Stochastic Financial Models 14

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1 Optional sampling theorem

Definition. Let $(X_t)_{t\geq 0}$ be an (either discrete- or continuous-time) adapted process and T a stopping time. The *stopped process* $(X_{t\wedge T})_{t\geq 0}$ is defined by

$$X_{t\wedge T} = \begin{cases} X_t & \text{if } t \leq T \\ X_T & \text{if } t > T \end{cases}$$

Remark. Recall the notation $a \wedge b = \min\{a, b\}$ for real numbers a, b.

For the rest of the lecture, **time is discrete.**

Proposition. *Let $(X_n)_{n\geq 0}$ be an adapted process and and T a stopping time. Then the stopped process $(X_{n\wedge T})_{n\geq 0}$ is X_0 plus a martingale transform.*

Proof. Note that

$$X_{n\wedge T} = X_0 + \sum_{k=1}^{n} \mathbb{1}_{\{k\leq T\}}(X_k - X_{k-1})$$

Since $\{k \leq T\} = \{T \leq k-1\}^c \in \mathcal{F}_{k-1}$ for all $k \geq 1$ the process $(\mathbb{1}_{\{n\leq T\}})_n$ is previsible. □

Corollary. *A stopped martingale is a martingale.*

Proof. This follows from the theorem that says the martingale transform of a bounded previsible process with respect to a martingale is again a martingale. □

Theorem (Optional stopping theorem)**.** *Let T be a stopping time and $(X_n)_{n\geq 0}$ be a martingale such that $(X_{n\wedge T})_n$ bounded and $T < \infty$ almost surely. Then*

$$\mathbb{E}(X_T) = X_0$$

Remark. Recall our convention that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ so X_0 is constant.

Proof. Let $M_n = X_{n\wedge T}$. Note $(M_n)_{n\geq 0}$ is a martingale so that

$$\mathbb{E}(X_{n\wedge T}) = \mathbb{E}(M_n|\mathcal{F}_0) = M_0 = X_0$$

for all non-random n, by the definition of martingale and the convention on $\mathcal{F}_0$.

Now by assumption there exists a constant $C > 0$ such that $|X_{n\wedge T}| \leq C$ a.s. for all n. Also, since T is a.s. finite we have $X_{n\wedge T} \to X_T$ a.s., and hence $|X_T| \leq C$ a.s. In particular, we have

$$|X_{n\wedge T} - X_T| \leq 2C\mathbb{1}_{\{T>n\}}$$

by the triangle inequality.

Combining the two observations above,

$$\begin{aligned}
|\mathbb{E}(X_T) - X_0| &= |\mathbb{E}(X_T - X_{n\wedge T})| \\
&\leq \mathbb{E}(|X_T - X_{n\wedge T}|) \\
&\leq 2C\mathbb{P}(T > n) \\
&\to 0
\end{aligned}$$

□

Remark. It turns out that we do not need to assume that T is finite nor do we need to assume that $(X_{n\wedge T})_n$ is bounded to get the conclusion. A much weaker version of the OST is

Theorem. (A more general optional stopping theorem). Let $(X_n)_n$ be a martingale and T a stopping time such that $(X_{n\wedge T})_n$ is uniformly integrable. Then $\mathbb{E}(X_T) = X_0$.

2 Examples of the optional stopping theorem

Let $(S_n)_{n\geq 0}$ be a simple symmetric random walk starting from $S_0 = 0$, i.e. $S_n = \xi_1 + \ldots + \xi_n$ where $(\xi_n)_{n\geq 1}$ are IID $\mathbb{P}(\xi_n = \pm 1) = \frac{1}{2}$.

Example 1.

- Fix integers $a, b > 0$ and let $T = \inf\{n \geq 0 : S_n \in \{-a, b\}$.
- By Markov Chains, $T < \infty$ almost surely.
- Let $p = \mathbb{P}(S_T = -a)$ and $q = \mathbb{P}(S_T = b)$.
- By optional stopping $S_0 = 0 = \mathbb{E}(S_T) = -ap + bq$
- $p = \frac{b}{a+b}$ and $q = \frac{a}{a+b}$
- Optional stopping is justified since $|S_{T\wedge n}| \leq \max\{a, b\}$ for all n.

Counterexample 2.

- Now let $\tau = \inf\{n \geq 0 : S_n = -a\}$.
- By Markov Chains, $\tau < \infty$ almost surely. So $S_\tau = -a$.
- $\mathbb{E}(S_\tau) = -a \neq 0 = S_0$ in apparent contradiction to the optional stopping theorem.
- But note that $S_{n\wedge\tau}$ is not bounded from above, so there is no a priori reason to believe that the optional stopping theorem is applicable.

tau the r.v. in previous example

Example 3. Our goal is to find the probability generating function $\mathbb{E}(z^\tau)$ for fixed $0 < z < 1$. *Claim*: the process $w^{S_n} z^n$ is a martingale iff $w + w^{-1} = 2z^{-1}$. Indeed, note

$$\frac{\mathbb{E}(w^{S_n} z^n | \mathcal{F}_{n-1})}{w^{S_{n-1}} z^{n-1}} = z\mathbb{E}(w^{\xi_n}) = \frac{z}{2}(w + w^{-1})$$

Let $M_n = w^{S_n} z^n$ where $w + w^{-1} = 2z^{-1}$. This is a martingale with $M_\tau = w^{-a} z^\tau$. We want to apply the optional stopping theorem to conclude

$$\mathbb{E}(M_\tau) = w^{-a}\mathbb{E}(z^\tau) = M_0 = 1$$

or

$$\mathbb{E}(z^\tau) = w^a.$$

But which value of w makes the above identity true? Given z, there are two possible solutions

$$w_\pm = \frac{1 \pm \sqrt{1 - z^2}}{z}$$

and $0 < w_- < 1$ while $w_+ > 1$. In particular, since $S_{n\wedge\tau} \geq -a$ for all n and $z < 1$, then

$$w_-^{S_{n\wedge\tau}} z^{n\wedge T} \leq w_-^{-a} \text{ for all } n$$

Hence the OST is applicable and the correct formula is with $w = w_-$, i.e.

$$\mathbb{E}(z^\tau) = w_-^a = \left(\frac{1 - \sqrt{1 - z^2}}{z}\right)^a.$$