Analysis of Algorithms
Homework 1

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Problem #1

Rank the following functions by order of growth. Further, partition the list into equivalence classes such that functions $f(n)$ and $g(n)$ are in the same class iff $f(n) \in \theta(g(n))$.

$\ln(\ln(x))$	$n2^n$	$n^{\lg(\lg(n))}$	$\ln n$	1
$(\lg(n))^{\lg(n)}$	e^n	$4^{\lg(n)}$	$(n+1)!$	$\sqrt{\lg(n)}$
n^3	$(\lg(n))^2$	$\lg n!$	2^{2^n}	$\frac{3}{2}^n$
$2^{\sqrt{2\lg(n)}}$	n	2^n	$n\lg(n)$	$2^{2^{n+1}}$
$2^{\lg(n)}$	$\sqrt{2}^{\lg(n)}$	n^2	$n!$	$(\lg(n))!$

Equivalence Classes, where $a \in \mathbb{R}$:

$$[\theta(1)] = \{1\},$$

$$[\theta(\sqrt{\lg(x)})] = \{\sqrt{\lg(n)}\},$$

$$[\theta(\ln(x))] = \{\ln(\ln(n)), ln(n)\},$$

$$[\theta(\lg(x))] = \{n\lg(n), \lg(n!)\},$$

$$[\theta(x)] = \{n\},$$

$$[\theta(a^{\sqrt{n}})] = \{2^{\sqrt{2\lg(n)}}\},$$

$$[\theta(x^2)] = \{n^2, (\lg(n))^2\},$$

$$[\theta(x^3)] = \{n^3\},$$

$$[\theta(a^{\lg(n)})] = \{2^{\lg(n)}, \sqrt{2}^{\lg(n)}, (\lg(n))^{\lg(n)}, 4^{\lg(n)}\},$$

$$[\theta(n^{\lg(n)})] = \{n^{\lg(\lg(n))}\},$$

$$[\theta(a^n)] = \{n2^n, e^n, 2^n, \frac{3}{2}^n\},$$

$$[\theta(x!)] = \{(\lg(n))!, n!, (n+1)!\}$$

$$[\theta(a^{a^x})] = \{2^{2^n}, 2^{2^{n+1}}\},$$

And the order of the functions, from highest to lowest, are:

$$2^{2^{n+1}}$$

$$2^{2^n}$$

$$(n+1)!$$

$$n!$$

$$e^n$$

$$n\,(2)^n$$

$$2^n$$

$$\frac{3}{2}^n$$

$$n^{\lg(\lg(n))}$$

$$(\lg(n))^{\lg(n)}$$

$$4^{\lg(n)}$$

$$\lg(n))!$$

$$n^3$$

$$n^2$$

$$n\lg(n)$$

$$\lg(n!)$$

$$2^{\lg(n)}, n\,(\text{These are functionally equivalent})$$

$$\sqrt{2}^{\lg(n)}$$

$$2^{\sqrt{2\lg(n)}}$$

$$\lg(n)^2$$

$$ln(n)$$

$$\sqrt{\lg(n)}$$

$$\ln(\ln(n))$$

$$1$$

Problem #2

Let:

$$a, b, c, k \in \mathbb{R}$$

We have the following functions:

$$\sqrt[k]{x}, a^x, x^c, \log_b x$$

We can translate them into:

$$x^{\frac{1}{k}}, a^x, x^c, \frac{\log x}{\log b}$$

For all constants, $\lim_{x \to \infty}$:

$$x^{\frac{1}{k}} < x^c, \frac{\log x}{\log b} < x^c, x^c < a^x, x^{\frac{1}{k}} < \frac{\log x}{\log b}$$

Which gives us:

$$x^{\frac{1}{k}} < \frac{\log x}{\log b} < x^c < a^x$$

Problem #3

For all three proofs, consider:

$$f(x) = n \implies f(x) \in \mathcal{O}(g(x))$$

$$N, c \in \mathbb{R}, N, c = 1$$

$$|f(x)| \leq c|g(x)|$$

a. Prove the following:

Using the class definition of Big O, prove that

$$n = \mathcal{O}(n^2)$$

Suppose:

$$x \geq N$$

$$N \leq x \implies x \leq x^2$$

$$x \leq x^2 \implies |x| \leq c|x^2|$$

Thus:

$$n = \mathcal{O}(n^2)$$

b. Prove the following:

Using the class definition of Big O, prove that

$$n^2 = \mathcal{O}(n^2)$$

Suppose:

$$1 \geq 1$$
$$1 \leq 1 \implies x \leq x$$
$$x \leq x \implies |x^2| \leq 1|x^2|$$

Thus:

$$n^2 = \mathcal{O}(n^2)$$

c. Prove the following:

Using the class definition of Big O, prove that

$$3n^2 + 5n = \mathcal{O}(n^2)$$

Suppose:

$$3x^2 + 5x \leq 3x^2 + 5x$$
$$3x^2 + 5x \leq 3x^2 + 5x^2$$
$$3x^2 + 5x \leq 8x^2$$

