Stochastic Financial Models 20

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1 Continuous-time finance

From discrete to continuous. Motivation

- Let $S_n = S_0 \xi_1 \cdots \xi_n$ be the stock price in the binomial model
- If we assume that the $(\xi_n)_n$ are IID, then $\log S_n = \log S_0 + X_1 + \ldots + X_n$ is a random walk
- Now time step n corresponds to time $t = n\delta$ where δ is very small.
- Let $\hat{S}_t = S_{t/\delta}$
- Then
$$\log \hat{S}_t = \log S_0 + \mu t + \sigma W_t$$
where
- $\mu = \mathbb{E}(X)/\delta$
- $\sigma^2 = \mathrm{Var}(X)/\delta$
- $W_t = \frac{X_1 + \ldots + X_{t/\delta} - \mu t}{\sigma}$

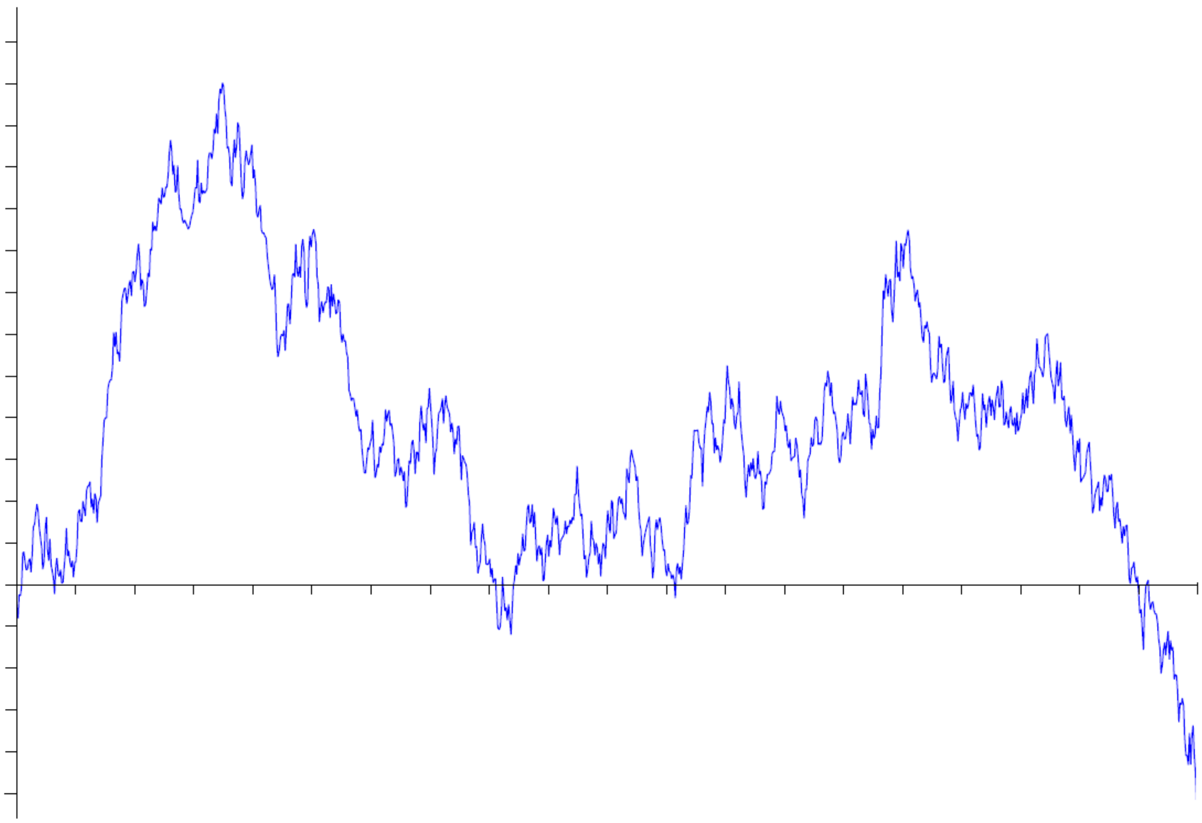
Properties of $(W_t)_{t=n\delta, n\geq 0}$

- $W_0 = 0$
- $\mathbb{E}(W_t - W_s) = 0, \quad \mathrm{Var}(W_t - W_s) = 0$ for all $0 \leq s \leq t$
- $W_t - W_s$ is independent of $(W_u)_{u<s}$ for all $0 \leq slet$
- and by the central limit theorem
$$W_t - W_s \approx N(0, t-s)$$
as $\delta \downarrow 0$ (that is, hold s, t fixed and let $m, n \uparrow \infty$, where $n = t/\delta$ and $m = t/\delta$)

2 Introduction to Brownian motion

Definition. A *Brownian motion* $(W_t)_{t\geq 0}$ is a stochastic process such that

- $t \mapsto W_t$ is continuous
- $W_0 = 0$
- $W_t - W_s$ is independent of $(W_u)_{0\leq u\leq s}$ for all $0 \leq s \leq t$.
- $W_t - W_s \sim N(0, t-s)$ for all $0 \leq s \leq t$.



