Analysis of Algorithms
Homework 1

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Problem #1

Rank the following functions by order of growth. Further, partition the list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class iff $f(n) \in \theta(g(n))$.

$\ln(\ln(x))$	$n2^n$	$n^{\lg(\lg(n))}$	$\ln n$	1
$(\lg(n))^{\lg(n)}$	e^n	$4^{\lg(n)}$	$(n+1)!$	$\sqrt{\lg(n)}$
n^3	$(\lg(n))^2$	$\lg n!$	2^{2^n}	$\frac{3}{2}^n$
$2^{\sqrt{2\lg(n)}}$	n	2^n	$n\lg(n)$	$2^{2^{n+1}}$
$2^{\lg(n)}$	$\sqrt{2}^{\lg(n)}$	n^2	$n!$	$(\lg(n))!$

Equivalence Classes, where $a \in \mathbb{R}$:

$$
\begin{gathered}
[\theta(1)] = \{1\}, \\
[\theta(\sqrt{\lg(x)})] = \{\sqrt{\lg(n)}\}, \\
[\theta(\ln(x))] = \{\ln(\ln(n)), ln(n)\}, \\
[\theta(\lg(x))] = \{n\lg(n), \lg(n!)\}, \\
[\theta(x)] = \{n\}, \\
[\theta(a^{\sqrt{n}})] = \{2^{\sqrt{2\lg(n)}}\}, \\
[\theta(x^2)] = \{n^2, (\lg(n))^2\}, \\
[\theta(x^3)] = \{n^3\}, \\
[\theta(a^{\lg(n)})] = \{2^{\lg(n)}, \sqrt{2}^{\lg(n)}, (\lg(n))^{\lg(n)}, 4^{\lg(n)}\}, \\
[\theta(n^{\lg(n)})] = \{n^{\lg(\lg(n))}\}, \\
[\theta(a^n)] = \{n2^n, e^n, 2^n, \frac{3}{2}^n\}, \\
[\theta(a^{a^x})] = \{2^{2^n}, 2^{2^{n+1}}\}, \\
[\theta(x!)] = \{n!, (\lg(n))!, (n+1)!\}
\end{gathered}
$$

Where the equivalence classes are ordered by growth, from $\theta(1)$, being the lowest growth, to $\theta(x!)$, being the highest growth.

Problem #2

Let:

$$a, b, c, k \in \mathbb{R}$$

We have the following functions:

$$\sqrt[k]{x}, a^x, x^c, \log_b x$$

We can translate them into:

$$x^{\frac{1}{k}}, a^x, x^c, \frac{\log x}{\log b}$$

For all constants, $\lim_{x \to \infty}$:

$$x^{\frac{1}{k}} < x^c, \frac{\log x}{\log b} < x^c, x^c < a^x, x^{\frac{1}{k}} < \frac{\log x}{\log b}$$

Which gives us:

$$x^{\frac{1}{k}} < \frac{\log x}{\log b} < x^c < a^x$$

Problem #3

For all three proofs, consider:

$$f(x) = n \implies f(x) \in \mathcal{O}(g(x))$$

$$N, c \in \mathbb{R}, N, c = 1$$

$$|f(x)| \leq c|g(x)|$$

a. Prove the following:

Using the class definition of Big O, prove that

$$n = \mathcal{O}(n^2)$$

Suppose:

$$x \geq N$$

$$N \leq x \implies x \leq x^2$$

$$x \leq x^2 \implies |x| \leq c|x^2|$$

Thus:

$$n = \mathcal{O}(n^2)$$

b. Prove the following:

Using the class definition of Big O, prove that

$$n^2 = \mathcal{O}(n^2)$$

Suppose:

$$x \geq N$$
$$N \leq x \implies x + x \leq x^2 + x$$
$$x + x \leq x^2 + x \implies |x^2| \leq |x^3|$$

Remember:

$$\mathcal{O}(x^2) \subset \mathcal{O}(x^3)$$

Thus, if:

$$n^2 = \mathcal{O}(n^3)$$

Then:

$$n^2 = \mathcal{O}(n^2)$$

c. Prove the following:

Using the class definition of Big O, prove that

$$3n^2 + 5n = \mathcal{O}(n^2)$$

Suppose:

$$3x^2 + 5x \leq 3x^2 + 5x$$
$$3x^2 + 5x \leq 3x^2 + 5x^2$$
$$3x^2 + 5x \leq 8x^2$$

Thus:

$$N = 8, C = 1$$
$$|3x^2 + 5x| \leq 1|8x^2|$$
$$3x^2 + 5x = \mathcal{O}(x^2)$$

Project #4

Big $\mathcal{O}$ proofs

a. Given that $\sum_{k=2}^{n} \frac{1}{k} \leq \ln(n) - \ln(1)$, using the class definition of $\mathcal{O}$, prove that $H_n \in \mathcal{O}(\ln(n))$

Problem #6

Consider the following recurrence:

$$f(0; a, b) = a$$

$$f(1; a, b) = b$$

$$f(n; a, b) = f(n-1; b, a+b)$$

a. Prove using mathematical induction that for any $n \in \mathbb{N}$ if $n > 1$ then

$$f(n; a, b) = f(n-1; a, b) + f(n-2; a, b)$$

Base Case, $n = 2$:

$$f(n; a, b) = f(n-1; a, b) + f(n-2; a, b)$$

Step, $n + 1$:

$$f(n+1; a, b) \implies (n+1) + f(n; a, b) = f(n-1; a, b) + f(n-2; a, b) + (n+1)$$