Part IB — Analysis and Topology

Based on lectures by Dr P. Russell

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Part I
Generalizing continuity and convergence

§1 Three Examples of Convergence

§1.1 Convergence in $\mathbb{R}$

Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. We say (x_n) *converges* to x and write $x_n \to x$ if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, n \geq N \quad |x_n - x| < \epsilon.$$

Useful fact: $\forall\, a, b \in \mathbb{R}$ $|a+b| \leq |a| + |b|$ (Triangle Inequality).

Bolzano-Weierstrass Theorem (BWT) A bounded sequence in $\mathbb{R}$ must have a convergent subsequence (Proof by interval bisection).

Recall: A sequence (x_n) in $\mathbb{R}$ is Cauchy if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, m, n \geq N \quad |x_m - x_n| < \epsilon.$$

Easy exercise Convergent $\implies$ Cauchy

General Principle of Convergence (GPC) Any Cauchy sequence in $\mathbb{R}$ converges.

> *Outline.* If (x_n) Cauchy then (x_n) bounded so by BWT has a convergent subsequence, say $x_{n_j} \to x$. But as (x_n) Cauchy, $x_n \to x$. □

§1.2 Convergence in $\mathbb{R}^2$

Remark 1. This all works in $\mathbb{R}^n$

Let (z_n) be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. What should $z_n \to z$ mean?

In $\mathbb{R}$: "As n gets large, z_n gets arbitrarily close to z."

What does 'close' mean in $\mathbb{R}^2$?

In $\mathbb{R}$: a, b close if $|a-b|$ small. In $\mathbb{R}^2$: Replace $|\cdot|$ by $\|\cdot\|$

Recall: If $z = (x, y)$ then $\|z\| = \sqrt{x^2 + y^2}$.

Triangle Inequality If $a, b \in \mathbb{R}^2$ then $\|a+b\| \leq \|a\| + \|b\|$.

Definition 1.1

Let (z_n) be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. We say (z_n) **converges** to z and .. $z_n \to z$ if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, n \geq N\ \ \|z_n - z\| < \epsilon$.

Equivalently, $z_n \to z$ iff $\|z_n - z\| \to 0$ (convergence in $\mathbb{R}$).

Example 1.1

Let $(z_n), (w_n)$ be sequences in $\mathbb{R}^2$ with $z_n \to z, w_n \to w$. Then $z_n + w_n \to z + w$.

Proof.

$$\begin{aligned}\|(z_n + w_n) - (z + w)\| &\leq \|z_n - z\| + \|w_n - w\| \\ &\to 0 + 0 = 0 \text{ (by results from IA).}\end{aligned}$$

□

In fact, given convergence in $\mathbb{R}$, convergence in $\mathbb{R}^2$ is easy:

Proposition 1.1

Let (z_n) be a sequence in $\mathbb{R}^2$ and let $z \in \mathbb{R}^2$. Write $z_n = (x_n, y_n)$ and $z = (x, y)$. Then $z_n \to z$ iff $x_n \to x$ and $y_n \to y$.

Proof. ($\implies$): $|x_n - x|, |y_n - y| \leq \|z_n - z\|$. So if $\|z_n - z\| \to 0$ then $|x_n - x| \to 0$ and $|y_n - y| \to 0$.

($\impliedby$): If $|x_n - x| \to 0$ and $|y_n - y| \to 0$ then $\|z_n - z\| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0$ by results in $\mathbb{R}$. □

Definition 1.2 (Bounded Sequence)

A sequence (z_n) in $\mathbb{R}^2$ is **bounded** if $\exists\, M \in \mathbb{R}$ s.t. $\forall\, n\ \ \|z_n\| \leq M$.

Theorem 1.1 (BWT in $\mathbb{R}^2$)

A bounded sequence in $\mathbb{R}^2$ must have a convergent subsequence.

Theorem 1.2 (GPC for $\mathbb{R}^2$)
Any Cauchy sequence in $\mathbb{R}^2$ converges.

Proof. Let (z_n) be a Cauchy sequence in $\mathbb{R}^2$. Write $z_n = (x_n, y_n)$. For all m, n, $|x_m - x_n| \leq \|z_m - z_n\|$ so (x_n) is a Cauchy sequence in $\mathbb{R}$, so converges by GPC. Similarly, (y_n) converges in $\mathbb{R}$. So by 1.1, (z_n) converges. $\square$

Thought for the day What about continuity? Let $f : \mathbb{R}^2 \to \mathbb{R}$. What does it mean for f to be continuous? (Simple modification of defn for $\mathbb{R} \to \mathbb{R}$).

What can we do with it?

Big theorem in IA: If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on a closed bounded interval then f is bounded and attains its bounds.