Thus:

$$N = 8, C = 1$$
$$|3x^2 + 5x| \leq 1|8x^2|$$
$$3x^2 + 5x = \mathcal{O}(x^2)$$

Project #4

Big $\mathcal{O}$ proofs

a. Given that $\sum_{k=2}^{n} \frac{1}{k} \leq \ln(n) - \ln(1)$, using the class definition of $\mathcal{O}$, prove that $H_n \in \mathcal{O}(\ln(n))$

Assuming $1 \geq 1$, then:

$$1 \leq 1 \tag{1}$$
$$n \leq n \tag{2}$$
$$\ln(n) \leq \ln(n) \tag{3}$$
$$\ln(n) - \ln(1) \leq \ln(n) - \ln(1) \tag{4}$$
$$|\ln(n) - \ln(1)| \leq 1|\ln(n)| \tag{5}$$

However, this only proves that $H_2 \in \mathcal{O}(\ln(n))$. To prove H_1, we need to add a one to each side:

$$\ln(n) - \ln(1) + 1 \leq \ln(n) + 1 \tag{6}$$
$$\ln(n) \leq \ln(n) - \ln(1) + 1 \leq \ln(n) + 1 \tag{7}$$
$$\ln(n+1) \leq \ln(n) - \ln(1) \leq \ln(n) \tag{8}$$

Which shows that $H_n \in \mathcal{O}(\ln(n))$

b. Given that $\sum_{k=2}^{n} \frac{1}{k} \geq \ln(n+1) - \ln(2)$, using the class definition of ω, prove that $H_n \in \omega(\ln n)$

Assuming $1 \leq 1$, then:

$$1 \geq 1 \tag{9}$$
$$n \geq n \tag{10}$$
$$\ln(n) \geq \ln(n) \tag{11}$$
$$\ln(n) - \ln(2) \geq \ln(n) - \ln(2) \tag{12}$$
$$|\ln(n) - \ln(2)| \geq 1|\ln(n) - \ln(2)| \tag{13}$$

We then need to add + 1 to both sides:

$$|\ln(n) - \ln(2) + 1| \geq 1|\ln(n) - \ln(2) + 1| \tag{14}$$
$$|\ln(n) - \ln(2) + 1| \geq 1|\ln(n) - \ln(2) + 1| \tag{15}$$
$$|\ln(n+1)| \geq |\ln(n) - \ln(2) + 1| \geq 1|\ln(n) - \ln(2) + 1| \tag{16}$$

Which then we can shift to

$$|\ln(n)| \geq |\ln(n+1) - \ln(2)| \geq 1|\ln(n+1) - \ln(2)|$$

Which shows that $H_n \in \omega(\ln(n))$

Problem #5

The smallest {n} that fib starts slowing down at is 30.

Problem #6

Consider the following recurrence:

$$f(0; a, b) = a$$

$$f(1; a, b) = b$$

$$f(n; a, b) = f(n-1; b, a+b)$$

a. Prove using mathematical induction that for any $n \in \mathbb{N}$ if $n > 1$ then $f(n; a, b) = f(n-1; a, b) + f(n-2; a, b)$.

Observe that, for $n = 2$:

$$\begin{aligned} f(2; a, b) &= f(1; b, a+b) && (17) \\ &= a + b && (18) \\ &= f(1; a, b) + f(0; a, b) && (19) \\ &= f(2-1; a, b) + f(2-2; a, b) && (20) \end{aligned}$$

Assume that $f(k; a, b) = f(k-1; a, b) + f(k-2; a, b)$ is true, where $0 \leq k < n$, then:

$$\begin{aligned} f(k+1; a, b) &= f(k; b, a+b) && (21) \\ &= f(k-1; b, a+b) + f(k-2; b, a+b) && (22) \\ &= f(k; a, b) + f(k-1; a, b) && (23) \end{aligned}$$

Which proves f(n; a, b) = f(n - 1; a, b) + f(n - 2; a, b)

b. Prove using the strong form of induction that for any $n \in \mathbb{N}, F_n = f(n; 0, 1)$. You should use the previous result in your proof.

Suppose $F_2, F_3, ..., F_k \implies F_{k+1}$

$$F_2 = f(2; 0, 1) = f(1; 0, 1) + f(0; 0, 1) = 1 + 0 = 1$$

$$F_3 = f(3; 0, 1) = f(2; 1, 1+0) = f(1; 1, 1+0+1) = 1+0+1 = 2$$

$$F_k = f(k-1; 0, 1) + f(k-2; 0, 1)$$

Assume that $F_k = f(k-1; 0, 1) + f(k-2; 0, 1)$, then

$$\begin{aligned} F_{k+1} &= f(k; 0, 1) + f(k-1; 0, 1) && (24) \\ &= f(k+1; 0, 1) && (25) \end{aligned}$$

Which proves that for all n, $F_n = f(n; 0, 1)$

Problem #7

Exceeds maximum recursion depth before slowing down at all