Graph Theory

Lectured by I. B. Leader, Michaelmas Term 2007

Last updated: Tue 21st Aug, 2012

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Course schedule

GRAPH THEORY (D) **24 lectures, Michaelmas term**

No specific prerequisites.

Introduction
Basic definitions. Trees and spanning trees. Bipartite graphs. Euler circuits. Elementary properties of planar graphs. Statement of Kuratowski's theorem. [3]

Connectivity and matchings
Matchings in bipartite graphs; Hall's theorem and its variants. Connectivity and Menger's theorem. [3]

Extremal graph theory
Long paths, long cycles and Hamilton cycles. Complete subgraphs and Turan's theorem. Bipartite subgraphs and the problem of Zarankiewicz. The Erdős-Stone theorem; *sketch of proof*. [5]

Eigenvalue methods
The adjacency matrix and the Laplacian. Strongly regular graphs. [2]

Graph colouring
Vertex and edge colourings; simple bounds. The chromatic polynomial. The theorems of Brooks and Vizing. Equivalent forms of the four colour theorem; the five colour theorem. Heawood's theorem for surfaces; the torus and the Klein bottle. [5]

Ramsey theory
Ramsey's theorem (finite and infinite forms). Upper bounds for Ramsey numbers. [3]

Probabilistic methods
Basic notions; lower bounds for Ramsey numbers. The model $G(n,p)$; graphs of large girth and large chromatic number. The clique number. [3]

Appropriate books

B.Bollobás *Modern Graph Theory.* Springer 1998 (£26.00 paperback).

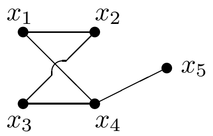
J.A.Bondy and U.S.R.Murty *Graph Theory with Applications.* Elsevier 1976
(available online via http://www.ecp6.jussieu.fr/pageperso/bondy/bondy.html).

R.Diestel *Graph Theory.* Springer 2000 (£38.50 paperback).

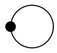
D.West *Introduction to Graph Theory.* Prentice Hall 1999 (£45.99 hardback).

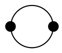
Chapter 1 : Introduction

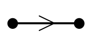
A **graph** is a pair (V, E), where V is a set and E is a subset of $V^{(2)} = \{\{x, y\} : x, y \in V, x \neq y\}$, the set of unordered pairs from V.

E.g., 

has $V = \{x_1, x_2, x_3, x_4, x_5\}$, $E = \{x_1x_2,\ x_1x_4,\ x_2x_3,\ x_3x_4,\ x_4x_5\}$.

Notes. 1. No loops: 

2. No multiple edges: 

3. No directed edges: 

Unless otherwise stated, V is *finite*.

We call $V = V(G)$ the **vertex set** of G, and $E = E(G)$ the **edge set** of G.

The **order** of G is $|G| = |V(G)|$, and the **size** is $e(G) = |E(G)|$.

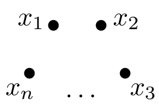
Often write, say, $x \in G$ to mean $x \in V(G)$.

Examples.

1. The **empty graph** E_n.

   $V = \{x_1, \ldots, x_n\}$, $E = \emptyset$.

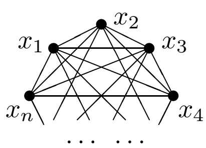
   So $|G| = n$, $e(G) = 0$.



2. The **complete graph** K_n.

   $V = \{x_1, \ldots, x_n\}$, $E = V^{(2)}$.

   So $|G| = n$, $e(G) = \binom{n}{2}$.

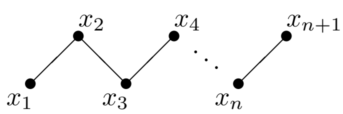


3. The **path** P_n of length n

   $V = \{x_1, \ldots, x_{n+1}\}$,

   $E = \{x_1x_2, \ldots, x_nx_{n+1}\} = \{x_i, x_{i+1} : 1 \leqslant i \leqslant n\}$.

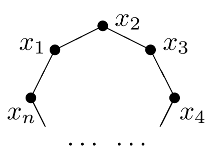
   So $|G| = n + 1$, $e(G) = n$.



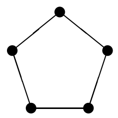
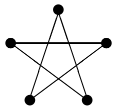
4. The **cycle** C_n of length n. n > 2 to prevent loops and multiple edges

   $V = \{x_1, \ldots, x_n\}$, $E = \{x_i, x_{i+1} : 1 \leqslant i \leqslant n - 1\} \cup \{x_nx_1\}$.

   So $|G| = n$, $e(G) = n$.



Graphs $G = (V, E)$ and $H = (V', E')$ are **isomorphic** if there is a bijection $f : V \to V'$ such that $xy \in E \iff f(x)f(y) \in E'$.

E.g.,  and  are isomorphic.

Say that $H = (V', E')$ is a **subgraph** of $G = (V, E)$ if $V' \subset V$ and $E' \subset E$.

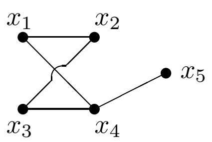
E.g., C_n is a subgraph of K_n.

For $xy \in E$, write $G - xy$ for the graph $(V, E \setminus \{xy\})$.

For $xy \notin E$, write $G + xy$ for the graph $(V, E \cup \{xy\})$.

If $xy \in E$, say x, y are **adjacent** or **neighbours**.

The **neighbourhood** of x is $\Gamma(x) = \{y \in V : xy \in E\}$, and the **degree** of x is $d(x) = |\Gamma(x)|$.

E.g., in  we have $\Gamma(x_4) = \{x_1, x_3, x_5\}$, so $d(x_4) = 3$.

If $V(G) = \{x_1, \ldots, x_n\}$, the **degree sequence** of G is $d(x_1), \ldots, d(x_n)$.

The **maximum degree** of G is $\Delta(G) = \max\limits_{1 \leqslant i \leqslant n} d(x_i)$.

The **minimum degree** of G is $\delta(G) = \min\limits_{1 \leqslant i \leqslant n} d(x_i)$.

E.g., the above example has degree sequence: 2, 2, 2, 3, 1. So $\delta(G) = 1$ and $\Delta(G) = 3$.

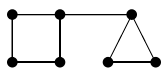
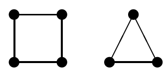
Say G is **regular of degree** k, or k**-regular**, if $d(x) = k$ for all $x \in V(G)$.

E.g., C_n is regular of degree 2, and K_n is $(n-1)$-regular.

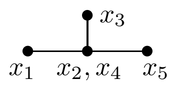
Unless otherwise stated, V is finite

In a graph G, an x-y **path** is a sequence $x_1, \ldots, x_k$ $(k \geqslant 1)$ of distinct vertices of G with $x_1 = x$, $x_k = y$ and $x_i x_{i+1} \in E$ for all $1 \leqslant i \leqslant k-1$. It has **length** $k - 1$.

Say G is **connected** if for all $x, y \in V$, there is an x-y path in G.

E.g.,  is connected, but  is not connected.

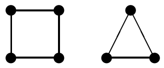
Write $x \sim y$ if there is an x-y path. Note, $\sim$ is an equivalence relation.

If $x_1 \ldots x_k$ and $x_k \ldots x_l$ are paths, then $x_1, \ldots, x_l$ need not be a path, e.g., 

– but it *contains* a path from x_1 to x_l. (E.g., choose minimal $1 \leqslant i \leqslant l$ for which there is some $k \leqslant j \leqslant l$ with $x_i = x_j$, and then $x_1, \ldots, x_i, x_{j+1}, \ldots, x_l$ is a path.)

The equivalence classes of $\sim$ are the **components** of G. So a connected graph has one component (or no vertices)

So the component of x is $\{y : \exists\ x\text{-}y \text{ path in } G\}$.

E.g.,  has two components.

A **walk** is a sequence $x_1, \ldots, x_k$ such that $x_i x_{i+1} \in E(G)$ for all $1 \leqslant i \leqslant k-1$.

Thus G has an x-y walk $\Longleftrightarrow$ G has an x-y path.

Trees

A graph is **acyclic** if it has no cycle.

A **tree** is a connected acyclic graph.

In a tree T, a vertex x with $d(x) = 1$ is called a **leaf** or **endvertex**.

Proposition 1. Let G be a graph. The following are equivalent.

(a) G is a tree.

(b) G is **minimal connected** (i.e., G connected, $G - xy$ disconnected for all $xy \in E$).

(c) G is **maximal acyclic** (i.e., G acyclic, $G + xy$ has a cycle for all $xy \notin E$).

Proof. (a) $\Rightarrow$ (b). If $G - xy$ is connected, then it has an x-y path P, but then Pyx is a cycle in G. ⨳

(b) $\Rightarrow$ (a). If G has a cycle C, choose $xy \in E(C)$. Then $G - xy$ is connected, because for all $a, b \in V$, we have an a-b path in G, and if it used edge xy then replace xy with $C - xy$ to obtain an a-b walk in $G - xy$. ⨳

For xy notin E,

(a) $\Rightarrow$ (c). We have x-y path P in G, so $G + xy$ contains the cycle Pyx.

(c) $\Rightarrow$ (a). If G is not connected, choose x, y in different components of G, then $G + xy$ is acyclic. ⨳ $\square$

Proposition 2. Let T be a tree ($|T| \geqslant 2$). Then T has a leaf.

Proof. Let $P = x_1 \ldots x_k$ be a longest path in T (which exists as we have a finite graph). Then $\Gamma(x_k) \subset P$ (by maximality of P), and $\Gamma(x_k) \cap P = \{x_{k-1}\}$, since T is acyclic. So $d(x_k) = 1$. $\square$

Remark. The proof actually shows there are $\geqslant 2$ leaves, as the same proof works for x_1.

Alternatively, if T has no leaf, "go for a walk". Choose some $x_1, x_2 \in E$ and then choose $x_3, x_4, x_5, \ldots$ as follows. Having chosen $x_1, \ldots, x_k$, choose $x_{k+1} \in \Gamma(x_k) \setminus \{x_{k-1}\}$. This must repeat (as T finite), giving a cycle. ⨳ $\square$.

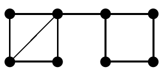
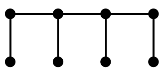
Proposition 3. Let T be a tree on n vertices ($n \geqslant 1$). Then $e(T) = n - 1$.

For G a graph, $W \subset V$, write $G[W]$ for the **subgraph spanned by** W. That is, $G[W]$ has vertex set W, edge set $E \cap W^{(2)}$. For $x \in V$, write $G - x$ for $G[V \setminus \{x\}]$.

Proof of 3. Induction on n. Done if $n = 1$.

Given T on n vertices ($n \geqslant 2$), choose a leaf x. Then $G-x$ is a tree, with $|G-x| = n-1$, so $e(G - x) = n - 2$, by induction. Thus $e(G) = e(G - x) + 1 = n - 1$. $\square$

For G a connected graph, a **spanning tree** of G is a subgraph T of G, with $V(T) = V(G)$, that is a tree.

E.g.,  has  as a spanning tree.

Clearly every connected G does have a spanning tree: just remove edges until we get a minimal connected graph.

For a non-inductive proof of Proposition 3 we'll show that any connected G has a spanning tree T on $n - 1$ edges (then done as if G is a tree, then the only spanning tree of G is G itself, e.g., by minimal connectedness).

For $x, y \in G$, the **distance** from x to y, $d(x, y)$ is the shortest length of any x-y path.

We construct our spanning tree in G, as follows.

Fix $x_0 \in G$, then for each $x \in V \setminus \{x_0\}$ choose a shortest x-x_0 path, say $xx' \ldots x_0$. (So $d(x', x_0) = d(x, x_0) - 1$.)

Let T consist of all the xx' for $x \in V \setminus \{x_0\}$. Then $e(T) = n - 1$.

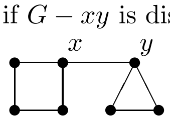
T is connected, as for all x, $xx'x'' \ldots$ forms a path to x_0.

T is acyclic. Suppose T has a cycle C. On C choose x at maximum distance from x_0, say $d(x, x_0) = k$. Then **both** neighbours of x are at distance $\leqslant k$ from x_0, $\#$ of construction. $\square$

Notes. 1. A **forest** is an acyclic graph.

Thus G is a forest $\iff$ every component of G is a tree.

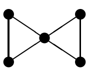
2. For G a connected graph, $xy \in E$, say xy is a **bridge** if $G - xy$ is disconnected.

Thus G is a tree $\implies$ every edge is a bridge. And, e.g., in 

xy is a bridge.

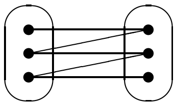
3. For G connected, $x \in G$, say x is a **cutvertex** if $G - x$ is disconnected.

Clearly, if G has a bridge then it has a cutvertex (for $|G| > 2$).

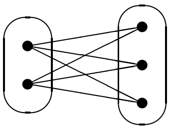
The converse is *false*, e.g., 

Bipartite Graphs

A graph G is **bipartite** on vertex classes V_1 and V_2 if V_1, V_2 **partition** V (i.e., $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$) and $E(G) \subset \{xy : x \in V_1, y \in V_2\}$ (i.e., no edges inside V_1, V_2).

E.g., a path: V_1  V_2

The **complete bipartite graph** $K_{n,m}$ has $|V_1| = n, |V_2| = m$, and $E = \{xy : x \in V_1, y \in V_2\}$. So $e(K_{n,m}) = nm$.

E.g., so $K_{2,3}$ is: 

Proposition 4. G bipartite $\iff$ G has no odd cycle.

A **circuit** in a graph G is a closed walk (i.e., a walk of the form $x_1 \ldots x_k$ where $x_1 = x_k$).

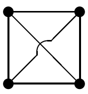
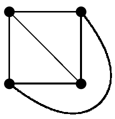
Note that if G has an odd circuit, then it has an odd cycle. Indeed, if $x_1, \ldots x_k x_1$ is an odd circuit and $x_i = x_j$ (some $1 \leqslant i \leqslant j \leqslant k$), then one of $x_i x_{i+1} \ldots x_j$ and $x_j x_{j+1} \ldots x_k x_1 \ldots x_i$ is an odd circuit. Then done by induction on k.
As k < length of original circuit and as we assume every odd circuit with length less than original circuit has an odd cycle (SPI). For base case we can consider odd circuit of length -3.

Proof of 4. ($\Rightarrow$) The vertices in a cycle must alternate between V_1 and V_2.

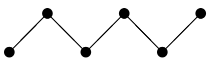
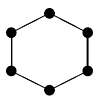
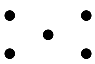
($\Leftarrow$) Wlog G is connected (as if each component of G is bipartite, then so is G). Fix $x_1 \in G$ and put $V_1 = \{x \in G : d(x, x_1) \text{ even}\}$, $V_2 = \{x \in G : d(x, x_1) \text{ odd}\}$. (The only possible choices.) If we had $x, y \in V_1$ or $x, y \in V_2$ with $xy \in E$, then xy together with shortest paths from x to x_1 and y to x_1 gives an odd circuit. $\times$ $\square$

Planar Graphs

A graph G is **planar** if it can be drawn in the plane without crossing edges. A **plane graph** is such a drawing.

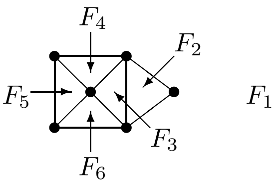
E.g.,  is not a plane graph, but  is. So K_4 is planar.

Examples.

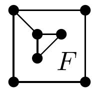
1. Any path is planar: 
2. Any cycle is planar: 
3. The empty graph is planar: 
4. K_4 is planar (as shown above).

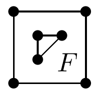
Which graphs are planar? How do we check if a given graph is not planar?

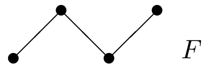
Given a graph G, $\mathbb{R}^2 - G$ (i.e., the plane with G removed) splits up into connected regions called **faces**. The **boundary** of a face consists of the vertices and edges that touch it.

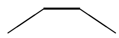
E.g., 

Here, the boundaries of F_2, F_3, F_4, F_5, F_6 consist of 3-cycles, and of F_1 consists of a 5-cycle.

Warning. 1. The boundary of a face need not be a cycle: 

2. The boundary of a face need not even be connected: 

3. The two faces on either side of an edge may be the same: 

Formal bit:  is officially 

Let $x, y \in \mathbb{R}^2$, $x \neq y$. A **polygonal arc** from x to y is a finite union of (closed) straight line segments, $\overline{x_1x_2} \cup \overline{x_2x_3} \cup \ldots \cup \overline{x_{n-1}x_n}$, with $x = x_1, y = x_n$, that are disjoint except for $\overline{x_{i-1}x_i} \cap \overline{x_ix_{i+1}} = \{x_i\}$.

