Analysis of Algorithms

Homework 4

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Question: 3

The expected case time complexity for the select algorithm is characterized by the following recurrence
$T(1) = 2$

$T(n) = (n+1) + \frac{1}{n}\sum_{q=1}^{n-1} T(q)$

Use techniques, including iteration, to solve the recurrence exactly.

Proof 3: We know that:

$$T(n) = (n+1) + \frac{1}{n}\sum_{q=1}^{n-1} T(q)$$

$$T(n+1) = (n+2) + \frac{1}{n+1}\sum_{q=1}^{n} T(q)$$

From that definition, we can assert that:

$$n\left(T(n) - (n+1)\right) = \sum_{q=1}^{n-1} T(q)$$

Thus:

$$\begin{aligned}
T(n+1) - T(n) &= (n+2) - (n+1) + \left(\frac{1}{n+1}\sum_{q=1}^{n} T(q) - \frac{1}{n}\sum_{q=1}^{n-1} T(q)\right) \\
&= (n+2) - (n+1) + \left(\frac{1}{n+1}\sum_{q=1}^{n-1} T(q) - \frac{1}{n}\sum_{q=1}^{n-1} T(q)\right) + \frac{1}{n+1}T(n) \\
&= 1 + \left(\frac{1}{n+1}\sum_{q=1}^{n-1} T(q) - \frac{1}{n}\sum_{q=1}^{n-1} T(q)\right) + \frac{1}{n+1}T(n) \\
&= 1 + \left(\frac{1}{n+1} - \frac{1}{n}\right)\sum_{q=1}^{n-1} T(q) + \frac{1}{n+1}T(n) \\
&= 1 - \frac{1}{n\,(n+1)}\sum_{q=1}^{n-1} T(q) + \frac{1}{n+1}T(n)
\end{aligned}$$

We can now substitute from the definition:

$$\begin{aligned}
T(n+1) - T(n) &= 1 - \frac{1}{n\,(n+1)}\left(n(T(n) - (n+1)\right) + \frac{1}{n+1}T(n) \\
&= 1 - \frac{T(n) - (n+1)}{(n+1)} + \frac{T(n)}{n+1} \\
&= 1 - \left(\frac{T(n)}{n+1} - 1\right) + \frac{T(n)}{n+1} \\
&= 1 + \frac{-T(n)}{n+1} + 1 + \frac{T(n)}{n+1} \\
&= 2
\end{aligned}$$

Which means we end up with, after subtracting $-T(n)$:

$$\begin{aligned}
T(n+1) &= T(n) + 2 \\
&= 2n
\end{aligned}$$

Proving that our select algorithm runs in $O(n)$ time

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