Algebraic Topology

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https://www.dpmms.cam.ac.uk/∼or257/teaching/notes/at.pdf

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Chapter 1

Introduction

Algebraic Topology is the art of turning existence questions in topology into existence questions in algebra, and then showing that the algebraic object cannot exist: this then implies that the original topological object cannot exist. This procedure usually has a loss of information, and if the algebraic object *does* exist it does not typically allow us to show that the original topological object does.

Many interesting topological problems can be expressed in the following form.

The extension problem. Let X be a topological space, $A \subset X$ a subspace, and $f : A \to Y$ a continuous function into a third space Y. Does there exist a continuous function $F : X \to Y$ so that $F|_A = f$?

The most basic instance of this problem arises when taking $X = D^n$ to be the n-dimensional disc, $A = Y = S^{n-1}$ to be the $(n-1)$-dimensional sphere, and $f = \mathrm{Id}_{S^{n-1}} : S^{n-1} \to S^{n-1}$ to be the identity. The problem then asks whether there is a continuous function from the disc to its boundary which fixes each point on the boundary. In the course we shall prove the following theorem.

Theorem. There is no continuous function $F : D^n \to S^{n-1}$ such that the composition

$$S^{n-1} \xrightarrow{\text{incl.}} D^n \xrightarrow{F} S^{n-1}$$

is the identity.

You may find it intuitively quite clear that there is no such F, but the difficulty in proving this is apparent: while it is obvious that various naïve choices of $F : D^n \to S^{n-1}$ do not work, we must show that *no* choice can work, and in principle there are a great many potential choices with very little mathematical structure to work with. The machinery of Algebraic Topology translates proving the theorem above to proving:

Theorem. There is no group homomorphism $F : \{0\} \to \mathbb{Z}$ such that the composition

$$\mathbb{Z} \xrightarrow{0} \{0\} \xrightarrow{F} \mathbb{Z}$$

is the identity.

I hope you will agree that this algebraic theorem is considerably easier to prove. There are several other classical results in mathematics which we shall be able to rephrase as an extension problem—or something like it—and hence solve using the machinery of Algebraic Topology. One is that the notion of dimension is well-defined:

Theorem. If there is a homeomorphism $\mathbb{R}^n \cong \mathbb{R}^m$, then $n = m$.

Another is the fundamental theorem of algebra:

Theorem. Any non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

This introduction has already mentioned several times the *machinery* of Algebraic Topology. It is a characteristic of this subject that much foundational work must be put in before any profit can be taken, but the foundational work then renders interesting and apparently difficult problems almost trivial. Take this as consolation during the drier parts of the course.

1.1 Some recollections and conventions

In order to write fewer words, we make the following convention.

Definition 1.1.1. A continuous function will be called a **map**.

We will need to repeatedly justify why various formulas that we write down do indeed define maps, i.e. are continuous. Almost all of these situations will be dealt with using the following convenient lemma.

Lemma 1.1.2 (Gluing lemma). *Let $f : X \to Y$ be a function between topological spaces, and $C, K \subset X$ be closed subsets such that $C \cup K = X$. Then f is continuous if and only if $f|_C : C \to Y$ and $f|_K : K \to Y$ are continuous.*

Proof. As a restriction of a continuous map is continuous, the condition is clearly necessary. To show it is sufficient, we use the characterisation of continuity in terms of closed sets: f is continuous if and only if for each closed subset $D \subset Y$, $f^{-1}(D)$ is closed.

In the situation of the lemma we have $f^{-1}(D) \cap C = (f|_C)^{-1}(D)$, which is closed in C as $f|_C$ is continuous, and so is closed in X because C is closed in X. Similarly $f^{-1}(D) \cap K$ is closed in X. But then $f^{-1}(D) = f^{-1}(D) \cap (C \cup K) = (f^{-1}(D) \cap C) \cup (f^{-1}(D) \cap K)$ is a finite union of closed sets, and so closed. □

Later in the course we will need the following standard lemma.

Lemma 1.1.3 (Lesbegue number lemma). *Let (X, d) be a metric space which is compact. For any open cover $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ of X there exists a $\delta > 0$ such that each ball $B_\delta(x) \subset X$ is contained entirely in some U_α.*

Proof. Suppose no such δ exists. Then for each $n \in \mathbb{N}$ there exist an $x_n \in X$ such that $B_{1/n}(x_n)$ is not contained entirely in some U_α. As X is compact the set $\{x_n\}_{n\in\mathbb{N}}$ has a limit point, y. This lies in some U_α, so there exists an $r > 0$ such that $B_r(y)$ also lies in U_α. But now let $N > 0$ be such that

(i) $\frac{1}{N} < \frac{r}{2}$, and

(ii) $d(x_N, y) < \frac{r}{2}$.

Then $B_{1/N}(x_N)$ is contained inside $B_r(y)$, and hence inside U_α, a contradiction. □

1.2 Cell complexes

A convenient way to construct topological spaces, which we shall see interacts well with the tools of Algebraic Topology, is the notion of attaching cells.

Definition 1.2.1. For a space X and a map $f : S^{n-1} \to X$, the space obtained by **attaching an n-cell to X along f** is the quotient space

$$X \cup_f D^n := (X \sqcup D^n)/\sim$$

where $\sim$ is the equivalence relation generated by $x \sim f(x)$ for $x \in S^{n-1} \subset D^n$.

Definition 1.2.2. A (finite) **cell complex** is a space X obtained by

(i) starting with a finite set X^0 with the discrete topology, called the **0-skeleton**,

(ii) having defined the $(n-1)$-skeleton X^{n-1}, form the **n-skeleton** X^n by attaching a collection of n-cells along finitely many maps $\{f_\alpha : S^{n-1} \to X^{n-1}\}_{\alpha \in I}$, i.e.

$$X^n = \left(X^{n-1} \sqcup \bigsqcup_{\alpha \in I} D^n_\alpha \right) / x \in S^{n-1}_\alpha \subset D^n_\alpha \sim f_\alpha(x) \in X^{n-1},$$

(iii) stop at some finite stage k, so $X = X^k$; this k is the **dimension** of X.

Chapter 2

Homotopy and the fundamental group

2.1 Homotopy

As we shall use the unit interval $[0,1] \subset \mathbb{R}$ repeatedly, we denote it by I.

Definition 2.1.1 (Homotopy)**.** Let $f, g : X \to Y$ be maps[1]. A **homotopy from** f **to** g is a map $H : X \times I \to Y$ such that

$$H(x,0) = f(x) \qquad \text{and} \qquad H(x,1) = g(x).$$

If such an H exists, we say that f **is homotopic to** g, and write $f \simeq g$. If we wish to record which homotopy between these maps we have in mind, then we write $f \simeq_H g$.

If $A \subset X$ is a subset, then we say that the homotopy H is **relative to** A if in addition

$$H(a,t) = f(a) = g(a) \text{ for all } a \in A \text{ and all } t \in I.$$

In this case we write $f \simeq g$ rel A.

Proposition 2.1.2. *For spaces X and Y and a subspace $A \subset X$, the relation "homotopic relative to A" on the set of maps from X to Y is an equivalence relation.*

Proof. We must verify three properties.

(i) For a map $f : X \to Y$, define a homotopy $H : X \times I \to Y$ by $H(x,t) = f(x)$. We may write this as the composition

$$H : X \times I \xrightarrow{proj} X \xrightarrow{f} Y$$

of continuous functions, so it is continuous. It satisfies $H(x,0) = f(x) = H(x,1) = H(x,t)$ for all x and t, so is a homotopy from f to f relative to any subset $A \subset X$. Thus $f \simeq f$ rel A.

(ii) Let $H : X \times I \to Y$ be a homotopy from f to g relative to A. Define $\bar{H}(x,t) = H(x, 1-t)$. This is continuous as the map $t \mapsto 1-t : I \to I$ is continuous (in fact, a homeomorphism). This new homotopy satisfies

$$\bar{H}(x,0) = H(x,1) = g(x) \quad \bar{H}(x,1) = H(x,0) = f(x)$$

$$\bar{H}(a,t) = H(a,1-t) = f(a) = g(a) \text{ for } a \in A$$

and so is a homotopy from g to f relative to A.

[1] Recall that a **map** is a continuous function.

(iii) Let $H : X \times I \to Y$ be a homotopy from f to g relative to A, and $H' : X \times I \to Y$ be a homotopy from g to h relative to A. then define a function $H'' : X \times I \to Y$ by the formula

$$H''(x,t) = \begin{cases} H(x,2t) & 0 \leq t \leq 1/2 \\ H'(x,2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

This function is well defined, as when $t = 1/2$ we have

$$H(x, 2 \cdot 1/2) = H(x,1) = g(x) = H'(x,0) = H'(x,(2 \cdot 1/2) - 1).$$

Furthermore, by the Gluing Lemma it is continuous, as it restricts to the continuous function $(x,t) \mapsto H(x,2t)$ on $X \times [0,1/2]$ and to the continuous function $(x,t) \mapsto H'(x,2t-1)$ on $X \times [1/2,1]$. □

Definition 2.1.3 (Homotopy equivalence)**.** A map $f : X \to Y$ is a **homotopy equivalence** if there exists a map $g : Y \to X$ such that $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. We call g a **homotopy inverse** to f.

We say X is **homotopy equivalent** to Y if a homotopy equivalence f exists, and write $X \simeq Y$.

Example 2.1.4. Let $X = S^1$, $Y = \mathbb{R}^2 \setminus \{0\}$, and $i : S^1 \to \mathbb{R}^2 \setminus \{0\}$ be the standard inclusion. Define a map

$$\begin{aligned} r : \mathbb{R}^2 \setminus \{0\} &\longrightarrow S^1 \\ x &\longmapsto \tfrac{x}{|x|}. \end{aligned}$$

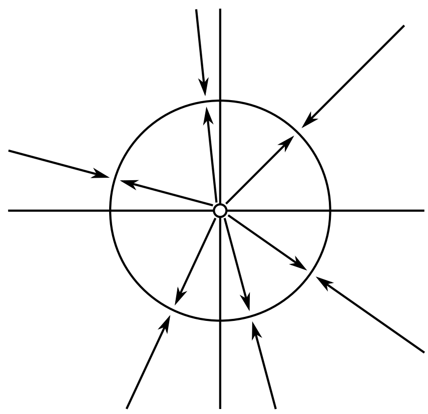


Figure 2.1 The retraction of $\mathbb{R}^2 \setminus \{0\}$ to S^1.

Then $r \circ i = \mathrm{Id}_{S^1}$, so we may take the constant homotopy. On the other hand the composition $i \circ r : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ is $x \mapsto \frac{x}{|x|}$. Define a homotopy

$$\begin{aligned} H : (\mathbb{R}^2 \setminus \{0\}) \times I &\longrightarrow \mathbb{R}^2 \setminus \{0\} \\ (x,t) &\longmapsto \tfrac{x}{t + (1-t) \cdot |x|}, \end{aligned}$$

which is well-defined as if $t+(1-t)\cdot|x| = 0$ then $|x| = \frac{t}{t-1} \leq 0$, which is impossible. This is a homotopy from $i \circ r$ to $\mathrm{Id}_{\mathbb{R}^2\setminus\{0\}}$, and so i is a homotopy equivalence with homotopy inverse r; $S^1 \simeq \mathbb{R}^2 \setminus \{0\}$.

Example 2.1.5. Let $Y = \mathbb{R}^n$, $X = \{0\}$, and $i : \{0\} \to \mathbb{R}^n$ be the inclusion. Let $r : \mathbb{R}^n \to \{0\}$ be the only map there is. Then $r \circ i = \mathrm{Id}_{\{0\}}$, and $i \circ r : \mathbb{R}^n \to \mathbb{R}^n$ is the constant map to 0. Define $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ by $H(x,t) = t \cdot x$, which is a homotopy from $i \circ r$ to $\mathrm{Id}_{\mathbb{R}^n}$. Thus i is a homotopy equivalence.

This property is central in the subject, and gets its own name.

Definition 2.1.6 (Contractible)**.** A space X is **contractible** if it homotopy equivalent to the 1-point space: $X \simeq \{*\}$.

We wish to say that the relation of homotopy equivalence between spaces is an equivalence relation, but first need a tool.

Lemma 2.1.7. *Let $f_0, f_1 : X \to Y$ be homotopic maps, and $g_0, g_1 : Y \to Z$ be homotopic maps. Then the maps $g_0 \circ f_0, g_1 \circ f_1 : X \to Z$ are homotopic.*

Proof. Let H be a homotopy from f_0 to f_1, and G be a homotopy from g_0 to g_1. We will show that both $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic to $g_0 \circ f_1$, and then use transitivity of the homotopy relation.

(i) $\mathbf{g_0 \circ f_0 \simeq g_0 \circ f_1}$: Let $H'' : X \times I \to Z$ be the map $g_0 \circ H$ (a composition of continuous functions, so continuous). This gives the desired homotopy.

(ii) $\mathbf{g_1 \circ f_1 \simeq g_0 \circ f_1}$: Let $H''' : X \times I \to Z$ be the map $H' \circ (f_1 \times \mathrm{Id}_I)$ (a composition of continuous functions, so continuous). This gives the desired homotopy. □

In particular, this lemma says that given maps

$$\begin{tikzcd} T \arrow[r, "i"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}$$

such that $f \simeq g$, then $h \circ f \simeq h \circ g$ and $f \circ i \simeq g \circ i$. This is the form in which we will generally use this lemma.

Proposition 2.1.8. *The relation of homotopy equivalence satisfies*

(i) $X \simeq X$ for any space X,

(ii) if $X \simeq Y$ then $Y \simeq X$,

(iii) if $X \simeq Y$ and $Y \simeq Z$ then $X \simeq Z$.

Proof.

(i) Let $f = g = \mathrm{Id}_X$ and all the homotopies be constant.

(ii) If $f : X \to Y$ is a homotopy equivalence with homotopy inverse g, then g is clearly also a homotopy equivalence (with homotopy inverse f).

(iii) Suppose that we have maps

$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \arrow[r, bend left, "f'"] & Z \arrow[l, bend left, "g'"] \end{tikzcd}$$

such that

$$f \circ g \simeq \mathrm{Id}_Y \quad f' \circ g' \simeq \mathrm{Id}_Z \quad g \circ f \simeq \mathrm{Id}_X \quad g' \circ f' \simeq \mathrm{Id}_Y.$$

Then applying Lemma 2.1.7 several times we obtain

$$(g \circ g') \circ (f' \circ f) = g \circ (g' \circ f') \circ f \simeq g \circ \mathrm{Id}_Y \circ f = g \circ f \simeq \mathrm{Id}_X$$

and

$$(f' \circ f) \circ (g \circ g') = f' \circ (f \circ g) \circ g' \simeq f' \circ \mathrm{Id}_Y \circ g' = f' \circ g' \simeq \mathrm{Id}_Z.$$

□

Definition 2.1.9 (Deformation retraction)**.** For a space X and a subspace A, a **deformation retraction** of X to A is a map $r : X \to A$ so that $r|_A = \mathrm{Id}_A$ and $i \circ r \simeq \mathrm{Id}_X$[2].

In particular, if there is a deformation retraction of X to A then the inclusion map $i : A \hookrightarrow X$ is a homotopy equivalence with homotopy inverse r: the homotopy $i \circ r \simeq \mathrm{Id}_X$ is part of the definition of deformation retraction, and $r \circ i = \mathrm{Id}_A$ so we can take the constant homotopy. i.e. the homotopy is the identity.

There is a related notion to that of a deformation retraction, which we introduce now.

Definition 2.1.10 (Retraction)**.** For a space X and a subspace A, a **retraction** of X to A is a map $r : X \to A$ so that $r|_A = \mathrm{Id}_A$.

Warning 2.1.11. A retraction need not be a homotopy equivalence! For example, if X is a non-empty space and $x_0 \in X$ a point, there is a retraction of X to the subspace $\{x_0\}$ given by sending every point of X to x_0. This will only be a homotopy equivalence if X is contractible.

2.2 Paths

Definition 2.2.1 (Paths and loops)**.** For a space X and points $x_0, x_1 \in X$, a **path from** x_0 **to** x_1 is a map $\gamma : I \to X$ such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. We write $\gamma : x_0 \rightsquigarrow x_1$. If $x_0 = x_1$ then we call γ a **loop** based at x_0.

[2]If we insist that $i \circ r \simeq \mathrm{Id}_X$ rel A then this is called a **strong deformation retraction**. Some authors call this stronger notion "deformation retraction", so beware.

If γ is a path from x_0 to x_1 and γ' is a path from x_1 to x_2, then we define a new path $\gamma \cdot \gamma' : I \to X$, the **concatenation** of γ and γ', by the formula

$$(\gamma \cdot \gamma')(t) = \begin{cases} \gamma(2t) & 0 \leq t \leq 1/2 \\ \gamma'(2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

(This is continuous by the Gluing Lemma.) This is a path from x_0 to x_2. We also define a path $\gamma^{-1} : T \to X$, the **inverse** of γ, by the formula

$$\gamma^{-1}(t) = \gamma(1-t),$$

which gives a path from x_1 to x_0. Finally, we define a path $c_{x_0} : I \to X$, the **constant path** at the point x_0, by $c_{x_0}(t) = x_0$.

If we define a relation $\sim$ on the space X by

$$x \sim y \Leftrightarrow \text{ there is a path in } X \text{ from } x \text{ to } y$$

then the above three constructions (concatenation, inverse, constant paths) show that this is an equivalence relation.

Definition 2.2.2 (Path components and π_0)**.** The equivalence classes of $\sim$ are called **path components**, and the set of equivalence classes is denoted $\pi_0(X)$. If there is a single equivalence class then we say X is **path connected**.

Proposition 2.2.3. *To a map $f : X \to Y$ there is a well-defined associated function*

$$\pi_0(f) : \pi_0(X) \longrightarrow \pi_0(Y)$$

given by $\pi_0(f)([x]) = [f(x)]$. Furthermore,

(i) if $f \simeq g$ then $\pi_0(f) = \pi_0(g)$,

(ii) for maps $A \xrightarrow{h} B \xrightarrow{k} C$ we have $\pi_0(k \circ h) = \pi_0(k) \circ \pi_0(h)$,

(iii) $\pi_0(\mathrm{Id}_X) = \mathrm{Id}_{\pi_0(X)}$.

Proof. We must show that the proposed function is well-defined, so let $[x] = [x'] \in \pi_0(X)$: this means that there is a path $\gamma : I \to X$ from x to x'. Then $f \circ \gamma : I \to Y$ is a path from $f(x)$ to $f(x')$, and so $[f(x)] = [f(x')] \in \pi_0(Y)$, as required.

Properties (ii) and (iii) are clear from the definition. For (i), let $H : X \times I \to Y$ be a homotopy from f to g. For $x \in X$ the map $H(x, -) : I \to Y$ is a path from $f(x)$ to $g(x)$, and so $[f(x)] = [g(x)] \in \pi_0(Y)$. Thus $\pi_0(f)([x]) = \pi_0(g)([x])$, and this holds for any x so $\pi_0(f) = \pi_0(g)$ as required. ☐

Corollary 2.2.4. *If $f : X \to Y$ is a homotopy equivalence then $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is a bijection.*

Proof. If $g : Y \to X$ is a homotopy inverse to f then we have

$$\pi_0(f) \circ \pi_0(g) = \pi_0(f \circ g) = \pi_0(\mathrm{Id}_Y) = \mathrm{Id}_{\pi_0(Y)}$$

and similarly

$$\pi_0(g) \circ \pi_0(f) = \pi_0(g \circ f) = \pi_0(\mathrm{Id}_X) = \mathrm{Id}_{\pi_0(X)}$$

so $\pi_0(g)$ is inverse to $\pi_0(f)$. □

Example 2.2.5. The space $\{-1, 1\}$ with the discrete topology is not contractible, i.e. is not homotopy equivalent to the one-point space $\{*\}$. This is because continuous paths in $\{-1, 1\}$ must be constant, so $\pi_0(\{-1, 1\}) = \{-1, 1\}$ has cardinality 2. But $\pi_0(\{*\}) = \{*\}$ has cardinality 1, so these are not in bijection.

Example 2.2.6. The space $\mathbb{R}^2$ does not retract (cf. Definition 2.1.10) on to the subspace $\{(-1, 0), (1, 0)\}$. If we write $i : \{(-1, 0), (1, 0)\} \to \mathbb{R}^2$ for the inclusion, and suppose $r : \mathbb{R}^2 \to \{(-1, 0), (1, 0)\}$ is a retraction, so $r \circ i = \mathrm{Id}_{\{(-1,0),(1,0)\}}$, then we obtain a factorisation

$$\pi_0(\{(-1, 0), (1, 0)\}) \xrightarrow{\pi_0(i)} \pi_0(\mathbb{R}^2) \xrightarrow{\pi_0(r)} \pi_0(\{(-1, 0), (1, 0)\})$$

of the identity map. By the previous example $\pi_0(\{(-1, 0), (1, 0)\})$ has cardinality 2, but $\mathbb{R}^2$ is path-connected so $\pi_0(\mathbb{R}^2)$ has cardinality 1. In particular, $\pi_0(i)$ cannot be injective, so $\pi_0(r) \circ \pi_0(i)$ cannot be the identity map.

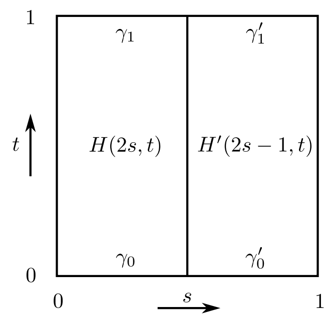
Definition 2.2.7. Two paths $\gamma, \gamma' : I \to X$ from x_0 to x_1 are **homotopic as paths** if they are homotopic relative to $\{0, 1\} \subset I$ in the sense of Definition 2.1.1. We write $\gamma \simeq \gamma'$ as paths from x_0 to x_1.

Lemma 2.2.8. *If $\gamma_0 \simeq \gamma_1$ as paths from x_0 to x_1, and $\gamma'_0 \simeq \gamma'_1$ as paths from x_1 to x_2, then $\gamma_0 \cdot \gamma'_0 \simeq \gamma_1 \cdot \gamma'_1$ as paths from x_0 to x_2.*

Proof. Let H be a relative homotopy from γ_0 to γ_1, and H' be a relative homotopy from γ'_0 to γ'_1. As $H(1, t) = x_1 = H'(0, t)$ for all t, these homotopies fit together to a function $H'' : I \times I \to X$ given by

$$H''(s, t) = \begin{cases} H(2s, t) & 0 \le s \le 1/2 \\ H'(2s - 1, t) & 1/2 \le s \le 1 \end{cases}$$

which is continuous by the Gluing Lemma.



This satisfies

$$H''(-,0) = (\gamma_0 \cdot \gamma_0')(-) \quad H''(-,1) = (\gamma_1 \cdot \gamma_1')(-) \quad H''(0,t) = x_0 \quad H''(1,t) = x_2$$

as required. □

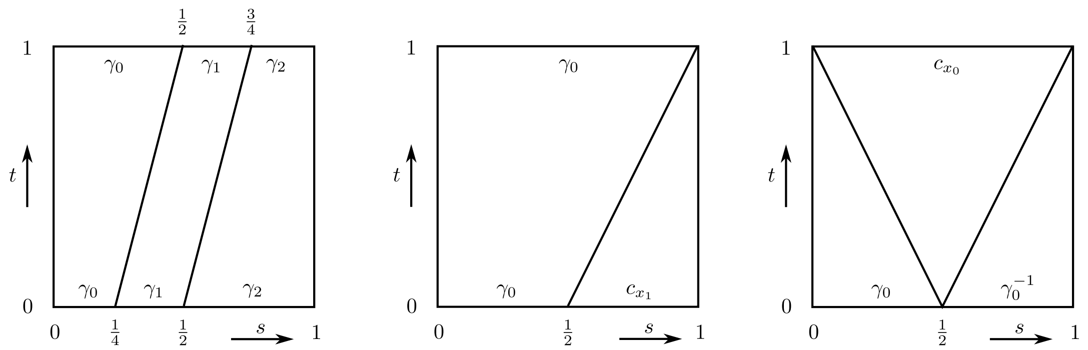
Proposition 2.2.9. *Let $\gamma_0 : x_0 \rightsquigarrow x_1$, $\gamma_1 : x_1 \rightsquigarrow x_2$, and $\gamma_2 : x_2 \rightsquigarrow x_3$ be paths in X. Then*

(i) $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2 \simeq \gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$ *as paths from x_0 to x_3,*

(ii) $\gamma_0 \cdot c_{x_1} \simeq \gamma_0 \simeq c_{x_0} \cdot \gamma_0$ *as paths from x_0 to x_1,*

(iii) $\gamma_0 \cdot \gamma_0^{-1} \simeq c_{x_0}$ *as paths from x_0 to x_0, and $\gamma_0^{-1} \cdot \gamma_0 \simeq c_{x_1}$ as paths from x_1 to x_1.*

Proof. First contemplate the following figures.



From these, we arrive at the following formulæ.

(i) The homotopy

$$H(s,t) = \begin{cases} \gamma_0(\frac{4s}{t+1}) & 0 \le s \le \frac{t+1}{4} \\ \gamma_1(4s-1-t) & \frac{t+1}{4} \le s \le \frac{t+2}{4} \\ \gamma_2(1-\frac{4(1-s)}{2-t}) & \frac{t+2}{4} \le s \le 1 \end{cases}$$

goes from $(\gamma_0 \cdot \gamma_1) \cdot \gamma_2$ to $\gamma_0 \cdot (\gamma_1 \cdot \gamma_2)$ as paths from x_0 to x_3.

(ii) The homotopy

$$H(s,t) = \begin{cases} \gamma_0(\frac{2s}{t+1}) & 0 \le s \le \frac{t+1}{2} \\ x_1 & \frac{t+1}{2} \le s \le 1, \end{cases}$$

goes from $\gamma_0 \cdot c_{x_1}$ to γ_0 as paths from x_0 to x_1.

(iii) The homotopy

$$H(s,t) = \begin{cases} \gamma_0(2s) & 0 \le s \le \frac{1-t}{2} \\ \gamma_0(1-t) & \frac{1-t}{2} \le s \le \frac{1+t}{2} \\ \gamma_0(2-2s) & \frac{1+t}{2} \le s \le 1 \end{cases}$$

goes from $\gamma_0 \cdot \gamma_0^{-1}$ to c_{x_0} as paths from x_0 to x_0. □

2.3 The fundamental group

Theorem 2.3.1. *Let X be a space and $x_0 \in X$ be a point. Let $\pi_1(X, x_0)$ denote the set of homotopy classes of loops in X based at x_0. Then the rule $[\gamma] \cdot [\gamma'] := [\gamma \cdot \gamma']$ and the element $e := [c_{x_0}]$ define a group structure on $\pi_1(X, x_0)$.*

We call the resulting group the **fundamental group of X based at** x_0.

Proof. Lemma 2.2.8 shows that the proposed composition law is well-defined, and Proposition 2.2.9 shows that the group axioms are satisfied. □

Definition 2.3.2. A **based space** is a pair (X, x_0) of a space X and a point $x_0 \in X$, called the **basepoint**. A **map of based spaces** $f : (X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$. A **based homotopy** is a homotopy relative to $\{x_0\} \subset X$ in the sense of Definition 2.1.1.

Proposition 2.3.3. *To a based map $f : (X, x_0) \to (Y, y_0)$ there is a well-defined associated function $\pi_1(f) : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ given by $\pi_1(f)([\gamma]) = [f \circ \gamma]$. It satisfies*

(i) $\pi_1(f)$ *is a group homomorphism,*

(ii) if f is based homotopic to f' then $\pi_1(f) = \pi_1(f')$,

(iii) for based maps $(A, a) \xrightarrow{h} (B, b) \xrightarrow{k} (C, c)$ we have $\pi_1(k \circ h) = \pi_1(k) \circ \pi_1(h)$,

(iv) $\pi_1(\mathrm{Id}_X) = \mathrm{Id}_{\pi_1(X,x_0)}$.

Proof. The proposed function $\pi_1(f)$ is well-defined, as if $\gamma \simeq \gamma'$ as paths then $f \circ \gamma \simeq f \circ \gamma'$ as paths.

(i) Note that $f \circ c_{x_0} = c_{y_0}$, so $\pi_1(f)$ preserves the unit. Also $f \circ (\gamma \cdot \gamma') = (f \circ \gamma) \cdot (f \circ \gamma')$, so $\pi_1(f)$ preserves the composition law.

(ii) if $f \simeq f'$ rel $\{x_0\}$ then $f \circ \gamma \simeq f' \circ \gamma$ rel $\{0, 1\}$, as $\gamma(\{0, 1\}) = \{x_0\}$.

(iii) $\pi_1(k \circ h)([\gamma]) = [k \circ h \circ \gamma] = \pi_1(k)([h \circ \gamma]) = \pi_1(k)(\pi_1(h)([\gamma]))$, for any $[\gamma] \in \pi_1(A, a)$.

(iv) $\pi_1(\mathrm{Id}_X)([\gamma]) = [\mathrm{Id}_X \circ \gamma] = [\gamma]$ for any $[\gamma] \in \pi_1(X, x_0)$. □

After having introduced the notation $\pi_1(f)$ for the group homomorphism induced by a based map f, will will now discard it in favour of the shorter notation f_*.

