# II. FIRST-ORDER LINEAR DIFFERENTIAL EQUATIONS

We now start to investigate differential equations proper. As noted in the Introduction, a differential equation is an equation involving derivatives of the dependent variable with respect to the independent variable(s).

Unlike regular, algebraic equations, the solution of a differential equation is a *function* that satisfies the equation. To obtain a unique solution requires specifying further suitable *boundary conditions*.

In this part of the course we shall consider *first-order differential equations*, where the highest derivative that appears is, for example, $dy/dx$.

## 1 Exponential function

As we shall see, the exponential function plays a key role in the solution of linear, first-order equations (we shall define linearity shortly). We therefore begin with a brief recap of the properties of the exponential function.

**Definition** (Exponential function). The exponential function is defined by the infinite series

$$\begin{aligned}\exp(x) &\equiv 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\,. \qquad (1)\end{aligned}$$

Using the binomial theorem (see Examples Sheet 1 for

an alternative approach) this can also be written as

$$\begin{aligned}\exp(x) &= \lim_{k\to\infty}\left(1+\frac{x}{k}\right)^k \qquad (2)\\ &= \lim_{k\to\infty}\left[1+k\left(\frac{x}{k}\right)+\frac{k(k-1)}{2!}\left(\frac{x}{k}\right)^2+\cdots\right]\\ &= 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\,.\end{aligned}$$

Differentiating the series definition (1) term by term, we see that

$$\begin{aligned}\frac{d\exp(x)}{dx} &= 1+2\times\frac{x}{2!}+3\times\frac{x^2}{3!}+\cdots\\ &= \exp(x)\,. \qquad (3)\end{aligned}$$

Since $\exp(0)=1$, we can alternatively *define* the exponential function as the unique solution of the differential equation

$$\frac{df}{dx} = f(x) \quad \text{with } f(0)=1.$$

It follows that

$$\int_1^{\exp(x)}\frac{dy}{y} = x\,. \qquad (4)$$

From this follows one of the main properties of exponentials:

$$\exp(x_1+x_2) = \exp(x_1)\exp(x_2)\,. \qquad (5)$$

To see this, note that, from Eq. (4),

$$x_1+x_2 = \int_1^{\exp(x_1)}\frac{dy}{y}+\int_1^{\exp(x_2)}\frac{dy}{y}\,.$$

If we now make the variable substitution $u=\exp(x_1)y$ in the second term on the right, we have

$$\begin{aligned}x_1+x_2 &= \int_1^{\exp(x_1)}\frac{dy}{y}+\int_{\exp(x_1)}^{\exp(x_1)\exp(x_2)}\frac{du}{u}\\ &= \int_1^{\exp(x_1)\exp(x_2)}\frac{dy}{y}\,. \qquad (6)\end{aligned}$$

However, from Eq. (4) we also have

$$x_1 + x_2 = \int_1^{\exp(x_1+x_2)} \frac{dy}{y} \,.$$

Since the right-hand side of this expression must equal the right-hand side of Eq. (6) for all $x_1$ and $x_2$, we establish the property (5). (It may also be shown rather more directly from the limit definition in Eq. 2.)

The property in Eq. (5) is reminiscent of powers. Combining with $\exp(0) = 1$, we can write

$$\exp(x) = e^x \,, \tag{7}$$

where the value of $e$ is

$$e = \exp(1) = \lim_{k\to\infty} \left(1 + \frac{1}{k}\right)^k = 2.718\ldots \,.$$

The inverse function of $\exp(x)$ is denoted $\ln(x)$, so that

$$\exp(\ln x) = e^{\ln x} = x \,.$$

This is sometime written as the logarithm to the base-$e$, i.e., $\ln x = \log_e x$ and referred to as the *natural logarithm* of $x$.

The natural logarithm allows us to write[1] (for real $a > 0$)

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a} \,,$$

from which it follows that

$$\frac{da^x}{dx} = (\ln a) e^{x \ln a} = (\ln a) a^x \,.$$

The exponential function plays a prominent role in the analysis of differential equations since it is an *eigenfunction* of the derivative operator.