For a graph G with vertex set $\{v_1, \ldots, v_n\}$, a **drawing** of G consists of distinct points $x_1, \ldots, x_n \in \mathbb{R}^2$ together with, for each $v_iv_j \in E$, a polygonal arc p_{ij} from x_i to x_j such that $p_{ij} \cap p_{kl} = \emptyset$ if i, j, k, l distinct, and $p_{ij} \cap p_{jk} = \{x_j\}$.

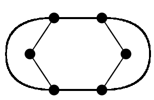
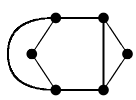
For $x, y \in \mathbb{R}^2 - G$, write $x \sim y$ if there exists a polygonal arc in $\mathbb{R}^2 - G$ from x to y. The components (equivalence classes) of $\sim$ are called the **faces**. The **boundary** of a face consists of G intersect ~~its closure.~~
the closure of the face

We'll assume various facts about $\mathbb{R}^2$, like "a cycle has two faces", or "a boundary of a face consists of vertices and (whole) edges". (Proved by induction on the total number of straight line segments in the drawing.)

End of formal bit!

Remarks. 1. Every tree is planar, with exactly one face. (Proof by induction, via removing a leaf.)

2. A planar graph may have genuinely different drawings.

E.g.,  has a face with a 6-cycle as boundary, but  doesn't.

Is the *number* of faces fixed? Yes:

Theorem 5 (Euler's Formula). Let G be a connected plane graph with n vertices, m edges and f faces. Then $n - m + f = 2$.

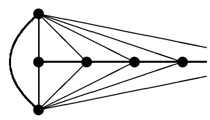
Note. We need G to be connected. E.g., E_n has n vertices, 0 edges, 1 face.

Proof of 5. If G has no cycles, then G is a tree (acyclic and connected), so $m = n - 1$, $f = 1$, so $n - m + f = 2$.

If G has a cycle, choose an edge e that is on a cycle. Then $G - e$ is connected (as e was on a cycle), and has n vertices, $m - 1$ edges, and $f - 1$ faces (as e was on a cycle). But by induction, $n - (m-1) + (f-1) = 2$, so $n - m + f = 2$. $\square$

Theorem 6. Let G be a plane graph on n vertices ($n \geqslant 3$) with m edges. Then $m \leqslant 3n - 6$.

Notes. 1. This is a linear bound on m – whereas in general a graph could have anything up to $\binom{n}{2} = \frac{n^2 - n}{2}$ edges.

2. This bound is best possible, e.g.:  $\cdots$ (line of $n - 2$ vertices)

Here, the number of edges is $(n-3) + 2(n-2) + 1 = 3n - 6$.

Proof of 6. Wlog, G connected. (If not, add edges to make it so.) Thus $n - m + f = 2$.

If we sum, for each face f, the number of edges in the boundary of f, we obtain $\geqslant 3f$, because each face has $\geqslant 3$ edges in its boundary (for $n \geqslant 3$ – theorem trivial for $n \leqslant 2$). But we also obtain $\leqslant 2m$ (since each edge counted $\leqslant 2$).

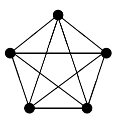
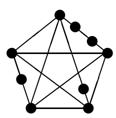
Thus $3f \leqslant 2m$, i.e. $f \leqslant \frac{2m}{3}$. So $n - m + \frac{2m}{3} \geqslant 2$, so $\frac{m}{3} \leqslant n - 2$. $\square$

Corollary 7. K_5 is not planar.

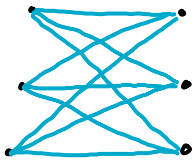
Proof. $n = 5$, $m = 10$, so $10 \leqslant 15 - 6$. $\lightning$ $\square$

Hence any graph containing K_5 is not planar. E.g., K_n, $n \geqslant 5$.

A **subdivision** of a graph G is obtained by replacing edges of G with disjoint paths.

E.g., K_5:  Subdivided K_5: 

So any subdivision of K_5 is not planar.

Proposition 8. $K_{3,3}$ is not planar. 

Remarks. 1. We cannot use Theorem 6, as $n = 6$, $m = 9$ holds.

2. So we cannot, e.g., connect up 3 houses to 3 utilities without pipes crossing.

Proof of 8. $K_{3,3}$ is triangle-free, so if drawn in the plane, we must have $\geqslant 4$ edges on the boundary of each face. So $4f \leqslant 2m$, i.e. $f \leqslant m/2$.

So $n - m + \dfrac{m}{2} \geqslant 2$, i.e. $m \leqslant 2(n-2)$. But $n = 6$, $m = 9$. ⨳ □

Remark. The **girth** of a graph is the length of a shortest cycle (with girth ∞ if the graph has no cycle).

The proof above yields: if G is planar, girth $\geqslant g$, then $m \leqslant \max\left(\dfrac{g}{g-2}(n-2), \underset{\substack{\uparrow \\ \text{silly bit}}}{n-1}\right)$.

$gf \leq 2m$ thus $n - m + \dfrac{2}{g}m \geq 2$ so ...

Corollary 9. If G contains a subdivision of K_5 or $K_{3,3}$, then G is not planar. □

Kuratowski's Theorem says that these are the *only* obstruction to being planar:

G planar $\iff G \not\supset$ subdivided K_5 or $K_{3,3}$.

So, to show G planar: draw it.

To show G not planar: find a subdivided K_5 or $K_{3,3}$.

Chapter 2 : Connectivity and Matchings

Let G be a bipartite graph, with vertex classes X and Y. A **matching** from X to Y is a set $\{xx' : x \in X\}$ of edges of G, such that $x \mapsto x'$ is injective. In other words, it consists of $|X|$ *independent* edges (i.e., no vertices in common).

When does G have a matching?

"Matchmaker" terminology: let $X = \{\text{boys}\}$, $Y = \{\text{girls}\}$, with x joined to y if x knows y. Can we pair up each boy with a girl he knows?

Clearly impossible if $d(x) = 0$ for some $x \in X$, or if there are (distinct) $x_1, x_2 \in X$ with $\Gamma(x_1) = \Gamma(x_2) = \{y\}$, some $y \in Y$.

For $A \subset X$, write $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$, then we clearly need $|\Gamma(A)| \geqslant |A|$ for all $A \subset X$.

Are there any other obstructions to a matching?

Theorem 1 (Hall's "Marriage" Theorem). Let G be a bipartite graph, with vertex classes X, Y. Then G has a matching from X to $Y \iff |\Gamma(A)| \geqslant |A|$ for all $A \subset X$. (This is "Hall's condition".)

Proof (1). $(\Rightarrow)$ Trivial.

$(\Leftarrow)$ Induction on $|X|$. $|X| = 1$ is trivial.

We have G, vertex classes X, Y, with $|X| > 1$, such that $|\Gamma(A)| \geqslant |A|$ for all $A \subset X$.

Question: do we have $|\Gamma(A)| > |A|$ for all $A \subset X$ $(A \neq \emptyset, X)$?

If yes, choose any $x \in X$ and any $y \in \Gamma(x)$. Let $G' = G - x - y$.

Claim. G' has a matching from $X \setminus \{x\}$ to $Y \setminus \{y\}$.

Proof of claim. We need $|\Gamma_{G'}(A)| \geqslant |A|$ for all $A \subset X \setminus \{x\}$ $(A \neq \emptyset)$.
But $|\Gamma_G(A)| \geqslant |A| + 1$, so $|\Gamma_{G'}(A)| \geqslant |A| + 1 - 1$.

If no, then there is some $A \subset X$ with $|\Gamma(A)| = |A|$. Let $G' = G[A \cup \Gamma(A)]$ and $G'' = G[(X \setminus A) \cup (Y \setminus \Gamma(A))]$.

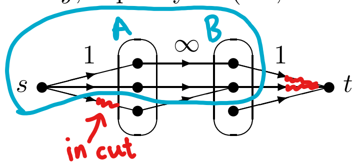
Claim 1. G' has a matching from A to $\Gamma(A)$.

Proof of claim. For $B \subset A$, have $|\Gamma_{G'}(B)| \geqslant |B|$, because $\Gamma_{G'}(B) = \Gamma_G(B)$.
So by induction G' has a matching from A to B

Claim 2. G'' has a matching from $X \setminus A$ to $Y \setminus \Gamma(A)$.

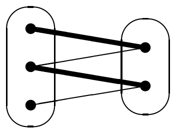
Proof of claim. For $B \subset X \setminus A$, consider $A \cup B$. Have $|\Gamma_G(A \cup B)| \geqslant |A| + |B|$, so $|\Gamma_G(A \cup B) \setminus \Gamma_G(A)| \geqslant |B|$, so $|\Gamma_{G''}(B)| \geqslant |B|$ so by induction G'' has a matching □
as $|\Gamma(A)| = |A|$.

Proof (2). Form a directed network as follows: add a source s joined to each $x \in X$ by an edge of capacity 1, and a sink t joined to each $y \in Y$ by an edge of capacity 1, and also direct each $xy \in E(G)$ from x to y, capacity ∞ (i.e., some huge number).



Then an integer-valued flow of size $|X|$ is precisely a matching from X to Y. So, by integrality theorem of max-flow min-cut, we just need to show that every cut has size $\geqslant |X|$. So given a cut $\{s\} \cup A \cup B$ ($A \subset X$, $B \subset Y$), wlog $\Gamma(A) \subset B$ (else capacity ∞), so capacity $= |X| - |A| + |B| \geqslant |X|$ as $|B| \geqslant |A|$, since $B \supset \Gamma(A)$. □

A **matching of deficiency** d in a bipartite graph, vertex classes X and Y, consists of $|X|-d$ independent edges.

E.g.,  – a matching of deficiency 1.

Corollary 2 (Defect Hall). Let G be a bipartite graph, vertex classes X, Y. Then G has a matching of deficiency d from X to Y $\iff$ $|\Gamma(A)| \geqslant |A| - d$ for all $A \subset X$.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Form G' by adding d new points to Y, each joined to all of X. Then $|\Gamma_{G'}(A)| \geqslant |A|$ for all $A \subset X$, so by Hall there is a matching in G', giving a matching of deficiency d in G. □

(In terms of boys and girls: add d imaginary girls to Y, known to all boys. Hall gives a matching, and at most d boys are paired with imaginary girls, so at least $|X| - d$ are paired with real girls.)

Let $S_1, \ldots, S_n$ be sets. A **transversal** for $S_1, \ldots, S_n$ consists of distinct points $x_1, \ldots, x_n$ with $x_i \in S_i$ for all i.

E.g., $\{1,2,3\}$, $\{3,4\}$, $\{4,5\}$, $\{5\}$ has a transversal $1,3,4,5$.

When is there a transversal?

Corollary 3. Sets $S_1, \ldots, S_n$ have a transversal $\iff$ $\left|\bigcup_{i \in A} S_i\right| \geqslant |A|$ for all $A \subset \{1, \ldots, n\}$.

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Wlog all S_i are finite. Form a bipartite graph as follows: $X = \{1, \ldots, n\}$, $Y = S_1 \cup \ldots \cup S_n$, with $i \in X$ joined to $j \in Y$ if $j \in S_i$. Thus a transversal is precisely a matching from X to Y. But for $A \subset X$, have $\left|\bigcup_{i \in A} S_i\right| \geqslant |A|$, i.e. $|\Gamma(A)| \geqslant |A|$, so done by Hall. □

Remarks. 1. Actually, corollary 3 is equivalent to Hall, since a matching in a bipartite graph G, vertex classes $\{x_1,\ldots,x_n\}$ and Y is exactly a transversal for $\Gamma(x_1),\ldots,\Gamma(x_n)$.

2. There is also a defect form: there exists a transversal for all but d of $S_1,\ldots,S_n \iff \left|\bigcup_{i\in A} S_i\right| \geqslant |A| - d$ for all $A \subset \{1,\ldots,n\}$.

A typical application of Hall

Let G be a finite group, H a subgroup of G. Have the left cosets $L_1,\ldots,L_k$ ($k = |G|/|H|$), say $g_1H,\ldots,g_kH$, and have the right cosets $R_1,\ldots,R_k$, say $Hg_1',\ldots,Hg_k'$.

Can we choose representatives of the left cosets that are also representatives of the right cosets – that is, $g_1,\ldots,g_k$ such that $g_1H,\ldots,g_kH$ are the left cosets and $Hg_1,\ldots,Hg_k$ are the right cosets? In other words, we seek a permutation π of $\{1,\ldots,k\}$ such that $L_i \cap R_{\pi(i)} \neq \emptyset\ \forall\, i$. ("Pair up each left coset with a right coset that it meets.")

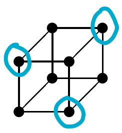
Thus we seek a matching from X to Y in G, where $X = \{L_1,\ldots,L_k\}$, $Y = \{R_1,\ldots,R_k\}$, and L_i is joined to R_j if $L_i \cap R_j \neq \emptyset$. Thus, by Hall, need to check that $|\Gamma(A)| \geqslant |A|\ \forall\, A \subset X$. But $\left|\bigcup_{i\in A} L_i\right| = |A||H|$, so $\bigcup_{i\in A} L_i$ must meet at least $|A|$ of the R_j, as $|R_j| = |H|$ for all j. So done. $\square$

Connectivity

Idea. How "connected" is a graph?

A tree is connected, but we can remove a point to disconnect it.

A cycle G is connected, and $G - x$ is connected for all $x \in G$ (although we can remove two points to disconnect it).

 — $G - x - y$ still connected $\forall\, x, y \in G$.

For G connected, $|G| > 1$, the **connectivity** of G, $\kappa(G)$, is the smallest $|S|$ such that $S \subset V(G)$ and $G - S$ is disconnected, or a single point (because of complete graphs).

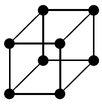
Say G is k**-connected** if $\kappa(G) \geqslant k$.

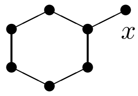
Thus G is k-connected $\iff$ no set of size $< k$ disconnects G (or makes it a single point).

Equivalently, G is k-connected $\iff |G| > k$ and no set of size $< k$ disconnects G.

Thus: G 1-connected $\iff G$ is connected ($|G| > 1$)
G 2-connected $\iff G$ has no cutvertex ($|G| > 2$)
G 3-connected $\iff G$ cannot be disconnected by removing 2 vertices ($|G| > 3$)

Examples.

1. Any tree T is *not* 2-connected ($|T| > 1$)
2. A cycle C_n is 2-connected, but not 3-connected.
3.  is 3-connected.
4. K_n is $(n-1)$-connected.

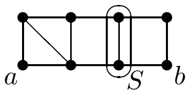
Warning. We can have $\kappa(G - x) > \kappa(G)$, e.g. 

Remark. Always have $\kappa(G) \leqslant \delta(G)$ = minimum degree in G: choose x with $d(x) = \delta(x)$ and remove $S = \Gamma(x)$. Then $|S| = \delta(x)$ and G is disconnected (or a single point).

We know: G connected $\Longrightarrow \forall\, a, b$ there is an a-b path in G.

It would be nice if: G k-connected $\Longrightarrow \forall\, a, b$ there are k *independent* (i.e., disjoint vertices apart from a, b) a-b paths.

For G connected, a, b distinct vertices of G, say $S \subset V(G) \setminus \{a, b\}$ **separates** a from b, or is an a-b **separator**, if a, b are in different components of $G - S$ (i.e., every a-b path meets S).

E.g., 

Theorem 4 (Menger's Theorem). Let G be a connected graph, and a, b distinct non-adjacent vertices of G. If all a-b separators have size $\geqslant k$ then there exists a family of k independent a-b paths.

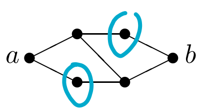
Remarks. 1. Converse trivial: any separator contains $\geqslant 1$ point from each of the k paths.

2. Equivalent form: minimum size of an a-b separator = maximum number of independent a-b paths.

3. Need "non-adjacent", else no separators!

4. Menger generalises Hall. Given bipartite G on X, Y, with $|\Gamma(A)| \geqslant |A|\ \forall\, A \subset X$, form G' by adding s (joined to all of X) and t (joined to all of Y). Then a matching is precisely a family of X independent s-t paths, so by Menger, it's enough to check that every separator has size $\geqslant |X|$. So let $S = A \cup B$ be a separator, where $A \subset X, B \subset Y$. Then $\Gamma(X \setminus A) \subset B$, so $|A| + |B| \geqslant |\Gamma(X \setminus A)| + |A| \geqslant |A| + |X \setminus A| = |X|$. $\square$

5. *Cannot* prove Menger by choosing one point on each path in a maximum-sized "family of independent a-b paths".

