Part IB — Geometry

Based on lectures by Prof. G. Paternain and notes by thirdsgames.co.uk

Lent 2022

Contents

§1 Surfaces

§1.1 Basic definitions

Definition 1.1

A **topological surface** is a topological space Σ such that

1. for all points $p \in \Sigma$, there exists an open neighbourhood $p \in U \subset \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology;
2. Σ is Hausdorff and second countable.

Definition 1.2 (Hausdorff)

A space X is **Hausdorff** if two points $p \neq q \in X$ have open neighbourhoods U, V such that $U \cap V = \varnothing$.

Definition 1.3 (Second Countable)

A space X is **second countable** if it has a countable base; there exists a countable family of open sets U_i, such that every open set is a union of some of the U_i.

Remark 1.

1. $\mathbb{R}^2$ is homeomorphic to the open disc $D(0,1) = \{x \in \mathbb{R}^2 : \|x\| < 1\}$.
2. The first part of the definition is important whilst the second part (Hausdorff and second countable) is a technical point. These topological requirements are typically not the purpose of considering topological spaces, but they are occasionally technical requirements to prove interesting theorems.
3. Note that subspaces of Hausdorff and second countable spaces are also Hausdorff and second countable. In particular, Euclidean space $\mathbb{R}^n$ is Hausdorff (as $\mathbb{R}^n$ is a metric space) and second countable (consider the set of balls $D(p,q)$ for points p with rational coordinates, and rational radii q). Hence, any subspace of $\mathbb{R}^n$ is implicitly Hausdorff and second countable.

Example 1.1

$\mathbb{R}^2$ is a topological surface. Any open subset of $\mathbb{R}^2$ is also a topological surface. For example, $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{(0,0)\} \cup \left\{\left(0, \frac{1}{n}\right) : n = 1, 2, \dots\right\}$ are topological surfaces.

Example 1.2

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. The graph of f, denoted Γ_f, is defined by

$$\Gamma_f = \Big\{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\Big\} \subset \mathbb{R}^3$$

with the subspace topology when embedded in $\mathbb{R}^3$.

Recall that the product topology on $X \times Y$ for X, Y topological spaces, has basic open sets $U \times V$, where $U \subset X$, $V \subset Y$ open. Also the product topology has the feature that $g : Z \to X \times Y$ is continuous iff $\pi_x \circ g : Z \to X$ and $\pi_y \circ f : Z \to Y$ are continuous[a].

Hence, any graph $\Gamma \subset X \times Y$ is homeomorphic to X if f is continuous. Indeed, the projection π_x projects each point in the graph onto the domain. The function $s : x \mapsto (x, f(x))$ is continuous as $\pi_x \circ s$ and $\pi_y \circ s$ are. So $\pi_x\,|_{\Gamma_f}$ and s are inverse homeomorphisms.

So, in our case, the graph Γ_f is homeomorphic to $\mathbb{R}^2$, and so is a topological surface.

[a] π_x, π_y are the canonical projections, $\pi_x : X \times Y \to X$

Remark 2. As a topological surface, Γ_f is independent of the function f. However, we will later introduce more ways to describe topological spaces that will ascribe new properties to Γ_f which do depend on f.

Example 1.3

The sphere:

$$S^2 = \Big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\Big\}$$

is a topological surface, when using the subspace topology in $\mathbb{R}^3$.

This is a subspace of $\mathbb{R}^3$ so is Hausdorff and second countable.

Consider the stereographic projection of S^2 onto $\mathbb{R}^2$ from the north pole $(0, 0, 1)$. The projection satisfies $\pi_+ : S^2 \setminus \{(0, 0, 1)\}$ and

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

Certainly, π_+ is continuous, since we do not consider the point $(0, 0, 1)$ to be in its domain. The inverse map is given by

$$(u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

This is also a continuous function. Hence π_+ is a homeomorphism.

Similarly, we can construct the stereographic projection from the south pole, $\pi_- : S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$.

$$(x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right).$$

This is a homeomorphism.

Hence, every point in S^2 lies either in the domain of π_+ or π_-, and hence sits in an open set $S^2 \setminus \{(0,0,1)\}$ or $S^2 \setminus \{(0,0,-1)\}$ which are homeomorphic to $\mathbb{R}^2$. So S^2 is a topological surface.

Remark 3. S^2 is compact by the Heine-Borel theorem; it is a closed bounded set in $\mathbb{R}^3$.

Example 1.4

The real projective plane is a topological surface.

The group $\mathbb{Z}_2$ acts on S^2 by homeomorphisms via the *antipodal map* $a : S^2 \to S^2$, mapping $x \mapsto -x$. So $\mathbb{Z}_2$ sits in the group of homeomorphisms of S^2, $\mathrm{Homeo}(S^2)$, as we can map $-1 \to a$.

Definition 1.4 (The Real Projective Plane)

The real projective plane, $\mathbb{RP}^2$, is the quotient of S^2 given by identifying every point x with its image $-x$ under a.

$$\mathbb{RP}^2 = {S^2}/{\mathbb{Z}_2} = {S^2}/{\sim}; \quad x \sim a(x)$$

Lemma 1.1

As a set, $\mathbb{RP}^2$ naturally bijects with the set of straight lines in $\mathbb{R}^3$ through the origin.

Proof. Any line through the origin intersects S^2 exactly in a pair of antipodal points $x, -x$. Similarly, pairs of antipodal points uniquely define a line through the origin. $\square$

Lemma 1.2

$\mathbb{RP}^2$ is a topological surface with the quotient topology.

Recall: Quotient topology : $q : X \to Y$ (q the quotient map), $V \subset Y$ is open iff

$q^{-1}V \subset X$ is open in X(i.e. iff q is continuous).

Proof. We must check that $\mathbb{RP}^2$ is Hausdorff since it is constructed by a quotient, not a subspace.
If $[p] \neq [m] \in \mathbb{RP}^2$, then $\pm p, \pm m \in S^2$ are distinct antipodal pairs. We can therefore construct distinct open discs[a] around p, m in S^2, and their antipodal images. These uniquely define open neighbourhoods of $[p], [q]$, which are disjoint, as for $q : S^2 \to \mathbb{RP}^2$, $q(B_\delta(p))$ is open since $q^{-1}(q(B_\delta(p))) = B_\delta(p) \cup (-B_\delta(p))$ is open.

Similarly, we can check that $\mathbb{RP}^2$ is second countable.
We know that S^2 is second countable, so let $\mathcal{U}_0$ be a countable base for the topology on S^2. Let $\overline{\mathcal{U}_0} = \{q(u) : u \in \mathcal{U}_0\}$. $q(u)$ is open as $q^{-1}(q(u)) = u \cup (-u)$ is open. $\overline{\mathcal{U}_0}$ is clearly countable since $\mathcal{U}_0$ is. Now, if $V \subset \mathbb{RP}^2$ is open, then by definition of quotient topology $q^{-1}(V)$ is open in S^2 hence $q^{-1}(V) = \cup_\alpha U_\alpha, U_\alpha \in \mathcal{U}_0$. $V = q(q^{-1}V) = q(\cup_\alpha U_\alpha) = \cup_\alpha q(U_\alpha), q(U_\alpha) \in \overline{\mathcal{U}_0}$.

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\overline{D}$ be a small (contained in an open hemisphere) closed disc, which is a neighbourhood of $p \in S^2$. The quotient map restricted to $\overline{D}$, written $q|_{\overline{D}} : \overline{D} \to q(\overline{D}) \subset \mathbb{RP}^2$, is a continuous function from a compact space to a Hausdorff space. Further, q is injective on $\overline{D}$ since the disc was contained entirely in a single hemisphere so it cannot contain antipodal points.
Recall from AT that the "topological inverse function theorem" (TIFT) states that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.[b]
So $q|_{\overline{D}}$ is a homeomorphism from $\overline{D}$ to $q(\overline{D})$. This then induces the homeomorphism $q|_D : D \to q(D)$ where D is an open disc, the interior of $\overline{D}$. So by construction, $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ which is homeomorphic to an open disc on S^2 and so to $\mathbb{R}^2$, concluding the proof. □

[a] Just take a ball in $\mathbb{R}^3$ and intersect with S^2

[b] A brief proof is we want to show the inverse function is continuous, so it maps closed sets to closed sets. Take a closed set inside compact space so its compact, apply the continuous function to it so the image is compact. A compact set in a Hausdorff space is closed.

Example 1.5

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$, and then we define the torus to be the product space $S^1 \times S^1$, with the subspace topology from $\mathbb{C}^2$ (which is identical to the product topology).

Lemma 1.3

The torus is a topological surface.

Proof. Consider the map $e : \mathbb{R}^2 \to S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ defined by

$$(s,t) \mapsto \left(e^{2\pi i s}, e^{2\pi i t}\right)$$

We have an equivalence relation on $\mathbb{R}^2$ given by translations by $\mathbb{Z}^2$ as e is constant under them. This induces a map $\hat{e}$ from $\mathbb{R}^2/\mathbb{Z}^2$.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"'] & S^1 \times S^1 \\
\mathbb{R}^2/\mathbb{Z}^2 \arrow[ur, "\hat{e}"] &
\end{tikzcd}$$

Under the quotient topology given by the quotient map q, $\mathbb{R}^2/\mathbb{Z}^2$ is a topological space. The map $[0,1]^2 \to \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is surjective, so $\mathbb{R}^2/\mathbb{Z}^2$ is compact. As e is constant on an equivalence class, $\hat{e}$ is a continuous map from a compact space to a Hausdorff space, and $\hat{e}$ is bijective, so $\hat{e}$ is a homeomorphism by TIFT.

We already have that $S^1 \times S^1$ is compact and Hausdorff (as a closed and bounded set in $\mathbb{C}^2$, equivalent to $\mathbb{R}^4$), so it suffices to show it is locally homeomorphic to $\mathbb{R}^2$.

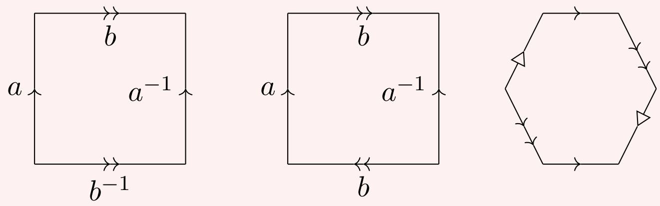
Similarly to the case of $S^2 \to \mathbb{RP}^2$, pick $[p] \in q(p), p \in \mathbb{R}^2$, then we can choose a small closed disc $\overline{D}(p) \subset \mathbb{R}^2$ such that $\overline{D}(p) \cap \left(\overline{D}(p) + (n,m)\right) = \varnothing$ for all nonzero $(n,m) \in \mathbb{Z}^2$. Hence $e|_{\overline{D}(p)}$ and $q|_{\overline{D}(p)}$ are injective. Now, restricting to the open disc as before, we can find an open disc neighbourhood of $[p] \in \mathbb{R}^2/\mathbb{Z}^2$. Since $[p]$ was chosen arbitrarily, $S^1 \times S^1$ is a topological surface. $\square$

Another viewpoint:
$\mathbb{R}^2/\mathbb{Z}^2$ is also given by imposing on $[0,1]^2$ the equivalence relation

$$\begin{aligned}
(x,0) &\sim (x,1) \quad \forall\, 0 \le x \le 1 \\
(0,y) &\sim (1,y) \quad \forall\, 0 \le y \le 1.
\end{aligned}$$

Example 1.6

Let P be a planar Euclidean polygon (including interior), with oriented edges. We will pair the edges, and without loss of generality we will assume that paired edges have the same Euclidean length.



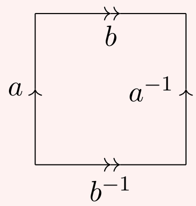
We can assign letter names to each edge pair, and denote a polygon by the sequence of edges found when traversing in a clockwise orientation. The edge pair name is inverted if the edge is traversed in the reverse direction. Note the difference between the annotations on the first two shapes above, due to the reversed direction of the edge.

If two edges $\{e, \hat{e}\}$ are paired, this defines a unique Euclidean isometry from e to $\hat{e}$ respecting the orientation, which will be written $f_{e\hat{e}} : e \to \hat{e}$.
The set of all such functions generate an equivalence relation on the polygon P, where we identify $x \in \partial P$ (a point on the boundary) with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4
$P/\sim$, with the quotient topology, is a topological surface.

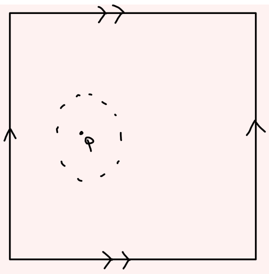
Example 1.7

Consider the torus, defined here as $T^2 = [0,1]^2/\sim$.

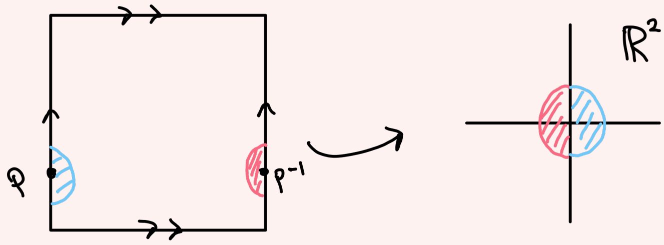


Let P be the polygon $[0,1]^2$.

If p is in the interior of P, then we pick $\delta > 0$ small s.t. $\overline{B_\delta(p)}$[a] lies in the interior of P. Arguing as before in $\mathbb{RP}^2$, the quotient map is injective on $\overline{B_\delta(p)}$ and is a homeomorphism on its interior.



Let p be on an edge, but not a vertex.



Let us say without loss of generality that $p = (0, y_0) \sim (1, y_0) = p'$. Let δ be sufficiently small that the closed half-discs U, V centred on p, p^{-1} with radius δ do not intersect any vertices.
Then we define a map from the union of the two half-discs to the disc $B(0, \delta) \subset \mathbb{R}^2$ via

$$U : (x, y) \underset{f_u}{\mapsto} (x, y - y_0)$$
$$V : (x, y) \underset{f_v}{\mapsto} (x - 1, y - y_0)$$

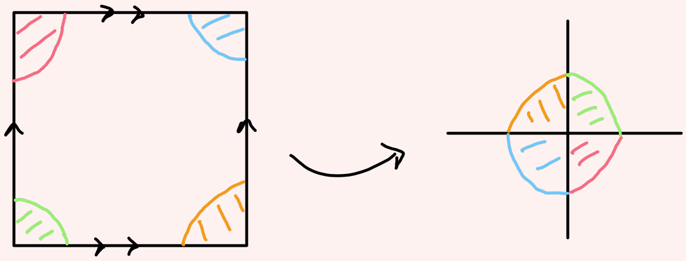
which will be a bijective map.

Recall the gluing lemma from Analysis and Topology: that if $X = A \cup B$ is a union of closed subspaces, and $f : A \to Y$, $g : B \to Y$ are continuous and $f|_{A \cap B} = g|_{A \cap B}$, they define a continuous map on X.

f_U, f_V are continuous on $U, V \subset [0, 1]^2$. By the definition of the quotient topology, $q \circ f_U$ and $q \circ f_V$ are also continuous ($q : [0, 1]^2 \to {[0, 1]^2}/{\sim}$).
In T^2, $1/2$-discs, $q \circ U, q \circ V$ overlap but our maps agree as they are compatible with the equivalence relation.
Hence, by the gluing lemma, f_U, f_V "glue" together to give a continuous map to an open neighbourhood of $[p] \in T^2$ to $\mathbb{R}^2$.

We can show that this is a homeomorphism using the usual process: pass to a closed disc, apply the topological inverse function theorem, then apply the result to the interior. If $[p] \in T^2$ lies in an edge on P, it has a neighbourhood homeomorphic to a disc.

Now it suffices to consider points p on a vertex. All four vertices of the square are identified to the same point in the torus as each vertex lies on two edges and so is identified to two other vertices.



and analogously we get that a vertex has a neighbourhood homeomorphic to a disc.

Thus, ${[0,1]^2}/{\sim}$ is a topological surface.

[a] The closure of $B_\delta(p)$

Example 1.8 (General Polygon)

We can generalise this proof to an arbitrary planar Euclidean polygon P, such as the hexagon above. The equivalence relation $x \sim f_{e\hat{e}}(x)$ induces an equivalence relation on the vertices of P, by considering the images of the vertices under all $f_{e\hat{e}}$. However, it is not necessarily the case that an equivalence class of vertices contains exactly four vertices, so quarter-discs are not necessarily applicable. Again, there are three types of point:

- interior points, for which a neighbourhood not intersecting the boundary is chosen;
- points on edges, for which a corresponding point exists and two half-discs can be glued to form the neighbourhood; and
- points on vertices. For this case, all vertices of the polygon have a neighbourhood which is a sector of a circle. Let there be r vertices in a given equivalence class. Let α be the sum of the angles of the sectors in a given class.
  Any sector can be identified with a given sector in the disc $B(0, \delta) \subset \mathbb{R}^2$, which we will choose to have angle α/r. Then, we can glue each sector together in $\mathbb{R}^2$, compatibly with the orientations of the edges and arrows, inducing a neighbourhood which is locally homeomorphic to a disc.