Is there a similar theorem for $\mathbb{R}^2 \to \mathbb{R}$. What do we replace 'closed bounded interval' by? We proved the theorem using BWT. Why did it work? Why did we need a closed bounded interval to make it work? What can we do in $\mathbb{R}^2$?

§1.3 Convergence of Functions

Let $X \subset \mathbb{R}$[1], let $f_n : X \to \mathbb{R}$ $(n \geq 1)$ and let $f : X \to \mathbb{R}$. What does it mean for f_n to converge to f.

Obvious idea:

Definition 1.3 (Pointwise convergence)
Say (f_n) **converges pointwise** to f and write $f_n \to f$ pointwise if $\forall\, x \in X$ $f_n(x) \to f(x)$ as $n \to \infty$.

Pros

- Simple
- Easy to check
- Defined in terms of convergence in $\mathbb{R}$

Cons

- Doesn't preserve 'nice' properties.
- 'Doesn't feel right'.

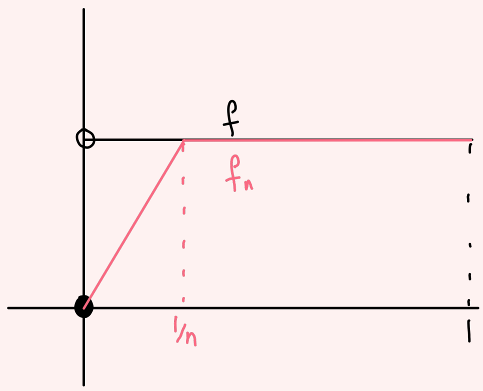
In all three examples, have $X = [0, 1]$, $f_n \to f$ pointwise.

[1]Mostly can think of $X = \mathbb{R}$ or some interval

Example 1.2 (Every f_n continuous but f not)

$$f_n(x) = \begin{cases} nx & x \leq \frac{1}{n} \\ 1 & x \geq \frac{1}{n} \end{cases}$$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$



Clearly f_n continuous for all n but f not. If $x = 0$, $\forall\, n$ $f_n(0) = 0 = f(0)$. If $x > 0$, for sufficiently large n $f_n(x) = 1 = f(x)$ so $f_n(x) \to f(x)$.

Example 1.3 (Every f_n integrable but f not)

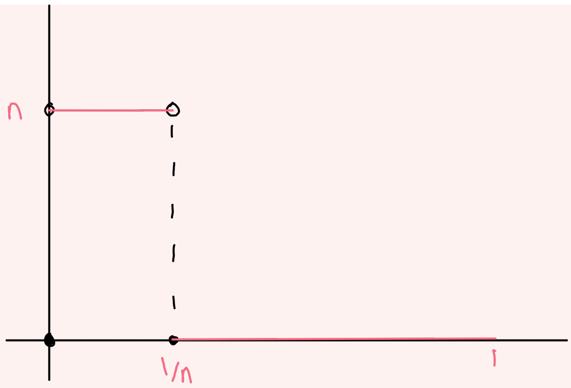
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

This is a non integrable[a] function so now we want to find f_n such that they converge pointwise to this. Enumerate the rationals in $[0, 1]$ as $q_1, q_2, \ldots$ For $n \geq 1$, set $f_n(x) = \mathbb{1}_{q_1,\ldots,q_n}$. f_n integrable as it is nonzero at finitely many points.

[a]N.B. As in IA 'integrable' means 'Riemann integrable'

Example 1.4 (Every f_n and f integrable but $\int_0^1 f_n \not\to \int_0^1 f$)

Let $f(x) = 0$ for all x, so $\int_0^1 f = 0$. Define f_n s.t. $\int_0^1 f_n = 1$ for all n.



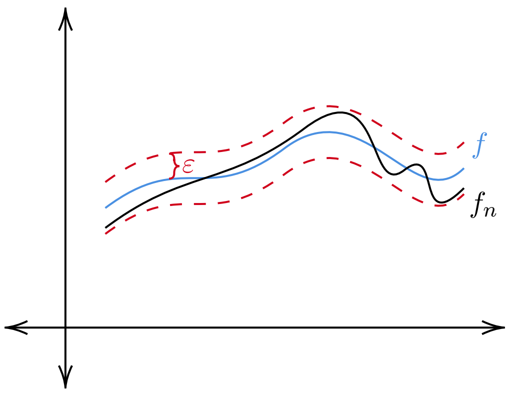
$$f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Better definition:

Definition 1.4 (Uniform convergence)

Let $X \subset \mathbb{R}$, $f_n : X \to \mathbb{R}$ $(n \geq 1)$, $f : X \to \mathbb{R}$. We say (f_n) **converges uniformly** to f and write $f_n \to f$ uniformly if $\forall\ \epsilon > 0\ \exists\ N\ \forall\ x \in X\ \forall\ n \geq N\ |f_n(x) - f(x)| < \epsilon$.

cf $f_n \to f$ pointwise: $\forall\ \epsilon > 0\ \forall\ x \in X\ \exists\ N\ \forall\ n \geq N\ |f_n(x) - f(x)| < \epsilon$. (We have swapped the $\forall\ x \in x$ and $\exists\ N$). Pointwise convergence allows for N to be a function of x whilst uniform convergence requires N to work for all x even the worst case. In particular, $f_n \to f$ uniformly $\implies$ $f_n \to f$ pointwise.