E.g., 

Proof (1). We may assume that $k \geqslant 2$, as the theorem is trivial for $k = 1$.

Let k be the minimum size of any a-b separator. We need k independent paths from a to b. If not possible, take a minimal counterexample (say minimal k, then minimal $e(G)$ for k). Let S be an a-b separator, $|S| = k$.

Suppose first that $S \not\subset \Gamma(a)$, $S \not\subset \Gamma(b)$.

Form G' from G by replacing the component of $G - S$ containing a by a single point a' joined to all of S. Then $e(G') < e(G)$, as $S \not\subset \Gamma(a)$. In G', there is no a'-b separator of size $< k$ (else the same set separates a and b in G), so by minimality of $e(G)$ we have k independent a'-b paths in G'. I.e., we have k paths $B_1, \ldots, B_k$ from b to S, disjoint except at b.

Similarly, we have k paths $A_1, \ldots, A_k$ from a to S, disjoint except at a. No A_i can meet a B_j except on S (else S is not a separator), so put the A_i and B_j together to form k independent a-b paths. ⋇

Now suppose that every a-b separator S of size k has $S \subset \Gamma(a)$ or $S \subset \Gamma(b)$.

We cannot have any $x \in \Gamma(a) \cap \Gamma(b)$, because if so then consider $G - x$. All a-b separators in $G - x$ have size $\geqslant k - 1$ (as with x, they separate a from b in G), so by minimality of k there exist $k - 1$ independent paths in $G - x$. Now add axb, and obtain k independent paths in G. ⋇

Choose a shortest a-b path, say $ax_1 \ldots x_r b$ ($r \geqslant 2$) and consider $G - x_1x_2$. In $G - x_1x_2$ have a separator S of size $k - 1$ (by minimality), so $S \cup \{x_1\}$ and $S \cup \{x_2\}$ are separators in G. Since $x_1 \notin \Gamma(b)$, we must have $S \cup \{x_1\} \subset \Gamma(a)$. And since $x_2 \notin \Gamma(a)$, we must have $S \cup \{x_2\} \subset \Gamma(b)$. So $S \subset \Gamma(a) \cap \Gamma(b)$, contradicting $\Gamma(a) \cap \Gamma(b) = \emptyset$. (If $S = \emptyset$ then $k = 1$.) ⋇ □

Proof (2). We'll apply vertex capacity form of max-flow min-cut. Form a directed network by replacing each edge xy by directed edges $\vec{xy}$ and $\vec{yx}$ and give each capacity 1. Then an integer-valued flow of size k is exactly a family of k independent a-b paths. So, by integrality form of max-flow min-cut, just need that every vertex cut set has size $\geqslant k$, i.e. every a-b separator has size $\geqslant k$. □

Corollary 5. Let G be a graph, $|G| > 1$. Then G is k-connected $\iff$ for all $a, b \in G$, there exist k independent a-b paths.

Remark. Sometimes also called "Menger's Theorem".

Proof. ($\Leftarrow$) Certainly G is connected, with $|G| > k$. Also, no set of size $< k$ can disconnect a from b (else choose a, b in different components).

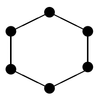
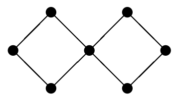
($\Rightarrow$) If a, b are non-adjacent, we are done by Menger (as G is k-connected, so no set of size $< k$ can separate a, b).

If a, b are adjacent, let $G' = G - ab$. Then G' is $(k-1)$-connected (as G is k-connected), so by Menger there exist $k - 1$ independent a-b paths in G. Now add edge ab as a k^{th} path. □

For G connected, $|G| > 1$, the **edge connectivity** $\lambda(G)$ of G is the smallest size of a set $W \subset E(G)$ such that $G - W$ is disconnected.

Say G is **k-edge-connected** if $\lambda(G) \geqslant k$.

Thus: G 1-edge-connected $\iff$ G connected ($|G| > 1$)
G 2-edge-connected $\iff$ G connected and has no bridge ($|G| > 1$)
(Note, different from 2-connected: while bridge $\Rightarrow$ cutvertex, the converse is false.)

E.g.,  has $\lambda(G) = 2$, and  has $\lambda(G) = 2$, $\kappa(G) = 1$.

We always have $\lambda(G) \leqslant \delta(G)$.

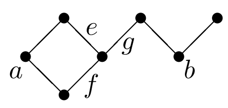
Theorem 6 (Edge version of Menger). Let G be a connected graph, and a, b distinct vertices of G. Then the minimum size of $W \subset E(G)$ separating a from b equals the maximum number of edge-disjoint a-b paths.

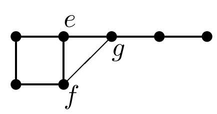
Idea. "View edges of G as vertices, and apply vertex Menger."

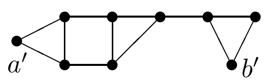
The **line graph** $L(G)$ of a graph G has vertex set $E(G)$, with e joined to f if they meet (share a vertex) in G.

Proof of 6. Form G' from $L(G)$ by adding new vertices a', b' to $L(G)$ with edges $a'e$ for each $e \in E(G) = V(L(G))$ with $a \in e$, and edges $b'e$ for each $e \in E(G)$ with $b \in e$.

Then $\exists$ a-b path in $G \iff \exists$ a'-b' path in G', and deleting edges in G corresponds to deleting vertices (not a', b') in G'. So done by vertex Menger applied to G'. $\square$

E.g., if G is 

then $L(G)$ is 

and G' is 

Corollary 7. Let G be a graph, $|G| > 1$. Then G is k-edge-connected $\iff$ for all distinct $a, b \in G$ there exist k edge-disjoint a-b paths.

Proof. ($\Leftarrow$) Obvious.

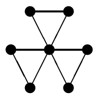
($\Rightarrow$) Edge form of Menger. $\square$

Remark. Or prove Theorem 6 and Corollary 7 by max-flow min-cut (usual edge capacity version).

Chapter 3 : Extremal Problems

An **Euler circuit**, or **Eulerian circuit**, in a graph G is a circuit passing through each edge exactly once. I.e., $x_1, \ldots, x_k$ ($x_k = x_1$) such that if $xy \in E$ then there is a unique $1 \leqslant i \leqslant k-1$ with $xy = x_i x_{i+1}$.

Say G is **Eulerian** if it has an Euler circuit.

E.g.,  is Eulerian. But any graph with a bridge is not Eulerian.

Which graphs are Eulerian?

Proposition 1. Let G be a connected graph. Then G is Eulerian $\Longleftrightarrow$ $d(x)$ is even for all $x \in G$.

Remark. Hence G is Eulerian $\Longleftrightarrow$ all degrees are even and at most one component contains an edge.

Proof of 1. ($\Rightarrow$) If an Eulerian circuit passes through x k times, then $d(x) = 2k$.

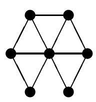
($\Leftarrow$) Use induction on $e(G)$. Done if $e(G) = 0$.

Given G connected, with $e(G) > 0$ and $d(x)$ even for all $x \in G$, suppose that G is not Eulerian and let C be a largest circuit in G with no edges repeated. Note that $e(C) > 0$ as G has a cycle (since $d(x) \geqslant 2$ for all $x \in G$, so G is not a tree).

Let H be a component of $G - E(C)$ with $e(H) > 0$. Then H is connected and $d_H(x)$ is even for all $x \in H$ (as $d_G(x)$ and $d_C(x)$ are even for all x). So, by the induction hypothesis, H has an Euler circuit C'. But now C and C' are edge disjoint circuits that share a vertex (we do have $V(H) \cap V(C) \neq \emptyset$ as G is connected), so we can combine them to form a circuit longer than C. $\divideontimes$ $\square$

Let G be a graph of order n. Say G is **Hamiltonian** if it has a cycle of length n (i.e., a cycle through all vertices). Such a cycle is called a **Hamilton cycle**.

There is no nice "$\Leftrightarrow$" condition known for Hamiltonicity. We can't have a parity type of condition, as G Hamiltonian $\Rightarrow$ $G + xy$ Hamiltonian.

E.g.,  is Hamiltonian (use the perimeter).

But any graph with a cutvertex is not Hamiltonian.

How "large" does a graph on n vertices have to be to ensure it is Hamiltonian?

Silly question. How many edges do we need to ensure that G is Hamiltonian?

This is silly because any x with $d(x) = 1$ stops G from being Hamiltonian, so take G to be the **complement** (i.e., $\overline{G} = (V, V^{(2)} \setminus E)$, for $G = (V, E)$) of $\{x_1x_2, x_1x_3, \ldots, x_1x_{n-1}\}$. Then $e(G) = \binom{n}{2} - (n-2)$, but G is not Hamiltonian.

Better question. What $\delta(G)$ ensures that G is Hamiltonian?

E.g., n even, take two disjoint copies of $K_{n/2}$. Then $\delta(G) = \dfrac{n}{2} - 1$, but G is not Hamiltonian.
E.g., n odd, take two copies of $K_{(n+1)/2}$, meeting at a single point. Then $\delta(G) = \dfrac{n-1}{2}$, but there is no Hamilton cycle.

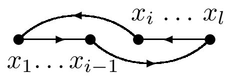
Theorem 2. Let G be a graph of order n ($n \geqslant 3$) Then $\delta(G) \geqslant \dfrac{n}{2} \Longrightarrow G$ is Hamiltonian.

Proof. G is connected, since if x, y are non-adjacent vertices then $\Gamma(x), \Gamma(y) \subset V \setminus \{x, y\}$, so $\Gamma(x) \cap \Gamma(y) \neq \emptyset$, as $|\Gamma(x)|, |\Gamma(y)| \geqslant n/2$ but $|V \setminus \{x, y\}| < n$.

Let $x_1 x_2 \ldots x_l$ be a longest path in G. (Note $l \geqslant 3$ since G is connected and $n \geqslant 3$.)

Wlog G has no cycle of length l, because: if $l = n$, we're done; and if $l < n$ then as G is connected, there exists $x \notin$ cycle adjacent to some $y \in$ cycle, which yields a path on $l + 1$ points.

Thus $x_l x_1 \notin E$. Moreover we cannot have $2 \leqslant i \leqslant l$ with $x_1 x_i, x_{i-1} x_l \in E$, or else we have a cycle:



Now, $\Gamma(x_1) \subset \{x_2, \ldots, x_l\}$ and $\Gamma(x_l) \subset \{x_1, \ldots, x_{l-1}\}$ (by maximality of the path), and, by above, $\Gamma(x_1)$ is disjoint from $\Gamma_+(x_l) = \{x_i : 2 \leqslant i \leqslant l : x_{i-1} \in \Gamma(x_l)\}$. But $|\Gamma(x_1)|,\ |\Gamma_+(x_l)| \geqslant n/2$ and $\Gamma(x_1),\ \Gamma_+(x_l) \subset \{x_2, \ldots, x_l\}$. $\lightning$ $\square$

Remark. We didn't use the full strength of $\delta(G) \geqslant n/2$. We used only that x, y non-adjacent $\Rightarrow d(x) + d(y) \geqslant n$.

Similarly,

Proposition 3. Let G be a graph of order n ($n \geqslant 3$). Then G connected and $\delta(G) \geqslant \dfrac{k}{2}$, where $k < n \Longrightarrow G$ has a path of length k.

Note. We do need $k < n$: e.g., $G = K_k$. And we do need G to be connected: e.g., two disjoint copies of K_k.

Proof of 3. Let $x_1 \ldots x_l$ be a longest path in G. ($l \geqslant 3$ since G is connected and $n \geqslant 3$.) Suppose that $l < k$. Then, as in the proof of Theorem 2: wlog G has no l-cycle, and thus $\Gamma(x_1)$ and $\Gamma_+(x_l)$ are disjoint subsets of $\{x_2, \ldots, x_l\}$, each of size $\geqslant k/2$. $\lightning$ $\square$

Theorem 4. Let G be a graph of order n with $e(G) > \dfrac{n(k-1)}{2}$. Then G contains a path of length k.

Remarks. 1. Equivalently, $G \not\supset P_k \ \Rightarrow\ e(G) \leqslant \dfrac{n(k-1)}{2}$.

2. This cannot be improved, if k divides n: e.g., n/k disjoint K_k.

Proof of 4. Use induction on n. Done if $n \leqslant 2$.

Given G, with $|G| = n \geqslant 3$ and $G \not\supset P_k$, we want $e(G) \leqslant \frac{1}{2}n(k-1)$.

Wlog, G is connected – if not, then components $G_1, \ldots, G_r$ of orders $n_1, \ldots, n_r$ have $e(G_i) \leqslant \frac{1}{2}n_i(k-1)$ by induction, whence $e(G) \leqslant \sum \frac{1}{2}n_i(k-1) = \frac{1}{2}n(k-1)$.

Thus G has a vertex x of degree $\leqslant \frac{1}{2}(k-1)$, by Proposition 3.
(Note: we may assume that $k < n$, as if $n \leqslant k$ then $e(G) \leqslant \frac{1}{2}n(k-1)$ trivially).

But then $G - x$ is on $n-1$ vertices and has no P_k, so $e(G-x) \leqslant \frac{1}{2}(n-1)(k-1)$ by induction, whence $e(G) \leqslant \frac{1}{2}(n-1)(k-1) + d(x) \leqslant \frac{1}{2}n(k-1)$. $\square$

Remark. Both Theorem 2 and Theorem 4 are *extremal* results. They answer "how large can a graph be with a certain property?" Often this property is non-containment of a given graph. E.g., how big can $e(G)$ be if G is triangle-free?

Turán's Theorem

How many edges guarantee a K_r?

Equivalently, how many edges can a graph on n vertices have if it does not have K_r as a subgraph?

For $r = 3$, we would try $G = K_{a,b}$ where $a + b = n$. Take $a = b = n/2$ if n is even, and $a = (n+1)/2$, $b = (n-1)/2$ if n is odd.

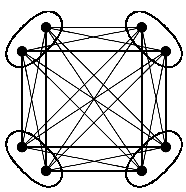
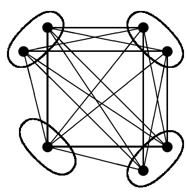
Say G is **k-partite** on classes $V_1, \ldots, V_k$ if $V_1, \ldots, V_k$ partition V and $G[V_i] = \emptyset$ for all i.

So G $(r-1)$-partite $\Rightarrow G \not\supset K_r$ (else $\geqslant 2$ vertices of K_r in some V_i).

If in addition $e(G) = \{xy : x \in V_i, y \in V_j, \text{ some } i \neq j\}$, then say G is **complete k-partite**.

The **Turán graph** $T_k(n)$ is the complete k-partite graph on n vertices, with vertex classes $V_1, \ldots, V_k$, where $|V_1|, \ldots, |V_k|$ are as equal as posisble.

(Integers are "as equal as possible" if $|a_i - a_j| \leqslant 1 \; \forall \, i, j$.)

E.g. $T_4(8)$:  and $T_4(7)$: 

Certainly $T_{r-1}(n) \not\supset K_r$ (as $T_{r-1}(n)$ is $(r-1)$-partite), and $T_{r-1}(n)$ is maximal K_r-free: if we add any edge then we make a K_r.

If k divides n then all classes have size n/k, so $d(x) = n - n/k = n(1 - 1/k)$ for all x. In general, the classes have size $\lceil n/k \rceil$ or $\lfloor n/k \rfloor$, so all degrees are $n - \lceil n/k \rceil$ or $n - \lfloor n/k \rfloor$.

Notes. 1. To obtain $T_k(n-1)$ from $T_k(n)$, remove a point from a largest class, i.e. a point of minimum degree.

2. And to obtain $T_k(n)$ from $T_k(n-1)$, add a point to a smallest class.

Theorem 5 (Turán's Theorem). Let G be a graph on n vertices. Then $e(G) > e(T_{r-1}(n))$ $\implies G \supset K_r$.

Remarks. 1. This is best possible, as $T_{r-1}(n) \not\supset K_r$.

2. If we knew G was $(r-1)$-partite, done by some kind of AM-GM inequality. But there's no reason why G should be $(r-1)$-partite: e.g. C_5 is K_3-free, but not bipartite.