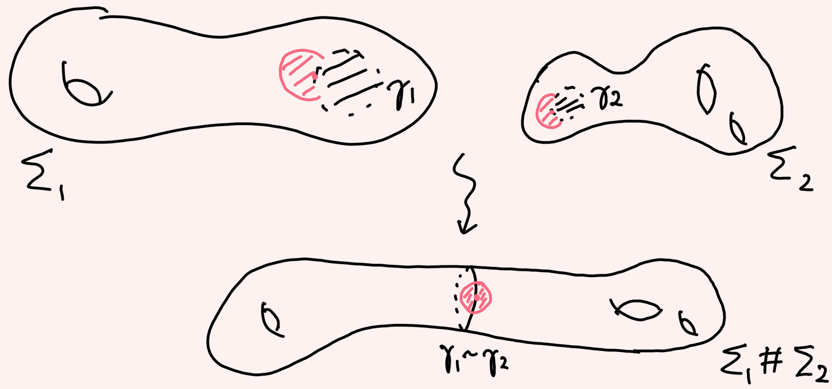
  If $r = 1$, we have an equivalence class comprising a single vertex, which gives a single sector. For r to be one, the two edges attached to this vertex must be paired and have the same direction (either both inwards or outwards from the vertex). This quotient space is simply a cone, which is homeomorphic to a disc as required.

We can also show that the quotient space is Hausdorff and second countable. By construction, two distinct points in the quotient space can be separated by open neighbourhoods by selecting a sufficiently small radius such that the discs considered in the derivation above are disjoint. For second countability, consider

- discs in the interior of P with rational centres and radii;
- for each edge of P, consider an isometry $e \to [0, \ell]$ where ℓ is the length of e, taking discs on e which are centred at rational values in $[0, \ell]$; and
- for each vertex, consider discs centred at these vertices with rational radii.

Example 1.9 (Connected Sums)

Given topological surfaces Σ_1, Σ_2 we can remove an open disc from each and glue the resulting boundary circles.



Explicitly, take $\Sigma_1 \setminus D_1 \amalg$ [a]$\Sigma_2 \setminus D_2$ and impose a quotient relation by identifying $\theta \in \partial D_1 \sim \theta \in \partial D_2$ where θ is an angle parametrising $S^1 = \partial D_i$, ∂D_i is the boundary of D_i. The result $\Sigma_1 \# \Sigma_2$ is called the **connected sum** of Σ_1, Σ_2.

In principle this depends on many choices and takes some effort to prove that it is well-defined.

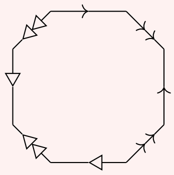
[a]Disjoint Union

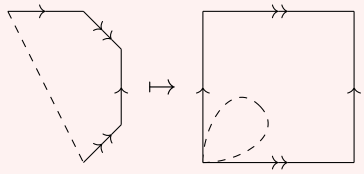
Lemma 1.5

The connected sum $\Sigma_1 \# \Sigma_2$ is a topological surface.

Proof. Not proved in this course, if you want to learn more try 'Introduction to topological manifolds' by Jack Lee. □

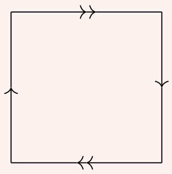
Example 1.10

Consider the following octagon.  The associated quotient space ${}^{P}\!/_{\sim}$ can be seen to be homeomorphic to a surface with two holes, known as a double torus. All vertices are identified as the same vertex in the quotient space. We can cut the octagon along a diagonal, leaving two topological surfaces which are homeomorphic to a torus.



Thus, the connected sum of the two half-octagons are the connected sum of two toruses.

Example 1.11

Consider the following square.  This is homeomorphic to the real projective plane $\mathbb{RP}^2$. This is because we identify points on the boundary with their antipodes, when interpreting the square as the closed disc $B(0, 1)$. The real projective plane was constructed by identifying points on the unit sphere with their antipodes. Thus, we can construct a homeomorphism by considering only points in the upper hemisphere (taking antipodes as required), and then orthographically projecting onto the xy plane. Under this transformation, points on the boundary are identified with their antipodes as required.

§1.2 Subdivisions

Definition 1.5 (Subdivision)

A **subdivision** of a compact topological surface Σ comprises

1. a finite subset $V \subset \Sigma$ of vertices;

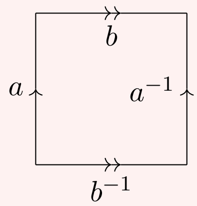
2. a finite subset of edges $E = \{e_i : [0,1] \to \Sigma\}$ s.t. 1) each e_i is a continuous injection on its interior and $e_i^{-1}V = \{0,1\}$, the endpoints. 2) e_i, e_j have disjoint images except perhaps at their endpoints.
3. we require that each connected component of $\Sigma \setminus (\cup_i e_i[0,1] \cup V)$ is homeomorphic to an open disc called a **face**. In particular, the closure of a face has boundary $\overline{F} \setminus F$ lying in $(\cup_i e_i[0,1] \cup V)$.

Definition 1.6 (Triangulation)
We say that a subdivision is a **triangulation** if each closed face (closure of a face) contains exactly three edges, and two closed faces are disjoint, meet at exactly one edge or just one vertex.

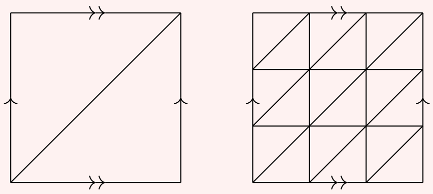
Example 1.12
A cube displays a subdivision of S^2. A tetrahedron displays a triangulation of S^2.

Example 1.13

We can display subdivisions of surfaces constructed from polygons.

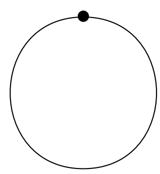


This is a subdivision of a torus with one vertex, two edges, and one face. We can construct additional subdivisions of a torus, for example:



The first of these examples is not a triangulation, since the two faces meet in more than one edge. The second is a triangulation.

Remark 4. The following is a very degenerate subdivision of S_2.



[1] This

[1]This is not a circle, its a 2-sphere.

has one vertex, no edges, and one face.

§1.3 Euler classification

Definition 1.7 (Euler Characteristic)

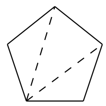
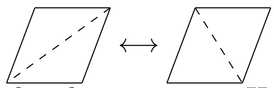
The **Euler characteristic** of a subdivision is

$$\#\text{[a]}V - \#E + \#F$$

[a] The number/size of the set

Theorem 1.1
1. Every compact topological surface has a subdivision (and indeed triangulations).
2. The Euler characteristic is invariant under choice of subdivision, and is topologically invariant of the surface (depends only on the homeomorphism type of Σ).

Hence, we might say that a surface has a particular Euler characteristic, without referring to subdivisions. We write this $\chi(\Sigma)$.

Remark 5. It is not trivial to prove part (i). For part (ii), note that subdivisions can be converted into triangulations by constructing triangle fans.  Triangulations can be related by local moves, such as  It is easy to check that both of these moves do not change the Euler characteristic. However, it is hard to make this argument rigorous, and it does not give much explanation for why the result is true. In Part II Algebraic Topology, a more advanced definition of the Euler characteristic is given, which admits a more elegant proof.

Proof. No proof will be given. □

Example 1.14

The Euler characteristic of S^2 is $\chi(S^2) = 2$.

Example 1.15

For the torus, $\chi(T^2) = 0$.

Example 1.16

If Σ_1, Σ_2 are compact surfaces, then the connected sum $\Sigma_1 \# \Sigma_2$ can be constructed by removing a face of a triangulation, then gluing together the boundary circles (three edges) in a way that matches the edges.

Then the connected sum inherits a subdivision, and we can find that it has Euler characteristic $\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$, where the remaining term corresponds to the two faces that were removed; the changes of three vertices and three edges cancel each other.

In particular, a surface Σ_g with g holes can be written $\#_{i=1}^{g} T^2$, so $\chi(\Sigma_g) = 2 - 2g$. We call g the **genus** of Σ.

§2 Abstract smooth surfaces

§2.1 Charts and atlases

Recall that if Σ is a topological surface, any point lies in an open neighbourhood homeomorphic to a disc.

Definition 2.1 (Chart)

A pair (U, φ), where U is an open set in Σ and $\varphi\colon U \to V$ is a homeomorphism to an open set $V \subset \mathbb{R}^2$, is called a **chart** for Σ. If $p \in U$, we might say that (U, φ) is a chart for Σ *at* p.

Definition 2.2 (Local parameterisation)

The inverse $\sigma = \varphi^{-1}\colon V \to U$ is known as a **local parametrisation** for the surface.

Definition 2.3 (Atlas)

A collection of charts $\{(U_i, \varphi_i)_{i\in I}\}$ whose domains cover Σ ($\cup_{i\in I} U_i = \Sigma$) is known as an **atlas** for Σ.

Example 2.1

If $Z \subset \mathbb{R}^2$ is closed, $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas containing one chart, $(\mathbb{R}^2 \setminus Z, \varphi = \mathrm{id})$.

Example 2.2

For S^2, there is an atlas with two charts, which are the two stereographic projections from the poles.

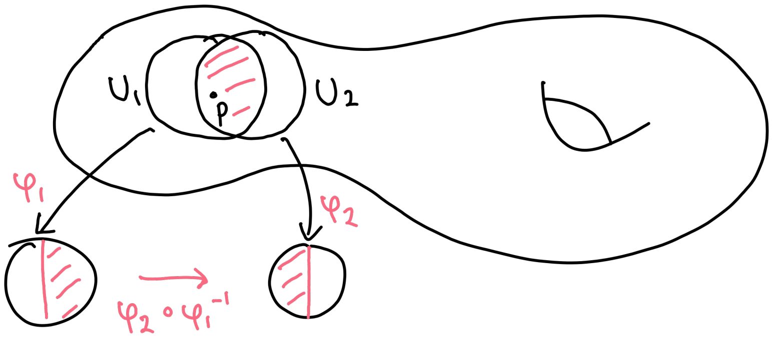
Definition 2.4 (Transition Map)

Let (U_i, φ_i) be charts containing the point $p \in \Sigma$, for $i = 1, 2$. Then the map

$$\varphi_2 \circ \varphi_1^{-1}\Big|_{\varphi_1(U_1\cap U_2)} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

converts between the corresponding charts, and is called a **transition map** between charts. This is a homeomorphism of open sets in $\mathbb{R}^2$.

Recall from Analysis and Topology that if $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$ are open, then a



continuous map $f\colon V \to V'$ is called *smooth* if it is infinitely differentiable. Equivalently, it is smooth if continuous partial derivatives of all orders in all variables exist at all points.

Definition 2.5 (Diffeomorphism)

A homeomorphism $f : V \to V'$ is called a **diffeomorphism** if it is smooth and it has a smooth inverse.

Definition 2.6 (Abstract Smooth Surface)

An **abstract smooth surface** Σ is a topological space with an atlas of charts $\{(U_i, \varphi_i)_{i\in I}\}$ s.t. all transition maps are diffeomorphisms.

Remark 6. We could not simply consider a smoothness condition for Σ itself without appealing to atlases, since Σ is an arbitrary topological space and could have almost any topology.

Example 2.3

The atlas of two charts with stereographic projections gives S^2 the structure of an abstract smooth surface.

Example 2.4

For the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$, recall that we obtained charts from (the inverses of) the projection restricted to small discs in $\mathbb{R}^2$.

The transition maps for this atlas are all translations[a] of $\mathbb{R}^2$. Hence T^2 inherits the structure of an abstract smooth surface. Explicitly, let us define $e\colon \mathbb{R}^2 \to T^2$ by $(t, s) \mapsto (e^{2\pi it}, e^{2\pi is})$, then consider the atlas

$$\Big\{(e(D_\varepsilon(x, y)), e^{-1} \text{ on this image})\Big\}$$

for $\varepsilon < \frac{1}{3}$. These are charts on T^2, and the transition maps are (restricted to appropriate domains) translations in $\mathbb{R}^2$. Hence T^2, via this atlas, has the structure of an abstract smooth surface.

[a] If small discs intersect in $\mathbb{R}^2/\mathbb{Z}^2$ then they have points which are integer translations?

Remark 7. The definition of a topological surface is a notion of structure. One can observe a topological space and determine whether it is a topological surface. Conversely, to be an abstract smooth surface is to have a specific set of data; that is, we must provide charts for the surface in order to see that it is indeed an abstract smooth surface.

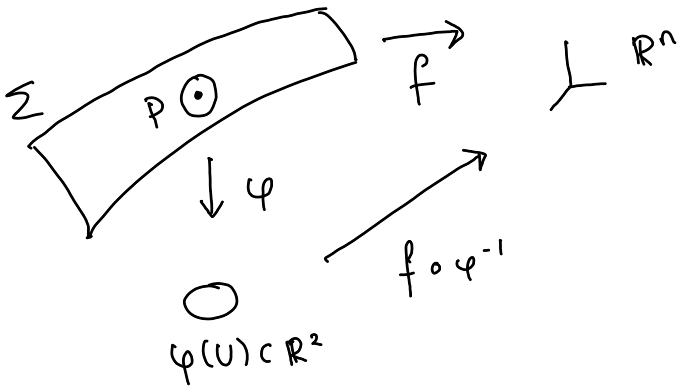
Definition 2.7 (Smooth Function)

Let Σ be an abstract smooth surface, and $f\colon \Sigma \to \mathbb{R}^n$ be a continuous map. We say that f is **smooth** at $p \in \Sigma$ if, for all charts (U, φ) of p[a] belonging to the smooth atlas for Σ, the map

$$f \circ \varphi^{-1} \colon \underbrace{\varphi(U)}_{\subset \mathbb{R}^2} \to \mathbb{R}^n$$

is smooth at $\varphi(p) \in \mathbb{R}^2$.

[a] $p \in U$



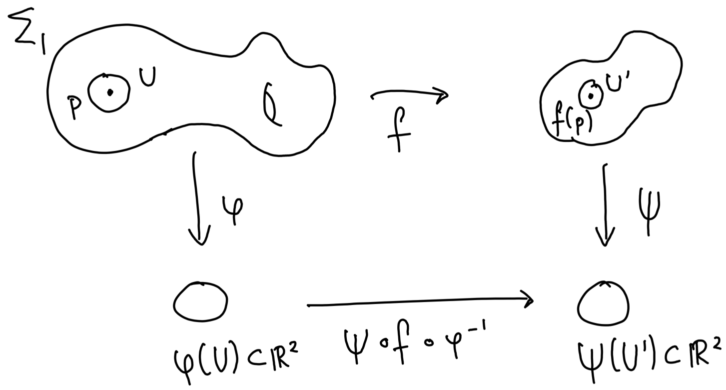
Remark 8. Note that the choice of chart and atlas was arbitrary, but smoothness of f at p is independent of the choice of chart, since the transition maps between two such charts are diffeomorphisms.

$$f \circ \varphi_1^{-1} = f \circ \varphi_2^{-1} \circ (\varphi_2 \circ \varphi_1^{-1})$$

$(\varphi_2 \circ \varphi_1^{-1})$ is a transition map and so is a diffeomorphism. So by chain rule follows!

Definition 2.8 (Smooth Function between Surfaces)

Let Σ_1, Σ_2 be abstract smooth surfaces.
Then a map $f\colon \Sigma_1 \to \Sigma_2$ is **smooth** if it is 'smooth in the local charts'. Given a chart (U, φ) at p and a chart (U', ψ) at $f(p)$, with $f(U) \subset U'$, the map $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$.



Remark 9. Smoothness of f at p does not depend on the choice of chart, provided that the charts all belong to the same atlas.

Definition 2.9 (Diffeomorphic Surfaces)

Two surfaces Σ_1, Σ_2 are **diffeomorphic** if $\exists\, f\colon \Sigma_1 \to \Sigma_2$ which is smooth and has smooth inverse.

Remark 10. Often, we convert from a given smooth atlas for an abstract smooth surface Σ to the *maximal compatible* smooth atlas. That is, we consider the atlas with the maximal possible set of charts, all of which have transition maps that are diffeomorphisms. This can be accomplished formally by use of Zorn's lemma.

§3 Smooth surfaces in $\mathbb{R}^3$

§3.1 Definitions and equivalent characterisations

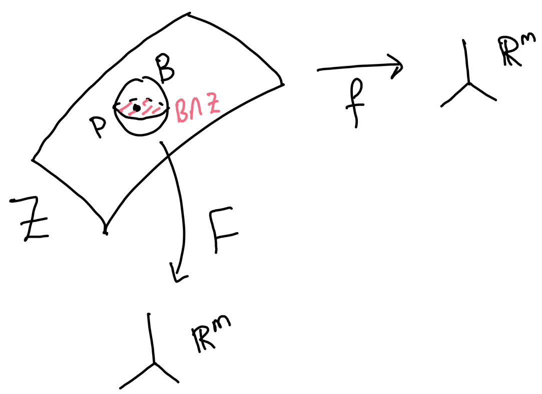
Recall that if $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$, then $f : V \to V'$ is smooth if it is infinitely differentiable.

Definition 3.1 (Smooth Function on $\mathbb{R}^n$)

If Z is an arbitrary subset of $\mathbb{R}^n$, we say that $f : Z \to \mathbb{R}^m$ is **smooth** at $p \in Z$ if $\exists$ an open ball $p \in B \subset \mathbb{R}^n$ and a smooth map $F : B \to \mathbb{R}^m$ which extends f such that they agree on $B \cap Z$[a]. In other words, f is locally the restriction of a smooth map defined on an open set.

