Analysis of Algorithms
Homework 3

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Problem #2

$$\begin{aligned} search([1,2,3],2) &= searchHelp([1,2,3],2,0,2) && \text{(1a)}\\ \hat{m} &= 0 + \lfloor (2-0)/2 \rfloor && \text{(1b)}\\ \hat{m} &= 2 && \text{(1c)}\\ v &< a[\hat{m}] && \text{(1d)}\\ &= searchHelp([1,2,3],2,0,2) && \text{(1e)} \end{aligned}$$

Which results in an infinite loop, as the final searchHelp call is identical to the initial. This is because the $\hat{m}$ calculation is returning 2, the length of the array minus one, but the binary search operation is not incrementing or decrementing $\hat{m}$ when recursively calling it.

Problem #3

a. Strassen's Algorithm computation:

$$\begin{pmatrix} 1 & 3 \\ 7 & 5 \end{pmatrix} \begin{pmatrix} 6 & 8 \\ 4 & 2 \end{pmatrix}$$

$$\begin{aligned} M_1 &= (1+5)(6+2) && \text{(2a)}\\ M_2 &= (7+5)(6) && \text{(2b)}\\ M_3 &= (1)(8-2) && \text{(2c)}\\ M_4 &= (5)(4-6) && \text{(2d)}\\ M_5 &= (1+3)(2) && \text{(2e)}\\ M_6 &= (7-1)(6+8) && \text{(2f)}\\ M_7 &= (3-5)(4+2) && \text{(2g)} \end{aligned}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} M_1 + M_4 - M_5 + M_7 & M_3 + M_5 \\ M_2 + M_4 & M_1 - M_2 + M_3 + M_6 \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}$$

$$C = \begin{bmatrix} 18 & 14 \\ 62 & 66 \end{bmatrix}$$

b. Strassen's Algorithm pseudo-code:

$$\text{strassen}(A, B) = \begin{cases} \text{A x B}, & \text{if a.rows} = 1 \\ \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\text{where } \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} &= A \\
\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} &= B \\
P_1 &= \text{strassen}(A_{11}, (B_{12} - B_{22})) \\
P_2 &= \text{strassen}((A_{11} + A_{12}), B_{22}) \\
P_3 &= \text{strassen}((A_{21} + A_{22}), B_{11}) \\
P_4 &= \text{strassen}(A_{22}, (B_{21} - B_{11})) \\
P_5 &= \text{strassen}((A_{11} + A_{22}), (B_{11} + B_{22})) \\
P_6 &= \text{strassen}((A_{12} - A_{22}), (B_{21} + B_{22})) \\
P_7 &= \text{strassen}((A_{11} - A_{21}), (B_{11} + B_{12})) \\
C_{11} &= P_5 + P_4 - P_2 + P_6 \\
C_{12} &= P_1 + P_2 \\
C_{21} &= P_3 + P_4 \\
C_{22} &= P_5 + P_1 - P_3 - P_7
\end{aligned}$$

c. Verify that $C_{2,1} = A_{21}B_{11} + A_{22}B_{21}$

$$\begin{aligned}
C_{2,1} &= P_3 + P_4 \\
&= (A_{21} + A_{22})B_{11} + A_{22}(B_{21} - B_{11}) \\
&= A_{21}B_{11} + A_{22}B_{11} + A_{22}B_{21} - A22B_{11} \\
&= A_{21}B_{11} + A_{22}B_{21}
\end{aligned}$$

d. Verify that $C_{2,2} = A_{21}B_{12} + A_{22}B_{22}$

$$C_{22} = P_5 + P_1 - P_3 - P_7$$

$$= (A_{11}+A_{22})(B_{11}+B_{22})+A_{11}(B_{12}-B_{22})-(A_{21}+A_{22})B_{11}-(A_{11}-A_{21})(B_{11}+B_{12})$$

$$= (A_{11}+A_{22})(B_{11}+B_{22})+A_{11}B_{12}-A_{11}B_{22}-A_{21}B_{11}+A_{22}B_{11}-(A_{11}-A_{21})(B_{11}+B_{12})$$

$$= A_{11}B_{11}+A_{11}B_{22}+A_{22}B_{11}+A_{22}B_{22}+A_{11}B_{12}-A_{11}B_{22}-A_{21}B_{11}+A_{22}B_{11}-(A_{11}-A_{21})(B_{11}+B_{12})$$
$$= A_{11}B_{11}+A_{22}B_{11}+A_{22}B_{22}+A_{11}B_{12}-A_{21}B_{11}+A_{22}B_{11}-A_{11}B_{11}-A21B_{11}-A_{11}B_{12}-A_{21}B_{12}$$
$$= A_{22}B_{11} + A_{22}B_{22} - A_{21}B_{11} - A_{22}B_{11} - A21B_{11} + A_{21}B_{12}$$
$$= A_{22}B_{22} - A_{21}B_{11} + A21B_{11} + A_{21}B_{12}$$
$$= A_{22}B_{22} + A_{21}B_{12}$$