3. It looks like the proof has to be fiddly, as $e(T_{r-1}(n))$ is a complicated function of n and r.

Proof. Key idea: we'll strengthen the theorem to make it easier to prove. We'll prove: $|G| = n$, $e(G) = e(T_{r-1}(n))$, $G \not\supset K_r \implies G \cong T_{r-1}(n)$. (This certainly implies Theorem 5, by maximality of $T_{r-1}(n)$.)

Use induction on n: $n \leqslant r-1$ is trivial.

We have G with $|G| = n$, $e(G) = e(T_{r-1}(n))$, $G \not\supset K_r$.

Claim. $\delta(G) \leqslant \delta(T_{r-1}(n))$.

Proof of claim. We have $e(G) = e(T_{r-1}(n))$ and so $\displaystyle\sum_{y \in G} d_G(y) = \sum_{y \in T_{r-1}(n)} d_{T_{r-1}(n)}(y)$.

But the $d_{T_{r-1}(n)}(y)$ are as equal as possible, so $\delta(G) \leqslant \delta(T_{r-1}(n))$, as claimed.

Choose $x \in G$ with $d(x) = \delta(G)$ and let $G' = G - x$. Then $|G'| = n-1$, $G' \not\supset K_r$, and $e(G') = e(G) - \delta(G) \geqslant e(T_{r-1}(n)) - \delta(T_{r-1}(n)) = e(T_{r-1}(n-1))$ (by note 1 above). Thus $\delta(G) = \delta(T_{r-1}(n))$ and $G' \cong T_{r-1}(n-1)$ by the induction hypothesis.

Let G' have vertex classes $V_1, \ldots, V_{r-1}$. We cannot have x joined to a point in every V_i, else $G \supset K_r$, so $\Gamma(x) \cap V_i = \emptyset$ for some i. But $d(x) = n - 1 - \min |V_i|$ (by note 2 above). So $\Gamma(x) = \bigcup_{j \neq i} V_j$, for some i with $|V_i|$ minimal. Thus G is complete $(r-1)$-partite, with vertex classes V_j $(j \neq i)$ and $V_i \cup \{x\}$. Thus $G \cong T_{r-1}(n)$. $\square$

There are many other proofs of Turán.

The Problem of Zarankiewicz

"Bipartite version of Turán's Theorem" : how many edges can a bipartite graph G (with n vertices in each class) have if $G \not\supset K_{t,t}$?

Write $Z(n,t)$ for this maximum. How large is $Z(n,t)$?

Theorem 6. Let $t \geqslant 2$. Then $Z(n,t) \leqslant n^{2-1/t}t^{1/t} + nt$. In particular, $Z(n,t) \leqslant 2n^{2-1/t}$ for n sufficiently large (i.e., for all $n \geqslant n_0(t)$).

Proof. Let G be bipartite, vertex classes X, Y, where $|X| = |Y| = n$, with $G \not\supset K_{t,t}$. Let degrees in X be $d_1, \ldots, d_n$. We'll see that the average degree d is $\leqslant n^{1-1/t}t^{1/t} + t$.

Wlog each $d_i \geqslant t-1$. (If $d_i \leqslant t-2$, add an edge.)

For each t-set A in Y, how many $x \in X$ have $A \subset \Gamma(x)$? At most $t-1$, as $G \not\supset K_{t,t}$.

Thus, the number of (x, A) with $x \in X$, $A \subset \Gamma(x)$, $|A| = t$ is $\leqslant (t-1)\binom{n}{t}$.

But this number also equals $\sum \binom{d_i}{t}$, so $\sum \binom{d_i}{t} \leqslant (t-1)\binom{n}{t}$.

Now, the function $\binom{x}{t} = \dfrac{x(x-1)\ldots(x-t+1)}{t!}$ is a convex function of x for $x \geqslant t-1$. (E.g., put $y = x-t+1$, then $\binom{x}{t} = \dfrac{(y+t-1)\ldots y}{t!}$, which is a non-negative linear combination of powers of y.)

Thus $\sum \binom{d_i}{t} \geqslant n\binom{d}{t}$, so $n\binom{d}{t} \leqslant (t-1)\binom{n}{t}$, and so $\dfrac{n(d-t+1)^t}{t!} \leqslant \dfrac{(t-1)n^t}{t!}$.

Thus $d - t + 1 \leqslant n^{1-1/t}(t-1)^{1/t}$, whence $d \leqslant n^{1-1/t}t^{1/t} + t$. $\square$

Is this the right value? Does $Z(n,t)$ actually grow as $n^{2-1/t}$ (t fixed)?

$t = 2$. G bipartite, $G \not\supset K_{2,2} = C_4$. Can $e(G)$ be as large as $cn^{3/2}$? Linear $e(G)$ is easy, e.g. a $2n$-cycle. But $n^{1.01}$? In fact, there are examples of G with $e(G) = cn^{3/2}$ (coming from algebra – projective planes).

$t = 3$. Here, $cn^{5/3}$ is correct (but harder).

$t = 4$ is unknown!

**** Non-examinable section ****

The Erdős-Stone Theorem

For a fixed graph H, write $\mathrm{Ex}(n, H) = \max\{e(G) : |G| = n, G \not\supset H\}$.

E.g., Turán's Theorem says: $\mathrm{Ex}(n, K_r) \sim \left(1 - \dfrac{1}{r-1}\right)\binom{n}{2}$.

Or, more precisely, Turán says: $\dfrac{\mathrm{Ex}(n, K_r)}{\binom{n}{2}} \to 1 - \dfrac{1}{r-1}$ as $r \to \infty$.

Note. $\dfrac{e(G)}{\binom{n}{2}}$ is called the **density** of G.

And Theorem 4 says: $\mathrm{Ex}(x, P_k) \sim \dfrac{n(k-1)}{2}$.

How does $\mathrm{Ex}(n, H)$ behave for general H (as $n \to \infty$)?

Turán says: $\dfrac{e(G)}{\binom{n}{2}} > 1 - \dfrac{1}{r-1} \implies G \supset K_r$. But what if $\dfrac{e(G)}{\binom{n}{2}} > 1 - \dfrac{1}{r-1} + 0.001$?

Write $K_r(t)$ for $T_r(rt)$. ("K_r blown up by t".)

Remarkably, $\dfrac{e(G)}{\binom{n}{2}} > 1 - \dfrac{1}{r-1} + 0.001 \implies G \supset K_r(t)$ for any t (for n large enough).

In general, we have the...

Erdős-Stone Theorem. For all r, ϵ, t, $\dfrac{e(G)}{\binom{n}{2}} > 1 - \dfrac{1}{r-1} + \epsilon \implies G \supset K_r(t)$ for $n > n_0(r, \epsilon, t)$.

Sketch proof. We have G, average degree $\geqslant \left(1 - \dfrac{1}{r-1} + \epsilon\right) n$.

1. Get $H \subset G$ (H large) with $\delta(H) \geqslant \left(1 - \dfrac{1}{r-1} + \epsilon\right) n'$ ($n' = |H|$).

   (Similar to: average degree $d \Rightarrow$ can get $\delta \geqslant d/2$.)

2. By induction on r, $H \supset K_{r-1}(t')$, some t' large. Write K for the $K_{r-1}(t')$.

3. Have lots of points in $H - K$, each joined to $\geqslant t$ of each class of K (by $\delta(H)$).

4. Get $\geqslant t$ of these points joined to the same t-set in each class of K (by pigeonhole principle). □

For given H, choose least r with H r-partite. (E.g., H = Petersen graph: not bipartite as it has a 5-cycle, but it is 3-partite.) So then $H \subset K_r(t)$, some t.

Then $T_{r-1}(n) \not\supset H$ (as $T_{r-1}(n)$ is $(r-1)$-partite), so $\dfrac{\mathrm{Ex}(n,H)}{\binom{n}{2}} \geqslant 1 - \dfrac{1}{r-1}$.

But Erdős-Stone says $\dfrac{\mathrm{Ex}(n,H)}{\binom{n}{2}} \geqslant 1 - \dfrac{1}{r-1} + \epsilon \implies G \supset$ any $K_r(t)$, so $G \supset H$.

Conclusion: $\dfrac{\mathrm{Ex}(n,H)}{\binom{n}{2}} \to 1 - \dfrac{1}{r-1}$ (where r is least such that H r-partite).

Remark. If H bipartite, this says $\dfrac{\mathrm{Ex}(n,H)}{\binom{n}{2}} \to 0$. But what is the growth speed of $\mathrm{Ex}(n, H)$?

Unknown for most H. (E.g., $H = C_{2n}$, $n \geqslant 6$).

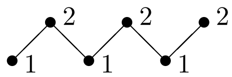
**** End of non-examinable section ****

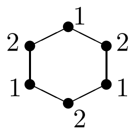
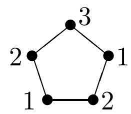
Chapter 4 : Colourings

An r-**colouring** of a graph G is a function $c : V(G) \to [r] = \{1, \ldots, r\}$, such that $c(x) \neq c(y)$ whenever x, y are adjacent.

The **chromatic number** $\chi(G)$ of G is the smallest r such that G has an r-colouring.

Examples.

1. $\chi(P_n) = 2$: 

2. $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$:  

3. $\chi(K_n) = n$ (as all vertices need a different colour).

4. $\chi(E_n) = 1$.

5. T a tree $\Rightarrow \chi(T) = 2$. (E.g., induction: remove a leaf.)

6. $\chi(K_{m,n}) = 2$ (one colour for each class).

Clearly any bipartite graph is 2-colourable, and conversely if c is a 2-colouring of G, then $X = \{x : c(x) = 1\}$, $Y = \{y : c(y) = 2\}$ show that G is bipartite. Similarly, G r-colourable $\iff G$ r-partite.

$\left(\text{So Erdős-Stone corollary says: } \dfrac{\mathrm{Ex}(n, H)}{\binom{n}{2}} \to 1 - \dfrac{1}{\chi(H) - 1} \text{ as } r \to \infty.\right)$

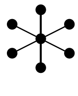
If $|G| = n$ then trivially $\chi(G) \leqslant n$. We can improve this.

Proposition 1. Let G be a graph with maximum degree Δ. Then $\chi(G) \leqslant \Delta + 1$.

Note. This is best possible: e.g., G a complete graph or odd cycle.

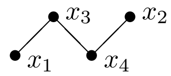
Proof. Order $V(G)$ as $x_1, \ldots, x_n$ and colour $x_1, \ldots, x_n$ in turn. When we come to colour x_i, we find $\leqslant \Delta$ colours used already on neighbours of x_i, so there is $\geqslant 1$ colour we can use for x_i. $\square$

Remarks. 1. We can have $\chi(G)$ much less than Δ.

E.g., $K_{1,n}$ is 2-colourable:  (also called a **star**)

2. Could view proof of Proposition 1 as an application of the **greedy algorithm**: for a given ordering $x_1, \ldots, x_n$, colour each in turn, always using the smallest colour available.

Warning. 1. Greedy algorithm may use *more* than $\chi(G)$ colours.

E.g., in 

greedy gives 3 colours.

2. $G \supset K_r \Rightarrow \chi(G) \geqslant r$, but $G \not\supset K_r \not\Rightarrow \chi(G) < r$. E.g., $C_5 \not\supset C_3$, but not 2-colourable.

In fact, there is no simple formula for $\chi(G)$.

Colouring Planar Graphs

How many colours do we need to colour a planar graph?

3 isn't enough, e.g. K_4.

Proposition 2 (Six Colour Theorem). G planar $\Longrightarrow$ G is 6-colourable.

Proof. Use induction on $|G|$. Done if $|G| \leqslant 6$.

We have planar G, $|G| > 6$.

Claim. $\delta(G) \leqslant 5$.

Proof of claim. We have $e(G) \leqslant 3n - 6$ (as G is planar), so $\sum_{x \in G} d(x) \leqslant 6n - 12$, so some x has $d(x) \leqslant 5$.

Choose x such that $d(x) \leqslant 5$. Then $G - x$ is planar, so by induction it has a 6-colouring. Then $\Gamma(x)$ receives $\leqslant 5$ colours, so we can colour x with the remaining colour. $\square$

Theorem 3 (Five Colour Theorem). G planar $\Longrightarrow$ G is 5-colourable.

Proof. Use induction on $|G|$. Done if $|G| \leqslant 5$.

Given planar G, $|G| > 5$, we have $\delta(G) \leqslant 5$ (as before), so choose $x \in G$ with $d(x) \leqslant 5$. By the induction hypothesis, we have a 5-colouring of $G - x$, so done unless $d(x) = 5$ and all 5 colours appear in $\Gamma(x)$.

Say $\Gamma(x)$ is $x_1, \ldots, x_5$ (clockwise) with x_i having colour i for each i.

Question: is there a 1-3 path from x_1 to x_3? (A 1-3-**path** is a path on which colours are alternately 1 and 3.)

If no, let H be the 1-3-**component** of x_1 (i.e., all vertices we can reach from x_1 by a 1-3-path). So $x_3 \notin H$. Swap colours 1 and 3 on H. This is still a colouring of $G - x$, but we now have x_1 of colour 3, so we can use colour 1 for x.

If yes, we have a 1-3-path P from x_1 to x_3, but then there is no 2-4-path from x_2 to x_4 (as it would have to meet P, as G is planar). So finish as above: swap 2, 4 on the 2-4-component of x_2 and then use colour 2 for x. $\square$

Remark. The i-j-paths in the proof are called **Kempe chains**.

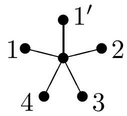
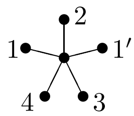
Suppose we wanted to colour the *faces* of a plane graph such that distinct faces sharing an edge get different colours – i.e., "colouring a plane map".

Given a plane graph G, form the **dual graph** G' by taking a point for each face, and joining two points if they share an edge. So G' planar and a colouring G' corresponds exactly to colouring the faces of G. Thus Theorem 3 tells us that every planar map is 5-colourable.

**** Non-examinable section ****

Theorem (Four Colour Theorem). G planar $\Longrightarrow G$ is 4-colourable.

"Proof". Use induction on $|G|$. Done if $|G| \leqslant 4$.

> Given planar G, $|G| > 4$, we have $\delta(G) \leqslant 5$ (as usual), so choose $x \in G$ with $d(x) \leqslant 5$. We can 4-colour $G-x$ by induction, so done unless either $d(x) = 4$ or 5, with all colours appearing in $\Gamma(x)$.
>
> If $d(x) = 4$, proceed as in the Five Colour Theorem. If there is no 1-3-path from x_1 to x_3, swap the colours 1 and 3 on the 1-3-component of x_1 and use colour 1 for x. And if there is a 1-3-path from x_1 to x_3, then there is no 2-4-path from x_2 to x_4, so swap colours 2 and 4 on the 2-4-component of x_2 and use colour 2 for x.
>
> If $d(x) = 5$, we could have 
>
> or 
>
>
> First picture. If there is no 2-4-path from x_2 to x_4 then done (swap colours 2 and 4 on the 2-4-component of x_2).
>
> If there is a 2-4-path from x_2 to x_4 then there is no 1-3-path from x_3 to x_1 or $x_{1'}$, so done (swap colours 1 and 3 on the 1-3-component of x_3).
>
> Second picture. If there is no 2-4-path from x_2 to x_4 then done, as before.
>
> So wlog there is a 2-4-path from x_2 to x_4 (so there is no 1-3-path from x_1 to x_3). Similarly, wlog there is a 2-3-path from x_2 to x_3 (so there is no 1-4-path from $x_{1'}$ to x_4). So swap colours 1 and 3 on the 1-3-component of x_1, and colours $1'$ and 4 on the $1'$-4-component of $x_{1'}$. This leaves colour 1 for use at x. $\square$

The above "proof" was given by Kempe in 1879.

In 1890, Heawood spotted a mistake – *but where is the mistake?*

The theorem was eventually proved in 1976 by Appel and Haken. In the proof of Theorem 3, we used the fact that "a vertex of degree 1, 2, 3, 4 or 5" forms an **unavoidable** (every planar graph contains $\geqslant$ one) set of **reducible** (cannot be present in a minimal counterexample) configurations. Appel and Haken found an unavoidable set of about 1900 reducible configurations for the Four Colour Theorem (using computers a lot).

