Part IB — Methods

Based on lectures by Dr E P Shellard and notes by thirdsgames.co.uk

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§3.1 Waves on an elastic string

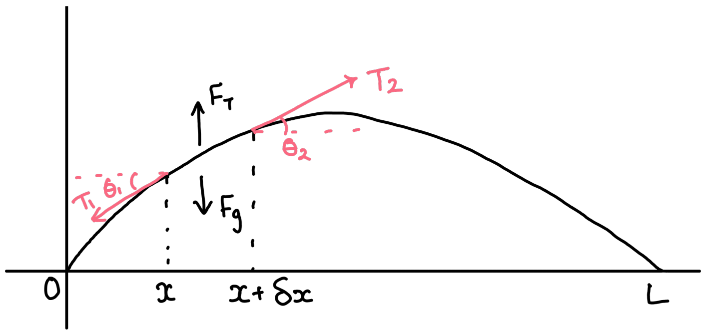
Consider a small displacement $y(x,t)$ on a stretched string with fixed ends at $x = 0$ and $x = L$, that is, with boundary conditions

$$y(0,t) = y(L,t) = 0. \tag{3.1}$$

and initial conditions

$$y(x,0) = p(x),\ \frac{\partial y}{\partial t}(x,0) = q(x) \tag{3.2}$$

We derive the equation of motion governing the motion of the string by balancing forces on a string segment $(x, x+\delta x)$ and take the limit as $\delta x \to 0$.



Let T_1 be the tension force acting to the left at angle θ_1 from the horizontal. Analogously, let T_2 be the rightwards tension force at angle θ_2. We assume at any point on the string that $\left|\frac{\partial y}{\partial x}\right| \ll 1$, so the angles of the forces, θ_1, θ_2 are small. In the x dimension,

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 \implies T_1 \approx T_2 = T \text{ by small angle approximation}$$

So the tension T is a constant independent of x up to an error of order $O\left(\left|\frac{\partial y}{\partial x}\right|^2\right)$. In the y dimension, since the θ are small,

$$F_T = T_2 \sin\theta_2 - T_1 \sin\theta_1 \approx T\left(\left.\frac{\partial y}{\partial x}\right|_{x+\delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}\right) \approx T\frac{\partial^2 y}{\partial x^2}\delta x$$

By $F = ma$,

$$F_T + F_g = (\mu\,\delta x)\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2}\delta x - g\mu\delta x$$

where F_g is the gravitational force and μ is the mass per unit length (linear mass density). We define the wave speed as

$$c = \sqrt{\frac{T}{\mu}} \text{ (a constant)}$$

and find

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} - g = c^2\frac{\partial^2 y}{\partial x^2} - g \tag{3.3}$$

We often assume gravity is negligible to produce the pure wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}. \tag{3.4}$$

The 1D wave equation is then $\ddot{y} = c^2 y''$.

§3.2 Separation of variables

We wish to solve the wave equation eq. (3.4) subject to boundary conditions eq. (3.1) and initial conditions eq. (3.2). Consider a possible solution of seperable form (ansatz):

$$y(x,t) = X(x)T(t) \tag{3.5}$$

Substituting into the wave equation eq. (3.4),

$$\frac{1}{c^2}\ddot{y} = y'' \implies \frac{1}{c^2}X\ddot{T} = X''T.$$

Then

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X}$$

However, $\frac{\ddot{T}}{T}$ depends only on t and $\frac{X''}{X}$ depends only on x. Thus, both sides must be equal to some *separation constant* $-\lambda$.

$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

Hence,

$$X'' + \lambda X = 0 \tag{3.6}$$

$$\ddot{T} + \lambda c^2 T = 0. \tag{3.7}$$

§3.3 Boundary conditions and normal modes

We will begin by first solving the spatial ODE eq. (3.6). One of $\lambda > 0, \lambda < 0, \lambda = 0$ must be true. The boundary conditions eq. (3.1) restrict the possible λ.

1. First, suppose $\lambda < 0$. Take $\chi^2 = -\lambda$. Then,

   $$X(x) = Ae^{\chi x} + Be^{-\chi x} = \tilde{A}\cosh(\chi x) + \tilde{B}\sinh(\chi x).$$

   The boundary conditions are $x(0) = x(L) = 0$, so only the trivial solution is possible: $\tilde{A} = \tilde{B} = 0$.

2. Now, suppose $\lambda = 0$. Then

   $$X(x) = Ax + B.$$

   Again, the boundary conditions impose $A = B = 0$ giving only the trivial solution.

3. Finally, the last possibility is $\lambda > 0$.

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right)$$

The boundary conditions give

$$A = 0; \quad B\sin\left(\sqrt{\lambda}L\right) = 0 \implies \sqrt{\lambda}L = n\pi.$$

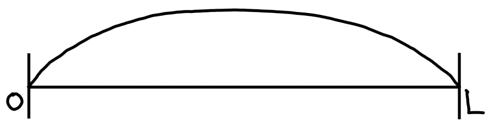
The following are the eigenfunctions and eigenvalues.

$$X_n(x) = B_n \sin\frac{n\pi x}{L}; \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 \ (n > 0) \tag{3.8}$$

