III. NONLINEAR, FIRST-ORDER DIFFERENTIAL EQUATIONS

Having studied linear, first-order differential equations, we now consider *nonlinear* first-order equations. In this case, the dependent variable (e.g., $y(x)$) appears nonlinearly. Such equations have a rich phenomenology.

In general, a first-order differential equation takes the form

$$Q(x, y)\frac{dy}{dx} + P(x, y) = 0\,, \tag{1}$$

for general, non-trivial functions P and Q. (This is not the most general form, since dy/dx could also appear nonlinearly, but we will not consider such cases here.)

1 Separable equations

Definition (Separable equation). A first-order differential equation is *separable* if it can be written in the form

$$q(y)dy = p(x)dx\,,$$

and so all the terms involving y explicitly can be collected to one side of the equation, and all the terms involving x explicitly can be collected to the other.

Separable equations can be solved directly by integration:

$$\int q(y)\,dy = \int p(x)\,dx\,.$$

Example. Consider

$$\left(x^2y - 3y\right)\frac{dy}{dx} - 2xy^2 = 4x\,.$$

Rearranging,

$$\frac{dy}{dx} = \frac{4x + 2xy^2}{x^2y - 3y} = \left(\frac{2x}{x^2 - 3}\right)\left(\frac{2 + y^2}{y}\right)\,.$$

Therefore

$$
\begin{aligned}
& \frac{y}{2+y^2}dy = \frac{2x}{x^2-3}dx\,, \\
\Rightarrow \quad & \frac{1}{2}\ln\left(y^2+2\right) = \ln\left(x^2-3\right) + C\,, \\
\Rightarrow \quad & \left(y^2+2\right)^{1/2} = A\left(x^2-3\right)\,,
\end{aligned}
$$

where C is an arbitrary integration constant and $A = e^C$.

2 Exact equations

Definition (Exact equation). Equation (1) is an *exact equation* if and only if the differential $P(x,y)dx + Q(x,y)dy$ is *exact*, i.e., there exists a function $f(x,y)$ such that

$$df = P(x,y)dx + Q(x,y)dy\,.$$

It follows that if Eq. (1) is exact, $df = 0$ and so $f(x,y) =$ const. is the solution. This is generally an implicit relation between x and y, which satisfies the differential equation.

If $P(x,y)dx + Q(x,y)dy$ is an exact differential of f, then $df = P(x,y)dx + Q(x,y)dy$. However, from the chain rule in differential form,

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\,,$$

so we must have

$$\frac{\partial f}{\partial x} = P \quad \text{and} \quad \frac{\partial f}{\partial y} = Q\,. \tag{2}$$

Solving these equations determines the function $f(x,y)$ (see the example below).

It follows from Eq. (2) that

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial P}{\partial y} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}\,.$$

Since mixed second partial derivatives commute, we must have

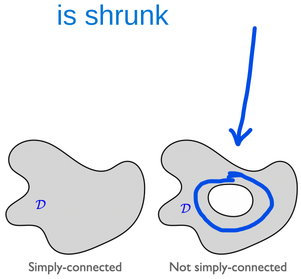
$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{3}$$

if $P(x, y)dx + Q(x, y)dy$ is exact.

The converse is not necessarily true: it is possible for Eq. (3) to hold but $Pdx + Qdy$ not to be exact. Equation (3) is therefore a *necessary but not sufficient condition* for the differential to be exact. However, if it holds throughout some *simply-connected* domain, then it can be shown that the differential is exact in that domain.

Definition (Simply-connected domain). A domain $\mathcal{D}$ is simply-connected if it is path-connected (i.e., every pair of points can be connected by a path in $\mathcal{D}$) and any closed curve can be continuously shrunk to a point in $\mathcal{D}$ without leaving $\mathcal{D}$.

This curve will leave D as it is shrunk



Examples. In 2D, a disk is simply-connected, but a disk with a hole in the middle is not (see the figure to the right for more general examples). The 2D surface of a sphere in 3D is simply-connected, but that of a torus (e.g., a ring doughnut) is not.

Theorem. If

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

throughout a simply connected domain $\mathcal{D}$, then $Pdx + Qdy$ is an exact differential of a single-valued function $f(x, y)$ in $\mathcal{D}$, i.e., there exists a single-valued function $f(x, y)$ in $\mathcal{D}$ such that $df = Pdx + Qdy$.