**** End of non-examinable section ****

We have $\chi(G) \leqslant \Delta + 1$ for all graphs G, and we can have equality – for example K_n and C_{odd}. We will show that in fact $\chi(G) \leqslant \Delta$ for all connected G except K_n and C_{odd}.

Remark. If G is connected and not regular, we can certainly colour with Δ colours.

Indeed, choose x_n with $d(x_n) < \Delta$. Then choose x_{n-1} adjacent to x_n (as G connected). Then choose $x_{n-2} \in G - \{x_n, x_{n-1}\}$ adjacent to $\{x_n, x_{n-1}\}$ (as G connected). And so on. Then run the greedy algorithm on the order $x_1, \ldots, x_n$.

Each x_i has a forward edge, i.e. $x_i x_j \in E$, some $j > i$ (for all $i \neq n$), so uses $\leqslant \Delta$ colours. (The presence of a forward edge means that when choosing the colour of a vertex, we have already chosen colours for at most $\Delta - 1$ of its neighbours.) □

Proposition 4 (Brooks' Theorem). Let G be connected, but not a complete graph or odd cycle. Then $\chi(G) \leqslant \Delta$.

Proof. Wlog G is regular (by the remark above), and $\Delta \geqslant 3$ (as $\Delta = 1$ is trivial, and $\Delta = 2$ $\Rightarrow G$ is a cycle).

Let G be a minimal counterexample (i.e., $|G|$ minimal). Wlog G is 2-connected: as if x is a cutvertex, let $G_1, \ldots, G_k$ be the components of $G - x$ together with x; then each G_i is Δ-colourable (by remark above, as $d_{G_i}(x) < \Delta$ for all i), so G itself is Δ-colourable.

Case 1. G is 3-connected.

We want an ordering such that every vertex has a forward edge, and x_n has two neighbours of the same colour.

Choose any x_n. There must be some $x_1, x_2 \in \Gamma(x_n)$ with $x_1 x_2 \notin E$ (or else $x_n \cup \Gamma(x_n)$ forms a $K_{\Delta+1}$, whence we must have a point of degree $> \Delta$, as G is connected but not itself a complete graph ⨳).

Now, $G - \{x_1, x_2\}$ is connected (as G is 3-connected) so order its vertices $x_3, \ldots, x_n$ (as in the remark above) such that for all $3 \leqslant i \leqslant n-1$, there is $j > i$ for which $x_i x_j \in E(G)$. Then run the greedy algorithm on the ordering $x_1, \ldots, x_n$. This uses $\leqslant \Delta$ colours. □

Case 2. G is not 3-connected.

Choose a separator $\{x, y\}$ (i.e., $G - \{x, y\}$ is disconnected), and let $G_1, \ldots, G_k$ be the components of $G - \{x, y\}$ together with x and y. Then each G_i has a Δ-colouring (by the remark above, as $d_{G_i}(x) \leqslant \Delta - 1$ for all i).

If $xy \in E$, then x and y have different colours in the colouring of G_i for each i, so we can recolour and combine to form a Δ-colouring of G.

So suppose $xy \notin E$. If each G_i has at least one of $d_{G_i}(x)$, $d_{G_i}(y)$ being $\leqslant \Delta - 2$, we can recolour to ensure x and y have different colours in G_i.

So done, unless some G_i has $d_{G_i}(x) = d_{G_i}(y) = \Delta - 1$, say $i = 1$. Then we must have $k = 2$, with $d_{G_2}(x) = d_{G_2}(y) = 1$ (as $d(x), d(y) \leqslant \Delta$). Let $\Gamma_{G_2}(x) = \{u\}$, $\Gamma_{G_2}(y) = \{v\}$. Then $\{x, v\}$ is a separator, *not* of this form. □

The Chromatic Polynomial

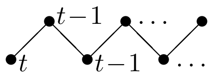
We are carrying more information on G than the number $\chi(G)$. For any graph G and $t = 1, 2, 3, \ldots$, let $P_G(t)$ be the number of t-colourings of G. (So $\chi(G)$ equals the least t such that $P_G(t) > 0$.)

Examples.

$P_{K_n}(t) = t(t-1)(t-2)\ldots(t-n+1)$.

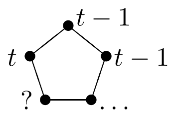
$P_{E_n}(t) = t^n$.

$P_{P_n}(t) = t(t-1)^n$:



And in general $P_{T_n}(t) = t(t-1)^{n-1}$ for any tree T on n vertices (by induction)

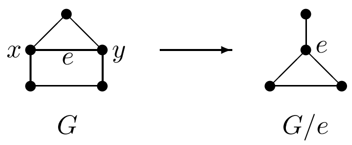
$P_{C_n}(t) = ..?$



Is $P_G(t)$ always a polynomial?

For a graph G and $e = xy \in E(G)$, the **contraction** of G by e, written G/e, is formed from G by replacing x and y by a new vertex e, joined to all points that were joined to x or y.

E.g.,



Lemma 5. Let G be a graph, and $e \in E$. Then $P_G = P_{G-e} - P_{G/e}$.

Remark. Called the **deletion-contraction relation** or **cut-fuse relation**.

Proof. The t-colourings of $G - e$ in which x and y get different colours correspond exactly with the t-colourings of G.

And the t-colourings of $G - e$ in which x and y get the same colour correspond exactly with the t-colourings of G/e.

Thus $P_{G-e}(t) = P_G(t) + P_{G/e}(t)$. $\square$

Note. We could not use Lemma 5 (and a base case of $G = E_n$) to define P_G, as it's not clear that it gives a unique function P_G for all G.

Proposition 6. $P_G(t)$ is a polynomial in t.

Proof. Use induction on $e(G)$. $P_{E_n}(t) = t^n$, so $e(G) = 0$ is done.

Given G, $e(G) > 0$, choose $e \in E$. Then P_{G-e} and $P_{G/e}$ are polynomials, by induction. So $P_G = P_{G-e} - P_{G/e}$ is also a polynomial. $\square$

For a tree T, we had $P_T(t) = t^n - (n-1)t^{n-1} + \dots$

Proposition 7. Let G be a graph on n vertices with m edges. Then the leading terms of $P_G(t)$ are $t^n - mt^{n-1} + \dots$

Proof. Use induction on $e(G)$. $P_{E_n}(t) = t^n$, so $e(G) = 0$ is done.

Given G, $e(G) > 0$, choose $e \in E$. Then, by induction, $P_{G-e} = t^n - (m-1)t^{n-1} + \dots$ and $P_{G/e} = t^{n-1} + \dots$. So $P_G(t) = t^n - mt^{n-1} + \dots$. $\square$

Remarks. 1. We can get other information about G from its chromatic polynomial. For example, it turns out that $P_G(t) = t^n - mt^{n-1} + \left(\binom{m}{2} - \#\text{triangles of } G\right) t^{n-2} + \dots$.

2. Since P_G is a polynomial, we can talk about $P_G(t)$ for any real or complex t.

3. Four Colour Theorem says: planar G has $P_G(4) > 0$. I.e., P_G has no root at 4. No polynomial-style direct proof is known that $P_G(4) \neq 0$ for all planar G. It is known that $P_G(\varphi + 2) \neq 0$ where $\varphi = (1+\sqrt{5})/2$.

Edge Colouring

A k-**edge-colouring** of G is a map $c : E(G) \to \{1, \dots, n\}$ such that $c(e) \neq c(f)$ whenever e, f share a vertex.

The smallest such k is called the **edge-chromatic number** or **chromatic index** of G, written $\chi'(G)$. (Thus $\chi'(G) = \chi(L(G))$.)

E.g., $\chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$

But also, $\chi'(G)$ can be far from $\chi(G)$. E.g., $K_{1,n}$ (a star) has $\chi(G) = 2$, $\chi'(G) = n$.

Clearly $\chi'(G) \geqslant \Delta(G)$ for all G. We can have $\chi'(G) > \Delta(G)$, for example C_{odd}.

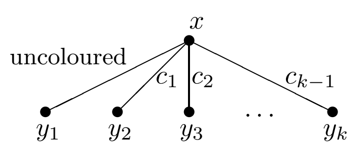
But very surprisingly:

Theorem 8 (Vizing's Theorem). For any G, we have $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$.

Proof. Let $\Delta = \Delta(G)$. We must show that $\chi'(G) \leqslant \Delta + 1$, i.e. that G can be $(\Delta+1)$-edge-coloured. Use induction on $e(G)$. Done if $e(G) = 0$.

Given G, $e(G) > 0$, choose $e \in E(G)$. We have a $(\Delta+1)$-edge-colouring of $G - e$ (by induction). Let e be xy_1. At each vertex, some colour is not being used (as $\Delta + 1 > d(y)$ for all $y \in G$).

Choose a maximal sequence of distinct vertices $y_1, \dots, y_k$ as follows. Given edge xy_i, choose a colour c_i missing at y_i. If there is a new edge from x with colour c_i, let xy_{i+1} be this edge.



This must stop, as G is finite. And it must stop because either c_k is not used at x, or $c_k = c_j$ for some $j < k$.

If c_k is not used at x, then recolour by giving xy_i colour c_i for all $1 \leqslant i \leqslant k$. This is a $(\Delta+1)$-edge-colouring of G.

If $c_k = c_j$ for some $j < k$, wlog $j = 1$, because we can recolour xy_i with colour c_i for $1 \leqslant i \leqslant j-1$, leaving xy_j as the uncoloured edge. Let c be a colour not used at x.

If there is no $c - c_1$ path from y_1 to x, then swap c and c_1 on all edges of the $c - c_1$ component of y_1. This leaves c missing at y_1 (since c_1 was missing at y_1 previously) and at x, so we can give xy_1 colour c, so done.

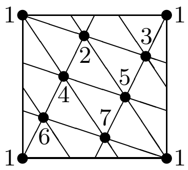
Similarly, if there is no $c - c_1$ path from y_k to x (recall $c_k = c_1$), then swap c and c_1 on the $c - c_1$ component of y_k. We can now use colour c for xy_k and colour c_i for xy_i $(1 \leqslant i \leqslant k-1)$, so done.

Otherwise, the $c - c_1$ component of x (call it H) contains y_1 and y_k, but H is connected and has $\Delta(H) \leqslant 2$ (as H is 2-edge-coloured) and so is a path or cycle, but $d_H(x) = d_H(y_1) = d_H(y_k) = 1$. ⨳ $\square$

Graphs on Surfaces

We know that G drawn on the plane or a sphere has $\chi(G) \leqslant 5$ (well, actually $\leqslant 4$). What about G drawn on other surfaces?

E.g., we can draw K_7 on a torus:



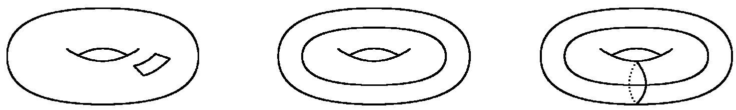
(with edges identified).

In general?

The **surface of genus** g (or the **compact orientable surface of genus** g) consists of a sphere with g handles attached.

For plane/sphere, $n - m + f = 2$ for G connected, and $n - m + f \geqslant 2$ for any planar G (add edges to make G connected).

G on a torus?



$n - m + f = 2$ $\qquad$ $n - m + f = 1$ $\qquad$ $n - m + f = 0$

Fact. For G on a surface of genus g, we have $n - m + f \geqslant 2 - 2g$. (This is the **Euler characteristic**, $E = 2 - 2g$.)

So $n - m + f \geqslant E$, and $3f \leqslant 2m$ (for $m \geqslant 3$) as usual.

Thus $n - m + \dfrac{2m}{3} \geqslant E$, and so $m \leqslant 3(n - E)$.

Theorem 9 (Heawood's Theorem). Let G be a graph drawn on a surface of Euler characteristic $E \leqslant 0$.

Then $\chi(G) \leqslant H(E) = \left\lfloor \dfrac{7 + \sqrt{49 - 24E}}{2} \right\rfloor$.

Note. Tantalizingly, $H(2) = 4$, so "almost" have the Four Colour Theorem (but need $E = 2$).

Proof. Let G be a graph drawn on the surface, with $\chi(G) = k$. We want $k \leqslant H(E)$. Choose minimal G with $\chi(G) = k$. Then $\delta(G) \geqslant k - 1$ (by minimality of G) and $n \geqslant k$.

We know $m \leqslant 3(n - E)$, so the sum of all degrees $= 2m \leqslant 6(n - E)$, and so $\delta(G) \leqslant 6(n - E)/n = 6 - 6E/n$.

Thus $k - 1 \leqslant \delta(G) \leqslant 6 - 6E/n \leqslant 6 - 6E/k$ (as $n \geqslant k$ and $E \leqslant 0$, which is why this fails for the Four Colour Theorem).

So $k^2 - k \leqslant 6k - 6E$, i.e. $k^2 - 7k + 6E \leqslant 0$. Whence result. $\square$

Remark. Equality holds. We can draw $K_{H(E)}$ on a surface of characteristic E – this took 75 years to prove!

Chapter 5 : Ramsey Theory

Can we find order in enough disorder?

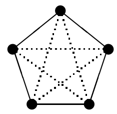
Suppose we 2-edge-color K_6, e.g., $c : E(K_6) \to \{1, 2\}$. Can we always find a monochromatic K_3 (i.e., a K_3 on which c is constant)?

Choose $x \in V(K_6)$. We have $d(x) = 5$, so there are at least 3 edges from x with the same colour – say xy_1, xy_2, xy_3 have colour 1. If any edge y_iy_j $(i \neq j)$ also has colour 1, then xy_iy_j is a colour-1 triangle. But if all edges y_iy_j are colour 2, then $y_1y_2y_3$ is a colour-2 triangle.

What about K_4? Is there an n such that K_n 2-edge-coloured $\Rightarrow \exists$ a monochromatic K_4?

In general, write $R(s)$ for the smallest n (if it exists) such that whenever K_n is 2-coloured (we now mean "edge-coloured"), there exists a monochromatic K_s (i.e., a K_s on which $c : E(K_n) \to \{1, 2\}$ is constant).

Aim. To show that $R(s)$ exists for all s (and find out roughly how fast $R(s)$ grows).

E.g., the above shows that $R(3) \leqslant 6$. In fact, $R(3) = 6$, because of 

Idea. "To go from monochromatic K_3 to monochromatic K_4, first try to get a red K_3 or blue K_4."

For $s, t \geqslant 2$, write $R(s, t)$ for the smallest n (if it exists) such that whenever K_n is 2-coloured, we have either a red K_s or a blue K_t.

So: $R(s, s) = R(s)$,

$R(s, t) = R(t, s)$,

$R(s, 2) = s$.

Equivalently, $R(s, t)$ is the smallest n (if it exists) such that every graph G on n vertices has either $K_s \subset G$ or $K_t \subset \overline{G}$.

Theorem 1 (Ramsey's Theorem). $R(s, t)$ exists for all s, t. And moreover, we have $R(s, t) \leqslant R(s - 1, t) + R(s, t - 1)$, for $s, t \geqslant 3$.

Proof. It is enough to show that if both $R(s - 1, t)$ and $R(s, t - 1)$ exist, then we have $R(s, t) \leqslant R(s - 1, t) + R(s, t - 1)$, as then $R(s, t)$ exists for all s, t by induction on $s + t$.

So let $a = R(s - 1, t)$, $b = R(s, t - 1)$. Choose a 2-colouring of K_{a+b}, and choose $x \in K_{a+b}$. Then $d(x) = a + b - 1$, so we have either $\geqslant a$ red edges or $\geqslant b$ blue edges incident with x.

If $\geqslant a$ are red, then consider the K_a spanned by the endpoints of the edges from x. By the definition of a, this K_a contains either a red K_{s-1} or a blue K_t.

If $\geqslant b$ are blue, similarly. $\square$

Remarks. 1. So any graph on n vertices has $K_s \subset G$ or $K_s \subset \overline{G}$ for n large enough.

2. Very few of the **Ramsey numbers** $R(s,t)$ are known exactly. (See later.)

Corollary 2. Let $s,t \geqslant 2$. Then $R(s,t) \leqslant \binom{s+t-2}{s-1}$. In particular, $R(s) \leqslant 2^{2s}$.

Proof. Induction on $s+t$. Done if $s=2$ or $t=2$. Given $s,t \geqslant 3$,

$$R(s,t) \leqslant R(s-1,t) + R(s,t-1) \leqslant \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}. \qquad \square$$

What about more colours?

Write $R_k(s_1,\ldots,s_k)$ for the smallest n (if it exists) such that whenever K_n is k-coloured, there exists a K_{s_i} of colour i, for some $1 \leqslant i \leqslant k$.