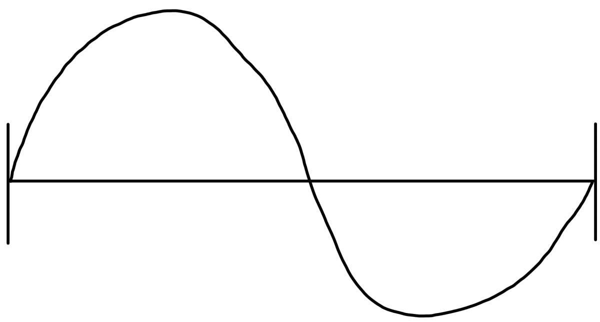
These are also called the **normal modes** of the system because the spatial shape in x does not change in time, but the amplitude may vary.
The fundamental mode is the lowest frequency of vibration, given by

$$n = 1 \implies \lambda_1 = \frac{\pi^2}{L^2}$$

The second mode is the first overtone, and is given by



$$n = 2 \implies \lambda_2 = \frac{4\pi^2}{L^2}$$



§3.4 Initial conditions and temporal solutions

Substituting λ_n into the time ODE eq. (3.7),

$$\ddot{T} + \frac{n^2\pi^2c^2}{L^2}T = 0.$$

Hence,

$$T_n(t) = C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}. \tag{3.9}$$

Therefore, a specific solution of the wave equation, eq. (3.4), satisfying the boundary conditions, eq. (3.1), is (absorbing the B_n into the C_n, D_n):

$$y_n(x,t) = T_n(t)X_n(x) = \left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$$

Exercise 3.1. Verify it's a solution.

Since the wave equation eq. (3.4) is linear (and b.cs eq. (3.1) are homogenous) we can add the solutions (the y_n) together to find **general string solution**

$$y(x,t) = \sum_{n=1}^{\infty}\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}. \tag{3.10}$$

By construction, this $y(x,t)$ satisfies the boundary conditions, so now we can impose the initial conditions eq. (3.2):

$$y(x,0) = p(x) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}$$

We can find the C_n using standard Fourier series techniques eq. (1.12), since this is exactly a half-range sine series. Further,

$$\frac{\partial y(x,0)}{\partial t} = q(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} D_n \sin\frac{n\pi x}{L}$$

Again we can solve for the D_n in a similar way. Using eq. (1.12):

$$\begin{aligned} C_n &= \frac{2}{L}\int_0^L p(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \\ D_n &= \frac{2}{n\pi c}\int_0^L q(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x \end{aligned} \tag{3.11}$$

Hence eq. (3.11) is the solution to eq. (3.4) satisfying eqs. (3.1) and (3.2).

Example 3.1

Consider the initial condition of a see-saw wave parametrised by ξ, and let $L = 1$. This can be visualised as plucking the string at position ξ.

$$y(x,0) = p(x) = \begin{cases} x(1-\xi) & 0 \le x < \xi \\ \xi(1-x) & \xi \le x < 1 \end{cases}$$

We also define

$$\frac{\partial y(x,0)}{\partial t} = q(x) = 0$$

The Fourier series eq. (1.8) for p is given by

$$C_n = \frac{2\sin n\pi\xi}{(n\pi)^2}; \quad D_n = 0$$

Hence the solution to the wave equation is

$$y(x,t) = \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} \sin n\pi\xi \sin n\pi x \cos n\pi ct$$

Take $\xi = \frac{1}{2}, C_{2m} = 0, C_{2m-1} = \frac{2(-1)^{m+1}}{((2m-1)\pi)^2}$ (odd only), e.g. Guitar has $\frac{1}{4} \le \xi \le \frac{1}{3}$, Violin $\xi \approx \frac{1}{7}$.

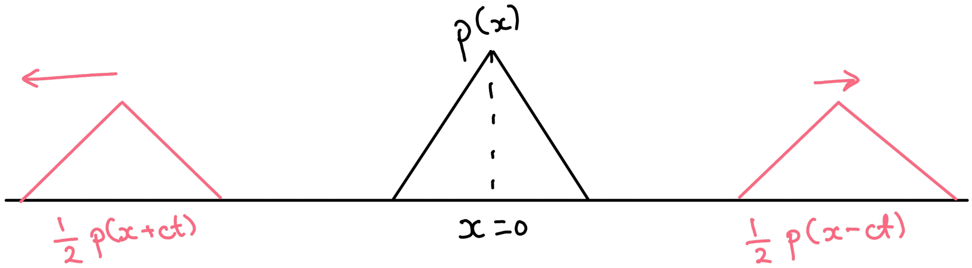
Solution in characterstic coordinates

Recall sine/cosine summation identities (before eq. (1.1)) which means our general solution eq. (3.10) becomes

$$\begin{aligned} y(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty}\left[C_n \sin\frac{n\pi}{L}(x-ct) + D_n \cos\frac{n\pi}{L}(x-ct) + C_n \sin\frac{n\pi}{L}(x+ct) + D_n\cos\frac{n\pi}{L}(x+ct)\right] \\ &\equiv f(x-ct) + g(x+ct) \end{aligned} \tag{3.12}$$

The standing wave solution eq. (3.10) is made up of a right-moving wave (along characteristic $x - ct = \eta$, η a constant) and a left-moving wave ($x + ct = \xi$, ξ a constant) i.e. a general solution with arbitrary f, g (see later).

Special case: $q(x) = 0$ in eq. (3.1) $\implies$ $f = g = \frac{1}{2}D$ at $t = 0$.



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§3.5 Separation of variables methodology

A general strategy for solving higher-dimensional partial differential equations is as follows.

1. Obtain a linear PDE system, using boundary and initial conditions.
2. Separate variables to yield decoupled ODEs.
3. Impose homogeneous boundary conditions to find eigenvalues and eigenfunctions.
4. Use these eigenvalues (constants of separation) to find the eigenfunctions in the other variables.
5. Sum over the products of separable solutions to find the general series solution.
6. Determine coefficients for this series using the initial conditions.