[a] $F\mid_{B\cap Z} = f\mid_{B\cap Z}$

Remark 11. This is useful as it may be difficult to take partial derivatives on Z, as when you consider a small deviation from a point p that deviation might not lie in Z.



Definition 3.2 (Diffeormorpishms in $\mathbb{R}^n, \mathbb{R}^m$)

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. We say that X and Y are **diffeomorphic** if $\exists\, f : X \to Y$ smooth with smooth inverse.

Definition 3.3 (Smooth Surface in $\mathbb{R}^3$)

A **smooth surface in $\mathbb{R}^3$** is a subset of $\Sigma \subset \mathbb{R}^3$ s.t. $\forall\, p \in \Sigma$, $\exists$ an open subset $p \in U \subset \Sigma$ that is diffeomorphic to an open set in $\mathbb{R}^2$.

In other words, for all $p \in \Sigma$, there exists an open ball $p \in B \subset \mathbb{R}^3$ such that if $U = B \cap \Sigma$ and there exists a map $F : B \to V \subset \mathbb{R}^2$ smooth s.t. $F|_U : U \to V$ is a homeomorphism, and the inverse map $V \to U \subset \Sigma \subset \mathbb{R}^3$ is smooth.

So we have two notions of smoothness, one abstract and one based on the ambient space and we need to reconcile them.

Definition 3.4 (Allowable Parameterisation)

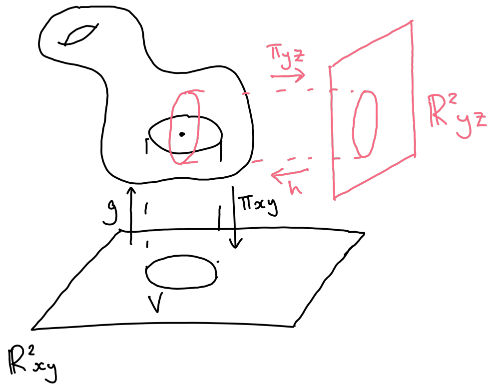
Let $\sigma\colon V \to U$ where $V \subset \mathbb{R}^2$ is open and $U \subset \Sigma \subset \mathbb{R}^3$ is open in Σ, such that σ is a smooth homeomorphism and $D\sigma|_x$ has rank 2 for all $x \in V$. Then σ is called an **allowable parametrisation**. If $\sigma(0) = p$, we say that σ is an allowable parametrisation *near* p.

Theorem 3.1

For a subset $\Sigma \subset \mathbb{R}^3$, the following are equivalent (TFAE).

1. Σ is a smooth surface in $\mathbb{R}^3$;
2. Σ is locally the graph of a smooth function, over one of the three coordinate planes: for all $p \in \Sigma$ there exists an open ball $p \in B \subset \mathbb{R}^3$ and an open set $V \subset \mathbb{R}^2$ such that
$$\Sigma \cap B = \{(x, y, g(x, y)) \colon g\colon V \to \mathbb{R} \text{ smooth}\}$$
or one of the other coordinate planes;
3. Σ is locally cut out by a smooth function with non-zero derivative: for all $p \in \Sigma$ there exists an open ball $p \in B \subset \mathbb{R}^3$ and a smooth function $f\colon B \to \mathbb{R}$ such that
$$\Sigma \cap B = f^{-1}(0); \quad Df\Big|_x \neq 0 \;\forall\, x \in B.$$
4. Σ is locally the image of an allowable parametrisation near all points.

Remark 12. Part (2) implies that if Σ is a smooth surface in $\mathbb{R}^3$, each $p \in \Sigma$ belongs to a chart (U, φ) where φ is (the restriction of) one of the three coordinate plane projections $\pi_{xy}, \pi_{yz}, \pi_{xz}$ from $\mathbb{R}^3$ to $\mathbb{R}^2$. Consider the transition map between two such charts.



If the two charts are based on the same projection such as π_{xy}, then the transition map is the identity. If they are based on different projections π_{xy} and π_{yz}, then the transition map is

$$(x, y) \mapsto (x, y, g(x, y)) \mapsto (y, g(x, y))$$

which has inverse

$$(y, z) \mapsto (h(y, z), y, z) \mapsto (h(y, z), y)$$

Hence all of the transition maps between such charts are smooth. This gives Σ the structure of an abstract smooth surface.

Some of the relations given in the above theorem are easy to prove, but others come as a result of the inverse function theorem.

§3.2 Inverse and implicit function theorems

Theorem 3.2 (Inverse Function Theorem)
Let $U \subset \mathbb{R}^n$ be open, and $f : U \to \mathbb{R}^n$ be continuously differentiable. Let $p \in U$ and $f(p) = q$. Suppose $Df|_p$ is invertible.
Then there is an open neighbourhood V of q and a differentiable map $g : V \to \mathbb{R}^n$ and $g(q) = p$ with image an open neighbourhood $U' \subset U$ of p such that $f \circ g = \mathrm{id}_V$ and $g \circ f = \mathrm{id}_{U'}$. If f is smooth, then g is also.

Remark 13. The chain rule then implies that $Dg|_q = \left(Df|_p\right)^{-1}$.

The inverse function theorem concerns functions $\mathbb{R}^n \to \mathbb{R}^n$, where $Df|_p$ is an isomorphism. If we have a map $\mathbb{R}^n \to \mathbb{R}^m$ for $n > m$, then we can discuss the behaviour when $Df|_p$ is surjective. The derivative $Df|_p$ is an $m \times n$ matrix, so if it has full rank, up to the permutation of coordinates we have that the last m columns are linearly independent.

Theorem 3.3 (Implicit Function Theorem)

Let $p = (x_0, y_0)$ be a point in an open set $U \subset \mathbb{R}^k \times \mathbb{R}^\ell$. Let $f\colon U \to \mathbb{R}^\ell$ be a continuously differentiable map s.t. $p \mapsto 0$ and $\left(\frac{\partial f_i}{\partial y_j}\right)_{\ell\times\ell}$ is an isomorphism at p.
Then there is an open neighbourhood V of x_0 in $\mathbb{R}^k$ and a continuously differentiable map $g\colon V \to \mathbb{R}^\ell$ with $x_0 \mapsto y_0$ such that if $(x, y) \in U \cap (V \times \mathbb{R}^\ell)$, then $f(x, y) = 0 \iff y = g(x)$. If f is smooth, so is g.

Proof. Let $F\colon U \to \mathbb{R}^k \times \mathbb{R}^\ell$ be defined by $(x, y) \mapsto (x, f(x, y))$. Then note that

$$DF = \begin{pmatrix} I & * \\ 0 & \frac{\partial f_i}{\partial y_j} \end{pmatrix}$$

hence DF is an isomorphism at (x_0, y_0). By the Inverse function theorem, F is locally invertible near $F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$.
Consider an open neighbourhood $(x_0, 0) \in V \times V' \subset \mathbb{R}^k \times \mathbb{R}^\ell$, where V, V' open, on which this continuously differentiable inverse $G\colon V \times V' \to U' \subset U \subset \mathbb{R}^k \times \mathbb{R}^\ell$ exists, such that $F \circ G = \mathrm{id}_{V\times V'}$.
Then,

$$G(x, y) = (\varphi(x, y), \psi(x, y)) \implies F \circ G(x, y) = (\varphi(x, y), f(\varphi(x, y), \psi(x, y))) = (x, y)$$

Hence $\varphi(x, y) = x$. We have $f(x, \psi(x, y)) = y$ when $(x, y) \in V \times V'$. This gives $f(x, y) = 0 \iff y = \psi(x, 0)$[a].
We then define $g\colon V \to \mathbb{R}^\ell$ by $x \mapsto \psi(x, 0)$. □

[a] Here y is in V' so it is in the image of f, i.e. it is $f(a, b)$ for some a, b. So $y = 0$ gives us $f(a, b) = 0$. We see that $f(a, \psi(a, 0)) = 0$ so $b = \psi(a, 0)$ defines our surface in U

Example 3.1

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be smooth and $f(x_0, y_0) = 0$, and suppose $\frac{\partial f}{\partial y} \neq 0$ at (x_0, y_0). Then there exists a smooth map $g\colon (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ with $g(x_0) = y_0$ and $f(x, y) = 0 \iff y = g(x)$ for (x, y) in some open neighbourhood of (x_0, y_0).

Since $f(x, g(x)) = 0$ in this open neighbourhood, we can differentiate that expression to find

$$\begin{aligned} f_x(x) + f_y(g(x))g'(x) &= 0 \\ g'(x) &= \frac{-f_x}{f_y} \end{aligned}$$

noting that $f_y \neq 0$ in some neighbourhood near (x_0, y_0). Note that the level set $f(x, y) = 0$ is implicitly defined by g, which is a function for which we have an integral expression.

Example 3.2

Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a smooth map with $f(x_0, y_0, z_0) = 0$. Consider the level set $\Sigma = f^{-1}(0)$, assuming that $Df \neq 0$ at (x_0, y_0, z_0). Permuting coordinates if necessary, we can assume $\frac{\partial f}{\partial z} \neq 0$ at this point. Then there exists an open neighbourhood V of (x_0, y_0) and a smooth function $g\colon V \to \mathbb{R}$ such that $(x_0, y_0) \mapsto z_0$ with the property that for an open set $(x_0, y_0, z_0) \in U$, the set $f^{-1}(0) \cap U$ is the graph of the function g, which is $\{(x, y, g(x, y))\colon (x, y) \in V\}$.

§3.3 Conditions for smoothness

We now prove Theorem 3.1, relating equivalent conditions for smoothness of a surface Σ.

Proof. First, we show that (b) implies all of the other conditions. If Σ is locally a graph $\{(x, y, g(x, y)) : (x, y) \in V\}$, we find a chart from the coordinate plane projection π_{xy} of that graph. Since this projection is smooth and defined on an open neighbourhood of points of Σ, this shows that Σ is a smooth surface in $\mathbb{R}^3$ so (b) $\implies$ (a).

Further, since Σ is locally the given graph, it is cut out by the function $f(x, y, z) = z - g(x, y)$ and note $\frac{\partial f}{\partial z} = 1 \neq 0$ so (b) $\implies$ (c).

Finally, the local parametrisation $\sigma(x, y) = (x, y, g(x, y))$ is allowable; g is smooth so σ is smooth, the partial derivatives of σ are linearly independent as $\sigma_x = (1, 0, g_x)$, $\sigma_y = (0, 1, g_y)$ which is injective/full rank and σ is injective where required. Thus (b) $\implies$ (d).

Now, we show (a) implies (d). This is simply part of the definition of being a smooth surface in $\mathbb{R}^3$, being locally diffeomorphic to $\mathbb{R}^2$. In particular, at $p \in \Sigma$, Σ is locally diffeomorphic to $\mathbb{R}^2$ and the inverse of such a local diffeomorphism is an allowable parametrisation.

We have already shown (c) implies (b); this was example 3.2.

Finally, we must prove (d) implies (b), and then the result will hold. Let $p \in \Sigma$ and V be an open set in $\mathbb{R}^2$ with an allowable parametrisation to Σ, $\sigma : V \to U \subset \Sigma$ s.t. $\sigma(0) = p$. Write $\sigma = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v))$, we have

$$D\sigma = \begin{pmatrix} \frac{\partial \sigma_1}{\partial u} & \frac{\partial \sigma_1}{\partial v} \\ \frac{\partial \sigma_2}{\partial u} & \frac{\partial \sigma_2}{\partial v} \\ \frac{\partial \sigma_3}{\partial u} & \frac{\partial \sigma_3}{\partial v} \end{pmatrix}.$$

This is injective and so has rank 2, hence there exist two rows defining an invertible matrix. Suppose those are the first two rows and consider $\varphi = \pi_{xy} \circ \sigma \colon V \to \mathbb{R}^2$. $D(\pi_{xy} \circ \sigma)|_0$ is an isomorphism.
Let us apply the Inverse function theorem. Hence Σ is locally a graph of $(x, y, \sigma_3(\varphi^{-1}(x, y)))$[a], so (d) $\implies$ (b). $\square$

[a] This is true as $(x, y) = \varphi(u, v)$ and so $\sigma_3(\varphi^{-1}(x, y)) = \sigma_3(u, v)$ therefore $(x, y, \sigma_3(\varphi^{-1}(x, y))) = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v)) \in \Sigma$

Example 3.3 (Ellipsoid)

The ellipsoid $E \subset \mathbb{R}^3$ is $f^{-1}(0)$ for $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$. For all $p \in E = f^{-1}(0)$, $Df|_p \neq 0$ (as $p \neq 0$), so E is a smooth surface in $\mathbb{R}^3$.

Example 3.4

The unit sphere S^2 in $\mathbb{R}^3$ is $f^{-1}(0)$ for $f(x, y, z) = x^2 + y^2 + z^2 - 1$. For any point on S^2, $Df \neq 0$, so S^2 is a smooth surface.

Example 3.5 (Surface of revolution)

Let $\gamma \colon [a, b] \to \mathbb{R}^3$ be a smooth map with image in the xz plane, so

$$\gamma(t) = (f(t), 0, g(t))$$

such that γ is injective, $\gamma' \neq 0$, and $f > 0$. The *surface of revolution* of γ about z has allowable parametrisation

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

where $(u, v) \in (a, b)$[a] $\times (\theta, \theta + 2\pi)$ for a fixed θ.

Left to the reader to check that σ is homeomorphic to its image.

Note that $\sigma_u = (f_u \cos v, f_u \sin v, g_u)$ and $\sigma_v = (-f \sin v, f \cos v, 0)$, and we can check $\|\sigma_u \times \sigma_v\|^2 = f^2((f')^2 + (g')^2)$ which is nonzero on γ, so this really is an allowable parametrisation.

[a] We use an open set so that the surface doesn't have a boundary.

Example 3.6

The orthogonal group $O(3)$ acts on S^2 by diffeomorphisms. Indeed, any $A \in O(3)$ defines a linear (hence smooth) map $\mathbb{R}^3 \to \mathbb{R}^3$ preserving S^2. Hence, the induced map on S^2 is by a homeomorphism which is smooth according to the above definition. This is analogous to the action of the Möbius group on $S^2 = \mathbb{C} \cup \{\infty\}$.

§3.4 Orientability

Definition 3.5 (Orientation-Preserving Map)

Let V, V' be open sets in $\mathbb{R}^2$. Let $f \colon V \to V'$ be a diffeomorphism. Then at every point $x \in V$, $Df|_x \in GL(2, \mathbb{R})$; it is invertible since f is a diffeomorphism.
Let $GL^+(2, \mathbb{R})$ be the subgroup of matrices with positive determinant.

We say that f is **orientation-preserving** if $Df|_x \in GL^+(2, \mathbb{R}) \; \forall \, x \in V$.

Definition 3.6 (Orientable)

An abstract smooth surface Σ is **orientable** if it admits an atlas $\{(U_i, \varphi_i)\}$ where the transition maps are all orientation-preserving diffeomorphisms of of open sets of $\mathbb{R}^2$. A choice of such an atlas is an **orientation** of Σ; Σ can be called **oriented** when such an orientation is given.

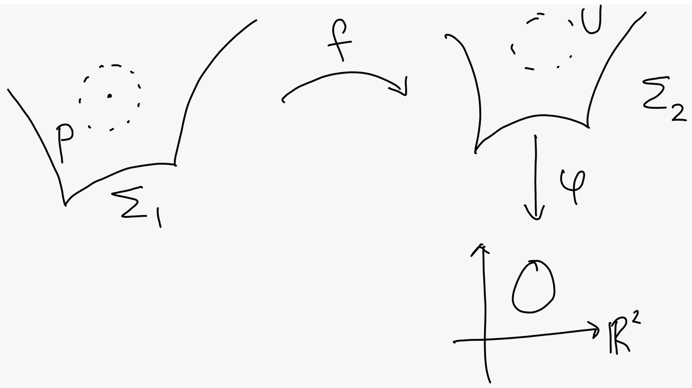
Remark 14. If we have an atlas where all the transition maps have det < 0, then we can find transition maps and so an atlas with det > 0, e.g. composing $(x, y) \mapsto (y, x)$ with the transition maps should work.

Remark 15. An orientable atlas belongs to a maximal compatible oriented smooth atlas.

Lemma 3.1

If Σ_1 and Σ_2 are diffeomorphic abstract smooth surfaces, then Σ_1 is orientable iff Σ_2 is orientable.

Proof. Let $f \colon \Sigma_1 \to \Sigma_2$ be a diffeomorphism. Suppose Σ_2 is orientable and equipped with an oriented atlas.

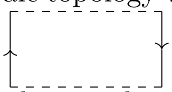


Consider the atlas on Σ_1 given by $(f^{-1}(U), \varphi \circ f|_{f^{-1}(U)})$, where (U, ψ) is a chart at $f(p)$ in the oriented atlas for Σ_2. Then, the transition function[a] between two such charts is exactly the transition function between charts in the Σ_2 atlas. $\square$

[a] $(\varphi_1 \circ f) \circ (\varphi_2 \circ f)^{-1} = \varphi_1 \circ \varphi_2^{-1}$

Remark 16.

1. There is no sensible classification of the set of all smooth or topological surfaces. For instance, $\mathbb{R}^2 \setminus Z$ for a closed set Z can be shown to yield uncountably many types of homeomorphisms.
   However, compact smooth surfaces may be classified by their Euler characteristic and their orientability, up to diffeomorphism. This theorem will not be proven in this course.

