

CM 2110 Calculus and Statistical Distributions

Dr. Priyanga D. Talagala

2020-05-29

Contents

Course Syllabus	5
Pre-requisites	5
Learning Outcomes	5
Outline Syllabus	5
Method of Assessment	6
Recommended Texts	6
Lecturer	6
Schedule	6
 1 Statistical Distributions	 1
Recap: CM 1110-Probability	1
1.1 Random Variable	3
1.2 Probability Mass Function	7
1.3 Probability Density Function	11
1.4 Cumulative Distribution Function	15
1.5 Descriptive properties of distributions	19
1.6 Models for discrete distributions	19
1.7 Models for continuous distributions	19
 2 Estimations	 21
2.1 Point Estimation	21
2.2 Interval Estimation	21

3 Hypothesis Testing	23
3.1 Null and alternative hypotheses	23
3.2 Errors in testing hypotheses-type I and type II error	23
3.3 Significance level, size, power of a test	23
3.4 Formulation of hypotheses	23
3.5 Methods of testing hypotheses	23
4 Design of Experiments	25
4.1 Introduction to experimental design	25
4.2 Basic principles of experimental design	25
4.3 Completely randomized design	25

Course Syllabus

Pre-requisites

CM 1110

Remark:

This course module contains two main sections: (1) mathematics and (2) statistics. This syllabus is designed for the statistics section. Lectures for mathematics section and statistics section are conducted by two lecturers as two separate sub modules (1.5 hour lectures/Week). End Semester Examination is conducted as a single examination.

Learning Outcomes

On successful completion of this module, students will be able to plan more carefully the design of experiment in advance which provide evidence for or against theories of cause and effect and make inferences about population characteristics based on sample information and thereby solve data analysis problems in different application domains. (R(<https://cran.r-project.org/>) and RStudio are also freely available to install on your own computer). Get the Open Source Edition of RStudio Desktop. RStudio allows you to run R in a more user-friendly environment.

Outline Syllabus

- Functions of Several Variables
- Linear Algebra
- Coordinate Systems & Vectors
- Differential Equations
- **Statistical Distributions**
- **Estimation**

- Hypothesis Testing
- Design of Experiments

Method of Assessment

- Mid-semester examination
- End-semester examination

Recommended Texts

- Casella, G., & Berger, R. L. (2002). Statistical inference (Vol. 2, pp. 337-472). Pacific Grove, CA: Duxbury.
- Mood, A.M., Graybill, F.A. and Boes, D.C. (2007): Introduction to the Theory of Statistics, 3rd Edn. (Reprint). Tata McGraw-Hill Pub. Co. Ltd.
- Montgomery, D. C. (2017). Design and analysis of experiments. John Wiley & sons.

Lecturer

Dr. Priyanga D. Talagala

Schedule

Lectures:

- Friday [9.15 am - 10.45 am]

Tutorial:

- Friday [11.00 am - 12.30 pm]

Consultation time:

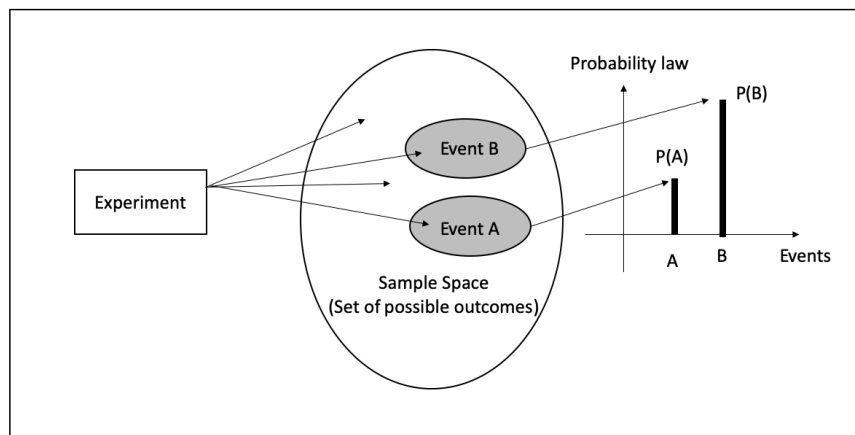
- Friday [8.15 am to 9.00 am]