Example 3.2

We will solve the wave equation instead in characteristic coordinates. Recall the sine and cosine summation identities:

$$
\begin{aligned}
y(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty}\left[\left(C_n \sin\frac{n\pi}{L}(x-ct) + D_n\cos\frac{n\pi}{L}(x-ct)\right)\right. \\
&\quad \left. + \left(C_n \sin\frac{n\pi}{L}(x+ct) - D_n\cos\frac{n\pi}{L}(x+ct)\right)\right] \\
&= f(x-ct) + g(x+ct)
\end{aligned}
$$

The standing wave solution can be interpreted as a superposition of a right-moving wave and a left-moving wave. A special case is $q(x) = 0$, implying $f = g = \frac{1}{2}p$. Then,

$$
y(x,t) = \frac{1}{2}[p(x-ct) + p(x+ct)]
$$

§3.6 Energy of oscillations

A vibrating string has kinetic energy due to its motion.

$$
\text{Kinetic energy} = \frac{1}{2}\mu\int_0^L \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x
$$

It has potential energy given by

$$
\text{Potential energy} = T\Delta x = T\int_c^T \left(\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} - 1\right)\mathrm{d}x \approx \frac{1}{2}T\int_0^L \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x
$$

assuming that the disturbances on the string are small, that is, $\left|\frac{\partial y}{\partial x}\right| \ll 1$. The total energy on the string, given $c^2 = T/\mu$, is given by

$$E = \frac{1}{2}\mu \int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right] \mathrm{d}x$$

Substituting the solution, using the orthogonality conditions,

$$\begin{aligned} E = \frac{1}{2}\mu \sum_{n=1}^{\infty} \int_0^L \Bigg[&-\left(\frac{n\pi c}{L} C_n \sin\frac{n\pi ct}{L} + \frac{n\pi c}{L} D_n \cos\frac{n\pi ct}{L}\right)^2 \sin^2\frac{n\pi x}{L} \\ &+ c^2\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)^2 \frac{n^2\pi^2}{L^2} \cos^2\frac{n\pi x}{L}\Bigg] \mathrm{d}x \\ = \frac{1}{4}\mu \sum_{n=1}^{\infty} &\frac{n^2\pi^2c^2}{L}\left(C_n^2 + D_n^2\right) \end{aligned}$$

which is an analogous result to Parseval's theorem. This is true since

$$\int \cos^2\frac{n\pi x}{L}\,\mathrm{d}x = \frac{1}{2}$$

and $\cos^2 + \sin^2 = 1$. We can think of this energy as the sum over all the normal modes of the energy in that specific mode. Note that this quantity is constant over time.

§3.7 Wave reflection and transmission

The travelling wave has left-moving and right-moving modes. A *simple harmonic* travelling wave is

$$y = \mathrm{Re}\left[Ae^{i\omega(t-x/c)}\right] = A\cos\left[\omega(t - x/c) + \phi\right]$$

where the phase ϕ is equal to $\arg A$, and the wavelength λ is $2\pi c/\omega$. In further discussion, we assume only the real part is used. Consider a density discontinuity on the string at $x = 0$ with the following properties.

$$\mu = \begin{cases} \mu_- & \text{for } x < 0 \\ \mu_+ & \text{for } x > 0 \end{cases} \implies c = \begin{cases} c_- = \sqrt{\frac{T}{\mu_-}} & \text{for } x < 0 \\ c_+ = \sqrt{\frac{T}{\mu_+}} & \text{for } x > 0 \end{cases}$$

assuming a constant tension T. As a wave from the negative direction approaches the discontinuity, some of the wave will be reflected, given by $Be^{i\omega(t+x/c_-)}$, and some of the wave will be transmitted, given by $De^{i\omega(t-x/c_+)}$. The boundary conditions at $x = 0$ are

1. y is continuous for all t (the string does not break), so

$$A + B = D \tag{$*$}$$

2. The forces balance, $T\left.\frac{\partial y}{\partial x}\right|_{x=0^-} = T\left.\frac{\partial y}{\partial x}\right|_{x=0^+}$ which means $\frac{\partial y}{\partial x}$ must be continuous for all t. This gives
$$\frac{-i\omega A}{c_-} + \frac{i\omega B}{c_-} = \frac{-i\omega D}{c_+} \tag{$\dagger$}$$

We can eliminate B from $(*)$ by subtracting $\frac{c_-}{i\omega}(\dagger)$.

$$2A = D + D\frac{c_-}{c_+} = \frac{D}{c_+}(c_+ + c_-)$$

Hence, given A, we have the solution for the transmitted amplitude and reflected amplitude to be

$$D = \frac{2c_+}{c_- + c_+}A; \quad B = \frac{c_+ - c_-}{c_- + c_+}$$

In general A, B, D are complex, hence different phase shifts are possible.

There are a number of limiting cases, for example

1. If $c_- = c_+$ we have $D = A$ and $B = 0$ so we have full transmission and no reflection.
2. (Dirichlet boundary conditions) If $\frac{\mu_+}{\mu_-} \to \infty$, this models a fixed end at $x = 0$. We have $\frac{c_+}{c_-} \to 0$ giving $D = 0$ and $B = -A$. Notice that the reflection has occurred with opposite phase, $\phi = \pi$.
3. (Neumann boundary conditions) Consider $\frac{\mu_+}{\mu_-} \to 0$, this models a free end. Then $\frac{c_+}{c_-} \to \infty$ giving $D = 2A$, $B = A$. This gives total reflection but with the same phase.

