Part IB — Geometry

Based on lectures by Prof. G. Paternain and notes by thirdsgames.co.uk

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Contents

§1 Surfaces

§1.1 Basic definitions

Definition 1.1

A **topological surface** is a topological space Σ such that

1. for all points $p \in \Sigma$, there exists an open neighbourhood $p \in U \subset \Sigma$ such that U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology;
2. Σ is Hausdorff and second countable.

Definition 1.2 (Hausdorff)

A space X is **Hausdorff** if two points $p \neq q \in X$ have open neighbourhoods U, V such that $U \cap V = \varnothing$.

Definition 1.3 (Second Countable)

A space X is **second countable** if it has a countable base; there exists a countable family of open sets U_i, such that every open set is a union of some of the U_i.

Remark 1.

1. $\mathbb{R}^2$ is homeomorphic to the open disc $D(0,1) = \{x \in \mathbb{R}^2 : \|x\| < 1\}$.
2. The first part of the definition is important whilst the second part (Hausdorff and second countable) is a technical point. These topological requirements are typically not the purpose of considering topological spaces, but they are occasionally technical requirements to prove interesting theorems.
3. Note that subspaces of Hausdorff and second countable spaces are also Hausdorff and second countable. In particular, Euclidean space $\mathbb{R}^n$ is Hausdorff (as $\mathbb{R}^n$ is a metric space) and second countable (consider the set of balls $D(p,q)$ for points p with rational coordinates, and rational radii q). Hence, any subspace of $\mathbb{R}^n$ is implicitly Hausdorff and second countable.

Example 1.1

$\mathbb{R}^2$ is a topological surface. Any open subset of $\mathbb{R}^2$ is also a topological surface. For example, $\mathbb{R}^2 \setminus \{0\}$ and $\mathbb{R}^2 \setminus \{(0,0)\} \cup \left\{ \left(0, \frac{1}{n}\right) : n = 1, 2, \ldots \right\}$ are topological surfaces.

Example 1.2

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. The graph of f, denoted Γ_f, is defined by

$$\Gamma_f = \left\{(x, y, f(x, y)) : (x, y) \in \mathbb{R}^2\right\} \subset \mathbb{R}^3$$

with the subspace topology when embedded in $\mathbb{R}^3$.

Recall that the product topology on $X \times Y$ for X, Y topological spaces, has basic open sets $U \times V$, where $U \subset X$, $V \subset Y$ open. Also the product topology has the feature that $g : Z \to X \times Y$ is continuous iff $\pi_x \circ g : Z \to X$ and $\pi_y \circ f : Z \to Y$ are continuous[a].

Hence, any graph $\Gamma \subseteq X \times Y$ is homeomorphic to X if f is continuous. Indeed, the projection π_x projects each point in the graph onto the domain. The function $s : x \mapsto (x, f(x))$ is continuous as $\pi_x \circ s$ and $\pi_y \circ s$ are. So $\pi_x |_{\Gamma_f}$ and s are inverse homeomorphisms.

So, in our case, the graph Γ_f is homeomorphic to $\mathbb{R}^2$, and so is a topological surface.

[a] π_x, π_y are the canonical projections, $\pi_x : X \times Y \to X$

Remark 2. As a topological surface, Γ_f is independent of the function f. However, we will later introduce more ways to describe topological spaces that will ascribe new properties to Γ_f which do depend on f.

Example 1.3

The sphere:

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\right\}$$

is a topological surface, when using the subspace topology in $\mathbb{R}^3$.

This is a subspace of $\mathbb{R}^3$ so is Hausdorff and second countable.

Consider the stereographic projection of S^2 onto $\mathbb{R}^2$ from the north pole $(0, 0, 1)$. The projection satisfies $\pi_+ : S^2 \setminus \{(0, 0, 1)\}$ and

$$(x, y, z) \mapsto \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right).$$

Certainly, π_+ is continuous, since we do not consider the point $(0, 0, 1)$ to be in its domain. The inverse map is given by

$$(u, v) \mapsto \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right).$$

This is also a continuous function. Hence π_+ is a homeomorphism.

Similarly, we can construct the stereographic projection from the south pole, $\pi_- : S^2 \setminus \{(0,0,-1)\} \to \mathbb{R}^2$.

$$(x, y, z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right).$$

This is a homeomorphism.

Hence, every point in S^2 lies either in the domain of π_+ or π_-, and hence sits in an open set $S^2 \setminus \{(0,0,1)\}$ or $S^2 \setminus \{(0,0,-1)\}$ which are homeomorphic to $\mathbb{R}^2$. So S^2 is a topological surface.

Remark 3. S^2 is compact by the Heine-Borel theorem; it is a closed bounded set in $\mathbb{R}^3$.

Example 1.4

The real projective plane is a topological surface.

The group $\mathbb{Z}_2$ acts on S^2 by homeomorphisms via the *antipodal map* $a : S^2 \to S^2$, mapping $x \mapsto -x$. So $\mathbb{Z}_2$ sits in the group of homeomorphisms of S^2, $\text{Homeo}(S^2)$, as we can map $-1 \to a$.

Definition 1.4 (The Real Projective Plane)

The real projective plane, $\mathbb{RP}^2$, is the quotient of S^2 given by identifying every point x with its image $-x$ under a.

$$\mathbb{RP}^2 = {S^2}/{\mathbb{Z}_2} = {S^2}/{\sim}; \quad x \sim a(x)$$

Lemma 1.1

As a set, $\mathbb{RP}^2$ naturally bijects with the set of straight lines in $\mathbb{R}^3$ through the origin.