Chapter 1

Statistical Distributions

Recap: CM 1110-Probability

Axioms of probability



- **Probability** of an event quantifies the **uncertainty**, randomness, or the possibility of occurrence the event.
- The probability of event E is usually denoted by $P(E)$.
- Mathematically, the function $P(\cdot)$ is a set function defined from sample space (Ω) to $[0, 1]$ interval, satisfying the following properties.
- These are called the ‘**axioms of probability**’.

- **Axiom 1:** For any event A , $P(A) \geq 0$
- **Axiom 2:** $P(\Omega) = 1$
- **Axiom 3:**
 - (a) If A_1, A_2, \dots, A_k is a finite collection of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \dots \cup A_k) = \sum_{i=1}^k P(A_i)$$

- (b) If A_1, A_2, \dots is an infinite collection of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup \dots) = \sum_{i=1}^{\infty} P(A_i)$$

NOTE

- Axioms 1 and 2 imply that for any event E , $0 \leq P(E) \leq 1$.
- $P(E) = 1 \iff$ the event E is certain to occur.
- $P(E) = 0 \iff$ the event E cannot occur.

Methods for determining Probability

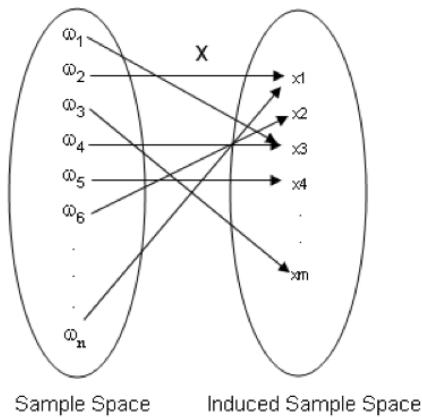
- There are several ways for determining the probability of events.
- Usually we use the following methods to obtain the probability of events.
 - Classical method
 - Relative frequency method (Empirical approach)
 - Subjective method
 - **Using probability models**

1.1 Random Variable

- Some sample spaces contain quantitative (numerical) outcomes, others contain qualitative outcomes.
- Often it is convenient to work with sample spaces containing numerical outcomes.
- A function that maps the original sample space into the real numbers is called a ‘random variable’.
- This is more useful when the original sample space contains qualitative outcomes.

Definition 1: Random Variable

Let Ω be a sample space. Let X be a function from Ω to \Re (*i.e.* $X : \Omega \rightarrow \Re$). Then X is called a random variable.



- A random variable assigns a real number to each outcome of a sample space.
- In other words, to each outcome of an experiment or a sample point ω_i , of the sample spaces, there is a unique real number x_i , known as the value of the random variable X .
- The range of the random variable is called the *induced sample space*.
- *A note on notation:* Random variables will always denoted with uppercase letters and the realized values of the random variable (or its range) will be denoted by the corresponding lowercase letters. Thus, the random variable X can take the value x .
- Each outcome of a sample space occurs with a certain probability. Therefore, each possible value of a random variable is associated with a probability.
- Any events of a sample space can be written in terms of a suitably defined random variable.

1.1.1 Types of Random Variables

- A random variable is of two types
 - Discrete Random Variable
 - Continuous Random Variable

1.1.1.1 Discrete Random Variable

- If the induced sample space is discrete, then the random variable is called a **discrete random variable**.

Example 01 Consider the experiment of tossing a coin. Express the following events using a suitably defined random variable

$H =$ The event of getting a head

$T =$ The event of getting a tail

Example 02

Consider the experiment of rolling of a die. Express the following events using a suitably defined random variable

$A =$ *The event that the number faced up is less than 5*

$B =$ *The event that the number faced up is even*

$C =$ *The event that the number faced up is 2 or 5*

Example 03

Consider the experiment of tossing a coin 10 times. Then the sample space Ω contains $2^{10} = 1024$ outcomes. Each outcome is a sequence of 10 H's and T's.