Corollary 3. $R_k(s_1,\ldots,s_k)$ exists for all $k \geqslant 1$ and $s_1,\ldots,s_k \geqslant 2$.

Proof ("Turquoise Spectacles"). Induction on k, the number of colours. Done if $k=1$. (Or if $k=2$, by Ramsey's Theorem.)

Given $k \geqslant 2$ and $s_1,\ldots,s_k$, let $n = R(s_1, R_{k-1}(s_2,\ldots,s_k))$. Then any k-colouring of K_n may be viewed as a 2-colouring, with colours 1 and "2 or 3 or … or k".

So, by the choice of n, we have either a K_{s_1} coloured 1 (so done), or a $K_{R_{k-1}(s_2,\ldots,s_k)}$ coloured with colours 2, 3, … , k (i.e., $(k-1)$-coloured), so done by the definition of $R_{k-1}(s_2,\ldots,s_n)$. $\square$

What about r-sets?

E.g., $r=3$. Colour each *triangle* red/blue – do we get a 4-set all of whose triangles are the same colour? (We are asking for a much denser monochromatic structure.)

For X a set and $r = 1,2,\ldots$, write $X^{(r)} = \{A \subset X : |A| = r\}$. Unless otherwise stated, $X = [n] = \{1,\ldots,n\}$.

Write $R^{(r)}(s,t)$ for the smallest n (if it exists) such that whenever $X^{(r)}$ is 2-coloured (i.e., we have $c : X^{(r)} \to \{1,2\}$), there exists a red s-set (i.e., $A \subset X$, $|A| = s$, with $c(B) = 1$ for all $B \in A^{(r)}$) or a blue t-set.

So: $R^{(2)}(s,t) = R(s,t)$,

$R^{(1)}(s,t) = s+t-1 = (s-1)+(t-1)+1$ (pigeonhole principle),

$R^{(r)}(s,t) = R^{(r)}(t,s)$,

$R^{(r)}(s,r) = R^{(r)}(r,s) = s$.

Theorem 4 (Ramsey for r-sets). Let $r \geqslant 1$, $s,t \geqslant r$. Then $R^{(r)}(s,t)$ exists.

Idea. In the proof of Theorem 1 ($r=2$), we used the case $r=1$.

Proof. Induction on r. Done if $r = 1$ (pigeonhole principle) or $r = 2$ (Theorem 1).

Given $r > 1$, use induction on $s + t$. Done if $s = r$ or $t = r$. So, suppose $s, t > r$.

Claim. $R^{(r)}(s,t) \leqslant R^{(r-1)}\big(R^{(r)}(s-1,t), R^{(r)}(s,t-1)\big) + 1$.

Proof of claim. Let $a = R^{(r)}(s-1,t)$, $b = R^{(r)}(s,t-1)$, and $n = R^{(r-1)}(a,b) + 1$.

Given a 2-colouring c of $X^{(r)}$, choose $x \in X$ and let $Y = X \setminus \{x\}$. Then c induces a 2-colouring c' of $Y^{(r-1)}$ by $c'(A) = c(A \cup \{x\})$, for each $A \in Y^{(r-1)}$. So by the definition of $R^{(r-1)}(a,b)$, we have either a red a-set for c' or a blue b-set for c'.

By symmetry, wlog we have a red a-set Z for c', i.e. $A \cup \{x\}$ is red for all $A \in Z^{(r-1)}$. But by the definition of a, Z contains either a red $(s-1)$-set for c or a blue t-set for c.

If a blue t-set, then we are done.

If a red $(s-1)$-set, then add x and obtain a red s-set. $\square$

Remarks. 1. Similarly for k colours – e.g., by "turquoise spectacles".

2. The bounds we get on $R^{(s,t)}$ are quite large: "to get $R^{(r)}$, iterate $R^{(r-1)}$ about $s+t$ times".

Define $f_1, f_2, f_3, \ldots : \mathbb{N} \to \mathbb{N}$ as follows:

$$f_1(x) = 2x, \text{ and for } n \geqslant 2,\ f_n(x) = \underbrace{f_{n-1}(f_{n-1}(f_{n-1} \ldots (f_{n-1}}_{x \text{ times}}(x)) \ldots)) \ .$$

So $f_2(x) = 2^x$, $f_3(x) = 2^{2^{\cdot^{\cdot^{2}}}} \Big\} x \text{ times}$. And $f_4(x)$..?

Well, $f_4(1) = 2$, $f_4(2) = 2^2 = 4$, $f_3(2) = 2^{2^{2^2}} = 65536$, $f_4(4) = 2^{2^{\cdot^{\cdot^{2}}}} \Big\} 65536 \text{ times}$

Then our bound on $R^{(r)}(s,t)$ is of the form $f_r(s+t)$.

(These sorts of bounds are often a feature of such double induction proofs.)

Infinite Ramsey Theory

Given a 2-colouring c of $\mathbb{N}^{(2)}$, can we always find an infinite monochromatic subset? (I.e., $M \subset \mathbb{N}$, M infinite, with c constant on $M^{(2)}$?)

Examples. 1. Colour ij red if $i + j$ is even, and blue if odd. Take, e.g., $M = \{n : n \text{ even}\}$.

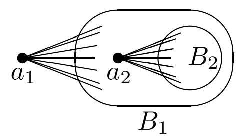
2. Colour ij red if $\max\{n : 2^n \text{ divides } i + j\}$ is even, and blue otherwise. Take, e.g., $M = \{2^2, 2^4, 2^6, 2^8, \ldots\}$.

3. Colour ij red if $i + j$ has an even number of (distinct) prime factors, and blue if odd.

Note. Asking for an infinite red set is more than asking for arbitrarily large finite red sets. For example, consider the colouring for which all edges within the sets $\{1, 2\}$, $\{3, 4, 5\}$, $\{6, 7, 8, 9\}$, $\{10, 11, 12, 13, 14\}$, $\{15, 16, 17, 18, 19, 20\}$,... are coloured red, and all other edges are coloured blue. Then there are arbitrarily large finite red sets, but no infinite red set.

Theorem 5 (Infinite Ramsey). Let $\mathbb{N}^{(2)}$ be 2-coloured. Then there exists an infinite monochromatic $M \subset \mathbb{N}$.

Proof. Choose $a_1 \in \mathbb{N}$. Then there exists infinite $B_1 \subset \mathbb{N} \setminus \{a_1\}$ such that all edges from a_1 to B_1 have the same colour, say c_1. Choose $a_2 \in B_1$. Then there exists infinite $B_2 \in B_1 \setminus \{a_2\}$ such that all edges from a_2 to B_2 are the same colour, say c_2. Continue inductively.



We obtain points $a_1, a_2, \ldots$ in $\mathbb{N}$ and colours $c_1, c_2, \ldots$ such that $a_i a_j$ has colour c_i (for $i < j$). We must have infinitely many of the c_i the same colour, say $c_{i_1} = c_{i_2} = \ldots$. Then $M = \{a_{i_1}, a_{i_2}, \ldots\}$ is infinite monochromatic. $\square$

Remarks. 1. Similarly for k colours. (E.g., by "turquoise spectacles".)

2. This is called a **2-pass proof**.

3. In the third example above (prime factors), no explicit example is known.

Example. Any sequence in $\mathbb{R}$ (or any totally ordered set) has a monotonic subsequence. Indeed, given sequence $x_1, x_2, \ldots$, 2-colour $\mathbb{N}^{(2)}$ by giving ij the colour "up" if $x_i \leqslant x_j$ and "down" if $x_i > x_j$. Now apply infinite Ramsey.

What about r-sets?

E.g., $r = 3$. 2-colour $\mathbb{N}^{(3)}$ by giving ijk $(i < j < k)$ the colour red if i divides $j + k$, and blue if not. We can take $M = \{1, 2, 4, 8, 16, \ldots\}$.

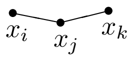
Theorem 6 (Infinite Ramsey for r-sets). Let $r \geqslant 1$, and let $\mathbb{N}^{(r)}$ be 2-coloured. Then there exists an infinite monochromatic subset $M \subset \mathbb{N}$.

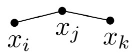
Proof. Induction on r. Done if $r = 1$ (or if $r = 2$, by Theorem 5).

Given $r > 1$, and given a 2-colouring c of $\mathbb{N}^{(r)}$, choose $a_1 \in \mathbb{N}$. This induces a 2-colouring c' of $(\mathbb{N} \setminus \{a_1\})^{(r-1)}$ by $c'(F) = c(F \cup \{a_1\})$. So there exists an infinite monochromatic $B_1 \subset \mathbb{N} \setminus \{a_1\}$ for c', by the induction hypothesis. I.e., we have some colour c_1 such that every r-set of the form $\{a_1\} \cup F$, for $F \subset B_1$, has colour c_1.

Choose $a_2 \in B_1$. This induces a 2-colouring of $(B_1 \setminus \{a_2\})^{(r-1)}$, so we get an infinite $B_2 \subset B_1 \setminus \{a_2\}$ and a colour c_2 such that every r-set of the form $\{a_2\} \cup F$, for $F \subset B_2$, has colour c_2. Continue inductively.

We obtain points $a_1, a_2, \ldots$ in $\mathbb{N}$ and colours $c_1, c_2, \ldots$ such that any r-set $a_{i_1} \ldots a_{i_r}$ (for $i_1 < \ldots < i_r$) has colour c_i. But infinitely many of the c_i agree, say $c_{i_1} = c_{i_2} = \ldots$. Then $M = \{a_{i_1}, a_{i_2}, \ldots\}$ is monochromatic. $\square$

Example. We saw that given points $(1, x_1), (2, x_2), \ldots$ in $\mathbb{R}^2$, we can find a subsequence such that the induced function (i.e., piecewise linear, through these dots) is monotonic. In fact, we can guarantee that the induced function is convex or concave. Colour $\mathbb{N}^{(3)}$ by giving ijk the colour "convex" if 

and "concave" if 

and apply Theorem 6.

Exact Ramsey Numbers

Very few non-trivial $(s, t, \geqslant 3)$ of the $R(s, t)$ are known exactly:

$R(3,3) = 6$, $R(3,4) = 9$, $R(3,5) = 14$, $R(3,6) = 18$, $R(3,7) = 23$, $R(3,8) = 28$, $R(3,9) = 36$.

$R(4,4) = 18$, $R(4,5) = 25$. Known that $43 \leqslant R(5,5) \leqslant 49$

For example, to show $R(4,4) > 17$, 2-colour $\mathbb{Z}_{17}^{(2)}$ by giving ij colour red if $i - j$ is a square mod 17, and blue if not. (Have to check that there is no monochromatic K_4.)

For more than two colours, the only number is $R_3(3,3,3) = 17$.

For r-sets, the only known number is $R^{(3)}(4,4) = 13$.

This is hard because was are asking "exactly how much disorder" guarantees a given amount of order – hard to analyse.

"Put it on a computer?"

To show, for example, $R(5,5) > 43$, need to examine $\binom{43}{5}$ 5-sets, in each of $2^{\binom{43}{2}}$ colourings. But $2^{\binom{43}{2}} > 2^{800} > 10^{250}$, so no chance.

Chapter 6 : Random Graphs

How fast does $R(s)$ grow?

We know $R(s) \leqslant 4^s$. What about a lower bound?

It's easy to see that $R(s) > (s-1)^2$: take $s-1$ copies of K_{s-1}, then colour the edges within each K_{s-1} red, and those between different copies blue.

It was believed (in the 1940s) that perhaps $R(s) \sim cs^2$. However…

Theorem 1 (Erdős, 1947). Let $s \geqslant 3$. Then $R(s) > 2^{s/2}$.

Proof. Choose a colouring of K_n at random, by taking each edge to be red or blue with probability 1/2 each (independently). Then $\mathbb{P}(\text{a fixed } s\text{-set is monochromatic}) = 2(\frac{1}{2})^{\binom{s}{2}}$.

The number of s-sets is $\binom{n}{s}$, so $\mathbb{P}(\exists \text{ a monochromatic } s\text{-set}) \leqslant \binom{n}{s} 2^{1-\binom{s}{2}}$.

Thus we must have $R(s) > n$ if $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$, i.e. if $\binom{n}{s} < 2^{\binom{s}{2}-1}$.

But $\binom{n}{s} \leqslant \dfrac{n^s}{s!}$ and $s! \geqslant 2^{\frac{s}{2}+1}$ (for $s \geqslant 3$, by induction on s).

So done if $n^s \leqslant 2^{s^2/2}$, i.e. if $n \leqslant 2^{s/2}$ □

Remarks. 1. The above is a **random graphs** argument.

2. We could rewrite it as: there are $2^{\binom{n}{2}}$ colourings, and a given s-set is monochromatic in $2 \times 2^{\binom{n}{s}-\binom{s}{2}}$ of them, so done if $\binom{n}{s} \times 2^{1+\binom{n}{2}-\binom{s}{2}} < 2^{\binom{n}{2}}$.

(But this is a bad viewpoint for later, when we won't have all graphs equally likely.)

3. The proof gives no hint as to how to construct such a colouring.

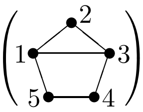
4. No construction giving an exponential lower bound on $R(s)$ is known!

So, to find an actual graph G on 10^6 points with no K_{40} in G or $\overline{G}$, the best thing to do is just toss a coin for each edge.

We have now $\sqrt{2}^s \leqslant R(s) \leqslant 4^s$. No better bounds ($\sqrt{2} \to \sqrt{2}+\epsilon$ or $4 \to 4-\epsilon$) are known. (There exists a heuristic argument for each of $\sqrt{2}^s$, 2^s, 4^s.)

The **probability space** $G(n,p)$ is defined on the set of all graphs on $\{1,\dots,n\}$ as follows. We choose each edge to be present with probability p, and absent with probability $1-p$.

So, e.g., in the proof of Theorem 1, we worked inside $G(n, \frac{1}{2})$.

Example. In $G(5,p)$, $\mathbb{P}$ (

) $= p^6(1-p)^4$.

It can be useful to look at p other than $1/2$.

Recall Zarankiewicz. We had $Z(n,t) \leqslant 2n^{2-1/t}$ (for t fixed, n large).

What about a lower bound? (Hopefully better than the trivial lower bound.)

Could: choose a random bipartite graph G, vertex classes X, Y (with $|X| = |Y| = n$), choosing each edge independently with probability p, and choosing p to make the expected number of $K_{t,t}$ less than 1.

Now, the number of $K_{t,t}$ is $\binom{n}{t}^2$, and $\mathbb{P}(\exists \text{ fixed } K_{t,t} \subset G) = p^{t^2}$.

So the expected number of $K_{t,t}$ in G is $\binom{n}{t}^2 p^{t^2} < \dfrac{1}{4} n^{2t} p^{t^2}$.

Take $p = n^{-2/t}$ to get $\mathbb{E}(\#K_{t,t} \text{ in } G) < 1/4$. Whence $\mathbb{P}(G \text{ has no } K_{t,t}) > 3/4$.

Also, $\mathbb{E}(\#\text{edges}) = pn^2$, so we would get $\mathbb{P}(e(G) > \frac{1}{2}pn^2) > 1/2$, so there exists G, with no $K_{t,t}$, with $e(G) > \frac{1}{2}pn^2 = \frac{1}{2}n^{2-2/t}$. $\square$

We can do better, however.

Theorem 2. $Z(n,t) > \dfrac{1}{2} n^{2-2/(t+1)}$.

Idea. If a graph G has m edges and r copies of $K_{t,t}$, remove an edge from each copy of $K_{t,t}$ to obtain a graph with $\geqslant m - r$ edges and no $K_{t,t}$, showing $Z(n,t) > m - r$.

Proof. Choose a random bipartite graph G, vertex classes X, Y ($|X| = |Y| = n$), by taking each edge independently with probability p. Let $M = e(G)$, and $R =$ the number of $K_{t,t}$ in G.

As before, $\mathbb{E}(M) = pn^2$ and $\mathbb{E}(R) = \binom{n}{t}^2 p^{t^2}$.

So, using linearity of expectation, $\mathbb{E}(M - R) = pn^2 - \binom{n}{t}^2 p^{t^2} \geqslant pn^2 - \dfrac{1}{2} n^{2t} p^{t^2}$.

Taking $p = n^{-2/(t+1)}$, we have $\mathbb{E}(M - R) \geqslant n^{2-2/(t+1)} - \dfrac{1}{2} n^{2t - 2t^2/(t+1)} = \dfrac{1}{2} n^{2-2/(t+1)}$.