[1] While rational powers of $a$ are defined directly in terms of repeated multiplication and roots, e.g., $a^{2/3}$ is the (positive) cube root of $a \times a$, this equation essentially defines what it means to raise a number to an irrational power. An alternative approach is to define $a^x$ in terms of the limit of a sequence of terms $a^{x_n}$, where the $x_n$ are rational but have $x$ as their limit.

**Definition** (Eigenfunction). An *eigenfunction* of the derivative operator is a function that is unchanged, up to a multiplicative scaling by the *eigenvalue*, under the action of the operator. That is,

$$\frac{df}{dx} = \lambda f(x)\,,$$

where $f(x)$ is the eigenfunction and $\lambda$ is the eigenvalue[2].

The eigenfunctions of $d/dx$ are the functions $e^{\lambda x}$ since

$$\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}\,.$$

## 2 First-order linear differential equations

Differential equations of this form have the following properties.

- **Linear** – a differential equation is linear if the dependent variable, $y$ say, and its derivatives only appear linearly.
- **First order** – a differential equation is first order if the higest derivative that appears is first order, i.e., $dy/dx$.

### 2.1 Homogeneous, first-order linear differential equations

We shall initially specialise further to consider *homogeneous* equations with *constant coefficients*.

- **Homogeneous** – a differential equation in which all terms involve the dependent variable (e.g., $y$) or its derivatives, so that $y = 0$ is a solution.
- **Constant coefficients** – a differential equation has constant coefficients if the independent variable (e.g., $x$) does not appear explicitly.

[2]The terminology "eigen" is from the German for "own".

**Example**. Consider the first-order, linear, homogeneous differential equation

$$5\frac{dy}{dx} - 3y = 0\,. \tag{8}$$

Let us try a solution $y = Ae^{\lambda x}$; then

$$\frac{dy}{dx} = A\lambda e^{\lambda x} = \lambda y\,,$$

so to be a solution we require $5\lambda - 3 = 0$. This is an example of a *characteristic equation* and the solution is $\lambda = 3/5$. Since this is a linear, homogeneous equation, the solution $y = Ae^{3x/5}$ holds for any $A$.

Generally, for any linear, homogeneous differential equation (so not necessarily first order), any constant multiple of a solution is also a solution.

Moreover, it can be shown that an $n$th-order linear differential equation has precisely $n$ independent solutions. Specialising to the case of the first-order linear equation (8), we see that $y = Ae^{3x/5}$ is the *general solution*. A specific unique solution is obtained by specifying a suitable boundary condition for the dependent variable. For example, the value of $y$ at $x = 0$ determines the constant $A$.

#### 2.1.1 Discrete equations

It is interesting to compare the solution of Eq. (8) with that of a related *discrete equation*. A discrete equation involves a function evaluated at a discrete set of points.

Suppose we have $5dy/dx - 3y = 0$ and the boundary condition $y(0) = y_0$. We know that the solution is

$$y(x) = y_0 e^{3x/5}\,.$$

Consider now an approximate solution to this differential equation, whereby we approximate the derivative by

a finite difference. In particular, consider discretising the equation at points $\{x_n\}$, spaced by $h$. The values of $y$ at these points are $\{y_n\}$ and we approximate the derivative at $x_n$ by

$$\left.\frac{dy}{dx}\right|_{x_n} \approx \frac{y_{n+1} - y_n}{h} \,.$$

(This is called the *forward Euler scheme* – it is not particularly good for numerical analysis and better schemes do exist, but it is fine to illustrate the key idea here.) The original equation (8) in discrete form becomes

$$5\left(\frac{y_{n+1} - y_n}{h}\right) - 3y_n \approx 0 \quad \Rightarrow \quad y_{n+1} \approx \left(1 + \frac{3h}{5}\right) y_n \,. \tag{9}$$

The final relation in Eq. (9) is an example of a *recurrence relation*. If we apply this repeatedly, we find

$$y_n = \left(1 + \frac{3h}{5}\right) y_{n-1} = \left(1 + \frac{3h}{5}\right)^2 y_{n-2} = \left(1 + \frac{3h}{5}\right)^n y_0 \,.$$