§3.8 Wave equation in plane polar coordinates

Consider the two-dimensional wave equation for $u(r, \theta, t)$ given by

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

with boundary conditions at $r = 1$ on a unit disc given by

$$u(1, \theta, t) = 0$$

and initial conditions for $t = 0$ given by

$$u(r, \theta, 0) = \phi(r, \theta); \quad \frac{\partial u}{\partial t} = \psi(r, \theta)$$

Suppose that this equation is separable. First, let us consider temporal separation. Suppose that

$$u(r, \theta, t) = T(t)V(r, \theta)$$

Then we have

$$\ddot{T} + \lambda c^2 T = 0; \quad \nabla^2 V + \lambda V = 0$$

In plane polar coordinates, we can write the spatial equation as

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \lambda V = 0$$

We will perform another separation, supposing

$$V(r,\theta) = R(r)\Theta(\theta)$$

to give

$$\Theta'' + \mu\Theta = 0; \quad r^2 R'' + rR' + \left(\lambda r^2 - \mu\right)R = 0$$

where λ, μ are the separation constants. The polar solution is constrained by periodicity $\Theta(0) = \Theta(2\pi)$, since we are working on a disc. We also consider only $\mu > 0$. The eigenvalue is then given by $\mu = m^2$, where $m \in \mathbb{N}$.

$$\Theta_m(\theta) = A_m \cos m\theta + B_m \sin m\theta$$

Or, in complex exponential form,

$$\Theta_m(\theta) = C_m e^{im\theta}; \quad m \in \mathbb{Z}$$

§3.9 Bessel's equation

We can solve the radial equation (in the previous subsection) by converting it first into Sturm-Liouville form, which can be accomplished by dividing by r.

$$\frac{\mathrm{d}}{\mathrm{d}r}(rR') - \frac{m^2}{r} = -\lambda r R$$

where $p(r) = r, q(r) = \frac{m^2}{r}, w(r) = r$, with self-adjoint boundary conditions with $R(1) = 0$. We will require R is bounded at $R(0)$, and since $p(0) = 0$ there is a regular singular point at $r = 0$. This particular equation for R is known as Bessel's equation. We will first substitute $z \equiv \sqrt{\lambda}r$, then we find the usual form of Bessel's equation,

$$z^2\frac{\mathrm{d}^2 R}{\mathrm{d}z^2} + z\frac{\mathrm{d}R}{\mathrm{d}z} + (z^2 - m^2)R = 0$$

We can use the method of Frobenius by substituting the following power series:

$$R = z^p \sum_{n=0}^{\infty} a_n z^n$$

to find

$$\sum_{n=0}^{\infty} \left[a_n(n+p)(n+p-1)z^{n+p} + (n+p)z^{n+p} + z^{n+p+2} + m^2 z^{n+p}\right] = 0$$

Equating powers of z, we can find the indicial equation

$$p^2 - m^2 = 0 \implies p = m, -m$$

The regular solution, given by $p = m$, has recursion relation

$$(n+m)^2 a_n + a_{n-2} - m^2 a_n = 0$$

which gives

$$a_n = \frac{-1}{n(n+2m)} a_{n-2}$$

Hence, we can find

$$a_{2n} = a_0 \frac{(-1)^n}{2^{2n} n!(n+m)(n+m-1)\dots(m+1)}$$

If, by convention, we let

$$a_0 = \frac{1}{2^m m!}$$

we can then write the *Bessel function of the first kind* by

$$J_m(z) = \left(\frac{z}{2}\right)^m \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{z}{2}\right)^{2n}$$

§3.10 Asymptotic behaviour of Bessel functions

If z is small, the leading-order behaviour of $J_m(z)$ is

$$J_0(z) \approx 1$$
$$J_m(z) \approx \frac{1}{m!} \left(\frac{z}{2}\right)^m$$

Now, let us consider large z. In this case, the function becomes oscillatory;

$$J_m(z) \approx \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

§3.11 Zeroes of Bessel functions

We can see from the asymptotic behaviour that there are infinitely many zeroes of the Bessel functions of the first kind as $z \to \infty$. We define j_{mn} to be the nth zero of J_m, for $z > 0$. Approximately,

$$\cos\left(z - \frac{m\pi}{2} - \frac{\pi}{4}\right) = 0 \implies z - \frac{m\pi}{2} - \frac{\pi}{4} = n\pi - \frac{\pi}{2}$$

Hence

$$z \approx n\pi + \frac{m\pi}{2} - \frac{\pi}{4} \equiv \tilde{j}_{mn}$$

§3.12 Solving the vibrating drum

Recall that the radial solutions become

$$R_m(z) = R_m(\sqrt{\lambda}x) = AJ_m(\sqrt{\lambda}x) + BY_m(\sqrt{\lambda}x)$$

Imposing the boundary condition of boundedness at $r = 0$, we must have $B = 0$. Further imposing $r = 1$ and $R = 0$ gives $J_m(\sqrt{\lambda}) = 0$. These zeroes occur at $j_{mn} \approx n\pi + \frac{m\pi}{2} - \frac{\pi}{4}$. Hence, the eigenvalues must be j_{mn}^2. Therefore, the spatial solution is

$$V_{mn}(r, \theta) = \Theta_m(\theta)R_{mn}(\sqrt{\lambda_{mn}}r) = (A_{mn}\cos m\theta + B_{mn}\sin m\theta)J_m(j_{mn}r)$$

The temporal solution is

$$\ddot{T} = -\lambda c T \implies T_{mn}(t) = \cos(j_{mn}ct), \sin(j_{mn}ct)$$

Combining everything together, the full solution is

$$\begin{aligned} u(r, \theta, t) &= \sum_{n=1}^{\infty} J_0(j_{0n}r)(A0n\cos j_{0n}ct + C_{0n}\sin j_{0n}ct) \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)\cos j_{mn}ct \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(C_{mn}\cos m\theta + D_{mn}\sin m\theta)\sin j_{mn}ct \end{aligned}$$

Now, we impose the boundary conditions

$$u(r, \theta, 0) = \phi(r, \theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)$$

and

$$\frac{\partial u}{\partial t}(r, \theta, 0) = \psi(r, \theta) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} j_{mn}cJ_m(j_{mn}r)(C_{mn}\cos m\theta + D_{mn}\sin m\theta)$$

We need to find the coefficients by multiplying by J_m, cos, sin and using the orthogonality relations, which are

$$\int_0^1 J_m(j_{mn}r)J_m(j_{mk}r)r\,\mathrm{d}r = \frac{1}{2}\left[J_m'(j_{mn})\right]^2\delta_{nk} = \frac{1}{2}[J_{m+1}(j_{mn})]^2\delta_{nk}$$

by using a recursion relation of the Bessel functions. We can then integrate to obtain the coefficients A_{mn}.

$$\int_0^{2\pi}\mathrm{d}\theta\cos p\theta\int_0^1 r\,\mathrm{d}r\,J_p(j_{pq}r)\phi(r,\theta) = \frac{\pi}{2}[J_{p+1}(j_{pq})]^2A_{pq}$$

where the $\frac{\pi}{2}$ coefficient is 2π for $p = 0$. We can find analogous results for the B_{mn}, C_{mn}, D_{mn}.