Aside: an inexact differential on a non-simply-connected domain

Consider the differential $Pdx + Qdy$ with

$$P = -\frac{y}{x^2 + y^2} \quad \text{and} \quad Q = \frac{x}{x^2 + y^2}\,.$$

We have

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2} = \frac{\partial Q}{\partial x}\,,$$

so that Eq. (3) is satisifed away from the origin, $(x, y) = (0, 0)$. It follows that the differential is exact in a simply-connected region excluding the origin. Indeed, a suitable potential is $\theta(x, y)$, where θ is the polar angle of plane-polar coordinates, with

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta\,.$$

This follows since $\tan\theta = y/x$ gives, for example,

$$\frac{1}{\cos^2\theta}\frac{\partial\theta}{\partial x} = -\frac{y}{x^2} \quad \Rightarrow \quad \frac{\partial\theta}{\partial x} = -\frac{y}{x^2+y^2} = P.$$

However, the differential is *not* exact in the non-simply-connected region $0 < x^2 + y^2 \le 1$ (which excludes the origin) since it cannot be written as the differential of a *single-valued* function throughout this domain. Rather, the potential θ changes by 2π in traversing any closed path that encircles the origin once.

Example. Consider

$$\begin{aligned} 6y(y-x)\frac{dy}{dx} + 2x - 3y^2 &= 0 \qquad (4) \\ \Rightarrow \quad \left(2x - 3y^2\right)dx + 6y(y-x)dy &= 0\,, \end{aligned}$$

so that

$$P(x, y) = 2x - 3y^2 \quad \text{and} \quad Q(x, y) = 6y^2 - 6xy\,.$$

It follows that

$$\frac{\partial P}{\partial y} = -6y = \frac{\partial Q}{\partial x}\,,$$

and so the differential $Pdx + Qdy$ is exact in any simply-connected domain.

Furthermore, the solution $f(x, y) =$ const. must satisfy the two equations

$$\frac{\partial f}{\partial x} = 2x - 3y^2 = P\,; \quad \frac{\partial f}{\partial y} = 6y^2 - 6xy = Q\,. \qquad (5)$$

If we integrate the first equation with respect to x, remembering that y is being held constant in the partial derivative $\partial f/\partial x$, we have

$$f(x, y) = x^2 - 3xy^2 + h(y)\,,$$

for some *function* $h(y)$. In general, this term is a function of y. If we take a partial derivative with respect to x keeping y constant, $h(y)$ will make no contribution.

Taking the partial derivative of this $f(x, y)$ with respect to y and comparing to the second equation in (5), we have

$$\begin{aligned} -6xy + \frac{dh}{dy} &= 6y^2 - 6xy \\ \Rightarrow \quad \frac{dh}{dy} &= 6y^2 \\ \Rightarrow \quad h(y) &= 2y^3 + C\,, \end{aligned}$$

for some constant C.

Therefore the solution to Eq. (4) is

$$f(x, y) = x^2 - 3xy^2 + 2y^3 = \text{const.}$$

This can, of course, be verified by direct substitution.

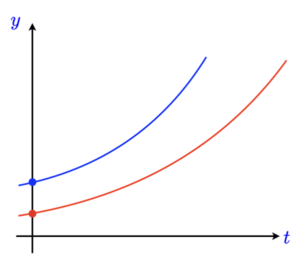
3 Solution curves and isoclines

It is not always possible to solve nonlinear equations explicitly, but we can gain insight into the "flow" of solutions using various graphical methods. We shall introduce some of these methods in this section.

3.1 Solution curves

Consider a first-order differential equation of the form

$$\frac{dy}{dt} = f(t, y)\,.$$



Each initial condition, e.g., specifying $y(0) = y_0$ at $t = 0$, will generate a distinct *solution curve* (or trajectory); see the figure to the right.

Example. Consider the nonlinear equation

$$\frac{dy}{dt} = t(1 - y^2)\,. \tag{6}$$

This equation is separable:

$$\frac{dy}{1-y^2} = t dt \,,$$

and can be integrated (using partial fractions is helpful) to obtain

$$\frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| = \frac{1}{2}t^2 + C \,,$$

where C is a constant. Therefore

$$y = \frac{A - e^{-t^2}}{A + e^{-t^2}} \,, \tag{7}$$

for some further constant A (with $A = e^{2C}$ for $|y| < 1$, and $A = -e^{2C}$ for $|y| > 1$). This *general solution* of the differential equation produces a family of solution curves, parameterised by A.

We can express the parameter A in terms of, say, the value $y(0) = y_0$ using

$$y(0) = \frac{A-1}{A+1} \quad \Rightarrow \quad A = \frac{1+y_0}{1-y_0} \,.$$

The solution curves given by Eq. (7) are plotted in Fig. 1.

In this example, we could solve the differential equation exactly. However, let us now consider whether we can understand the key properties of the family of solution curves *without* solving the equation explicitly. This is important since we may not be able to solve a given (nonlinear) differential equation in closed form.

We first note, from Eq. (6), that $\dot{y} = 0$ for all t if $y = \pm 1$. There are therefore two constant solutions, $y = \pm 1$.