Express the following events in terms of a suitably defined random variable.

$D =$ The event that the number of heads is 5

$E =$ The event that the number of tails is less than 4

1.1.1.2 Continuous Random Variable

- If the induced sample space is continuous, then the random variable is called a **continuous random variable**.

Example 04

Consider the experiment of measuring the lifetime (in hours) of a randomly selected bulb. Express the following events in terms of a suitably defined random variable.

F = The event that the lifetime is less than 300 hours

G = The event that the lifetime is 1000 hours

1.2 Probability Mass Function

Definition 2: Discrete density function of a discrete random variable

If X is a discrete random variable with distinct values $x_1, x_2, \dots, x_n, \dots$, then the function, denoted by $f_X(\cdot)$ and defined by

$$f_X(x) = \begin{cases} P(X = x) & \text{if } x = x_j, j = 1, 2, \dots, n, \dots \\ 0 & \text{if } x \neq x_j \end{cases} \quad (1.1)$$

is defined to be the discrete density function of X .

- The values of a discrete random variable are often called *mass points*.
- $f_X(x)$ denotes the *mass* associated with the *mass point* x_j .
- **Probability mass function** *discrete frequency function* and *probability function* are other terms used in place of *discrete density function*
- Probability function gives the measure of probability for different values of X .

1.2.1 Properties of a Probability Mass Function

- Let X be a discrete random variable with probability mass function $f_X(x)$. Then,

1. For any $x \in \mathfrak{R}$, $0 \leq f_X(x) \leq 1$.
2. Let E be an event and $I = \{X(\omega) : \omega \in E\}$. Then $P(E) = P(X \in I) = \sum_{x \in I} f_X(x)$.
3. Let $R = \{X(\omega) : \omega \in \Omega\}$. Then $\sum_{x \in \mathfrak{R}} f_X(x) = 1$.

1.2.2 Representations of Probability Mass Functions

Example 05

Consider the experiment of tossing a fair coin. Let

$$X = \begin{cases} 0 & \text{if the outcome is a Tail} \\ 1 & \text{if the outcome is a Head} \end{cases} \quad (1.2)$$

Find the probability mass function of X . Is X discrete or continuous?

1.2.2.1 Using a table

1.2. PROBABILITY MASS FUNCTION. STATISTICAL DISTRIBUTIONS

1.2.2.2 Using a function

1.2.2.3 Using a graph

1.3 Probability Density Function

- Let X be a continuous random variable.
- Then, it is not possible to define a pmf f_x with properties mentioned in Section 1.2. **Why?**
- Instead, we can find a function f_x with the some different properties.
- Probability density function (pdf) of a continuous random variable is a function that describes the relative likelihood for this random variable to occur at a given point.

1.3.1 Properties of a Probability Density Function

Let X be a continuous random variable with probability density function f_x . Then,

1. For any $x \in \mathfrak{R}$, $f_X(x) \geq 0$.
2. Let E be an event and $I = \{X(\omega) : \omega \in E\}$. Then $P(E) = P(X \in I) = \int_I f_X(x)dx$.
3. Let $R = \{X(\omega) : \omega \in \Omega\}$. Then $\int_{\mathfrak{R}} f_X(x)dx = 1$.

1.3.2 Existence of pdf

- To see the existence of such a function, consider a continuous random variable X ,
- Suppose that we have a very large number of observations, N , of X , measured to high accuracy (large number of decimal places).
- consider the following grouped frequency table and the histogram constructed from those data.
- The height of the bar on a class interval of this histogram is equal to the relative frequency per unit in that class interval.