Example 3.3

Consider an initial radial profile $u(r,\theta,0) = \phi(r) = 1 - r^2$. Then, $m = 0, B_{mn} = 0$ for all m and $A_{mn} = 0$ for all $m \neq 0$. Then

$$\frac{\partial u}{\partial t}(r,0,0) = 0$$

hence $C_{mn}, D_{mn} = 0$. We just now need to find

$$A_{0n} = \frac{2}{J_0(j_{0n})^2}\int_0^1 J_0(j_{0n}r)(1-r)^2r\,\mathrm{d}r = \frac{2}{J_0(j_{0n})^2}\frac{J_2(j_{0n})}{j_{0n}^2} \approx \frac{J_2(j_{0n})}{n}\text{ as } n \to \infty$$

Then the approximate solution is

$$u(r,\theta,t) = \sum_{n=1}^{\infty} A_{0n}J_0(j_{0n}r)\cos j_{0n}ct$$

The fundamental frequency is $\omega_d = j_{01}c\frac{2}{d} \approx 4.8\frac{c}{d}$ where d is the diameter of the drum. Comparing this to a string with length d, this has a fundamental frequency of $\omega_s = \frac{\pi c}{d} \approx 0.77\omega_d$.

§3.13 Diffusion equation derivation with Fourier's law

In a volume V, the overall heat energy Q is given by

$$Q = \int_V c_V\rho\theta\,\mathrm{d}V$$

where c_V is the specific heat of the material, ρ is the mass density, and θ is the temperature. The rate of change due to heat flow is

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_V c_V\rho\frac{\partial\theta}{\partial t}\,\mathrm{d}V$$

Fourier's law for heat flow is

$$q = -k\boldsymbol{\nabla}\theta$$

where q is the heat flux. We will integrate this over the surface $S = \partial V$, giving

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_S q \cdot \hat{n}\,\mathrm{d}S$$

The negative sign is due to the normals facing outwards. This is exactly

$$-\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_S (-k\boldsymbol{\nabla}\theta) \cdot \hat{n}\,\mathrm{d}S = \int_V -k\nabla^2\theta\,\mathrm{d}V$$

Equating these two forms for $\frac{\mathrm{d}Q}{\mathrm{d}t}$, we find

$$\int_V (c_V\rho\frac{\partial\theta}{\partial t} - k\nabla^2\theta)\,\mathrm{d}V = 0$$

Since V was arbitrary, the integrand must be zero. So we have

$$\frac{\partial\theta}{\partial t} - \frac{k}{c_V\rho}\nabla^2\theta = 0$$

Let $D = \frac{k}{c_V\rho}$ be the diffusion constant. Then we have the diffusion equation

$$\frac{\partial\theta}{\partial t} - D\nabla^2\theta = 0$$

§3.14 Diffusion equation derivation with statistical dynamics

We can derive this equation in another way, using statistical dynamics. Gas particles diffuse by scattering every fixed time step Δt with probability density function $p(\xi)$ of moving by a displacement ξ. On average, we have

$$\langle\xi\rangle = \int p(\xi)\xi\,\mathrm{d}\xi = 0$$

since there is no bias the direction in which any given particle is travelling. Suppose that the probability density function after $N\Delta t$ time is described by $P_{N\Delta t}(x)$. Then, for the next time step,

$$P_{(N+1)\Delta t}(x) = \int_{-\infty}^{\infty} p(\xi)P_{N\Delta t}(x-\xi)\,\mathrm{d}\xi$$

Using the Taylor expansion,

$$P_{(N+1)\Delta t}(x) \approx \int_{-\infty}^{\infty} p(\xi)\left[P_{N\Delta t}(x) + P'_{N\Delta t}(x)(-\xi) + P''_{N\Delta t}(x)\frac{\xi^2}{2} + \cdots\right]\mathrm{d}\xi$$

$$\approx P_{N\Delta t}(x) - P'_{N\Delta t}(x) \langle \xi \rangle + P''_{N\Delta t}(x) \frac{\langle \xi^2 \rangle}{2} + \cdots$$
$$\approx P_{N\Delta t}(x) + P''_{N\Delta t}(x) \frac{\langle \xi^2 \rangle}{2} + \cdots$$

since $\int p(\xi)\,\mathrm{d}\xi = 1$. Identifying $P_{N\Delta t}(x) = P(x, N\Delta t)$, we can write

$$P(x, (N+1)\Delta t) - P(x, N\Delta t) = \frac{\partial^2}{\partial x^2} P(x, N\Delta t) \frac{\langle \xi^2 \rangle}{2}$$

Assuming that the variance $\frac{\langle \xi^2 \rangle}{2}$ is proportional to $D\Delta t$, then for small Δt, we find

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

which is exactly the diffusion equation.

