Stochastic Financial Models 11

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1 Conditional expectation

Set up: Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Given a sigma-algebra $\mathcal{G} \subseteq \mathcal{F}$, how to define $\mathbb{E}(X|\mathcal{G})$?
Motivation. Conditional expectation given an event

$$\mathbb{E}(X|G) = \frac{\mathbb{E}(X\mathbb{1}_G)}{\mathbb{P}(G)}$$

where X is integrable (i.e. $\mathbb{E}(|X|) < \infty$) and $\mathbb{P}(G) > 0$.
Motivation. Conditional expectation given a discrete random variable.

Suppose Y takes values in $\{y_1, y_2, \ldots\}$ and X in integrable. Let

$$f(y) = \mathbb{E}(X|Y = y)$$

Then we define

$$\mathbb{E}(X|Y) = f(Y)$$

Note that in this set-up $\mathbb{E}(X|Y)$ is $\sigma(Y)$-measurable. Also, it satisfies the *Projection property:* For any $\sigma(Y)$-measurable random event G we have

$$\mathbb{E}[X\mathbb{1}_G] = \mathbb{E}[\mathbb{E}[X|Y]\mathbb{1}_G].$$

Proof of the projection property for conditional expectation given a discrete random variable: By measurability there exists a subset $B \subseteq \{y_1, y_2, \ldots\}$ such that $G = \{Y \in B\}$. By the law of total probability

$$\begin{aligned}
\mathbb{E}[f(Y)\mathbb{1}_{\{Y\in B\}}] &= \sum_i \mathbb{P}(Y = y_i)\mathbb{E}(X|Y = y_i)\mathbb{1}_{\{y_i\in B\}} \\
&= \sum_{i:y_i\in B} \mathbb{E}(X\mathbb{1}_{\{Y=y_i\}}) \\
&= \mathbb{E}[X\mathbb{1}_{\{Y\in B\}}]
\end{aligned}$$

since

$$\sum_{i:y_i\in B} \mathbb{1}_{\{Y=y_i\}} = \mathbb{1}_{\{Y\in B\}}$$

We now use the projection property as defining property of conditional expectation:

Definition. The conditional expectation of an integrable random variable X given a sigma-algebra $\mathcal{G}$ is any $\mathcal{G}$-measurable integrable random variable Z such that

$$\mathbb{E}(X\mathbb{1}_G) = \mathbb{E}(Z\mathbb{1}_G)$$

for all events $G \in \mathcal{G}$.

Proposition (Existence and uniqueness of conditional expectations)**.** *Let X be integrable and $\mathcal{G}$ be a sigma-algebra. There exists a unique conditional expectation of X given $\mathcal{G}$.*

Proof. Existence requires some analysis. But uniqueness is straight-forward. Let Z_0, Z_1 be two conditional expectations of X given $\mathcal{G}$. By definition, this means for all $G \in \mathcal{G}$ we have

$$\mathbb{E}[Z_0\mathbb{1}_G] = \mathbb{E}[X\mathbb{1}_G] = \mathbb{E}[Z_1\mathbb{1}_G] \tag{$*$}$$

Now note $\{Z_0 < Z_1\}$ is in $\mathcal{G}$ since Z_1 and Z_0 are both $\mathcal{G}$-measurable by definition. Of course

$$(Z_1 - Z_0)\mathbb{1}_{\{Z_0<Z_1\}} \geq 0$$

But by equation $(*)$ we have

$$\mathbb{E}[(Z_1 - Z_0)\mathbb{1}_{\{Z_0<Z_1\}}] = 0$$

so by the pigeon-hole principle we have $(Z_1 - Z_0)\mathbb{1}_{\{Z_0<Z_1\}} = 0$ almost surely. That is to say, we have $Z_1 - Z_0 \leq 0$ almost surely. Now by symmetry we also have $Z_1 - Z_0 \geq 0$ almost surely, and hence $Z_1 = Z_0$ almost surely as claimed. □

Notation: The conditional expectation of X given $\mathcal{G}$ is denoted $\mathbb{E}(X|\mathcal{G})$. In the special case where $\mathcal{G} = \sigma(Y)$ for a random variable Y, we write $\mathbb{E}(X|Y)$ for $\mathbb{E}(X|\sigma(Y))$.

Remark. Note that we have already checked that our new definition of $\mathbb{E}(X|Y)$ agrees with our old definition in the case where Y is discrete.

The following gives an interpretation of conditional expectation given a sigma-algebra:

Proposition (Mean squared error minimisation)**.** *Suppose X is square-integrable and $\mathcal{G}$ a sigma-algebra. Then $\mathbb{E}(X|\mathcal{G})$ minimises the quantity*

$$\mathbb{E}[(X - Z)^2]$$

among all $\mathcal{G}$-measurable square-integrable Z.

Sketch of proof. By measure theory, the following extended projection property holds true. For any square-integrable $\mathcal{G}$-measurable random variable Y we have

$$\mathbb{E}[XY] = \mathbb{E}\left(\mathbb{E}[X|\mathcal{G}]Y\right)$$

Now given Z, let $Y = \mathbb{E}[X|\mathcal{G}] - Z$.

$$\begin{aligned}
\mathbb{E}[(X - Z)^2] &= \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}] + Y)^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2] + 2\mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])Y] + \mathbb{E}[Y^2] \\
&= \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2] + \mathbb{E}[Y^2] \\
&\geq \mathbb{E}[(X - \mathbb{E}[X|\mathcal{G}])^2]
\end{aligned}$$

since Y is $\mathcal{G}$-measurable, where we have used the extension of the projection property discussed above. □

Remark. The above proof may look familiar – this is exactly how the Rao–Blackwell theorem from IB Statistics is proven.