Interval	Class boundaries	Class frequency	Height of the bar	Area of the bar
I_1	$x_1 - \delta x/2, x_1 + \delta x/2$	n_1	$\frac{n_1}{\delta x * N}$	$\frac{n_1}{N}$
I_2	$x_2 - \delta x/2, x_2 + \delta x/2$	n_2	$\frac{n_2}{\delta x * N}$	$\frac{n_2}{N}$
\vdots	\vdots	\vdots	\vdots	\vdots
I_k	$x_k - \delta x/2, x_k + \delta x/2$	n_k	$\frac{n_k}{\delta x * N}$	$\frac{n_k}{N}$
Total				

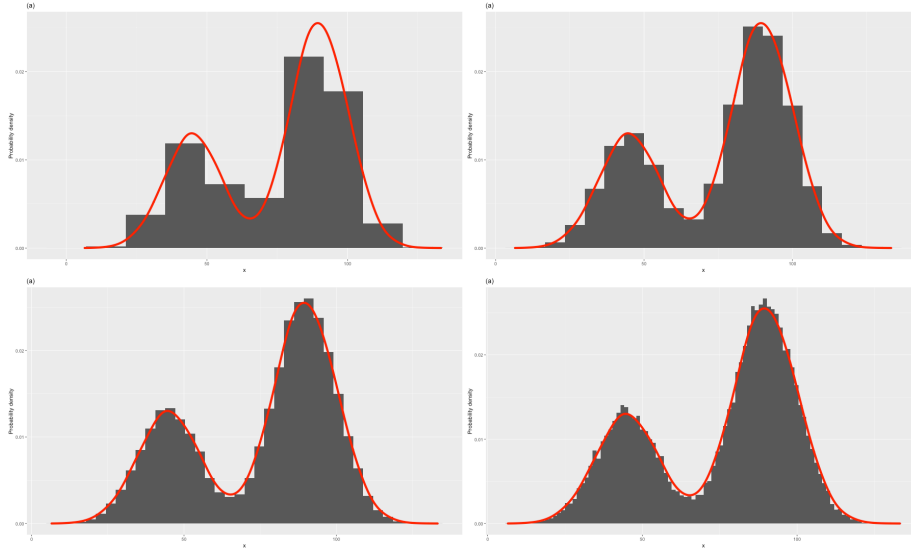


Figure 1.1: Histograms with different class intervals and a possible model for the pdf

- Then, for the i^{th} interval,

$$P(x_i - \frac{\delta x}{2} \leq X \leq x_i + \frac{\delta x}{2}) \approx \text{Area of the bar}$$

and therefore

$$\text{Height of the bar} \approx \frac{\text{Area of the bar}}{\delta x} \approx \frac{P(x_i - \frac{\delta x}{2} \leq X \leq x_i + \frac{\delta x}{2})}{\delta x}$$

- Therefore, the height of a bar represents the *probability density* in that class interval.
- When $\delta x \rightarrow 0$, it will allow us to approximate the histogram by a smooth curve as in Figure 1.1 (d).
- As the area under each histogram is 1, the area under the curve is also 1
- For any point x ,

$$\text{The height of the curve} \approx \lim_{\delta x \rightarrow 0} \frac{P(x_i - \frac{\delta x}{2} \leq X \leq x_i + \frac{\delta x}{2})}{\delta x}$$

will represent the **the probability density at point x** .

1.3. PROBABILITY DENSITY FUNCTION STATISTICAL DISTRIBUTIONS

- Let the above smooth curve be denoted by f_X .
- Then, f_X has the properties mentioned in Section 1.3.1.
- The function is called the **probability density function of X** .
- **NOTE** Here $f_X(x)$ represents **Probability density at point x** . Not *Probability at point x* .

