Analysis of Algorithms

Homework 6

Thomas Schollenberger (tss2344)

December 5, 2022

Question: 1

Verfify that $\{u_1, u_2\}$ is an orthogonal set and find the projection of y onto span$\{u_1, u_2\}$.

$$u_1 = \begin{bmatrix} -4 \\ -1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, y = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

Answer to 1: $u_1(u_2) = -4(0) + -1(1) + 1(1) = 0$

$$A = \begin{bmatrix} -4 & 0 \\ -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 18 & 0 \\ 0 & 2 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 21 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 18 & 0 & 21 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{7}{6} \\ 0 & 1 & \frac{5}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{6} \\ \frac{5}{2} \end{bmatrix}$$

$$y_{proj} = A \begin{bmatrix} \frac{7}{6} \\ \frac{5}{2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 14 \\ 4 \\ 11 \end{bmatrix}$$

□

Question: 2

Let $u_1, \ldots, u_p$ be an orthogonal basis for a subspace W of $\mathcal{R}^n$, and let $T : \mathcal{R}^n \to \mathcal{R}^n$ be defined by $T(\mathrm{x}) = \mathrm{proj}_W\, \mathrm{x}$. Show that T is a linear transformation

Answer to 2: T is a linear transformation because it is a linear combination of the basis vectors.

$$T\left(\sum_{i=1}^{n} c_i u_i\right) = \sum_{i=1}^{n} c_i \mathrm{proj}_W u_i = \sum_{i=1}^{n} c_i u_i$$

□

Question: 3

Use the Gram-Schmidt process to find an orthogonal basis for the column space of

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}$$

Answer to 3: $u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$,

$u_2 = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 5 \\ 5 \end{bmatrix}$,

$u_3 = \begin{bmatrix} 5 \\ 1 \\ 3 \\ 2 \\ 8 \end{bmatrix}$

$v_1 = u_1$ $v_2 = u_2 - \frac{u_2(v_1)}{v_1(v_1)} v_1$ $v_3 = u_3 - \frac{u_3(v_1)}{v_1(v_1)} v_1 - \frac{u_3(v_2)}{v_2(v_2)} v_2$

$v_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 11 \\ 4 \\ 5 \\ 5 \end{bmatrix}$

$v_3 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\{v_1, v_2, v_3\}$ is an orthogonal basis for the column space of A □

Question: 4

Let W be the span of the first two columns of the matrix from question 3, and let $x \in W$ be defined below. Use your answer from question 3 to decompose x into a vector $\hat{x} \in W$ and a vector $z \in W^{\perp}$.

$$x = \begin{bmatrix} 3 \\ 9 \\ 0 \\ -9 \\ 3 \end{bmatrix}$$

Answer to 4:

$$x = \hat{x} + z$$

$$\hat{x} = \frac{1}{2} \begin{bmatrix} 1 \\ 11 \\ 4 \\ 5 \\ 5 \end{bmatrix}$$

$$z = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$\square$