Thus there exists a graph with $m - r \geqslant \dfrac{1}{2} n^{2-2/(t+1)}$, so $Z(n,t) > \dfrac{1}{2} n^{2-2/(t+1)}$. $\square$

Remark. The above proof is called **modifying a random graph**.

Graphs with Large $\chi(G)$

To make $\chi(G) \geqslant k$ – can ensure this by $G \supset K_k$.

To have $\chi(G) \geqslant k$ – need not have $G \supset K_k$. E.g., $G = C_5$.

In fact, we can have $\chi(G)$ much larger than the **clique number** $= \max\{k : G \supset K_k\}$, because:

(a) the graph G in Theorem 1 (e.g., the red edges) is on $n = 2^{s/2}$ vertices, with no $K \subset G$ and no **independent set** (i.e., a set with no edges) of size s. But in any colouring, each colour class is an independent set, so $\chi(G) \geqslant \frac{2^{s/2}}{s-1}$ – much more than the clique number, which is $\leqslant s-1$.

(b) We can construct a graph G to be triangle-free (i.e., $K_3 \not\subset G$) but with $\chi(G)$ large (although this is not easy).

Could we have large girth (length of shortest cycle) and still have $\chi(G)$ large, for example, with girth $\geqslant 10$, $\chi(G) \geqslant 100$? This seems unlikely, however…

Theorem 3. For all k, g, there exists a graph G with $\chi(G) \geqslant k$ and $\text{girth}(G) \geqslant g$.

Idea. Try to find G on n vertices such that the number of short cycles (length $\leqslant g$) is $\leqslant n/2$, and every independent set has size $\leqslant n/2k$. Then done by removing a vertex in each short cycle to obtain a graph H with $\text{girth}(H) \geqslant g$ and $\chi(H) \geqslant (n/2)/(n/2k) = k$.

Proof. Choose a random $G \in G(n,p)$, where $p = n^{-1+1/g}$.

Let x_i be the number of i-cycles in G, $3 \leqslant i \leqslant g-1$, and let x be the number of cycles of length $< g$, so $x = x_3 + \ldots + x_{g-1}$.

Then $\mathbb{E}(x_i) = (\#\text{ possible } i\text{-cycles}) \times \mathbb{P}(\text{given } i\text{-cycle is present}) \leqslant n^i p^i$.

So $\mathbb{E}(x) \leqslant \displaystyle\sum_{i=3}^{g-1} (np)^i = \sum_{i=3}^{g-1} n^{i/g} \leqslant g n^{(g-1)/g} = n \frac{g}{n^{1/g}} < \frac{n}{4}$ for n large (as $g/n^{1/g} \to 0$).

Thus $\mathbb{P}(x \leqslant n/2) > 1/2$ (else $\mathbb{P}(x \geqslant n/2) \geqslant 1/2$, whence $\mathbb{E}(x) \geqslant n/4$ $\times\!\!\!\times$).

Let $t = n/2k$ (for n a multiple of $2k$) and let y be the number of t-sets that are independent. Then

$$\begin{aligned}
\mathbb{E}(y) &= (\#t\text{-sets}) \times \mathbb{P}(\text{given } t\text{-set independent}) = \binom{n}{t}(1-p)^{\binom{t}{2}} \\
&\leqslant n^t e^{-p\binom{t}{2}} \quad (\text{because } 1 - x \leqslant e^{-x}\,\forall\, x) \\
&\leqslant \exp\left(\frac{n}{2k}\log n - \frac{n^2}{8k^2}\, n^{-1+1/g}\right) \\
&\to 0 \text{ as } n \to \infty \quad (\text{because } n\log n - n n^{1/g} \to -\infty).
\end{aligned}$$

So $\mathbb{E}(y) < 1/2$ (for n large). So $\mathbb{P}(y = 0) > 1/2$.

Thus there exists $G \in G(n,p)$ with $x \leqslant n/2$, $y = 0$, so done. $\square$

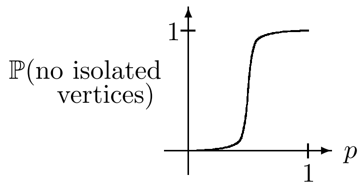
The Structure of a Random Graph

What does a "typical" random graph $G \in G(n,p)$ look like?

How do the properties vary as p varies?

For example, does G have no isolated vertices?

We would expect a gradual increase of $\mathbb{P}$(no isolated vertices) as we increase p. But in fact we get a **threshold effect**:



This jump or threshold happens below $p =$ constant, since if $p =$ constant then $\mathbb{P}$(a given vertex is isolated) is exponentially small, whence $\mathbb{P}(\exists$ isolated vertex) is also small.

So, where does that jump happen?

Probability digression/reminder

Let X be a random variable, taking values in $\{0, 1, 2, \ldots\}$.

To show $\mathbb{P}(X = 0)$ is large, it is enough to show that the mean $\mu = \mathbb{E}(X)$ is small. Indeed, for any t, we have $\mu \geqslant \mathbb{P}(X \geqslant t)t$ whence $\mathbb{P}(X \geqslant t) \leqslant \mu/t$ (known as **Markov's inequality**), so $\mathbb{P}(X \geqslant 1) \leqslant \mu$, so $\mathbb{P}(X = 0) \geqslant 1 - \mu$.

To show $\mathbb{P}(X = 0)$ is small, it is not enough to show that μ is large. E.g., take $X = 0$ with probability 999/1000, and $X = 10^{10}$ with probability 1/1000.

So instead look at the variance $V = \mathrm{Var}(X) = \mathbb{E}((X - \mu)^2) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$. For any t, $\mathbb{P}(|X - \mu| \geqslant t) = \mathbb{P}(|X - \mu|^2 \geqslant t^2) \leqslant V/t^2$, by Markov (known as **Chebyshev's inequality**).

Thus $\mathbb{P}(|X - \mu| \geqslant \mu) \leqslant V/\mu^2$. So in particular $\mathbb{P}(X = 0) \leqslant V/\mu^2$.

Conclusion: to show $\mathbb{P}(X = 0)$ small, show V/μ^2 small.

Suppose X is the number of some events A that occur. Then $\mu = \mathbb{E}(X) = \sum_n \mathbb{P}(A)$.

Variance? $\mathbb{E}^2(X) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B)$, and $\mathbb{E}(X^2) = \sum_{A,B} \mathbb{P}(A \cap B) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B|A)$.

(Using, e.g., $X = \sum_A I_A$, so $X^2 = \sum_{A,B} I_A I_B = \sum_{A,B} I_{A \cap B}$.)

So variance $V = \mathrm{Var}(X) = \sum_{A,B} \mathbb{P}(A)(\mathbb{P}(B|A) - \mathbb{P}(B))$.

(Note: no contribution to this from independent A, B.)

The phrase **almost surely** means "with probability $\to 1$ as $n \to \infty$".

Theorem 4. Let λ be constant, and let $G \in G(n,p)$ where $p = \lambda \dfrac{\log n}{n}$.

If $\lambda < 1$ then almost surely G has an isolated vertex. While if $\lambda > 1$, then almost surely G has no isolated vertex.

Remark. Theorem 4 tells us that $p = \log n / n$ is a threshold for the property of having an isolated vertex.

Proof of 4. Let X be the number of isolated vertices of G.

Then $\mu = \mathbb{E}(X) = n(1-p)^{n-1} = \dfrac{n}{1-p}(1-p)^n$.

For $\lambda > 1$, we have $\mu \leqslant \dfrac{n}{1-p}e^{-pn} = \dfrac{n}{1-p}e^{-\lambda \log n} = \dfrac{n^{1-\lambda}}{1-p} \to 0$ as $n \to \infty$, so certainly $\mathbb{P}(X=0) \to 1$.

For $\lambda < 1$, we have $1 - p \geqslant e^{-(1+\delta)p}$, for any δ (p small enough).

So $\mu \geqslant \dfrac{n}{1-p}e^{-(1+\delta)pn} = \dfrac{n}{1-p}e^{-(1+\delta)\lambda \log n} = \dfrac{n^{1-(1+\delta)\lambda}}{1-p}$.

Choosing δ such that $(1+\delta)\lambda < 1$, we have $\mu \to \infty$ as $n \to \infty$.

$$\mathrm{Var}(X) = \underbrace{n(1-p)^{n-1}(1-(1-p)^{n-1})}_{n \text{ terms in which } A = B} + \underbrace{n(n-1)(1-p)^{n-1}((1-p)^{n-2}-(1-p)^{n-1})}_{n(n-1) \text{ terms in which } A \neq B}$$

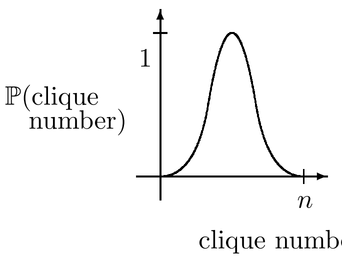
$$\leqslant \mu + n(n-1)(1-p)^{n-1}p(1-p)^{n-2}$$

$$\leqslant \mu + \frac{p}{1-p}n^2(1-p)^{2n-2} = \mu + \frac{p}{1-p}\mu^2.$$

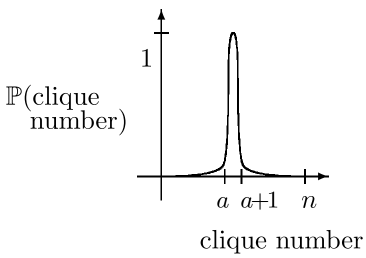
Thus $\dfrac{V}{\mu^2} \leqslant \dfrac{1}{\mu} + \dfrac{p}{1-p} \to 0$ as $n \to \infty$. So $\mathbb{P}(X=0) \to 0$ as $n \to \infty$. $\square$

A different kind of threshold effect is the clique number of a random graph. Let $0 < p < 1$ be fixed. What is the distribution of the clique number of $G \in G(n,p)$?

We'd guess:



but in fact:



It turns out that, almost surely, the clique number is a or $a+1$ (some a).

Theorem 5. Let $0 < p < 1$ be fixed, and let d be a real number with $\binom{n}{d} p^{\binom{d}{2}} = 1$.

Then, almost surely, $G \in G(n,p)$ has clique number equal to $\lceil d \rceil$, $\lfloor d \rfloor$, or $\lfloor d \rfloor - 1$.

Remark. With more work, we could check that only two values occur.

Sketch proof of 5. Fix an integer k, and let X be the number of K_k in G.

So $\mu = \mathbb{E}(X) = \binom{n}{k} p^{\binom{k}{2}}$.

Task: if $k \leqslant d-1$, then almost surely $X \neq 0$, while if $k \geqslant d+1$ then almost surely $X = 0$.

If $k \geqslant d+1$, then $\mu \to 0$ (check), so $\mathbb{P}(X=0) \to 1$.

If $k \leqslant d-1$, then $\mu \to \infty$ (check).

Now, $V = \binom{n}{k} p^{\binom{k}{2}} \sum_{s=2}^{k} \underbrace{\binom{k}{s}\binom{n-k}{k-s}}_{\#B \text{ meeting } A \text{ in } s \text{ points}} \left(p^{\binom{k}{2}-\binom{s}{2}} - p^{\binom{k}{2}}\right)$.

So, $\dfrac{V}{\mu^2} \leqslant \dfrac{1}{\mu} \sum_{s=2}^{k} \binom{k}{s}\binom{n-k}{k-s} p^{\binom{k}{2}-\binom{s}{2}}$.

Check that the first and last terms dominate – i.e., sum $\leqslant$ (first + last)$\times$ constant.

So, $\dfrac{V}{\mu^2} \leqslant \text{constant} \times \left(\binom{k}{2}\binom{n-k}{k-2} p^{\binom{k}{2}-1} + 1\right)$, whence $V/\mu^2 \to 0$.

Thus $\mathbb{P}(X=0) \to 0$. $\square$

Chapter 7 : Algebraic Methods

The **diameter** of a graph G is $\max\{d(x,y) : x, y, \in G\}$.

So, e.g., G has diameter $1 \iff G$ is complete.

What about diameter 2? How many vertices can G have if G has diameter 2 and maximum degree Δ?

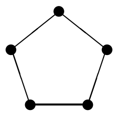
Expanding from a point x, we see that $V(G) = \{x\} \cup \Gamma(x) \cup \Gamma(\Gamma(x))$. And $|\Gamma(x)| \leqslant \Delta$, so $|G| \leqslant 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$.

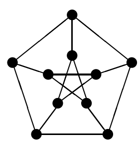
If $|G| = 1 + \Delta^2$, then G must be regular.

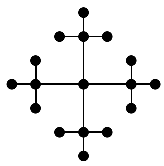
A k-regular graph of diameter 2 on $n = 1 + k^2$ vertices is called a **Moore graph**, or **Moore graph of diameter** 2.

Equivalently, a k-regular graph is a Moore graph $\iff$ for all $x \neq y$ there exists a unique path of length $\leqslant 2$ from x to y (for $k \neq 1$).

Examples.

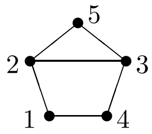
1. $k = 2$. C_5 : 

2. $k = 3$. Petersen Graph : 

3. $k = 4$. Try  and add further required edges. But this fails!

4. $k > 4$. Ought to keep failing. Why?

Let G be a graph on vertex set $[n] = \{1, \ldots, n\}$.
The **adjacency matrix** of G is the $n \times n$ matrix A with $A_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$.

E.g., for 

we find $A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$

Note that A is always real and symmetric.

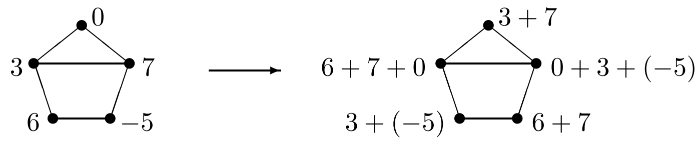
The matrix A contains all of the information of G.

E.g., look at A^2. $(A^2)_{ij} = \sum_k A_{ik}A_{kj}$ = the number of walks of length 2 from i to j.

Similarly, $(A^3)_{ij}$ = the number of walks of length 3 from i to j.

We have a linear map $x \mapsto Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^n$. Thus $(Ax)_i = \sum_j A_{ij}x_j$.

E.g., so



I.e., add the values at neighbours. So if $x = (6, 3, 7, -5, 0)$ then $Ax = (-2, 13, -2, 13, 10)$.

E.g., if A is k-regular, then $(1, 1, \ldots, 1) \mapsto (k, k, \ldots, k)$.

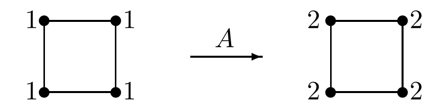
Since A is real symmetric, it is diagonalisable: it has a basis of eigenvectors, say $e_1, e_2, \ldots, e_n$ with corresponding eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. (We can take the $e_1, \ldots, e_n$ to be an orthonormal basis.)

Often write $\lambda_{\max} = \lambda_1$ and $\lambda_{\min} = \lambda_n$. Note that $\sum \lambda_i = 0$, as $\operatorname{trace}(A) = 0$. So $\lambda_{\max} > 0$ and $\lambda_{\min} < 0$ (unless $G = E_n$).

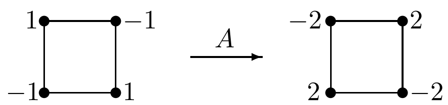
To find eigenvalues, we often don't need lots of calculation.

Example. C_4 has $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$.

So $\operatorname{rank}(A) = 2$, so we have 0 as an eigenvalue twice. The other two?



so 2 is an eigenvalue.



so -2 is an eigenvalue (or use $\sum = 0$)

Thus the eigenvalues are $2, 0, 0, -2$.

Let $e_1, \ldots, e_n$ be an orthonormal basis of eigenvectors. Take $x \in \mathbb{R}^n$, say $x = \sum c_i e_i$, with $||x|| = 1$, i.e., $\sum c_i^2 = 1$. Then $Ax = \sum c_i \lambda_i e_i$, so $\langle x, Ax \rangle = \sum \lambda_i c_i^2$.