§3.15 Similarity solutions

The characteristic relation between the variance and time suggests that we seek solutions with a dimensionless parameter. If we can a change of variables of the form $\theta(\eta) = \theta(x, t)$, then it will likely be easier to solve. Consider

$$\eta \equiv \frac{x}{2\sqrt{Dt}}$$

Then,

$$\frac{\partial \theta}{\partial t} = \frac{\partial \eta}{\partial t} \frac{\partial \theta}{\partial \eta} = \frac{-1}{2} \frac{x}{\sqrt{D} t^{3/2}} \theta' = \frac{-1}{2} \frac{\eta}{t} \theta'$$

and

$$D \frac{\partial^2 \theta}{\partial x^2} = D \frac{\partial}{\partial x} \left(\frac{\partial \eta}{\partial x} \frac{\partial \theta}{\partial \eta} \right) = D \frac{\partial}{\partial x} \left(\frac{1}{2\sqrt{Dt}} \theta' \right) = \frac{D}{4Dt} \theta'' = \frac{1}{4t} \theta''$$

Substituting into the diffusion equation,

$$\theta'' = -2\eta\theta'$$

Let $\psi = \theta'$. Then

$$\frac{\psi'}{\psi} = -2\eta \implies \ln \psi = -\eta^2 + \text{constant}$$

Then, choosing a constant of $c\frac{2}{\sqrt{\pi}}$,

$$\psi = c\frac{2}{\sqrt{\pi}} e^{-\eta^2} \implies \theta(\eta) = c\frac{2}{\sqrt{\pi}} \int_0^\eta e^{-u^2}\,\mathrm{d}u = c\,\mathrm{erf}(\eta) = c\,\mathrm{erf}(\frac{x}{2\sqrt{Dt}})$$

where

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2}\,\mathrm{d}u$$

This describes discontinuous initial conditions that spread over time.

§3.16 Heat conduction in a finite bar

Suppose we have a bar of length $2L$ with $-L \leq x \leq L$ and initial temperature

$$\theta(x,0) = H(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq L \\ 0 & \text{if } -L \leq x < 0 \end{cases}$$

with boundary conditions $\theta(L,t) = 1$, $\theta(-L,t) = 0$. Currently the boundary conditions are not homoegeneous, so Sturm-Liouville theory cannot be used directly. If we can identify a steady-state solution (time-independent) that reflects the late-time behaviour, then we can turn it into a homoegeneous set of boundary conditions. We will try a solution of the form

$$\theta_s(x) = Ax + B$$

since this certainly satisfies the diffusion equation. To satisfy the boundary conditions,

$$A = \frac{1}{2L}; \quad B = \frac{1}{2}$$

Hence we have a solution

$$\theta_s = \frac{x+L}{2L}$$

We will subtract this solution from our original equation for θ, giving

$$\hat\theta(x,t) = \theta(x,t) - \theta_s(x)$$

with homogeneous boundary conditions

$$\hat\theta(-L,t) = \hat\theta(L,t) = 0$$

and initial conditions

$$\theta(x,0) = H(x) - \frac{x+L}{2L}$$

We will now separate variables in the usual way. We will consider the ansatz

$$\hat\theta(x,t) = X(x)T(t) \implies X'' = -\lambda X; \dot T = -D\lambda T$$

The boundary conditions imply $\lambda > 0$ and give the Fourier modes $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$. For $\cos\sqrt{\lambda}L = 0$, we require $\sqrt{\lambda_m} = \frac{m\pi}{2L}$ for m odd. Also, $\sin\sqrt{\lambda}L = 0$ gives $\sqrt{\lambda_n} = \frac{n\pi}{L}$ for n even. Since $\hat\theta$ is odd due to our initial conditions, we can take

$$X_n = B_n \sin\frac{n\pi x}{L}; \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

Substituting into $\dot T = -D\lambda T$, we have

$$T_n(t) = c_n \exp\left(-\frac{Dn^2\pi^2}{L^2}t\right)$$

In general, the solution is

$$\hat\theta(x,t) = \sum_{n=1}^\infty b_n \sin\frac{n\pi x}{L}\exp\left(-\frac{Dn^2\pi^2}{L^2}t\right)$$

§3.17 Particular solution to diffusion equation

Recall that

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \exp\left(-\frac{Dn^2\pi^2}{L^2}t\right)$$

At $t = 0$, we have a pure Fourier sine series. We can then impose the initial conditions, to give

$$b_n = \frac{1}{L}\int_{-L}^{L} \hat{\phi}(x,0)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

where

$$\hat{\phi}(x,0) = H(x) - \frac{x+L}{2L}$$

Hence, we can use the half-range sine series and find

$$b_n = \underbrace{\frac{2}{L}\int_{0}^{L}\left(H(x) = \frac{1}{2}\right)\sin\frac{n\pi x}{L}\,\mathrm{d}x}_{\text{square wave/2}} - \underbrace{\frac{2}{L}\frac{x}{2L}\sin\frac{n\pi x}{L}\,\mathrm{d}x}_{\text{sawtooth}/2L}$$

which gives

$$b_n = \frac{2}{(2m-1)\pi} - \frac{(-1)^{n+1}}{n\pi}$$

where $n = 2m-1$, and the first term vanishes for n even. For n odd or even, we find the same result

$$b_n = \frac{1}{n\pi}$$

Hence

$$\hat{\theta}(x,t) = \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi x}{L}e^{-D\frac{n^2\pi^2}{L^2}t}$$

For the inhomogeneous boundary conditions,

$$\theta(x,t) = \frac{x+L}{2L} + \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin\frac{n\pi x}{L}e^{-D\frac{n^2\pi^2}{L^2}t}$$

The similarity solution $\frac{1}{2}\left(1+\operatorname{erf}(\frac{x}{2\sqrt{Dt}})\right)$ is a good fit for early t, but it does not necessarily satisfy the boundary conditions, so for large t it is a bad approximation.

§3.18 Laplace's equation

Laplace's equation is

$$\nabla^2 \phi = 0$$

This equation describes (among others) steady-state heat flow, potential theory $F = -\boldsymbol{\nabla}\phi$, and incompressible fluid flow $v = \boldsymbol{\nabla}\phi$. The equation is solved typically on a domain D, where boundary conditions are specified often on the boundary surface. The Dirichlet boundary conditions fix ϕ on the boundary surface ∂D. The Neumann boundary conditions fix $\hat{n} \cdot \boldsymbol{\nabla}\phi$ on ∂D.

