $\mathcal{O}(D_3) \in K_0(\mathbb{P}_\Sigma)$ and $\mathcal{O}_{D_1}(D_3) \in K_0^c(\mathbb{P}_\Sigma)$, they give solutions $\Psi = (\Psi_d)_{d \in C^\circ}$ and $\Phi = (\Phi_c)_{c \in C}$ (rather transcendental).
- Plugging them into the formula above yields

$$\langle \Phi, \Psi \rangle_{GKZ} = \chi(\mathcal{O}(D_3), \mathcal{O}_{D_1}(D_3)) = 1$$

Thank you!