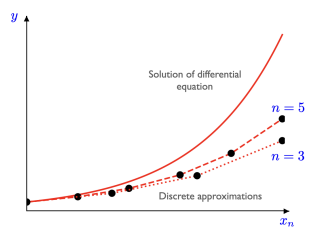
If we now suppose that $x_0 = 0$ and $x_n = nh = x$, i.e., we take $n$ steps to go from $x = 0$ to the point of interest, $x$, we can write

$$y_n = y_0 \left(1 + \frac{3}{5}\frac{x}{n}\right)^n \,.$$

In the limit as $n \to \infty$, we expect this to agree with the exact solution $y(x) = y_0 e^{3x/5}$. This is indeed the case since, recalling Eq. (2), we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_0 \left(1 + \frac{3x/5}{n}\right)^n = y_0 \exp(3x/5) \,.$$

As shown in the figure to the right, the larger $n$ is, the larger the value of $y_n$ at the given point $x$.



#### 2.1.2 Series solution

At this point, it is useful to introduce and illustrate a powerful technique for solving differential equations that we shall have much more to say about later in the course. The idea is to look for solutions in the form of an infinite power series,[3] i.e.,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \,. \tag{10}$$

Substituting this expansion into the differential equation determines the coefficients $a_n$.

**Example**. Consider again

$$5\frac{dy}{dx} - 3y = 0 \,. \tag{11}$$

Differentiating Eq. (10) gives

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \,.$$

Multiplying Eq. (11) by $x$ (for convenience), we have terms involving

$$\begin{aligned} xy' &= \sum_{n=1}^{\infty} n a_n x^n \,, \\ xy &= \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{m=1}^{\infty} a_{m-1} x^m \,, \end{aligned}$$

where in the second line we have let $m = n + 1$. Relabelling $m \to n$ in this summation, and substituting into Eq. (11) (after multiplying through by $x$), we find

$$\begin{aligned} & 5\sum_{n=1}^{\infty} n a_n x^n - 3\sum_{n=1}^{\infty} a_{n-1} x^n = 0 \\ \Rightarrow \quad & \sum_{n=1}^{\infty} x^n \left(5 n a_n - 3 a_{n-1}\right) = 0 \,. \end{aligned}$$

[3] We shall see later that not all differential equations admit solutions of this form, but significant classes of equations do (or at least a close generalisation).

Since this must hold for all $x$, we must have

$$5na_n - 3a_{n-1} = 0 \quad \Rightarrow \quad a_n = \frac{3}{5n}a_{n-1} \quad (n \geq 1)\,.$$

Iterating this recursion relation, we have

$$a_n = \frac{3}{5n}a_{n-1} = \left(\frac{3}{5}\right)^2 \frac{1}{n(n-1)}a_{n-2} = \cdots = \left(\frac{3}{5}\right)^n \frac{1}{n!}a_0\,,$$

so that

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{3x}{5}\right)^n = a_0 e^{3x/5}\,.$$

In the final term we have identified the infinite series as the power series expansion of $\exp(3x/5)$; see Eq. (1).

### 2.2 Forced (inhomogeneous) equations

So far we have considered homogeneous differential equations. However, differential equations can also involve terms that are explicit functions of the independent variable and do not include the dependent variable nor its derivatives. Such equations are called *inhomogeneous* or *forced* equations and $y = 0$ is no longer a (trivial) solution.

We shall consider two simple, but important, types of forcing terms that break homogeneity.

#### 2.2.1 Constant forcing

Constant forcing involves introducing a constant term in a differential equation.

**Example**. Consider

$$5\frac{dy}{dx} - 3y = 10\,,$$

where now we have added the constant term on the right-hand side. The general method for solving such forced, linear equations is as follows.

