Stochastic Financial Models 16

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1 Dynamic programming principle

Given

- A sequence $(\xi_n)_{n\geq 1}$ of independent random variables generating a filtration $(\mathcal{F}_n)_{n\geq 0}$
- A function $G(\cdot,\cdot,\cdot,\cdot)$
- An initial condition X_0

for any previsible $(U_n)_{n\geq 1}$ construct the controlled Markov process by

$$X_n^U = \begin{cases} X_0 & \text{if } n = 0 \\ G(n, X_{n-1}^U, U_n, \xi_n) & \text{if } n \geq 1 \end{cases}$$

Now given

- A non-random time horizon $N > 0$
- Suitably integrable functions $f(\cdot,\cdot)$ and $g(\cdot)$

we seek to maximise

$$\mathbb{E}\left[\sum_{k=1}^{N} f(k, U_k) + g(X_N^U)\right]$$

Theorem (The dynamic programming principle)**.** *Let V solve the Bellman equations:*

$$V(N,x) = g(x) \quad \textit{for all } x$$
$$V(n-1,x) = \sup_u \{f(n,u) + \mathbb{E}[V(n, G(n,x,u,\xi_n))]\} \quad \textit{for all } 1 \leq n \leq N, \textit{ and } x$$

and suppose for each n and x there is an optimal solution $u^*(n,x)$ to the maximisation problem, so that

$$V(n-1,x) = f(n, u^*(n,x)) + \mathbb{E}[V(n, G(n,x,u^*(n,x),\xi_n))] \quad \textit{for all } 1 \leq n \leq N, \textit{ and } x$$

Fix the initial condition $X_0^* = X_0$ *and let*

$$\begin{aligned} U_n^* &= u^*(n, X_{n-1}^*), \\ X_n^* &= G(n, X_{n-1}^*, U_n^*, \xi_n) \text{ for all } 1 \le n \le N \end{aligned}$$

so that $X_n^{U^*} = X_n^*$ *for all* $0 \le n \le N$*. Then* $(U_n^*)_{1\le n\le N}$ *is the optimal control and* V *is the value function.*

Proof. Fix X_0 and let $(U_n)_{1\le n\le N}$ be a previsible control, and consider the associated controlled process $(X_n^U)_{0\le n\le N}$. Let

$$M_n^U = \sum_{k=1}^n f(k, U_k) + V(n, X_n^U)$$

Claim: $(M_n)_{0\le n\le N}$ is a supermartigale.

Indeed, this process is adapted and integrable (by assumption). Now using the 'fix known quantities and average over independent quantities' property of conditional expectation, we have by the Bellman equation that

$$\begin{aligned} &\mathbb{E}[M_n^U - M_{n-1}^U | \mathcal{F}_{n-1}] \\ &= f(n, U_n) + \mathbb{E}[V(n, X_n^U)|\mathcal{F}_{n-1}] - V(n-1, X_{n-1}^U) \\ &= \{f(n,u) + \mathbb{E}[V(n, G(n,x,u,\xi_n))] - V(n-1,x)\} \big|_{u=U_n, x=X_{n-1}^U} \\ &\le 0 \end{aligned}$$

with equality if $U_n = u^*(n, X_{n-1}^U)$.

Hence, using $V(N,x) = g(x)$ for all x, we have by the tower property that

$$\begin{aligned} \mathbb{E}\left[\sum_{k=n+1}^N f(k,U_k) + g(X_N^U) \middle| X_n^U\right] &= \mathbb{E}\left[M_N^U - \sum_{k=1}^n f(k,U_k) \middle| X_n^U\right] \\ &= \mathbb{E}\left[\mathbb{E}\left(M_N^U - \sum_{k=1}^n f(k,U_k) \middle| \mathcal{F}_n\right) \middle| X_n^U\right] \\ &\le \mathbb{E}\left[M_n^U - \sum_{k=1}^n f(k,U_k) \middle| X_n^U\right] \\ &= V(n, X_n^U) \end{aligned}$$

with equality if $U = U^*$. This shows

$$V(n,x) = \max_{(U_k)_{k+1\le n\le N}} \mathbb{E}\left[\sum_{k=n+1}^N f(k,U_k) + g(X_N^U) \middle| X_n^U = x\right]$$

as claimed. $\square$

Remark. The above proof uses an argument sometimes called the *martingale principle of optimal control.*

2 Optimal investment

Given a market with interest rate r and d risky assets with prices $(S_n)_{n\geq 0}$

- consider an investor who, between time $n-1$ and time n holds θ_n shares and consumes C_n cash, where $0 \leq C_n \leq X_{n-1}$.
- wealth evolves as $X_n = (1+r)(X_{n-1} - C_n) + \theta_n^\top [S_n - (1+r)S_{n-1}]$

To have a tractable problem, we make some simplifying assumptions:

- assume $d = 1$
- assume $S_n = S_{n-1}\xi_n$ where $(\xi_n)_n$ are independent
- Note $X_n = G(n, X_{n-1}, \binom{C_n}{\eta_n}, \xi_n))$ where $\eta_n = S_{n-1}\theta_n$ and

$$G(n, x, \tbinom{c}{\eta}, v) = (1+r)(x-c) + \eta(\xi - (1+r))$$

  so the wealth is a controlled Markov process with two-dimensional controls $u = \binom{c}{\eta}$.

Given a time horizon N, a natural goal is to

$$\text{maximise } \mathbb{E}\left[\sum_{k=1}^{N} U(C_k) + U(X_N)\right]$$

where U is the investor's utility function

The Bellman equation is

$$\begin{aligned} V(N, x) &= U(x) \\ V(n-1, x) &= \max_{c,\eta} \mathbb{E}\left[U(c) + V(n, (1+r)(x-c) + \eta(\xi - (1+r)))\right] \end{aligned}$$

- Generally, intractable
- but suppose the utility is CRRA: $U(x) = \frac{1}{1-R}x^{1-R}$ for $x > 0$, where $R > 0, R \neq 1$.