§3.19 Laplace's equation in three-dimensional Cartesian coordinates

In $\mathbb{R}^3$ with Cartesian coordinates, Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

We seek separable solutions in the usual way:

$$\phi(x, y, z) = X(x)Y(y)Z(z)$$

Substituting,

$$X''YZ + XY''Z + XYZ'' = 0$$

Dividing by XYZ as usual,

$$\begin{aligned} \frac{X''}{X} &= \frac{-Y''}{Y} - \frac{Z''}{Z} = -\lambda_\ell \\ \frac{Y''}{Y} &= \frac{-Z''}{Z} - \frac{X''}{X} = -\lambda_m \\ \frac{Z''}{Z} &= \frac{-X''}{X} - \frac{Y''}{Y} = -\lambda_n = \lambda_\ell + \lambda_m \end{aligned}$$

From the eigenmodes, our general solution will be of the form

$$\phi(x, y, z) = \sum_{\ell, m, n} a_{\ell m n} X_\ell(x) Y_m(y) Z_n(z)$$

Consider steady ($\frac{\partial \phi}{\partial t} = 0$) heat flow in a semi-infinite rectangular bar, with boundary conditions $\phi = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$; and $\phi = 1$ at $z = 0$ and $\phi \to 0$ as $z \to \infty$. We will solve for each eigenmode successively. First, consider $X'' = -\lambda_\ell X$ with $X(0) = X(a) = 0$. This gives

$$\lambda_\ell = \frac{l^2 \pi^2}{a^2}; \quad X_\ell = \sin \frac{\ell \pi x}{a}$$

where $\ell > 0, \ell \in \mathbb{N}$. By symmetry,

$$\lambda_m = \frac{m^2\pi^2}{b^2}; \quad Y_m = \sin\frac{m\pi y}{b}$$

For the z mode,

$$Z'' = -\lambda_n Z = (\lambda_\ell + \lambda_m)Z = \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right)Z$$

Since $\phi \to 0$ as $z \to \infty$, the growing exponentials must vanish. Therefore,

$$Z_{\ell m} = \exp\left[-\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}\pi z\right]$$

Thus the general solution is

$$\phi(x,y,z) = \sum_{\ell,m} a_{\ell m}\sin\frac{\ell\pi x}{a}\sin\frac{m\pi y}{b}\exp\left[-\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}\pi z\right]$$

Now, we will fix $a_{\ell m}$ using $\phi(x,y,0) = 1$ using the Fourier sine series.

$$a_{\ell m} = \frac{2}{b}\int_0^b \frac{2}{a}\int_0^a \underbrace{1\sin\frac{\ell\pi x}{a}}_{\text{square wave}}\underbrace{\sin\frac{m\pi y}{b}}_{\text{square wave}}\,\mathrm{d}x\,\mathrm{d}y$$

So only the odd terms remain, giving

$$a_{\ell m} = \frac{4a}{a(2k-1)\pi}\cdot\frac{4b}{b(2p-1)\pi}$$

where $\ell = 2k-1$ is odd and $m = 2p-1$ is odd. Simplifying,

$$a_{\ell m} = \frac{16}{\pi^2 \ell m} \quad \text{for } \ell, m \text{ odd}$$

So the heat flow solution is

$$\phi(x,y,z) = \sum_{\ell,m\text{ odd}} \frac{16}{\pi^2\ell m}\sin\frac{\ell\pi x}{a}\sin\frac{\ell\pi y}{b}\exp\left[-\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}\pi z\right]$$

As z increases, every contribution but the lowest mode will be very small. So low ℓ, m dominate the solution.

§3.20 Laplace's equation in plane polar coordinates

In plane polar coordinates, Laplace's equation becomes

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Consider a separable form of the answer, given by

$$\phi(r, \theta) = R(r)\Theta(\theta)$$

We then have

$$\Theta'' + \mu\Theta = 0; \quad r(rR')' - \mu R = 0$$

The polar equation can be solved easily by considering periodic boundary conditions. This gives $\mu = m^2$ and the eigenmodes

$$\Theta_m(\theta) = \cos m\theta, \sin m\theta$$

The radial equation is *not* Bessel's equation, since there is no second separation constant. We simply have

$$r(rR')' - m^2 R = 0$$

We will try a power law solution, $r = \alpha r^\beta$. We find

$$\beta^2 - m^2 = 0 \implies \beta = \pm m$$

So the eigenfunctions are

$$R_m(r) = r^m, r^{-m}$$

which is one regular solution at the origin and one singular solution. In the case $m = 0$, we have

$$(rR') = 0 \implies rR' = \text{constant} \implies R = \log r$$

So

$$R_0(r) = \text{constant}, \log r$$

The general solution is therefore

$$\phi(r, \theta) = \frac{a_0}{2} + c_0 \log r + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta) r^m + \sum_{m=1}^{\infty} (c_m \cos m\theta + d_m \sin m\theta) r^{-m}$$

Example 3.4

Consider a soap film on a unit disc. We wish to solve Laplace's equation with a vertically distorted circular wire of radius $r = 1$ with boundary conditions $\phi(1, \theta) = f(\theta)$. The z displacement of the wire produces the $f(\theta)$ term. We wish to find $\phi(r, \theta)$ for $r < 1$, assuming regularity at $r = 0$. Then, $c_m = d_m = 0$ and the solution is of the form

$$\phi(r, \theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta) r^m$$

At $r = 1$,

$$\phi(1, \theta) = f(\theta) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos m\theta + b_m \sin m\theta)$$

which is exactly the Fourier series. Thus,

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos m\theta \, \mathrm{d}\theta \,; \quad b_m = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin m\theta \, \mathrm{d}\theta$$

We can see from the equation that high harmonics are confined to have effects only near $r = 1$.

