Stochastic Financial Models 5

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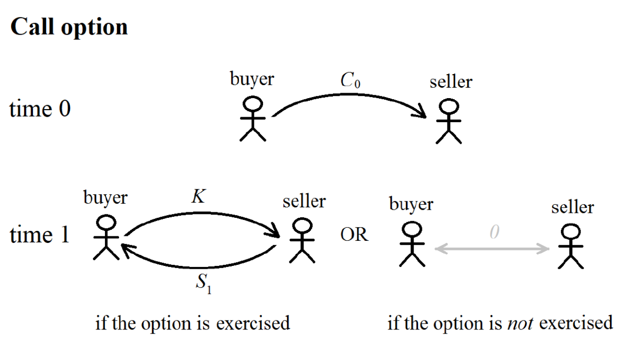
1 Contingent claims

In the context of a one-period model a *contingent claim* is just another name for an asset with a random payout at time 1.

- interest rate r and d risky assets with time t price vector S_t, for $t \in \{0, 1\}$. These are thought of as ‘fundamental’ assets.
- We introduce a $(d + 1)$st risky asset with time-1 payout Y.
- Often $Y = g(S_1)$ for some function g, but not always.
- The problem is to find a ‘reasonable’ time-0 price for the claim

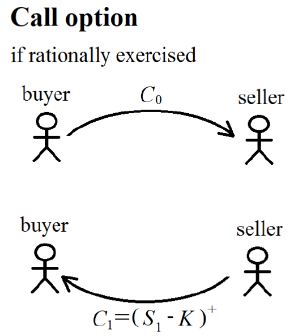
Example

Definition. A *call option* is the right, but not the obligation, to buy a certain asset at a certain price (called the strike) at a certain time in the future (the maturity date).



- If $S_1 > K$ it is rational to receive the payout $S_1 - K$.
- If $S_1 \leq K$ it is rational to let the call expire unexercised.
- The payout is $(S_1 - K)^+ = g(S_1)$

- notation: $x^+ = \max\{x, 0\}$ is the positive part of the real number x.



2 Indifference pricing

Consider an investor with initial wealth X_0 and concave, increasing utility function U. She is offered to buy a contingent claim with payout Y. How much should she pay?

- Let
$$\mathcal{X} = \{(1+r)X_0 + \theta^\top [S_1 - (1+r)S_0] : \theta \in \mathbb{R}^d\}$$
be the set of time-1 wealths attainable from trading the original market.

- The agent would prefer to buy one share of the contingent claim with time-1 payout Y for time-0 price π iff there exists an $X^* \in \mathcal{X}$ such that
$$\mathbb{E}[U(X^* + Y - (1+r)\pi)] \geq \mathbb{E}[U(X)]$$
for all $X \in \mathcal{X}$.

Assumption. In the examples from this course, we will assume that the data of the problem is such that any given utility maximisation problem has a solution.

Definition. An *indifference* (or *reservation*) price of the claim with payout Y is any solution π of
$$\max_{X \in \mathcal{X}} \mathbb{E}[U(X + Y - (1+r)\pi)] = \max_{X \in \mathcal{X}} \mathbb{E}[U(X)]$$

3 Properties of indifference prices

Theorem. *Under our assumptions*[1]*, indifference prices exist and are unique.*

[1]For the technically minded, we will assume the random variable $U(X+Y+x)$ is integrable for all $X \in \mathcal{X}$, $x \in \mathbb{R}$, and possible payouts Y, and that for x, Y there exists $X^* \in \mathcal{X}$ such that $\mathbb{E}[U(X^*+Y+x)] = \max_{X \in \mathcal{X}} \mathbb{E}[U(X+Y+x)]$

Proof. Next time.

Notation. For a fixed initial wealth X_0 and utility function U, we will let $\pi(Y)$ denote the (unique) indifference price of a contingent claim with payout Y.

Theorem (Indifference prices are increasing)**.** *If* $Y_0 \leq Y_1$ *almost surely with* $\mathbb{P}(Y_0 < Y_1) > 0$ *then*

$$\pi(Y_0) < \pi(Y_1)$$

Proof. Next time.

Theorem (Indifference prices are concave)**.** *Given random variable* Y_0, Y_1 *and* $0 < p < 1$, *we have*

$$\pi(pY_1 + (1-p)Y_0) \geq p\ \pi(Y_1) + (1-p)\pi(Y_0)$$

Proof. Next time.

Definition. The marginal utility price of a claim with payout Y is

$$\pi_0(Y) = \frac{\mathbb{E}[U'(X^*)Y]}{(1+r)\mathbb{E}[U'(X^*)]}.$$

where $X^* \in \mathcal{X}$ is such that $\mathbb{E}[U(X^*)] = \max_{X\in\mathcal{X}} \mathbb{E}[U(X)]$.

Note that our first marginal utility pricing theorem (from last time) says

$$\pi_0(a + b^\top S_1) = \frac{a}{1+r} + b^\top S_0$$

for any $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

Theorem (Marginal utility price is larger than indifference price)**.**

$$\pi(Y) \leq \pi_0(Y)$$

Proof. Next time.

Theorem (Convergence of indifference prices to marginal utility prices)**.**

$$\lim_{\varepsilon\to 0} \frac{\pi(\varepsilon Y)}{\varepsilon} = \pi_0(Y)$$

Proof. Next time.