Stochastic Financial Models 8

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1 Arbitrage

Recall the set-up

- one risk-free asset with interest rate r
- d risky assets with time-t price S_t for $t \in \{0, 1\}$

Definition. An *arbitrage* is a portfolio $\varphi \in \mathbb{R}^d$ such that

$$\varphi^\top [S_1 - (1+r)S_0] \geq 0 \text{ almost surely}$$

Depends on $\mathbb{P}$ only through events of measure 0.

and

$$\mathbb{P}\big(\varphi^\top [S_1 - (1+r)S_0] > 0\big) > 0.$$

Arbitrage and utility maximisation

Fix initial wealth X_0 and strictly increasing utility function U, consider the problem

$$\text{maximise } \mathbb{E}[U(X)] \text{ over } X \in \mathcal{X}$$

where

$$\mathcal{X} = \{(1+r)X_0 + \theta^\top [S_1 - (1+r)S_0] : \quad \theta \in \mathbb{R}^d\}$$

- Suppose φ is an arbitrage.
- Given $X \in \mathcal{X}$ consider
$$X^* = X + \varphi^\top [S_1 - (1+r)S_0]$$
- Note $X^* \in \mathcal{X}$ also, but
$$U(X^*) \geq U(X) \text{ almost surely}$$
and
$$\mathbb{P}\big(U(X^*) > U(X)\big) > 0$$
- Hence
$$\mathbb{E}[U(X^*)] > \mathbb{E}[U(X)]$$

- Since $X \in \mathcal{X}$ was arbitrary, there cannot be a maximiser!

Why arbitrages are bad for theory

- Suppose φ is an arbitrage.
- From above, an investor would prefer the portfolio $(n+1)\varphi$ to $n\varphi$ for any n.
- As n gets large, the assumption that an agent can trade with no price impact becomes more and more unrealistic.

Comments

- The definition of arbitrage does not depend on the agent's initial wealth X_0 or utility function U.
- However, it does depend on the agent's *beliefs* through the probability measure $\mathbb{P}$.
- Agents with equivalent beliefs will agree on the set of arbitrage portfolios.

2 Fundamental theorem of asset pricing

Things we know so far

- If there exists an optimal solution to a utility maximisation problem, then there exists risk-neutral measure.
- If there exists an optimal solution to a utility maximisation problem, then there exists no arbitrage.

Theorem (FTAP)**.** *A market model has no arbitrage if and only if there exists a risk-neutral measure.*

Proof of the easy direction. Let φ be such that

$$\mathbb{P}\big(\varphi^\top[S_1 - (1+r)S_0] \geq 0\big) = 1.$$

Suppose there exists a risk-neutral measure $\mathbb{Q}$. By equivalence

$$\mathbb{Q}\big(\varphi^\top[S_1 - (1+r)S_0] \geq 0\big) = 1.$$

However

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\{\varphi^\top[S_1 - (1+r)S_0]\} &= \varphi^\top \mathbb{E}^{\mathbb{Q}}[S_1 - (1+r)S_0] \\ &= 0\end{aligned}$$

by the definition of risk-neutrality.

By the pigeon-hole principle

$$\mathbb{Q}\big(\varphi^\top [S_1 - (1+r)S_0] > 0\big) = 0.$$

Again by equivalence

$$\mathbb{P}\big(\varphi^\top [S_1 - (1+r)S_0] > 0\big) = 0.$$

Hence φ is not an arbitrage. □

Proof of the harder direction of the FTAP. Assume that there is no arbitrage. For easier notation, let $\xi = S_1 - (1+r)S_0$.

We also assume without loss that

$$\mathbb{E}[e^{-\theta^\top \xi}] < \infty$$

for all $\theta \in \mathbb{R}^d$. (Otherwise, we replace $\mathbb{P}$ with the equivalent measure $\widetilde{\mathbb{P}}$ with density

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}} \propto e^{-\|\xi\|^2}$$

and note by equivalence there is no $\widetilde{\mathbb{P}}$-arbitrage.)

Consider the problem of maximising $\mathbb{E}[U(\theta^\top \xi)]$ and $U(x) = -e^{-x}$. We will show that the assumption of no arbitrage implies that there exists an optimal solution.

Let $(\theta_n)_n$ be a sequence such that

$$\mathbb{E}[U(\theta_n^\top \xi)] \to \sup\{\mathbb{E}[U(\theta^\top \xi)] : \quad \theta \in \mathbb{R}^d\}$$

Case: $(\theta_n)_n$ is bounded. Then by the Bolzano–Weierstrass theorem, there exists a convergent subsequence. By passing to that subsequence, we assume $\theta_n \to \theta_0$.

By continuity

$$\mathbb{E}[U(\theta_n^\top \xi)] \to \mathbb{E}[U(\theta_0^\top \xi)]$$

Hence θ_0 is a maximiser. We are done since $U'(\theta_0^\top \xi)$ is proportional to the density of a risk-neutral measure.

Case: every maximising sequence $(\theta_n)_n$ is unbounded. (next time)