

Analysis of Algorithms

Homework 1

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Problem #1

Rank the following functions by order of growth. Further, partition the list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class iff $f(n) \in \theta(g(n))$.

$\ln(\ln(x))$	$n2^n$	$n^{\lg(\lg(n))}$	$\ln n$	1
$(\lg(n))^{\lg(n)}$	e^n	$4^{\lg(n)}$	$(n+1)!$	$\sqrt{\lg(n)}$
n^3	$(\lg(n))^2$	$\lg n!$	2^{2^n}	$\frac{3^n}{2}$
$2^{\sqrt{2 \lg(n)}}$	n	2^n	$n \lg(n)$	$2^{2^{n+1}}$
$2^{\lg(n)}$	$\sqrt{2^{\lg(n)}}$	n^2	$n!$	$(\lg(n))!$

Equivalence Classes, where $a \in \mathbb{R}$:

$$\begin{aligned}
 [\theta(1)] &= \{1\}, \\
 [\theta(\sqrt{\lg(x)})] &= \{\sqrt{\lg(n)}\}, \\
 [\theta(\ln(x))] &= \{\ln(\ln(n)), \ln(n)\}, \\
 [\theta(\lg(x))] &= \{n \lg(n), \lg(n!)\}, \\
 [\theta(x)] &= \{n\}, \\
 [\theta(a^{\sqrt{n}})] &= \{2^{\sqrt{2 \lg(n)}}\}, \\
 [\theta(x^2)] &= \{n^2, (\lg(n))^2\}, \\
 [\theta(x^3)] &= \{n^3\}, \\
 [\theta(a^{\lg(n)})] &= \{2^{\lg(n)}, \sqrt{2^{\lg(n)}}, (\lg(n))^{\lg(n)}, 4^{\lg(n)}\}, \\
 [\theta(n^{\lg(n)})] &= \{n^{\lg(\lg(n))}\}, \\
 [\theta(a^n)] &= \{n2^n, e^n, 2^n, \frac{3^n}{2}\}, \\
 [\theta(x!)] &= \{(\lg(n))!, n!, (n+1)!\}
 \end{aligned}$$

$$[\theta(a^{a^x})] = \{2^{2^n}, 2^{2^{n+1}}\},$$

And the order of the functions, from highest to lowest, are:

$$2^{2^{n+1}}$$

$$2^{2^n}$$

$$(n+1)!$$

$$n!$$

$$e^n$$

$$n(2)^n$$

$$2^n$$

$$3^n$$

$$2$$

$$n^{\lg(n)}$$

$$(\lg(n))^{\lg(n)}$$

$$4^{\lg(n)}$$

$$\lg(n)!$$

$$n^3$$

$$n^2$$

$$n \lg(n)$$

$$\lg(n!)$$

$$2^{\lg(n)}, n \text{ (These are functionally equivalent)}$$

$$\sqrt{2^{\lg(n)}}$$

$$2^{\sqrt{2 \lg(n)}}$$

$$\lg(n)^2$$

$$\ln(n)$$

$$\sqrt{\lg(n)}$$

$$\ln(\ln(n))$$

$$1$$

Problem #2

Let:

$$a, b, c, k \in \mathbb{R}$$

We have the following functions:

$$\sqrt[k]{x}, a^x, x^c, \log_b x$$

We can translate them into:

$$x^{\frac{1}{k}}, a^x, x^c, \frac{\log x}{\log b}$$

For all constants, $\lim_{x \rightarrow \infty}$:

$$x^{\frac{1}{k}} < x^c, \frac{\log x}{\log b} < x^c, x^c < a^x, x^{\frac{1}{k}} < \frac{\log x}{\log b}$$

Which gives us:

$$x^{\frac{1}{k}} < \frac{\log x}{\log b} < x^c < a^x$$

Problem #3

For all three proofs, consider:

$$f(x) = n \implies f(x) \in \mathcal{O}(g(x))$$

$$N, c \in \mathbb{R}, N, c = 1$$

$$|f(x)| \leq c|g(x)|$$

a. Prove the following:

Using the class definition of Big O, prove that

$$n = \mathcal{O}(n^2)$$

Suppose:

$$x \geq N$$

$$N \leq x \implies x \leq x^2$$

$$x \leq x^2 \implies |x| \leq c|x^2|$$

Thus:

$$n = \mathcal{O}(n^2)$$

b. Prove the following:

Using the class definition of Big O, prove that

$$n^2 = \mathcal{O}(n^2)$$

Suppose:

$$1 \geq 1$$

$$1 \leq 1 \implies x \leq x$$

$$x \leq x \implies |x^2| \leq 1|x^2|$$

Thus:

$$n^2 = \mathcal{O}(n^2)$$

c. Prove the following:

Using the class definition of Big O, prove that

$$3n^2 + 5n = \mathcal{O}(n^2)$$

Suppose:

$$3x^2 + 5x \leq 3x^2 + 5x$$

$$3x^2 + 5x \leq 3x^2 + 5x^2$$

$$3x^2 + 5x \leq 8x^2$$

Thus:

$$N = 8, C = 1$$

$$|3x^2 + 5x| \leq 1|8x^2|$$

$$3x^2 + 5x = \mathcal{O}(x^2)$$

Project #4

Big O proofs

a. Given that $\sum_{k=2}^n \frac{1}{k} \leq \ln(n) - \ln(1)$, using the class definition of \mathcal{O} , prove that $H_n \in \mathcal{O}(\ln(n))$

Problem #5

The smallest $\{n\}$ that fib starts slowing down at is 30.

Problem #6

Consider the following recurrence:

$$\begin{aligned}f(0; a, b) &= a \\f(1; a, b) &= b \\f(n; a, b) &= f(n-1; b, a+b)\end{aligned}$$

a. Prove using mathematical induction that for any $n \in \mathbb{N}$ if $n > 1$ then $f(n; a, b) = f(n-1; a, b) + f(n-2; a, b)$.

Observe that, for $n = 2$:

$$f(2; a, b) = f(1; b, a+b) \quad (1)$$

$$= a + b \quad (2)$$

$$= f(1; a, b) + f(0; a, b) \quad (3)$$

$$= f(2-1; a, b) + f(2-2; a, b) \quad (4)$$

Assume that $f(k; a, b) = f(k-1; a, b) + f(k-2; a, b)$ is true, where $0 \leq k < n$, then:

$$f(k+1; a, b) = f(k; b, a+b) \quad (5)$$

$$= f(k-1; b, a+b) + f(k-2; b, a+b) \quad (6)$$

$$= f(k; a, b) + f(k-1; a, b) \quad (7)$$

Which proves $f(n; a, b) = f(n-1; a, b) + f(n-2; a, b)$

b. Prove using the strong form of induction that for any $n \in \mathbb{N}$, $F_n = f(n; 0, 1)$. You should use the previous result in your proof.

Suppose $F_2, F_3, \dots, F_k \implies F_{k+1}$

$$F_2 = f(2; 0, 1) = f(1; 0, 1) + f(0; 0, 1) = 1 + 0 = 1$$

$$F_3 = f(3; 0, 1) = f(2; 1, 1+0) = f(1; 1, 1+0+1) = 1 + 0 + 1 = 2$$

$$F_k = f(k-1; 0, 1) + f(k-2; 0, 1)$$

Assume that $F_k = f(k-1; 0, 1) + f(k-2; 0, 1)$, then

$$F_{k+1} = f(k; 0, 1) + f(k-1; 0, 1) \quad (8)$$

$$= f(k+1; 0, 1) \quad (9)$$

Which proves that for all n , $F_n = f(n; 0, 1)$

Problem #7

Exceeds maximum recursion depth before slowing down at all