To proceed further, it is helpful to consider the *slope field* of the differential equation.

3.2 Slope field and isoclines

In the differential equation

$$\frac{dy}{dt} = f(t, y) \,,$$

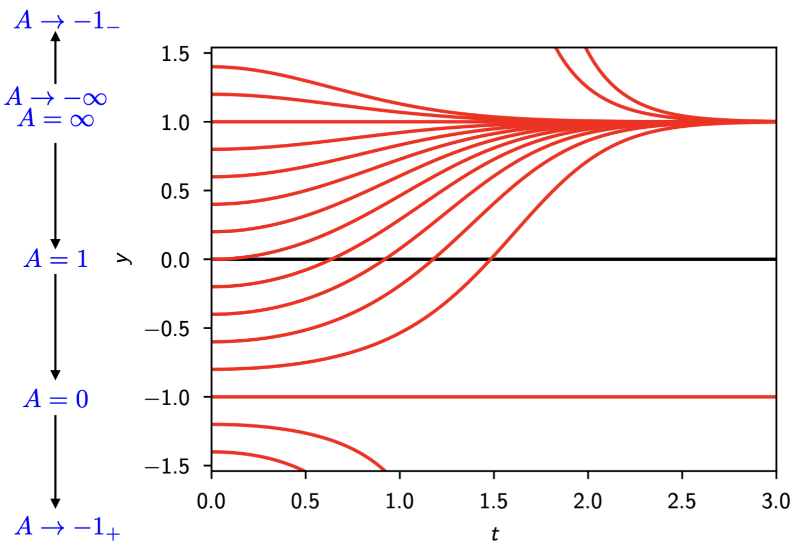


Figure 1: Solution curves $y(t)$ of the differential equation (6), as given by Eq. (7). The variation of the initial value y_0 with the parameter A is also indicated. Note that A changes discontinuously through $y_0 = 1$. Also, the solution curves are discontinuous where $e^{-t^2} = -A$, for A in the range $-1 < A < 0$.

the function $f(t, y)$ on the right-hand side determines the gradient (or slope) of the solution curve through the point (t, y). The *slope field* represents these gradients by short straight-line segments, one centred at each point (a regular grid in the t–y plane is often used), with gradient $f(t, y)$.

By construction, the slope field at a given point is tangent to the solution curve through that point. It therefore tells us the direction in which the solution curve flows.

It is often helpful to supplement the slope field with *isoclines*, which are curves along which $f(t, y)$ is constant.

For the example above, Eq. (6), we have

$$f(t, y) = t\left(1 - y^2\right)$$

so that, for $t > 0$, $\dot{y} < 0$ for $|y| > 1$ and $\dot{y} > 0$ for $|y| < 1$. The isoclines have

$$t\left(1 - y^2\right) = D \quad \Rightarrow \quad y^2 = 1 - D/t\,,$$

where D is a constant that parameterises the isoclines. Along each isocline, the slope field is constant; see Fig. 2.

By drawing curves through the slope field, we can construct approximations to the solution curves even if we cannot determine their functional form exactly.

Finally, note that if $f(t, y)$ is a single-valued function, the solution curves cannot cross in the t–y plane.

4 Fixed (equilibrium) points and stability

In this section, we consider the properties of *fixed points* or *equilibrium points* of differential equations. The analysis of fixed points typically reveals many important properties of the solution of the differential equation.

Definition (Fixed/equilibrium point). A *fixed point* or *equilibrium point* of a differential equation $dy/dt = f(t, y)$ is a constant solution, $y = c$. This corresponds to $dy/dt = 0$ for all t.

In the specific example above, Eq. (6), we have $f(t, y) = t(1 - y^2)$ and so there are fixed points at $y = \pm 1$. From consideration of the solution curves shown in Fig. 1, it is clear that these two fixed points have qualitatively different character.

Specifically, the solution curves converge towards $y = 1$ as t increases, while they diverge from $y = -1$. For these reasons, the fixed point $y = 1$ is said to be a *stable fixed point* while $y = -1$ is an *unstable fixed point*.

Definition (Stability of fixed points). A fixed point $y = c$ is *stable* if whenever y is deviated slightly from c, $y \to c$ as $t \to \infty$. A fixed point is *unstable* if the devitation grows in magnitude as $t \to \infty$.