Proposition 2.3.4 (Change of basepoint)**.** *Let $u : x_0 \rightsquigarrow x_1$ be a path in X. It induces a group isomorphism*

$$\begin{aligned} u_\# : \pi_1(X, x_0) &\longrightarrow \pi_1(X, x_1) \\ [\gamma] &\longmapsto [u^{-1} \cdot \gamma \cdot u] \end{aligned}$$

satisfying

(i) if $u \simeq u'$ as paths from x_0 to x_1 then $u_\# = u'_\#$,

(ii) $(c_{x_0})_\# = \mathrm{Id}_{\pi_1(X,x_0)}$,

(iii) *if* $v : x_1 \rightsquigarrow x_2$ *is a path then* $(u \cdot v)_\# = v_\# \circ u_\#$,

(iv) *if* $f : X \to Y$ *sends* x_0 *to* y_0 *and* x_1 *to* y_1 *then*

$$(f \circ u)_\# \circ f_* = f_* \circ u_\# : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_1),$$

or in other words the square

$$\begin{tikzcd} \pi_1(X,x_0) \arrow[r, "f_*"] \arrow[d, "u_\#"] & \pi_1(Y,y_0) \arrow[d, "(f\circ u)_\#"] \\ \pi_1(X,x_1) \arrow[r, "f_*"] & \pi_1(Y,y_1) \end{tikzcd}$$

commutes*: going either way around it from the top left corner to the bottom right corner yields the same homomorphism.*

(v) *if* $x_1 = x_0$ *then* $u_\#$ *is the automorphism of* $\pi_1(X, x_0)$ *given by conjugation by* $[u] \in \pi_1(X, x_0)$.

Proof. That $u_\#$ is a group homomorphism is a consequence of invoking Proposition 2.2.9 several times. Properties (i) – (iii) and (v) are by now standard arguments. For (iv) we calculate

$$\begin{aligned} ((f \circ u)_\# \circ f_*)([\gamma]) &= (f \circ u)_\#([f \circ \gamma]) = [(f \circ u)^{-1} \cdot (f \circ \gamma) \cdot (f \circ u)] \\ &= [f \circ (u^{-1} \cdot \gamma \cdot u)] = f_*([u^{-1} \cdot \gamma \cdot u]) = f_*(u_\#([\gamma])). \end{aligned}$$ □

Warning 2.3.5. By this proposition the fundamental groups based at different points in a path connected space X are isomorphic, *but they are not canonically isomorphic*! One has to *choose* a path between their basepoints to obtain an isomorphism.

Thus it makes sense to say "the fundamental group is trivial / abelian / has group-theoretic property $\mathcal{P}$" without referring to a basepoint. But it does not make sense to have an element $a_0 \in \pi_1(X, x_0)$ and say "let $a_1 \in \pi_1(X, x_1)$ be the corresponding element".

If $H : X \times I \to Y$ is a homotopy from f to g, and $x_0 \in X$ is a basepoint, then $u := H(x_0, -) : I \to Y$ is a path from $f(x_0)$ to $g(x_0)$ in Y. We can ask about the relationship between the three homomorphisms

$$\begin{aligned} f_* &: \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0)) \\ g_* &: \pi_1(X, x_0) \longrightarrow \pi_1(Y, g(x_0)) \\ u_\# &: \pi_1(Y, f(x_0)) \longrightarrow \pi_1(Y, g(x_0)), \end{aligned}$$

which fit into the diagram

$$\begin{tikzcd} & \pi_1(Y,f(x_0)) \arrow[d, "u_\#", "\cong"'] \\ \pi_1(X,x_0) \arrow[ur, "f_*"] \arrow[r, "g_*"] & \pi_1(Y,g(x_0)). \end{tikzcd} \tag{2.3.1}$$

Lemma 2.3.6. *$u_\# \circ f_* = g_*$, or in other words the triangle* (2.3.1) *commutes.*

Proof. For a loop $\gamma : I \to X$ based at x_0 consider the composition

$$F : I \times I \xrightarrow{\gamma \times \mathrm{Id}_I} X \times I \xrightarrow{H} Y.$$

Consider the path $\ell^+ : I \to I \times I$ from $(0,1)$ to $(1,1)$ given by $s \mapsto (s,1)$, and the path $\ell^- : I \to I \times I$ from $(0,1)$ to $(1,1)$ given by concatenating the paths

$$s \mapsto (0, 1-s) \quad s \mapsto (s, 0) \quad s \mapsto (1, s).$$

These are paths between the same points, and we can define a homotopy

$$L(s,t) = t \cdot \ell^-(s) + (1-t) \cdot \ell^+(s)$$

between them, using the vector space structure on $\mathbb{R}^2 \supset I \times I$ and the fact that $I \times I$ is convex. This shows that $\ell^+ \simeq \ell^-$ as paths from $(0,1)$ to $(1,1)$.

Hence $F \circ \ell^+ \simeq F \circ \ell^-$ as paths from $F(0,1) = H(x_0, 1) = g(x_0)$ to $F(1,1) = H(x_0, 1) = g(x_0)$. But $[F \circ \ell^+] = [g \circ \gamma]$ and $[F \circ \ell^-] = [u^{-1} \cdot (f \circ \gamma) \cdot u]$, as required. □

Theorem 2.3.7. *Let $f : X \to Y$ be a homotopy equivalence, and $x_0 \in X$. Then*

$$f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof. Let $g : Y \to X$ be a homotopy inverse to f, so $f \circ g \simeq_H \mathrm{Id}_Y$ and $g \circ f \simeq_{H'} \mathrm{Id}_X$. Let $u' : I \to X$ be the path $u'(t) = H'(x_0, 1-t)$, going from x_0 to $g \circ f(x_0)$. Then applying Lemma 2.3.6 gives, (let f, g in lemma be id and $g \circ f$).

$$u'_\# = (g \circ f)_* = g_* \circ f_* : \pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g \circ f(x_0)),$$

which is an isomorphism: thus the map f_* we are studying is injective, and the map

$$g_* : \pi_1(Y, f(x_0)) \longrightarrow \pi_1(X, g \circ f(x_0)) \tag{2.3.2}$$

is surjective. If can show that g_* is also injective then we are done, as then it is an isomorphism and so $f_* = (g_*)^{-1} \circ u'_\#$ is also an isomorphism.

To show that g_* is injective we consider the path $u : I \to Y$ given by $u(t) = H(f(x_0), 1-t)$, going from $f(x_0)$ to $f \circ g \circ f(x_0)$. Applying Lemma 2.3.6 again gives

$$u_\# = (f \circ g)_* = f_* \circ g_* : \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g \circ f(x_0)) \longrightarrow \pi_1(Y, f \circ g \circ f(x_0)),$$

which is an isomorphism. (Note the second map here is induced by f but is not the map f_* we are studying, as the basepoints involved are different.) Thus the map (2.3.2) is also injective, as required. □

Definition 2.3.8. A space X is **simply connected** if it is path connected and its fundamental group is trivial ~~an~~ for some (and hence every) basepoint.

Example 2.3.9. Any contractible space is simply connected.

Lemma 2.3.10. *A space X is simply connected if and only if for each pair of points $x_0, x_1 \in X$ there is a unique homotopy class of path between them.*

Proof. Suppose X is simply connected, and let $x_0, x_1 \in X$. As X is path connected there exists a path from x_0 to x_1. If γ and γ' are two such paths, then $\gamma^{-1} \cdot \gamma'$ is a loop based at x_0 and so $[\gamma^{-1} \cdot \gamma'] \in \pi_1(X, x_0) = \{[c_{x_0}]\}$. Thus there is a homotopy of paths $\gamma^{-1} \cdot \gamma' \simeq c_{x_0}$. Preconcatenating with γ, there is a homotopy of paths from γ' to γ, as required.

Now suppose X has the property described in the statement of the lemma. In particular there exists a path between any pair of points, so X is path connected. Furthermore, if γ is a loop based at x_0 then so is c_{x_0}, and so these are homotopic as paths. Thus $[\gamma] = [c_{x_0}] \in \pi_1(X, x_0)$, and so $\pi_1(X, x_0)$ is trivial. □

$\gamma' \simeq \gamma \cdot \gamma^{-1} \cdot \gamma' \simeq \gamma\, c_{x_0} \simeq \gamma.$ (all rel end points)

Chapter 3

Covering spaces

3.1 Covering spaces

Definition 3.1.1. A **covering space** of a space X is a pair $(\widetilde{X}, p)$ of a space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ such that for any $x \in X$ there is an open neighbourhood U of x such that $p^{-1}(U)$ is a disjoint union of open sets of $\widetilde{X}$ each of which is mapped homeomorphically onto U. In this situation we call p a **covering map**.

For such an open neighbourhood U, we shall usually write $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$ where each $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. We call these open sets U **evenly covered**.

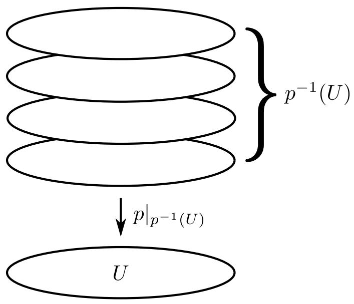


Figure 3.1 The generic picture of an evenly covered set U.

Example 3.1.2. A homeomorphism is a covering map.

Example 3.1.3. If $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ are covering maps, so is the product $p \times q : \widetilde{X} \times \widetilde{Y} \to X \times Y$.

Example 3.1.4. Let $S^1 \subset \mathbb{C}$ be the set of complex numbers of modulus 1, and consider the map $p : \mathbb{R} \to S^1$ given by $p(t) = e^{2\pi i t}$. For the set $U_{y>0} := \{x + iy \in S^1 \mid y > 0\}$ we have

$$p^{-1}(U_{y>0}) = \coprod_{j \in \mathbb{Z}} (j, j + \tfrac{1}{2}),$$

the disjoint union of $\mathbb{Z}$-many open intervals. Furthermore

$$p|_{(j,j+\frac{1}{2})} : (j, j + \tfrac{1}{2}) \longrightarrow U_{y>0}$$

is a continuous bijection, with inverse

$$\begin{aligned} U_{y>0} &\longrightarrow (j, j + \tfrac{1}{2}) \\ x + iy &\longmapsto j + \arccos(x)/2\pi. \end{aligned}$$

Similarly with $U_{y<0}$, $U_{x>0}$ and $U_{x<0}$, and these four open sets cover S^1. Thus p is a covering map.

Example 3.1.5. Let $S^1 \subset \mathbb{C}$, and $p : S^1 \to S^1$ be given by $p(z) = z^n$. Let $\eta := e^{2\pi i/n}$, and for $y \in S^1$ let ξ be some nth root of y. Then

$$p^{-1}(y) = \{\xi, \xi \cdot \eta, \xi \cdot \eta^2, \dots, \xi \cdot \eta^{n-1}\}$$

and $p^{-1}(S^1 \setminus \{y\})$ is the complement of this set. Let

$$V_0 := \{z \in S^1 \mid 0 < \arg(z/\xi) < 2\pi/n\}$$

and $V_i := V_0 \cdot \eta^i$ for $i = 0, 1, 2, \dots, n-1$, so $p^{-1}(S^1 \setminus \{y\}) = \coprod_{i=0}^{n-1} V_i$. Then $p|_{V_i} : V_i \to S^1 \setminus \{y\}$ is a homeomorphism.

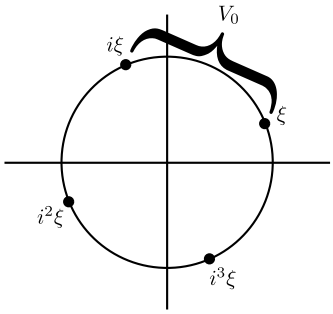


Figure 3.2 The set V_0 for $n = 4$.

Example 3.1.6. Let $S^2 \subset \mathbb{R}^3$ be the unit sphere, and $\sim$ be the equivalence relation on S^2 generated by $x \sim -x$, with quotient space $\mathbb{RP}^2 := S^2/\sim$ (the real projective plane) and quotient map $p : S^2 \to \mathbb{RP}^2$. Let $V := \{(x, y, z) \in S^2 \mid z \neq 0\}$, an open set, and $U = p(V)$. Then $p^{-1}(U) = V$, and so $U \subset \mathbb{RP}^2$ is open.

Furthermore, $V = V_{z<0} \coprod V_{z>0}$, and we claim that $p|_{V_{z>0}} : V_{z>0} \to U$ is a homeomorphism (similarly for $V_{z<0}$ of course). To see this, we shall construct an inverse. By the definition of the quotient topology, a continuous map $U \to Y$ for some space Y is the same as a continuous map $V \to Y$ which in constant on $\sim$-equivalence classes. Thus define the map

$$\begin{aligned} t : V &\longrightarrow V_{z>0} \\ (x, y, z) &\longmapsto \begin{cases} (x, y, z) & \text{if } z > 0, \\ (-x, -y, -z) & \text{if } z < 0. \end{cases} \end{aligned}$$

This is clearly continuous and constant on $\sim$-equivalence classes, and so defines a continuous function $\bar{t} : U \to V_{z>0}$. It is inverse to $p|_{V_{z>0}}$, and so p is a covering map.

Our interest in covering spaces comes from the following important property they have with respect to paths and homotopies.

Definition 3.1.7 (Lifting)**.** Let $p : \widetilde{X} \to X$ be a covering space, and $f : Y \to X$ be a map. A **lift** of f along p is a map $\tilde{f} : Y \to \widetilde{X}$ such that $p \circ \tilde{f} = f$.

Lemma 3.1.8 (Uniqueness of lifts lemma)**.** *Let $p : \widetilde{X} \to X$ be a covering space, $f : Y \to X$ be a map, and $\tilde{f}_0, \tilde{f}_1 : Y \to \widetilde{X}$ be two lifts of f. Then the set*

$$S := \{y \in Y \mid \tilde{f}_0(y) = \tilde{f}_1(y)\}$$

is both open and closed in Y. Thus if Y is connected then either $S = \emptyset$ or $S = Y$.

Proof. Let us first show that S is open. Let $y \in S$ and U be an open neighbourhood of $f(s) \in X$ which is evenly covered, so $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$. Then $\tilde{f}_0(y)$ and $\tilde{f}_1(y)$ are equal so lie in the same V_β. Thus, N open. on $N := \tilde{f}_0^{-1}(V_\beta) \cap \tilde{f}_1^{-1}(V_\beta)$ we have

$$p|_{V_\beta} \circ \tilde{f}_0|_N = f|_N = p|_{V_\beta} \circ \tilde{f}_1|_N$$

but $p|_{V_\beta}$ is a homeomorphism, so $\tilde{f}_0|_N = \tilde{f}_1|_N$ and $y \in N \subset S$. Thus S is open.

Let us now show that S is closed. Let $y \in \overline{S}$ and suppose $\tilde{f}_0(y) \neq \tilde{f}_1(y)$. Let U be an open neighbourhood of $f(y) \in X$ which is evenly covered; $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$. Thus $\tilde{f}_0(y) \in V_\beta$ and $\tilde{f}_1(y) \in V_\gamma$ for some $\beta, \gamma \in I$. As each V_α is mapped homeomorphically to U, and $\tilde{f}_0(y) \neq \tilde{f}_1(y)$, we must have $\beta \neq \gamma$. Now $\tilde{f}_0^{-1}(V_\beta) \cap \tilde{f}_1^{-1}(V_\gamma)$ is an open neighbourhood of y, so muct intersect S by the definition of closure. But then V_β and V_γ must intersect, a contradiction. □

Lemma 3.1.9 (Homotopy lifting lemma)**.** *Let $p : \widetilde{X} \to X$ be a covering space, $H : Y \times I \to X$ be a homotopy from f_0 to f_1, and $\tilde{f}_0$ be a lift of f_0. Then there exists a unique homotopy $\widetilde{H} : Y \times I \to \widetilde{X}$ such that*

(i) $\widetilde{H}(-, 0) = \tilde{f}_0(-)$, *and*

(ii) $p \circ \widetilde{H} = H$.

Proof. Let $\{U_\alpha\}_{\alpha \in I}$ be an open cover of X by sets which are evenly covered, and write $p^{-1}(U_\alpha) = \coprod_{\beta \in I_\alpha} V_\beta$ with $p|_{V_\beta} \xrightarrow{\sim} U_\alpha$.

Then $\{H^{-1}(U_\alpha)\}_{\alpha \in I}$ is an open cover of $Y \times I$. For each $y_0 \in Y$ this restricts to an open cover of the compact space $\{y_0\} \times I$, and by the Lesbegue number lemma there is a $N = N(y_0)$ such that each path

$$H|_{\{y_0\} \times [i/N, (i+1)/N]} : \{y_0\} \times [\tfrac{i}{N}, \tfrac{i+1}{N}] \longrightarrow X$$

lies entirely inside some U_α. In fact, as $\{y_0\} \times I$ is compact there is an open neighbourhood W_{y_0} of y_0 such that $H(W_{y_0} \times [\frac{i}{N}, \frac{i+1}{N}])$ lies entirely inside some U_α for each i. We are now in the situation depicted in Figure 3.3.

We obtain a lift $\widetilde{H}|_{W_{y_0} \times I}$ of $H|_{W_{y_0} \times I}$ as follows:

(i) We have $H|_{W_{y_0} \times [0, \frac{1}{N}]} : W_{y_0} \times [0, \frac{1}{N}] \longrightarrow U_\alpha$ and a lift $\tilde{f}_0|_{W_{y_0}} : W_{y_0} \to \widetilde{X}$ with image in some V_β which is mapped homeomorphically to U_α via $p|_{V_\beta}$, so we can let

$$\widetilde{H}|_{W_{y_0} \times [0, \frac{1}{N}]} := (p|_{V_\beta})^{-1} \circ H|_{W_{y_0} \times [0, \frac{1}{N}]} : W_{y_0} \times [0, \tfrac{1}{N}] \longrightarrow V_\beta \subset \widetilde{X}.$$

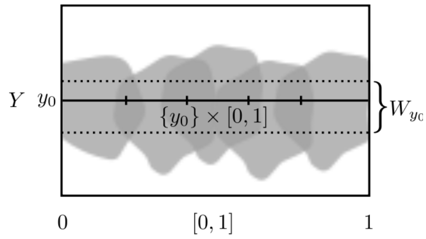


Figure 3.3

(ii) Proceed in the same way, lifting the (short) homotopy $H|_{W_{y_0}\times[\frac{1}{N},\frac{2}{N}]}$ starting at $\widetilde{H}|_{W_{y_0}\times\{\frac{1}{N}\}}$, and so on.

At the end of this process, we obtain a map $\widetilde{H}|_{W_{y_0}\times I}$ lifting $H|_{W_{y_0}\times I}$ and extending $\tilde{f}_0|_{W_{y_0}}$.

We may do this for every point $y_0 \in Y$, so it is now enough to check that on $(W_{y_0} \times I) \cap (W_{y_1} \times I) = (W_{y_0} \cap W_{y_1}) \times I$ the two lifts we have constructed agree. The two lifts do agree on $(W_{y_0} \cap W_{y_1}) \times \{0\}$, as here they both restrict to $f_0|_{W_{y_0}\cap W_{y_1}}$, and so by Lemma 3.1.8 they agree on an open and closed subset of $(W_{y_0} \cap W_{y_1}) \times I$ containing $(W_{y_0} \cap W_{y_1}) \times \{0\}$. Restricting to each $\{y\} \times I$ we must obtain an open and closed subset containing $\{y\} \times \{0\}$, and so the whole of $\{y\} \times I$ (as I is connected): thus the two maps agree on the whole of $(W_{y_0} \cap W_{y_1}) \times I$. □

The following corollary is obtained by taking $Y = \{*\}$ in the homotopy lifting lemma.

Corollary 3.1.10 (Path lifting)**.** *Let $p : \widetilde{X} \to X$ be a covering space, $\gamma : I \to X$ be a path, and $\tilde{x}_0 \in \widetilde{X}$ be such that $p(\tilde{x}_0) = \gamma(0)$. Then there exists a unique path $\tilde{\gamma} : I \to \widetilde{X}$ such that*

(i) $\tilde{\gamma}(0) = \tilde{x}_0$, and

(ii) $p \circ \tilde{\gamma} = \gamma$.

Corollary 3.1.11. *Let $p : \widetilde{X} \to X$ be a covering space, $\gamma, \gamma' : I \to X$ be paths from x_0 to x_1, and $\tilde{\gamma}, \tilde{\gamma}' : I \to \widetilde{X}$ be lifts starting at $\tilde{x}_0 \in p^{-1}(x_0)$. If $\gamma \simeq \gamma'$ as paths, then $\tilde{\gamma} \simeq \tilde{\gamma}'$ as paths; in particular $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$.*

Proof. Let $H : I \times I \to X$ be a homotopy of paths from γ to γ', and lift it starting at $\tilde{\gamma}$ to obtain a homotopy $\widetilde{H} : I \times I \to \widetilde{X}$. Then

(i) $\widetilde{H}(-, 1)$ is a lift of γ' starting at $\tilde{x}_0$, so is equal to $\tilde{\gamma}'$, by uniqueness of lifts

(ii) $H(0, -)$ and $H(1, -)$ are constant paths, so their lifts $\widetilde{H}(0, -)$ and $\widetilde{H}(1, 0)$ are too, Note constant lifts work and by uniqueness are thus the lift.

and hence $\widetilde{H}$ is a homotopy of paths from $\tilde{\gamma}$ to $\tilde{\gamma}'$. □

Corollary 3.1.12. *Let $p : \widetilde{X} \to X$ be a covering space, and X be path connected. Then the sets $p^{-1}(x)$ for $x \in X$ are all in bijection.* with each other

Proof. Let $\gamma : I \to X$ be a path from x_0 to x_1, and for each $y_0 \in p^{-1}(x_0)$ let $\tilde{\gamma}_{y_0}$ be the unique lift of γ starting at y_0. Define the function

$$\begin{aligned}\gamma_* : p^{-1}(x_0) &\longrightarrow p^{-1}(x_1)\\ y_0 &\longmapsto \tilde{\gamma}_{y_0}(1).\end{aligned}$$

Similarly, the inverse path γ^{-1} defines a function $(\gamma^{-1})_* : p^{-1}(x_1) \to p^{-1}(x_0)$. Now for any $y_0 \in p^{-1}(x_0)$ we calculate

$$\begin{aligned}(\gamma^{-1})_* \circ \gamma_*(y_0) &= \text{endpoint of lift of } \gamma^{-1} \text{ starting at } \tilde{\gamma}_{y_0}(1)\\ &= \text{endpoint of lift of } \gamma^{-1} \cdot \gamma \text{ starting at } y_0\\ &= \text{endpoint of lift of } c_{x_0} \text{ starting at } y_0\\ &= y_0,\end{aligned}$$

so $(\gamma^{-1})_*$ is a left inverse for γ. The analogous calculation with the other composition shows it is also a right inverse. ☐

Definition 3.1.13. Say a covering map $p : \widetilde{X} \to X$ is an n**-sheeted cover** (for $n \in \mathbb{N} \cup \{\infty\}$) if each $p^{-1}(x)$ has cardinality n.

We now come to the first connection between covering spaces and the fundamental group.

Lemma 3.1.14. *Let $p : \widetilde{X} \to X$ be a covering map, $x_0 \in X$, and $\tilde{x}_0 \in p^{-1}(x_0)$. Then the map $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective.*

Proof. Let $\gamma : I \to \widetilde{X}$ be a loop based at $\tilde{x}_0$ such that $p_*([\gamma]) = [c_{x_0}]$, so $p \circ \gamma \simeq c_{x_0}$ as paths. Let H be a homotopy of paths from $p \circ \gamma$ to c_{x_0}, and lift it starting at the lift γ of $p \circ \gamma$. This gives a homotopy of paths $\widetilde{H}$ from γ to a lift of c_{x_0}, which must be $c_{\tilde{x}_0}$, and hence $[\gamma] = [c_{\tilde{x}_0}]$. ☐

In the proof of Corollary 3.1.12 we constructed, for a covering map $p : \widetilde{X} \to X$ and path $\gamma : I \to X$ from x_0 to x_1, a function $\gamma_* : p^{-1}(x_0) \to p^{-1}(x_1)$. By Corollary 3.1.11 this function only depends on the homotopy class of γ as a path. In particular, this construction defines a (right) action $\bullet$ of the group $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$.

Lemma 3.1.15. *Let $p : \widetilde{X} \to X$ be a covering map, and X be path connected.*

(i) $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ if and only if $\widetilde{X}$ is path connected.

(ii) The stabiliser of $y_0 \in p^{-1}(x_0)$ is the subgroup

$$\mathrm{Im}(p_* : \pi_1(\widetilde{X}, y_0) \to \pi_1(X, x_0)) \leq \pi_1(X, x_0).$$

(iii) If $\widetilde{X}$ is path connected then there is a bijection

$$\text{cosets } \frac{\pi_1(X, x_0)}{p_*\pi_1(\widetilde{X}, y_0)} \longrightarrow p^{-1}(x_0)$$

induced by acting on the point y_0.

Proof. For (i), let us first show that if $\widetilde{X}$ is path connected then $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$. Let $y_0, y_0' \in p^{-1}(x_0)$, and let $\gamma : I \to \widetilde{X}$ be a path between them, which exists by hypothesis. Then $p \circ \gamma$ is a loop in X based at x_0, and γ is the lift of it starting at y_0. Thus

$$y_0 \bullet [p \circ \gamma] = y_0',$$

as required.

Now let us show that if $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ then $\widetilde{X}$ is path connected. Suppose, for a contradiction, that $y_0, z_0 \in \widetilde{X}$ lie in different path components. Choose a path $\gamma : I \to X$ from $p(y_0)$ to $p(z_0)$, and lift it starting at y_0. This gives a path in $\widetilde{X}$ from y_0 to a z_0', and $p(z_0') = p(z_0)$. As z_0' is in the path component of y_0, it is *not* in the path component of z_0. But as $\pi_1(X, x_0)$ acts transitively on $p^{-1}(x_0)$ there is a $[\gamma'] \in \pi_1(X, x_0)$ such that $z_0 \bullet [\gamma'] = z_0'$, and so the lift of γ' starting at z_0 ends at z_0', which is a contradiction.

For (ii), suppose that $y_0 \bullet [\gamma] = y_0$, so the lift $\tilde{\gamma}$ of γ starting at y_0 also ends at y_0. Thus $\tilde{\gamma}$ is a loop based at y_0, and $\gamma = p \circ \tilde{\gamma}$, so

$$[\gamma] = p_*([\tilde{\gamma}]) \in \mathrm{Im}(p_* : \pi_1(\widetilde{X}, y_0) \to \pi_1(X, x_0)).$$

Conversely, if $[\gamma'] \in \pi_1(\widetilde{X}, y_0)$ then γ' is a (and so the) lift of $p \circ \gamma'$ starting at y_0. It ends at y_0 too, and so $y_0 \bullet [p \circ \gamma'] = y_0$.

For (iii), we simply apply the Orbit-Stabiliser theorem from the theory of group actions. □

Definition 3.1.16. We say that a covering map $p : \widetilde{X} \to X$ is a **universal cover** if $\widetilde{X}$ is simply connected.

The following connection between universal covers and the fundamental group is immediate from this definition and Lemma 3.1.15.

Corollary 3.1.17. *If a covering map $p : \widetilde{X} \to X$ is a universal cover, then each $\tilde{x}_0 \in p^{-1}(x_0)$ determines a bijection*

$$\begin{aligned} \ell : \pi_1(X, x_0) &\xrightarrow{\sim} p^{-1}(x_0) \\ [\gamma] &\longmapsto \tilde{x}_0 \bullet [\gamma]. \end{aligned}$$

We wish to use this bijection to understand the fundamental group, but it must be used carefully, as $\pi_1(X, x_0)$ is a group but $p^{-1}(x_0)$ does not have a group structure. Unravelling the definitions shows that the multiplication law on $p^{-1}(x_0)$ induced by this bijection can be described as follows: for $y_0, z_0 \in p^{-1}(x_0)$, the product $\ell(\ell^{-1}(y_0) \cdot \ell^{-1}(z_0))$ is obtained by

(i) choose a path $\tilde{\gamma} : I \to \widetilde{X}$ from $\tilde{x}_0$ to z_0 (this is unique up to homotopy, by Lemma 2.3.10),

(ii) let γ be the lift of the loop $p \circ \tilde{\gamma}$ starting at y_0,

(iii) then $\ell(\ell^{-1}(y_0) \cdot \ell^{-1}(z_0)) = \gamma(1)$.

3.2 The fundamental group of the circle and applications

We will now use this description to compute the group $\pi_1(S^1, 1)$.

