Stochastic Financial Models 13

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1 Discrete-time martingales

Example.

- Let $X_1, X_2, \ldots$ be independent with $\mathbb{E}(X_n) = 0$ for all n.
- Let $S_0 = 0$ and $S_n = X_1 + \ldots + X_n$.

Then $(S_n)_{n \geq 0}$ is a martingale in the filtration generated by $(X_n)_{n \geq 1}$ since

- S_n is integrable: $\mathbb{E}(|S_n|) \leq \mathbb{E}(|X_1|) + \ldots + \mathbb{E}(|X_n|) < \infty$
- S_n is clearly $\mathcal{F}_n$ measurable (since it is a function of $X_1, \ldots, X_n$)
- $\mathbb{E}(S_n - S_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(X_n|\mathcal{F}_{n-1}) = \mathbb{E}(X_n) = 0$ by the independence of X_n and $\mathcal{F}_{n-1}$.

Note that in this example $(S_n)_{n \geq 0}$ and $(X_n)_{n \geq 1}$ generate the same filtration

Definition. A discrete-time process $(H_n)_{n \geq 1}$ is *previsible* (or *predictable*) with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$ iff H_n is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$.

Remark. The index set for a previsible process is usually $\{1, 2, \ldots\}$.
Remark. Let $X_n = H_{n+1}$. Then $(H_n)_{n \geq 1}$ is previsible if and only if $(X_n)_{n \geq 0}$ is adapted.

Definition. The *martingale transform* of a previsible process $(H_n)_{n \geq 1}$ with respect to an adapted process $(X_n)_{n \geq 0}$ is the process defined by

$$Y_n = \sum_{k=0}^{n} H_k (X_k - X_{k-1})$$

Theorem. *The martingale transform of a bounded previsible process with respect to a martingale is a martingale.*

Proof. Let $(H_n)_{n\geq 1}$ be bounded and previsible and $(X_n)_{n\geq 0}$ a martingale, and let $(Y_n)_{n\geq 0}$ be the martingale transform. Note that $(Y_n)_{n\geq 0}$ is adapted since each term of the formula defining Y_n is $\mathcal{F}_n$-measurable by the adaptedness of (X_n) and the previsibility of (H_n). Integrability follows from by the triangle inequality

$$\mathbb{E}(|Y_n|) \leq \mathbb{E}\left(\sum_{k=1}^{n} |H_k||X_k - X_{k-1}|\right) \leq C \sum_{k=1}^{n} \mathbb{E}(|X_k - X_{k-1}|) < \infty$$

and the integrability of (X_n) (from the definition of martingale), where $C > 0$ is the constant such that $|H_k| \leq C$ a.s. for all k (from the assumption of boundedness of (H_n))

Now

$$\begin{aligned} \mathbb{E}(Y_n - Y_{n-1}|\mathcal{F}_{n-1}) &= \mathbb{E}[H_n(X_n - X_{n-1})|\mathcal{F}_{n-1}] \\ &= H_n \mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) \\ &= 0 \end{aligned}$$

by taking out what is known, and the martingale property of $(X_n)_{n\geq 0}$. $\square$

Important example from finance. Consider a market

- with a risk-free asset with interest rate r
- and d risky assets with time n prices $(S_n)_{n\geq 0}$.

and investor who

- holds the portfolio $\theta_n \in \mathbb{R}^d$ of risky assets during the time interval $(n-1, n]$,
- and the rest of his wealth is held in the risk-free asset.
- Suppose the investor is *self-financing*: his changes in wealth are explained by the changes in asset prices (but not by consumption or non-market income)

$$X_n = (1+r)X_{n-1} + \theta_n^\top [S_n - (1+r)S_{n-1}]$$

Definition. The investor's *discounted* wealth at time n is $\frac{X_n}{(1+r)^n}$. The *discounted* asset prices at time n are $\frac{S_n}{(1+r)^n}$.

Proposition. *A self-financing investor's discounted wealth is the initial wealth plus the martingale transform of the portfolio process with respect to the discounted risky asset prices.*

Proof. It is easy to see by induction that

$$\frac{X_n}{(1+r)^n} = X_0 + \sum_{k=1}^{n} \theta_k^\top \left(\frac{S_k}{(1+r)^k} - \frac{S_{k-1}}{(1+r)^{k-1}}\right)$$

$\square$

2 Stopping times

Definition. A *stopping time* for a filtration $(\mathcal{F}_t)_{t\geq 0}$ is a random variable T valued in $\{0, 1, 2, \ldots, +\infty\}$ (discrete-time) or $[0, +\infty]$ (continuous time) such that

$$\{T \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0$$

Example.

- Let $(X_n)_{n\geq 0}$ be a discrete-time adapted process.
- Let $T = \inf\{n \geq 0 : X_n > 0\}$
- Convention: $\inf \emptyset = \infty$.
- Then T is a stopping time.

Note $\{T \leq n\} = \cup_{k=0}^{n} \{X_k > 0\} \in \mathcal{F}_n$ since $\{X_k > 0\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ for all $k \leq n$. (Recall that the sigma-algebra $\mathcal{F}_n$ is closed under finite unions.)

Possible counter-example.

- Let $(X_n)_{n\geq 0}$ be an adapted process.
- Let $T = \sup\{n \geq 0 : X_n > 0\}$
- Then T is a *not* a stopping time in general.

Note $\{T \leq n\} = \cap_{k=n+1}^{\infty} \{X_k \leq 0\}$ so the event $\{T \leq n\}$ generally contains information about the future.