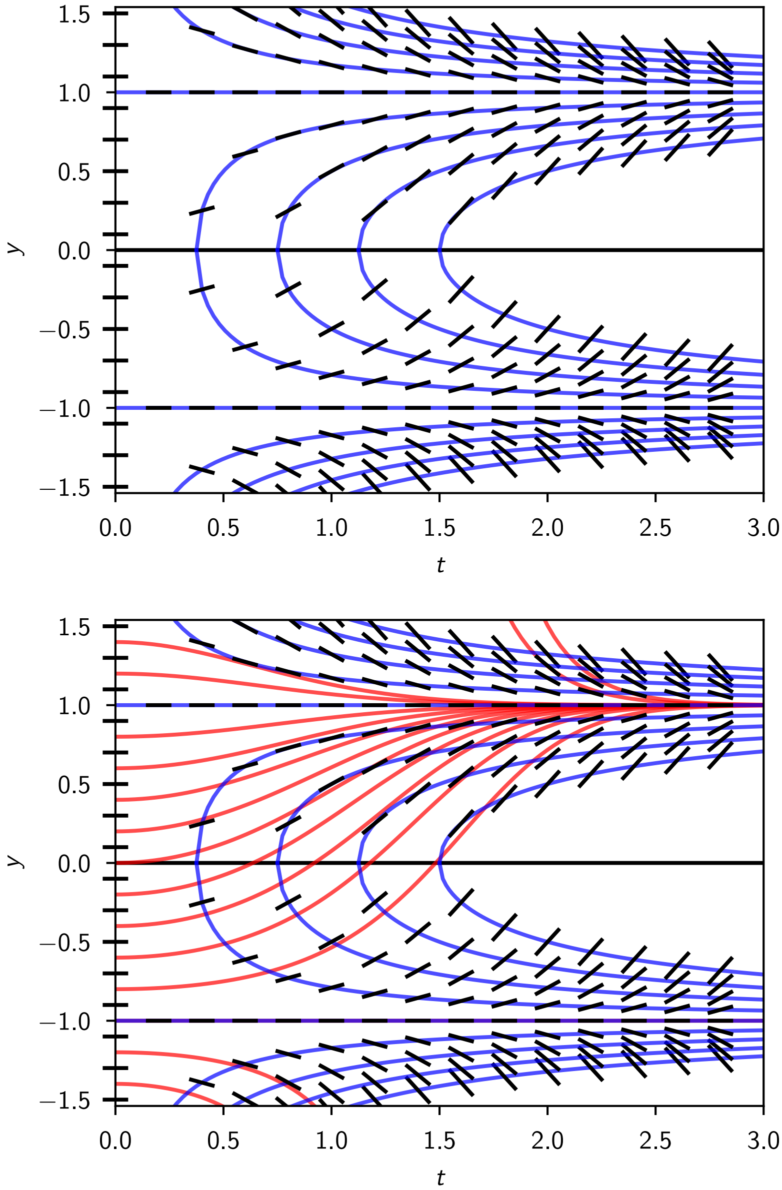


Figure 2: *Top*: isoclines (blue) and the slope field (black sticks) along these for the differential equation (6). *Bottom*: the solution curves (red) are tangent to the slope field everywhere.

4.1 Perturbation analysis and stability

To determine the stability of a fixed point we can use "perturbation analysis". This involves considering the form of the differential equation in the vicinity of the fixed point $y = c$.

Suppose that $y = c$ is a fixed point of the first-order differential equation $dy/dt = f(t, y)$, so that $f(t, c) = 0$ for all t. Consider a small perturbation from the fixed point, which we can write as

$$y(t) = c + \epsilon(t)\,,$$

where $\epsilon(t)$ is a small perturbation. Substituting into the differential equation, we have

$$\begin{aligned}
\frac{d\epsilon}{dt} &= f(t, c + \epsilon) \\
&= f(t, c) + \epsilon \frac{\partial f}{\partial y}(t, c) + O(\epsilon^2) \\
&= \epsilon \frac{\partial f}{\partial y}(t, c) + O(\epsilon^2)\,,
\end{aligned}$$

where we have performed a Taylor expansion in passing to the second line, and used $f(t, c) = 0$ in the third. Sufficiently close to $y = c$ (i.e., for ϵ suitably small), we can approximate the evolution of ϵ with

$$\frac{d\epsilon}{dt} \approx \left[\frac{\partial f}{\partial y}(t, c)\right] \epsilon\,. \tag{8}$$

This differential equation is *linear* and so is generally much simpler to solve than the original (nonlinear) equation. We can use Eq. (8) to study how the perturbation grows with time and hence determine the nature of the fixed point.

Note that if $\partial f/\partial y = 0$ at the fixed point, we must retain higher-order terms in the Taylor expansion of $f(t, c + \epsilon)$ to determine stability.