Theorem 3.2.1. *Let $u : I \to S^1$ be the loop given by $t \mapsto e^{2\pi i t}$, which is based at $1 \in S^1 \subset \mathbb{C}$. Then there is a group isomorphism $\pi_1(S^1, 1) \cong (\mathbb{Z}, +)$ sending $[u]$ to 1.*

Proof. Recall from Example 3.1.4 that we constructed a covering space $p : \mathbb{R} \to S^1$ via $p(t) = e^{2\pi i t}$. The space $\mathbb{R}$ is contractible, and so simply connected, so this is a universal cover. As a set we have $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$, and we may choose $\tilde{x}_0 = 0 \in \mathbb{Z} = p^{-1}(1)$ as a basepoint. Acting on this basepoint then gives a bijection

$$\ell : \pi_1(S^1, 1) \longrightarrow \mathbb{Z}.$$

To compute $\ell^{-1}(m)$ we can choose the path $\tilde{u}_m : I \to \mathbb{R}$ given by $\tilde{u}_n(t) = mt$, which goes from 0 to m, so that $\ell^{-1}(m) = [p \circ \tilde{u}_m]$. Then we find that

$$\begin{aligned}
\ell(\ell^{-1}(n) \cdot \ell^{-1}(m)) &= \text{endpoint of lift of } [p \circ \tilde{u}_m] \text{ starting at } n \\
&= \text{endpoint of } t \mapsto n + mt : I \to \mathbb{R} \\
&= n + m
\end{aligned}$$

Thus the multiplication law on $\mathbb{Z} = p^{-1}(1)$ induced by ℓ is the usual addition of integers. Furthermore, $\ell([u]) = 1$. □

In the following theorem, we consider $D^2 \subset \mathbb{C}$ to be the set of complex numbers of modulus at most 1.

Theorem 3.2.2. *The disc D^2 does not retract to $S^1 \subset D^2$.*

Proof. Suppose that $r : D^2 \to S^1$ is a retraction, and let $i : S^1 \to D^2$ be the inclusion. By definition of retraction, $r \circ i = \mathrm{Id}_{S^1}$, and so taking fundamental groups based at $1 \in S^1 \subset D^2$ we obtain a factorisation

$$\mathbb{Z} \cong \pi_1(S^1, 1) \xrightarrow{i_*} \pi_1(D^2, 1) \xrightarrow{r_*} \pi_1(S^1, 1) \cong \mathbb{Z}$$

of the identity map. But D^2 is contractible, and so $\pi_1(D^2, 1) = \{[c_1]\}$ is the trivial group. This is a contradiction, as the identity map of $\mathbb{Z}$ cannot factor through a trivial group. □

Corollary 3.2.3 (Brouwer's fixed point theorem, 1909)**.** *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. Suppose f is a map without fixed points. Then define a function $r : D^2 \to S^1$ by sending x to the first point on S^1 hit by the ray from $f(x)$ through x. (This definition makes sense only because we know that $f(x) \neq x$.) One may convince oneself that r is continuous if f is. If $x \in S^1 \subset D^2$ then such a ray will first hit S^1 at x, and so $r(x) = x$. Thus r is a continuous retraction of D^2 to S^1, but this is impossible by Theorem 3.2.2. □

Theorem 3.2.4 (Fundamental theorem of algebra). *Every nonconstant polynomial over $\mathbb{C}$ has a root in $\mathbb{C}$.*

Proof. Suppose not, and let $p(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be a (monic) polynomial having no root. Fix

$$r > \begin{cases} |a_1| + |a_2| + \cdots + |a_n| \\ 1 \end{cases}$$

and note that on the circle $|z| = r$ we have the estimate

$$|z|^n = r^{n-1} \cdot r > |z|^{n-1} \cdot (|a_1| + |a_2| + \cdots + |a_n|) \geq |a_1 z^{n-1} + \cdots + a_n|$$

Thus for $t \in [0,1]$ the polynomial

$$p_t(z) := z^n + t \cdot (a_1 z^{n-1} + \cdots + a_n)$$

has no roots on the circle $|z| = r$.

Consider the homotopy of loops in $S^1 \subset \mathbb{C}$

$$F(s,t) = \frac{p_t(r \cdot e^{2\pi i s})/p_t(r)}{|p_t(r \cdot e^{2\pi i s})/p_t(r)|},$$

which is well-defined as the loop $s \mapsto p_t(r \cdot e^{2\pi i s})$ is never 0. When $t = 0$ this gives the loop $s \mapsto e^{2\pi i n s}$, which represents $n \in \pi_1(S^1, 1)$. When $t = 1$ this gives the loop

$$f_r(s) := \frac{p(r \cdot e^{2\pi i s})/p(r)}{|p(r \cdot e^{2\pi i s})/p(r)|}$$

in $S^1 \subset \mathbb{C}$, so we must have $[f_r] = n \in \pi_1(S^1, 1)$.

As p is assumed to have no roots, by varying r we see that f_r is homotopic to f_0, which is constant: thus $[f_r] = 0 \in \pi_1(S^1, 1)$. Hence $n = 0$, and so the polynomial p is constant. $\square$

3.3 The construction of universal covers

It practice we can often directly construct a universal cover of a given space X, but for theoretical work we need a *general* construction.

Observation 1. Suppose that $p : \widetilde{X} \to X$ is a universal cover, $x_0 \in X$ is a basepoint, and $U \ni x_0$ is a neighbourhood which is evenly covered, so $p^{-1}(U) = \coprod_{\alpha \in I} V_\alpha$ and each $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism. Fix some α. Then for any loop $\gamma : I \to U$ based at x_0 there is a lift to a loop $\tilde{\gamma} : I \to V_\alpha$ based at $\tilde{x}_0 = V_\alpha \cap p^{-1}(x_0)$. Thus the homomorphism

$$p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$$

sends $[\tilde{\gamma}]$ to $[\gamma]$. But $\widetilde{X}$ is simply connected, being a universal cover, so $[\tilde{\gamma}] = [c_{\tilde{x}_0}]$ and so $[\gamma] = [c_{x_0}]$.

Thus if X has a universal cover, then any point $x \in X$ has a neighbourhood U_x so that any loop in U_x based at x is contractible in X. This property is called being

i.e. $\pi_1(U_x, x) \xrightarrow{(\mathrm{incl})_*} \pi_1(X, x)$ is trivial.

semilocally simply connected, and we have just argued that it is necessary in order for a space to have a universal cover.

Observation 2. Suppose that $p : \widetilde{X} \to X$ is a universal cover and $x_0 = p(\tilde{x}_0)$ is a basepoint. As $\widetilde{X}$ is simply connected, for any point $y \in \widetilde{X}$ there is a unique homotopy class of paths $[\alpha]$ from $\tilde{x}_0$ to y, and $[p \circ \alpha]$ gives a preferred homotopy class of path from x_0 to $p(y)$. Thus we can recover y as the endpoint of the lift of $[p \circ \alpha]$ starting at $\tilde{x}_0$, which defines a bijection

$$\widetilde{X} \longrightarrow \left\{ \begin{array}{l} \text{homotopy classes of paths in } X \\ \text{starting at } x_0 \text{ and ending anywhere} \end{array} \right\}.$$

We may use this bijection in order to construct a universal cover, when we don't yet have one.

Theorem 3.3.1. *Let X be path connected, locally path connected*[1]*, and semilocally simply connected. Then there exists a covering map $p : \widetilde{X} \to X$ with $\widetilde{X}$ simply connected.*

Proof. (Non examinable.) As a set, define

$$\widetilde{X} := \left\{ \begin{array}{l} \text{homotopy classes of paths in } X \\ \text{starting at } x_0 \text{ and ending anywhere} \end{array} \right\}$$

with the function $p : \widetilde{X} \to X$ given by $p([\gamma]) = \gamma(1)$. We must construct a topology on $\widetilde{X}$, show that p is continuous in this topology, show that p is a covering map in this topology, and show that $\widetilde{X}$ is simply connected.

Let us first construct the topology on $\widetilde{X}$. Consider the set

$$\mathcal{U} := \left\{ U \subset X \;\middle|\; \begin{array}{l} U \text{ is open in } X, \text{ is path connected, and} \\ \pi_1(U, x) \to \pi_1(X, x) \text{ is trivial for all } x \in U \end{array} \right\}.$$

Claim: $\mathcal{U}$ is a *basis* for the topology on X.

Proof of claim. Let V be an open neighbourhood of a point $x \in X$.

(i) As X is semilocally simply connected there is a neighbourhood $U' \ni x$ such that $\pi_1(U', x) \to \pi_1(X, x)$ is trivial.

(ii) As X is locally path connected there is an open neighbourhood $x \in U \subset V \cap U'$ which is path connected.

(iii) The map $\pi_1(U, x) \to \pi_1(X, x)$ factors through the trivial map $\pi_1(U', x) \to \pi_1(X, x)$, so is trivial.

[1]Recall from Example Sheet 1 that X is **locally path connected** if for every point $x \in X$ and every neighbourhood $U \ni x$, there exists a smaller neighbourhood $x \in V \subset U$ such that V is path connected.

$$\begin{tikzcd} \pi_1(U,x) \arrow[rr] \arrow[dr] & & \pi_1(X,x) \\ & \pi_1(U',x) \arrow[ur, "\text{trivial}"'] & \end{tikzcd}$$

(iv) Let $y \in U$ be another point, and $u : I \to U$ a path from x to y. Then we have a commutative square

$$\begin{tikzcd} \pi_1(U,y) \arrow[r] \arrow[d, "u_\#"] & \pi_1(X,y) \arrow[d, "u_\#"] \\ \pi_1(U,x) \arrow[r, "\text{trivial}"] & \pi_1(X,x) \end{tikzcd}$$

where the vertical maps are isomorphisms, so the top map is trivial too.

Thus U is a neighbourhood of x contained in V which is in the collection $\mathcal{U}$, and hence $\mathcal{U}$ is indeed a basis for the topology on X. $\square$

Note that if $x, y \in U \in \mathcal{U}$ then there exists a path from x to y in U, and all such paths are homotopic in X. For a $[\alpha] \in \widetilde{X}$ and a $U \in \mathcal{U}$ such that $\alpha(1) \in U$, define

$$(\alpha, U) := \left\{ [\beta] \in \widetilde{X} \;\middle|\; \begin{array}{l} [\beta] = [\alpha \cdot \alpha'] \text{ for some path} \\ \alpha' \text{ in } U \text{ starting at } \alpha(1) \end{array} \right\}.$$

Claim: These sets form a basis for a topology on $\widetilde{X}$.

Proof of claim. We must check that $[\beta] \in (\alpha_0, U_0) \cap (\alpha_1, U_1)$ has a neighbourhood of this form.

Let $W \subset U_0 \cap U_1$ be a neighbourhood of $\beta(1) \in U_0 \cap U_1$ in the collection $\mathcal{U}$, which exists as $\mathcal{U}$ is a basis for the topology on X. Then (β, W) is a neighbourhood of $[\beta]$, and it is enough to show that $(\beta, W) \subset (\alpha_0, U_0) \cap (\alpha_1, U_1)$. If $[\gamma] \in (\beta, W)$ then it is a concatenation of β and a path in W, but then it is a concatenation of α_0 and a path in U_0 too; similarly, it is a concatenation of α_1 and a path in U_1. $\square$

We give $\widetilde{X}$ the topology generated by this basis.

Let us now show that p is continuous in this topology; as $\mathcal{U}$ is a basis for the topology on X, it is enough to show that $p^{-1}(U) \subset \widetilde{X}$ is open for each $U \in \mathcal{U}$. But if $[\alpha] \in p^{-1}(U)$, so α is a path starting at x_0 and ending in U, then $[\alpha] \in (\alpha, U) \subset p^{-1}(U)$.

Let us now show that p is a covering map. We first claim that each map

$$p|_{(\alpha,U)} : (\alpha, U) \to U$$

is a homeomorphism. It is surjective as U is path connected, and hence every point in U may be reached by a path from $\alpha(1)$. If $[\beta], [\beta'] \in (\alpha, U)$ are homotopy classes of path which end at the same point, and each may be obtained from α by concatenation of a path in U, then they differ by concatenation of a loop in U. As every loop in U is contractible in X, it follows that $[\beta] = [\beta']$ and so the map is injective. Finally $p((\gamma, V)) = V$, so p is an open map, and hence $p|_{(\alpha,U)}$ is too.

We now claim that $p^{-1}(U)$ is partitioned into sets of the form (α, U). We have already show that it is covered by sets of this form, so it remains to show that any two such sets are either disjoint or equal. Thus suppose $[\gamma] \in (\alpha, U) \cap (\beta, U)$. Then $[\gamma] = [\alpha \cdot \alpha'] = [\beta \cdot \beta']$ for paths α' and β' in U, and so $[\alpha] = [\beta \cdot \beta' \cdot (\alpha')^{-1}]$ and $\beta' \cdot (\alpha')^{-1}$ is a path in U, so $[\alpha] \in (\beta, U)$. But then if $[\delta] \in (\alpha, U)$, so $[\delta] = [\alpha \cdot \alpha'']$, then $[\delta] = [\beta \cdot \beta' \cdot (\alpha')^{-1} \cdot \alpha'']$

and so $[\delta] \in (\beta, U)$. Thus $(\alpha, U) \subset (\beta, U)$, and the reverse inclusion follows by the same argument.

Finally, we must show that $\widetilde{X}$ is simply connected. The fundamental observation is that if $\gamma : I \to X$ is a path starting at x_0, then its lift to $\widetilde{X}$ starting at $[c_{x_0}] \in \widetilde{X}$ is the path

$$s \longmapsto [t \mapsto \gamma(st)] : I \longrightarrow \widetilde{X},$$

which ends at the point $[\gamma] \in \widetilde{X}$. Thus if a loop γ in X based at x_0 lifts to a loop in $\widetilde{X}$ based at $[c_{x_0}]$ then $[\gamma] = [c_{x_0}]$, which shows that $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) = \{e\} \leq \pi_1(X, x_0)$. But then by Lemma 3.1.14 the group $\pi_1(\widetilde{X}, \tilde{x}_0)$ must be trivial. □

3.4 The Galois correspondence

We have seen in Lemma 3.1.14 that if $p : \widetilde{X} \to X$ is a path connected covering space, $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$, then $p_* : \pi_1(\widetilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$ is injective, and we obtain a subgroup $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) \leq \pi_1(X, x_0)$.

If $\tilde{x}_0' \in p^{-1}(x_0)$ is another point, then choose a path γ from $\tilde{x}_0$ to $\tilde{x}_0'$. Then $p \circ \gamma$ is a loop in X based at x_0, so we have an element $[p \circ \gamma] \in \pi_1(X, x_0)$, and

$$[p \circ \gamma]^{-1} \cdot p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) \cdot [p \circ \gamma] = p_*(\pi_1(\widetilde{X}, \tilde{x}_0'))$$

i.e. the subgroups obtained from these two basepoints in $\widetilde{X}$ are conjugate. Thus fixing a based space (X, x_0) we obtain functions

$$\left\{ \begin{array}{c} \text{based covering maps} \\ p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0) \end{array} \right\} \longrightarrow \{\text{subgroups of } \pi_1(X, x_0)\}$$

and

$$\left\{\text{covering maps } p : \widetilde{X} \to X\right\} \longrightarrow \left\{ \begin{array}{c} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(X, x_0) \end{array} \right\}.$$

We wish to show that dividing out by an appropriate equivalence relation on the left hand side turn both of these functions into bijections.

Proposition 3.4.1 (Surjectivity). *Suppose that X is path connected, locally path connected, and semilocally simply connected. Then for any subgroup $H \leq \pi_1(X, x_0)$ there is a based covering map $p : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\widetilde{X}, \tilde{x}_0)) = H$.*

Note that taking H to be the trivial group, this proposition implies the existence of a universal cover. Thus it requires all the technical hypotheses of Theorem 3.3.1.

Proof. Let $q : \overline{X} \to X$ be the universal cover constructed in Theorem 3.3.1, whose underlying set consists of the homotopy classes of paths in X starting at x_0. Define a relation $\sim_H$ on $\overline{X}$ by

$$[\gamma] \sim_H [\gamma'] \text{ if } \gamma(1) = \gamma'(1) \text{ and } [\gamma \cdot (\gamma')^{-1}] \in H \subset \pi_1(X, x_0).$$

(i) $[\gamma] \sim_H [\gamma]$ as $[c_{x_0}] \in H$,

(ii) if $[\gamma] \sim_H [\gamma']$ then $[\gamma \cdot (\gamma')^{-1}] \in H$, and so $[\gamma' \cdot \gamma^{-1}] \in H$, and hence $[\gamma'] \sim_H [\gamma]$,

(iii) if $[\gamma] \sim_H [\gamma'] \sim_H [\gamma'']$, then $[\gamma \cdot (\gamma')^{-1}], [\gamma' \cdot (\gamma'')^{-1}] \in H$ and so the product $[\gamma \cdot (\gamma'')^{-1}] \in H$ too.

Thus $\sim_H$ is an equivalence relation. (We could have defined $\sim_H$ for any subset H of $\pi_1(X, x_0)$, but this argument shows that such a $\sim_H$ is an equivalence relation if and only if H is a subgroup.)

Let us define $\widetilde{X}_H := \overline{X}/\sim_H$ to be the quotient space, and $p_H : \widetilde{X}_H \to X$ to be the induced map. Note that if $[\gamma] \in (\alpha, U)$ and $[\gamma'] \in (\beta, U)$ have $[\gamma] \sim_H [\gamma']$ then (α, U) and (β, U) are identified by $\sim_H$, as $[\gamma \cdot \eta] \sim_H [\gamma' \cdot \eta]$ for η a path in U. Thus p_H is a covering map.

It remains to show that $(p_H)_*(\pi_1(\widetilde{X}_H, [[c_{x_0}]])) = H$. If $[\gamma] \in H$ then the lift if $\bar{\gamma}$ of γ to $\overline{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$, so the lift to $\widetilde{X}_H$ ends at $[[\gamma]] = [[c_{x_0}]]$ so is a loop. This shows the containment $\supseteq$. On the other hand, if $[\gamma] = (p_H)_*([\tilde{\gamma}])$ for $\tilde{\gamma}$ a loop in $\widetilde{X}_H$, then the lift of $\tilde{\gamma}$ to $\overline{X}$ starting at $[c_{x_0}]$ ends at $[\gamma]$. As $\tilde{\gamma}$ is a loop in $\widetilde{X}_H$, $[\gamma]$ must be identified with $[c_{x_0}]$, so lies in H; this shows the containment $\subseteq$. $\square$

Proposition 3.4.2 (Based uniqueness). *If* Let (X, x0) satisfy the conditions of universal covers

$$p_1 : (\widetilde{X}_1, \tilde{x}_1) \longrightarrow (X, x_0) \quad \text{and} \quad p_2 : (\widetilde{X}_2, \tilde{x}_2) \longrightarrow (X, x_0)$$

are path connected covering spaces, then there is a based homeomorphism

$$h : (\widetilde{X}_1, \tilde{x}_1) \longrightarrow (\widetilde{X}_2, \tilde{x}_2)$$

such that $p_2 \circ h = p_1$ *if and only if* $(p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$.

Proof. If h exists, we have $(p_1)_* = (p_2)_* \circ h_*$ and h_* is an isomorphism, so $(p_1)_*$ and $(p_2)_*$ have the same image.

For the converse direction, let $H := (p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1))$; it is then enough to give a based homeomorphism $h : (\widetilde{X}_H, [[c_{x_0}]]) \to (\widetilde{X}_1, \tilde{x}_1)$. We begin with the map

$$r : (\overline{X}, [c_{x_0}]) \longrightarrow (\widetilde{X}_1, \tilde{x}_1)$$

given by sending the point $[\gamma]$ to the end point of the lift $\tilde{\gamma}$ of γ to $\widetilde{X}_1$ starting at $\tilde{x}_1$.

Now

$$\begin{aligned} r([\gamma]) = r([\gamma']) &\iff \tilde{\gamma} \text{ and } \tilde{\gamma}' \text{ end at the ssame point in } \widetilde{X}_1 \\ &\iff [\gamma' \cdot \gamma^{-1}] \in \pi_1(\widetilde{X}_1, \tilde{x}_1) = H \\ &\iff [\gamma] \sim_H [\gamma'] \end{aligned}$$

and so the map r descends to a continuous bijection

$$q : (\widetilde{X}_H, [[c_{x_0}]]) \longrightarrow (\widetilde{X}_1, \tilde{x}_1)$$

of covering spaces over (X, x_0). This map is also open, as both covering maps to X are local homeomorphisms, and so q is a homeomorphism. $\square$

Proposition 3.4.3 (Unbased uniqueness). *If*

$$p_1 : \widetilde{X}_1 \longrightarrow X \quad \text{and} \quad p_2 : \widetilde{X}_2 \longrightarrow X$$

are path connected covering spaces, then there is a homeomorphism

$$h : \widetilde{X}_1 \longrightarrow \widetilde{X}_2$$

such that $p_2 \circ h = p_1$ if and only if $(p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1))$ is conjugate to $(p_2)_*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$ for some $\tilde{x}_i \in p_i^{-1}(x_0)$.

Proof. If h exists, choose a $\tilde{x}_1 \in p_1^{-1}(x_0)$ and let $\tilde{x}_2 = h(\tilde{x}_1)$. With these basepoints everything in sight is based, and so $(p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1)) = (p_2)_*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$.

Conversely, suppose we have chosen $\tilde{x}_1$ and $\tilde{x}_2$ such that

$$[\gamma]^{-1} \cdot (p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1)) \cdot [\gamma] = (p_2)_*(\pi_1(\widetilde{X}_2, \tilde{x}_2))$$

for some $[\gamma] \in \pi - 1(X, x_0)$. If we lift γ starting at $\tilde{x}_1$ then it ends at some $\tilde{x}_1' \in \widetilde{X}_1$, and

$$(p_1)_*(\pi_1(\widetilde{X}_1, \tilde{x}_1')) = (p_2)_*(\pi_1(\widetilde{X}_2, \tilde{x}_2)).$$

Then the previous proposition gives a based homeomorphism $h : (\widetilde{X}_1, \tilde{x}_1') \to (\widetilde{X}_2, \tilde{x}_2)$. $\square$

Chapter 4

Some group theory

4.1 Free groups and presentations

Let $S = \{s_\alpha\}_{\alpha \in I}$ be a set, called the **alphabet**, and let $S^{-1} = \{s_\alpha^{-1}\}_{\alpha \in I}$ (we suppose that $S \cap S^{-1} = \emptyset$). A **word** in the alphabet S is a (possibly empty) finite sequence

$$(x_1, x_2, \ldots, x_n)$$

of elements of $S \cup S^{-1}$. A word is **reduced** if it contains no subwords

$$(s_\alpha, s_\alpha^{-1}) \quad \text{or} \quad (s_\alpha^{-1}, s_\alpha).$$

An **elementary reduction** of a word $(x_1, \ldots, x_i, s_\alpha, s_\alpha^{-1}, x_{i+3}, \ldots, x_n)$ means replacing it by $(x_1, \ldots, x_i, x_{i+3}, \ldots, x_n)$, and similarly with words containing $(s_\alpha^{-1}, s_\alpha)$.

Definition 4.1.1 (Free group)**.** The **free group** on the alphabet S, $F(S)$, is the set of reduced (possibly empty) words in S. The group operation is given by concatenating words and then applying elementary reductions until the word is reduced.

It is not really clear that this group operation is well-defined, or that it is associative. It is clear though that it is unital

$$() \cdot (x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) \cdot ()$$

and that it has inverses

$$(x_1, x_2, \ldots, x_n) \cdot (x_n^{-1}, x_{n-1}^{-1}, \ldots, x_1^{-1}) = ().$$

In Section 4.2 we give an alternative description of $F(S)$ which is obviously a group, and show it agrees with that of Definition 4.1.1.

By construction there is a function $\iota : S \to F(S)$ given by sending the element s_α to the reduced word (s_α).

Lemma 4.1.2 (Universal property of free groups)**.** *For any group H, the function*

$$\left\{ \begin{array}{l} \textit{group homomorphisms} \\ \varphi : F(S) \to H \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \textit{functions} \\ \phi : S \to H \end{array} \right\},$$

given by precomposing with ι, is a bijection.

Proof. Given a function $\phi : S \to H$ we want a homomorphism φ such that $\varphi((s_\alpha)) = \phi(s_\alpha)$. But there is a unique way to do this, by defining, on not necessarily reduced words,

$$\varphi((s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})) = \phi(s_{\alpha_1})^{\epsilon_1} \cdots \phi(s_{\alpha_n})^{\epsilon_n}.$$

Note that if $(s_{\alpha_1}^{\epsilon_1}, \dots, s_{\alpha_n}^{\epsilon_n})$ is not reduced, so contains for example $(s_\alpha, s_\alpha^{-1})$, then the product $\phi(s_{\alpha_1})^{\epsilon_1} \cdots \phi(s_{\alpha_n})^{\epsilon_n}$ contains $\phi(s_\alpha) \cdot \phi(s_\alpha)^{-1} = 1$ and so we may reduce the word $(s_{\alpha_1}^{\epsilon_1}, \dots, s_{\alpha_n}^{\epsilon_n})$ without changing the value of φ on it. As the group operation in $F(S)$ is given by concatenation and reduction of words, this shows that φ is a homomorphism. □

Definition 4.1.3 (Presentation)**.** Let S be a set and $R \subset F(S)$ be a subset. The group $\langle S \,|\, R \rangle$ is defined to be the quotient $F(S)/\langle\langle R \rangle\rangle$ of (S) by the smallest normal subgroup containing R. More concretely,

$$\langle\langle R \rangle\rangle = \{(r_1^{\epsilon_1})^{g_1} \cdot (r_2^{\epsilon_2})^{g_2} \cdots (r_n^{\epsilon_n})^{g_n} \,|\, r_i \in R, \epsilon_i \in \{\pm 1\}, g_i \in F(S)\}.$$

We call $\langle S \,|\, R \rangle$ the group with generators S and relations R, and call the data (S, R) a **presentation** of this group. If S and R are both finite, we call it a **finite presentation**.

Informally, we consider elements of $\langle S \,|\, R \rangle$ as words in the alphabet S, up to the relation that we may apply elementary reductions (or the converse) and cancel out any subword lying in R (or the converse).

Lemma 4.1.4 (Universal property of group presentations)**.** *For any group H, the function*

$$\left\{ \begin{array}{l} \textit{group homomorphisms} \\ \psi : \langle S \,|\, R \rangle \to H \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \textit{functions } \phi : S \to H \textit{ such} \\ \textit{that } \varphi(r) = 1 \textit{ for each } r \in R \end{array} \right\},$$

given by sending ψ to the composition $\phi : S \xrightarrow{\iota} F(S) \xrightarrow{quot.} \langle S \,|\, R \rangle \xrightarrow{\psi} H$, is a bijection

Proof. If ψ and ψ' give the same functions $\phi = \phi' : S \to H$, then by Lemma 4.1.2 the two homomorphisms

$$\begin{tikzcd} F(S) \arrow[r, "\text{quot.}"] & \langle S \,|\, R \rangle \arrow[r, shift left, "\phi"] \arrow[r, shift right, "\phi'"'] & H \end{tikzcd}$$

are equal, and as the quotient map is surjective it follows that $\psi = \psi'$.

On the other hand, let a function $\phi : S \to H$ be given such that the induced map $\varphi : F(S) \to H$ satisfied $\varphi(r) = 1$ for all $r \in R$. Then $R \subset \mathrm{Ker}(\varphi)$, and $\mathrm{Ker}(\varphi)$ is a normal subgroup of $F(S)$ so $\langle\langle R \rangle\rangle \subset \mathrm{Ker}(\varphi)$ too (as $\langle\langle R \rangle\rangle$ was the minimal normal subgroup containing R). Thus φ descends to a homomorphism $\psi : F(S)/\langle\langle R \rangle\rangle \to H$. □

Example 4.1.5 (The canonical[1] presentation)**.** Let G be a group, and $\phi = \mathrm{Id}_G : G \to G$ be the identity homomorphism. By Lemma 4.1.2 there is an induced homomorphism $\varphi : F(G) \to G$. This is certainly surjective, as the word (g) maps to $g \in G$, and we can let $R := \mathrm{Ker}(\varphi)$. By Lemma 4.1.4 there is an induced homomorphism

$$\psi : \langle G \,|\, R \rangle \longrightarrow G$$

which is an isomorphism (by the first isomorphism theorem).

By this example every group may be given a presentation, though of course this is quite useless for practical work.

---

[1] or "stupid"

Example 4.1.6. Let $G = \langle a, b \,|\, a\rangle$ and $H = \langle t \,|\, \rangle$, and let us show that these groups are isomorphic. First consider the function

$$\begin{aligned} \phi : \{a, b\} &\longrightarrow H \\ a &\longmapsto e \\ b &\longmapsto [t] \end{aligned}$$

which satisfies $\varphi(a) = e$ and so defines a homomorphism $\psi : G \to H$. Now consider the function

$$\begin{aligned} \phi' : \{t\} &\longrightarrow G \\ t &\longmapsto [b] \end{aligned}$$

which defines a homomorphism $\psi' : H \to G$.

Now $\psi \circ \psi'([t]) = \psi([b]) = [t]$, and as $[t]$ generates H it follows that $\psi \circ \psi' = \mathrm{Id}_H$. On the other hand $\psi' \circ \psi([b]) = \psi'([t]) = [b]$ and

$$\psi' \circ \psi([a]) = \psi'(1) = e = [a]$$

in the group G, as a is a relation. As $[a]$ and $[b]$ generate G, it follows that $\psi' \circ \psi = \mathrm{Id}_G$.

Example 4.1.7. Let $G = \langle a, b \,|\, ab^{-3}, ba^{-2}\rangle$. In this group we have $[a] = [b]^3$ and $[b] = [a]^2$, so putting these together $[a] = [a]^6$, and multiplying by $[a]^{-1}$ we get $[a]^5 = e$. Thus using $[b] = [a]^2$ we can write every word in a and b just using the letter a, and using $[a]^5 = e$ we may ensure that the largest power of a appearing is 4. Thus the group G has at most 5 elements, e, $[a]$, $[a]^2$, $[a]^3$, $[a]^4$. Lets guess that it might be $\mathbb{Z}/5$, and prove this.