2. There is a definition of orientation-preserving *homeomorphism* that does not rely on the determinant, but that instead relies on some algebraic topology which is not covered in this course. The Möbius band is the surface  where the dashed lines represent the absence of edges. It is provable that an abstract smooth surface is orientable $\iff$ it contains no subsurface (an open set) homeomorphic to the Möbius band. We can therefore say that a topological surface is orientable $\iff$ it contains no subsurface (an open set) homeomorphic to a Möbius band.

3. We can define other structures on an abstract smooth surface by considering smooth atlases such that if $\varphi_1\varphi_2^{-1}$ is a transition map, then $D(\varphi_1\varphi_2^{-1})$ at x belongs to a specific subgroup $G \leq GL(2, \mathbb{R})$. For example, defining $G = \{e\}$ leads to Euclidean surfaces. The group $GL(1, \mathbb{C})$ identified as a subgroup of $GL(2, \mathbb{R})$ yields the Riemann surfaces, also $D(\varphi_1\varphi_2^{-1}) \in GL(1, \mathbb{C}) \implies \varphi_1\varphi_2^{-1}$ holomorphic as being in $GL(1, \mathbb{C})$ implies that the Cauchy Riemann equations hold.

Example 3.7

For S^2 with the atlas of two stereographic projections, we can find the transition map to be

$$(u, v) \mapsto \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right)$$

on $\mathbb{R}^2 \setminus \{0\}$. This has positive determinant, so S^2 is orientable.

Note we only need to know the determinant at one point in $\mathbb{R}^2 \setminus \{0\}$ as $\mathbb{R}^2 \setminus \{0\}$ is a connected set so the determinant of the differential is a super continuous function, so if it has sign positive at one point it must have sign positive on the whole space. If its sign changed it must be 0 somewhere but we know its a diffeomorphism so the determinant of the differential can't be 0.

Example 3.8

For the torus T^2, we previously found an atlas such that the transition maps are translations of $\mathbb{R}^2$. Hence the torus is an oriented surface, and also a Euclidean surface.

For surfaces in $\mathbb{R}^3$, we'd like to have orientability dictated by some "ambient feature", i.e. we want to be able to just look at the surface and know if its orientable or not.

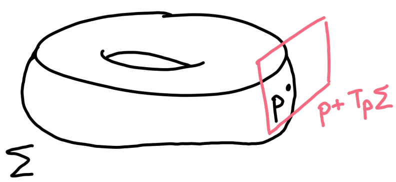
§3.5 Tangent planes

Recall that an *affine* subspace of a vector space is a translation of a linear subspace.

Definition 3.7 (Tangent Plane)

Let Σ be a smooth surface in $\mathbb{R}^3$, and $p \in \Sigma$. Let $\sigma \colon V \to U \subset \Sigma$ be an allowable parametrisation of Σ near p, so V is an open subset of $\mathbb{R}^2$ and U is open in Σ, such that $\sigma(0) = p$.

The **tangent plane** $T_p\Sigma$ of Σ at p is the image of $(D\sigma|_0) \subset \mathbb{R}^3$, which is a two-dimensional vector subspace of $\mathbb{R}^3$. The **affine tangent plane** is $p + T_p\Sigma$, which is an affine subspace of $\mathbb{R}^3$.



Remark 17. The affine tangent plane is the 'best' linear approximation to a surface Σ at a given point.

Lemma 3.2

$T_p\Sigma$ is well-defined, i.e. it's independent of the choice of allowable parametrisation near p.

Proof (i). Suppose $\sigma\colon V \to U$ and $\widetilde{\sigma}\colon \widetilde{V} \to \widetilde{U}$ are allowable parametrisations with $\sigma(0) = \widetilde{\sigma}(0) = p$. There exists a transition map $\sigma^{-1} \circ \widetilde{\sigma}$, which is a diffeomorphism of open sets in $\mathbb{R}^2$. Therefore,

$$\widetilde{\sigma} = \sigma \circ \underbrace{\left(\sigma^{-1} \circ \widetilde{\sigma}\right)}_{\text{diffeomorphism}}$$

Hence $D(\sigma^{-1} \circ \widetilde{\sigma})\big|_0$ is an isomorphism. Thus, the images of $D\,\widetilde{\sigma}\big|_0$ and $D\,\sigma\big|_0$ agree. $\square$

Proof (ii). Let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a smooth map such that γ has image inside Σ, and $\gamma(0) = p$. We will show that $\gamma'(0) \in T_p\Sigma$. If $\sigma\colon V \to U$ is an allowable parametrisation with $\sigma(0) = p$ as above, and ε is sufficiently small such that $\operatorname{Im}\gamma \subset U$, then $\gamma(t) = \sigma(u(t), v(t))$ for some smooth functions $u, v\colon (-\varepsilon, \varepsilon) \to V$. Then $\gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t)$ is in the image of $D\,\sigma\big|_t$. Thus, $T_p\Sigma = \operatorname{span}\{\gamma'(0)\colon \gamma \text{ as above}\}$. $\square$

Definition 3.8 (Normal Direction)

If Σ is a smooth surface in $\mathbb{R}^3$ and $p \in \Sigma$, the **normal direction** to Σ at p is $(T_p\Sigma)^\perp$, the Euclidean orthogonal complement to the tangent plane at p.

Remark 18. For all $p \in \Sigma$, there exist exactly two normalised normal vectors.

Definition 3.9 (Two-Sided)

A smooth surface in $\mathbb{R}^3$ is **two-sided** if it admits a continuous global choice of unit normal vector.

Lemma 3.3

A smooth surface in $\mathbb{R}^3$ is orientable (as an abstract smooth surface) iff it is two-sided (as a smooth surface in $\mathbb{R}^3$).

Proof. Let $\sigma\colon V \to U \subset \Sigma$ be an allowable parametrisation. Let $\sigma(0) = p$. We will define the positive unit normal with respect to σ at p to be the normal vector $n_\sigma(p)$ with the property that $\{\sigma_u, \sigma_v, n_\sigma(p)\}$ and $\{e_1, e_2, e_3\}$ are related by a positive determinant change of basis matrix (the two sets of vectors have the same orientation), where $\{e_1, e_2, e_3\}$ are the standard basis vectors. In other words,

$$n_\sigma(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

Consider an alternative parametrisation $\widetilde{\sigma}\colon \widetilde{V} \to \widetilde{U}$, where $\widetilde{\sigma}(0) = p$, such that $\widetilde{\sigma}$ belongs to the same oriented and smooth atlas as σ. Hence, $\sigma = \widetilde{\sigma} \circ \varphi$ for some transition map $\varphi = \widetilde{\sigma}^{-1} \circ \sigma$. Let

$$D\,\varphi\Big|_0 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

Hence,

$$\sigma_u = \alpha\widetilde{\sigma}_u + \gamma\widetilde{\sigma}_v; \quad \sigma_v = \beta\widetilde{\sigma}_u + \delta\widetilde{\sigma}_v$$

This gives

$$\sigma_u \times \sigma_v = \det\left(D\,\varphi\Big|_0\right)\widetilde{\sigma}_u \times \widetilde{\sigma}_v \tag{†}$$

The determinant here is positive since the charts in question belong to an oriented atlas. Thus the positive unit normal is intrinsic to the surface, it does not depend on the choice of parametrisation. The expression for $n_\sigma(p)$ is continuous since the cross product is continuous, hence Σ is two-sided.

Conversely, if Σ is two-sided, there exists a global continuous choice of normal vector, so we can consider the subatlas of the smooth atlas s.t we have a chart (U, φ) with $\varphi^{-1} = \sigma$ and $\{\sigma_u, \sigma_v, n\}$ is an oriented basis of $\mathbb{R}^3$. We can make $\{\sigma_u, \sigma_v, n\}$ have positive orientation by negating σ. By (†), the transition maps between such charts are orientation-preserving. Hence Σ is orientable. $\square$

Remark 19. Given $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ smooth with $\operatorname{Im}(\gamma) \subset \Sigma$ and $\gamma(0) = p$. $\gamma(t) = \sigma(u(t), v(t))$ so $\gamma'(0) = D\sigma|_0\,(u'(0), v'(0)) \in T_p\Sigma$. This gives that $T_p\Sigma = \operatorname{span}\{\gamma'(0)\colon \gamma \text{ as above}\} =$ "tangent vectors to curves in Σ".

Lemma 3.4

If Σ is a smooth surface in $\mathbb{R}^3$ and $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth map which preserves Σ setwise, then $DA|_p \in L(\mathbb{R}^3, \mathbb{R}^3)$ maps $T_p\Sigma$ to $T_{A(p)}\Sigma$ for $p \in \Sigma$.

Proof. Let $\gamma\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ be a smooth map such that its image lies on Σ, and $\gamma(0) = p$. Recall that $T_p\Sigma$ is spanned by $\gamma'(0)$ for such curves γ. Now, consider

$A \circ \gamma \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$, which also has image Σ, and

$$D\, A\Big|_{\gamma(0)} \circ D\, \gamma\Big|_0 = D\, A\Big|_p (\gamma'(0)) = D\, (A \circ \gamma)\Big|_0 \in T_{A(p)}\Sigma$$

□

Example 3.9 (Unit Sphere)

Let $S^2 \subset \mathbb{R}^3$ be the unit sphere. The normal vector at p is the line through the origin and p; indeed, since SO_3 acts transitively on S^2, it suffices to check at one point, such as the north pole. We can choose the outward-facing normal vector to be the positive normal, denoted $n(p)$. S^2 is two-sided by the construction of this normal vector, hence S^2 is orientable.

Alternatively, take any $\gamma : (-\varepsilon, \varepsilon) \to S^2$ with $\gamma(0) = p$. $\|\gamma(t)\|^2 = 1$, so differentiating at $t = 0$, $2\langle \gamma'(0), p\rangle = 0$ so $(T_pS^2)^{\perp} = \mathbb{R}p = \{xp : x \in \mathbb{R}\}$. Let $n(p) = p$, clearly a global continuous choice of normal vector so S^2 is 2-sided.

Example 3.10 (Möbius Band)

Walk around the unit circle in the xy-plane and take an open interval of length 1. Rotate this line in the cz-plane as we move around the circle, s.t. it has rotated by $\frac{\theta}{2}$ after moving an angle θ in the circle (see picture). After a full turn the segment returns to its original position but with end points inverted.
One embedding of the Möbius band in $\mathbb{R}^3$ is

$$\sigma(t, \theta) = \left(\left(1 - t\sin\frac{\theta}{2}\right)\cos\theta, \left(1 - t\sin\frac{\theta}{2}\right)\sin\theta, t\cos\frac{\theta}{2}\right)$$

where $(t, \theta) \in V_1 = \left\{t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \theta \in (0, 2\pi)\right\}$ or
$(t, \theta) \in V_2 = \left\{t \in \left(-\frac{1}{2}, \frac{1}{2}\right), \theta \in (-\pi, \pi)\right\}$.
This gives us the standard Möbius band parametrically, I don't think its worthwhile trying to understand why exactly it works or what the explanation even means.

We can check that if σ_i is σ on V_i, then σ_i is allowable. Further,

$$\sigma_t \times \sigma_\theta = \left(-\cos\theta\cos\frac{\theta}{2}, -\sin\theta\cos\frac{\theta}{2}, -\sin\frac{\theta}{2}\right) \equiv n_\theta$$

which is already normalised. As $\theta \to 0$ from above, $n_\theta \to (-1, 0, 0)$. As $\theta \to 2\pi$ from below, $n_\theta \to (1, 0, 0)$. Hence, the surface is not two-sided.

§4 Geometry of surfaces in $\mathbb{R}^3$

§4.1 First fundamental form

Let $\gamma\colon (a,b) \to \mathbb{R}^3$ be smooth. The **length** of γ is

$$L(\gamma) = \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

This result is independent of the choice of parametrisation. Let $s\colon (A,B) \to (a,b)$ be a monotonically increasing function, let $\tau(t) = \gamma(s(t))$ and $s \geq 0$. Then

$$L(\tau) = \int_A^B \|\tau'(t)\| \,\mathrm{d}t = \int_A^B \|\gamma'(s(t))\| |s'(t)| \,\mathrm{d}t = \int_a^b \|\gamma'(t')\| \,\mathrm{d}s = L(\gamma)$$

Lemma 4.1

If $\gamma\colon (a,b) \to \mathbb{R}^3$ is continuously differentiable and $\gamma'(t) \neq 0$, then γ can be parametrised by arc length, i.e. a parameter s.t. $\|\gamma'(s)\| = 1 \ \forall\, s$.

Proof. Left as an exercise. □

Let Σ be a smooth surface in $\mathbb{R}^3$, and let $\sigma\colon V \to U \subset \Sigma$ be an allowable parametrisation. If $\gamma\colon (a,b) \to U$ is smooth, then there exist functions $(u(t), v(t))\colon (a,b) \to V$ smooth s.t. $\gamma(t) = \sigma(u(t), v(t))$. Hence $\gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t)$, giving

$$\|\gamma'(t)\|^2 = Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2$$

for functions

$$E = \langle \sigma_u, \sigma_u \rangle\,; \quad F = \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle\,; \quad G = \langle \sigma_v, \sigma_v \rangle$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ represents the usual Euclidean inner product. Note that E, F, G depend only on σ and not on γ, also they are smooth functions on V.

Definition 4.1 (First fundamental Form)

The **first fundamental form** of Σ in the parametrisation σ is the expression

$$E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$$

This notation is designed to remind you that

$$L(\gamma) = \int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\,\mathrm{d}t$$

where $\gamma(t) = \sigma(u(t), v(t))$.

Remark 20. The Euclidean inner product on $\mathbb{R}^3$ provides an inner product on the subspace $T_p\Sigma$. Choosing a parametrisation σ, we can say $T_p\Sigma = \operatorname{Im} D\,\sigma|_0 = \operatorname{span}\{\sigma_u, \sigma_v\}$ where $\sigma(0) = p$. The first fundamental form is a symmetric bilinear form on the tangent spaces $T_p\Sigma$, varying smoothly in p. However, we choose to express this in a basis coming from the parametrisation σ. In particular, we can think about the matrix expression

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

This is an example of a **Riemannian metric**.

Example 4.1

The plane $\mathbb{R}^2_{xy} \subset \mathbb{R}^3$ has the parametrisation $(u, v) \mapsto (u, v, 0)$. Hence, $\sigma_u = e_1$ and $\sigma_v = e_2$, hence the first fundamental form is $\mathrm{d}u^2 + \mathrm{d}v^2$.

We could also use polar coordinates, using $\sigma(r, \theta) = (r\cos\theta, r\sin\theta, 0)$. This parametrises the plane without the origin. This gives $\sigma_r = (\cos\theta, \sin\theta, 0)$ and $\sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$. The first fundamental form is $\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

Definition 4.2 (Isometries)

Let Σ, Σ' be smooth surfaces in $\mathbb{R}^3$. We say that they are **isometric** if there exists a diffeomorphism $f\colon \Sigma \to \Sigma'$ that preserves the lengths of all curves. More formally, for every smooth curve $\gamma\colon (a, b) \to \Sigma$, $L_\Sigma(\gamma) = L_{\Sigma'}(f \circ \gamma)$.

Example 4.2

Let $\Sigma' = f(\Sigma)$ where $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a global isometry, or rigid motion, of $\mathbb{R}^3$; that is, $v \mapsto Av + b$ for an orthogonal matrix A. These isometries preserve the Euclidean inner product on $\mathbb{R}^3$, hence f preserves length and so it is an isometry.

$$\begin{aligned} \|(f \circ \gamma)'(t)\| &= \|A\gamma'(t)\| \\ &= \|\gamma'(t)\|. \end{aligned}$$

However, in the definition, we need not map all of $\mathbb{R}^3$ to itself, just $\Sigma \to \Sigma'$.

Often we're interested in local statements.

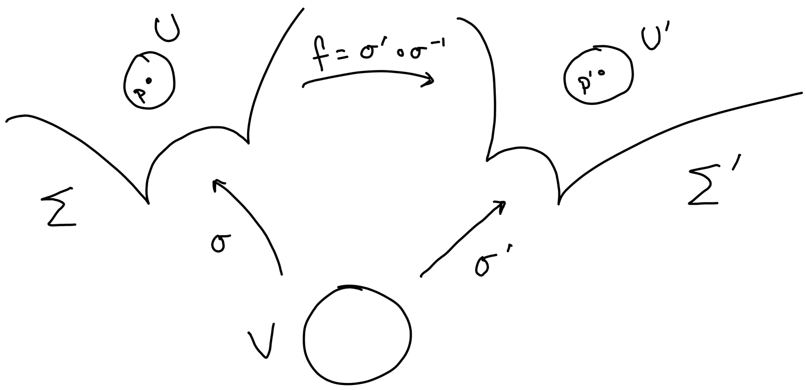
Definition 4.3 (Locally Isometric)

We say that Σ and Σ' are **locally isometric** near points $p \in \Sigma$ and $q \in \Sigma'$ if there exist open neighbourhoods U of p and V of q such that U and V are isometric.

We can also say that Σ and Σ' are locally isometric if they are locally isometric at all points; that is, each point of Σ is locally isometric to some point on Σ'.

Lemma 4.2

Smooth surfaces Σ, Σ' in $\mathbb{R}^3$ are locally isometric near $p \in \Sigma$ and $q \in \Sigma'$ iff there exist allowable parametrisations $\sigma\colon V \to U \subset \Sigma$ and $\sigma'\colon V \to U' \subset \Sigma'$ such that the first fundamental forms are equal in V ($E = E', F = F', G = G'$).