Example. For our specific example, Eq. (6), with $f(t, y) = t(1 - y^2)$, we have fixed points at $y = \pm 1$ and

$$\frac{\partial f}{\partial y} = -2yt = \begin{cases} -2t & \text{at } y = 1 \ , \\ 2t & \text{at } y = -1 \ . \end{cases}$$

Therefore, near $y = 1$,

$$\frac{d\epsilon}{dt} \approx -2t\epsilon \quad \Rightarrow \quad \epsilon = \epsilon_0 e^{-t^2} \, ,$$

with ϵ_0 a constant. As $t \to \infty$, $\epsilon(t) \to 0$ for any ϵ_0 and so $y(t) \to 1$. It follows that $y = 1$ is a *stable* fixed point.

On the other hand, near $y = -1$,

$$\frac{d\epsilon}{dt} \approx 2t\epsilon \quad \Rightarrow \quad \epsilon = \epsilon_0 e^{t^2} \, .$$

Now, $|\epsilon(t)| \to \infty$ as $t \to \infty$, so if $y(t)$ starts in the vicinity of the fixed point $y = -1$, it will diverge from there.[1] It follows that $y = -1$ is an *unstable* fixed point.

4.2 Autonomous systems and phase portraits

An *autonomous* system is a special case where dy/dt is determined only by y, so that the system does not depend on time explicitly.

Definition (Autonomous system). An autonomous system is descibed by a differential equation of the form

$$\frac{dy}{dt} = f(y) \, , \tag{9}$$

i.e., the derivative dy/dt is only (explicitly) dependent on y.

The analysis of the stability of the fixed points is more straightforward for autonomous systems. In particular,

[1] The linearised equation (8) assumes that ϵ is small and so we cannot really claim that $|\epsilon| \to \infty$ at late times. However, we can be sure that the perturbation grows as t increases.

if $y = c$ is a fixed point of Eq. (9), perturbation analysis leads to the particularly simple equation

$$\frac{d\epsilon}{dt} = \left[\frac{df}{dy}(c)\right] \epsilon = k\epsilon \,, \tag{10}$$

where k is a constant for a given fixed point. The solutions of this differential equation are of the form

$$\epsilon(t) = \epsilon_0 e^{kt} \,,$$

where ϵ_0 is a constant. It follows that the stability of the fixed point is determined by the sign of k.

Therefore, for the autonomous system (9), with fixed point $y = c$,

$$\text{if} \quad \begin{cases} f'(c) < 0 & \Rightarrow \quad \text{stable fixed point,} \\ f'(c) > 0 & \Rightarrow \quad \text{unstable fixed point.} \end{cases}$$

Example (Chemical kinetics). Consider a chemical reaction $A + B \to C + D$. Let us start with a_0 molecules of A, b_0 of B and no C nor D. Each reaction depletes the numbers of A and B molecules by one each and increases C and D similarly. We thus have:

	A	$+$	B	$\to$	C	$+$	D
Number of molecules	$a(t)$		$b(t)$		$c(t)$		$c(t)$
Intitial number of molecules	a_0		b_0		0		0

with $a(t) = a_0 - c(t)$ and $b(t) = b_0 - c(t)$.

We assume that the rate of reaction is proportional to ab (as would be appropriate for dilute gases or solutions) so that

$$\begin{aligned} \frac{dc}{dt} &= \lambda ab \\ &= \lambda(a_0 - c)(b_0 - c) \\ &= f(c) \,, \end{aligned} \tag{11}$$

where λ is a positive constant. We thus have an autonomous, first-order, nonlinear differential equation.

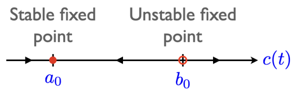
The fixed points are clearly $c = a_0$ and $c = b_0$. Let us assume that $a_0 < b_0$, in which case $c = a_0$ corresponds to having depleted all A molecules, while $c = b_0$ corresponds to the unphysical case of having depleted all B molecules (which requires $a < 0$).

We can analyse the stability by computing df/dc at the fixed points. We have

$$\frac{df}{dc} = \lambda(2c - a_0 - b_0) = \begin{cases} \lambda(a_0 - b_0) \text{ at } c = a_0 \ , \\ \lambda(b_0 - a_0) \text{ at } c = b_0 \ , \end{cases}$$

so that $c = a_0$ is a stable fixed point and $c = b_0$ is an unstable fixed point.

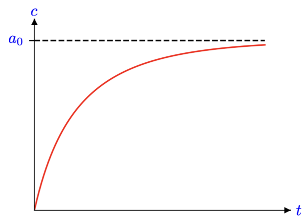
We can illustrate this behaviour with a 1D *phase portrait*, which is a plot of the dependent variable only, with arrows indicating the evolution with time. An example is shown to the right.



Finally, we note that we can easily find (exercise!) the exact solution to Eq. (11) with $c(0) = 0$:

$$c(t) = \frac{a_0 b_0 \left(1 - e^{-\lambda(b_0 - a_0)t}\right)}{b_0 - a_0 e^{-\lambda(b_0 - a_0)t}} \,.$$

This is plotted to the right.