Consider the function

$$\begin{aligned} \phi : \{a, b\} &\longrightarrow \mathbb{Z}/5 \\ a &\longmapsto 1 \\ b &\longmapsto 2. \end{aligned}$$

Then $\varphi(ab^{-3}) = 1 - 2 \cdot 3 = -5 = 0 \in \mathbb{Z}/5$ and $\varphi(ba^{-2}) = 2 - 2 = 0 \in \mathbb{Z}/5$ so by Lemma 4.1.4 this gives a homomorphism

$$\psi : \langle a, b \,|\, ab^{-3}, ba^{-2}\rangle \longrightarrow \mathbb{Z}/5.$$

This homomorphism sends $[a]$ to $1 \in \mathbb{Z}/5$ so is surjective, but we already argued that G had at most 5 elements, so it is also injective: thus ψ is an isomorphism.

4.2 Another view of free groups Not Lectured

Fix a set S and let W be the set of reduced words in the alphabet S, and $P(W)$ be the group of permutations of the set W.

Definition 4.2.1. For each $\alpha \in I$, define a function $L_\alpha : W \to W$ by the formula

$$L_\alpha(x_1, x_2, \ldots, x_n) = \begin{cases} (s_\alpha, x_1, x_2, \ldots, x_n) & \text{if } x_1 \neq s_\alpha^{-1} \\ (x_2, \ldots, x_n) & \text{if } x_1 = s_\alpha^{-1}. \end{cases}$$

Note that in the second case $x_2 \neq s_\alpha$, otherwise $(x_1, x_2, \ldots, x_n)$ would not be reduced.

Lemma 4.2.2. *L_α is a bijection, so represents an element of $P(W)$.*

Proof. Let $(x_1, \ldots, x_n) \in W$. If $x_1 = s_\alpha$ then $x_2 \neq s_\alpha^{-1}$, so $L_\alpha(x_2, \ldots, x_n) = (x_1, \ldots, x_n)$. If $x_1 \neq s_\alpha$ then $(s_\alpha^{-1}, x_1, \ldots, x_n)$ is a reduced word, and $L_\alpha(s_\alpha^{-1}, x_1, \ldots, x_n) = (x_1, \ldots, x_n)$. Thus L_α is surjective.

If $L_\alpha(x_1, x_2, \ldots, x_n) = L_\alpha(y_1, y_2, \ldots, y_m)$ and this reduced word starts with s_α then $x_1 \neq s_\alpha^{-1}$ and $y_1 \neq s_\alpha^{-1}$, and so $x_i = y_i$ for each i. If this reduced word does not start with s_α then $x_1 = y_1 = s_\alpha^{-1}$, and

$$(x_2, \ldots, x_n) = L_\alpha(x_1, x_2, \ldots, x_n) = L_\alpha(y_1, y_2, \ldots, y_m) = (y_2, \ldots, y_m),$$

so $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_m)$. □

We can now give an alternative definition of the free group, which is manifestly a group.

Definition 4.2.3 (Free group)**.** The free group $F(S)$ is the subgroup of $P(W)$ generated by the elements $\{L_\alpha\}_{\alpha \in I}$.

Lemma 4.2.4. *The function $\phi : F(S) \to W$ given by $\sigma \mapsto \sigma \cdot ()$ is a bijection.*

This identifies $F(S)$ with the set of reduced words W in the alphabet S, and shows that the group operation is given by concatenation of words followed by word reduction. Thus the definition given in Definition 4.1.1 is indeed a group.

Proof. If $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is a reduced word, with $\epsilon_i \in \{\pm 1\}$, then

$$(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n}) = L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n} \cdot () = \phi(L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}),$$

and so ϕ is surjective.

As the $\{L_\alpha\}_{\alpha \in I}$ generate $F(S)$, any element σ may be represented by a concatenation

$$\sigma = L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n} \in P(W).$$

As $L_\alpha \cdot L_\alpha^{-1} = \mathrm{Id}_W$ and $L_\alpha^{-1} \cdot L_\alpha = \mathrm{Id}_W$, if the word $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is not reduced then we can simplify $L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}$ while giving the same element $\sigma \in P(W)$. Thus we may suppose that any σ is represented by $L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}$ such that the associated word $(s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n})$ is reduced. But then

$$\phi(\sigma) = \sigma \cdot () = (s_{\alpha_1}^{\epsilon_1}, \ldots, s_{\alpha_n}^{\epsilon_n}),$$

from which we can recover $\sigma = L_{\alpha_1}^{\epsilon_1} \cdots L_{\alpha_n}^{\epsilon_n}$, which shows that ϕ is injective. □

4.3 Free products with amalgamation

Let H, G_1, and G_2 be groups, and let

$$G_1 \xleftarrow{i_1} H \xrightarrow{i_2} G_2$$

be homomorphisms. Suppose that we have presentations $G_i = \langle S_i \mid R_i \rangle$ (which is no loss of generality, by Example 4.1.5). Then the **free product** of G_1 and G_2 is

$$G_1 * G_2 := \langle S_1 \cup S_2 \mid R_1 \cup R_2 \rangle.$$

The functions $S_i \to S_1 \cup S_2 \to F(S_1 \cup S_2) \to G_1 * G_2$ and Lemma 4.1.4 give canonical homomorphisms

$$G_1 \xrightarrow{j_1} G_1 * G_2 \xleftarrow{j_2} G_2$$

and we define the **free product with amalgamation** over H to be the quotient

$$G_1 *_H G_2 := (G_1 * G_2)/\langle\langle j_1 i_1(h)(j_2 i_2(h))^{-1} \mid h \in H \rangle\rangle.$$

Note that the free product is the free product with amalgamation for $H = \{e\}$. By construction, the square

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"] & G_1 *_H G_2 \end{tikzcd}$$

commutes, that is $j_1 \circ i_1 = j_2 \circ i_2 : H \to G_1 *_H G_2$.

Lemma 4.3.1 (Universal property of amalgamated free products)**.** *For any group K the function*

$$\left\{ \begin{array}{l} \text{group homomorphisms} \\ \phi : G_1 *_H G_2 \to K \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{group homomorphisms } \phi_1 : G_1 \to K \\ \text{and } \phi_2 : G_2 \to K \text{ such that } \phi_1 \circ i_1 = \phi_2 \circ i_2 \end{array} \right\},$$

given by sending ϕ to $\phi_1 = \phi \circ j_1$ and $\phi_2 = \phi \circ j_2$, is a bijection.

Proof. Given ϕ_1 and ϕ_2, define $\hat{\phi} : F(S_1 \cup S_2) \to K$ via the function

$$\begin{aligned} \hat{\hat{\phi}} : S_1 \cup S_2 &\longrightarrow K \\ s \in S_i &\longmapsto \phi_i((s)). \end{aligned}$$

If $r \in R_1$ then $\hat{\phi}(r) = \phi_1(r) = e$ as $[r] = e \in G_1 = \langle S_1 \mid R_1 \rangle$; similarly for $r \in R_2$. Thus $\hat{\phi}$ descends to a homomorphism $\tilde{\phi} : G_1 * G_2 \to K$. Finally, for $h \in H$ we have

$$\tilde{\phi}(j_1 i_1(h)(j_2 i_2(h))^{-1}) = \tilde{\phi}(j_1 i_1(h))\tilde{\phi}(j_2 i_2(h))^{-1} = \phi_1 i_1(h)(\phi_2 i_2(h))^{-1} = e$$

and so $\langle\langle j_1 i_1(h)(j_2 i_2(h))^{-1} \mid h \in H \rangle\rangle$ lies in $\mathrm{Ker}(\tilde{\phi})$, so $\tilde{\phi}$ descends to a homomorphism $\phi : G_1 *_H G_2 \to K$ as required. $\square$

Chapter 5

The Seifert–van Kampen theorem

5.1 The Seifert–van Kampen theorem

Let X be a space, $A, B \subset X$ be subsets, and $x_0 \in A \cap B$ be a point. There is then a commutative diagram

$$\begin{tikzcd}
\pi_1(A \cap B, x_0) \arrow[r] \arrow[d] & \pi_1(A, x_0) \arrow[d] \\
\pi_1(B, x_0) \arrow[r] & \pi_1(X, x_0)
\end{tikzcd}$$

where all the homomorphisms are those induced on the fundamental group by the natural inclusions. By the universal property of amalgamated free products (Lemma 4.3.1) there is a corresponding homomorphism

$$\phi : \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \longrightarrow \pi_1(X, x_0).$$

The Seifert–van Kampen theorem concerns conditions under which this homomorphism is an isomorphism.

Theorem 5.1.1 (Seifert–van Kampen)**.** *Let X be a space, and $A, B \subset X$ be open subsets which cover X and such that $A \cap B$ is path connected. Then for any $x_0 \in A \cap B$ the homomorphism*

$$\phi : \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) \longrightarrow \pi_1(X, x_0)$$

is an isomorphism.

Before giving the proof of this theorem, we give some important examples.

Example 5.1.2 (Spheres)**.** Consider the sphere S^n for $n \geq 2$. This may be covered by the open sets

$$U := \{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_{n+1} > -1\}$$
$$V := \{(x_1, x_2, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < 1\}$$

the complements of the north and south poles. Each of these is open, and stereographic projection shows that they are homeomorphic to $\mathbb{R}^n$ and so contractible. Finally, cylindrical projection gives a homeomorphism

$$U \cap V \cong S^{n-1} \times (-1, 1)$$

and as $n \geq 2$ this is path connected. Thus by the Seifert–van Kampen theorem we have

$$\pi_1(S^n, x_0) \cong \{e\} *_{\pi_1(U \cap V, x_0)} \{e\}$$

and so $\pi_1(S^n, x_0) \cong \{e\}$. Thus the n-sphere is simply connected for $n \geq 2$.

Example 5.1.3 (Real projective space)**.** Recall that we have shown that the quotient map $p : S^n \to \mathbb{RP}^n$ is a covering map, with two sheets. If $n \geq 2$ then S^n is simply connected, by the previous example, and so this is the universal cover of $\mathbb{RP}^n$. By Lemma 3.1.15 (iii), there is then a bijection between the group $\pi_1(\mathbb{RP}^n, x_0)$ and the set $p^{-1}(x_0)$ with two elements: there is only one group of cardinality two, so $\pi_1(\mathbb{RP}^n, x_0) = \mathbb{Z}/2$.

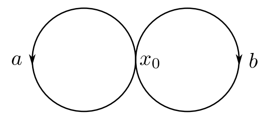
For based spaces (X, x_0) and (Y, y_0), the **wedge product** $X \vee Y$ is the quotient space of $X \sqcup Y$ by the relation generated by $x_0 \sim y_0$. This common point is taken to be the basepoint of $X \vee Y$.

Example 5.1.4 (Wedge of two circles)**.** Let the circle $S^1 \subset \mathbb{C}$ have the basepoint $1 \in S^1$, and consider the space $S^1 \vee S^1$ with basepoint x_0 the wedge point. Let us take the open cover

$$U := S^1 \vee (S^1 \setminus \{-1\}) \qquad V := (S^1 \setminus \{-1\}) \vee S^1.$$

As $S^1 \setminus \{-1\}$ deformation retracts to the basepoint $1 \in S^1$, notice that U deformation retracts to $S^1 \vee \{1\} \cong S^1$, V deformation retracts to $\{1\} \vee S^1$, and $U \cap V = (S^1 \setminus \{-1\}) \vee (S^1 \setminus \{-1\})$ deformation retracts to $\{x_0\}$. Thus by the Seifert–van Kampen theorem we have

$$\pi_1(S^1 \vee S^1, x_0) \cong \pi_1(S^1 \vee \{1\}, x_0) * \pi_1(\{1\} \vee S^1, x_0).$$



If we let a be the standard loop around the first circle and b be the standard loop around the second circle, as shown above, we thus have

$$\pi_1(S^1 \vee S^1, x_0) \cong \langle a, b \,|\, \rangle,$$

a free group on two generators.

This is our first example of a space with a really complicated fundamental group. Let us apply the theory of Section 3.4 to it, to investigate some of its covering spaces.

Example 5.1.5. The function

$$\begin{aligned} \phi : \{a, b\} &\longrightarrow \mathbb{Z}/3 \\ a &\longmapsto 1 \\ b &\longmapsto 1 \end{aligned}$$

induces a surjective homomorphism $\varphi : \pi_1(S^1 \vee S^1, *) = \langle a, b \,|\, \rangle \to \mathbb{Z}/3$. Thus $K := \mathrm{Ker}(\varphi)$ is an index 3 subgroup of $\pi_1(S^1 \vee S^1, *)$. By the results of Section 3.4 there is a corresponding unique based covering space

$$p : (\widetilde{X}, \tilde{x}_0) \longrightarrow (S^1 \vee S^1, *)$$

with $p_*(\pi_1(\widetilde{X}, \tilde{x_0})) = K \leq \pi_1(S^1 \vee S^1, *)$. Let us work out what it looks like.

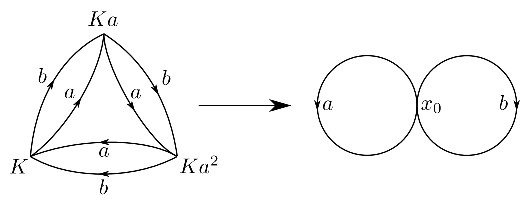
By Lemma 3.1.15 (iii), the group $\pi_1(S^1 \vee S^1, *)$ acting on the point $\tilde{x}_0 \in p^{-1}(*)$ induces a bijection

$$\frac{\pi_1(S^1 \vee S^1, *)}{p_*(\pi_1(\widetilde{X}, \tilde{x_0}))} \longrightarrow p^{-1}(*)$$

and so p is a 3-sheeted cover, because $p_*(\pi_1(\widetilde{X}, \tilde{x_0})) = K$ has index 3. The 3 elements of $p^{-1}(*)$ are thus identified with the right cosets of K, lets say $\tilde{x}_0 = K$, Ka, and Ka^2.

Each of the edges a and b in $S^1 \vee S^1$ has 3 lifts to edges in $\widetilde{X}$, and we wish to know where these go from and to. This is easy: if $x \in \pi_1(S^1 \vee S^1, *) = \langle a, b \,|\, \rangle$ is a word in a and b, then lifting it starting at the point $Ky \in p^{-1}(*)$ gives a path ending at Kyx.

Thus the lift of a starting at K ends at Ka, the lift starting at Ka ends at Ka^2, and the lift starting at Ka^2 ends at $Ka^3 = K$. Similarly, the lift of b starting at K ends at $Kb = Ka$, the lift starting at Ka ends at $Kab = Ka^2$, and the lift starting at Ka^2 ends at $Ka^2b = K$. Thus the covering space $\widetilde{X}$ looks like

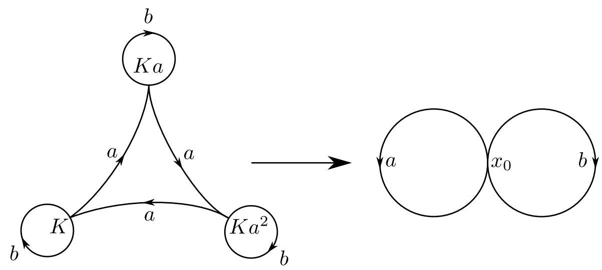


where the labels on the edges show to which edge in $S^1 \vee S^1$ they map to.

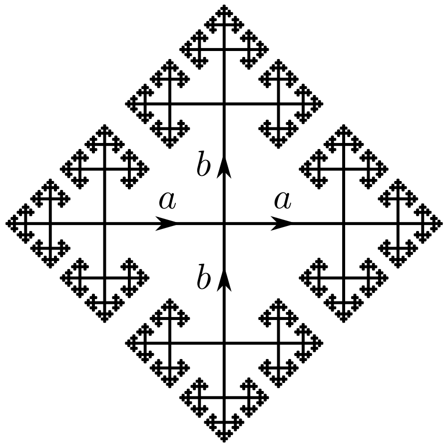
Example 5.1.6. The function

$$\begin{aligned} \phi : \{a, b\} &\longrightarrow \mathbb{Z}/3 \\ a &\longmapsto 1 \\ b &\longmapsto 0 \end{aligned}$$

induces a surjective homomorphism $\varphi : \pi_1(S^1 \vee S^1, *) = \langle a, b \,|\, \rangle \to \mathbb{Z}/3$. Thus $K := \mathrm{Ker}(\varphi)$ is an index 3 subgroup of $\pi_1(S^1 \vee S^1, *)$, and there is a corresponding covering space $p : (\widetilde{X}, \tilde{x}_0) \to (S^1 \vee S^1, *)$. The same analysis as in the last example gives the covering space



Example 5.1.7. The universal cover of $S^1 \vee S^1$ may be described as the infinite 4-valent tree (i.e. graph with no loops), where each vertex has a copy of the edge a coming in, a copy of the edge a going out, a copy of the edge b coming in, and a copy of the edge b



going out. This certainly describes *a* covering space of $S^1 \vee S^1$; it is easy to see that this tree is simply connected, by showing that it is in fact contractible.

Let us now move on to the proof of the Seifert–van Kampen theorem.

Proof of Theorem 5.1.1. First observe that we may assume that A and B are also path connected: if not, there is a path component of A or B which does not intersect $A \cap B$, and so this lies in a different path component of X to x_0. Hence loops based at x_0 cannot reach it, so removing it does not change any of $\pi_1(A, x_0)$, $\pi_1(B, x_0)$, $\pi_1(A \cap B, x_0)$ or $\pi_1(A \cup B, x_0)$.

The map ϕ is surjective. Let $\gamma : I \to X$ be a loop based at x_0, so $\{\gamma^{-1}(A), \gamma^{-1}(B)\}$ is an open cover of I; let $\frac{1}{n}$ be a Lesbegue number for this cover. Then each path $\gamma|_{[\frac{i}{n}, \frac{i+1}{n}]}$ has image entirely in A or entirely in B (or both). If subsequent path pieces $\gamma|_{[\frac{i}{n}, \frac{i+1}{n}]}$ and $\gamma|_{[\frac{i+1}{n}, \frac{i+2}{n}]}$ both lie in A or both lie in B then we may concatenate them. Proceeding in this way, we may express γ as a concatenation $\gamma_1 \cdot \gamma_2 \cdots \gamma_k$ where each $\gamma_i(0)$ and each $\gamma_i(1)$ lie in $A \cap B$.

Now for each point $\gamma_1(1), \gamma_2(1), \ldots, \gamma_{k-1}(1) \in A \cap B$ choose a path u_i from $\gamma_i(1)$ to x_0 in $A \cap B$. Then γ is homotopic to

$$(\gamma_1 \cdot u_1) \cdot (u_1^{-1} \cdot \gamma_2 \cdot u_2) \cdots (u_{k-1}^{-1} \cdot \gamma_k)$$

as a path, and so is a product of elements coming from $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$, as required.

The map ϕ is injective. By considering the description of the amalgamated free product, the group $\pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$ has the following description. It is generated by

(i) for each loop $\gamma : I \to A$, an element $[\gamma]_A$,

(ii) for each loop $\gamma : I \to B$, an element $[\gamma]_B$,

subject to the relations

(i) if $\gamma \simeq \gamma'$ as paths in A, then $[\gamma]_A = [\gamma']_A$, and similarly for B,

(ii) if $\gamma, \gamma' : I \to A$ then $[\gamma]_A \cdot [\gamma']_A = [\gamma \cdot \gamma']_A$, and similarly for B,

(iii) if $\gamma : I \to A \cap B$ is a loop then $[\gamma]_A = [\gamma']_B$.

Let $S_1, \dots, S_n$ be a sequence of elements of $\{A, B\}$, and $\gamma_i : I \to S_i$ be a sequence of loops based at x_0, and suppose that

$$\phi([\gamma_1]_{S_1} \cdot [\gamma_2]_{S_2} \cdots [\gamma_n]_{S_n}) = [c_{x_0}] \in \pi_1(X, x_0).$$

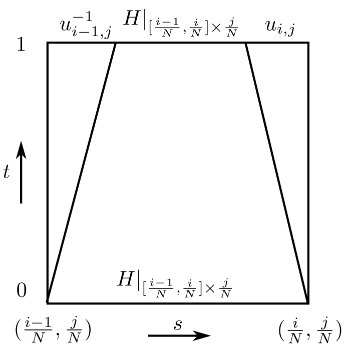
Temporary definition. For a sequence $\gamma_1, \gamma_2 \dots, \gamma_n$ of loops based at x_0, let $\gamma_1 \cdot \gamma_2 \cdots \gamma_n$, unbracketed, be the concatenation formed by cutting I into n pieces of length $\frac{1}{n}$, and running each path γ_i n times as fast.

By assumption $\gamma_1 \cdot \gamma_2 \cdots \gamma_n \simeq c_{x_0}$ as paths in X; let H be a homotopy between them. We can subdivide $I \times I$ into squares of side length $\frac{1}{N}$ for $n \mid N$ so that H sends each square entirely into A or entirely into B. Choose for each square a label A or B saying into which set it is mapped (some squares will admit either choice, and some not).

A	A	B	A
B	A	A	B
A	B	B	B
A	B	B	A

For each point $(\frac{i}{N}, \frac{j}{N}) \in I \times I$ choose a path $u_{i,j}$ from $H(\frac{i}{N}, \frac{j}{N})$ to x_0, such that if $(\frac{i}{N}, \frac{j}{N})$ is a corner of a square with label A then the path $u_{i,j}$ lies in A, and if it is a corner of a square with label B then the path $u_{i,j}$ lies in B (so if it is a corner of squares with both labels then the path lies in $A \cap B$).

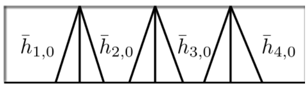
Step 1. For the path $H|_{[\frac{i-1}{N}, \frac{i}{N}] \times \frac{j}{N}}$ which ends at $(\frac{i}{N}, \frac{j}{N})$, there is a homotopy to the path $u_{i-1,j}^{-1} \cdot H|_{[\frac{i-1}{N}, \frac{i}{N}] \times \frac{j}{N}} \cdot u_{i,j}$ which ends at x_0 given by



This homotopy can be taken to lie in A if either the square above or below $[\frac{i-1}{N}, \frac{i}{N}] \times \frac{j}{N}$ is labelled A, and similarly for B, as the paths $u_{i-1,j}$ and $u_{i,j}$ then lie in the required subspace. Call this homotopy $h_{i,j}$, and write $\bar{h}_{i,j}$ for the reverse homotopy.

If $i-1=0$ or $i=N$ the homotopy can be suitably modified, as then the path $H|_{[\frac{i-1}{N}, \frac{i}{N}] \times \frac{j}{N}}$ already starts or ends at x_0.

Step 2. Form a homotopy by gluing together the homotopies $\bar{h}_{i,0}$ as shown.



This shows that each path γ_i in S_i may be homotoped (in S_i) to be the composition of N/n loops $\gamma_i^1 \cdot \gamma_i^2 \cdots \gamma_i^{N/n}$ given by

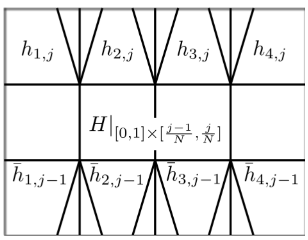
$$(\gamma_i|_{[0,\frac{n}{N}]} \cdot u_{(i-1)n+1,0}) \cdot (u^{-1}_{(i-1)n+1,0} \cdot \gamma_i|_{[\frac{n}{N},\frac{2n}{N}]} \cdot u_{(i-1)n+2,0}) \cdots (u^{-1}_{in-1,0} \cdot \gamma_i|_{[\frac{N-n}{N},1]})$$

and so $[\gamma_i]_{S_i} = [\gamma_i^1]_{S_i} \cdots [\gamma_i^{N/n}]_{S_i}$ in the amalgamated free product. Thus it is enough to show that

$$([\gamma_1^1]_{S_1} \cdots [\gamma_1^{N/n}]_{S_1}) \cdot ([\gamma_2^1]_{S_2} \cdots [\gamma_2^{N/n}]_{S_2}) \cdots ([\gamma_n^1]_{S_n} \cdots [\gamma_n^{N/n}]_{S_n})$$

is trivial in the amalgamated free product.

Step 3. We now form a homotopy by gluing together the pieces $H|_{[\frac{i-1}{N},\frac{i}{N}]\times[\frac{j-1}{N},\frac{j}{N}]}$ and the homotopies $h_{i,j}$ and $\bar{h}_{i,j}$ together. For each row $H|_{[0,1]\times[\frac{j-1}{N},\frac{j}{N}]}$ of the homotopy H we form a homotopy G_j by



Note that each column of the above maps entirely into A or entirely into B, and the points i/N along the top or bottom all map to x_0.

Step 4. As each column of the homotopy G_j lies entirely in A or B, and has its four vertices mapping to x_0, it follows that the loop given by the top of each column in homotopic to the loop given by the bottom of each column by a homotopy which lies entirely in A or entirely in B. Thus the sequence of composable loops along the bottom and the sequence of composable loops along the top represent the same element of the amalgamated free product. As the sequence

$$([\gamma_1^1]_{S_1} \cdots [\gamma_1^{N/n}]_{S_1}) \cdot ([\gamma_2^1]_{S_2} \cdots [\gamma_2^{N/n}]_{S_2}) \cdots ([\gamma_n^1]_{S_n} \cdots [\gamma_n^{N/n}]_{S_n})$$

arises as the bottom of G_0, this argument shows that it is equivalent in the amalgamated free product to the sequence of composable loops arising from the top of G_{N-1}. But by reversing the argument of Step 2, the sequence of composable loops arising from the top of G_{N-1} is a refinement of the loop $H(-,t)$, which was constant. $\square$

5.2 The effect on the fundamental group of attaching cells

Let $f : (S^{n-1}, *) \to (X, x_0)$ be a based map, and

$$Y := X \cup_f D^n = (X \sqcup D^n)/z \in S^{n-1} \subset D^n \sim f(z) \in X.$$

We let $y_0 = [x_0]$ be the equivalence class of x_0, so the inclusion $i : X \to Y$ is a based map. Thus there is an induced homomorphism

$$i_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0).$$

The space Y has an open cover by the interior of the disc

$$U := \mathrm{int}(D^n)$$

and the complement of the centre of the disc

$$V := X \cup_f (D^n \setminus \{0\}).$$

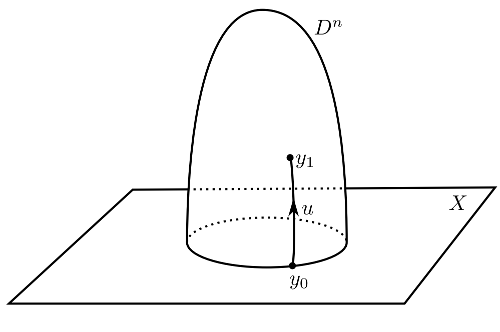
These satisfy $U \cap V \cong S^{n-1} \times (0,1)$, which is path connected as long as $n \geq 2$.

Theorem 5.2.1.

(i) If $n \geq 3$ then the map i_* is an isomorphism.

(ii) If $n = 2$ then the map i_* is surjective with kernel the normal subgroup generated by $[f] \in \pi_1(X, x_0)$.

Proof. Choose a point $y_1 \in U \cap V \subset \mathrm{int}(D^n)$, and a path u in $D^n \setminus \{0\}$ from y_1 to the basepoint $* \in S^{n-1} \subset D^n$ (which is identified with $x_0 \in X$).



If $n \geq 3$, then $U \cap V \simeq S^{n-1}$ is simply connected. By Seifert–van Kampen the map

$$\phi : \pi_1(U, y_0) *_{\pi_1(U \cap V, y_1)} \pi_1(V, y_1) \longrightarrow \pi_1(Y, y_1)$$

is an isomorphism, but both $\pi_1(U \cap V, y_1)$ and $\pi_1(U, y_1)$ are trivial and so the map $\pi_1(V, y_1) \to \pi_1(Y, y_1)$ is an isomorphism. Hence, the change-of-basepoint isomorphism given by u shows that $\pi_1(V, y_0) \to \pi_1(Y, y_0)$ is an isomorphism. Finally, the inclusion $(X, x_0) \to (V, y_0)$ is a based homotopy equivalence (by pulling $D^n \setminus \{0\}$ on to its boundary), and the theorem follows.

If $n = 2$ then $U \cap V \simeq S^1$ has fundamental group $\mathbb{Z}$. The map

$$\mathbb{Z} \cong \pi_1(U \cap V, y_1) \longrightarrow \pi_1(V, y_1)$$

is seen to send the generator to the loop $u_\#([f])$. Thus, as $\pi_1(U, y_1) = \{e\}$, the Seifert–van Kampen theorem shows that

$$\phi : \pi_1(V, y_1)/\langle\langle u_\#[f]\rangle\rangle \longrightarrow \pi_1(Y, y_1)$$

is an isomorphism. Under the change-of-basepoint isomorphism given by the path u, and again using that the inclusion $(X, x_0) \to (V, y_0)$ is a based homotopy equivalence, the theorem follows. □

Example 5.2.2. The standard cell structure on the torus T has two 1-cells a and b attached to a single vertex v, and a 2-cell attached along a loop given by $a \cdot b \cdot a^{-1} \cdot b^{-1}$. The 1-skeleton T^1 is thus a wedge of two circles, so $\pi_1(T^1, v) \cong \langle a, b \mid \rangle$, and attaching the 2-cell thus gives

$$\pi_1(T, v) \cong \langle a, b \mid a \cdot b \cdot a^{-1} \cdot b^{-1}\rangle.$$

One may easily check that this is a presentation of the abelian group $\mathbb{Z} \times \mathbb{Z}$.