1. Find any solution of the forced equation. This is called the *particular integral* and we shall denote it by $y_p(x)$. This may require some guesswork. For our example, we might spot that there is a solution with $y = \text{const.}$, in which case we would have

$$y_p(x) = -10/3\,.$$

2. Now write the general solution in the form

$$y(x) = y_p(x) + y_c(x)\,,$$

involving the particular integral and a *complementary function* $y_c(x)$. Since the differential equation is linear in $y$ and its derivatives, the complementary function must satisfy the homogeneous equation:

$$5\frac{dy_c}{dx} - 3y_c = 0 \quad \Rightarrow \quad y_c(x) = Ae^{3x/5}\,.$$

3. Combining, we have the full general solution

$$y(x) = -\frac{10}{3} + Ae^{3x/5}\,.$$

Any boundary conditions may now be applied (to the full solution $y$) to determine the constant $A$.

This method of solving linear, forced equations is general and is not restricted to first-order equations (with constant coefficients).

#### 2.2.2 Eigenfunction forcing

A second particularly simple form of forcing is when the forcing is an eigenfunction of the underlying differential operator.

**Example**. Consider a radioactive material, in which isotope $A$ decays into isotope $B$ at a rate proportional to the number $a(t)$ of remaining nuclei of $A$, and $B$ decays into $C$ at a rate proportional to the number $b(t)$ of remaining nuclei of $B$. Determine $b(t)$.

We have

$$\frac{da}{dt} = -k_a a\,,$$
$$\frac{db}{dt} = k_a a - k_b b\,,$$

where $k_a$ and $k_b$ are the appropriate rate constants. We can solve the first equation for $a(t)$ directly to obtain

$$a(t) = a_0 e^{-k_a t}\,,$$

where $a_0$ is the number of $A$ nuclei at $t = 0$. Substituting into the rate equation for $b(t)$ gives

$$\frac{db}{dt} + k_b b = k_a a_0 e^{-k_a t}\,. \tag{12}$$

The forcing term in Eq. (12), being an exponential function, is an eigenfunction of the differential operator on the left-hand side. This suggests we try a particular integral

$$b_p = Ce^{-k_a t},$$

for some suitable choice of the constant $C$. Substituting $b_p$ into Eq. (12) we obtain

$$-k_a C + k_b C = k_a a_0 \quad \Rightarrow \quad C = \frac{k_a}{k_b - k_a} a_0\,,$$

provided $k_a \neq k_b$.

As before, we then consider the general solution $b = b_c + b_p$, where $b_c$ is the solution of the homogeneous equation:

$$\frac{db_c}{dt} + k_b b_c = 0 \quad \Rightarrow \quad b_c = De^{-k_b t}\,,$$

for some constant $D$, and so

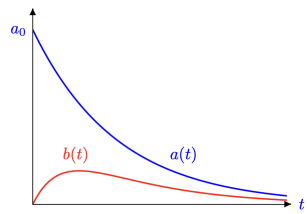
$$b(t) = \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + De^{-k_b t}\,.$$

A particular situation of interest is when $b = 0$ at $t = 0$, i.e., the isotope $B$ only appears due to decay of isotope

$A$. In this case,

$$b(t) = \frac{k_a}{k_b - k_a} a_0 \left(e^{-k_a t} - e^{-k_b t}\right) ,$$
$$\Rightarrow \quad \frac{b(t)}{a(t)} = \frac{k_a}{k_b - k_a} \left(1 - e^{(k_a - k_b)t}\right) .$$

Typical time variations of $a(t)$ and $b(t)$ are shown in the figure to the right. Analyses of this type allow rocks and other materials to be dated by measuring the ratio of isotopes (e.g., carbon dating).




The solution we obtained in this example is not valid for $k_a = k_b$. For this case, we could proceed by guessing a suitable alternative particular integral $b_p(t)$. However, we shall see below an alternative approach that will also inform the choice of a suitable particular integral.

### 2.3 Non-constant coefficients

So far we have considered linear, first-order differential equations with constant coefficients. We now drop the last restriction so that the coefficients of $y$ and $dy/dx$ are allowed to be functions of $x$.