Example (Population dynamics and the logistic equation). The logistic equation is a simple, but widely applicable, model of population dynamics. Suppose we have a population of size $y(t)$. Let the birth rate be αy, with α a positive constant. If we model the death rate as βy, with β a further positive constant, then the dynamics of the population is described by

$$\frac{dy}{dt} = (\alpha - \beta) y \,,$$

and grows or decays exponentially according to the sign of $\alpha - \beta$ (i.e., whether the birth rate exceeds the death rate or the other way around). Such a model is unrealistic, with populations often naturally regulating after early exponential growth.

We can improve the model by modifying the death rate. Suppose that to survive, members of the population must consume some resource that is limited. Let us assume that in a given time interval, the probability of a given member not finding the resource (and so dying) is proportional to y, since the rest of the population are also consuming the resource. Adding this to the death rate, we now have

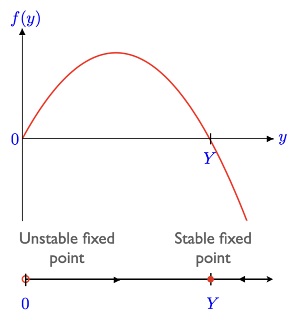
$$\begin{aligned}\frac{dy}{dt} &= (\alpha - \beta)y - \gamma y^2 \\ &= \lambda y \left(1 - \frac{y}{Y}\right) ,\end{aligned} \tag{12}$$

where $\lambda = \alpha - \beta$ and $Y = \lambda/\gamma$. This is the *differential logistic equation.*

The logistic equation is separable and can be easily solved exactly. However, let us reconstruct the behaviour from the phase portrait.

Equation (12) is autonomous with the derivative given by $f(y) = \lambda y(1 - y/Y)$. The fixed points are $y = 0$ and $y = Y$ and

$$\frac{df}{dy} = \lambda\left(1 - \frac{2y}{Y}\right) = \begin{cases} \lambda \text{ at } y = 0 \ , \\ -\lambda \text{ at } y = Y \ . \end{cases}$$



For $\lambda > 0$, we see that $y = 0$ is an unstable fixed point and $y = Y$ is a stable fixed point. A plot of $f(y)$ and the 1D phase portrait is shown to the right.

When the population is small ($y \ll Y$),

$$\frac{dy}{dt} \approx \lambda y$$

and there is exponential growth for $\lambda > 0$. However, as the population grows, the additional term in the death rate, $-\gamma y^2$, becomes important and the stable fixed point $y = Y$ is approached (exponentially).

4.3 Fixed points in discrete equations

We introduced discrete equations earlier as an approximation to differential equations. Now let us consider fixed points of discrete equations.

Consider a *first-order* discrete equation of the form

$$x_{n+1} = f(x_n)\,. \tag{13}$$

Definition (Fixed point of a discrete equation). A *fixed point* of a first-order discrete equation is a value of x_n such that $x_{n+1} = x_n$, i.e.,

$$f(x_n) = x_n\,.$$

We can investigate the stability of fixed points using a perturbation analysis, similiar to that used for differential equations. Suppose that x_f is a fixed point and x_n is close to x_f. If we write $x_n = x_f + \epsilon_n$, where the $\{\epsilon_n\}$ are small perturbations, the discrete equation (13) gives

$$\begin{aligned} x_f + \epsilon_{n+1} &= f(x_f + \epsilon_n) \\ &\approx f(x_f) + \epsilon_n \frac{df}{dx}(x_f) \\ \Rightarrow \quad \epsilon_{n+1} &\approx \epsilon_n \frac{df}{dx}(x_f)\,, \end{aligned}$$

where we have used $f(x_f) = x_f$ as x_f is a fixed point.

It follows that the *iterates* $\{x_n\}$ get closer to the fixed point or diverge from it according to the magnitude of df/dx at the fixed point. In particular,

$$\text{if} \quad \begin{cases} |f'(x_f)| < 1 & \Rightarrow \quad \text{stable fixed point,} \\ |f'(x_f)| > 1 & \Rightarrow \quad \text{unstable fixed point.} \end{cases}$$

Extended example (Logistic map). We illustrate these ideas with a discrete form of the differential logistic equation (12) called the *discrete logistic equation* or the *logistic map*:

$$x_{n+1} = r x_n (1 - x_n)\,. \tag{14}$$

This simple discrete equation has a remarkably rich phenomenology, some of which we shall explore in this extended example.