1.3.3 Calculation of Probability using pdf

- Let $c, d \in \Re$ such that $c \leq d$. Then,

$$P(c \leq X \leq d) = \int_c^d f_X(x) dx$$

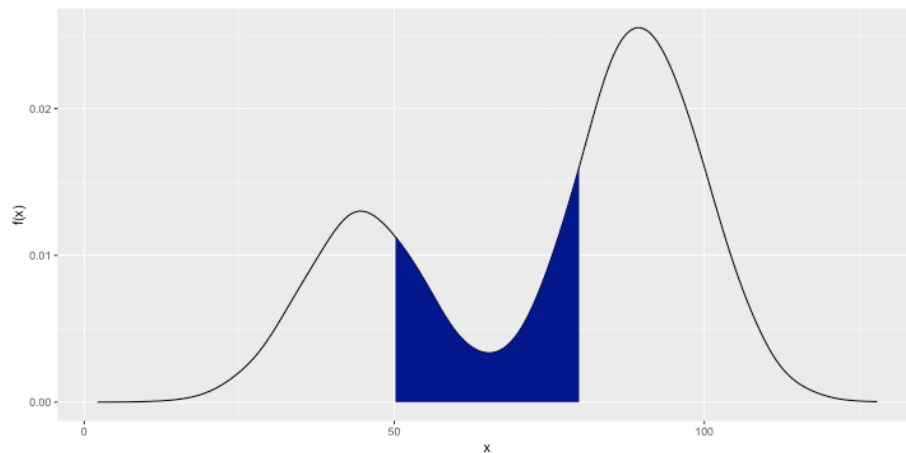


Figure 1.2: Histograms with different class intervals and a possible model for the pdf

- **NOTE:** if X is a continuous random variable with the p.d.f f_X , then for any $k \in \Re$,

$$P(X = k) = P(k \leq X \leq k) = \int_k^k f_X(x) dx = 0$$

- Therefore, for a continuous random variable X ,

$$P(c < X < d) = P(c \leq X < d) = P(c < X \leq d) = P(c \leq X \leq d) = \int_c^d f_X(x) dx$$

1.4 Cumulative Distribution Function

- There are many problems in which it is of interest to know the probability that the values of a random variable is less than or equal to some real number x .

Definition 3: Cumulative distribution function

The *cumulative distribution function* or *cdf* of a random variable X , denoted by $F_X(x)$, is defined by

$$F_x(x) = P(X \leq x), \text{ for all } x$$

- Therefore, if X is a discrete random variable, the cdf is given by,

$$F_X(x) = \sum_{t \leq x} f_X(t), \quad -\infty < x < \infty$$

where $f_X(t)$ is the value of the pmf of X at t .

1.4.1 Relationship between cdf and pdf

- If X is a continuous random variable, the cdf is given by,

$$F_X(x) = \int_{-\infty}^x f_X(t) dt \quad -\infty < x < \infty$$

where $f_X(t)$ is the value of the pdf of X at t . (Here t is a dummy integration variable).

- Conversely,

$$f_X(x) = \frac{dF_X(x)}{dx}$$

Example 06

An owner of a software engineering company is interested in knowing how many years his employees stay with his company. Let X be the number of years an employee will stay with the company. Over the years, he has established the following probability distribution:

1.4. CUMULATIVE DISTRIBUTION FUNCTION

x	1	2	3	4	5	6	7
$f_X(x) = P(X = x)$	0.1	0.05	0.1	?	0.3	0.2	0.1

1. Find $f_X(4)$

2. Find $P(X < 4)$

3. Find $P(X \leq 4)$

4. Draw the probability mass function of X

5. Draw the cumulative distribution function of X

1.4.2 Properties of a cumulative distribution function of a Discrete random variable

1.5 Descriptive properties of distributions

1.6 Models for discrete distributions

1.7 Models for continuous distributions

Chapter 2

Estimations

2.1 Point Estimation

2.1.1 Methods of finding point estimators

2.1.2 Methods of evaluating point estimators

2.2 Interval Estimation

2.2.1 Interpretation of confidence intervals

2.2.2 Methods of finding interval estimators

2.2.3 Methods of evaluating interval estimators

Chapter 3

Hypothesis Testing

- 3.1 Null and alternative hypotheses
- 3.2 Errors in testing hypotheses-type I and type II error
- 3.3 Significance level, size, power of a test
- 3.4 Formulation of hypotheses
- 3.5 Methods of testing hypotheses

3.5. METHODS OF TESTING HYPOTHESES3. HYPOTHESIS TESTING

Chapter 4

Design of Experiments

4.1 Introduction to experimental design

4.2 Basic principles of experimental design

4.3 Completely randomized design