3 Properties of Brownian motion

Theorem (Wiener 1923)**.** *Brownian motion exists.*

Remark. A Brownian motion is called a Wiener process in the US.

Theorem. *Brownian motion is a martingale in its filtration $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$.*

Proof. Brownian motion is integrable, adapted and

$$\mathbb{E}[W_t - W_s|\mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0$$

for $0 \leq s \leq t$ by the independence of $W_t - W_s$ and $\mathcal{F}_s$. $\square$

Theorem. *Brownian motion is a Markov process.*

Proof. Let g be a bounded function. Since W_s is $\mathcal{F}_s$ measurable and $W_t - W_s$ is independent of $\mathcal{F}_s$ for $0 \leq s \leq t$, we have

$$\begin{aligned}
\mathbb{E}[g(W_t)|\mathcal{F}_s] &= \mathbb{E}[g(W_t - W_s + W_s)|\mathcal{F}_s] \\
&= \mathbb{E}[g(W_t - W_s + x)]\big|_{x=W_s} \\
&= \mathbb{E}[g(W_t)|W_s]
\end{aligned}$$

□

Definition. A process $(X_t)_{t\geq 0}$ is *Gaussian* iff the random variables $X_{t_1}, \ldots, X_{t_n}$ are jointly normal for all $0 \leq t_1 \leq \ldots \leq t_n$, i.e. the random variable $\sum_{i=1}^n a_i X_{t_i}$ is normally distributed for all constants $a_1, \ldots, a_n$.

Theorem. *The following are equivalent*

1. $(W_t)_{t\geq 0}$ *is a Brownian motion*
2. $(W_t)_{t\geq 0}$ *is a Gaussian process such that*
   - $t \mapsto W_t$ *is continuous*
   - $\mathbb{E}[W_t] = 0$ *for all* $t \geq 0$
   - $\mathbb{E}[W_s W_t] = s$ *for all* $0 \leq s \leq t$

Proof. Suppose $(W_t)_{t\geq 0}$ is a Brownian motion. Fix $0 = t_0 \leq t_1 \leq \ldots \leq t_n$ and $a_1, \ldots, a_n$. Note

$$\sum_{i=1}^n a_i W_{t_i} = \sum_{i=1}^n b_i (W_{t_i} - W_{t_{i-1}})$$

where $b_k = \sum_{i=k}^n a_i$. Since $W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent normals, and the linear combination of independent normals is normal, we have that $(W_t)_{t\geq 0}$ is Gaussian with $\mathbb{E}[W_t] = \mathbb{E}[W_0] = 0$ and

$$\begin{aligned}
\mathbb{E}[W_s W_t] &= \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] \\
&= \mathrm{Var}(W_s) + \mathbb{E}(W_s)\mathbb{E}(W_t - W_s) \\
&= s.
\end{aligned}$$

for $0 \leq s \leq t$, since W_s and $W_t - W_s$ are independent.

Conversely, suppose $(W_t)_{t\geq 0}$ is a continuous Gaussian process such that $\mathbb{E}[W_t] = 0$ and $\mathbb{E}[W_s W_t] = s$ for all $0 \leq s \leq t$. Then for $t \geq 0$ we have $\mathrm{Var}(W_t) = \mathbb{E}(W_t^2) = t$ and hence for $0 \leq s \leq t$, we have

$$\begin{aligned}
\mathrm{Var}(W_t - W_s) &= \mathrm{Var}(W_t) + \mathrm{Var}(W_s) - 2\mathrm{Cov}(W_s, W_t) \\
&= t + s - 2s \\
&= t - s.
\end{aligned}$$

Finally for $0 \leq u \leq s \leq t$ we have

$$\begin{aligned} \mathrm{Cov}(W_u, W_t - W_s) &= \mathbb{E}[W_u W_t] - \mathbb{E}[W_u W_s] \\ &= u - u = 0 \end{aligned}$$

By Gaussianity, the increment is independent of $(W_u)_{0 \leq u \leq s}$. $\square$

Remark. We have used the standard fact that if the random vectors X and Y are jointly Gaussian and $\mathrm{Cov}(X, Y) = 0$, then it follows that X and Y are independent.

Theorem. *Let $(W_t)_{t \geq 0}$ be a Brownian motion. Then each of the following processes are also Brownian motions.*

1. $\tilde{W}_t = cW_{t/c^2}$, *for any constant* $c \neq 0$.
2. $\tilde{W}_t = W_{t+T} - W_T$ *for any constant* $T \geq 0$.
3. $\tilde{W}_0 = 0$ *and* $\tilde{W}_t = tW_{1/t}$ *for* $t > 0$.

Proof. Check that each process is a continuous mean-zero Gaussian process with the correct covariance. [For 3, we technically need the Brownian law of large number $\frac{W_s}{s} \to 0$ as $s \to \infty$ to prove continuity of $\tilde{W}$ at $t = 0$.] $\square$