Part IB — Analysis and Topology

Based on lectures by Dr P. Russell

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Contents

Part I
Generalizing continuity and convergence

§1 Three Examples of Convergence

§1.1 Convergence in $\mathbb{R}$

Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. We say (x_n) *converges* to x and write $x_n \to x$ if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, n \geq N \quad |x_n - x| < \epsilon.$$

Useful fact: $\forall\, a, b \in \mathbb{R}\ |a+b| \leq |a| + |b|$ (Triangle Inequality).

Bolzano-Weierstrass Theorem (BWT) A bounded sequence in $\mathbb{R}$ must have a convergent subsequence (Proof by interval bisection).

Recall: A sequence (x_n) in $\mathbb{R}$ is Cauchy if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, m, n \geq N \quad |x_m - x_n| < \epsilon.$$

Easy exercise Convergent $\implies$ Cauchy

General Principle of Convergence (GPC) Any Cauchy sequence in $\mathbb{R}$ converges.

Outline. If (x_n) Cauchy then (x_n) bounded so by BWT has a convergent subsequence, say $x_{n_j} \to x$. But as (x_n) Cauchy, $x_n \to x$. □

§1.2 Convergence in $\mathbb{R}^2$

Remark 1. This all works in $\mathbb{R}^n$

Let (z_n) be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. What should $z_n \to z$ mean?

In $\mathbb{R}$: "As n gets large, z_n gets arbitrarily close to z."

What does 'close' mean in $\mathbb{R}^2$?

In $\mathbb{R}$: a, b close if $|a-b|$ small. In $\mathbb{R}^2$: Replace $|\cdot|$ by $\|\cdot\|$

Recall: If $z = (x, y)$ then $\|z\| = \sqrt{x^2 + y^2}$.

Triangle Inequality If $a, b \in \mathbb{R}^2$ then $\|a+b\| \leq \|a\| + \|b\|$.

Definition 1.1

Let (z_n) be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. We say (z_n) **converges** to z and .. $z_n \to z$ if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, n \geq N\ \ \|z_n - z\| < \epsilon$.

Equivalently, $z_n \to z$ iff $\|z_n - z\| \to 0$ (convergence in $\mathbb{R}$).

Example 1.1

Let $(z_n), (w_n)$ be sequences in $\mathbb{R}^2$ with $z_n \to z, w_n \to w$. Then $z_n + w_n \to z + w$.

Proof.

$$\begin{aligned} \|(z_n + w_n) - (z + w)\| &\leq \|z_n - z\| + \|w_n - w\| \\ &\to 0 + 0 = 0 \text{ (by results from IA).} \end{aligned}$$

□

In fact, given convergence in $\mathbb{R}$, convergence in $\mathbb{R}^2$ is easy:

Proposition 1.1

Let (z_n) be a sequence in $\mathbb{R}^2$ and let $z \in \mathbb{R}^2$. Write $z_n = (x_n, y_n)$ and $z = (x, y)$. Then $z_n \to z$ iff $x_n \to x$ and $y_n \to y$.

Proof. ($\implies$): $|x_n - x|, |y_n - y| \le \|z_n - z\|$. So if $\|z_n - z\| \to 0$ then $|x_n - x| \to 0$ and $|y_n - y| \to 0$.

($\impliedby$): If $|x_n - x| \to 0$ and $|y_n - y| \to 0$ then $\|z_n - z\| = \sqrt{(x_n - x)^2 + (y_n - y)^2} \to 0$ by results in $\mathbb{R}$. □

Definition 1.2 (Bounded Sequence)

A sequence (z_n) in $\mathbb{R}^2$ is **bounded** if $\exists\, M \in \mathbb{R}$ s.t. $\forall\, n\ \ \|z_n\| \le M$.

Theorem 1.1 (BWT in $\mathbb{R}^2$)

A bounded sequence in $\mathbb{R}^2$ must have a convergent subsequence.

Theorem 1.2 (GPC for $\mathbb{R}^2$)

Any Cauchy sequence in $\mathbb{R}^2$ converges.

Proof. Let (z_n) be a Cauchy sequence in $\mathbb{R}^2$. Write $z_n = (x_n, y_n)$. For all m, n, $|x_m - x_n| \le \|z_m - z_n\|$ so (x_n) is a Cauchy sequence in $\mathbb{R}$, so converges by GPC. Similarly, (y_n) converges in $\mathbb{R}$. So by **??**, (z_n) converges. □

<u>Thought for the day</u> What about continuity? Let $f : \mathbb{R}^2 \to \mathbb{R}$. What does it mean for f to be continuous? (Simple modification of defn for $\mathbb{R} \to \mathbb{R}$).

What can we do with it?

Big theorem in IA: If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on a closed bounded interval then f is bounded and attains its bounds.

Is there a similar theorem for $\mathbb{R}^2 \to \mathbb{R}$. What do we replace 'closed bounded interval' by? We proved the theorem using BWT. Why did it work? Why did we need a closed bounded interval to make it work? What can we do in $\mathbb{R}^2$?

§2 Metric Spaces

§3 Topological Spaces

Part II
Generalizing differentiation