Proof. By definition, the first fundamental form of Σ determines the lengths of all curves on Σ that lie in $\sigma(V) = U$.

($\Longleftarrow$): If we have σ and σ' with equal fundamental forms, then $\sigma' \circ \sigma^{-1} : U \to U'$ is an isometry since given curve $\gamma(t)$

$$
\begin{aligned}
\sigma^{-1}(\gamma(t)) &= (u(t), v(t)) \\
\left\| \frac{d}{dt} \underbrace{\sigma' \circ \sigma^{-1}}_{f} \circ \gamma \right\|^2 &= \left\| \frac{d}{dt} \sigma'(u(t), v(t)) \right\|^2 \\
&= E'\dot{u}^2 + 2F'\dot{u}\dot{v} + G'\dot{v}^2 \\
&= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \\
&= \left\| \frac{d}{dt} \gamma(t) \right\|^2 \\
\therefore L(\sigma' \circ \sigma^{-1} \circ \gamma) &= L(\gamma).
\end{aligned}
$$

($\Longrightarrow$): We shall first show that the lengths of curves in U determine the first fundamental form of σ. Given $\sigma\colon V \to U$, without loss of generality let $V = B(0, \delta)$

for some $\delta > 0$, where $\sigma(0) = p$. Consider, for all $\varepsilon < \delta$, the curve

$$\gamma_\varepsilon \colon [0, \varepsilon] \to U; \quad t \mapsto \sigma(t, 0)$$

Then,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(\gamma_\varepsilon) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^\varepsilon \sqrt{E(t, 0)} \, \mathrm{d}t = \sqrt{E(\varepsilon, 0)}$$

Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} L(\gamma_\varepsilon) = \sqrt{E(0, 0)}$$

So we can determine E at p by looking at lengths of curves. We can similarly consider

$$\chi_\varepsilon \colon [0, \varepsilon] \to U; \quad t \mapsto \sigma(0, t)$$

which determines G. Finally, consider

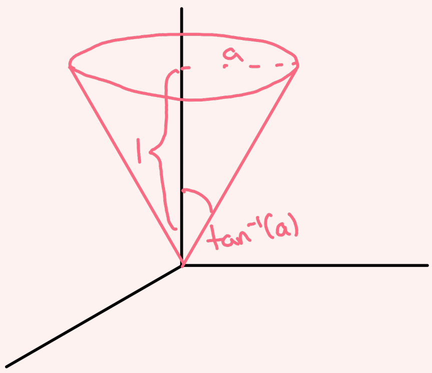
$$\lambda_\varepsilon \colon [0, \varepsilon] \to U; \quad t \mapsto \sigma(t, t)$$

which determines $\sqrt{(E + 2F + G)(0, 0)}$ which gives F implicitly.

So if $f : U \to U'$ is a local isometry take any allowable parametrisation $\sigma' : V \to U'$ then $= f^{-1} \circ \sigma'$ is s.t. the first fundamental form of σ, σ' agree. $\square$

Example 4.3

Consider the cone with angle $\arctan a$ to the vertical.



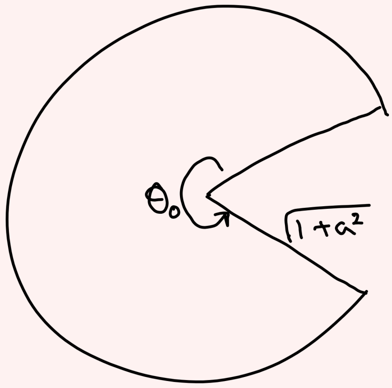
For $u > 0$ and $v \in (0, 2\pi)$, we define

$$\sigma(u, v) = (au \cos v, au \sin v, u).$$

This parametrises the cone excluding the line at $v = 0$.
The first fundamental form is

$$(1 + a^2)\,\mathrm{d}u^2 + a^2u^2\,\mathrm{d}v^2$$

Consider cutting the cone along the line $v = 0$ and flattening it into a plane sector.



The circumference of the sector is $2\pi a$ and the radius is $\sqrt{1 + a^2}$, hence the angle traced out by the sector is $\theta_0 = \frac{2\pi a}{\sqrt{1+a^2}}$. We can parametrise this subset of the plane by

$$\sigma(r, \theta) = \left(\sqrt{1 + a^2}r\cos\left(\frac{a\theta}{\sqrt{1 + a^2}}\right), \sqrt{1 + a^2}r\sin\left(\frac{a\theta}{\sqrt{1 + a^2}}\right), 0\right)$$

for $r > 0$ and $\theta \in (0, 2\pi)$. We can then check that the first fundamental form here is

$$(1 + a^2)\,\mathrm{d}r^2 + r^2a^2\,\mathrm{d}\theta^2$$

which matches the first fundamental form for the cone itself. Hence the cone and the plane are locally isometric.

However, the cone and plane are not globally isometric, since the two topological spaces are not homeomorphic, so no diffeomorphism that preserves lengths can be constructed. An intuitive way to think about this is that the cone doesn't include the origin so shrinking any curve on the cone to 0 leaves the cone whilst the same is not true in the plane, this is the notion of simple connectedness.

Example 4.4

The sphere of radius a, given by $\{x^2 + y^2 + z^2 = a^2\}$, has an open set with allowable parametrisation

$$\sigma(u, v) = (a\cos u\cos v, a\cos u\sin v, a\sin u)$$

where $u \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $v \in (0, 2\pi)$. This parametrises the complement of a half great circle. Here,

$$\sigma_u = (-a\sin u\cos v, -a\sin u\sin v, a\cos u); \quad \sigma_v = (-a\cos u\sin v, a\cos u\cos v, 0)$$

Hence,

$$E = a^2; \quad F = 0; \quad G = a^2\cos^2 u$$

which gives the first fundamental form as

$$a^2\,\mathrm{d}u^2 + a^2\cos^2 u\,\mathrm{d}v^2$$

Example 4.5

Consider the surface of revolution given by a curve

$$\eta(t) = (f(t), 0, g(t))$$

rotated about the z axis. The resulting surface has parametrisation

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

Hence,

$$\sigma_u = (f_u\cos v, f_u\sin v, g_u); \quad \sigma_v = (-f\sin v, f\cos v, 0)$$

which gives

$$(f_u^2 + g_u^2)\,\mathrm{d}u^2 + f^2\,\mathrm{d}v^2$$

Lemma 4.3

Let Σ be a smooth surface in $\mathbb{R}^3$, and let $p \in \Sigma$. Suppose we have two allowable parametrisations $\sigma\colon V \to U$ and $\sigma'\colon V' \to U$ s.t. $\sigma(0) = \sigma'(0) = p$ and U an open nbd of p. The two parametrisations differ by a transition map $f = {\sigma'}^{-1} \circ \sigma$ which is a diffeomorphism of open subsets of $\mathbb{R}^2$. There exist first fundamental forms for both parametrisations. Then,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (Df)^{\mathsf{T}} \begin{pmatrix} E' & F' \\ F' & G' \end{pmatrix} (Df)$$

Proof. By definition,

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \sigma_u \cdot \sigma_u & \sigma_u \cdot \sigma_v \\ \sigma_v \cdot \sigma_u & \sigma_v \cdot \sigma_v \end{pmatrix} = (D\sigma)^\intercal (D\sigma)$$

Since, $\sigma = \sigma' \circ F \implies D\sigma = D\sigma' Df$ so $(D\sigma)^\intercal(D\sigma) = (D\sigma = D\sigma' Df)^\intercal(D\sigma = D\sigma' Df) = (Df)^\intercal(D\sigma')^\intercal(D\sigma')(Df)$. $\square$

§4.2 Conformality

If $v, w \in \mathbb{R}^3$, we have $v \cdot w = \|v\| \cdot \|w\| \cdot \cos\theta$. This allows us to deduce the angle θ between two vectors given their dot product and lengths. This can also be done when v, w are in the tangent plane $T_p\Sigma$, and then we can express the angle in terms of the first fundamental form. Let σ be an allowable parametrisation for Σ near p, such that $D\,\sigma|_0$ evaluates to v at v_0 and w at w_0.

$$\begin{aligned} \cos\theta = \frac{v \cdot w}{\|v\| \cdot \|w\|} &= \frac{I(v_0, w_0)}{\sqrt{I(v_0, v_0)}\sqrt{I(w_0, w_0)}} \\ I(v_0, w_0) = v_0^\intercal \begin{pmatrix} E & F \\ F & G \end{pmatrix} w_0. \end{aligned}$$

where I denotes the first fundamental form of σ at zero.

Lemma 4.4

Let Σ be a smooth surface in $\mathbb{R}^3$, and let $\sigma \colon V \to U$ be an allowable parametrisation of Σ near p.
Then σ is *conformal* (preserves angles) iff $E = G$ and $F = 0$ in the first fundamental form.

Proof. ($\implies$): Consider curves $\gamma \colon t \mapsto (u(t), v(t))$ and $\tilde{\gamma} \colon t \mapsto (\tilde{u}(t), \tilde{v}(t))$ in V, where $\gamma(0) = \tilde{\gamma}(0) = 0 \in V$. Let σ be a parametrisation $V \to U \subset \Sigma$ such that $\sigma(0) = p \in \Sigma$. Then the curves $\sigma \circ \gamma$ and $\sigma \circ \tilde{\gamma}$ meet at angle θ on Σ, where

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F\left(\dot{u}\dot{\tilde{v}} + \dot{v}\dot{\tilde{u}}\right) + G\dot{v}\dot{\tilde{v}}}{\sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\sqrt{E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2}}$$

In particular, if σ is conformal, suppose $\gamma(t) = (t, 0)$ and $\tilde{\gamma}(t) = (0, t)$. Then, we have that the curves meet at $\frac{\pi}{2}$ in V, so they meet at $\frac{\pi}{2}$ in Σ, so we find that $\cos\theta = 0 \implies F = 0$ as $\dot{u} = \dot{\tilde{v}} = 1$ and $\dot{\tilde{u}} = \dot{v} = 0$.
Similarly, if $\gamma(t) = (t, t)$ and $\tilde{\gamma}(t) = (t, -t)$, we find $\cos\theta = 0 \implies E = G$.

($\impliedby$): Conversely, suppose there exists a parametrisation σ such that $E = G$ and

$F = 0$. Then, in this parametrisation, the first fundamental form is of the form $\rho\left(\mathrm{d}u^2 + \mathrm{d}v^2\right)$ for $\rho = E \colon V \to \mathbb{R}$. So

$$\cos\theta = \frac{\dot{u}\dot{\tilde{u}} + \dot{v}\dot{\tilde{v}}}{\sqrt{\dot{u}^2 + \dot{v}^2}\sqrt{\dot{\tilde{u}}^2 + \dot{\tilde{v}}^2}},$$

i.e. angles don't change.

Alternatively, the first fundamental form is a pointwise rescaling of the Euclidean fundamental form $\mathrm{d}u^2 + \mathrm{d}v^2$. Rescaling the plane does not change angles, so σ is conformal as required. □

Remark 21. Conformality in charts is historically important for cartography. The existence of conformal charts is closely connected to Riemann surfaces, which are topological surfaces locally modelled on $\mathbb{C}$ instead of $\mathbb{R}^2$.

§4.3 Area

Recall that a parallelogram spanned by vectors v, w has area $\|v \times w\| = \sqrt{\langle v, v\rangle \langle w, w\rangle - \langle v, w\rangle^2}$, where $\times$ denotes the cross product. Let $\sigma \colon V \to U \subset \Sigma$ be an allowable parametrisation with $\sigma(0) = p$, and consider $\sigma_u, \sigma_v \in T_p\Sigma$. The square of the area of the infinitesimal parallelogram spanned by σ_u, σ_v is given by

$$\left(\langle \sigma_u, \sigma_u\rangle \langle \sigma_v, \sigma_v\rangle - \langle \sigma_u, \sigma_v\rangle^2\right)^{1/2} = \sqrt{EG - F^2}.$$

Definition 4.4 (Area)

Let Σ be a smooth surface in $\mathbb{R}^3$, and $\sigma \colon V \to U \subset \Sigma$ an allowable parametrisation. Then,

$$\operatorname{area}(U) = \int_V \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v$$

Remark 22. This is independent of parametrisation. Indeed, suppose $\sigma \colon V \to U$ and $\widetilde{\sigma} \colon \widetilde{V} \to U$ are allowable. Then $\widetilde{\sigma} = \sigma \circ \varphi$ for some transition map $\varphi = \sigma^{-1} \circ \widetilde{\sigma} \colon \widetilde{V} \to V$. We know then that by Lemma 4.3

$$\begin{pmatrix} \widetilde{E} & \widetilde{F} \\ \widetilde{F} & \widetilde{G} \end{pmatrix} = (D\widetilde{\sigma})^\intercal (D\widetilde{\sigma}) = (D\varphi)^\intercal \begin{pmatrix} E & F \\ F & G \end{pmatrix} (D\varphi)$$

Hence by taking determinants,

$$\sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2} = |\det(D\varphi)| \sqrt{EG - F^2}$$

The usual change of variables formula for integration, combined with the fact that φ is a diffeomorphism, gives

$$\int_V \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \int_{\widetilde{V}} \sqrt{\widetilde{E}\widetilde{G} - \widetilde{F}^2}\,\mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}\,.$$

So $\mathrm{area}(U)$ is intrinsic and well-defined.

Note, we can compute the area of an open set $U \subset \Sigma$, not necessarily lying in a single parametrisation, by covering the set by a finite amount of open subsets which lie in single charts. For instance, if Σ is compact, we can compute the area of Σ itself.

Example 4.6

Consider the graph $\Sigma = \{(u, v, f(u, v)) \colon (u, v) \in \mathbb{R}^2\}$, where $f \colon \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. This has a global parametrisation $\sigma(u, v) = (u, v, f(u, v))$. Here, $\sigma_u = (1, 0, f_u)$ and $\sigma_v = (0, 1, f_v)$, hence

$$\sqrt{EG - F^2} = \sqrt{1 + f_u^2 + f_v^2}$$

Let $U_R \subset \Sigma$ be the part of the graph lying inside the disc $B(0, R) \subset \mathbb{R}^2$. Then

$$\mathrm{area}(U_R) = \int_{B(0,R)} \sqrt{1 + f_u^2 + f_v^2}\,\mathrm{d}u\,\mathrm{d}v \geq \pi R^2$$

with equality exactly when $f_u = f_v = 0$, which is when f is constant and U_R is contained inside a plane perpendicular to the z axis. Hence, the projection from Σ to $\mathbb{R}^2_{xy}$ is not area-preserving, unless Σ is a plane perpendicular to the z axis.

Example 4.7

Consider the sphere enclosed exactly by a cylinder. The cylindrically radial projection from the sphere to the cylinder is area-preserving. You will prove this in Sheet 2.

§4.4 Second fundamental form

Let's try to measure how much $\Sigma \subset \mathbb{R}^3$ deviates from its own tangent planes.

Let $\sigma \colon V \to U \subset \Sigma$ be allowable. By using Taylor's theorem, we can write

$$\begin{aligned}\sigma(u + h, v + \ell) = {} & \sigma(u, v) + h\sigma_u(u, v) + \ell\sigma_v(u, v) \\ & + \frac{1}{2}\Big(h^2\sigma_{uu}(u, v) + 2h\ell\sigma_{uv}(u, v) + \ell^2\sigma_{vv}(u, v)\Big) + O(h^3, \ell^3)\end{aligned}$$

where h, ℓ are small, and $(u+h, v+\ell) \in V$. Recall that if $p = \sigma(u,v)$, we have $T_p\Sigma = \langle \{\sigma_u, \sigma_v\} \rangle$. Hence, the orthogonal distance from $\sigma(u+h, v+\ell)$ to the affine tangent plane $T_p\Sigma + p$ is given by projection to the normal direction.

$$\langle n, \sigma(u+h, v+\ell) - \sigma(u,v) \rangle = \frac{1}{2}\Big(\langle n, \sigma_{uu}\rangle h^2 + 2\langle n, \sigma_{uv}\rangle h\ell + \langle n, \sigma_{vv}\rangle \ell^2\Big) + O(h^3, \ell^3)$$

Definition 4.5 (Second Fundamental Form)

The **second fundamental form** of Σ in the allowable parametrisation σ is the quadratic form

$$L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2$$

where

$$L = \langle n, \sigma_{uu}\rangle\,; \quad M = \langle n, \sigma_{uv}\rangle\,; \quad N = \langle n, \sigma_{vv}\rangle$$

and

$$n = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

We can write this as the matrix

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

which defines a quadratic form on $T_p\Sigma$ which varies smoothly in p.

Lemma 4.5

Let V be connected and $\sigma\colon V \to U \subset \Sigma$ be an allowable parametrisation such that the second fundamental form vanishes identically with respect to σ. Then U lies in an affine plane in $\mathbb{R}^3$.

Remark 23. The first fundamental form is a non-degenerate symmetric bilinear form on $T_p\Sigma$, whereas the second fundamental form may be degenerate.