Corollary 5.2.3. *For any group presentation $G := \langle S \mid R\rangle$ with S and R finite, there is a 2-dimensional based cell complex (X, x_0) with $\pi_1(X, x_0) \cong G$.*

Proof. Let Y be a wedge of $|S|$-many circles, each identified with an element of S. Let Y have the wedge point as basepoint, and call it y_0. For $s \in S$ let $x_s \in \pi_1(Y, y_0)$ be a loop which transverses the sth circle precisely once. Then sending s to x_s gives a group isomorphism

$$\langle S \mid \rangle \cong \pi_1(Y, y_0).$$

Each element $r \in R \subset \langle S \mid \rangle$ is represented by a word in the alphabet S, and so by an element of $\pi_1(Y, y_0)$. Let $\gamma_r' : I \to Y$ be a loop in Y such that $[\gamma_r']$ corresponds to r under the isomorphism above, and $\gamma_r : (S^1, 1) \to (Y, y_0)$ be the map from the circle obtained by gluing the endpoints of I together. By Theorem 5.2.1 attaching a 2-cell to Y along γ_r has the effect of dividing out by the normal subgroup generated by $[\gamma_r]$. Thus if we attach a 2-cell along γ_r for each $r \in R$ to form a cell complex X, with $x_0 = [y_0]$, then we have

$$\pi_1(X, x_0) \cong \pi_1(Y, y_0)/\langle\langle[\gamma_r], r \in R\rangle\rangle$$

But this is precisely the description of $\langle S \mid R\rangle$. □

The assumption that S and R are finite may be removed, by developing the Seifert–van Kampen theorem for covers by infinitely many open sets such that every possible intersection of them is path connected. Hence every group occurs as the fundamental group of a 2-dimensional cell complex (possibly having infinitely many cells).

5.3 A refinement of the Seifert–van Kampen theorem

The requirement in Theorem 5.1.1 that A and B be a pair of open sets does not fit with how one typically wants to use the theorem. In this section we shall describe weaker hypotheses that are usually more convenient.

Definition 5.3.1. A subset $A \subset X$ of a space X is called a **neighbourhood deformation retract** if there is an open neighbourhood $A \subset U \subset X$ and a strong[1] deformation retraction of U onto A.

Theorem 5.3.2. *Let X be a space, and $A, B \subset X$ be closed subsets which cover X and such that $A \cap B$ is both path connected and a neighbourhood deformation retract in both A and B. Then for any $x_0 \in A \cap B$ the homomorphism*

$$\phi : \pi_1(A, x_0) *_{\pi_1(A\cap B,x_0)} \pi_1(B, x_0) \longrightarrow \pi_1(X, x_0)$$

is an isomorphism.

Proof. Choose open neighbourhoods $A \cap B \subset U \subset A$ and $A \cap B \subset V \subset B$ which strongly deformation retract to $A \cap B$. Now the complement of $A \cup V$ in X is $B \setminus V$, which is closed as V was open in B; thus $A \cup V$ is open in X. Similarly $U \cup B$ is open in X. The strong deformation retraction of V to $A \cap B$ shows that the inclusion $A \to A \cup V$ is a homotopy equivalence, and similarly for $B \to U \cup B$.

Finally, $(A \cup V) \cap (U \cup B) = U \cup V$ and the strong deformation retractions of U to $A \cap B$ and V to $A \cap B$ glue together to give a (strong) deformation retraction of $U \cup V$ to $A\cap$V.

We may now apply Theorem 5.1.1 to the open cover given by $A \cup V$ and $U \cup B$, which by the above discussion has $(A \cup V) \cap (U \cup B) = U \cup V \simeq A \cap B$ path connected. The theorem then follows from noting that the maps

$$\begin{aligned} \pi_1(A, x_0) &\longrightarrow \pi_1(A \cup V, x_0) \\ \pi_1(B, x_0) &\longrightarrow \pi_1(U \cup B, x_0) \\ \pi_1(A \cap B, x_0) &\longrightarrow \pi_1(U \cup V, x_0) \end{aligned}$$

are all isomorphisms. □

Recall that for based spaces (X, x_0) and (Y, y_0) the wedge product $X \vee Y$ is the quotient space of $X \sqcup Y$ by the relation generated by $x_0 \sim y_0$. If the subspaces $\{x_0\} \subset X$ and $\{y_0\} \subset Y$ are neighbourhood deformation retracts[2] then it follows from Theorem 5.3.2 that

$$\pi_1(X \vee Y, [x_0]) \cong \pi_1(X, x_0) * \pi_1(Y, y_0).$$

[1] That is, a homotopy from Id_U to a map $r : U \to A$, which is constant when restricted to A.

[2] Some authors say that a based space (X, x_0) is **well-based** if $\{x_0\} \subset X$ is a neighbourhood deformation retract.

Example 5.3.3. The based space $(S^1, 1)$ has $\{1\} \subset S^1$ a neighbourhood deformation retract, and so

$$\pi_1(S^1 \vee S^1, [1]) \cong \pi_1(S^1, 1) * \pi_1(S^1, 1) \cong \mathbb{Z} * \mathbb{Z} \cong \langle a, b \,|\, \rangle$$

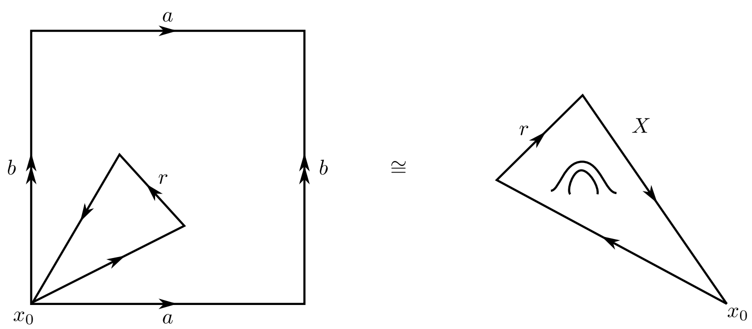
is a free group on two generators. By induction on n, for any finite n we thus have

$$\pi_1(\bigvee^n S^1, [1]) \cong \langle x_1, x_2, \ldots, x_n \,|\, \rangle,$$

a free group on n generators.

5.4 The fundamental group of a surface

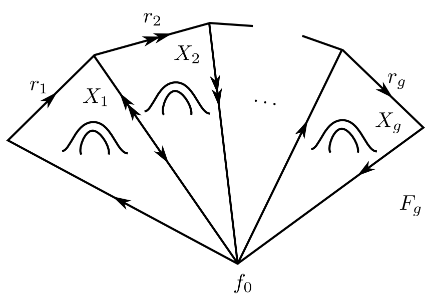
Example 5.4.1 (Orientable surfaces)**.** Consider the based space (X, x_0) obtained from the torus by removing an open triangle, as shown



By stretching the hole, we see that the inclusion

$$\iota : (S^1 \vee S^1, [1]) \longrightarrow (X, x_0)$$

of the loops a and b is a based homotopy equivalence. Thus we have $\pi_1(X, x_0) \cong \langle a, b \,|\, \rangle$. The loop r given by the boundary of the open disc we removed represents the class $[a, b] := aba^{-1}b^{-1} \in \pi_1(X, x_0)$, if we orient it as shown in the figure above.



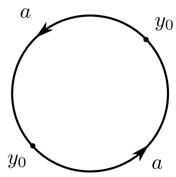
Gluing g copies of X together as shown above gives a based space (F_g, f_0). As the gluings are performed along intervals, which are contractible and are neighbourhood deformation retracts in X, Theorem 5.3.2 shows that there is an isomorphism

$$\pi_1(F_g, f_0) \cong \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \,|\, \rangle$$

where a_i and b_i are the loops a and b in the ith copy of X. The boundary of F_g is a circle, and the loop it represents is given by $r_1 \cdot r_2 \cdots r_g \in \pi_1(F_g, f_0)$, the product of the boundary loops on the individual copies of X. In terms of the generators a_i and b_i, the boundary loop is thus $\prod_{i=1}^{g}[a_i, b_i]$. Attaching D^2 to F_g gives a closed surface Σ_g, and by Theorem 5.2.1 we have

$$\pi_1(\Sigma_g) \cong \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid \prod_{i=1}^{g}[a_i, b_i]\rangle.$$

Example 5.4.2 (The projective plane again)**.** First recall from Example 5.1.3 that we have shown that $\pi_1(\mathbb{RP}^2, x_0)$ is $\mathbb{Z}/2$. Let us try to understand the generator of this group.



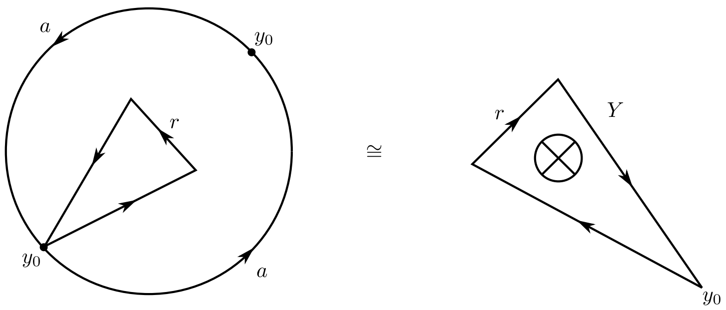
The description of $\mathbb{RP}^2$ as a quotient space of S^2 also shows that it may be obtained from D^2 (thought of as the upper hemisphere of S^2) by dividing out by the equivalence relation

$$x \in S^1 \subset D^2 \sim -x \in S^1 \subset D^2$$

given by identifying opposite points on the boundary of D^2. If we give D^2 the cell structure shown in the figure above then we see that there is an induced cell structure on $\mathbb{RP}^2$ having a single 0-cell, a single 1-cell, and a single 2-cell. If the loop around the 1-cell is called a, we see that the fundamental group of $\mathbb{RP}^2$ has the presentation

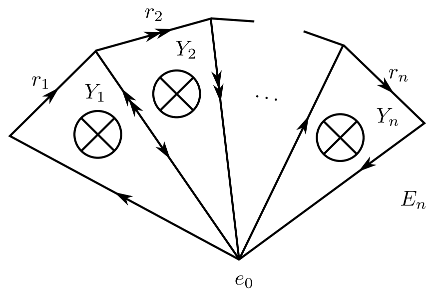
$$\pi_1(\mathbb{RP}^2, x_0) \cong \langle a \mid a^2\rangle.$$

Example 5.4.3 (Nonorientable surfaces)**.** Consider the based space (Y, y_0) obtained from $\mathbb{RP}^2$ by removing an open triangle, as shown



By stretching the hole, we see that Y is homotopy equivalent to S^1, and so $\pi_1(Y, y_0) \cong \langle a \mid \rangle$. The boundary loop r is now given by a^2.

Gluing n copies of Y together as shown gives a based space (E_n, e_0) with $\pi_1(E_n, e_0) \cong \langle a_1, a_2, \ldots, a_n \mid \rangle$, and with boundary loop given by $r_1 \cdot r_2 \cdots r_n = a_1^2 a_2^2 \cdots a_n^2$. Hence, if



we form the closed surface S_n by attaching a copy of D^2 to E_n along its boundary, we have

$$\pi_1(S_n) \cong \langle a_1, a_2, \ldots, a_n \mid a_1^2 a_2^2 \cdots a_n^2 \rangle.$$

Chapter 6

Simplicial complexes

6.1 Simplicial complexes

Definition 6.1.1. A finite set of points $a_0, a_1, \ldots, a_n \in \mathbb{R}^m$ are **affinely independent** if

$$\left\{\sum_{i=0}^{n} t_i a_i = 0 \text{ and } \sum_{i=0}^{n} t_i = 0\right\} \Leftrightarrow (t_0, t_1, \ldots, t_n) = 0.$$

Lemma 6.1.2. *The set $a_0, a_1, \ldots, a_n \in \mathbb{R}^m$ are affinely independent if and only if the vectors $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ are linearly independent.*

Proof. Suppose that the $a_0, a_1, \ldots, a_n \in \mathbb{R}^m$ are affinely independent, and that

$$\sum_{i=1}^{n} s_i(a_i - a_0) = 0.$$

Then $(-\sum_{i=1}^{n} s_i)a_0 + s_1 a_1 + \cdots + s_n a_n = 0$ and the sum of the coefficients is 0, so by definition of affinely independent we must have $s_i = 0$ for all i, so the vectors $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ are linearly independent.

Suppose now that the vectors $a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0$ are linearly independent, and that there are t_i with $\sum_{i=0}^{n} t_i a_i = 0$ and $\sum_{i=0}^{n} t_i = 0$. The second sum gives $t_0 a_0 = -\sum_{i=1}^{n} t_i a_i$, and so we may rewrite the first sum as $\sum_{i=1}^{n} t_i(a_i - a_0) = 0$. As the vectors $a_i - a_0$ are linearly independent, it follows that $t_i = 0$ for all $i \geq 1$, but then $t_0 = -\sum_{i=1}^{n} t_i$ is 0 too. $\square$

Definition 6.1.3. If $a_0, a_1, \ldots, a_n \in \mathbb{R}^m$ are affinely independent, they define an n-**simplex**

$$\sigma = \langle a_0, a_1, \ldots, a_n \rangle := \left\{\sum_{i=0}^{n} t_i a_i \mid \sum_{i=0}^{n} t_i = 1, t_i \geq 0\right\}$$

given by the convex hull of the points a_i. The a_i are called the **vertices** of σ, and are said to **span** σ.

If $x \in \langle a_0, a_1, \ldots, a_n \rangle$ then x can be written as $x = \sum_{i=0}^{n} t_i a_i$ for unique real numbers t_i (the uniqueness follows from the affine independence). The t_i are called the **barycentric coordinates** of x.

A **face** of a simplex $\sigma = \langle a_0, a_1, \ldots, a_n \rangle$ is a simplex τ spanned by a subset of $\{a_0, a_1, \ldots, a_n\}$; we write $\tau \leq \sigma$, and $\tau < \sigma$ if τ is a proper face of σ.

The **boundary** of a simplex σ, written $\partial\sigma$, is the union of all of its proper faces. Its **interior**, written $\mathring{\sigma}$, is $\sigma \setminus \partial\sigma$.

Lemma 6.1.4. *Let σ be a p-simplex in $\mathbb{R}^m$ and τ be a p-simplex in $\mathbb{R}^n$. Then σ and τ are homeomorphic.*

Proof. Let $\sigma = \langle a_0, a_1, \dots, a_p \rangle$ and $\tau = \langle b_0, b_1, \dots, b_p \rangle$. Define a function

$$\begin{aligned} h : \sigma &\longrightarrow \tau \\ \sum_{i=0}^{p} t_i a_i &\longmapsto \sum_{i=0}^{p} t_i b_i \end{aligned}$$

which is well-defined and a bijection, by the uniqueness of the barycentric coordinates t_i. As the vectors $a_i - a_0$ are linearly independent, we may extend h to an affine map $\hat{h} : \mathbb{R}^m \to \mathbb{R}^n$, so it is continuous. □

Definition 6.1.5. A **geometric** (or **euclidean**) **simplicial complex** in $\mathbb{R}^m$ is a finite set K of simplices in $\mathbb{R}^m$ such that

(i) if $\sigma \in K$ and $\tau \leq \sigma$, then $\tau \in K$,

(ii) if $\sigma, \tau \in K$ then $\sigma \cap \tau$ is either empty or is a face of both σ and τ.

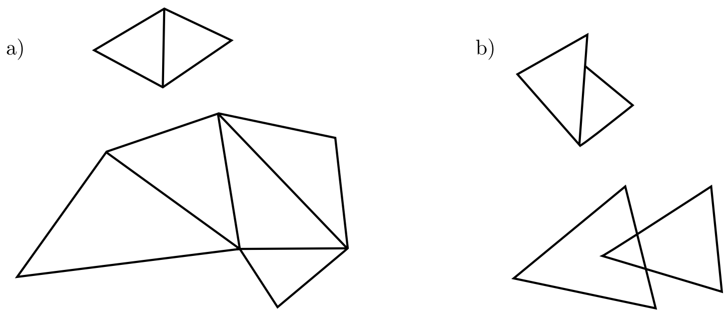


Figure 6.1 a) a geometric simplicial complex; b) not a geometric simplicial complex.

The **dimension** of K is the largest n for which K contains an n-simplex. The **polyhedron** $|K|$ associated to K is the subspace of $\mathbb{R}^m$ given by the union of all the simplices in K. The d**-skeleton** $K_{(d)}$ of a simplicial complex K is the sub simplicial complex given by the simplices of K of dimension at most d.

Note that a simplex is closed and bounded, so it is compact; as a polyhedron is a finite union of simplices, it follows that it is compact. As $\mathbb{R}^m$ is Hausdorff, any polyhedron is also Hausdorff.

Definition 6.1.6. A **triangulation** of a space X is a pair of a simplicial complex K and a homeomorphism $h : |K| \to X$.

Thus if X has a triangulation it must in particular be compact and Hausdorff; this is however not sufficient.

Example 6.1.7. The **standard n-simplex** $\Delta^n \subset \mathbb{R}^{n+1}$ is the simplex spanned by the basis vectors $e_1, e_2, \ldots, e_{n+1}$. It, along with all its faces, defines a simplicial complex.

Example 6.1.8. The **simplicial $(n-1)$-sphere** is the simplicial complex given by the proper faces of Δ^n, and all their faces. Its polyhedron is $\partial\Delta^n \subset \mathbb{R}^{n+1}$.

Example 6.1.9. In $\mathbb{R}^{n+1}$ consider the 2^{n+1} n-simplices given by $\langle \pm e_1, \pm e_2, \ldots, \pm e_{n+1}\rangle$, and let K be the simplicial complex given by these simplices and their faces. Define a function

$$\begin{aligned} h : |K| \subset \mathbb{R}^{n+1} &\longrightarrow S^n \\ x &\longmapsto \frac{x}{|x|} \end{aligned}$$

which is a bijection and continuous (it is the restriction of the map $\mathbb{R}^{n+1} \setminus \{0\} \to S^n$ given by the same formula) so is a homeomorphism (as the source is compact and the target is Hausdorff). This defines a triangulation of S^n.

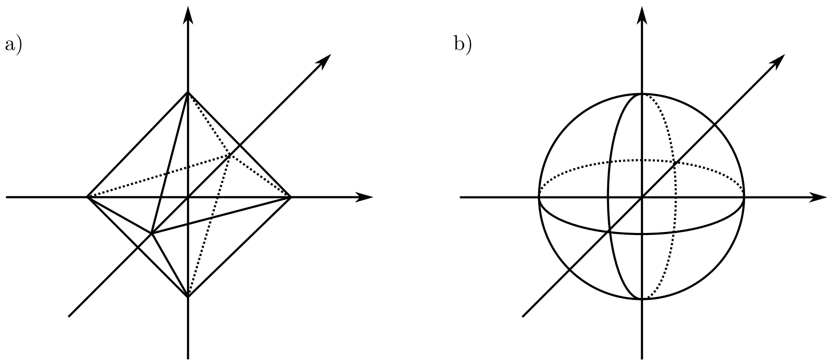


Figure 6.2 a) The polyhedron $|K|$; b) radial projection to the sphere.

Lemma 6.1.10. *Any point of the polyhedron $|K|$ lies in the interior of a unique simplex.*

Proof. If $x \in |K| = \cup_{\sigma \in K}\sigma$ then certainly x lies in some simplex σ, say with $\sigma = \langle a_0, a_1, \ldots, a_p\rangle$. Thus $x = \sum_{i=0}^{p} t_i a_i$. If $\tau \leq \sigma$ is the face given by $\{a_i \,|\, t_i > 0\}$ then we have $x = \sum_{i=0}^{n} s_i b_i$ with $\tau = \langle b_0, b_1, \ldots, b_n\rangle$ and $s_i > 0$ for all i, and so x is in $\mathring{\tau}$.

If $x \in \mathring{\tau}'$ too, then $\mathring{\tau} \cap \mathring{\tau}' \neq \emptyset$. Thus $\tau \cap \tau'$ is not empty, so is a face of both τ and τ'. But it contains an interior point of either simplex, so is not a proper face of either: thus $\tau = \tau \cap \tau' = \tau'$. $\square$

Definition 6.1.11. For a simplicial complex K, let V_K denote the set of 0-simplices of K, which we call the **vertices** of K.

Definition 6.1.12. A **simplicial map** f from K to L is a function $f : V_K \to V_L$ such that if $\sigma = \langle a_0, a_1, \ldots, a_p\rangle$ is a simplex of K, then the set $\{f(a_0), f(a_1), \ldots, f(a_p)\}$ spans a simplex of of L, which we call $f(\sigma)$. We will write $f : K \to L$ for a simplicial map.

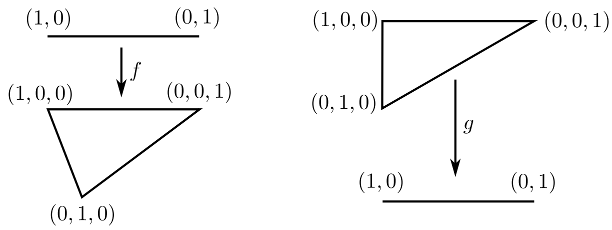
The reason for this perhaps surprising formulation is that the list $f(a_0), f(a_1), \ldots, f(a_p)$ *is allowed to have repeats.* Thus these elements to not need to be affinely independent, they just need to become so when we throw away repeated elements.

Example 6.1.13. The function

$$\begin{aligned} f : \Delta^1 \subset \mathbb{R}^2 &\longrightarrow \Delta^2 \subset \mathbb{R}^3 \\ (1,0) &\longmapsto (1,0,0) \\ (0,1) &\longmapsto (0,0,1) \end{aligned}$$

defines a simplicial map, as does the function

$$\begin{aligned} g : \Delta^2 \subset \mathbb{R}^3 &\longrightarrow \Delta^1 \subset \mathbb{R}^2 \\ (1,0,0) &\longmapsto (1,0) \\ (0,1,0) &\longmapsto (1,0) \\ (0,0,1) &\longmapsto (0,1). \end{aligned}$$



Lemma 6.1.14. *A simplicial map $f : K \to L$ induces a continuous map $|f| : |K| \to |L|$, and $|g \circ f| = |g| \circ |f|$.*

Proof. For a simplex $\sigma \in K$, $\sigma = \langle a_0, a_1, \ldots, a_p\rangle$, define

$$\begin{aligned} f_\sigma : \sigma &\longrightarrow |L| \\ \sum_{i=0}^{p} t_i a_i &\longmapsto \sum_{i=0}^{p} t_i f(a_i), \end{aligned}$$

which is linear in the barycentric coordinates t_i, so continuous.

If $\tau \leq \sigma$ then $f_\tau = f_\sigma|_\tau$, so for simplices σ and σ' we have $f_\sigma|_{\sigma\cap\sigma'} = f_{\sigma\cap\sigma'} = f_{\sigma'}|_{\sigma\cap\sigma'}$ and so the functions f_σ glue to a well-defined function

$$|f| : |K| = \bigcup_{\sigma\in K} \sigma \longrightarrow |L|,$$

which is continuous by the gluing lemma. The formula for f_σ shows that $|f|$ behaves as claimed under composition. $\square$

Note that we can recover the simplicial map f from the continuous map $|f|$ and the subsets $V_K \subset |K|$ and $V_L \subset |L|$. Thus a simplicial map from K to L could also be defined as a continuous map $|K| \to |L|$ which sends vertices to vertices and is linear on each simplex.

Definition 6.1.15. For a point $x \in |K|$,

(i) The (open) **star** of x is the union of the interiors of those simplices in K which contain x,

$$\mathrm{St}_K(x) = \bigcup_{x \in \sigma} \mathring{\sigma} \subset \mathbb{R}^m.$$

The complement of $\mathrm{St}_K(x)$ is the union of those simplices of K which do not contain x: this is a polyhedron, so closed in $|K|$, so $\mathrm{St}_K(x)$ is indeed open.

(ii) The **link** of x, $\mathrm{Lk}_K(x)$, is the union of those simplices of K which do not contain x, but are a face of a simplex which does contain x. This is a subpolyhedron of $|K|$, so is closed.

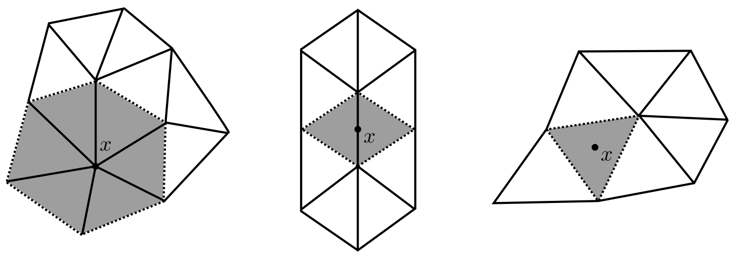


Figure 6.3 Three examples of points x with their open stars (in grey) and links (dotted).

6.2 Simplicial approximation

Definition 6.2.1. Let $f : |K| \to |L|$ be a continuous map. A **simplicial approximation** to f is a function $g : V_K \to V_L$ such that

$$f(\mathrm{St}_K(v)) \subset \mathrm{St}_L(g(v))$$

for every $v \in V_K$.

Lemma 6.2.2. *If g is a simplicial approximation to a map f, then g defines a simplicial map and f is homotopic to $|g|$. Furthermore, this homotopy may be supposed to be relative to the subspace $\{x \in |K| \,|\, f(x) = |g|(x)\}$ where the two maps already agree.*

Proof. To show that g defines a simplicial map, for each $\sigma \in K$ we must show that the images under g of the vertices of σ span a simplex in L. For $x \in \mathring{\sigma}$ we have $x \in \cap_{v \in V_\sigma} \mathrm{St}_K(v)$, and so

$$f(x) \in \bigcap_{v \in V_\sigma} f(\mathrm{St}_K(v)) \subset \bigcap_{v \in V_\sigma} \mathrm{St}_L(g(v)).$$

Thus if $\tau \in L$ is the unique simplex such that $f(x) \in \mathring{\tau}$, then every $g(v)$ is a vertex of τ, and so the collection $\{g(v) \mid v \in V_\sigma\}$ span a face of τ, so span a simplex of L.

We now show that f is homotopic to $|g|$. If $|L| \subset \mathbb{R}^m$, we *try* to define a homotopy by the formula

$$\begin{aligned} H : |K| \times I &\longrightarrow |L| \\ (x,t) &\longmapsto t \cdot f(x) + (1-t) \cdot |g|(x), \end{aligned}$$

where the linear interpolation takes place in $\mathbb{R}^m$. This certainly defines a continuous map to $\mathbb{R}^m$, and we must check that it lands in $|L| \subset \mathbb{R}^m$.

Let $x \in \mathring{\sigma} \subset |K|$ and suppose that $f(x) \in \mathring{\tau} \subset |L|$. If $\sigma = \langle a_0, a_1, \dots, a_p\rangle$ then by the argument above each $g(a_i)$ is a vetex of τ. Hence

$$|g|(x) = |g|\left(\sum_{i=0}^p t_i a_i\right) = \sum_{i=0}^p t_i g(a_i)$$

is a linear combination of vertices of τ, so lies in τ too. But then $f(x)$ and $|g|(x)$ both lie in τ, so the straight line between them does too. □

Definition 6.2.3. The **barycentre** of the simplex $\sigma = \langle a_0, a_1, \dots, a_p\rangle$ is the point $\hat{\sigma} = \frac{a_0+a_1+\cdots+a_p}{p+1}$. The (first) **barycentric subdivision** of a simplicial complex K is the simplicial complex

$$K' := \{\langle \hat{\sigma}_0, \hat{\sigma}_1, \dots, \hat{\sigma}_p\rangle \mid \sigma_i \in K \text{ and } \sigma_0 < \sigma_1 < \cdots < \sigma_p\}.$$

We define the rth barycentric subdivision $K^{(r)}$ inductively as $(K^{(r-1)})'$.

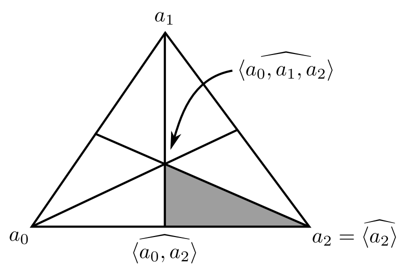


Figure 6.4 The barycentric subdivision of the 2-simplex $\langle a_0, a_1, a_2\rangle$, with the simplex $\langle \widehat{\langle a_2\rangle}, \widehat{\langle a_0, a_2\rangle}, \widehat{\langle a_0, a_1, a_2\rangle}\rangle$ indicated in grey.

It is not obvious that the collection of simplices K' described in this definition actually forms a simplicial complex. To see that it does is rather involved, and we collect the necessary arguments in the following proposition.

