Stochastic Financial Models 12

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1 Properties of conditional expectations

Theorem. *Supposing all conditional expectations are defined:*

- *additivity:* $\mathbb{E}(X+Y|\mathcal{G}) = \mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Y|\mathcal{G})$
- *'Pulling out a known factor': If X is $\mathcal{G}$-measurable, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$.*
- *tower property: If $\mathcal{H} \subseteq \mathcal{G}$ then*

$$\mathbb{E}[\mathbb{E}(X|\mathcal{G})|\mathcal{H}] = \mathbb{E}[\mathbb{E}(X|\mathcal{H})|\mathcal{G}] = \mathbb{E}(X|\mathcal{H})$$

- *If X is independent of $\mathcal{G}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$.*
- *positivity: If $X \geq 0$, then $\mathbb{E}(X|\mathcal{G}) \geq 0$.*
- *Jensen's inequality: If f is convex, then $\mathbb{E}[f(X)|\mathcal{G}] \geq f[\mathbb{E}(X|\mathcal{G})]$*
- *'Fix known quantity and average independent one': If X is independent of $\mathcal{G}$ and Y is $\mathcal{G}$-measurable, then*

$$\mathbb{E}[f(X,Y)|\mathcal{G}] = \mathbb{E}[f(X,y)|\mathcal{G}]\big|_{y=Y}$$

Example. Suppose X, Y are independent $N(0,1)$ random variables, and let $\mathcal{G} = \sigma(Y)$. Then

$$\mathbb{E}[f(X,Y)|\mathcal{G}] = \int f(x,Y)\varphi(x)dx$$

where φ is the probability density function of $N(0,1)$.

2 Filtrations, adaptedness and martingales

Definition. A *filtration* is a family $(\mathcal{F}_t)_{t\geq 0}$ of sigma-algebras such that $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $0 \leq s \leq t$.

Convention for this course: Unless otherwise specified, we will assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Definition. A *stochastic process* is a family $(X_t)_{t\geq 0}$ of random variables.

Definition. A stochastic process $(X_t)_{t\geq 0}$ is *adapted* to a filtration $(\mathcal{F}_t)_{t\geq 0}$ iff X_t is $\mathcal{F}_t$ measurable for all $t \geq 0$. The process is *integrable* if $\mathbb{E}(|X_t|) < \infty$ for all $t \geq 0$.

Remark. By our convention, if $(X_t)_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$, then X_0 is a constant, that is, not random.

The following definition is will be useful for examples.

Definition. The filtration $(\mathcal{F}_t)_{t\geq 0}$ *generated* by a process $(X_t)_{t\geq 0}$ is $\mathcal{F}_t = \sigma(X_s :\ 0 \leq s \leq t)$ for all $t \geq 0$. (i.e. the smallest fitration such that the process is adapted)

Definition. An adapted, integrable process $(X_t)_{t\geq 0}$ is a *martingale* with respect to a filtration $(\mathcal{F}_t)_{t\geq 0}$ iff

$$\mathbb{E}(X_t|\mathcal{F}_s) = X_s \text{ for all } 0 \leq s \leq t$$

Remark. By the rules of conditional expectations, an equivalent definition is this: An adapted, integrable process $(X_n)_{n\geq 0}$ is a martingale iff

$$\mathbb{E}(X_t - X_s|\mathcal{F}_s) = 0 \text{ for all } 0 \leq s \leq t.$$

Theorem. *An adapted, integrable discrete-time process $(X_n)_{n\geq 0}$ is a martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$ iff*

$$\mathbb{E}(X_n|\mathcal{F}_{n-1}) = X_{n-1} \textit{ for all } n \geq 1.$$

Proof. If $(X_n)_{n\geq 0}$ is a martingale, then we can use the definition with $s = n-1$ and $t = n$.

Now suppose the given condition holds for all $n \geq 1$. Note that for $k \geq 0$ we have

$$\begin{aligned}\mathbb{E}(X_{s+k}|\mathcal{F}_s) &= \mathbb{E}[\mathbb{E}(X_{s+k}|\mathcal{F}_{s+k-1})|\mathcal{F}_s] \\ &= \mathbb{E}[X_{s+k-1}|\mathcal{F}_s]\end{aligned}$$

by the tower property. Hence the martingale property is proven fixing s and using induction in t. □

Example. Given a filtration $(\mathcal{F}_t)_{t\geq 0}$ and an integrable random variable Y. Let $X_t = \mathbb{E}(Y|\mathcal{F}_t)$ for $t \geq 0$. Then $(X_t)_{t\geq 0}$ is a martingale.

- That X_t is integrable and $\mathcal{F}_t$-measurable is from the definition of conditional expectation.
- and $\mathbb{E}(X_t|\mathcal{F}_s) = \mathbb{E}[\mathbb{E}(Y|\mathcal{F}_t)|\mathcal{F}_s] = \mathbb{E}(Y|\mathcal{F}_s) = X_s$ by the tower property.