Proof. By definition,

$$\langle n, \sigma_u\rangle = 0 = \langle n, \sigma_v\rangle$$

Hence, by differentiating, we find

$$0 = \langle n_u, \sigma_u\rangle + \langle n, \sigma_{uu}\rangle = \langle n_v, \sigma_v\rangle + \langle n, \sigma_{vv}\rangle = \langle n_v, \sigma_u\rangle + \langle n, \sigma_{uv}\rangle$$

Some of these terms appear in the definition of the second fundamental form:

$$\begin{aligned}
L &= \langle n, \sigma_{uu} \rangle = - \langle n_u, \sigma_u \rangle \\
M &= \langle n, \sigma_{uv} \rangle = - \langle n_v, \sigma_u \rangle = - \langle n_u, \sigma_v \rangle \\
N &= \langle n, \sigma_{vv} \rangle = - \langle n_v, \sigma_v \rangle
\end{aligned}$$

If the second fundamental form vanishes, then n_u is orthogonal to σ_u, σ_v. Also $\langle n, n \rangle = 1$ so differentiating wrt u we get $2 \langle n, n_u \rangle = 0$ so n_u is orthogonal to n as well. Since $\{\sigma_u, \sigma_v, n\}$ form a basis for $\mathbb{R}^3$, we have $n_u = 0$. Similarly, $n_v = 0$, hence n is constant by the mean value theorem (V connected and by use of mean value inequality).

This implies that $\langle \sigma, n \rangle$ is constant as $\langle \sigma, n_u \rangle = \langle \sigma_u, n \rangle = \langle \sigma, n_v \rangle = \langle \sigma_v, n \rangle = 0$. So U is contained in a plane. □

Remark 24. The first fundamental form in parametrisation σ can be written $(D\sigma)^\intercal (D\sigma)$. We can similarly write the second fundamental form as

$$-(Dn)^\intercal (D\sigma) = \begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} n_u \cdot \sigma_u & n_u \cdot \sigma_v \\ n_v \cdot \sigma_u & n_v \cdot \sigma_v \end{pmatrix}.$$

Hence, if $\sigma \colon V \to U \subset \Sigma$ and $\widetilde{\sigma} \colon \widetilde{V} \to U \subset \Sigma$ are allowable parametrisations for an open set $U \subset \Sigma$ with transition map $\varphi \colon \widetilde{V} \to V$ given by $\varphi = \sigma^{-1} \circ \widetilde{\sigma}$, then we know the normals are the same up to a sign by Lemma 3.3 so

$$n_{\widetilde{\sigma}}(\widetilde{u}, \widetilde{v}) = \pm n_\sigma(\varphi(\widetilde{u}, \widetilde{v}))$$

In particular, if $\det(D\varphi) < 0$, we arrive at a negative sign. Thus

$$\begin{aligned}
\begin{pmatrix} \widetilde{L} & \widetilde{M} \\ \widetilde{M} & \widetilde{N} \end{pmatrix} &= -(Dn_{\widetilde{\sigma}}) D\widetilde{\sigma} \\
&= \pm (D\varphi)^\intercal \begin{pmatrix} L & M \\ M & N \end{pmatrix} (D\varphi)
\end{aligned}$$

The change in sign depends on whether the transition map preserves or reverses orientation. If we assume that $V, \widetilde{V}$ are connected, the determinant $\det(D\varphi)$ does not change sign.

Example 4.8

Consider the cylinder with allowable parametrisation

$$\sigma(u, v) = (a \cos u, a \sin u, v)$$

where $u \in (0, 2\pi), v \in \mathbb{R}$. Note that $\sigma_{uv} = \sigma_{vv} = 0$, hence $M = N = 0$. We can show that the second fundamental form is given by

$$\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}; \quad -a\,\mathrm{d}u^2$$

§4.5 Gauss maps

Definition 4.6 (Gauss Map)

Let Σ be a smooth oriented surface in $\mathbb{R}^3$. The **Gauss map** $n\colon \Sigma \to S^2$[a] is the map $p \mapsto n(p)$, where the normal vector is defined by the orientation of Σ and is normalised so it lies in the unit sphere.

[a] $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$

Lemma 4.6

The Gauss map is smooth.

Proof. Since smoothness is a local property, it suffices to check the smoothness of the map on an arbitrary parametrised part of Σ. Let $\sigma\colon V \to U \subset \Sigma$ be allowable and compatible with a chosen orientation. Then at $\sigma(u, v) = p \in \Sigma$ and

$$n(\sigma(u, v)) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

$n \circ \sigma : V \to S^2 \subseteq R^3$ which is smooth since σ is. Since σ is allowable, the denominator is non-vanishing. $\square$

Remark 25. If $\Sigma = F^{-1}(0)$ for some function $F\colon \mathbb{R}^3 \to \mathbb{R}$ with nonzero derivative DF at all points $x \in \Sigma$ (which was required for Σ to be a smooth surface in $\mathbb{R}^3$), then we can explicitly calculate the Gauss map to be

$$n(p) = \frac{\boldsymbol{\nabla} F}{\|\boldsymbol{\nabla} F\|}$$

Note that,

$$T_p\Sigma = T_{n(p)}S^2 = (n(p))^{\perp}$$

since the two planes are orthogonal to the same vector. More concretely, if $v \in T_p\Sigma$ is $\gamma'(0)$ where $\gamma\colon (-\varepsilon, \varepsilon) \to \Sigma$, $\gamma(0) = p$ for a smooth curve γ, we can apply the Gauss map to γ and find

$$n \circ \gamma\colon (-\varepsilon, \varepsilon) \to S^2; \quad (n \circ \gamma)(0) = n(p)$$

Then, by the chain rule,

$$D\, n\Big|_p (v) = D\, n\Big|_p (\gamma'(0)) = (n \circ \gamma)'(0) \in T_{n(p)} S^2 = T_p\Sigma$$

Thus, the derivative of the Gauss map is $D\, n|_p : T_p\Sigma \to T_p\Sigma$. This can be viewed as an endomorphism of a fixed (with respect to parametrisation choice) two-dimensional subspace of $\mathbb{R}^3$.

To summarise, let Σ be an oriented smooth surface in $\mathbb{R}^3$. Then,

1. The first fundamental form is a symmetric bilinear form $\langle \cdot\,, \cdot \rangle = \mathrm{I}_p \colon T_p\Sigma \times T_p\Sigma \to \mathbb{R}$, which is the restriction of the Euclidean inner product to this space $T_p\Sigma$. We can write $\mathrm{I}_p(v, w)$, where $v, w \in T_p\Sigma$.
2. The second fundamental form is also a symmetric bilinear form $\mathrm{II}_p \colon T_p\Sigma \times T_p\Sigma \to \mathbb{R}$, given by

$$\mathrm{II}_p(v, w) = \mathrm{I}_p \left(-D\, n\Big|_p (v), w \right)$$

   where n is the Gauss map.

If we choose an allowable parametrisation (which for the second fundamental form must be correctly oriented) $\sigma \colon V \to U \subset \Sigma$ near $p \in \Sigma$, and if

$$D\, \sigma\Big|_0 (\hat{v}) = v; \quad D\, \sigma\Big|_0 (\hat{w}) = w; \quad \sigma(0) = p$$

Then,

$$\mathrm{I}_p(v, w) = \hat{v}^\mathsf{T} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \hat{w}; \quad \mathrm{II}_p(v, w) = \hat{v}^\mathsf{T} \begin{pmatrix} L & M \\ M & N \end{pmatrix} \hat{w}$$

where E, F, G, L, M, N depend on the choice of σ. Note that the functions I_p and II_p are independent of σ.

Lemma 4.7

The derivative of the Gauss map is self-adjoint. More precisely, viewing $Dn|_p : T_p\Sigma \to T_p\Sigma$ as an endomorphism over the inner product space with the first fundamental form, this linear map satisfies

$$\mathrm{I}_p \left(Dn\Big|_p (v), w \right) = \mathrm{I}_p \left(v, Dn\Big|_p (w) \right)$$

for all $v, w \in T_p\Sigma$.

Proof. From expressions for local parametrisations, we can show that I_p and II_p are symmetric. Hence,

$$\mathrm{I}_p\left(D\, n\Big|_p (v), w\right) = -\,\mathrm{II}_p(v, w) = -\,\mathrm{II}_p(w, v) = \mathrm{I}_p\left(D\, n\Big|_p (w), v\right) = \mathrm{I}_p\left(v, D\, n\Big|_p (w)\right)$$

$\square$

Proof. Take σ a parametrisation with $\sigma(0) = p$. Then $\{\sigma_u, \sigma_v\}$ is a basis of $T_p\Sigma$. To prove self-adjoint it suffices to check that $\left\langle Dn|_p\,(\sigma_u), \sigma_v \right\rangle = \left\langle \sigma_u, Dn|_p\,(\sigma_v)\right\rangle$, equivalently $\langle n_u, \sigma_v\rangle = \langle \sigma_u, n_v\rangle$ as $n(p) = n(\sigma(u, v))$ so $n_u = Dn|_p\,(\sigma_u)$ by chain rule.

We have shown this earlier but to check

$$\langle n, \sigma_u\rangle = \langle n, \sigma_v\rangle = 0$$

Differentiate the first expression wrt v

$$\langle n_v, \sigma_u\rangle + \langle n, \sigma_{uv}\rangle = 0$$

Differentiate the second expression wrt u

$$\langle n_u, \sigma_v\rangle + \langle n, \sigma_{vu}\rangle = 0$$

(Recall $M = -\langle n_v, \sigma_u\rangle = -\langle n_u, \sigma_v\rangle$). $\square$

Let's try to find the matrix of $Dn|_p$ in the basis of $\{\sigma_u, \sigma_v\}$.

$$\begin{aligned} n_u &= Dn\Big|_p (\sigma_u) = a_{11}\sigma_u + a_{21}\sigma_v \\ n_v &= Dn\Big|_p (\sigma_v) = a_{12}\sigma_u + a_{22}\sigma_v \end{aligned}$$

Taking products with σ_u, σ_v

$$-\underbrace{\begin{pmatrix} L & M \\ M & N \end{pmatrix}}_{Q} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{P} \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A}$$

$$Q = -PA$$

$$Q^T = Q = -A^{\intercal} P^T = -A^{\intercal} P.$$

Note. A is not necessarily symmetric, it is if $\{\sigma_u, \sigma_v\}$ is orthonormal as this is when $P = I$.

If $v = D\sigma|_0(\hat{v})$, $w = D\sigma|_0(\hat{w})$

$$
\begin{aligned}
-\hat{v}^\mathsf{T}\begin{pmatrix} L & M \\ M & N \end{pmatrix} &= -v^\mathsf{T}\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\hat{w} \qquad (1) \\
&= I_p(v, -\left. Dn\right|_p(w)) \\
&= I_p(-\left. Dn\right|_p(v), w).
\end{aligned}
$$

Then the second fundamental form has an intrinsic form given by the symmetric bilinear form $\mathrm{II}_p\colon T_p\Sigma \times T_p\Sigma \to \mathbb{R}$, given by

$$
\mathrm{II}_p(v, w) = \mathrm{I}_p\left(-D\left. n\right|_p(v), w\right)
$$

where n is the Gauss map.

Remark 26. The *fundamental theorem of surfaces in* $\mathbb{R}^3$ states that a smooth oriented connected surface in $\mathbb{R}^3$ is determined completely, up to rigid motion, by the two fundamental forms.

§4.6 Gauss curvature

Definition 4.7 (Gauss Curvature)
Let Σ be a smooth surface in $\mathbb{R}^3$. The **Gauss curvature** $\kappa\colon \Sigma \to \mathbb{R}$ of Σ is the function defined by

$$
\kappa(p) = \det\left(D\left. n\right|_p\right)
$$

Remark 27. This is always well-defined, even if Σ is not oriented. This is because Σ is always locally orientable, we can always choose a local expression for n. If we replace n by $-n$, the determinant will not change as for 2×2 matrices, $\det(-A) = \det(A)$.

We can compute κ directly. Let Σ be a smooth surface in $\mathbb{R}^3$, and σ an allowable parametrisation for an open neighbourhood of a point p. Using eq. (1) we see that taking determinants:

$$
\begin{aligned}
LN - M^2 &= (EG - F^2)\kappa \\
\kappa = \det(A) &= \frac{LN - M^2}{EG - F^2}
\end{aligned}
$$

Example 4.9 (Cylinder)

For a cylinder $\{x^2 + y^2 = 1\}$, we saw previously that $\sigma(u, v) = (a\cos u, a\sin u, v)$ and the second fundamental form was $\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}$ so $\kappa(p) = 0 \forall\, p$.

We could have seen this without doing any calculations. The normal to the cylinder is always horizontal, so the Gauss map $n \colon \Sigma \to S^2$ has image which lies in the equator. So if $\gamma : (-\varepsilon, \varepsilon) \to \Sigma$ is a vertical curve, then $Dn|_p\,(\gamma'(0)) = (n \circ \gamma)'(0) = 0 \implies \det Dn|_p = 0$.

Definition 4.8 (Flat)

A smooth surface $\Sigma \subset \mathbb{R}^3$ with vanishing Gauss curvature everywhere on Σ is **flat**.

Remark 28. If $\sigma \colon V \to U$ is allowable, and n_σ is defined to be $n \circ \sigma \colon V \to S^2$, then

$$Dn_\sigma\Big|_0 : \sigma_u \mapsto (n_\sigma)_u; \quad \sigma_v \mapsto (n_\sigma)_v$$

In particular, $\kappa(p) = \kappa(\sigma(0))$ vanishes if and only if $(n_\sigma)_u \times (n_\sigma)_v = 0$. Usually, we will write n to denote n_σ. In this case, the condition for flatness is that $n_u \times n_v = 0$.

Example 4.10

If Σ is the graph of a smooth function f, then it is easy to check that $E = 1 + f_u^2, G = 1 + f_v^2, F = f_u f_v$ and so $EG - F^2 = 1 + f_u^2 + f_v^2$ and

$$L = \frac{f_{uu}}{\sqrt{EG - F^2}}; \qquad M = \frac{f_{uv}}{\sqrt{EG - F^2}}; \qquad N = \frac{f_{vv}}{\sqrt{EG - F^2}}$$

$$\kappa = \frac{f_{uu} f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

Hence, the curvature depends on the derivative and the Hessian of f.

1. If $f(u, v) = \frac{u^2+v^2}{2}$, at $(0, 0)$ we find $\kappa(0, 0, 0) = 1$.
2. If $f(u, v) = \frac{u^2-v^2}{2}$, $\kappa(0, 0, 0) = -1$.

For instance, let $f(u, v) = \sqrt{r^2 - u^2 - v^2}$. Here, the graph is a piece of a sphere of radius r. We can find

$$f_{uu}\Big|_0 = f_{vv}\Big|_0 = \frac{-1}{r}; \quad f_{uv}\Big|_0 = 0 \implies \kappa(0, 0, r) = \frac{1}{r^2}$$

Since $O(3)$ acts transitively on S^2, and the fundamental forms are preserved by such global isometries, $\kappa = \frac{1}{r^2}$ everywhere on the sphere of radius r.

Example 4.11

Let Σ be the smooth surface given by $\{z = x^2 + y^2\}$. We claim that, for the inward facing choice of orientation, the image of the Gauss map is the open northern hemisphere. Note that Σ is invariant under rotations about the z axis. Also, we can show that if R is a rotation, $n \circ R = R \circ n$. Therefore, it suffices to consider an arbitrary point with $y = 0$.

Here, $\Sigma = F^{-1}(0)$ for the function $F(x, y, z) = z - x^2 - y^2$, which has nonvanishing derivative at the points $p \in \Sigma$. Hence, at $p = (x, 0, x^2)$, we have

$$n(p) = \frac{\boldsymbol{\nabla} F}{\|\boldsymbol{\nabla} F\|} = \frac{(-2x, 0, 1)}{\sqrt{1 + 4x^2}}$$

We can check explicitly that this map has image which an arc lying in the open northern hemisphere.

§4.7 Elliptic, hyperbolic, and parabolic points

Definition 4.9 (Conics)

Let Σ be a smooth surface in $\mathbb{R}^3$ and $p \in \Sigma$. We say that p is

1. **elliptic** if $\kappa(p) > 0$;
2. **hyperbolic** if $\kappa(p) < 0$;
3. **parabolic** if $\kappa(p) = 0$.

Lemma 4.8

In a sufficiently small neighbourhood of an elliptic point p, Σ lies entirely on one side of $p + T_p\Sigma$. If p is hyperbolic, Σ lies on both sides of $p + T_p\Sigma$.

Proof. Let σ be a local parametrisation near p. Here,

$$\kappa = \frac{LN - M^2}{EG - F^2}$$

The denominator is always positive, since it is the determinant of a positive definite symmetric bilinear form I_p. Hence, the sign of κ depends on the sign of $LN - M^2$.

Recall that if $w = h\sigma_u + \ell\sigma_v \in T_p\Sigma$, then $\frac{1}{2}\,\mathrm{II}_p(w,w)$ measures the signed distance from $\sigma(h,l)$ to $p + T_p\Sigma$ ($\sigma(0) = p$) measured via the inner product with positive normal

$$\frac{1}{2}\left(Lh^2 + 2Mhl + Nl^2\right) + o(h^3, l^3).$$

If p is elliptic, then $\begin{pmatrix} L & M \\ M & n \end{pmatrix}$ has eigenvalues of the same sign, so it is either positive or negative definite[a] at p. So in a neighbourhood of p, this signed distance only has one sign locally.