Proposition 6.2.4. *The barycentric subdivision K' of a simplicial complex K is indeed a simplicial complex, and $|K'| = |K|$.*

Proof.
If $\sigma_0 < \sigma_1 < \cdots < \sigma_p$ then the $\hat{\sigma}_i$ are affinely independent. Suppose that

$$\sum_{i=0}^{p} t_i \hat{\sigma}_i = 0 \quad \text{and} \quad \sum_{i=0}^{p} t_i = 0.$$

Let $j = \max\{i \mid t_i \neq 0\}$; then

$$\hat{\sigma}_j = -\sum_{i=0}^{j-1} \frac{t_i}{t_j} \hat{\sigma}_i \in \sigma_{j-1},$$

so $\hat{\sigma}_j$ lies in a proper face of σ_j, which is impossible.

K' is a simplicial complex. Let $\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_p \rangle \in K'$. A face is given by omitting some $\hat{\sigma}_j$'s from $\sigma_0 < \sigma_1 < \cdots < \sigma_p$, but omitting some $\hat{\sigma}_j$'s from such a sequence still gives a sequence of proper faces.

Let $\sigma' = \langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_p \rangle$ and $\tau' = \langle \hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_q \rangle$, and consider $\sigma' \cap \tau'$. This lies inside $\sigma_p \cap \tau_q$, which is a simplex δ of K, and there are simplices

$$\sigma'' = \langle \hat{\sigma}_0 \cap \delta, \hat{\sigma}_1 \cap \delta, \ldots, \hat{\sigma}_p \cap \delta \rangle \quad \text{and} \quad \tau'' = \langle \hat{\tau}_0 \cap \delta, \hat{\tau}_1 \cap \delta, \ldots, \hat{\tau}_q \cap \delta \rangle$$

of K' (where if $\hat{\sigma}_i \cap \delta = \emptyset$ we ignore it). Now $\sigma' \cap \tau' = \sigma'' \cap \tau'' \subset \delta$, which reduces to the case that everything lies within a single simplex of K, namely δ.

We now proceed by induction on the dimension of δ, in cases:

(i) **σ'' and τ'' contain $\hat{\delta}$:** let $\bar{\sigma}''$ be the face of σ'' obtained by removing $\hat{\delta}$, and similarly $\bar{\tau}''$. Then $\sigma'' \cap \tau''$ is the simplex spanned by $\bar{\sigma}'' \cap \bar{\tau}''$ and $\hat{\delta}$, and $\bar{\sigma}'' \cap \bar{\tau}''$ lies in the boundary $\partial\delta$. This is smaller dimensional than δ, so by induction we may assume that $\bar{\sigma}'' \cap \bar{\tau}''$ is a simplex of K', but then $\sigma'' \cap \tau''$ is too.

(ii) **σ'' or τ'' does not contain $\hat{\delta}$:** then $\sigma'' \cap \tau'' \subset \partial\delta$ lies in a smaller dimensional simplex, so by induction we may assume it is a simplex of K'.

$|\mathbf{K}'| = |\mathbf{K}|$. Note that $\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_p \rangle \subseteq \sigma_p$, so $|K'| \subseteq |K|$. On the other hand, for $x \in \sigma = \langle a_0, a_1, \ldots, a_p \rangle$ written as $x = \sum_{i=0}^{p} t_i a_i$, we may reorder the a_i so that $t_0 \geq t_1 \geq \cdots \geq t_p$. Then

$$\begin{aligned} x &= (t_0 - t_1)a_0 + 2(t_1 - t_2)\frac{a_0 + a_1}{2} + 3(t_2 - t_3)\frac{a_0 + a_1 + a_2}{3} + \cdots \\ &= (t_0 - t_1)\widehat{\langle a_0 \rangle} + 2(t_1 - t_2)\widehat{\langle a_0, a_1 \rangle} + 3(t_2 - t_3)\widehat{\langle a_0, a_1, a_2 \rangle} + \cdots \\ &\in \left\langle \widehat{\langle a_0 \rangle}, \widehat{\langle a_0, a_1 \rangle}, \ldots, \widehat{\langle a_0, a_1, \ldots, a_p \rangle} \right\rangle \end{aligned}$$

which is a simplex of K'. $\square$

We now consider the following construction, which compares K and K'. First note that the vertices of K' are precisely the barycentres of simplices of K, so there is a bijection $V_{K'} \cong K$. Thus if we choose a function $K \to V_K$ which sends each simplex σ to a vertex v_σ of σ, then we obtain a function

$$\begin{aligned} g : V_{K'} &\longrightarrow V_K \\ \hat{\sigma} &\longmapsto v_\sigma. \end{aligned}$$

If $\langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_p \rangle$ is a simplex of K' then $\sigma_0 < \sigma_1 < \cdots < \sigma_p$, and so all v_{σ_i} are vertices of σ_p, and so $\langle v_{\sigma_0}, v_{\sigma_1}, \ldots, v_{\sigma_p} \rangle$ is a simplex of K. Thus g defines a simplicial map.

Furthermore, if the point $\hat{\sigma}$ lies in a simplex $\tau' = \langle \hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_p \rangle \in K'$, then $\hat{\sigma} \in \tau_p$ and so σ is a face of τ_p. In particular, $v_\sigma \in \sigma \subset \tau_p$. Thus $\mathring{\tau}' \subset \mathring{\tau}_p \subset \mathrm{St}_K(v_\sigma)$, and hence

$$\mathrm{St}_{K'}(\hat{\sigma}) \subset \mathrm{St}_K(v_\sigma),$$

so the simplicial map g is a simplicial approximation to $\mathrm{Id} : |K'| \to |K|$.

Definition 6.2.5. The **mesh** of K is

$$\mu(K) := \max\{\|v_0 - v_1\| \mid \langle v_0, v_1 \rangle \in K\}.$$

We will use the mesh as a measure of how fine a triangulation is, and will need to know that sufficiently many barycentric subdivisions will make the mesh as small as we like.

Lemma 6.2.6. *Suppose that K has dimension at most n. Then $\mu(K^{(r)}) \leq (\frac{n}{n+1})^r \cdot \mu(K)$, and so $\mu(K^{(r)}) \to 0$ as $r \to \infty$.*

Proof. This will follow by indutcion from the case $r = 1$. Let $\langle \hat{\tau}, \hat{\sigma} \rangle \in K'$, so $\tau < \sigma$ as simplices of K. Firstly we estimate

$$\|\hat{\tau} - \hat{\sigma}\| \leq \max\{\|v - \hat{\sigma}\| \mid v \text{ a vertex of } \sigma\},$$

as the euclidean distance to a point on a simplex is maximised at the vertices. Now write $\sigma = \langle v_0, v_1, \ldots, v_m \rangle$, with $m \leq n$, and suppose that v_0 is chosen so that $\|v_0 - \hat{\sigma}\|$ is maximal. Then we calculate

$$\begin{aligned} \|v_0 - \hat{\sigma}\| &= \|v_0 - \tfrac{1}{m+1} \sum_{i=0}^{m} v_i\| \\ &= \|\tfrac{m+1}{m+1} v_0 - \tfrac{1}{m+1} \sum_{i=0}^{m} v_i\| \\ &= \tfrac{1}{m+1} \|\sum_{i=1}^{m} (v_i - v_0)\| \\ &\leq \tfrac{1}{m+1} \sum_{i=1}^{m} \|v_i - v_0\| \\ &\leq \tfrac{m}{m+1} \cdot \mu(\sigma) \\ &\leq \tfrac{n}{n+1} \cdot \mu(K). \end{aligned}$$

□

Theorem 6.2.7 (Simplicial approximation theorem)**.** *Let $f : |K| \to |L|$ be a continuous map. Then there is a simplicial map $g : K^{(r)} \to L$ for some r, such that g is a simplicial approximation to f.*

Furthermore, if N is a subcomplex of K and $f|_{|N|} : |N| \to |L|$ is already simplicial, then we may choose g to agree with $f|_{|N|}$ on $V_N \subset V_K \subset V_{K^{(r)}}$.

Proof. Consider the open cover $\{f^{-1}(\mathrm{St}_L(w))\}_{w \in V_L}$ of $|K|$. This has a Lesbegue number δ (with respect to the euclidean metric on $|K|$), and by Lemma 6.2.6 we may choose an r such that $\mu(K^{(r)}) < \delta$. Then for each $v \in V_{K^{(r)}}$ we have

$$\mathrm{St}_{K^{(r)}}(v) \subset B_{\mu(K^{(r)})}(v) \subset f^{-1}(\mathrm{St}_L(w))$$

for some $w \in V_L$, and setting $g(v) = w$ we have

$$f(\mathrm{St}_{K^{(r)}}(v)) \subset \mathrm{St}_L(g(v)).$$

Thus g is a simplicial approximation to f.

For the addendum, note that if $v \in V_N$ then $f(v)$ is a vertex of L, so by increasing r we may in addition suppose that

$$\mathrm{St}_{K^{(r)}}(v) \subset B_{\mu(K^{(r)})}(v) \subset f^{-1}(\mathrm{St}_L(f(v)))$$

for each $v \in V_N$, and hence choose $g(v) := f(v)$ for these v. $\square$

Chapter 7

Homology

7.1 Simplicial homology

Definition 7.1.1. Let K be a simplicial complex, and $O_n(K)$ be the free abelian group with basis given by the symbols

$$\{[v_0, v_1, \dots, v_n] \mid v_0, v_1, \dots v_n \text{ span a simplex of } K\}.$$

Note that the v_i are considered to be *ordered*, and they could span a simplex of dimension less than n (i.e. the list could have repeats).

Let $T_n(K) \leq O_n(K)$ be the subgroup generated by the elements

(i) $[v_0, v_1, \dots, v_n]$ if the sequence contains repeated vertices,

(ii) $[v_0, v_1, \dots, v_n] - \mathrm{sign}(\sigma) \cdot [v_{\sigma(0)}, v_{\sigma(1)}, \dots, v_{\sigma(n)}]$ for a permutation σ of $\{0, 1, \dots, n\}$.

Then we define $C_n(K) := O_n(K)/T_n(K)$ to be the quotient group.

Lemma 7.1.2. *There is a (non-canonical) isomorphism* $C_n(K) \cong \mathbb{Z}^{\#\ n\text{-simplices of } K}$.

Proof. Choose a total order $\prec$ on V_K. Then each n-simplex $\sigma \in K$ determines a canonical ordered simplex $[\sigma]$ given by ordering the vertices a_i of σ so that $a_0 \prec a_1 \prec \cdots \prec a_n$. This defines a homomorphism

$$\begin{aligned} \phi : \mathbb{Z}^{n\text{-simplices of } K} &\longrightarrow O_n(K) \\ \sigma &\longmapsto [\sigma] \end{aligned}$$

and so a homomorphism

$$\begin{aligned} \phi' : \mathbb{Z}^{n\text{-simplices of } K} &\longrightarrow C_n(K) \\ \sigma &\longmapsto [[\sigma]]\,. \end{aligned}$$

For each symbol $[a_0, a_1, \dots, a_n]$ there is a unique permutation τ of $\{0, 1, \dots, n\}$ such that $a_{\tau(0)} \prec a_{\tau(1)} \prec \cdots \prec a_{\tau(n)}$, and we write $\mathrm{sign}([a_0, a_1, \dots, a_n]) := \mathrm{sign}(\tau)$. Using this we define a homomorphism

$$\begin{aligned} \rho : O_n(K) &\longrightarrow \mathbb{Z}^{n\text{-simplices of } K} \\ [a_0, a_1, \dots, a_n] &\longmapsto \begin{cases} \mathrm{sign}([a_0, a_1, \dots, a_n]) \cdot \langle a_0, a_1, \dots, a_n \rangle & \text{if there are no repeats} \\ 0 & \text{if there are repeats.} \end{cases} \end{aligned}$$

By construction $T_n(K) \leq \mathrm{Ker}(\rho)$, and so ρ induces a homomorphism

$$\rho' : C_n(K) \longrightarrow \mathbb{Z}^{n\text{-simplices of } K}.$$

Now it is easy to see that $\rho' \circ \phi'(\sigma) = \sigma$, and so $\rho' \circ \phi' = \mathrm{Id}$. Furthermore if $[a_0, a_1, \ldots, a_n]$ has no repeats then

$$\begin{aligned}\phi' \circ \rho'([a_0, a_1, \ldots, a_n]) &= \mathrm{sign}([a_0, a_1, \ldots, a_n]) \cdot [\langle a_0, a_1, \ldots, a_n \rangle] \\ &= \mathrm{sign}(\tau) \cdot [a_{\tau(0)}, a_{\tau(1)}, \ldots, a_{\tau(n)}]\end{aligned}$$

where τ is such that $a_{\tau(0)} \prec a_{\tau(1)} \prec \cdots \prec a_{\tau(n)}$, but in the quotient $C_n(K)$ this is equivalent to $[a_0, a_1, \ldots, a_n]$. As symbols $[a_0, a_1, \ldots, a_n]$ *with* repeats are trivial in $C_n(K)$, it follows that $\phi' \circ \rho' = \mathrm{Id}$. □

We now define a homomorphism

$$\begin{aligned} d_n : O_n(K) &\longrightarrow O_{n-1}(K) \\ [v_0, v_1, \ldots, v_n] &\longmapsto \sum_{i=0}^{n} (-1)^i \cdot [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n], \end{aligned}$$

where $[v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$ denotes the sequence obtained by removing v_i.

Lemma 7.1.3. *The homomorphism d_n sends $T_n(K)$ into $T_{n-1}(K)$.*

Proof. First note that $d_n([v_0, v_1, \ldots, v_n] - \mathrm{sign}(\sigma)[v_{\sigma(0)}, v_{\sigma(1)}, \ldots, v_{\sigma(n)}])$ is

$$\sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n] - \sum_{i=0}^{n} (-1)^i \mathrm{sign}(\sigma) [v_{\sigma(0)}, v_{\sigma(1)}, \ldots, \hat{v}_{\sigma(i)}, \ldots, v_{\sigma(n)}]. \tag{7.1.1}$$

We must show that this is trivial in $O_{n-1}(K)/T_{n-1}(K)$.

If $\sigma = (j, j+1)$, so $\mathrm{sign}(\sigma) = -1$, then we compute

$$\begin{aligned}\sum_{i=0}^{n} (-1)^i \mathrm{sign}(\sigma) [v_{\sigma(0)}, \ldots, \hat{v}_{\sigma(i)}, \ldots, v_{\sigma(n)}] &= \sum_{i=0}^{j-1} (-1)^{i+1} [v_0, \ldots, \hat{v}_i, \ldots, v_{j-1}, v_{j+1}, v_j, \ldots, v_n] \\ &\quad + (-1)^{j+1} [v_0, \ldots, v_{j-1}, v_j, v_{j+2}, \ldots, v_n] \\ &\quad + (-1)^{j+2} [v_0, \ldots, v_{j-1}, v_{j+1}, v_{j+2}, \ldots, v_n] \\ &\quad + \sum_{i=j+2}^{n} (-1)^{i+1} [v_0, \ldots, v_{j-1}, v_{j+1}, v_j, \ldots, \hat{v}_i, \ldots v_n].\end{aligned}$$

Now the terms in the first sum satisfy

$$[v_0, \ldots, \hat{v}_i, \ldots, v_{j-1}, v_{j+1}, v_j, \ldots, v_n] \equiv -[v_0, \ldots, \hat{v}_i, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, v_n] \mod T_{n-1}(K)$$

and those in the second sum satisfy

$$[v_0, \ldots, v_{j-1}, v_{j+1}, v_j, \ldots, \hat{v}_i, \ldots v_n] \equiv -[v_0, \ldots, v_{j-1}, v_j, v_{j+1}, \ldots, \hat{v}_i, \ldots v_n] \mod T_{n-1}(K)$$

so in $O_{n-1}(K)/T_{n-1}(K)$ we get

$$\sum_{i=0}^{n} (-1)^i \mathrm{sign}(\sigma) [v_{\sigma(0)}, v_{\sigma(1)}, \ldots, \hat{v}_{\sigma(i)}, \ldots, v_{\sigma(n)}] \equiv \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_n]$$

as required. This establishes that the element (7.1.1) is trivial in $O_{n-1}(K)/T_{n-1}(K)$ whenever σ is a transposition: as any permutation σ is a product of transpositions of the form $(j, j+1)$, it follows for arbitrary σ.

Now suppose that we have $[v_0, v_1, \ldots, v_n]$ with $v_j = v_{j+1}$. Then

$$\begin{aligned} d_n([v_0, v_1, \ldots, v_n]) = &\sum_{i=0}^{j-1} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_j, v_{j+1}, \ldots, v_n] \\ &+ (-1)^j [v_0, \ldots, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n] \\ &+ (-1)^{j+1} [v_0, \ldots, \ldots, v_j, v_{j+2}, \ldots, v_n] \\ &+ \sum_{i=j+2}^{n} (-1)^i [v_0, \ldots, v_j, v_{j+1}, \ldots, \hat{v}_i, \ldots, v_n] \end{aligned}$$

but the middle two terms cancel out, and the outer two terms contain repeated entries so lie in $T_{n-1}(K)$. In general, if $[v_0, v_1, \ldots, v_n]$ with $v_j = v_k$ then applying $\sigma = (j, k+1)$ reduces to the case $k = j+1$. $\square$

The consequence of this lemma is that $d_n : O_n(K) \to O_{n-1}(K)$ induces a homomorphism

$$d_n : C_n(K) \longrightarrow C_{n-1}(K).$$

Lemma 7.1.4. *The homomorphism $d_{n-1} \circ d_n : C_n(K) \to C_{n-2}(K)$ is trivial.*

Proof. We compute, at the level of $O_n(K)$,

$$\begin{aligned} d_{n-1} \circ d_n([v_0, \ldots, v_n]) &= d_{n-1} \left(\sum_{i=0}^{n} (-1)^i \cdot [v_0, \ldots, \hat{v}_i, \ldots, v_n] \right) \\ &= \sum_{i=0}^{n} (-1)^i \left(\sum_{k=0}^{i-1} (-1)^k [v_0, \ldots, \hat{v}_k, \ldots \hat{v}_i, \ldots, v_n] \right. \\ &\qquad \left. + \sum_{k=i}^{n-1} (-1)^k [v_0, \ldots, \hat{v}_i, \ldots \hat{v}_{k+1}, \ldots, v_n] \right) \end{aligned}$$

The coefficient of $[v_0, \ldots, \hat{v}_a, \ldots \hat{v}_b, \ldots, v_n]$ is $(-1)^a(-1)^b$ from $k = a$ and $i = b$ plus $(-1)^a(-1)^{b-1}$ from $i = a$ and $k+1 = b$. These cancel out, so $d_{n-1} \circ d_n([v_0, \ldots, v_n]) = 0$. the symbols $[v_0, \ldots, v_n]$ generate $C_n(K)$, so $d_{n-1} \circ d_n = 0$. $\square$

The consequence of this lemma is that we have the containment $\mathrm{Im}(d_n) \subset \mathrm{Ker}(d_{n-1})$. Homology is the measurement of how different these two groups are.

Definition 7.1.5. The ith **simplicial homology group** of a simplicial complex K is

$$H_i(K) := \frac{\mathrm{Ker}(d_n : C_n(K) \to C_{n-1}(K))}{\mathrm{Im}(d_{n+1} : C_{n+1}(K) \to C_n(K))}.$$

Example 7.1.6 (Homology of the simplicial circle)**.** Consider the simplicial complex K inside $\mathbb{R}^3$ given by all proper faces of the standard 2-simplex $\Delta^2 \subset \mathbb{R}^3$. If we write $\{e_1, e_2, e_3\}$ for the standard basis of $\mathbb{R}^3$, then K has simplices

$$\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_1, e_2\rangle, \langle e_1, e_3\rangle, \langle e_2, e_3\rangle.$$

Choosing the order $e_1 \prec e_2 \prec e_3$ on the vertices, we obtain isomorphisms

$$\begin{aligned} C_0(K) &\cong \mathbb{Z}\{[e_1], [e_2], [e_3]\} \\ C_1(K) &\cong \mathbb{Z}\{[e_1, e_2], [e_1, e_3], [e_2, e_3]\}, \end{aligned}$$

so both groups are free abelian of rank three, and have the bases shown. With respect to these isomorphisms, we have

$$d_1([e_i, e_j]) = [e_j] - [e_i],$$

so in the given bases d_1 is given by the matrix

$$\begin{pmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Hence

$$H_0(K) = \mathbb{Z}\{[e_1], [e_2], [e_3]\}/\langle [e_i] - [e_j] \mid i, j \in \{1, 2, 3\}\rangle \cong \mathbb{Z}$$

and

$$H_1(K) = \operatorname{Ker}(d_1) = \mathbb{Z}\{[e_1, e_2] - [e_1, e_3] + [e_2, e_3]\} \cong \mathbb{Z}.$$

Also $H_i(K) = 0$ for $i \geq 2$, as K has no simplices of dimension 2 or higher.

Example 7.1.7 (Homology of the 2-simplex)**.** Now consider the simplicial complex L in $\mathbb{R}^3$ given by the standard 2-simplex Δ^2 in $\mathbb{R}^3$ along with all its faces. The simplicial complex L is similar to K from the previous example, but has a 2-simplex as well. Thus we have

$$\begin{tikzcd} C_2(L) \arrow[r, "d_2"] & C_1(L) \arrow[r, "d_1"] & C_0(L) \end{tikzcd}$$

$$[e_1, e_2, e_3] \longmapsto [e_1, e_2] - [e_1, e_3] + [e_2, e_3]$$

where $C_1(L) = C_1(K)$ and $C_0(L) = C_0(K)$, and the formula for d_1 is the same as for K.

Now d_2 is injective, so $H_2(L) = 0$. Also $\operatorname{Im}(d_2) = \mathbb{Z}\{[e_1, e_2] - [e_1, e_3] + [e_2, e_3]\} = \operatorname{Ker}(d_1)$ and so $H_1(L) = 0$. Finally, we obtain $H_0(L) \cong \mathbb{Z}$ just as in the previous example.

7.2 Some homological algebra

Definition 7.2.1. A **chain complex** is a sequence C_0, C_1, $C_2 \ldots$ of abelian groups equipped with homomorphisms $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$. We write it as $C_\bullet$, and call the d_n the **differentials** of $C_\bullet$. The nth **homology** of $C_\bullet$ is

$$H_n(C_\bullet) := \frac{\operatorname{Ker}(d_n : C_n \to C_{n-1})}{\operatorname{Im}(d_{n+1} : C_{n+1} \to C_n)}.$$

We write $Z_n(C_\bullet) := \mathrm{Ker}(d_n)$, and call this the abelian group of n-**cycles** in $C_\bullet$, and $B_n(C_\bullet) := \mathrm{Im}(d_{n+1})$, and call this the abelian group of n-**boundaries** in $C_\bullet$.

A **chain map** $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that $f_n \circ d_{n+1} = d_{n+1} \circ f_{n+1}$; in other words, such that the square

$$\begin{tikzcd} C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "d_{n+1}"] & D_{n+1} \arrow[d, "d_{n+1}"] \\ C_n \arrow[r, "f_n"] & D_n \end{tikzcd}$$

commutes.

A **chain homotopy** from $f_\bullet$ to $g_\bullet$ (two chain maps from $C_\bullet$ to $D_\bullet$) is a sequence of homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = d_{n+1} \circ h_n + h_{n-1} \circ d_n : C_n \to D_n.$$

Note that in the above definitions we use the same symbol, d_n, for the differential in any chain complex. Which one we mean in any given situation will be clear from context.

Lemma 7.2.2. *A chain map $f_\bullet : C_\bullet \to D_\bullet$ induces a homomorphism*

$$\begin{aligned} f_* : H_n(C_\bullet) &\longrightarrow H_n(D_\bullet) \\ [x] &\longmapsto [f_n(x)]. \end{aligned}$$

Furthermore, if $f_\bullet$ and $g_\bullet$ are chain homotopic then $f_* = g_*$.

Proof. We need to check that f_* is well-defined.

(i) Let $[x] \in H_n(C_\bullet) = \frac{Z_n(C_\bullet)}{B_n(C_\bullet)}$ be represented by $x \in Z_n(C_\bullet)$. The element $f_n(x) \in D_n$ is a cycle, as $d_n(f_n(x)) = f_{n-1}(d_n(x)) = f_{n-1}(0) = 0$, because x is a cycle in C_n. Thus $f_n(x)$ does represent a homology class.

(ii) If $[x] = [y]$, then $x - y \in B_n(C_\bullet)$, and so $x - y = d_{n+1}(z)$ for some $z \in C_{n+1}$. Then $f_n(x) - f_n(y) = f_n(d_{n+1}(z)) = d_{n+1}(f_{n+1}(z))$ is a boundary, so $[f_n(x)] = [f_n(y)] \in H_n(D_\bullet)$.

Now suppose that $f_\bullet$ and $g_\bullet$ are chain homotopic via a chain homotopy $h_\bullet$, and let $x \in Z_n(C_\bullet)$. We have

$$g_n(x) - f_n(x) = d_{n+1}(h_n(x)) + h_{n-1}(d_n(x))$$

but x is a cycle so $d_n(x) = 0$. Thus $g_n(x) - f_n(x)$ is a boundary, but then $[g_n(x)] = [f_n(x)]$. ☐

Just as we did with the notion of homotopy of maps between spaces, we check that

(i) being chain homotopic defines an equivalence relation on the set of chain maps from $C_\bullet$ to $D_\bullet$, written $f_\bullet \simeq g_\bullet$,

(ii) if $a_\bullet : A_\bullet \to C_\bullet$ is a chain map, and $f_\bullet \simeq g_\bullet : C_\bullet \to D_\bullet$, then $f_\bullet \circ a_\bullet \simeq g_\bullet \circ a_\bullet$, and similarly with precomposition.

Definition 7.2.3. A chain map $f_\bullet : C_\bullet \to D_\bullet$ is a **chain homotopy equivalence** if there is a chain map $g_\bullet : D_\bullet \to C_\bullet$ and chain homotopies $f_\bullet \circ g_\bullet \simeq \mathrm{Id}_{D_\bullet}$ and $g_\bullet \circ f_\bullet \simeq \mathrm{Id}_{C_\bullet}$.

Lemma 7.2.4. *If $f_\bullet : C_\bullet \to D_\bullet$ is a chain homotopy equivalence, then $f_* : H_n(C_\bullet) \to H_n(D_\bullet)$ is an isomorphims for all n.*

Proof. We have $f_* \circ g_* = (f_\bullet \circ g_\bullet)_* = (\mathrm{Id}_{D_\bullet})_* = \mathrm{Id}_{H_n(D_\bullet)}$, and vice versa. □

7.3 Elementary calculations

Returning to simplicial complexes, note that for a simplicial complex K we have defined in Section 7.1 a chain complex $C_\bullet(K)$, which we call the **simplicial chain complex of** K. We want to show that a simplicial map $f : K \to L$ induces a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$.

Lemma 7.3.1. *Let $f : K \to L$ be a simplicial map. Then the formula*

$$\begin{aligned} f_n : C_n(K) &\longrightarrow C_n(L) \\ [a_0, a_1, \dots, a_n] &\longmapsto [f(a_0), f(a_1), \dots, f(a_n)] \end{aligned}$$

defines a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$, and hence a homomorphism $f_* : H_n(K) \to H_n(L)$.

Proof. We first need that $f_n(T_n(K)) \subset T_n(L)$, but this is clear. Then we compute

$$\begin{aligned} f_{n-1}(d_n([a_0, a_1, \dots, a_n])) &= f_{n-1}\left(\sum_{i=0}^{n} (-1)^i \cdot [a_0, \dots, \hat{a}_i, \dots, a_n]\right) \\ &= \sum_{i=0}^{n} (-1)^i \cdot [f(a_0), \dots, \widehat{f(a_i)}, \dots, f(a_n)] \\ &= d_n(f_n([a_0, a_1, \dots, a_n])). \end{aligned}$$

□

Definition 7.3.2. Say a simplicial complex K is **cone** with **cone point** $v_0 \in V_K$ if $|K| = \mathrm{St}_K(v_0) \cup \mathrm{Lk}_K(v_0)$. In other words, every simplex of K is a face of a simplex which contains v_0.

Proposition 7.3.3. *If K is a cone with cone point v_0, then the inclusion $i : \{v_0\} \to K$ induces a chain homotopy equivalence $i_\bullet : C_\bullet(\{v_0\}) \to C_\bullet(K)$. Hence*

$$H_n(K) = \begin{cases} \mathbb{Z}\{[v_0]\} & n = 0 \\ 0 & \textit{else.} \end{cases}$$

Proof. The (only) map $r : V_k \to \{v_0\}$ defines a simplicial map $r : K \to \{v_0\}$. We will show that $r_\bullet$ is a chain homotopy inverse to $i_\bullet$. One direction is easy: the composition $r_\bullet \circ i_\bullet : C_\bullet(\{v_0\}) \to C_\bullet(\{v_0\})$ is equal to the identity map.

