Stochastic Financial Models 21

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1 Properties of Brownian motion

Theorem (Wiener 1923)**.** *Brownian motion exists.*

Remark. A Brownian motion is called a Wiener process in the US.

Theorem. *Brownian motion is a martingale in its filtration $\mathcal{F}_t = \sigma(W_s : 0 \le s \le t)$.*

Proof. Brownian motion is integrable, adapted and

$$\mathbb{E}[W_t - W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0$$

for $0 \le s \le t$ by the independence of $W_t - W_s$ and $\mathcal{F}_s$. □

Theorem. *Brownian motion is a Markov process.*

Proof. Since W_s is $\mathcal{F}_s$ measurable and $W_t - W_s$ is independent of $\mathcal{F}_s$ for $0 \le s \le t$, we have

$$\begin{aligned}\mathbb{E}[g(W_t)|\mathcal{F}_s] &= \mathbb{E}[g(W_t - W_s + W_s)|\mathcal{F}_s] \\ &= \mathbb{E}[g(W_t - W_s + x)]\big|_{x=W_s} \\ &= \mathbb{E}[g(W_t)|W_s]\end{aligned}$$

□

Definition. A process $(X_t)_{t\ge0}$ is *Gaussian* iff the random variables $X_{t_1}, \ldots, X_{t_n}$ are jointly normal for all $0 \le t_1 \le \ldots \le t_n$, i.e. the random variable $\sum_{i=1}^n a_i X_{t_i}$ is normally distributed for all constants $a_1, \ldots, a_n$.

Theorem. *The following are equivalent*

1. $(W_t)_{t\ge0}$ *is a Brownian motion*
2. $(W_t)_{t\ge0}$ *is a Gaussian process such that*
   - $t \mapsto W_t$ *is continuous*

- $\mathbb{E}[W_t] = 0$ *for all* $t \geq 0$
- $\mathbb{E}[W_s W_t] = s$ *for all* $0 \leq s \leq t$

Proof. Suppose $(W_t)_{t\geq 0}$ is a Brownian motion. Fix $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ and $a_1, \ldots, a_n$. Note

$$\sum_{i=1}^{n} a_i W_{t_i} = \sum_{i=1}^{n} b_i (W_{t_i} - W_{t_{i-1}})$$

where $b_k = \sum_{i=k}^{n} a_i$. Since $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent normals, and the linear combination of independent normals is normal, we have that $(W_t)_{t\geq 0}$ is Gaussian with $\mathbb{E}[W_t] = \mathbb{E}[W_0] = 0$ and

$$\begin{aligned}
\mathbb{E}[W_s W_t] &= \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] \\
&= \mathrm{Var}(W_s) + \mathbb{E}(W_s)\mathbb{E}(W_t - W_s) \\
&= 0.
\end{aligned}$$

for $0 \leq s \leq t$, since W_s and $W_t - W_s$ are independent.

Conversely, suppose $(W_t)_{t\geq 0}$ is a continuous Gaussian process such that $\mathbb{E}[W_t] = 0$ and $\mathbb{E}[W_s W_t] = s$ for all $0 \leq s \leq t$. Then for $0 \leq u \leq s \leq t$ we have

$$\begin{aligned}
\mathrm{Cov}(W_u, W_t - W_s) &= \mathbb{E}[W_u W_t] - \mathbb{E}[W_u W_s] \\
&= u - u = 0
\end{aligned}$$

By normality, the increment $W_t - W_s$ is independent of W_u. By Gaussianity, the increment is independent of $(W_u)_{0\leq u\leq s}$.

Theorem. *Let $(W_t)_{t\geq 0}$ be a Brownian motion. Then each of the following processes are also Brownian motions.*

1. $\tilde{W}_t = cW_{t/c^2}$, *for any constant* $c \neq 0$.
2. $\tilde{W}_t = W_{t+T} - W_T$ *for any constant* $T \geq 0$.
3. $\tilde{W}_0 = 0$ *and* $\tilde{W}_t = tW_{1/t}$ *for* $t > 0$.

Proof. Check that each process is a continuous mean-zero Gaussian process with the correct covariance. [For 3, we technically need the Brownian law of large number $\frac{W_s}{s} \to 0$ as $s \to \infty$ to prove continuity of $\tilde{W}$ at $t = 0$.] □

2 Reflection principle

Theorem. *Let $(W_t)_{t\geq 0}$ be a Brownian motion, and $T_a = \inf\{t \geq 0 : W_t = a\}$. Then $T_a < \infty$ almost surely.*

Proof. Consider the case $a > 0$. (The case $a < 0$ is similar.) We must show

$$\sup_{t\geq 0} W_t > a \text{ almost surely}$$

By Brownian scaling, for any $c > 0$ and $0 < a < b$, we have

$$\begin{aligned}\mathbb{P}(a < \sup_{t\geq 0} W_t < b) &= \mathbb{P}(a < \sup_{t\geq 0} cW_{t/c^2} < b) \\ &= \mathbb{P}(a/c < \sup_{s\geq 0} W_s < b/c) \quad \text{letting } t/c^2 = \\ &\to 0\end{aligned}$$

by sending $c \uparrow \infty$. Since $Z = \sup_{t\geq 0} W_t \geq W_0 = 0$, we have shown that $Z \in \{0, +\infty\}$ almost surely.