Conversely, if p is hyperbolic, then $\mathrm{II}_p(w,w)$ is indefinite so takes both signs in a neighbourhood of p. $\square$

[a]As the matrix is symmetric

Remark 29. We cannot conclude anything about parabolic points *a priori*. For instance, the cylinder is flat (all points are parabolic), and the surface lies on one side of the tangent plane at every point. Consider also the *monkey saddle* defined by

$$\sigma(u,v) = (u, v, u^3 - 3v^2u)$$

which has a parabolic point at the origin, but Σ lies on both sides of the tangent plane in every open neighbourhood of the origin. At $p = \sigma(0,0)$, the Gauss curvature vanishes, but the surface lies locally on both sies of the tangent plane.

Proposition 4.1

Let Σ be a compact smooth surface in $\mathbb{R}^3$. Then Σ has an elliptic point.

The idea of the proof is as follows: Take a plane and move it towards the surface until it touches it, then the surface lies on one side of this plane so we have an elliptic/parabolic point. If we instead use a bowl and not a plane, then the surface must curve away from the bowl and so must have an elliptic point.

Proof. Since Σ is compact, it is closed and bounded as a subset of $\mathbb{R}^3$. Hence, for R' sufficiently large, Σ lies entirely within $\overline{B(0,R')}$. Let R be the minimal such R'[a]. Up to a global isometry of $\mathbb{R}^3$, there exists a point $p = (0,0,R) \in \Sigma$ on the sphere $S^2(R)$ of radius R. Here, $T_p\Sigma = T_pS^2$. Locally near p, we can view Σ as the graph of a smooth function $f\colon V \to \mathbb{R}$ on the x, y coordinates with the property that $f - \sqrt{R^2 - u^2 - v^2} \leq 0$. This expresses the fact that Σ lies underneath the sphere of radius R.

We can now consider the Taylor series of f. Note that $(0,0)$ is a maximum point of f, hence $f_u = f_v = 0$ at 0. Let $F(u,v) = f(u,v) - \sqrt{R^2 - u^2 - v^2}$, $F(u,v) \leq 0$. We can see that $F_u = F_v = 0$ and $F_{uu} = f_{uu} + \frac{1}{R}$, $F_{uv} = f_{uv}$, $F_{vv} = f_{vv} + \frac{1}{R}$ at

$(0,0)$. As F has a local maximum at 0 we know that its Hessian must be negative semi-definite for sufficiently small u, v:

$$\left(f_{uu} + \frac{1}{R}\right)u^2 + 2f_{uv}uv + \left(f_{vv} + \frac{1}{R}\right)v^2 \leq 0$$
$$\frac{1}{2}\left(f_{uu}u^2 + 2f_{uv}uv + f_{vv}v^2\right) \leq -\frac{1}{2R}(u^2+v^2)$$
$$\begin{pmatrix} L & M \\ M & N \end{pmatrix} = \begin{pmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{pmatrix}$$

Hence, the second fundamental form is locally negative definite near $(0,0)$. $E = G = 1$ and $F = 0$, which follows from our previous calculation of the first fundamental form for graphs. Hence, $\kappa(p) > 0$, so p is elliptic as required. In particular, the curvature at this point is greater than that of the sphere. □

[a] The distance of points from the origin is a cts fcn on a compact space so extremum exists and is achieved

§4.7.1 Gauss curvature and area

Theorem 4.1

Let Σ be a smooth surface in $\mathbb{R}^3$, and let $p \in \Sigma$ such that $\kappa(p) \neq 0$. Let U be an open neighbourhood of p, and consider a decreasing sequence $p \in A_i \subset U$ of open neighbourhoods that 'shrink to p', in the sense that for all $\varepsilon > 0$, $A_i \subset B(p, \varepsilon)$ for sufficiently large i. Then,

$$|\kappa(p)| = \lim_{i\to\infty} \frac{\text{area}_{S^2}(n(A_i))}{\text{area}_\Sigma(A_i)}$$

In other words, the Gauss curvature is an infinitesimal measure of how much the Gauss map n distorts area.

Remark 30. Around hyperbolic points, the signed area of $n(A_i)$ is reversed, since curves γ reverse direction under n. We can alternatively define the *signed area* of $n(A_i)$ to be the area of $n(A_i)$ if $\kappa > 0$ and the negation of this area if $\kappa < 0$. The above theorem holds when $\kappa = 0$, but this will not be proven.

Proof. This is all 'local' so let $\sigma : V \to U \subset \Sigma$ be an allowable parametrisation near $p \in \Sigma$. Using σ, we can define the open sets $\sigma^{-1}(A_i) = V_i \subset V$. Since the A_i shrink to p, we have that $\bigcap_i V_i = \{(0,0)\}$. We have

$$\text{area}_\Sigma(A_i) = \int_{V_1} \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \int_{V_i} \|\sigma_u \times \sigma_v\|\,\mathrm{d}u\,\mathrm{d}v$$

Consider $n \circ \sigma : V \to S^2 \subset \mathbb{R}^3$, by chain rule:

$$D(n \circ \sigma)\Big|_0 = Dn\Big|_p \circ D\sigma\Big|_0 .$$

σ allowable so $D\sigma|$ rank 2 and as $\kappa(p) \neq 0$, det $Dn|_p \neq 0$ so it too is rank 2, thus $D(n \circ \sigma)|_0$ rank 2. Thus $n \circ \sigma$ defines an allowable parametrisation for an open neighbourhood of $n((0,0))$ by the inverse function theorem. Therefore,

$$\text{area}_{S^2}(n(A_i)) = \int_{V_i} \|n_u \times n_v\| \, \mathrm{d}u \, \mathrm{d}v$$

for sufficiently large i such that $\sigma^{-1}A_i = V_i$ lies in the open neighbourhood of $(0,0)$ where $n \circ \sigma$ is a diffeomorphism.

$$\|n_u \times n_v\| = \|Dn(\sigma_u) \times Dn(\sigma_v)\|$$

Recall from last lecture

$$\begin{aligned}
Dn(\sigma_u) &= a_{11}\sigma_u + a_{21}\sigma_v \\
Dn(\sigma_v) &= a_{12}\sigma_u + a_{22}\sigma_v \\
\implies Du(\sigma_u) \times Du(\sigma_v) &= (a_{11}\sigma_u + a_{21}\sigma_v) \times a_{12}\sigma_u + a_{22}\sigma_v \\
&= \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{\det(Dn) = \kappa(p)} \sigma_u \times \sigma_v \\
\therefore \int_{V_i} \|n_u \times n_v\| \, \mathrm{d}u \, \mathrm{d}v &= \int_{V_i} \|Dn(\sigma_u) \times Dn(\sigma_v)\| \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_{V_i} |\det(Dn)| \cdot \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \\
&= \int_{V_i} |\kappa(u,v)| \cdot \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v
\end{aligned}$$

As κ is continuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that $|\kappa(u,v) - \kappa(0,0)| < \varepsilon$ for all $(u,v) \in B((0,0),\delta)$. In particular, for sufficiently large i, we have $\forall\ (u,v) \in V_i$

$$|\kappa(u,v)| \in (|\kappa(p)| - \varepsilon, |\kappa(p)| + \varepsilon)$$

Hence,

$$\begin{aligned}
(|\kappa(p)| - \varepsilon) \int_{V_i} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v &\leq \int_{V_i} |\kappa(u,v)| \cdot \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \\
&\leq (|\kappa(p)| + \varepsilon) \int_{V_i} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v
\end{aligned}$$

In other words,

$$|\kappa(p)| - \varepsilon \leq \frac{\text{area}_{S^2}(n(A_i))}{\text{area}_\Sigma(A_i)} \leq |\kappa(p)| + \varepsilon$$

Letting $i \to \infty$ gives the result as required. □

Theorem 4.2 (*theorema Egregium*)

The Gauss curvature of a smooth surface in $\mathbb{R}^3$ is isometry invariant. In other words, if $f\colon \Sigma_1 \to \Sigma_2$ is a diffeomorphism of surfaces in $\mathbb{R}^3$ which is an isometry, then $\kappa(p) = \kappa(f(p))$ for all p.

Remark 31. Isometries rely on only the first fundamental form, but Gauss curvature is defined using both fundamental forms. We can do a direct proof by simply differentiating the formula and rearranging until the result follows. This proof is given in Part II.

Alternatively, we can consider a different question: are some allowable parametrisations of a smooth surface in $\mathbb{R}^3$ 'better' than others in some way? If we have a parametrisation $\sigma\colon V \to U \subset \Sigma$, this defines certain distinguished curves, which are the images of $\sigma(t,0)$ and $\sigma(0,t)$. In this sense, looking for a 'best' parametrisation is equivalent to looking for 'best' distinguished curves near a point. This leads to the study of geodesics. We will later show that every smooth surface in $\mathbb{R}^3$ admits local parametrisations such that the first fundamental form has form $\mathrm{d}u^2 + G\,\mathrm{d}v^2$, so $E = 1$ and $F = 0$. We will also see (on an example sheet) that if such a local parametrisation exists, then κ can be expressed as a function just of G. This allows us to approach the proof of the *theorema egregium* from a more conceptual way, since we have expressed κ in terms of the first fundamental form alone.

Theorem 4.3 (Gauss-Bonnet theorem)

If Σ is a compact smooth surface in $\mathbb{R}^3$, then

$$\int_\Sigma \kappa \,\mathrm{d}A_\Sigma{}^{a} = 2\pi\chi(\Sigma)^{b}$$

[a] Locally $\sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v$

[b] Euler Characteristic

The LHS is geometric and the RHS is topological!

§5 Geodesics

§5.1 Definitions

Recall that we defined, for a smooth curve $\gamma\colon [a,b] \to \mathbb{R}^3$,

$$\text{length}(\gamma) = L(\gamma) = \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

Definition 5.1 (Energy)
The **energy** of γ is given by

$$E(\gamma) = \int_a^b \|\gamma'(t)\|^2 \,\mathrm{d}t$$

Consider $\Omega_{pq} = \{\text{all smooth curves } \gamma : [a,b] \to \mathbb{R}^3 : \gamma(a) = p, \gamma(b) = q\}$ then $E : \Omega_{pq} \to \mathbb{R}$. In fact what we really want is given $\Sigma \subset \mathbb{R}^3$, $\gamma : [a,b] \to \Sigma$. Then we want to find the critical points of E exactly like in variational principles.

Definition 5.2 (One-Parameter Variation)
Let $\gamma\colon [a,b] \to \Sigma$, where Σ is a smooth surface in $\mathbb{R}^3$. A **one-parameter variation** (with fixed endpoints) of γ is a smooth map $\Gamma\colon (-\varepsilon,\varepsilon) \times [a,b] \to \Sigma$, such that if $\gamma_s = \Gamma(s,\,\cdot\,)$[a], then $\gamma_0(t) = \gamma(t)$, and $\gamma_s(a)$ and $\gamma_s(b)$ are independent of s.

[a] I.e. $\gamma_s(t) = \Gamma(s,t)$

Definition 5.3 (Geodesic)
A smooth curve $\gamma\colon [a,b] \to \Sigma$ is a **geodesic** if, for every variation (γ_s) of γ with fixed endpoints as above, we have $\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = 0$. In other words, γ is a critical point of the energy functional on curves from $\gamma(a)$ to $\gamma(b)$.

A geodesic is the path that a free particle would follow if the only force acting on it was the one that kept it on the surface. E.g take a sphere to a place with no gravity, put a marble on it and give it a kick. The path it follows will be a geodesic.

§5.2 The geodesic equations

Let γ have image contained within the image of an allowable parametrisation $\sigma\colon V \to U$. Then, for sufficiently small s, we can write $\gamma_s(t) = \sigma(u(s,t), v(s,t))$. Suppose that the first fundamental form, with respect to σ, is

$$E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$$

Let

$$\begin{aligned} R &= E(u(s,t),v(s,t))\dot{u}^2 + 2F(u(s,t),v(s,t))\dot{u}\dot{v} + G(u(s,t),v(s,t))\dot{v}^2 \\ &= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \end{aligned}$$

where $\dot{u} = \frac{\partial u}{\partial t}$, $\dot{v} = \frac{\partial v}{\partial t}$. By definition,

$$E(\gamma_s) = \int_a^b R\,\mathrm{d}t$$

where R depends on s. Hence,

$$\begin{aligned} \frac{\partial R}{\partial s} &= \Big(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\Big)\frac{\partial u}{\partial s} + \Big(E_v\dot{v}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\Big)\frac{\partial v}{\partial s} \\ &\quad + 2(E\dot{u} + F\dot{v})\frac{\partial \dot{u}}{\partial s} + 2(F\dot{u} + G\dot{v})\frac{\partial \dot{v}}{\partial s} \end{aligned}$$

This gives

$$\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = \int_a^b \frac{\partial R}{\partial s}\,\mathrm{d}t\,.$$

Note $\frac{\partial \dot{u}}{\partial s} = \frac{\partial^2 u}{\partial s \partial t}$, $\frac{\partial \dot{v}}{\partial s} = \frac{\partial^2 v}{\partial s \partial t}$ and so we can integrate by parts. Note that $\frac{\partial u}{\partial s}$ and $\frac{\partial v}{\partial s}$ vanish at a, b as the endpoints are fixed. Hence,

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = \int_a^b \left(A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s}\right)\mathrm{d}t$$

where

$$\begin{aligned} A &= E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2 - 2\frac{\partial}{\partial t}(E\dot{u} + F\dot{v}) \\ B &= E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2 - 2\frac{\partial}{\partial t}(F\dot{u} + G\dot{v}) \end{aligned}$$

Note that we have absolute freedom for choosing the "variational vector field"

$$w(t) = \left(\frac{\partial u}{\partial s}(0,t), \frac{\partial v}{\partial s}(0,t)\right),$$

which are the $\frac{\partial u}{\partial s}, \frac{\partial v}{\partial s}$ in $\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = \int_a^b \frac{\partial R}{\partial s}\,\mathrm{d}t$.

Corollary 5.1

A smooth curve $\gamma\colon [a,b] \to \Sigma$ with image in $\operatorname{Im}\sigma$ is a geodesic iff $A = B = 0$, i.e. it satisfies the *geodesic equations*:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}\Big(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\Big)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}\Big(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\Big)$$

Note that these equations are evaluated at $s = 0$, so no choice of variation is required.

Remark 32.

1. If $w(t)$ with $w(a) = w(b) = 0$ then

$$\gamma_s(t) = \sigma((u(t), v(t)) + sw(t))$$

for s small enough is a variation of γ with fixed endpoints and variational vector field w.

2. Recall Q10, Sheet 4 of IA Analysis:

$$\int_a^b \underbrace{f(x)}_{\text{cont}} g(x)\, dx = 0 \quad \forall\, g : [a,b] \to \mathbb{R} \text{ s.t. } g(a) = g(b).$$
$$\implies f \equiv 0.$$

This justifies why $A = B = 0$

3. The best way to think about the geodesic equations is via the Euler-Lagrange equations of the Lagrangian, $L(u, v, \dot{u}, \dot{v}) = \frac{1}{2}(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)$ (purely kinetic energy). Recall from Variational Principles that the E-L eqns are $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}$ where $q_1 = u, q_2 = v$. These are the geodesic equations.

Remark 33. Solving a differential equation is a local procedure. The original definition of the geodesic seems to be a global property. However, we can always consider a sub-curve of γ to also be a geodesic, since its variations are variations of γ. So the definition can be thought of as local.

§5.3 Equivalent characterisation of geodesics

We have so far restricted our analysis to the first fundamental form, without considering its embedding in $\mathbb{R}^3$. Intuitively, we know that straight lines in $\mathbb{R}^2$ are not just locally shortest but also locally straightest. We would expect this to hold for other surfaces as well. We can charaterise this notion via stating that the change in the tangent vector to a curve is as small as it could be, subject to the constraint that it lies on the surface.

Proposition 5.1

Let Σ be a smooth surface in $\mathbb{R}^3$. A smooth curve $\gamma\colon [a,b] \to \Sigma$ is a geodesic iff $\ddot{\gamma}(t)$ is everywhere normal to the surface Σ.

Remark 34. This proposition makes use of the tangent plane, a notion that exists only because we have an embedding in $\mathbb{R}^3$.

Proof. The property of being a geodesic as we previously defined is a local property, and so is the condition in the proposition. Hence, we may work entirely within an allowable parametrisation $\sigma\colon V \to U$. Suppose $\gamma(t) = \sigma(u(t), v(t))$. Hence,

$$\dot{\gamma} = \sigma_u \dot{u} + \sigma_v \dot{v}$$

$\ddot{\gamma}$ is normal to Σ when it is orthogonal to the tangent plane, which is spanned by σ_u, σ_v. This is true iff

$$\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u \dot{u} + \sigma_v \dot{v}), \sigma_u \right\rangle = 0 = \left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\sigma_u \dot{u} + \sigma_v \dot{v}), \sigma_v \right\rangle$$

We will prove the first equality. This can be rewritten

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_u \rangle - \left\langle \sigma_u \dot{u} + \sigma_v \dot{v}, \frac{\mathrm{d}}{\mathrm{d}t}\sigma_u \right\rangle = 0$$

Note that $\langle \sigma_u, \sigma_u \rangle = E$ and $\langle \sigma_u, \sigma_v \rangle = F$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_{uu}\dot{u} + \sigma_{uv}\dot{v} \rangle = 0$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) - \left[\dot{u}^2 \langle \sigma_u, \sigma_{uu} \rangle + \dot{u}\dot{v}(\langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{uu} \rangle) + \dot{v}^2 \langle \sigma_v \sigma_{uv} \rangle\right] = 0$$

Note that $E_u = 2\langle \sigma_u, \sigma_{uu} \rangle$, $F_u = \langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{uu} \rangle$, and $G_u = 2\langle \sigma_v, \sigma_{uv} \rangle$. This gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}\left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2\right)$$

which is the first of the geodesic equations. By symmetry, we find the second geodesic equation similarly. $\square$

Corollary 5.2

If $\gamma : [a,b] \to \Sigma$ is a geodesic, then $|\dot{\gamma}(t)|$ is a constant, so geodesics are parametrised proportional to arc length (i.e. has constant speed).