For the other direction, define a homomorphism

$$\begin{aligned} h_n : O_n(K) &\longrightarrow O_{n+1}(K) \\ [a_0, \ldots, a_n] &\longmapsto [v_0, a_0, \ldots, a_n] \end{aligned}$$

and observe that it takes $T_n(K)$ into $T_{n+1}(K)$, and hence gives a homomorphism $h_n : C_n(K) \to C_{n+1}(K)$. If $n > 0$ then

$$\begin{aligned} (h_{n-1} \circ d_n + d_{n+1} \circ h_n)([a_0, \ldots, a_n]) &= \sum_{i=0}^{n} (-1)^i [v_0, a_0, \ldots, \hat{a}_i, \ldots, a_n] \\ &\quad + [a_0, \ldots, a_n] \\ &\quad + \sum_{i=0}^{n} (-1)^{i+1} [v_0, a_0, \ldots, \hat{a}_i, \ldots, a_n] \\ &= [a_0, \ldots, a_n] = (\mathrm{Id} - i_n \circ r_n)([a_0, \ldots, a_n]). \end{aligned}$$

If $n = 0$ then

$$\begin{aligned} (h_{-1} \circ d_0 + d_1 \circ h_0)([a_0]) &= d_1([v_0, a_0]) = [a_0] - [v_0] \\ &= (\mathrm{Id} - i_0 \circ r_0)([a_0]). \end{aligned}$$

Thus $h_\bullet$ is a chain homotopy from $i_\bullet \circ r_\bullet$ to $\mathrm{Id}_{C_\bullet(K)}$. □

Corollary 7.3.4. *Let Δ^n be the standard simplex in $\mathbb{R}^{n+1}$, and L be the simplicial complex given by Δ^n and all its faces. Then*

$$H_i(L) = \begin{cases} \mathbb{Z} & i = 0 \\ 0 & \textit{else.} \end{cases}$$

Proof. Any vertex of L is a cone point. □

Corollary 7.3.5. *Let K be the simplicial complex given by all the proper faces of $\Delta^n \subset \mathbb{R}^{n+1}$. Then for $n \geq 2$ we have*

$$H_i(K) = \begin{cases} \mathbb{Z} & i = 0 \textit{ or } n-1 \\ 0 & \textit{else.} \end{cases}$$

Proof. Note that K is the $(n-1)$-skeleton of the simplicial complex L from the previous corollary, so $C_i(K) = C_i(L)$ for $0 \leq i \leq n-1$, and $C_i(K)$ vanishes in higher degrees. Thus the two chain complexes are

$$\begin{tikzcd} C_0(L) \arrow[d, equal] & C_1(L) \arrow[l, "d_1^L"'] \arrow[d, equal] & \cdots \arrow[l, "d_2^L"'] & C_{n-2}(L) \arrow[l, "d_{n-2}^L"'] \arrow[d, equal] & C_{n-1}(L) \arrow[l, "d_{n-1}^L"'] \arrow[d, equal] & C_n(L) \arrow[l, "d_n^L"'] \\ C_0(K) & C_1(K) \arrow[l, "d_1^K"'] & \cdots \arrow[l, "d_2^K"'] & C_{n-2}(K) \arrow[l, "d_{n-2}^K"'] & C_{n-1}(K) \arrow[l, "d_{n-1}^K"'] & 0, \arrow[l] \end{tikzcd}$$

where we have indicated which complex the differentials belong to.

Now for $0 \leq i \leq n-1$ the two chain complexes are equal, and we deduce that $H_i(K) = H_i(L)$ for $0 \leq i \leq n-2$. In particular in this range the homology if $\mathbb{Z}$ in degree 0 and 0 otherwise.

Near the top we have $C_n(L) = \mathbb{Z}\{[e_1, \ldots, e_{n+1}]\}$. Firstly $H_n(L) = 0 = \mathrm{Ker}(d_n^L)$ so d_n^L is injective, and secondly $H_{n-1}(L) = 0$ so $\mathrm{Ker}(d_{n-1}^L) = \mathrm{Im}(d_n^L)$. Thus we have

$$\mathrm{Ker}(d_{n-1}^K) = \mathrm{Ker}(d_{n-1}^L) = \mathrm{Im}(d_n^L) \cong \mathbb{Z}$$

and so $H_{n-1}(K) \cong \mathbb{Z}$. □

Now that we have some example computations of the homology of certain simplicial complexes, it is interesting to ask what it *means*.

Lemma 7.3.6. *There is an isomorphism* $H_0(K) \cong \mathbb{Z}^{\#\text{ path components of } |K|}$.

Proof. First note that every path component of a polyhedron contains a vertex (as every point lies in an open simplex, so can be joined by a straight line to any vertex of that simplex). By definition $H_0(K) = \mathbb{Z}^{V_K}/\mathrm{Im}(d_1)$. If vertices v and w lies in the same path component, we can choose a path between them and then find a simplicial approximation to this path. This gives a sequence $v = v_0, v_1, \ldots, v_n = w$ of vertices such that each adjacent pair span a 1-simplex. Then $[v_i] - [v_{i-1}] = d_1([v_{i-1}, v_i])$ so summing over all i gives $[w] - [v] = d_1([v_0, v_1] + [v_1, v_2] + \cdots + [v_{n-1}, v_n])$, so $[v] = [w] \in H_0(K)$. □

7.4 The Mayer–Vietoris sequence

We will develop a tool akin to the Seifert–van Kampen Theorem, which will allow us to compute the homology of a simplicial complex by decomposing it into pieces, computing their homology, and then assembling them again using an algebraic machine.

Definition 7.4.1. We say that a pair of homomorphisms of abelian groups

$$A \xrightarrow{f} B \xrightarrow{g} C$$

are **exact at** B if $\mathrm{Im}(f) = \mathrm{Ker}(g)$. More generally, we say that a collection of homomorphisms

$$\cdots \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow A_{i+2} \longrightarrow A_{i+3} \longrightarrow A_{i+4} \longrightarrow \cdots$$

is **exact** if it is exact at A_i whenever there is both a map entering and a map exiting A_i. A **short exact sequence of abelian groups** is an exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0,$$

that is, f is injective (exactness at A), g is surjective (exactness at C), and $\mathrm{Im}(f) = \mathrm{Ker}(g)$ (exactness at B).

Chain maps $i_\bullet : A_\bullet \to B_\bullet$ and $j_\bullet : B_\bullet \to C_\bullet$ form a **short exact sequence of chain complexes** if for each n

$$0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \longrightarrow 0$$

is a short exact sequence of abelian groups.

Theorem 7.4.2. *If* $0 \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0$ *is a short exact sequence of chain complexes, then there are natural homomorphisms* $\partial_* : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ *such that*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(A_\bullet) \arrow[r, "i_*"] & H_n(B_\bullet) \arrow[r, "j_*"] & H_n(C_\bullet) \arrow[dll, "\partial_*"] & \\
 & H_{n-1}(A_\bullet) \arrow[r, "i_*"] & H_{n-1}(B_\bullet) \arrow[r, "j_*"] & H_{n-1}(C_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

is exact.

Let us defer the proof of this theorem.

Theorem 7.4.3 (Mayer–Vietoris)**.** *Let K be a simplicial complex, M and N be subcomplexes, and $L = N \cap M$. Suppose that M and N cover K (i.e. every simplex of K lies in M or N). Let us write*

$$\begin{tikzcd}
L \arrow[r, hook, "j"] \arrow[d, hook, "i"] & N \arrow[d, hook, "l"] \\
M \arrow[r, hook, "k"] & K
\end{tikzcd}$$

for the inclusion maps between these simplicial complexes. Then there are natural homomorphisms $\partial_* : H_n(K) \to H_{n-1}(L)$ *such that*

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_*"] & H_n(L) \arrow[r, "i_*+j_*"] & H_n(M) \oplus H_n(N) \arrow[r, "k_*-l_*"] & H_n(K) \arrow[dll, "\partial_*"] & \\
 & H_{n-1}(L) \arrow[r, "i_*+j_*"] & H_{n-1}(M) \oplus H_{n-1}(N) \arrow[r, "k_*-l_*"] & H_{n-1}(K) \arrow[r] & \cdots
\end{tikzcd}$$

$$\cdots \xrightarrow{i_*+j_*} H_0(M) \oplus H_0(N) \xrightarrow{k_*-l_*} H_0(K) \longrightarrow 0$$

is exact.

Proof. Note that for each n the sequence of abelian groups

$$0 \longrightarrow C_n(L) \xrightarrow{i_n+j_n} C_n(M) \oplus C_n(N) \xrightarrow{k_n-l_n} C_n(K) \longrightarrow 0$$

is exact:

(i) i_n and j_n are both injective,

(ii) every simplex of K lies in M or N, so $k_n - l_n$ is surjective,

(iii) if $(x, y) \in \operatorname{Ker}(k_n - l_n)$ then $k_n(x) = l_n(y)$, so x and y are both sums of simplices in both M and N, so lie in L and are equal.

Thus the chain maps $i_\bullet + j_\bullet : C_\bullet(L) \to C_\bullet(M) \oplus C_\bullet(N)$ and $k_\bullet - l_\bullet : C_\bullet(M) \oplus C_\bullet(N) \to C_\bullet(K)$ form a short exact sequence of chain complexes, so Theorem 7.4.2 applies, giving the result. $\square$

Proof of Theorem 7.4.2. The proof consists of what is affectionately known as "diagram-chasing", that is, following the logical consequences of an element being at a certain place in a commutative diagram.

Constructing ∂_* (The Snake Lemma). Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "d"] & B_n \arrow[r, "j_n"] \arrow[d, "d"] & C_n \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "i_{n-1}"] & B_{n-1} \arrow[r, "j_{n-1}"] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

and let $[x] \in H_n(C_\bullet)$ be a homology class, represented by a cycle $x \in C_n$. As j_n is surjective, there is a $y \in B_n$ such that $j_n(y) = x$. Then $d(y) \in B_{n-1}$ satisfies

$$j_{n-1}(d(y)) = d(j_n(y)) = d(x) = 0$$

as x was a cycle. By exactness of the bottom row at B_{n-1}, we thus have $d(y) = i_{n-1}(z)$ for some $z \in A_{n-1}$.

Now we have $i_{n-2}(d(z)) = d(i_{n-1}(z)) = d(d(y)) = 0$, and i_{n-2} is injective, so $d(z) = 0$. Thus z representes a homology class $\partial_*([x]) := [z] \in H_{n-1}(A_\bullet)$.

∂_* is well-defined. If $[x] = [x'] \in H_n(C_\bullet)$ then $x - x' = d(a)$ for some $a \in C_{n+1}$. Proceeding as above for x', we get a $y' \in B_n$. As j_{n+1} is surjective, let $b \in B_{n+1}$ be such that $j_{n+1}(b) = a$, then

$$j_n(y - y') = x - x' = d(a) = d(j_{n+1}(b)) = j_n(d(b)),$$

so by exactness at B_n we have $y - y' = d(b) + i_n(c)$ for some $c \in A_n$. Thus

$$d(y) - d(y') = d(i_n(c)) = i_{n-1}(d(c)),$$

so by injectivity of i_{n-1} we get $z - z' = d(c)$. Thus $[z] = [z'] \in H_{n-1}(A_\bullet)$.

∂_* is a homomorphism. Given $[x], [x'] \in H_n(C_\bullet)$, we can choose $y + y'$ as the lift to B_n of $x + x' \in C_n$. It is then immediate that $\partial_n([x + x']) = [z + z']$, so ∂_* is a homomorphism.

The sequence is exact at $H_n(C_\bullet)$. First let $[x] \in \mathrm{Im}(j_*)$. Then we can write $x = j_n(y)$ for $y \in B_n$ a cycle. The element y is then a lift of x, and it satisfies $d(y) = 0$. Thus $\partial_*([x]) = 0$.

Now suppose that $\partial_*([x]) = 0$. Thus the z obtained by the Snake Lemma construction satisfies $z = d(t)$ for some $t \in A_n$, and we have $j_n(y - i_n(t)) = x$. But $d(y - i_n(t)) = d(y) - d(i_n(t)) = d(y) - i_{n-1}(d(t)) = d(y) - i_{n-1}(z)$ which is 0 by the way we chose z. Thus $y - i_n(t)$ is a cycle, and $[x] = j_*([y - i_n(t)])$, so $[x] \in \mathrm{Im}(j_*)$.

The sequence is exact at $H_n(B_\bullet)$. It is clear that $\mathrm{Im}(i_*) \subset \mathrm{Ker}(j_*)$, as $j_n \circ i_n = 0$. Let y be a cycle so that $j_*([y]) = 0$, so $j_n(y) = d(a)$ for some $a \in C_{n+1}$. Let $b \in B_{n+1}$ be such that $j_{n+1}(b) = a$. Then

$$j_n(y - d(b)) = d(a) - d(j_{n+1}(b)) = 0$$

and so $y - d(b) = i_n(t)$ for some $t \in A_n$. As y is a cycle, so is $i_n(t)$, and as i_n is injective it follows that t is a cycle. But then $i_*([t]) = [y - d(b)] = [y]$, so $\mathrm{Im}(i_*) \supset \mathrm{Ker}(j_*)$.

The sequence is exact at $H_n(A_\bullet)$. If $\partial_*([x]) = [z]$ as constructed via the Snake Lemma, then $i_n(x) = d(y)$, so $i_*([x]) = 0$. Thus $\mathrm{Im}(\partial_*) \subset \mathrm{Ker}(i_*)$. Let z be a cycle so that $i_*([z]) = 0$, that is $i_n(z) = d(y)$. Then $d(j_{n+1}(y)) = j_n(d(y)) = j_n(i_n(z)) = 0$, so $j_{n+1}(y)$ is a cycle. By construction $\partial_*([j_{n+1}(y)]) = [z]$, so $\mathrm{Im}(\partial_*) \supset \mathrm{Ker}(i_*)$. □

7.5 Continuous maps and homotopy invariance

In this section we want to show that to each map $f : |K| \to |L|$ we may associate a homomorphism $f_* : H_i(K) \to H_i(L)$, even when f does not come from a simplical map, and further that if $f \simeq f'$ then $f_* = f'_*$.

Definition 7.5.1. Simplicial maps $f, g : K \to L$ are **contiguous** if for each $\sigma \in K$ the simplices $f(\sigma)$ and $g(\sigma)$ are faces of some simplex $\tau \in L$.

Lemma 7.5.2. *If $f, g : K \to L$ are both simplicial approximations to the same map, then they are contiguous.*

Proof. Let $F : |K| \to |L|$ be a continuous map which f and g are simplicial approximations to. If $x \in \mathring{\sigma} \subset |K|$ and $F(x) \in \mathring{\tau} \subset |L|$, then as in the proof of Lemma 6.2.2 we have that $f(\sigma)$ and $g(\sigma)$ are both faces of τ. □

Lemma 7.5.3. *If $f, g : K \to L$ are contiguous, then $f_\bullet \simeq g_\bullet : C_\bullet(K) \to C_\bullet(L)$ and so $f_* = g_*$.*

Proof. Choose an ordering $\prec$ on V_K, so we can represent basis elements for $C_n(K)$ by ordered simplices $[a_0, \ldots, a_n]$ such that $a_0 \prec a_1 \prec \cdots \prec a_n$. Define a homomorphism $h_n : C_n(K) \to C_{n+1}(L)$ on basis elements by

$$h_n([a_0, \ldots, a_n]) = \sum_{i=0}^{n} (-1)^i [f(a_0), \ldots, f(a_i), g(a_i), \ldots, g(a_n)].$$

Direct calculation shows that $d \circ h_n + h_{n-1} \circ d = g_n - f_n$, so the h_n provide a chain homotopy from $f_\bullet$ to $g_\bullet$. □

Let K' be the barycentric subdivision of a simplicial complex K, and for each vertex $\hat{\sigma} \in V_{K'}$ choose a vertex $a(\hat{\sigma})$ of σ, to obtain a function $a : V_{K'} \to V_K$.

Lemma 7.5.4. *The function $a : V_{K'} \to V_K$ defines a simplicial approximation to the identity map. Furthermore, every simplicial approximation $g : K' \to K$ to the identity map is of this form.*

Proof. The first claim is that $\mathrm{St}_{K'}(\hat{\sigma}) \subseteq \mathrm{St}_K(a(\hat{\sigma}))$. To see this, suppose that $\hat{\sigma}$ is a vertex of a simplex $\tau = \langle \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_p \rangle$ of K', say $\hat{\sigma} = \hat{\sigma}_i$. Then $a(\hat{\sigma})$ is a vertex of σ_i, and hence a vertex of σ_p. In particular $\mathring{\tau} \subset \mathring{\sigma}_p \subset \mathrm{St}_K(a(\hat{\sigma}))$.

If $g : K' \to K$ is a simplicial approximation to the identity map then we have $\mathring{\sigma} \subseteq \mathrm{St}_K(g(\hat{\sigma}))$, but the open simplices in $\mathrm{St}_K(g(\hat{\sigma}))$ are precisely those whose closure contains $g(\hat{\sigma})$, so $g(\hat{\sigma})$ must be a vertex of σ. □

Proposition 7.5.5. *For $a : K' \to K$ a simplicial approximation to the identity map, the induced map $a_* : H_n(K') \to H_n(K)$ is an isomorphism for all n.*

Proof. First consider the case where K consists of a single p-simplex, σ, and all its faces. Note that K' is a cone with cone point $\hat{\sigma}$, and we have seen that K is also a cone, with any vertex as cone point. By Proposition 7.3.3 we can thus calculate the homology of either of these simplicial complexes, and we find that it is 0 in strictly positive degrees, and $\mathbb{Z}$ in degree zero, generated by any vertex. Thus by direct computation, any simplicial approximation to the identity $a : K' \to K$ induces an isomorphism on homology in every degree.

Now let us prove the proposition by double induction on (i) the dimension of K, and (ii) the number of simplices of K of maximal dimension. If σ is a simplex of K of maximal dimension, then it is not a proper face of any simplex and so $L := K \setminus \{\sigma\}$ is also a simplicial complex. Let S be the simplicial complex given by σ and all its faces, and $T = S \cap L$. Any simplicial approximation to the identity $a : K' \to K$ sends L' into L and S' into S (so also T' into T). We thus obtain a map between Mayer–Vietoris sequences

$$\begin{tikzcd}
H_n(T') \arrow[r] \arrow[d, "(a|_{T'})_*"] & H_n(S') \oplus H_n(L') \arrow[r] \arrow[d, "(a|_{S'})_* \oplus (a|_{L'})_*"] & H_n(K') \arrow[r] \arrow[d, "a_*"] & H_{n-1}(T') \arrow[r] \arrow[d, "(a|_{T'})_*"] & H_{n-1}(S') \oplus H_{n-1}(L') \arrow[d, "(a|_{S'})_* \oplus (a|_{L'})_*"] \\
H_n(T) \arrow[r] & H_n(S) \oplus H_n(L) \arrow[r] & H_n(K) \arrow[r] & H_{n-1}(T) \arrow[r] & H_{n-1}(S) \oplus H_{n-1}(L)
\end{tikzcd}$$

in which every square commutes.

Now S is a simplex, so the proposition holds for it, T has strictly smaller dimension than K, so the proposition holds for it, and L has either strictly smaller dimension than K, or else has fewer simplices of maximal dimension, so the proposition also holds for it. Thus all the vertical maps in the diagram above are known to be isomorphisms except the middle one. An easy diagram-chase (known as the Five Lemma[1]) shows that the middle map must then be an isomorphism too. □

We will now use the isomorphisms of this proposition to identify $H_n(K)$ with $H_n(K')$, and so by iteration to identify $H_n(K)$ with $H_n(K^{(r)})$. Note that by Lemmas 7.5.2 and 7.5.3 all simplicial approximations to the identity $a : K' \to K$ induce *the same* isomorphism $a_* : H_*(K') \xrightarrow{\sim} H_*(K)$. We call this isomorphism ν_K, and we write $\nu_{K,r}$ for the composition $\nu_{K^{(r-1)}} \circ \cdots \circ \nu_{K'} \circ \nu_K$ and, for $r \geq s$, $\nu_{K,r,s}$ for the composition $\nu_{K^{(r-1)}} \circ \cdots \circ \nu_{K^{(s+1)}} \circ \nu_{K^{(s)}}$.

Proposition 7.5.6. *To each continuous map $f : |K| \to |L|$ there is an associated homomorphism $f_* : H_*(K) \to H_*(L)$ given by $f_* = s_* \circ \nu_{K,r}^{-1}$, where $s : K^{(r)} \to L$ is a simplicial approximation to f, which exists for some r by the Simplicial Approximation Theorem.*

(i) This homomorphism does not depend on r or s.

(ii) If $g : |M| \to |K|$ is another continuous map, then $(f \circ g)_* = f_* \circ g_*$.

[1] See Example Sheet 4, Q3

Proof. For (i), let $s : K^{(r)} \to L$ and $t : K^{(q)} \to L$ be simplicial approximations to f, with $r \geq q$. Choose $a : K^{(r)} \to K^{(q)}$ a simplicial approximation to the identity. Then $s, t \circ a : K^{(r)} \to L$ are both simplicial approximations to f, so are contiguous by Lemma 7.5.2, and hence induce the same homomorphism on homology by Lemma 7.5.3. Thus $s_* = t_* \circ a_*$, but $a_* = \nu_{K,r,q}$, so $s_* \circ \nu_{K,r}^{-1} = t_* \circ \nu_{K,r,q} \circ \nu_{K,r}^{-1} = t_* \circ \nu_{K,q}^{-1}$.

For (ii), let $s : K^{(r)} \to L$ be a simplicial approximation to f, and $t : M^{(q)} \to K^{(r)}$ be a simplicial approximation to g. Then $s \circ t$ is a simplicial approximation to $f \circ g$, so

$$\begin{aligned}(f \circ g)_* &= (s \circ t)_* \circ \nu_{M,q}^{-1}\\ &= (s_* \circ \nu_{K,r}^{-1}) \circ (\nu_{K,r} \circ t_* \circ \nu_{M,q}^{-1}) = f_* \circ g_*.\end{aligned}$$ □

Corollary 7.5.7. *If $f : |K| \to |L|$ is a homeomorphism, then $f_* : H_n(K) \to H_n(L)$ is an isomorphism for every n.*

For some applications it is convenient to have a chain-level description of the map $\nu_K^{-1} : H_i(K) \to H_i(K')$, that is, a chain map $s_\bullet : C_\bullet(K) \to C_\bullet(K')$ which induces ν_K^{-1} on homology. Morally speaking it is clear what such a map should do: it should send a simplex σ of K to the sum of all the simplices of the barycentric subdivision σ'. However there are signs involved. We begin by letting $s_0([v_0]) = [\widehat{\langle v_0 \rangle}]$, and supposing that $s_0, s_1, \ldots, s_{n-1}$ have been defined, satisfying $d_i \circ s_i = s_{i-1} \circ d_i$ for $i < n$, then for an n-simplex we let

$$s_n(\sigma) = [\hat{\sigma}, s_{n-1} d_n(\sigma)].$$

The notation $[\hat{\sigma}, -]$ must be understood to be linear in $-$, and expanded out to obtain a linear combination of ordered simplices. So for example

$$s_1([v_0, v_1]) = [\widehat{\langle v_0, v_1 \rangle}, \widehat{\langle v_1 \rangle} - \widehat{\langle v_0 \rangle}] = [\widehat{\langle v_0, v_1 \rangle}, \widehat{\langle v_1 \rangle}] - [\widehat{\langle v_0, v_1 \rangle}, \widehat{\langle v_0 \rangle}].$$

Proposition 7.5.8. *The maps s_n define a chain map $s_\bullet$, and $s_* = \nu_K^{-1}$.*

Proof. We calculate

$$\begin{aligned}d_n(s_n([v_0, \ldots, v_n])) &= d_n([\widehat{\langle v_0, \ldots, v_n \rangle}, s_{n-1} d_n([v_0, \ldots, v_n])])\\ &= s_{n-1} d_n([v_0, \ldots, v_n]) - [\widehat{\langle v_0, \ldots, v_n \rangle}, d_{n-1} s_{n-1} d_n([v_0, \ldots, v_n])]\end{aligned}$$

and as $d_{n-1} \circ s_{n-1} = s_{n-2} \circ d_{n-1}$ and $d_{n-1} \circ d_n = 0$ the second term vanishes, so the s_n define a chain map.

Let $\prec$ be a total order of V_K, and let $a : K \cong V_{K'} \to V_K$ send each simplex to its smallest vertex (with respect to $\prec$). This defines a simplicial approximation to the identity $a : K' \to K$. Certainly $a_0 \circ s_0$ is the identity, so suppose that $a_i \circ s_i$ is the identity for $i < n$. If $[v_0, \ldots, v_n]$ is a simplex with $v_0 \prec \cdots \prec v_n$ then

$$\begin{aligned}a_n(s_n([v_0, \ldots, v_n])) &= a_n([\widehat{\langle v_0, \ldots, v_n \rangle}, s_{n-1} d_n([v_0, \ldots, v_n])])\\ &= [v_0, a_{n-1} s_{n-1} d_n([v_0, \ldots, v_n])]\end{aligned}$$

and as $a_{n-1} \circ s_{n-1}$ is the identity this is $[v_0, d_n([v_0, \ldots, v_n])]$. Expanding out $d_n([v_0, \ldots, v_n])$, the first term is $[v_1, \ldots, v_n]$ and the other all have a v_0 in them, so when v_0 is added they become zero in $C_n(K)$. Thus we obtain $[v_0, \ldots, v_n]$, as required. □

To obtain the strongest applications of simplicial homology theory, we wish to know that continuous maps between polyhedra which are homotopic induce equal homomorphisms on homology. We first show that maps of polyhedra which are sufficiently close induce equal homomorphisms in homology.

Lemma 7.5.9. *For a simplicial complex L in $\mathbb{R}^m$, there is a $\epsilon = \epsilon(L) > 0$ such that if $f, g : |K| \to |L|$ satisfy $|f(x) - g(x)| < \epsilon$ for all $x \in |K|$, then $f_* = g_* : H_n(K) \to H_n(L)$.*

Proof. The sets $\{\mathrm{St}_L(w)\}_{w \in V_L}$ give an open cover of $|L|$, which is compact, so by the Lesbegue Number Lemma (Lemma 1.1.3) there is a $\epsilon > 0$ such that each ball of radius 2ϵ in $|L|$ lies in some $\mathrm{St}_L(w)$.

Let $f, g : |K| \to |L|$ be as in the statement, using this ϵ. By the Lesbague Number Lemma applied to the open cover $\{f^{-1}(B_\epsilon(y))\}_{y \in |L|}$ of $|K|$, there is a $\delta > 0$ such that each $f(B_\delta(x))$ is contained in some $B_\epsilon(y)$, and hence also each $g(B_\delta(x))$ is contained in $B_{2\epsilon}(y)$.

Choose $r \gg 0$ so that $\mu(K^{(r)}) < \frac{1}{2}\delta$. Then for each $v \in V_{K^{(r)}}$ the diameter of $\mathrm{St}_{K^{(r)}}(v)$ is less than δ, so $f(\mathrm{St}_{K^{(r)}}(v))$ and $g(\mathrm{St}_{K^{(r)}}(v))$ both lie in some $\mathrm{St}_L(w)$. Setting $s(v) = w$, we obtain $s : V_{K^{(r)}} \to V_L$ which is a simplicial approximation to both f and g. Thus $f_* := s_* \circ \nu_{K,r}^{-1} =: g_*$. □

Theorem 7.5.10. *If $f \simeq g : |K| \to |L|$, then $f_* = g_*$.*

Proof. Let $H : |K| \times I \to |L|$ be a homotopy from f to g. As $|K| \times I$ is compact, H is uniformly continuous. Thus for $\epsilon = \epsilon(L)$ provided by the previous lemma, there is a $\delta > 0$ such that

$$|s - t| < \delta \Rightarrow |H(x,s) - H(x,t)| < \epsilon \quad \forall x \in |K|.$$

Thus choose $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that $t_i - t_{i-1} < \delta$ for all i, and let $f_i(x) = H(x, t_i)$. Then by the previous lemma $(f_{i-1})_* = (f_i)_*$ for all i, but $f_0 = f$ and $f_k = g$. □

This allows us to define and work with homology for a larger class of spaces than just polyhedra.

Definition 7.5.11. A **h-triangulation** of a space X is a simplicial complex K and a homotopy equivalence $g : |K| \to X$. We define the **homology** of this h-triangulated space to be $H_n(X) := H_n(K)$.

Note that in particular a triangulation is an h-triangulation.

Lemma 7.5.12. *The homology of an h-triangulated space does not depend on the choice of h-triangulation.*

Proof. Let $\bar{g} : |\bar{K}| \to X$ be another h-triangulation, $f : X \to |K|$ be a homotopy inverse to g, and $\bar{f} : X \to |\bar{K}|$ be a homotopy inverse to $\bar{g}$. Then $f \circ \bar{g} : |\bar{K}| \to |K|$ is a homotopy equivalence with homotopy inverse $\bar{f} \circ g$. Thus

$$(f \circ \bar{g})_* \circ (\bar{f} \circ g)_* = \mathrm{Id}_{H_n(K)}$$

and vice-versa, so $(f \circ \bar{g})_* : H_n(\bar{K}) \to H_n(K)$ is an isomorphism. □

If $g : |K| \to X$ and $\bar{g} : |\bar{K}| \to \bar{X}$ are h-triangulated spaces, and $h : X \to \bar{X}$ is a continuous map, then we obtain a continuous map

$$|K| \xrightarrow{\sim} X \xrightarrow{h} \bar{X} \xrightarrow{\sim} |\bar{K}|$$

and hence a homomorphism $H_n(K) \to H_n(\bar{K})$. Identifying these groups with $H_n(X)$ and $H_n(\bar{X})$ respectively, we obtain a homomorphism

$$h_* : H_n(X) \to H_n(\bar{X}).$$

One can check that this is independent of choice of h-triangulations of either space, and that if $h \simeq h' : X \to \bar{X}$ then $h_* = h'_*$.