We can relate the logistic map to the differential logistic equation by approximating dy/dt in the latter with a finite difference in the time step Δt:

$$\frac{y_{n+1} - y_n}{\Delta t} = \lambda y_n \left(1 - \frac{y_n}{Y}\right) ,$$

so that

$$\begin{aligned} y_{n+1} &= y_n + \lambda \Delta t y_n \left(1 - \frac{y_n}{Y}\right) \\ &= (1 + \lambda\Delta t) y_n \left[1 - \left(\frac{\lambda\Delta t}{Y(1+\lambda\Delta t)}\right) y_n\right] . \end{aligned}$$

If we write

$$x_n = \left(\frac{\lambda\Delta t}{Y(1+\lambda\Delta t)}\right) y_n \quad \text{and} \quad r = (1 + \lambda\Delta t) ,$$

we recover the logistic map (14).

We are interested in non-negative iterates, $x_n > 0$. If $0 \le x_n \le 1$, the map ensures that $x_{n+1} \ge 0$ for $r > 0$. Moreover, if $r < 4$, we are ensured that $x_{n+1} \le 1$ also.

The fixed points of the logistic map satisfy $x_n = f(x_n)$, where $f(x_n) = r x_n (1 - x_n)$, and so are given by

$$x_n = 0 \quad \text{or} \quad x_n = 1 - \frac{1}{r} .$$

The fixed point at $x_n = 1 - 1/r$ is only in the physical range for $r \ge 1$.

To assess stability, we use

$$\frac{df}{dx} = r(1 - 2x) = \begin{cases} r \text{ at } x = 0 \ , \\ 2 - r \text{ at } x = 1 - 1/r \ . \end{cases}$$

We see that:

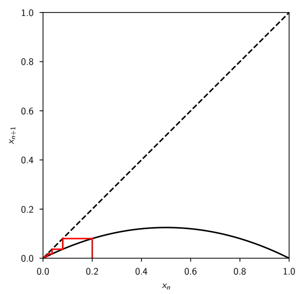
- $x_n = 0$ is a stable fixed point for $0 < r < 1$ and is unstable for $r > 1$;

- $x_n = 1 - 1/r$ is a stable fixed point for $1 < r < 3$, and is unstable for $r > 3$.

We can illustrate the evolution of the iterates using what is sometimes called a *cobweb diagram*. The idea is to plot the function $f(x)$ and the line $y = x$, so that the fixed points of $x_{n+1} = f(x_n)$ are given by the intersection of these. Starting at some initial value, say x_0, on the x-axis, we perform the following steps:

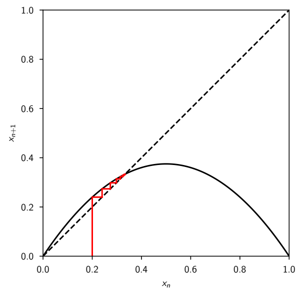
1. draw a vertical line from $(x_0, 0)$ to where it meets $y = f(x)$ at the point $(x_0, f(x_0))$;
2. draw a horizontal line from this point to where it meets $y = x$, so that the x value at the intersection is $f(x_0)$, i.e., x_1;
3. from this point, draw a vertical line to where it meets $y = f(x)$, followed by a horizontal line from there to the intersection with $y = x$, at which point the x value is x_2; and
4. repeat this sequence of vertical and horizontal lines as many times as required.

We now illustrate the behaviour of the logistic map with cobweb diagrams for different ranges of r.

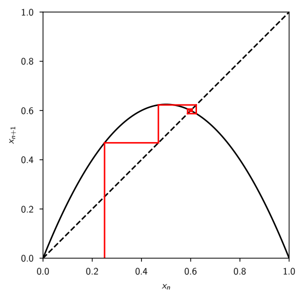


$\boldsymbol{0 < r < 1}$. In this case we have only a stable fixed point in the range $0 \leq x_n \leq 1$ and this is at $x_n = 0$. The iterates rapidly converge to this point, as shown in the diagram to the right (which has $r = 0.5$).

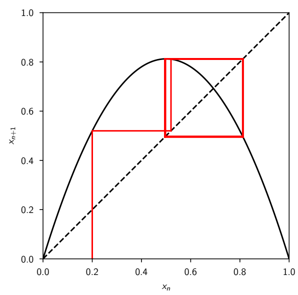
$1 < r < 2$. In this case we have an unstable fixed point, $x_n = 0$, and a stable fixed point, $x_n = 1 - 1/r$, which occurs to the left of the maximum of $f(x)$ at $x = 1/2$. Convergence to the stable fixed point is monotonic (see diagram to the right for $r = 1.5$).