Consider the general form of a first-order linear differential equation:

$$a(x)\frac{dy}{dx} + b(x)y = c(x) .$$

Dividing through by $a(x)$, we obtain the *standard form*:

$$\frac{dy}{dx} + p(x)y = f(x) . \tag{13}$$

We can always solve equations of this form by multiplying through by an *integrating factor* $\mu(x)$:

$$\mu\frac{dy}{dx} + (\mu p)y = \mu f . \tag{14}$$

The idea is to choose the integrating factor so that the left-hand side is the derivative $d(\mu y)/dx$ and the equation can be integrated directly. From the product rule, we require

$$\frac{d\mu}{dx} = \mu p \quad \Rightarrow \quad \frac{1}{\mu}\frac{d\mu}{dx} = p\,.$$

Integrating with respect to $x$ we have

$$\int p\,dx = \int \frac{1}{\mu}\frac{d\mu}{dx}\,dx = \ln\mu\,.$$

Therefore, the integrating factor is

$$\mu(x) = \exp\left[\int^x p(u)du\right]\,, \tag{15}$$

which is unique up to an irrelevant constant factor.

Since, by construction, Eq. (14) is equivalent to

$$\frac{d}{dx}(\mu y) = \mu f\,,$$

we have

$$\mu(x)y(x) = \int^x \mu(u)f(u)du\,,$$

from which $y(x)$ can be determined straightforwardly.

**Example**. Consider

$$x\frac{dy}{dx} + (1-x)y = 1\,,$$

or, in standard form,

$$\frac{dy}{dx} + \left(\frac{1}{x} - 1\right)y = \frac{1}{x}\,.$$

It follows that $p(x) = 1/x - 1$ and so, from Eq. (15),

$$\begin{aligned}\mu(x) &= \exp\left[\int^x p(u)\,du\right]\\ &= \exp\left[\int^x \left(\frac{1}{u} - 1\right)\,du\right]\\ &= \exp\left(\ln x - x\right)\\ &= xe^{-x}\,.\end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{d}{dx}\left(xe^{-x}y\right) = e^{-x} \\ \Rightarrow \quad & xe^{-x}y = -e^{-x} + C \\ \Rightarrow \quad & y = -\frac{1}{x} + \frac{C}{x}e^{x}\,, \end{aligned}$$

where $C$ is a constant to be determined from initial or boundary conditions.

In particular, if we require $y(x)$ to be finite as $x \to 0$ we must have $C = 1$, and so

$$y = \frac{e^x - 1}{x}\,.$$

#### 2.3.1 Radioactive decay example revisited

It is instructive to reconsider the example of radioactive decay discussed above, now using the method of an integrating factor. In particular, this will allow us to handle easily the case $k_a = k_b$.

Recall Eq. (12), which we repeat here for convenience:

$$\frac{db}{dt} + k_b b = k_a a_0 e^{-k_a t}\,.$$

This is already in standard form, with $p(t) = k_b$. It follows that the integrating factor

$$\mu(t) = \exp\left[\int^t k_b du\right] = e^{k_b t}\,,$$

and so multiplying through in Eq. (12), we have

$$\frac{d}{dt}\left(e^{k_b t}b\right) = k_a a_0 e^{(k_b - k_a)t}\,. \tag{16}$$

We now consider the two cases, $k_a \neq k_b$ and $k_a = k_b$, separately.

1. If $k_a \neq k_b$, the right-hand side of Eq. (16) still varies with $t$. Integrating, we find

$$\begin{aligned} e^{k_b t} b &= \frac{k_a}{k_b - k_a} a_0 e^{(k_b - k_a)t} + D\,, \\ \Rightarrow \quad b(t) &= \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + D e^{-k_b t}\,, \end{aligned}$$

   exactly as before.

2. If $k_a = k_b = k$, the right-hand side of Eq. (16) is independent of $t$. Integrating in this case, we have

$$\begin{aligned} e^{kt} b &= k a_0 t + D\,, \\ \Rightarrow \quad b(t) &= k a_0 t e^{-kt} + D e^{-kt}\,. \end{aligned}$$

Note that in the case $k_a = k_b = k$, an appropriate particular integral of Eq. (12) is

$$b_p(t) = k a_0 t e^{-kt}\,,$$

rather than being simply $b_p(t) \propto e^{-kt}$.