So,

$$\left.\begin{aligned} \min_{||x||=1} \langle x, Ax \rangle = \lambda_{\min} &\quad \text{(take } c_n = 1 \text{, all other } c_i = 0\text{)} \\[2ex] \max_{||x||=1} \langle x, Ax \rangle = \lambda_{\max} &\quad \text{(take } c_1 = 1 \text{, all other } c_i = 0\text{)} \end{aligned}\right\} \quad (*)$$

Proposition 1. For any graph G,

(i) λ an eigenvalue $\Longrightarrow |\lambda| \leqslant \Delta$

(ii) For G connected: Δ is an eigenvalue $\Longleftrightarrow G$ is regular

(iii) For G connected: $-\Delta$ is an eigenvalue $\Longleftrightarrow G$ is regular and bipartite

(iv) $\lambda_{\max} \geqslant \delta$.

Proof. (i) Choose an eigenvector x for λ, and choose i with $|x_i|$ maximal. Wlog, $x_i = 1$. But then $|(Ax)_i| = |\sum_{j \in \Gamma(i)} x_j| \leqslant \Delta \times 1$. So $|\lambda| \leqslant \Delta$.

(ii) ($\Leftarrow$) Let $x = (1, \ldots, 1)$, then $Ax = (\Delta, \ldots, \Delta)$.

($\Rightarrow$) From (i) we must have $d(i) = \Delta$ and that all $j \in \Gamma(i)$ also have $x_j = 1$. So we can repeat for each $j \in \Gamma(i)$, then for each $k \in \Gamma(j)$, etc. We obtain $d(k) = \Delta$ for all k (as G is connected).

(iii) ($\Leftarrow$) Choose $x = (\underbrace{1, 1, \ldots,}_{\text{on } X} \underbrace{-1, -1, \ldots}_{\text{on } Y})$.

($\Rightarrow$) From (i) we must have $d(j) = \Delta$ and $x_j = -1$ for all $j \in \Gamma(i)$. Repeat for each $k \in \Gamma(j)$, etc. We obtain $d(j) = \Delta$ for all $j \in G$. and for all $jk \in E$ we have $x_j = 1, x_k = -1$ or $x_j = -1, x_k = 1$. Thus G is regular and has no odd cycle.

(iv) Let $x = (1, \ldots, 1)$. Then $(Ax)_i \geqslant \delta$ for all i, so $\langle Ax, x \rangle \geqslant \delta \langle x, x \rangle = \delta n$. Hence $\lambda_{\max} \geqslant \delta$, by $(*)$ (immediately before Proposition). $\square$

Remark. In (ii), if Δ is an eigenvalue, it has multiplicity 1 (as it has the unique eigenvector $(1, \ldots, 1)$).

Eigenvalues can link in to other graph parameters. We know that $\chi(G) \leqslant \Delta + 1$. We can strengthen this to

Proposition 2. For any graph G, $\chi(G) \leqslant \lambda_{\max} + 1$.

Proof. Induction on $|G|$. Done if $|G| = 1$.

Given G with $|G| > 1$, choose $v \in G$ with $d(v) = \delta(G)$.

Claim. $\lambda_{\max}(G - v) \leqslant \lambda_{\max}$.

Then done: colour $G - v$ by induction, and we can colour v as $d(v) \leqslant \lambda_{\max}$.

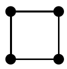
Proof of claim. Let B be A with row and column v removed – wlog, the last row and column.

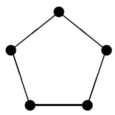
For any $x = (x_1, \ldots, x_{n-1})$, let $y = (x_1, \ldots, x_{n-1}, 0)$. Then $(Ay).y = (Bx).x$, and so $\lambda_{\max}(G - v) \leqslant \lambda_{\max}$, by $(*)$. $\square$

Towards Moore Graphs

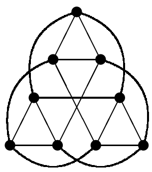
A graph G is **strongly regular** with parameters (k, a, b) if G is k-regular, with any two adjacent points having exactly a common neighbours, and any two non-adjacent points having exactly b common neighbours.

Examples.

C_4:  has $(2, 0, 2)$.

C_5:  has $(2, 0, 1)$

– in general a Moore graph of degree k is strongly regular with $(k, 0, 1)$.

 has $(4, 1, 2)$.

A huge constraint on such G is given by...

Theorem 3 (Rationality condition for strongly regular graph). Let G be a graph on n vertices, strongly regular with parameters (k, a, b), with $b \geqslant 1$.

Then the the numbers $\frac{1}{2}\left(n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}}\right)$ are integers.

Proof. G is connected, as $b \geqslant 1$. So k is an eigenvalue with multiplicity 1. What are the other eigenvalues?

We have $(A^2)_{ij} = \begin{cases} k & \text{if } i = j \\ a & \text{if } i \neq j,\ ij \in E \\ b & \text{if } i \neq j,\ ij \notin E \end{cases}$.

Thus $A^2 = kI + aA + b(J - I - A)$, where J has all entries equal to 1.

So $A^2 + (b-a)A + (b-k)I = bJ$.

For λ an eigenvalue ($\lambda \neq k$) with eigenvector x, we have $(1, \ldots, 1).x = 0$, since the eigenvectors are orthogonal. I.e., $Jx = 0$.

Thus $(\lambda^2 + (b-a)\lambda + (b-k))x = 0$, so $\lambda^2 + (b-a)\lambda + (b-k) = 0$.

So eigenvalues not equal to k are $\lambda = \frac{1}{2}\left(a - b \pm \sqrt{(b-a)^2 + 4(k-b)}\right)$ – say λ, μ, with multiplicities r, s. So $r + s = n - 1$, and $\lambda r + \mu s = -k$, as the eigenvalues sum to 0 (e.g., consider the trace). Solving for r, s, we get the numbers in the theorem. $\square$

Finally,

Corollary 4. If there exists a Moore graph of degree k, then $k \in \{2, 3, 7, 57\}$.

Remark. $k = 2 : C_5$

$k = 3 :$ Petersen graph

$k = 7 :$ exists

$k = 57 :$ unknown

Proof of 4. By Theorem 3, with $n = k^2 + 1$, $a = 0$, $b = 1$, we must have

$$\frac{1}{2}\left(k^2 \pm \frac{k^2 - 2k}{\sqrt{1 + 4(k-1)}}\right) \in \mathbb{Z}.$$

So either $k^2 - 2k = 0$ or $1 + 4(k-1) = 4k - 3$ is a perfect square.

If $k^2 - 2k = 0$, then $k = 2$. Done.

If $4k - 3 = t^2$, then t divides $k^2 - 2k = \left(\frac{t^2+3}{4}\right)^2 - 2\left(\frac{t^2+3}{4}\right)$.

So t divides $(t^2 + 3)^2 - 8(t^2 + 3) = t^4 - 2t^2 + 15$.

Thus t divides 15, so $t = 1, 3, 5, 15$, giving $k = 1$ (not possible), $3, 7, 57$. $\square$

1. Construct a 3-regular graph on 8 vertices. Is there a 3-regular graph on 9 vertices?

2. How many spanning trees does K_4 have?

3. Prove that every connected graph has a vertex that is not a cutvertex.

4. Let G be a graph on n vertices, $G \neq K_n$. Show that G is a tree if and only if the addition of any edge to G produces exactly 1 new cycle.

5. Let $n \geq 2$, and let $d_1 \leq d_2 \ldots \leq d_n$ be a sequence of integers. Show that there is a tree with degree sequence $d_1, \ldots, d_n$ if and only if $d_1 \geq 1$ and $\sum d_i = 2n - 2$.

6. Let $T_1, \ldots, T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex in all the T_i.

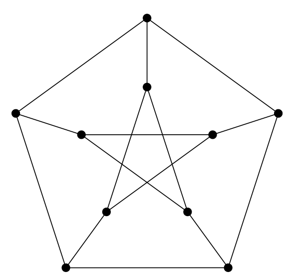
7. Show that every graph of average degree d contains a subgraph of minimum degree at least $d/2$.

8. The *clique number* of a graph G is the maximum order of a complete subgraph of G. Show that the possible clique numbers for a regular graph on n vertices are $1, 2, \ldots, \lfloor n/2 \rfloor$ and n.

9. Let G be a graph on vertex set V. Show that there is a partition $V_1 \cup V_2$ of V such that in each of $G[V_1]$ and $G[V_2]$ all vertices have even degree.

10. For which n and m is the complete bipartite graph $K_{n,m}$ planar?

11. Prove that the Petersen graph (shown) is not planar.



12. The *square* of a graph G has vertex set that of G and edge set $\{xy : d_G(x, y) \leq 2\}$. For which n is the square of the n-cycle planar?

13. Prove that every planar graph has a drawing in the plane in which every edge is a straight-line segment.

$^{+}$14. The group of all isomorphisms from a graph G to itself is called the *automorphism group* of G. Show that every finite group is the automorphism group of some graph. Is every group the automorphism group of some (possibly infinite) graph?

1. For which n and m is the complete bipartite graph $K_{n,m}$ Hamiltonian? Is the Petersen graph Hamiltonian?

2. Let G be a graph of order n with $e(G) > \binom{n}{2} - (n-2)$. Prove that G is Hamiltonian.

3. Let G be a bipartite graph with vertex classes X, Y. Show that if G has a matching from X to Y then there exists $x \in X$ such that every edge incident with x extends to a matching from X to Y.

4. Let G be a connected bipartite graph with vertex classes X, Y. Show that every edge of G extends to a matching from X to Y if and only if $|\Gamma(A)| > |A|$ for every $A \subset X$, $A \neq \emptyset, X$.

5. Let A be a matrix with each entry 0 or 1. Prove that the minimum number of rows and columns containing all the 1s of A equals the the maximum number of 1s that can be found with no two in the same row or column.

6. An $n \times n$ *Latin square* (resp. $r \times n$ *Latin rectangle*) is an $n \times n$ (resp. $r \times n$) matrix, with each entry from $\{1, \ldots, n\}$, such that no two entries in the same row or column are the same. Prove that every $r \times n$ Latin rectangle may be extended to an $n \times n$ Latin square.

$^+$7. Let G be a (possibly infinite) bipartite graph, with vertex classes X, Y, such that $|\Gamma(A)| \geq |A|$ for every $A \subset X$. Give an example to show that G need not contain a matching from X to Y. Show however that if G is countable and $d(x) < \infty$ for every $x \in X$ then G does contain a matching from X to Y. Does this remain true if G is uncountable?

8. Show that we always have $\kappa(G) \leq \lambda(G)$. For any positive integers $k \leq l$, construct a graph G with $\kappa(G) = k$ and $\lambda(G) = l$.

9. For a set $B \subset V(G)$ and a vertex a not in B, an *a-B fan* is a family of $|B|$ paths from a to B, any two meeting only at a. Show that a graph G (with $|G| > k$) is k-connected if and only if there is an a-B fan for every $B \subset V(G)$ with $|B| = k$ and every vertex a not in B.

10. Let G be a k-connected graph ($k \geq 2$), and let $x_1, \ldots, x_k$ be vertices of G. Show that there is a cycle in G containing all the x_i.

11. For each $r \geq 3$, construct a graph G such that G does not contain K_r but G is not $(r-1)$-partite.

12. Let G be a graph of order n that does not contain an even cycle. Prove that each vertex x of G with $d(x) \geq 3$ is a cutvertex, and deduce that G has at most $\lfloor 3(n-1)/2 \rfloor$ edges. Give (for each n) a graph for which equality holds. How does this bound compare with the maximum number of edges of a graph of order n containing no *odd* cycles?

13. A *deleted* K_r consists of a K_r from which an edge has been removed. Show that if G is a graph of order n ($n \geq r+1$) with $e(G) > e(T_{r-1}(n))$ then G contains a deleted K_{r+1}.

14. A *bowtie* consists of two triangles meeting in one vertex. Show that if G is a graph of order n ($n \geq 5$) with $e(G) > \lfloor n^2/4 \rfloor + 1$ then G contains a bowtie.

$^+$15. Let G be an r-regular graph on $2r+1$ vertices. Prove that G is Hamiltonian.

1. What is the chromatic number of the Petersen graph? What is its edge-chromatic number?

2. What is $\chi'(K_{n,n})$? What is $\chi'(K_n)$?

3. Let G be a graph with chromatic number k. Show that $e(G) \geq \binom{k}{2}$.

4. Show that, for any graph G, there is an ordering of the vertices of G for which the greedy algorithm uses only $\chi(G)$ colours.

5. For each $k \geq 3$, find a bipartite graph G, with an ordering $v_1, v_2, \ldots, v_n$ of its vertices, for which the greedy algorithm uses k colours. Give an example with $n = 2k-2$. Is there an example with $n = 2k-3$?

6. Let G be a bipartite graph of maximum degree Δ. Must we have $\chi'(G) = \Delta$?

7. Find the chromatic polynomial of the n-cycle.

8. Let G be a graph on n vertices, with $p_G(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \ldots + a_1 t + a_0$. Show that the a_i alternate in sign (in other words, $a_i \leq 0$ if $n-i$ is odd and $a_i \geq 0$ if $n-i$ is even). Show also that if G has m edges and c triangles then $a_{n-2} = \binom{m}{2} - c$.

9. An *acyclic orientation* of a graph G is an assignment of a direction to each edge of G in such a way that there is no directed cycle. Show that the number of acyclic orientations of G is precisely $|p_G(-1)|$.

10. Let G be a plane graph in which every face is a triangle. Show that the faces of G may be 3-coloured, unless $G = K_4$.

11. Can $K_{4,4}$ be drawn on the torus? What about $K_{5,5}$?

12. A *minor* of a graph G is any graph that may be obtained from a subgraph of G by successively contracting edges – equivalently, a graph H on vertex-set $\{v_1, \ldots, v_r\}$ is a minor of G if we can find disjoint connected subgraphs $S_1, \ldots, S_r$ of G such that whenever $v_i v_j \in E(H)$ there is an edge from S_i to S_j. Show that for any k there is an n such that every graph G with $\chi(G) \geq n$ has a K_k minor. Writing $c(k)$ for the least such n, show that $c(k+1) \leq 2c(k)$. [Hint: choose $x \in G$, and look at the sets $\{y \in G : d(x,y) = t\}$.] Show that $c(k) = k$ for $1 \leq k \leq 4$, and explain why $c(5) = 5$ would imply the 4-Colour Theorem.

13. Let G be a countable graph in which every finite subgraph can be k-coloured. Show that G can be k-coloured.

$^+$14. Construct a triangle-free graph of chromatic number 1526.

1. Show that $R(3,4) \leq 9$. By considering the graph on $\mathbb{Z}_8$ (the integers modulo 8) in which x is joined to y if $x - y = \pm 1$ or ± 2, show that $R(3,4) = 9$.

2. By considering the graph on $\mathbb{Z}_{17}$ in which x is joined to y if $x - y$ is a square modulo 17, show that $R(4,4) = 18$.

3. Show that $R_3(3,3,3) \leq 17$.

4. Let A be a set of $R^{(4)}(n,5)$ points in the plane, with no three points of A collinear. Prove that A contains n points forming a convex n-gon.

5. Let A be an infinite set of points in the plane, with no three points of A collinear. Prove that A contains an infinite set B such that no point of B is a convex combination of other points of B.

6. Show that every graph G has a partition of its vertex-set as $X \cup Y$ such that the number of edges from X to Y is at least $\frac{1}{2}e(G)$. Give three proofs: by induction, by choosing an optimal partition, and by choosing a random partition.

7. In a *tournament* on n players, each pair play a game, with one or other player winning (there are no draws). Prove that, for any k, there is a tournament in which, for any k players, there is a player who beats all of them. [Hint: consider a random tournament.] Exhibit such a tournament for $k = 2$.

8. Let X denote the number of copies of K_4 in a random graph G chosen from $G(n,p)$. Find the mean and the variance of X. Deduce that $p = n^{-2/3}$ is a threshold for the existence of a K_4, in the sense that if $pn^{2/3} \to 0$ then almost surely G does not contain a K_4, while if $pn^{2/3} \to \infty$ then almost surely G does contain a K_4.

9. Find the eigenvalues of K_n. Find the eigenvalues of $K_{n,m}$.

10. Prove that the matrix J (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph G if and only if G is regular and connected.

11. Let G be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that G is k-regular, for some k, and that the number of vertices of G is $1 + k^2/2$. Show also that k must belong to the set $\{2, 4, 14, 22, 112, 994\}$.

12. Let the infinite subsets of $\mathbb{N}$ be 2-coloured. Must there exist an infinite set $M \subset \mathbb{N}$ all of whose infinite subsets have the same colour?

$^+$13. Let A be an uncountable set, and let $A^{(2)}$ be 2-coloured. Must there exist an uncountable monochromatic set in A?