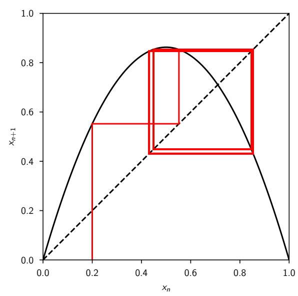
$2 < r < 3$. In this case we still have an unstable fixed point, $x_n = 0$, and a stable fixed point, $x_n = 1 - 1/r$, but the stable fixed point occurs to the right of the maximum of $f(x)$ at $x = 1/2$. Convergence to the stable fixed point is now oscillatory (see diagram to the right for $r = 2.5$).



$3 < r < 1 + \sqrt{6}$. For $r > 3$, the two fixed points are both unstable. For $3 < r < 1 + \sqrt{6} \approx 3.44949$, for almost all starting points the iterates approach oscillations between two values on either side of the fixed point $x_n = 1 - 1/r$, so that $x_{n+2} = x_n$. This is an example of a stable *limit cycle of period 2*. The values in the limit cycle are stable fixed points of the map for the second iterates, $x_{n+2} = f[f(x_n)]$, which follows from iterating the logistic map twice. An example cobweb diagram is shown to the right for $r = 3.25$.



$1 + \sqrt{6} < r < 3.54409$. In this range, for almost all starting points the iterates oscillate between four values with $x_{n+4} = x_n$. This is an example of a stable *limit cycle of period 4*. An example is given to the right for $r = 3.451$. Note that the range of r over which the limit cycle of period 4 is attained is shorter than for the period-2 cycle.



$3.54409 < r < 3.56995$. As r moves through this range, a stable limit cycle of period 8, then 16, then 32, etc., is reached. The length of each cycle falls rapidly and the ratio of succssive intervals asymptotically approaches a

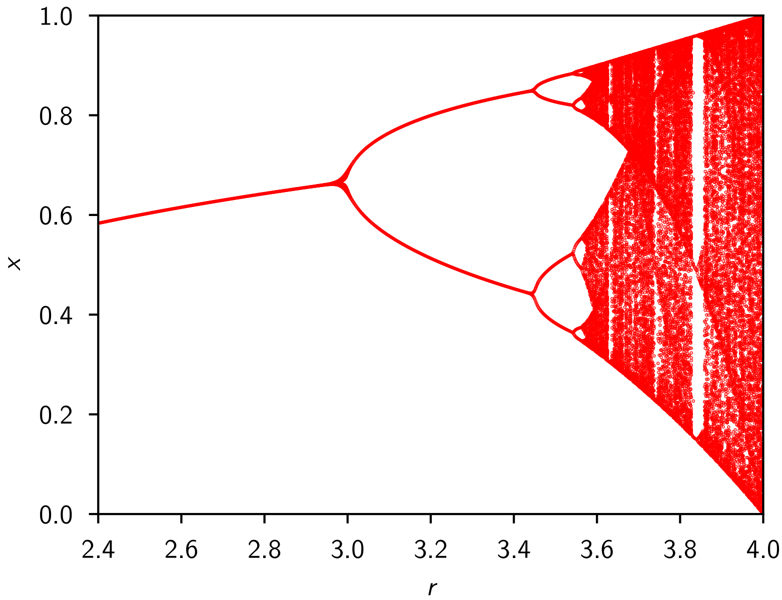
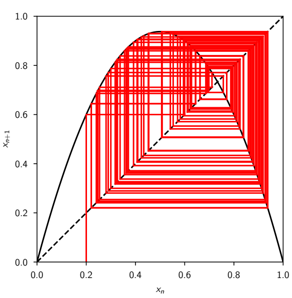


Figure 3: Bifurcation diagram for the logistic map, Eq. (14). This plots the asymptotic values of the iterates (for nearly all starting values) as r is varied. Convergence to a stable fixed point is indicated by the single branch for $r < 3$. Beyond this, the diagram bifurcates to a limit cycle of period 2, then 4, etc., of rapidly decreasing length, before chaotic behaviour ensues for $r > 3.56995$. For certain limited ranges of r beyond this, there are "islands of stability" where the map is non-chaotic.

constant (the *Feigenbaum constant*, with value approximately 4.66920). This is an example of a *period-doubling cascade*. The cascade ends at $r \approx 3.56995$.

$\boldsymbol{r > 3.56995}$. Beyond the end of the period-doubling cascade, the map becomes *chaotic*. For almost all inital conditions, the iterates no longer converge to oscillation amongst a finite number of values. Instead, the behavior is chaotic, with extreme sensitivity to the intial conditions (the example to the right has $r = 3.75$). There are, however, a few "islands of stability" – small ranges of r where the behaviour is non-chaotic and instead the iterates reach oscillation in a limit cycle.



The period-doubling cascade, the onset of chaotic behaviour and the islands of stability are illustrated in the *bifurcation diagram* in Fig. 3. This shows the asymptotic value(s) of the iterates as a function of r for nearly all starting values, with the limit cycles appearing as a finite number of branches.