No need multiple-integrals, no need Feynman's trick

\begin{align}J&=\int\_0^1 \frac{\ln(1+x-x^2)}{x}dx\overset{\text{IBP}}=-\int\_0^1 \frac{(1-2x)\ln x}{1+x(1-x)}dx\\

&=-\int\_0^{\frac{1}{2}} \frac{(1-2x)\ln x}{1+x(1-x)}dx-\int\_{\frac{1}{2}}^1 \frac{(1-2x)\ln x}{1+x(1-x)}dx\\

&\overset{u=x(1-x)}=-\int\_0^{\frac{1}{4}}\frac{\ln\left(\frac{1-\sqrt{1-4u}}{2}\right)}{1+u}du+\int\_0^{\frac{1}{4}}\frac{\ln\left(\frac{1+\sqrt{1-4u}}{2}\right)}{1+u}du\\

&=-\int\_0^{\frac{1}{4}}\frac{\ln\left(\frac{1-\sqrt{1-4u}}{1+\sqrt{1-4u}}\right)}{1+u}du\overset{z=\sqrt{1-4u}}=2\int\_0^1 \frac{z\ln\left(\frac{1-z}{1+z}\right)}{z^2-5}dz\\

&=\int\_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{z-\sqrt{5}}dz+\int\_0^1 \frac{\ln\left(\frac{1-z}{1+z}\right)}{z+\sqrt{5}}dz\\

&\overset{x=\frac{1-z}{1+z}}=-2\int\_0^1 \frac{\ln x}{(1+x)\left((\sqrt{5}+1)x+\sqrt{5}-1\right)}+2\int\_0^1 \frac{\ln x}{(1+x)\left((\sqrt{5}-1)x+\sqrt{5}+1\right)}\\

&=-\frac{\sqrt{5}+1}{\sqrt{5}-1}\int\_0^1 \frac{\ln x}{\frac{\sqrt{5}+1}{\sqrt{5}-1}x+1}dx-\frac{\sqrt{5}-1}{\sqrt{5}+1}\int\_0^1 \frac{\ln x}{\frac{\sqrt{5}-1}{\sqrt{5}+1}x+1}dx+2\int\_0^1 \frac{\ln x}{1+x}dx

\end{align}

Let ,

\begin{align}J&=\underbrace{-\rho\int\_0^1 \frac{\ln x}{\rho x+1}dx}\_{u=\rho x}-\underbrace{\frac{1}{\rho}\int\_0^1 \frac{\ln x}{\frac{x}{\rho}+1}dx}\_{u=\frac{1}{\rho}}+2\int\_0^1 \frac{\ln x}{1+x}dx\\

&=-\int\_0^\rho \frac{\ln\left(\frac{u}{\rho}\right)}{1+u}du-\int\_0^{\frac{1}{\rho}} \frac{\ln\left(\rho u\right)}{1+u}du+2\int\_0^1 \frac{\ln x}{1+x}dx\\

&=-\int\_0^\rho \frac{\ln u}{1+u}du-\int\_0^{\frac{1}{\rho}} \frac{\ln u}{1+u}du+\ln^2\rho+2\int\_0^1 \frac{\ln x}{1+x}dx\\

&=\int\_\rho^1 \frac{\ln u}{1+u}du-\underbrace{\int\_1^{\frac{1}{\rho}} \frac{\ln u}{1+u}du}\_{z=\frac{1}{u}}+\ln^2\rho\\

&=\int\_\rho^1 \frac{\ln u}{1+u}du+\int\_\rho^1\frac{\ln z}{z(1+z)}+\ln^2\rho\\

&=\int\_\rho^1\frac{\ln u}{u}du+\ln^2\rho=\left[\frac{\ln^2 u}{2}\right]\_\rho^1+\ln^2\rho=\frac{1}{2}\ln^2\rho\\

\end{align}

Moreover,

Let , then,

\begin{align}=\frac{\left(\sqrt{5}+1\right)^2}{\left(\sqrt{5}-1\right)\left(\sqrt{5}+1\right)}=\frac{\left(\sqrt{5}+1\right)^2}{4}=\varphi^2\end{align}

Therefore,

\begin{align}J=\frac{1}{2}\ln^2\rho=\boxed{2\ln^2\varphi}\end{align}