# Stochastic Financial Models 19

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## 1 Pricing and hedging European claims

**Definition.** A *European* contingent claim is an asset that pays an $\mathcal{F}_N$-measurable amount $Y$ at a fixed maturity date $N$.

Consider the binomial model with $a < r < b$ and a European claim with time $N$ payout $Y$. Assume that the filtration is generated by the $(S_n)_n$. This means there exists a function $f_N$ such that

$$Y = f_N(S_0, \ldots, S_N)$$

Since there is only one risk-neutral measure $\mathbb{Q}$, the unique time-$n$ no-arbitrage price of the claim

$$\begin{aligned} \pi_n &= \frac{1}{(1+r)^{N-n}} \mathbb{E}^{\mathbb{Q}}[Y|\mathcal{F}_n] \\ &= f_n(S_0, \ldots, S_n) \end{aligned}$$

where the function $f_n$ exists by measurability.

**Theorem.** *The wealth process starting from $X_0 = \pi_0$ employing the trading strategy $(\theta_n)_{1 \leq n \leq N}$ defined by*

$$\theta_n = \frac{f_n(S_0, \ldots, S_{n-1}, S_{n-1}(1+b)) - f_n(S_0, \ldots, S_{n-1}, S_{n-1}(1+a))}{S_{n-1}(b-a)}$$

*is such that $X_n = \pi_n$ for all $0 \leq n \leq N-1$ and $X_N = Y$.*

*Proof.* For each $n$, there is a unique $\mathcal{F}_{n-1}$-measurable solution $(x_{n-1}, b_n)$ to the equation

$$(1+r)x_{n-1} + b_n[S_n - (1+r)S_{n-1}] = \pi_n$$

i.e. the pair of equations

$$\begin{aligned} (1+r)x_{n-1} + b_n S_{n-1}(b-r) &= f_n(S_0, \ldots, S_{n-1}, S_{n-1}(1+b)) \\ (1+r)x_{n-1} + b_n S_{n-1}(a-r) &= f_n(S_0, \ldots, S_{n-1}, S_{n-1}(1+a)) \end{aligned}$$

This solution is $b_n = \theta_n$ and

$$\begin{aligned} x_{n-1} &= \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(\pi_n|\mathcal{F}_{n-1}) \\ &= \frac{1}{(1+r)^{N-n+1}}\mathbb{E}^{\mathbb{Q}}(Y|\mathcal{F}_{n-1}) \\ &= \pi_{n-1} \end{aligned}$$

by the tower property. Hence, if $X_0 = \pi_0$ and

$$X_n = (1+r)X_{n-1} + \theta_n[S_n - (1+r)S_{n-1}]$$

for $1 \le n \le N$, we have by induction that $X_n = \pi_n$ for all $0 \le n \le N-1$ and $X_N = Y$. $\square$

A European claim is often called *plain vanilla* if its payout of the form $Y = g(S_N)$ for some function $g$. For instance a call option with payout $Y = (S_N - K)^+$ is a vanilla contingent claim. Otherwise, a claim whose payout depends on the entire path of the underlying asset price is called *exotic*.

In the case of the binomial model, the risky asset price is Markovian under $\mathbb{Q}$. Hence, for vanilla claims, we have

$$\pi_n = V(n, S_n)$$

where the function is defined by

$$\begin{aligned} V(n,s) &= \frac{1}{(1+r)^{N-n}}\mathbb{E}^{\mathbb{Q}}[g(S_N)|S_n = s] \\ &= \frac{1}{(1+r)^{N-n}}\sum_{k=0}^{N-n}\binom{N-n}{k}q^k(1-q)^{N-n-k}g\big(s(1+b)^k(1+a)^{N-n-k}\big) \end{aligned}$$

for all $0 \le n \le N$ we note

$$\begin{aligned} V(N,s) &= g(s) \\ V(n-1,s) &= \frac{1}{1+r}\Big(q\, V(n, s(1+b)) + (1-q)V(n, s(1+a))\Big) \text{ for } 1 \le n \le N \end{aligned}$$

# 2 American claims

**Definition.** Given an adapted process $(Y_n)_{0 \le n \le N}$, an *American* contingent claim is a contract that pays its owner $Y_n$ if the owner chooses to exercise the contract at time $n$.

**Example.** An American put gives its owner the right, but not the obligation, to *sell* a certain stock for a fixed strike price $K$ for at *any time* up to the expiry $N$. The payout if exercised at time $n$ is $(K - S_n)^+$.

The time-$n$ price of an American claim in a binomial model with unique risk neutral measure $\mathbb{Q}$ can be calculated as

$$\pi_n = \max_{n \le T \le N} \mathbb{E}^{\mathbb{Q}} \left[ \frac{1}{(1+r)^{T-n}} Y_T | \mathcal{F}_n \right]$$

where the maximum is over stopping times $T$.

By the dynamic programming principle

$$\begin{aligned} \pi_N &= Y_N \\ \pi_{n-1} &= \max \left\{ Y_{n-1}, \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}(\pi_n | \mathcal{F}_{n-1}) \right\} \end{aligned}$$

An optimal stopping time is

$$T^* = \min\{0 \le n \le N : \pi_n = Y_n\}$$

but it need not be unique.

Note that $\pi_n \ge Y_n$ for all $0 \le n \le N$. That is, the price of the American claim always dominates the current available payout of the claim.

Also $\pi_{n-1} \ge \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}(\pi_n | \mathcal{F}_{n-1})$ for all $1 \le n \le N$. So the discounted price process $\big((1+r)^{-n}\pi_n\big)_{0 \le n \le N}$ is a supermartingale.

However, on the event $\{n \le T^*\}$ we have $\pi_{n-1} = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}(\pi_n | \mathcal{F}_{n-1})$, so the discounted price process is a martingale up to the optimal stopping time. That means we can find the hedging strategy just as in the case of European claims, by finding the unique $\mathcal{F}_{n-1}$ measurable $\theta_n$ such that

$$(1+r)\pi_{n-1} + \theta_n[S_n - (1+r)S_{n-1}] = \pi_n.$$