7.6 Homology of spheres and applications

As a first application, we can use homology to upgrade our proof of Brouwer's fixed point theorem (Corollary 3.2.3) to all dimensions.

Lemma 7.6.1. *The sphere S^{n-1} is triangulable, and for $n \geq 2$ we have*

$$H_i(S^{n-1}) \cong \begin{cases} \mathbb{Z} & i = 0, n-1 \\ 0 & \textit{else.} \end{cases}$$

Proof. If $\Delta^n \subset \mathbb{R}^{n+1}$ is the standard n-simplex, then its boundary $\partial\Delta^n$ is homeomorphic to S^{n-1}. This boundary is the polyhedron of the simplicial complex of Corollary 7.3.5, where its homology was computed. ☐

Theorem 7.6.2 (Brouwer)**.**

(i) There is no retraction of D^n to S^{n-1},

(ii) Any continuous map $f : D^n \to D^n$ has a fixed point.

Proof. The argument we gave in Corollary 3.2.3 shows that (i) implies (ii). To prove (i), suppose that $n \geq 2$, let $r : D^n \to S^{n-1}$ be a proposed retraction, and $i : S^{n-1} \to D^n$ be the inclusion, and consider

$$\mathbb{Z} \cong H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

This is the identity map, as $r \circ i = \mathrm{Id}_{S^{n-1}}$. But D^n is contractible, so is homotopy equivalent to the polyhedron of the simplicial complex consisting of a single vertex. As $n - 1 > 0$, it follows that $H_{n-1}(D^n) = 0$, which is a contradiction. ☐

Recall that in Example 6.1.9 we gave a different triangulation of S^n, using the simplicial complex K having simplices

$$\langle \pm e_1, \pm e_2, \ldots, \pm e_{n+1} \rangle$$

in $\mathbb{R}^{n+1}$, and all their faces. By Lemma 7.5.12 we must also have

$$H_i(K) \cong \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else,} \end{cases}$$

as different h-triangulations of S^n have the same homology.

Lemma 7.6.3. *The element*

$$x := \sum_{\underline{a} \in \{-1,1\}^{n+1}} a_1 a_2 \cdots a_{n+1} [a_1 e_1, \ldots, a_{n+1} e_{n+1}] \in C_n(K)$$

is a cycle, and generates $H_n(K) \cong \mathbb{Z}$.

Proof. When we apply the differential d to x, the simplex $[a_1 e_1, \ldots, \widehat{a_i e_i}, \ldots a_{n+1} e_{n+1}]$ appears twice, with $a_i = 1$ and $a_i = -1$, and these have opposite coefficients so cancel out. Also, this element is not divisible by any integer, so generates $\mathbb{Z} \cong H_n(K)$. □

The reflection $r_i : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ in the ith coordinate defines a simplicial map from K to itself, and this satisfies

$$\begin{aligned} r_i(x) &= \sum_{\underline{a} \in \{-1,1\}^{n+1}} a_1 a_2 \cdots a_{n+1} [a_1 e_1, \ldots, a_{i-1} e_{i-1}, -a_i e_i, a_{i+1} e_{i+1}, a_{n+1} e_{n+1}] \\ &= - \sum_{\underline{a} \in \{-1,1\}^{n+1}} a_1 a_2 \cdots a_{n+1} [a_1 e_1, \ldots, a_{n+1} e_{n+1}] = -x \end{aligned}$$

so $(r_i)_* : H_n(S^n) \to H_n(S^n)$ is multiplication by -1. The antipodal map $a : S^n \to S^n$ may be written as $r_1 r_2 \cdots r_{n+1}$, so $a_* : H_n(S^n) \to H_n(S^n)$ is multiplication by $(-1)^{n+1}$.

Corollary 7.6.4. *If n is even then the antipodal map $a : S^n \to S^n$ is not homotopic to the identity*[2].

[2]See Example Sheet 1, Q1

7.7 Homology of surfaces, and their classification

Definition 7.7.1. A **surface** is a Hausdorff topological space in which each point has an open neighbourhood homeomorphic to an open subset of $\mathbb{R}^2$. A **triangulated surface** is a surface X equipped with a triangulation $h : |K| \to X$.

Example 7.7.2 (Orientable surfaces)**.** Let K be the simplicial complex shown in Figure 7.1, which is a triangulation of the torus with an open disc removed, X.

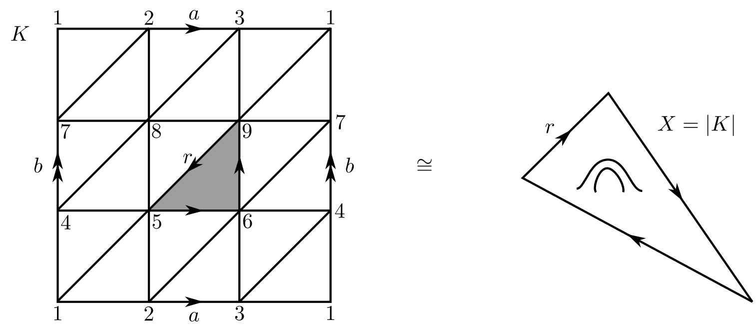
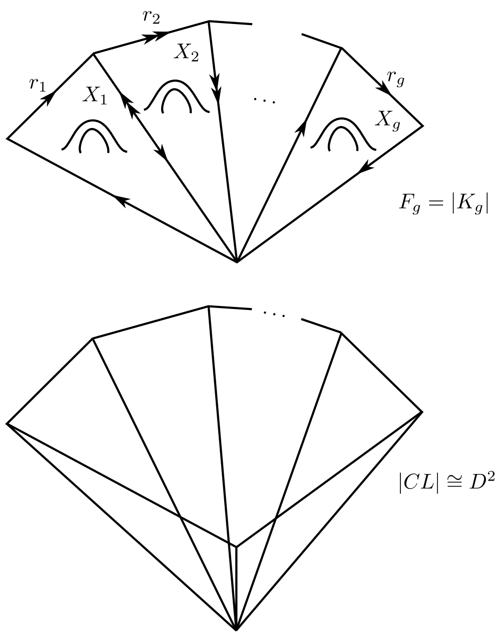
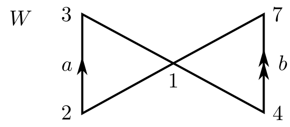


Figure 7.1 A triangulation of X, the torus with an open disc removed.

Its boundary is identified with the boundary of a 2-simplex. We can glue g copies of K together along an edge to obtain a simplicial complex K_g which triangulates the surface F_g from Example 5.4.1, with boundary given by a sub-simplicial complex L. We can then glue on the cone CL on L to obtain a triangulation of the surface Σ_g from Example 5.4.1.



Homology of K. The subcomplex $W \subset K$ given by



is such that the inclusion $|W| \hookrightarrow |K|$ is a homotopy equivalence, and hence the inclusion induces an isomorphism on homology. By an application of Mayer–Vietoris, decomposing W into two circles glued together at 1, we obtain $H_0(K) = \mathbb{Z}\{[1]\}$ and $H_1(K) = \mathbb{Z}\{a, b\}$ where

$$a = [1,2] + [2,3] + [3,1] \qquad\qquad b = [1,4] + [4,7] + [7,1],$$

and $H_2(K) = 0$. Furthermore, the boundary cycle $r = [6,5]+[9,6]+[5,9]$ of K represents the same homology class as $a + b - a - b$, so the zero homology class.

Homology of K_g. We can decompose $K_g = K_{g-1} \cup K$ where $K_{g-1} \cap K$ is a 1-simplex, Δ^1. The Mayer–Vietoris sequence is then

$$\begin{array}{l} 0 \longrightarrow H_2(K_{g-1}) \longrightarrow H_2(K_g) \\ \xrightarrow{\partial_*} H_1(\Delta^1) \xrightarrow{i_*+j_*} H_1(K) \oplus H_1(K_{g-1}) \xrightarrow{k_*-l_*} H_1(K_g) \\ \xrightarrow{\partial_*} H_0(\Delta^1) \xrightarrow{i_*+j_*} H_0(K) \oplus H_0(K_{g-1}) \xrightarrow{k_*-l_*} H_0(K_g) \longrightarrow 0 \end{array}$$

and Δ^1 is contractible so $H_1(\Delta^1) = 0$ and $H_0(\Delta^1) = \mathbb{Z}$ generated by any vertex. Thus we have

$$\begin{array}{l} 0 \longrightarrow 0 \longrightarrow H_2(K_g) \\ \xrightarrow{\partial_*} 0 \xrightarrow{i_*+j_*} \mathbb{Z}^2 \oplus H_1(K_{g-1}) \xrightarrow{k_*-l_*} H_1(K_g) \\ \xrightarrow{0} \mathbb{Z} \xrightarrow{i_*+j_*} \mathbb{Z} \oplus H_0(K_{g-1}) \xrightarrow{k_*-l_*} H_0(K_g) \longrightarrow 0 \end{array}$$

and so by induction

$$\begin{aligned} H_0(K_g) &= \mathbb{Z} \text{ generated by any vertex} \\ H_1(K_g) &= \mathbb{Z}^{2g} \text{ generated by } a_1, b_1, \ldots, a_g, b_g \\ H_2(K_g) &= 0. \end{aligned}$$

The boundary cycle $r_1 + \cdots + r_g$ represents the same homology class as $a_1 + b_1 - a_1 - b_1 + \cdots + a_g + b_g - a_g - b_g$, so is zero.

Homology of $K_g \cup CL$. The Mayer–Vietoris sequence for this decomposition is

$$
\begin{tikzcd}
 & 0 \arrow[r] & H_2(K_g \cup CL) \arrow[dll, "\partial_*"] \\
H_1(L) \arrow[r, "i_*+j_*"] & H_1(K_g) \oplus H_1(CL) \arrow[r, "k_*-l_*"] & H_1(K_g \cup CL) \arrow[dll, "\partial_*"] \\
H_0(L) \arrow[r, "i_*+j_*"] & H_0(K_g) \oplus H_0(CL) \arrow[r, "k_*-l_*"] & H_0(K_g \cup CL) \arrow[r] & 0
\end{tikzcd}
$$

The simplicial complex L is a triangulation of S^1, and CL is contractible, so we obtain

$$
\begin{tikzcd}
 & 0 \arrow[r] & H_2(K_g \cup CL) \arrow[dll, "\partial_*"] \\
\mathbb{Z} \arrow[r, "i_*"] & \mathbb{Z}^{2g} \oplus 0 \arrow[r, "k_*-l_*"] & H_1(K_g \cup CL) \arrow[dll, "0"] \\
\mathbb{Z} \arrow[r, "{(1,1)}"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "k_*-l_*"] & H_0(K_g \cup CL) \arrow[r] & 0
\end{tikzcd}
$$

and the map $i_* : H_1(L) \to H_1(K_g)$ sends the generator to the boundary cycle $r_1 + \cdots + r_g$, which is the zero homology class, so this map is zero. We deduce that

$$
\begin{aligned}
&H_0(K_g \cup CL) = \mathbb{Z} \text{ generated by any vertex} \\
&H_1(K_g \cup CL) = \mathbb{Z}^{2g} \text{ generated by } a_1, b_1, \ldots, a_g, b_g \\
&H_2(K_g \cup CL) = \mathbb{Z}.
\end{aligned}
$$

The group $H_2(K_g \cup CL)$ is generated by any 2-cycle x such that $\partial_*(x) \in H_1(L) \cong \mathbb{Z}$ is a generator.

Example 7.7.3 (Nonorientable surfaces)**.** Let K be the simplicial complex shown in Figure 7.2, which is a triangulation of the projective plane with an open disc removed, Y.

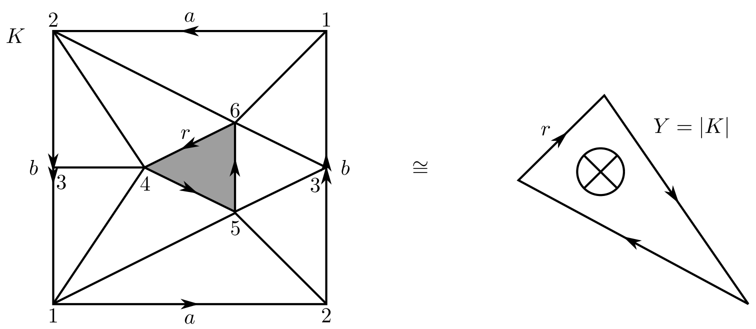
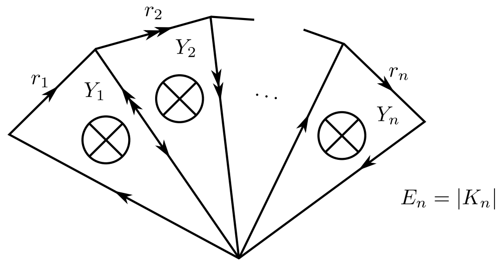
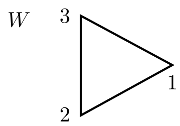


Figure 7.2 A triangulation of Y, the projective plane with an open disc removed.

Its boundary is identified with the boundary of a 2-simplex. We can glue n copies of K together along an edge to obtain a simplicial complex K_n which triangulates the surface E_n from Example 5.4.3, with boundary given by a sub-simplicial complex L. We can then glue on the cone CL on L to obtain a triangulation of the surface S_n from Example 5.4.3.



Homology of K. The subcomplex $W \subset K$ given by



is such that the inclusion $|W| \hookrightarrow |K|$ is a homotopy equivalence, and hence the inclusion induces an isomorphism on homology. We obtain $H_0(K) = \mathbb{Z}\{[1]\}$ and $H_1(K) = \mathbb{Z}\{u\}$ where

$$u = a + b = [1,2] + [2,3] + [3,1],$$

and $H_2(K) = 0$. Furthermore, the boundary cycle $r = [4,5]+[5,6]+[6,4]$ of K represents the same homology class as $2u$.

Homology of K_n. We can decompose $K_n = K_{n-1} \cup K$ where $K_{n-1} \cap K$ is a 1-simplex, Δ^1. The Mayer–Vietoris sequence is then

$$0 \longrightarrow 0 \oplus H_2(K_{n-1}) \longrightarrow H_2(K_n) \xrightarrow{\partial_*}$$
$$H_1(\Delta^1) \xrightarrow{i_* + j_*} H_1(K) \oplus H_1(K_{n-1}) \xrightarrow{k_* - l_*} H_1(K_n) \xrightarrow{\partial_*}$$
$$H_0(\Delta^1) \xrightarrow{i_* + j_*} H_0(K) \oplus H_0(K_{n-1}) \xrightarrow{k_* - l_*} H_0(K_n) \longrightarrow 0$$

and Δ^1 is contractible so $H_1(\Delta^1) = 0$ and $H_0(\Delta^1) = \mathbb{Z}$ generated by any vertex. Thus by induction

$$\begin{aligned} H_0(K_n) &= \mathbb{Z} \text{ generated by any vertex} \\ H_1(K_n) &= \mathbb{Z}^n \text{ generated by } u_1, \dots, u_n \\ H_2(K_n) &= 0. \end{aligned}$$

The boundary cycle $r_1 + \cdots + r_n$ represents the same homology class as $2u_1 + 2u_2 + \cdots + 2u_n$.

Homology of $K_n \cup CL$. The Mayer–Vietoris sequence for this decomposition is

$$\begin{tikzcd}
& 0 \oplus H_2(CL) \arrow[r] & H_2(K_n \cup CL) \arrow[dll, "\partial_*"] & \\
H_1(L) \arrow[r, "i_*+j_*"] & H_1(K_n) \oplus H_1(CL) \arrow[r, "k_*-l_*"] & H_1(K_n \cup CL) \arrow[dll, "\partial_*"] & \\
H_0(L) \arrow[r, "i_*+j_*"] & H_0(K_n) \oplus H_0(CL) \arrow[r, "k_*-l_*"] & H_0(K_n \cup CL) \arrow[r] & 0
\end{tikzcd}$$

The simplicial complex L is a triangulation of S^1, and CL is contractible, so we obtain

$$\begin{tikzcd}
& 0 \arrow[r] & H_2(K_n \cup CL) \arrow[dll, "\partial_*"] & \\
\mathbb{Z} \arrow[r, "i_*"] & \mathbb{Z}^n \oplus 0 \arrow[r, "k_*-l_*"] & H_1(K_n \cup CL) \arrow[dll, "0"] & \\
\mathbb{Z} \arrow[r, "(1,1)"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "k_*-l_*"] & H_0(K_n \cup CL) \arrow[r] & 0
\end{tikzcd}$$

and the map $i_* : H_1(L) \to H_1(K_g)$ sends the generator to the boundary cycle $r_1+\cdots+r_n$, which is the $2u_1 + 2u_2 + \cdots + 2u_n$. Thus this map is $1 \mapsto (2, 2, \ldots, 2)$, which is injective. We deduce that

$H_0(K_n \cup CL) = \mathbb{Z}$ generated by any vertex
$H_1(K_n \cup CL) = \mathbb{Z}^n/\mathbb{Z}\{(2, 2, \ldots, 2)\}$ generated by $u_1, \ldots, u_n$ subject to $2u_1 + \cdots + 2u_n = 0$
$H_2(K_n \cup CL) = 0$.

Note that the change of basis

$$e_1 = u_1 + u_2 + \cdots + u_n \qquad e_i = u_i \;\; 2 \le i \le n$$

gives an isomorphism of abelian groups $\mathbb{Z}^n/\mathbb{Z}\{(2, 2, \ldots, 2)\} \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{n-1}$.

To summarise these two examples, as well as Lemma 7.6.1 in dimension 2, the (triangulable) surfaces S^2, Σ_g for $g \ge 1$, and S_n for $n \ge 1$, all have non-isomorphic first homology groups:

$$H_1(S^2) = 0 \qquad H_1(\Sigma_g) \cong \mathbb{Z}^{2g} \qquad H_1(S_n) \cong \mathbb{Z}/2 \oplus \mathbb{Z}^{n-1}.$$

Thus no two of these surfaces are homeomorphic, or even homotopy equivalent. This proves one half of the following theorem. The other half is much more elementary, but quite long: it can be found for example in Maunder's *Algebraic Topology.*

Theorem 7.7.4 (Classification of triangulable surfaces)**.** *Any triangulable surface is homeomorphic to one of*

$$\begin{gathered} S^2, \Sigma_1, \Sigma_2, \Sigma_3, \ldots \\ S_1, S_2, S_3, \ldots \end{gathered}$$

and no two of these are homeomorphic.

We have been using the terms "orientable surface" for one obtained by gluing together copies of the torus, and "nonorientable surface" for one obtained by gluing together copies of the projective plane. These are in fact intrinsic properties of surfaces, and not just a feature of how we choose to construct them: this may be seen homologically by

$$H_2(S^2) = H_2(\Sigma_g) = \mathbb{Z} \quad \text{but} \quad H_2(S_n) = 0.$$

This difference accounts for our partition in Theorem 7.7.4 of surfaces into two series, the orientable ones $(S^2, \Sigma_1, \Sigma_2, \Sigma_3, \ldots)$ having second homology $\mathbb{Z}$, and the nonorientable ones $(S_1, S_2, S_3, \ldots)$ having second homology 0.

7.8 Rational homology, Euler and Lefschetz numbers

Definition 7.8.1. For a simplicial complex K, define the **rational n-chains**, $C_n(K;\mathbb{Q})$, by letting $O_n(K;\mathbb{Q})$ be the $\mathbb{Q}$-vector space with basis the oriented simplices, and $T_n(K;\mathbb{Q})$ be the sub vector space spanned by the usual collection of "trivial" simplices. We define $d_n : C_n(K;\mathbb{Q}) \to C_{n-1}(K;\mathbb{Q})$ by the usual formula, and write

$$H_n(K;\mathbb{Q}) := \frac{\operatorname{Ker}(d_n : C_n(K;\mathbb{Q}) \to C_{n-1}(K;\mathbb{Q}))}{\operatorname{Im}(d_{n+1} : C_{n+1}(K;\mathbb{Q}) \to C_n(K;\mathbb{Q}))}$$

for the nth **rational homology** of K. It is a $\mathbb{Q}$-vector space.

Everything we did for ordinary homology remains true for rational homology (induced maps, subdivision isomorphism, homotopy invariance, Mayer–Vietoris sequence). The following lemma gives some idea of what rational homology measures. Its proof is on Example Sheet 4.

Lemma 7.8.2. *If $H_n(K) \cong \mathbb{Z}^r \oplus F$ for F a finite abelian group, then $H_n(K;\mathbb{Q}) \cong \mathbb{Q}^r$.*

We thus know the homology of several spaces. For spheres, with $n > 0$,

$$H_i(S^n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0, n \\ 0 & \text{else.} \end{cases}$$

For orientable surfaces

$$H_i(\Sigma_g;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0, 2 \\ \mathbb{Q}^{2g} & i = 1 \\ 0 & \text{else.} \end{cases}$$

For nonorientable surfaces

$$H_i(S_n;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0 \\ \mathbb{Q}^{n-1} & i = 1 \\ 0 & \text{else.} \end{cases}$$

Definition 7.8.3. Let X be a polyhedron and $f : X \to X$ be a continuous map. The **Lefschetz number** of f is

$$L(f) := \sum_{i\geq 0}(-1)^i \mathrm{Tr}\left(f_* : H_i(X;\mathbb{Q}) \to H_i(X;\mathbb{Q})\right).$$

The **Euler characteristic** of X is

$$\chi(X) = L(\mathrm{Id}_X) = \sum_{i\geq 0}(-1)^i \dim_{\mathbb{Q}} H_i(X;\mathbb{Q}).$$

Example 7.8.4. Computing the Euler characteristic for the examples we have gives

$$\chi(S^n) = \begin{cases} 2 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad \chi(\Sigma_g) = 2 - 2g \qquad \chi(S_n) = 2 - n.$$

Example 7.8.5. Recall that the antipodal map $a : S^n \to S^n$ has $a_* : H_n(S^n) \to H_n(S^n)$ given by multiplication by $(-1)^{n+1}$. The same is true in rational homology, and so

$$L(a) = 1 + (-1)^n \cdot (-1)^{n+1} = 0.$$

Lemma 7.8.6. *Let V be a finite-dimensional vector space and $W \leq V$ be a subspace, Let $A : V \to V$ be a linear map such that $A(W) \subset W$. Let $B = A|_W : W \to W$ be the restriction and $C : V/W \to V/W$ be the induced map. Then*

$$\mathrm{Tr}(A) = \mathrm{Tr}(B) + \mathrm{Tr}(C).$$

Proof. Let $e_1, e_2, \ldots, e_r$ be a basis for W, and extend it to a basis $e_1, e_2, \ldots, e_n$ of V. The matrix of A with respect to this basis is of the form

$$\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$$

where X is the matrix for B with respect to the basis $e_1, e_2, \ldots, e_r$ and Z is the matrix for C with respect to the basis $[e_{r+1}], \ldots, [e_n]$. Now $\mathrm{Tr}\left(\begin{smallmatrix} X & Y \\ 0 & Z \end{smallmatrix}\right) = \mathrm{Tr}(X) + \mathrm{Tr}(Z)$, as required. ☐

Corollary 7.8.7. *For a chain map $f_\bullet : C_\bullet(K;\mathbb{Q}) \to C_\bullet(K;\mathbb{Q})$, we have*

$$\sum_{i\geq 0}(-1)^i \mathrm{Tr}(f_* : H_i(K;\mathbb{Q}) \to H_i(K;\mathbb{Q})) = \sum_{i\geq 0}(-1)^i \mathrm{Tr}(f_i : C_i(K;\mathbb{Q}) \to C_i(K;\mathbb{Q})).$$

Proof. There are exact sequence

$$0 \longrightarrow B_i(K;\mathbb{Q}) \longrightarrow Z_i(L;\mathbb{Q}) \longrightarrow H_i(K;\mathbb{Q}) \longrightarrow 0$$

and

$$0 \longrightarrow Z_i(K;\mathbb{Q}) \longrightarrow C_i(L;\mathbb{Q}) \longrightarrow B_{i-1}(K;\mathbb{Q}) \longrightarrow 0.$$

Let f_i^H, f_i^B, f_i^Z, and f_i^C be the maps induced by $f_\bullet$ on homology, boundaries, cycles, and chains respectively. Then

$$\begin{aligned} L(|f|) &= \sum_{i\geq 0}(-1)^i \mathrm{Tr}(f_i^H) \\ &= \sum_{i\geq 0}(-1)^i \left(\mathrm{Tr}(f_i^Z) - \mathrm{Tr}(f_i^B)\right) \\ &= \sum_{i\geq 0}(-1)^i \left(\mathrm{Tr}(f_i^C) - \mathrm{Tr}(f_{i-1}^B) - \mathrm{Tr}(f_i^B)\right) \\ &= \sum_{i\geq 0}(-1)^i \mathrm{Tr}(f_i^C). \end{aligned}$$

□

Corollary 7.8.8. *We have $\sum_{i\geq 0}(-1)^i \cdot \#i$-simplices of $K = \chi(|K|)$, and in particular this number does not depend on the triangulation of the polyhedron $|K|$.*

Theorem 7.8.9 (Lefschetz)**.** *Let $f : X \to X$ be a continuous map from a polyhedron to itself. If $L(f) \neq 0$ then f has a fixed point.*

Proof. We will show the contrapositive to the theorem: if f has no fixed points then $L(f) = 0$. Supposing that f has no fixed points, let

$$\delta := \inf\{|x - f(x)| \ : x \in X\}.$$

As X is compact, $\delta > 0$. Let K be a triangulation of X with $\mu(K) < \delta/2$, and choose $g : K^{(r)} \to K$ a simplicial approximation to f. For $v \in K^{(r)}$ we have $f(v) \in f(\mathrm{St}_{K^{(r)}}(v)) \subset \mathrm{St}_K(g(v))$, and so $|f(v) - g(v)| < \delta/2$. But $|f(v) - v| \geq \delta$, and so $|g(v) - v| \geq \delta/2$. Thus if $v \in \sigma \in K$ then $g(v) \notin \sigma$.

The map f_* is defined to be $g_* \circ (\nu_{K,r})_*^{-1}$. Using the subdivision map $s_\bullet$ from Proposition 7.5.8, iterated, the map f_* is induced by the chain map $g_\bullet \circ s_\bullet^{(r)} : C_\bullet(K;\mathbb{Q}) \to C_\bullet(K;\mathbb{Q})$. Thus by Lemma 7.8.7 we have

$$L(f) = \sum_{i\geq 0}(-1)^i \mathrm{Tr}(g_i \circ s_i^{(r)} : C_i(K;\mathbb{Q}) \to C_i(K;\mathbb{Q})).$$

If $\sigma \in K$ is an i-simplex then $s_i^{(r)}(\sigma)$ is a sum of simplices inside σ, and so $g_i(s_i^{(r)}(\sigma))$ is a sum of simplices disjoint from σ. Thus the matrix for $g_i \circ s_i^{(r)}$ with respect to the basis of simplices has zeroes on the diagonal, so has trace zero. □

Example 7.8.10. If X is a contractible polyhedron then $L(f) = 1$ for any $f : X \to X$, so any f has a fixed point. This recovers Brouwer's fixed point theorem, but is far more general.

Example 7.8.11. let $h : |K| \to G$ be a triangulation of a topological group G which is *connected* and *nontrivial.* The for $g \neq 1 \in G$ multiplication by g is a homeomorphism with no fixed points, so $0 = L(- \cdot g)$. On the other hand, choosing a path from g to $1 \in G$ gives a homotopy from $- \cdot g$ to the identity, so

$$0 = L(- \cdot g) = L(- \cdot 1) = L(\mathrm{Id}_G) = \chi(G).$$

Among the surfaces, only Σ_1 and S_2 have Euler characteristic zero, so no other surface admits a topological group structure. The torus $\Sigma_1 = S^1 \times S^1$ does admit a group structure, as $\mathbb{R}^2/\mathbb{Z}^2$. The surface S_2 is the Klein bottle, and does *not* admit a group structure. One way to see this is to use Q13 on Example Sheet 1, where the fundamental group of the Klein bottle is computed and shown to be non-abelian: it is an easy exercise to show that the fundamental group of a topological group (based at the identity element) is always abelian.

Example 7.8.12. Let $f : S_3 \to S_3$ be such that $f \circ f = \mathrm{Id}_{S_3}$. We have

$$H_i(S_3; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0 \\ \mathbb{Q}^2 & i = 1 \\ 0 & \text{else,} \end{cases}$$

and $f_* : H_0(S_3; \mathbb{Q}) \to H_0(S_3; \mathbb{Q})$ is the identity map, as this group is generated by any vertex of a triangulation of S_3. The linear map

$$f_* : \mathbb{Q}^2 \cong H_1(S_3; \mathbb{Q}) \longrightarrow H_1(S_3; \mathbb{Q}) \cong \mathbb{Q}^2$$

satisfies $(f_*)^2 = \mathrm{Id}$, so by elementary linear algebra its eigenvalues are all ± 1. Thus its trace is one of -2, 0, or 2, and so

$$L(f) = 1 - \mathrm{Tr}(f_* : H_1(S_3; \mathbb{Q}) \longrightarrow H_1(S_3; \mathbb{Q})) \in \{-1, 1, 3\}.$$

Hence f must have a fixed point.