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\dot\gamma,\dot\gamma\rangle = 2\left\langle \underbrace{\dot\gamma}_{\text{tangent to }\Sigma}, \underbrace{\ddot\gamma}_{\text{normal to }\Sigma}\right\rangle = 0$$

□

§5.3.1 Length vs Energy

Energy is sensitive to reparametrisation. If $f, g\colon [a,b] \to \mathbb{R}$ are smooth, the Cauchy-Schwarz inequality gives that

$$\left(\int_a^b fg\,\mathrm{d}t\right)^2 \leq \int_a^b f^2\,\mathrm{d}t \cdot \int_a^b g^2\,\mathrm{d}t$$

Let us apply this to $f = |\dot\gamma|$, $g = 1$ to find

$$(L(\gamma))^2 = \left(\int_a^b |\dot\gamma(t)|\cdot 1\,\mathrm{d}t\right)^2 \leq \left(\int_a^b |\dot\gamma(t)|^2\,\mathrm{d}t\right)\left(\int_a^b 1\,\mathrm{d}t\right) = E(\gamma)(b-a).$$

Since equality holds only when the two functions are proportional, we must have that $\|\gamma'(t)\|$ is constant for the equality to hold. In other words, γ must be parametrised proportional to arc length.

Corollary 5.3

A smooth curve $\gamma : [a,b] \to \Sigma \subset \mathbb{R}^3$ that has constant speed and locally minimises length is a geodesic.

Proof. Need to prove γ is a critical point of E.

Let $\tau : [a,b] \to \Sigma$ be any other curve connecting $\gamma(a)$ to $\gamma(b)$.

$$E(\gamma) = \frac{(L(\gamma))^2}{b-a} \leq \frac{(L(\tau))^2}{b-a} \leq E(\tau)$$

Thus γ is a critical point of E and hence a geodesic. □

Remark 35. We would like geodesics to be a local property, but not necessarily global length minimisers. For example, all arcs of great circles will be shown to be geodesics, even if large arcs are not global length minimisers between fixed endpoints.

While geodesics might not be global minimisers they are always local minimisers (see Wilson's book for proof).

Example 5.1 (Geodesic on planes)

The plane $\mathbb{R}^2$ has parametrisation $\sigma(u, v) = (u, v, 0)$ and first fundamental form $\mathrm{d}u^2 + \mathrm{d}v^2$. The geodesic equations here are

$$\ddot{u} = 0; \quad \ddot{v} = 0$$

In particular, the geodesics on the plane are given by

$$u(t) = \alpha t + \beta; \quad v(t) = \gamma t + \delta$$

This is a straight line, parametrised at constant speed.

Example 5.2 (Geodesics on unit sphere)

Consider the unit sphere with parametrisation

$$\begin{aligned} \sigma(u, v) &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \\ \sigma_\varphi &= (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0) \\ \sigma_\theta &= (\cos\theta\cos\varphi, -\cos\theta\sin\varphi, -\sin\theta) \end{aligned}$$

This has first fundamental form

$$E = \sin^2\theta, F = 0, G = 1$$

We have Lagrangian

$$L(\theta, \varphi, \dot{\theta}, \dot{\varphi}) = \frac{1}{2}\left(\dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2\right)$$

Euler-Lagrange

$$\begin{aligned} \frac{\partial L}{\partial \dot{\theta}} &= \dot{\theta}, \frac{\partial L}{\partial \dot{\varphi}} = \sin^2\dot{\varphi} \\ \frac{\partial L}{\partial \varphi} &= 0, \frac{\partial L}{\partial \theta} = \dot{\varphi}^2 \sin\theta\cos\theta \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\partial L}{\partial x} \\ \implies \frac{d}{dt}(\dot{\varphi}\sin^2\theta) &= 0, \ddot{\theta} = \dot{\varphi}^2\sin\theta\cos\theta \end{aligned} \tag{$\dagger$}$$

This gives right away that the equator $t \mapsto (t, \frac{\pi}{2})$ is a geodesic with speed 1. In fact all great circles parametrised with constant speed are geodesics. We can prove this by integrating ($\dagger$), but we can see this by geometrically noticing that such curves have $\ddot{\gamma}$ normal to $T_{\gamma(t)}S^2$.

Since geodesics solve a 2nd order ODE prescribing a point $p \in \Sigma$ and a direction $v \in T_p\Sigma$ determines the geodesic completely. Thus great circles are all possible geodesics, as there exists a great circle for all p, v.

Note that γ between p, q as in the picture does not minimise length.

§5.4 Surfaces of revolution

This is an important example.

Consider the surface of revolution given by $\eta(u) = (f(u), 0, g(u))$ in the xz-plane rotated about the z axis, where η is smooth and injective, and $f(u) > 0$.

Definition 5.4

A circle obtained by rotating a point of η is called a *parallel*. A curve optained by rotating η itself by a fixed angle about the z axis is called a *meridian*.

Lemma 5.1

A parallel given by $u = u_0$ is a geodesic when parametrised at constant speed iff $f'(u_0) = 0$.

Proof. Consider the allowable parametrisation

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

where $u \in (a, b)$ and $v \in (0, 2\pi)$. The first fundamental form is

$$\left[(f')^2 + (g')^2\right] \mathrm{d}u^2 + f^2\,\mathrm{d}v^2$$

If wlog we choose to parametrise η by arc length, this becomes

$$\mathrm{d}u^2 + f^2\,\mathrm{d}v^2$$

The Lagrangian for the geodesic is

$$\begin{aligned} L &= \frac{1}{2}\left(\dot{u}^2 + f^2\dot{v}^2\right) \\ \frac{\partial L}{\partial u} &= ff'\dot{v}^2, \frac{\partial L}{\partial \dot{u}} = \dot{u}, \frac{\partial L}{\partial v} = 0, \frac{\partial L}{\partial \dot{v}} = f^2\dot{v} \\ \text{E-L eqns} \implies \ddot{u} &= ff'\dot{v}^2, \frac{d}{dt}\left(f^2\dot{v}\right) = 0. \end{aligned} \tag{†}$$

We also know that geodesics travel with constant speed so $\dot{u}^2 + f^2\dot{v}^2$ is a non-zero constant. This is an example of a "completely integrable" problem, it has the same number of degrees of freedom as conserved quantities, 2.

Meridians: Let $v = v_0$, if $u(t) = t + u_0$, the map $t \mapsto (t + u_0, v_0)$ is a geodesic with speed 1 through (u_0, v_0) as it satisfies (†). As isometries map geodesics to geodesics (Sheet 3) all meridians are geodesics.

Parallels: Let $u = u_0$ then as $\dot{u}^2 + f^2\dot{v}^2 = a$ for some $a \neq 0$, $f^2\dot{v}^2 = a - u_0^2$. From (†) we need $ff'\dot{v}^2 = 0$ so $f'(u_0) = 0$. □

Let's look at the conserved quantity $f^2\dot{v}$ in more detail.

Proposition 5.2 (Clairaut's relation)

Consider a curve $\gamma(t)$ on Σ, making angle θ with the parallel of radius $\rho = f$. If γ is a geodesic, then $\rho\cos\theta$ is constant along γ.

Proof. Let $\gamma(t) = \sigma(u(t), v(t))$, so $\dot{\gamma} = \sigma_u\dot{u} + \sigma_v\dot{v}$. The tangent vector to the parallel is $\sigma_v = (-f\sin v, f\cos v, 0)$. By the earlier discussion on angles in terms of the first fundamental form,

$$\cos\theta = \frac{\langle \sigma_v, \sigma_u\dot{u} + \sigma_v\dot{v}\rangle}{\|\sigma_v\| \cdot \|\sigma_u\dot{u} + \sigma_v\dot{v}\|}$$

Assume γ is parametrised by arc length, so $\|\dot{\gamma}\| = 1$, so $\|\sigma_u\dot{u} + \sigma_v\dot{v}\| = 1$. Using that $F = 0, G = f^2$ we get

$$\cos\theta = \frac{f^2\dot{v}}{f} = f\dot{v}.$$

So if γ a geodesic then $\rho\cos\theta = f^2\dot{v}$ is a constant. □

This is just another way to write the conservation law arising from $\frac{\partial L}{\partial v} = 0$.

Example 5.3 (Ellisoid of revolution)

Usually, for a surface of revolution, we take the assumption that η never intersects the z-axis, or that f is positive. This ensures that all points on the surface are locally smooth. However, we can allow η to meet the z-axis orthogonally, as in the ellipsoid or sphere.

Consider an ellipsoid of revolution. $\rho\cos\theta$ is constant along a geodesic γ. Suppose that at some point γ intersects a parallel of radius ρ_0 at angle θ_0, and that γ is not a meridian (so $\cos\theta \neq 0$). Hence $\theta_0 \in [0, \frac{\pi}{2})$. In particular, $0 < c = \rho\cos\theta \leq \rho$ for $c = \rho_0\cos\theta_0$ so ρ is bounded below by c. A geodesic which is not a meridian is therefore 'trapped' between parallels with radius c. In particular, any geodesic through a pole is a meridian.

§5.5 Local existence of geodesics

It is difficult to solve the geodesic equations globally. We can often intead prove local results about any geodesics that may arise.

Recall Picard's theorem from Analysis and Topology. Let $I = [t_0 - a, t_0 + a] \subset \mathbb{R}$, $B = \{x\colon \|x - x_0\| \leq b\} \subset \mathbb{R}^n$, and $f\colon I \times B \to \mathbb{R}^n$ that is continuous, and Lipschitz in the second variable.

$$\|f(t, x_1) - f(t, x_2)\| \leq N\|x_1 - x_2\|$$

Then the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x)$ with $x(t_0) = x_0$ has a unique solution for some time interval $|t - t_0| < h$, where $h = \min\left\{a, \frac{b}{s}\right\}$ where $s = \sup\|f\|$. Further, if f is smooth in all parameters, then the solution to the differential equation is smooth and depends smoothly on the initial condition.

Recall the geodesic equations:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) &= \frac{1}{2}\left(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right) \\
\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) &= \frac{1}{2}\left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right)
\end{aligned}$$

We can write this as

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} = R(u, v, \dot{u}, \dot{v})$$

where R is composed of smooth functions of $u, v, \dot{u}, \dot{v}$. The matrix on the left hand side is invertible, and the inverse map $A \mapsto A^{-1}$ on matrices is smooth. Hence, we can write the geodesic equations in the form

$$\ddot{u} = A(u, v, \dot{u}, \dot{v}); \quad \ddot{v} = B(u, v, \dot{u}, \dot{v})$$

In the usual way we can turn second-order equations into first-order equations by introducing $p = \dot{u}, q = \dot{v}$, and we find

$$\dot{u} = p; \quad \dot{v} = q; \quad \dot{p} = A(u, v, p, q); \quad \dot{q} = B(u, v, p, q)$$

This is a system of first-order ordinary differential equations as governed by Picard's theorem. Since A, B are smooth, a local bound on $\|DA\|$ and $\|DB\|$ will give the required Lipschitz condition.

Corollary 5.4

Let Σ be a smooth surface in $\mathbb{R}^3$. For $p \in \Sigma$ and $v \in T_p\Sigma$, then there exists $\varepsilon > 0$ and a unique geodesic $\gamma\colon (-\varepsilon, \varepsilon) \to \Sigma$ such that

$$\gamma(0) = p; \quad \dot{\gamma}(0) = v$$

Moreover, γ depends smoothly on p, v.

The local existence of geodesics gives rise to allowable parametrisations of Σ with 'nice' properties in terms of the first fundamental form. Let $p \in \Sigma$, and consider a geodesic arc γ starting at p and parametrised by arc length. At each point $\gamma(t)$ for small $t > 0$, we can consider a geodesic arc γ_t starting at $\gamma(t)$, and $\gamma_t'(0)$ is orthogonal to $\gamma'(t)$, and also parametrised by arc length. Now, we define $\sigma(u,v) = \gamma_v(u)$, which is defined for $u \in [0, \varepsilon)$ and $v \in [0, \delta)$.

Lemma 5.2

For ε, δ sufficiently small, $\sigma\colon (u,v) \mapsto \gamma_v(u)$ defines an allowable parametrisation of an open set in Σ, taking the interior of the domain.

Proof. Smoothness follows from the addendum to Picard's theorem above. At the origin $(0,0)$, by construction we have σ_u, σ_v orthogonal. Hence, they stay linearly independent for sufficiently small ε, δ. So $D\sigma$ has full rank, and (on a smaller set if necessary) σ is injective. So σ is allowable. $\square$

Corollary 5.5

Any smooth surface Σ in $\mathbb{R}^3$ admits local parametrisations for which the first fundamental form has form $\mathrm{d}u^2 + G(u,v)\,\mathrm{d}v^2$, so $E = 1$ and $F = 0$.

Proof. Consider the parametrisation $\sigma(u,v) = \gamma_v(u)$. For v_0 fixed, the curve $u \mapsto \gamma_{v_0}(u)$ is a geodesic parametrised at unit speed, so $E = 1$. One of the geodesic equations is

$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}\left(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2\right)$$

and consider $v(t) = v_0$, $u(t) = t$. $E_v = \dot{v} = 0$ and $\dot{u} = 1$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}F = 0 \implies F_u\dot{u} = 0 \implies F_u = 0$$

So F is independent of u. At $u = 0$, then by construction of γ_v as being orthogonal to γ at $\gamma(v)$, we see $F = 0$. $\square$

These coordinates are called *geodesic normal coordinates*. Note that by fixing u and letting v vary, the curve obtained is typically not a geodesic, except for $u = 0$ which is γ itself. In these coordinates, we can also find

$$G(0, v) = 1; \quad G_u(0, v) = 0$$

The first result holds since σ_v has unit length at $u = 0$. The second result holds because $u = 0$ yields a geodesic with arc length parametrisation, and then we can use one of the geodesic equations to find

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}\left(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2\right) \implies 0 = \frac{1}{2}G_u(0, v)$$

§5.6 Surfaces of constant curvature

In the example sheets, we show that for a smooth surface Σ in $\mathbb{R}^3$ with allowable parametrisation for which $E = 1$ and $F = 0$, we have the following result for the Gauss curvature.

$$\kappa = \frac{-\left(\sqrt{G}\right)_{uu}}{\sqrt{G}}$$

If $a\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a dilation $a(x, y, z) = (ax, ay, az)$, then

$$\kappa_{a(\Sigma)} = \frac{1}{a^2}\kappa_\Sigma$$

since E, F, G rescale by a^2, and L, N, M rescale by a. This matches the results previously found for spheres of varying radii. By dilating, to understand surfaces of constant curvature it suffices to consider surfaces with constant curvature ± 1 or 0.

Proposition 5.3
Let Σ be a smooth surface in $\mathbb{R}^3$. Then,

1. if $\kappa \equiv 0$, then Σ is locally isometric to $(\mathbb{R}^2, \mathrm{d}u^2 + \mathrm{d}v^2)$;
2. if $\kappa \equiv 1$, then Σ is locally isometric to $(S^2, \mathrm{d}u^2 + \cos^2 u\, \mathrm{d}v^2)$.

Proof. Σ admits an allowable parametrisation with $E = 1$ and $F = 0$ by using geodesic normal coordinates, so

$$\kappa = \frac{-\sqrt{G}_{uu}}{\sqrt{G}}; \quad G(0, v) = 1; \quad G_u(0, v) = 0$$

If $\kappa \equiv 0$, we have $\sqrt{G}_{uu} = 0$, so $\sqrt{G} = A(v)u + B(v)$, and the boundary conditions give $A \equiv 0, B \equiv 1$. In particular, $G \equiv 1$. The fundamental form then is $\mathrm{d}u^2 + \mathrm{d}v^2$, which is that of $\mathbb{R}^2$.

If $\kappa \equiv 1$, we find $\left(\sqrt{G}\right)_{uu} + \sqrt{G} = 0$ so $\sqrt{G} = A(v)\sin u + B(v)\cos u$. The boundary conditions then imply that $A \equiv 0, B \equiv 1$ and hence the fundamental form is $\mathrm{d}u^2 + \cos^2 u\,\mathrm{d}v^2$. This matches the first fundamental form of a sphere with parametrisation

$$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

$\square$

Remark 36. If $\kappa \equiv -1$, we will find the first fundamental form $\mathrm{d}u^2 + \cosh^2 u\,\mathrm{d}v^2$. There exists an object known as the tractoid, which is a smooth surface in $\mathbb{R}^3$, and has this first fundamental form. We could alternatively choose not to embed this surface in $\mathbb{R}^3$.

In fact, the change of variables $v = e^v \tanh u, w = e^v \operatorname{sech} u$ turns the fundamental form $\mathrm{d}u^2 + \cosh^2 u\,\mathrm{d}v^2$ into $\frac{\mathrm{d}V^2 + \mathrm{d}W^2}{W^2}$, which is a 'standard' presentation of the first fundamental form, which we will see more of later.