Stochastic Financial Models 5

Michael Tehranchi

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1 Contingent claims

In the context of a one-period model a *contingent claim* is just another name for an asset with a random payout at time 1.

- interest rate r and d risky assets with time t price vector S_t, for $t \in \{0, 1\}$. These are thought of as 'fundamental' assets.

- We introduce a $(d+1)$st risky asset with time-1 payout Y.

- Often $Y = g(S_1)$ for some function g, but not always.

- The problem is to find a 'reasonable' time-0 price for the claim

Example

**Definition.** A *call option* is the right, but not the obligation, to buy a certain asset at a certain price (called the strike) at a certain time in the future (the maturity date).

Call option
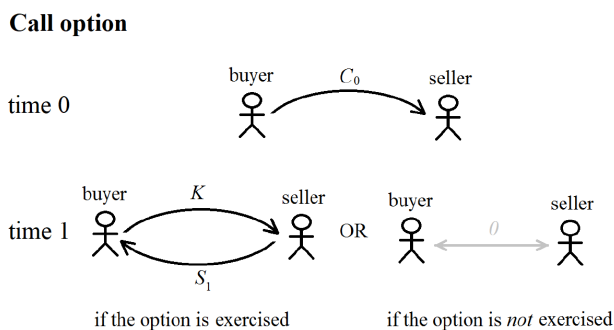
- If $S_1 > K$ it is rational to receive the payout $S_1 - K$.

- If $S_1 \le K$ it is rational to let the call expire unexercised.

- The payout is $(S_1 - K)^+ = g(S_1)$

- notation: $x^+ = \max\{x, 0\}$ is the positive part of the real number x.

Call option

if rationally exercised

buyer C_0 seller

buyer   seller

$C_1 = (S_1 - K)^+$
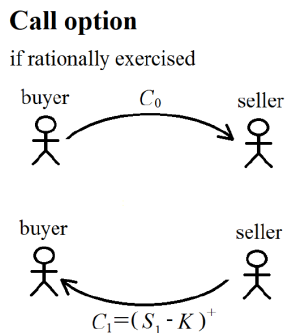
2   Indifference pricing

Consider an investor with initial wealth X_0 and concave, increasing utility function U. She is offered to buy a contingent claim with payout Y. How much should she pay?

- Let

$$\mathcal{X} = \{(1+r)X_0 + \theta^\top[S_1 - (1+r)S_0] : \theta \in \mathbb{R}^d\}$$

be the set of time-1 wealths attainable from trading the original market.

- The agent would prefer to buy one share of the contingent claim with time-1 payout Y for time-0 price π iff there exists an $X^* \in \mathcal{X}$ such that

$$\mathbb{E}[U(X^* + Y - (1+r)\pi)] \geq \mathbb{E}[U(X)]$$

for all $X \in \mathcal{X}$.

Assumption. In the examples from this course, we will assume that the data of the problem is such that any given utility maximisation problem has a solution.

Definition. An *indifference* (or *reservation*) price of the claim with payout Y is any solution π of

$$\max_{X \in \mathcal{X}} \mathbb{E}[U(X + Y - (1+r)\pi)] = \max_{X \in \mathcal{X}} \mathbb{E}[U(X)]$$

3   Properties of indifference prices

Theorem. *Under our assumptions[1], indifference prices exist and are unique.*

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[1]For the technically minded, we will assume the random variable $U(X + Y + x)$ is integrable for all $X \in \mathcal{X}$, $x \in \mathbb{R}$, and possible payouts Y, and that for x, Y there exists $X^* \in \mathcal{X}$ such that $\mathbb{E}[U(X^* + Y + x)] = \max_{X \in \mathcal{X}} \mathbb{E}[U(X + Y + x)]$

Proof. Next time.

Notation. For a fixed initial wealth X_0 and utility function U, we will let $\pi(Y)$ denote the (unique) indifference price of a contingent claim with payout Y.

Theorem (Indifference prices are increasing). *If $Y_0 \leq Y_1$ almost surely with $\mathbb{P}(Y_0 < Y_1) > 0$ then*

$$\pi(Y_0) < \pi(Y_1)$$

Proof. Next time.

Theorem (Indifference prices are concave). *Given random variable Y_0, Y_1 and $0 < p < 1$, we have*

$$\pi(pY_1 + (1-p)Y_0) \geq p\ \pi(Y_1) + (1-p)\pi(Y_0)$$

Proof. Next time.

Definition. The marginal utility price of a claim with payout Y is

$$\pi_0(Y) = \frac{\mathbb{E}[U'(X^*)Y]}{(1+r)\mathbb{E}[U'(X^*)]}.$$

where $X^* \in \mathcal{X}$ is such that $\mathbb{E}[U(X^*)] = \max_{X \in \mathcal{X}} \mathbb{E}[U(X)]$.

Note that our first marginal utility pricing theorem (from last time) says

$$\pi_0(a + b^\top S_1) = \frac{a}{1+r} + b^\top S_0$$

for any $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

Theorem (Marginal utility price is larger than indifference price).

$$\pi(Y) \leq \pi_0(Y)$$

Proof. Next time.

Theorem (Convergence of indifference prices to marginal utility prices).

$$\lim_{\varepsilon \to 0} \frac{\pi(\varepsilon Y)}{\varepsilon} = \pi_0(Y)$$

Proof. Next time.