Let $\hat{Z} = \sup_{t\geq 1}(W_t - W_1)$. Note Z and $\hat{Z}$ have the same distribution, so $\hat{Z} \in \{0, +\infty\}$ almost surely.

Note that $\{\hat{Z} = \infty\} = \{Z = \infty\}$ since $\sup_{0\leq t\leq 1} W_t$ is finite by the continuity of Brownian motion. Hence

$$\begin{aligned}p = \mathbb{P}(Z = 0) &= \mathbb{P}(Z = 0, \hat{Z} = 0) \\ &\leq \mathbb{P}(W_1 \leq 0, \hat{Z} = 0) \quad \text{as Z = 0 implies W\_t <= 0 for all t} \\ &= \frac{1}{2}\mathbb{P}(\hat{Z} = 0) = \frac{1}{2}p\end{aligned}$$

so $p = 0$. Hence $\sup_{t\geq 0} W_t = \infty$ almost surely. □

Theorem. *Let $(W_t)_{t\geq 0}$ be a Brownian motion and T a finite stopping time. The process $W_{t+T} - W_T$ is also a Brownian motion independent of $(W_t)_{0\leq t\leq T}$*

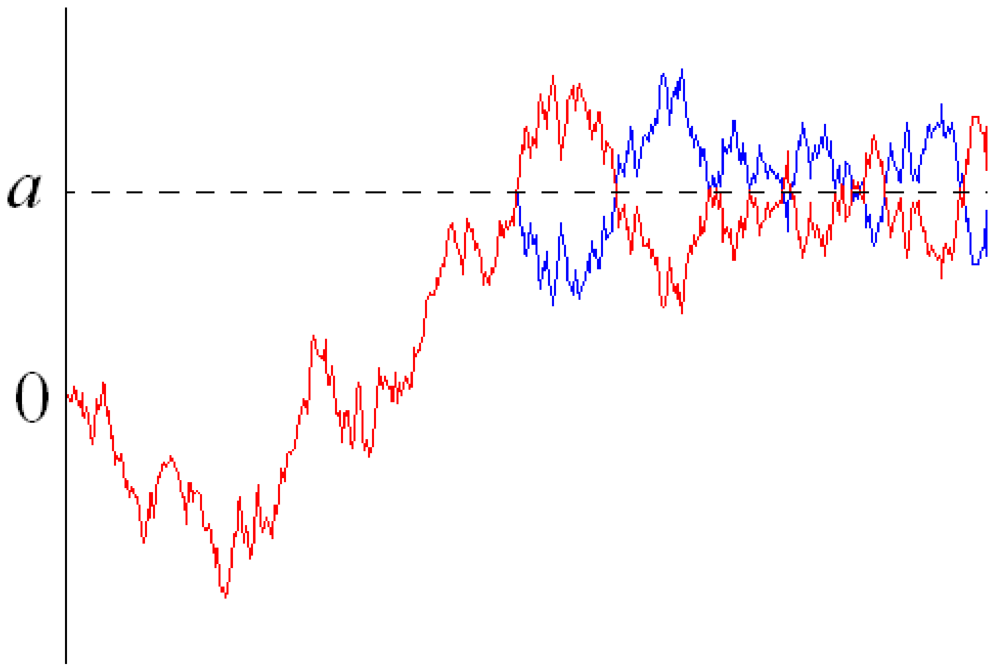
Proof. Omitted. The idea is Brownian motion is a *strong* Markov process. □

Applying this with the finite stopping time T_a together with the symmetry of Brownian motion, we have

Theorem (Reflection principle)**.** *Let $(W_t)_{t\geq 0}$ be a Brownian motion and let*

$$\tilde{W}_t = \begin{cases} W_t & \text{if } 0 \leq t < T_a \\ 2a - W_t & \text{if } t \geq T_a \end{cases}$$

Then $(\tilde{W}_t)_{t\geq 0}$ is a Brownian motion.



Reflection principle: Key formula

$$\boxed{\mathbb{P}(\max_{0\le s\le t} W_s \ge a, W_t \le b) = \mathbb{P}(W_t \ge 2a-b) \text{ for } a \ge 0, b \le a}$$

Proof. We have

$$\begin{aligned}\mathbb{P}(\max_{0\le s\le t} W_s \ge a, W_t \le b) &= \mathbb{P}(\tilde{W}_t \ge 2a-b)\\ &= \mathbb{P}(W_t \ge 2a-b)\end{aligned}$$

□