§3.21 Laplace's equation in cylindrical polar coordinates

In cylindrical coordinates,

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{4^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

With $\phi = R(r)\Theta(\theta)Z(z)$, we find

$$\Theta'' = -\mu\Theta; \quad Z'' = \lambda Z; \quad r(rR')' + (\lambda r^2 - \mu)R = 0$$

The polar equation can be easily solved by

$$\mu_m = m^2; \quad \Theta_m(\theta) = \cos m\theta, \sin m\theta$$

The radial equation is Bessel's equation, giving solutions

$$R = J_m(kr), Y_m(kr)$$

Setting boundary conditions in the usual way, defining $R = 0$ at $r = a$ means that

$$J_m(ka) = 0 \implies k = \frac{j_{mn}}{a}$$

The radial solution is

$$R_{mn}(r) = J_m\left(\frac{j_{mn}}{a}r\right)$$

We have eliminated the Y_n term since we require $r = 0$ to give a finite ϕ. Finally, the z equation gives

$$Z'' = k^2 Z \implies Z = e^{-kz}, e^{kz}$$

We typically eliminate the e^{kz} mode due to boundary conditions, such as $Z \to 0$ as $z \to \infty$. The general solution is therefore

$$\phi(r, \theta, z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(a_{mn}\cos m\theta + b_{mn}\sin m\theta)J_m\left(\frac{j_{mn}}{a}r\right)e^{-fracj_{mn}ra}$$

§3.22 Laplace's equation in spherical polar coordinates

In spherical polar coordinates,

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

We will consider the *axisymmetric case*; supposing that there is no ϕ dependence. We seek a separable solution of the form

$$\Phi(r, \theta) = R(r)\Theta(\theta)$$

which gives

$$(\sin\theta\Theta')' + \lambda\sin\theta\Theta = 0; \quad (r^2R')' - \lambda R = 0$$

Consider the substitution $x = \cos\theta, \frac{\mathrm{d}x}{\mathrm{d}\theta} = -\sin\theta$ in the polar equation. This gives $\frac{\mathrm{d}\Theta}{\mathrm{d}\theta} = -\sin\theta\frac{\mathrm{d}\Theta}{\mathrm{d}x}$ and hence

$$-\sin\theta\frac{\mathrm{d}}{\mathrm{d}x}\left[-\sin^2\theta\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right] + \lambda\sin\theta\Theta = 0 \implies \frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}\Theta}{\mathrm{d}x}\right] + \lambda\Theta = 0$$

This gives Legendre's equation, so it has solutions of eigenvalues $\lambda_\ell = \ell(\ell+1)$ and eigenfunctions

$$\Theta_\ell(\theta) = P_\ell(x) = P_\ell(\cos\theta)$$

The radial equation then gives

$$(r^2R')' - \ell(\ell+1)R = 0$$

We will seek power law solutions: $R = \alpha r^\beta$. This gives

$$\beta(\beta+1) - \ell(\ell+1) = 0 \implies \beta = \ell, \beta = -\ell - 1$$

Thus the radial eigenmodes are

$$R_\ell = r^\ell, r^{-\ell-1}$$

Therefore the general axisymmetric solution for spherical polar coordinates is

$$\Phi(r,\theta) = \sum_{\ell=0}^{\infty}(a_\ell r^\ell + b_\ell r^{-\ell-1})P_\ell(\cos\theta)$$

The a_ℓ, b_ℓ are determined by the boundary conditions. Orthogonality conditions for the P_ℓ can be used to determine coefficients. Consider a solution to Laplace's equation on the unit sphere with axisymmetric boundary conditions given by

$$\Phi(1,\theta) = f(\theta)$$

Given that we wish to find the interior solution, $b_n = 0$ by regularity. Then,

$$f(\theta) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(\cos\theta)$$

By defining $f(\theta) = F(\cos\theta)$,

$$F(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x)$$

We can then find the coefficients in the usual way, giving

$$a_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} F(x)P_\ell(x)\,\mathrm{d}x$$

§3.23 Generating function for Legendre polynomials

Consider a charge at $r_0 = (x,y,z) = (0,0,1)$. Then, the potential at a point P becomes

$$\begin{aligned}
\Phi(r) &= \frac{1}{|r-r_0|} = \frac{1}{(x^2+y^2+(x-1)^2)^{1/2}} \\
&= \frac{1}{(r^2(\sin^2\phi+\cos^2\phi)\sin^2\theta + r^2\cos^2\theta - 2r\cos\theta + 1)^{1/2}} \\
&= \frac{1}{(r^2\sin^2\theta + r^2\cos^2\theta - 2r\cos\theta + 1)^{1/2}} \\
&= \frac{1}{(r^2 - 2r\cos\theta + 1)^{1/2}}
\end{aligned}$$

$$= \frac{1}{(r^2 - 2r\overline{x} + 1)^{1/2}}$$

where

$\overline{x} \equiv \cos\theta$. This function Φ is a solution to Laplace's equation where $r \neq r_0$. Note that we can represent any axisymmetric solution as a sum of Legendre polynomials. Now,

$$\frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x) r^\ell$$

With the normalisation condition for the Legendre polynomials $P_\ell(1) = 1$, we find

$$\frac{1}{1-r} = \sum_{\ell=0}^{\infty} a_\ell r^\ell$$

Using the geometric series expansion, we arrive at $a_\ell = 1$. This gives

$$\frac{1}{\sqrt{r^2 - 2rx + 1}} = \sum_{\ell=0}^{\infty} P_\ell(x) r^\ell$$

which is the generating function for the Legendre polynomials.

Part II
Inhomogenous ODEs and Greens Functions