MATH 3311, FALL 2025: HOMEWORK 7

For a prime p, a **Sylow p-subgroup** of a finite group G is a p-subgroup $P \leq G$ such that $p \nmid |G/P|$. We have already seen Sylow Theorem A, which basically tells us that Sylow p-subgroups always exist. Here are the remaining two:

Theorem 1 (Sylow theorem B)**.** *Let G be a finite group and let $\mathrm{Syl}_p(G)$ be the set of Sylow p-subgroups of G. Then G acts* transitively *on $\mathrm{Syl}_p(G)$ via conjugation.*

Theorem 2 (Sylow theorem C)**.** *Let n_p be the number of Sylow p-subgroups of a finite group G. Then:*

(1) n_p divides $[G : P]$ for any Sylow p-subgroup $P \leq G$.
(2) $n_p \equiv 1 \pmod p$.

Definition 1. A group G is **simple** if it has no non-trivial, proper *normal* subgroups.

(1) Show that no group of order 100 or 150 is simple.
Hint: What can you say about the Sylow 5-subgroups in both cases?
Let G be an arbitrary group of order 100. Let $P \in \mathrm{Syl}_5(G)$, which we know exists by Sylow A By Sylow C, $|\mathrm{Syl}_5(G)| = n_5 \mid [G : P] = 100/25 = 4$ and $|\mathrm{Syl}_5(G)| = n_5 \equiv 1 \pmod 5$. The only n_p satisfying this is $n_p = 1$. So P is a unique Sylow p-subgroup. Therefore P is normal, and G cannot be simple
Now, Let H be an arbitrary group of order 150. Let $Q \in \mathrm{Syl}_5(G)$, which we know exists by Sylow A. Again, by Sylow C, $|\mathrm{Syl}_5(H)| = m_5 \mid [G : Q] = 150/25 = 6$ and $|\mathrm{Syl}_5(H)| = m_5 \equiv 1 \pmod 5$. The only m_p satisfying this is $m_p = 1$. So Q is normal, and H cannot be simple.

(2) Show that any simple finite *abelian* group is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some prime p.
Hint: Consider Sylow p-subgroups for various primes p. You might also want to use something from Homework 4.

(3) Let G be a group and $H \leq G$ be a subgroup. Consider the action of H on the set of cosets G/H via left multiplication. Show that we have

$$(G/H)^H = N_G(H)/H \subset G/H.$$

$$X^G = \{x \in X : \ g \cdot x = x \text{ for all } g \in G\}.$$

$(G/H)^H = \{gH \in G/H : \ \forall g' \in G, g' \cdot gH = gH\}$

(4) Let G be a finite group, $H \trianglelefteq G$ a normal subgroup and $P \in \mathrm{Syl}_p(H)$ a Sylow p-subgroup of H. Show that $G = HN_G(P)$.
Hint: Note that, if $g \in G$, then gPg^{-1} is also in $\mathrm{Syl}_p(H)$.

(5) Let G be a finite group with $P \in \mathrm{Syl}_p(G)$. Show that, for any subgroup $M \leq G$ containing $N_G(P)$, we have $M = N_G(M)$.
In other words, every subgroup containing $N_G(P)$ is its own normalizer.
Hint: Note that M is normal in $N_G(M)$, and apply the previous problem.

(6) Let G be a simple group of order 60, and let n_2 be the number of Sylow 2-subgroups of G.

(a) Show that $n_2 = 5$ or $n_2 = 15$.
(b) If $n_2 = 15$, show that there exist two distinct Sylow 2-subgroups P_1 and P_2 of G such that

$$|P_1 \cap P_2| = 2.$$

Hint: For part (b), think about the possibilities for n_5, and what this says for how many elements can be contained in the union of all Sylow 2-subgroups.

(7) Let G be as in the previous problem. Suppose that one of the following is true:
(a) $n_2 = 5$;
(b) G admits a subgroup $H \le G$ of index 5.
Show that G is isomorphic to a subgroup of S_5 of index 2.

(8) With the same notation as the previous two problems, suppose that $n_2 = 15$, and let $P_1, P_2 \in \mathrm{Syl}_2(G)$ be two Sylow 2-subgroups such that $|P_1 \cap P_2| = 2$. Show that the normalizer $N_G(P_1 \cap P_2)$ has index 5 in G. Conclude that any simple group G of order 60 is isomorphic to a subgroup of S_5 of index 2.
Hint: Note that both P_1 and P_2 are contained in $N_G(P_1 \cap P_2)$.

(9) Suppose that we have $n, m \in \mathbb{Z}$ and consider the homomorphism

$$\mathbb{Z} \xrightarrow{a \mapsto (a,a)} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}.$$

(a) Show that the kernel of this homomorphism is generated by the least common multiple of n and m.
(b) Show that the image of the homomorphism contains $d\mathbb{Z}/n\mathbb{Z} \times d\mathbb{Z}/m\mathbb{Z}$ where $d = \gcd(n, m)$.
(c) Conclude that, if n and m are relatively prime, there is a natural isomorphism of groups

$$\mathbb{Z}/nm\mathbb{Z} \xrightarrow{\simeq} \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}.$$