Stochastic Financial Models 15

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1 Submartingales and supermartingales

Definition. An integrable adapted process $(X_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ (in either discrete- or continuous-time) is a submartingale if and only if

$$\mathbb{E}(X_t|\mathcal{F}_s) \geq X_s \text{ for all } 0 \leq s \leq t$$

An integrable adapted $(X_t)_{t\geq 0}$ is called *supermartingale* with respect to a filtration iff $(-X_t)_{t\geq 0}$ is a submartingale.

For the rest of the time, we work in **discrete-time**.

Remark. In discrete-time, a *submartingale* is an integrable adapted process $(X_n)_{n\geq 0}$ such that

$$\mathbb{E}(X_n|\mathcal{F}_{n-1}) \geq X_{n-1} \text{ for all } n \geq 1$$

by the tower property and the posivity of conditional expectation.

Theorem. *The martingale transform of a non-negative bounded previsible process with respect to a submartingale is a submartingale.*

Proof. Let $(H_n)_{n\geq 1}$ be non-negative, bounded and previsible, and $(X_n)_{n\geq 0}$ a submartingale, and let $(Y_n)_{n\geq 0}$ be the martingale transform. Integrability of $(Y_n)_n$ follows from the boundedness of $(H_n)_n$ and integrability of $(X_n)_n$. The adaptedness of $(Y_n)_n$ follows from the adaptedness of both $(H_n)_n$ and $(X_n)_n$.

Now

$$\begin{aligned}\mathbb{E}(Y_n - Y_{n-1}|\mathcal{F}_{n-1}) &= \mathbb{E}[H_n(X_n - X_{n-1})|\mathcal{F}_{n-1}] \\ &= H_n\mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1}) \\ &\geq 0\end{aligned}$$

by taking out what is known, and the submartingale property of $(X_n)_{n\geq 0}$. □

Theorem. *Let $(X_n)_{n\geq 0}$ be a submartingale and $S \leq T$ are stopping times. Let*

$$M_n = X_{n\wedge T} - X_{n\wedge S}.$$

Then $(M_n)_{n\geq 0}$ is a submartingale.

Proof. Note

$$M_n = \sum_{k=1}^{n} \mathbb{1}_{\{S<k\leq T\}}(X_k - X_{k-1}).$$

Also $H_n = \mathbb{1}_{\{S<k\leq T\}} = \mathbb{1}_{\{S\leq n-1\}} - \mathbb{1}_{\{T\leq n-1\}}$ is bounded and $\mathcal{F}_{n-1}$-measurable. Hence $(M_n)_n$ is the martingale transform of a non-negative bounded previsible process with respect to a submartingale. □

Theorem (Optional sampling theorem)**.** *Let $(X_n)_{n\geq 0}$ be a submartingale and $S \leq T$ are bounded stopping times, then*

$$\mathbb{E}(X_T) \geq \mathbb{E}(X_S)$$

Proof. Let $M_n = X_{n\wedge T} - X_{n\wedge S}$. Now pick a constant N such that $T \leq N$ a.s. The conclusion follows from $\mathbb{E}(M_N) \geq M_0 = 0$ since $M_N = X_T - X_S$. □

2 Controlled Markov processes

Definition. A *Markov process* $(X_t)_{t\geq 0}$ with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ (in either discrete- or continuous-time) is an adapted process such that

$$\mathbb{P}(X_t \in A|\mathcal{F}_s) = \mathbb{P}(X_t \in A|X_s)$$

for all $0 \leq s \leq t$, (measurable) sets A, and where $(\mathcal{F}_t)_{t\geq 0}$.

We now work in discrete time. To check that a process $(X_n)_n$ is a martingale, we need only check

$$\mathbb{P}(X_n \in A|\mathcal{F}_{n-1}) = \mathbb{P}(X_n \in A|X_{n-1})$$

for all $n \geq 1$.

A useful way of to think about a Markov process is as **random dynamical system.** A Markov process valued in $\mathcal{X}$ can be constructed with

- Initial condition $X_0 = x$
- A function $G : \mathbb{N} \times \mathcal{X} \times \mathcal{V} \to \mathcal{X}$
- An sequence $(\xi_n)_{n\geq 1}$ of independent $\mathcal{V}$-valued random variable
- Then we construct the process recursively

  $$X_n = G(n, X_{n-1}, \xi_n)$$

  for $n \geq 1$.

Example. A simple symmetric random walk on $\mathbb{Z}$ starting at $X_0 = 0$ can be constructed as follows

- Let $\mathcal{V} = \{-1, 1\}$
- Let $(\xi_n)_{n\geq 1}$ be an IID sequence such that $\mathbb{P}(\xi_n = \pm 1) = 1/2$.
- Let $G(n, x, v) = x + v$ for all n.
- Then $X_n = G(n, X_{n-1}, \xi_n)$ for $n \geq 1$.

A **controlled Markov process** is built from

- Initial condition $X_0 = x$
- A previsible process $(U_n)_{n\geq 1}$
- A function $G : \mathbb{N} \times \mathcal{X} \times \mathcal{U} \times \mathcal{V} \to \mathcal{X}$
- A sequence $(\xi_n)_{n\geq 1}$ of independent $\mathcal{V}$-valued random variables
- Then we construct the process recursively

  $$X_n^U = G(n, X_{n-1}^U, U_n, \xi_n)$$

  for $n \geq 1$.

3 Stochastic optimal control

A typical problem that we will encounter is this. Given a controlled Markov process $(X_n^U)_{n\geq 0}$ and a (non-random) time horizon N we wish to

$$\text{maximise } \mathbb{E}\left[\sum_{k=1}^{N} f(k, U_k) + g(X_N^U) \middle| X_0 = x\right]$$

over previsible controls $(U_k)_{1\leq k\leq N}$, where the controlled Markov process evolves as $X_n = G(n, X_{n-1}, U_n, \xi_n)$ for $n \geq 1$ for a given function G and independent sequence $(\xi_n)_n$.

Definition. The system of equations

$$\begin{aligned} V(N, x) &= g(x) \quad \text{for all } x \\ V(n-1, x) &= \sup_u \{f(n, u) + \mathbb{E}[V(n, G(n, x, u, \xi_n))]\} \quad \text{for all } x, 1 \leq n \leq N \end{aligned}$$

is called the *Bellman equation* for the problem.

Definition. The *value function* for the problem is

$$V(n, x) = \sup_{(U_k)_{n+1\leq n\leq N}} \mathbb{E}\left[\sum_{k=n+1}^{N} f(k, U_k) + g(X_N^U) \middle| X_n^U = x\right].$$

The dynamic programming principle: Under some assumptions, the solution to the Bellman equation is the value function. (details in next lecture)