Equivalently, $f_n \to f$ uniformly if for sufficiently large n $f_n - f$ is bounded and $\sup_{x \in X} |f_n - f| \to 0$.

Theorem 1.3 (A uniform limit of cts functions is cts)

Let $X \subset \mathbb{R}$, let $f_n : X \to \mathbb{R}$ be continuous ($n \geq 1$) and let $f_n \to f : X \to \mathbb{R}$ uniformly. Then f is cts.

Proof. Let $x \in X$. Let $\epsilon > 0$. As $f_n \to f$ uniformly, we can find N s.t. $\forall\ n \geq N\ \forall\ y \in X\ |f_n(y) - f(y)| < \epsilon$. In particular, $\forall\ y \in X\ |f_N(y) - f(y)| < \epsilon$. As f_N is cts, we can find $\delta > 0$ s.t. $\forall\ y \in X,\ |y - x| < \delta \implies |f_N(y) - f_N(x)| < \epsilon$. Now let $y \in X$ with $|y - x| < \delta$. Then

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)|^{a} \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Hence f is cts. □

[a] The core of this proof is this inequality.

Remark 2. This is often called a '3ϵ proof' (or an $\frac{\epsilon}{3}$ proof).

Theorem 1.4

Let $f_n : [a, b] \to \mathbb{R}$ ($n \geq 1$) be integrable and let $f_n \to f : [a, b] \to \mathbb{R}$ uniformly. Then f is integrable and $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$.

Proof. As $f_n \to f$ uniformly, we can pick n suff. large s.t. $f_n - f$ is bounded. Also f_n is bounded (as integrable). So by triangle inequality, $f = (f - f_n) + f_n$ is bounded. Let $\epsilon > 0$. As $f_n \to f$ uniformly there is some N s.t. $\forall\ n \geq N\ \forall\ x \in [a, b]$ we have $|f_n(x) - f(x)| < \epsilon$.
In particular, $\forall\ x \in [a, b]\ |f_N(x) - f(x)| < \epsilon$.

By Riemann's criterion, there is some dissection $\mathcal{D}$ of $[a, b]$ for which $S(f_n, \mathcal{D}) - s(f_n, \mathcal{D}) < \epsilon$. Let $\mathcal{D} = \{x_0, x_1, x_2, \ldots, x_k\}$ where $a = x_0 < x_1 < \cdots < x_k = b$. Now

$$\begin{aligned} S(f, \mathcal{D}) &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x) \\ &\leq \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} (f_N(x) + \epsilon) \\ &= \sum_{i=1}^{k} (x_i - x_{i-1}) \left(\left(\sup_{x \in [x_{i-1}, x_i]} f_N(x) \right) + \epsilon \right) \\ &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f_N(x) + \sum_{i=1}^{k} (x_i - x_{i-1}) \epsilon \end{aligned}$$

$$= S(f_N, \mathcal{D}) + (b-a)\epsilon.$$

That is $S(f,\mathcal{D}) \le S(f_N,\mathcal{D}) + (b-a)\epsilon$. Similarly $s(f,\mathcal{D}) \ge s(f_N,\mathcal{D}) - (b-a)\epsilon$. Hence

$$\begin{aligned} S(f,\mathcal{D}) - s(f,\mathcal{D}) &\le S(f_N,\mathcal{D}) - s(f_N,\mathcal{D}) + 2(b-a)\epsilon \\ &< (2(b-a)+1)\epsilon \end{aligned}$$

But $2(b-a)+1$ is a constant so $(2(b-a)+1)\epsilon$ can be made arbitrarily small. Hence by Riemann's criterion, f is integrable over $[a,b]$.

Now, for any n suff. large that $f_n - f$ is bounded,

$$\begin{aligned} \left| \int_a^b f_n - \int_a^b f \right| &= \left| \int_a^b (f_n - f) \right| \\ &\le \int_a^b |f_n - f| \\ &\le (b-a) \sup_{x\in[a,b]} |f_n - f| \\ &\to 0 \text{ as } n \to \infty \text{ since } f_n \to f \text{ uniformly.}^{a} \end{aligned}$$

□

[a]Note we said that $f_n \to f$ uniformly if $\sup |f_n - f| \to 0$.

§2 Metric Spaces

§3 Topological Spaces

Part II
Generalizing differentiation