e. Solve this recurrence, assuming $n = 2^m$

$$T(1) = 1$$
$$T(n) = 7T(\frac{n}{2}) + \frac{9}{2}n^2$$

$$T(n) = 7T(\frac{n}{2} + \frac{9}{2}n^2) \tag{3a}$$
$$= 7[7T(\frac{\frac{n}{2}}{2}) + \frac{9}{2}\frac{n}{2}^2] + \frac{9}{2}n^2 \tag{3b}$$
$$= 7[7[7T(\frac{\frac{\frac{n}{2}}{2}}{2}) + \frac{9}{2}\frac{n}{4}^2] + \frac{9}{2}\frac{n}{2}^2] + \frac{9}{2}n^2 \tag{3c}$$
$$= 7[7[7T(2^{m-3}) + \frac{9}{2}(2^{m-2})^2] + \frac{9}{2}(2^{m-1})^2] + \frac{9}{2}(2^m)^2 \tag{3d}$$
$$= 7^3T(2^{m-3}) + 7^2\frac{9}{2}(2^{m-2})^2 + 7\frac{9}{2}(2^{m-1})^2 + \frac{9}{2}(2^m)^2 \tag{3e}$$
$$= 7^kT(2^{m-k}) + 7^{k-1}\frac{9}{2}(2^{m-(k-1)})^2 + 7^{k-2}\frac{9}{2}(2^{m-(k-2)})^2 + 7^{k-3}\frac{9}{2}(2^{m-(k-3)})^2 \tag{3f}$$
$$= 7^kT(2^{m-k}) + \sum_{i=0}^{k-1} 7^i\frac{9}{2}(2^{m-(i)})^2 \tag{3g}$$
$$= 7^{\lg n} - 6n^2 \tag{3h}$$

f. Modify Strassen's algorithm to work for a n x n, where n is not an exact power of 2

To achieve this, we would need to add zeros to the n x n matrix so it could become a $2n$ x $2n$ matrix. From there, we would be performing 7 operations on a 2n matrix, which would make it run in $\theta(2n^{\lg(7)})$, which is equivalent to $\theta(n^{\lg(7)})$

g. Show how to multiply complex numbers $a + bi$ and $c + di$ using only three multiplications of real numbers. The algorithm should take the real numbers a, b, c, and d as input and produce the real component $ac - bd$ and the imaginary component $ad + bc$ separately.

mult(a, b, c, d) = (real, imaginary), where:

$$\begin{aligned} mul_1 &= a(c) \\ mul_2 &= b(d) \\ mul_3 &= (a + b)(c + d) \\ \text{real} &= mul_1 - mul_2 \\ \text{imaginary} &= mul_3 - mul_1 - mul_2 \end{aligned}$$

Problem #4

Consider the recurrence $T(1) = 0$, $T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + n$.

a. Let $D(n) = T(n+1) - T(n)$. It is a fact that $D(1) = 2$, $D(n) = D(\lfloor n/2 \rfloor) + 1$. Prove using the strong form of induction that for any $n \in N$, if $n \geq 1$ then $D(n) = \lfloor \lg(n) \rfloor + 2$.

Proof by Strong Induction:

Base Cases:

$$D(n) = T(n + 1) - T(n)$$

$$D(1) = 2$$

$$D(n) = D(\lfloor n/2 \rfloor) + 1$$

This means that $\lfloor n \rfloor = 2^k$, $D(n/2) = \lfloor \lg(n) \rfloor + D(1)$, thus:

$$\begin{aligned} D(n) &= D(\lfloor n/2 \rfloor) + 1 \\ D(n) &= \lfloor \lg(n) \rfloor + 1 + 1 \\ D(n) &= \lfloor \lg(n) \rfloor + 2 \end{aligned}$$

For all $n \in \mathbb{N}$, where $n \geq 1$

b. Then prove that $T(n) - T(1) = \sum_{k=1}^{n-1} D(k)$, and show that an immediate consequence is that $T(n) = \sum_{k=1}^{n-1}(\lfloor \lg(n) \rfloor + 2)$

We know that

$$D(n) = \lfloor \lg(n) \rfloor + 2$$

$$D(n) = T(n+1) - T(n)$$
$$D(n-1) = T(n) - T(n-1)$$

Furthermore

$$T(n) - T(1) = T(n) - 0 = T(n)$$
$$T(n) = \lfloor n \lg(n) \rfloor + n$$

Which follows that

$$\lfloor n \lg(n) \rfloor + n - T(1) = \sum_{k=1}^{n-1} D(k)$$

$$\lfloor n \lg(n) \rfloor + n = \sum_{k=1}^{n-1} (\lfloor \lg(n) \rfloor + 2)$$

c. Now show that $T(n) = \sum_{k=1}^{n-1} (\lfloor \lg(n) \rfloor + 2)$, implies that $T(n) = O(n \log(n))$

Recall that

$$\sum_{k=1}^{n} \lg(n) = O(n \lg(n))$$

Which implies that

$$T(n) = \sum_{k=1}^{n-1} (\lfloor \lg(n) \rfloor + 2)$$
$$T(n) = O(n \log(n))$$