Proof. Any line through the origin intersects S^2 exactly in a pair of antipodal points $x, -x$. Similarly, pairs of antipodal points uniquely define a line through the origin. □

Lemma 1.2

$\mathbb{RP}^2$ is a topological surface with the quotient topology.

Recall: Quotient topology : $q : X \to Y$ (q the quotient map), $V \subset Y$ is open iff

$q^{-1}V \subset X$ is open in X(i.e. iff q is continuous).

Proof. We must check that $\mathbb{RP}^2$ is Hausdorff since it is constructed by a quotient, not a subspace.
If $[p] \neq [m] \in \mathbb{RP}^2$, then $\pm p, \pm m \in S^2$ are distinct antipodal pairs. We can therefore construct distinct open discs[a] around p, m in S^2, and their antipodal images. These uniquely define open neighbourhoods of $[p], [q]$, which are disjoint, as for $q : S^2 \to \mathbb{RP}^2$, $q(B_\delta(p))$ is open since $q^{-1}(q(B_\delta(p))) = B_\delta(p) \cup (-B_\delta(p))$ is open.

Similarly, we can check that $\mathbb{RP}^2$ is second countable.
We know that S^2 is second countable, so let $\mathcal{U}_0$ be a countable base for the topology on S^2. Let $\overline{\mathcal{U}_0} = \{q(u) : u \in \mathcal{U}_0\}$. $q(u)$ is open as $q^{-1}(q(u)) = u \cup (-u)$ is open. $\overline{\mathcal{U}_0}$ is clearly countable since $\mathcal{U}_0$ is. Now, if $V \subset \mathbb{RP}^2$ is open, then by definition of quotient topology $q^{-1}(V)$ is open in S^2 hence $q^{-1}(V) = \cup_\alpha U_\alpha, U_\alpha \in \mathcal{U}_0$. $V = q(q^{-1}V) = q(\cup_\alpha U_\alpha) = \cup_\alpha q(U_\alpha), q(U_\alpha) \in \overline{\mathcal{U}_0}$.

Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\overline{D}$ be a small (contained in an open hemisphere) closed disc, which is a neighbourhood of $p \in S^2$. The quotient map restricted to $\overline{D}$, written $q|_{\overline{D}} : \overline{D} \to q(\overline{D}) \subset \mathbb{RP}^2$, is a continuous function from a compact space to a Hausdorff space. Further, q is injective on $\overline{D}$ since the disc was contained entirely in a single hemisphere so it cannot contain antipodal points.
Recall from AT that the "topological inverse function theorem" (TIFT) states that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.[b]
So $q|_{\overline{D}}$ is a homeomorphism from $\overline{D}$ to $q(\overline{D})$. This then induces the homeomorphism $q|_D : D \to q(D)$ where D is an open disc, the interior of $\overline{D}$. So by construction, $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ which is homeomorphic to an open disc on S^2 and so to $\mathbb{R}^2$, concluding the proof. $\square$

[a] Just take a ball in $\mathbb{R}^3$ and intersect with S^2

[b] A brief proof is we want to show the inverse function is continuous, so it maps closed sets to closed sets. Take a closed set inside compact space so its compact, apply the continuous function to it so the image is compact. A compact set in a Hausdorff space is closed.

Example 1.5

Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$, and then we define the torus to be the product space $S^1 \times S^1$, with the subspace topology from $\mathbb{C}^2$ (which is identical to the product topology).

Lemma 1.3

The torus is a topological surface.

Proof. Consider the map $e : \mathbb{R}^2 \to S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ defined by

$$(s,t) \mapsto \left(e^{2\pi is}, e^{2\pi it}\right)$$

We have an equivalence relation on $\mathbb{R}^2$ given by translations by $\mathbb{Z}^2$ as e is constant under them. This induces a map $\hat{e}$ from ${\mathbb{R}^2}/{\mathbb{Z}^2}$.

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "e"] \arrow[d, "q"'] & S^1 \times S^1 \\
{\mathbb{R}^2}/{\mathbb{Z}^2} \arrow[ur, "\hat{e}"] &
\end{tikzcd}$$

Under the quotient topology given by the quotient map q, ${\mathbb{R}^2}/{\mathbb{Z}^2}$ is a topological space. The map $[0,1]^2 \to \mathbb{R}^2 \to {\mathbb{R}^2}/{\mathbb{Z}^2}$ is surjective, so ${\mathbb{R}^2}/{\mathbb{Z}^2}$ is compact. So $\hat{e}$ is a continuous map from a compact space to a Hausdorff space, and $\hat{e}$ is bijective, so $\hat{e}$ is a homeomorphism by TIFT.

We already have that $S^1 \times S^1$ is compact and Hausdorff (as a closed and bounded set in $\mathbb{C}^2$, equivalent to $\mathbb{R}^4$), so it suffices to show it is locally homeomorphic to $\mathbb{R}^2$.

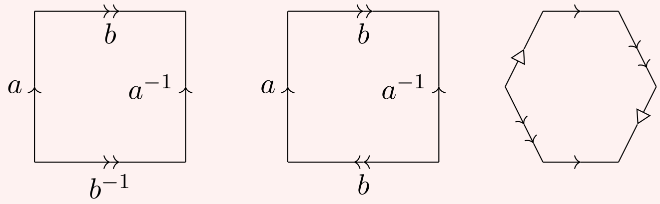
Similarly to the case of $S^2 \to \mathbb{RP}^2$, pick $[p] \in q(p), p \in \mathbb{R}^2$, then we can choose a small closed disc $\overline{D}(p) \subset \mathbb{R}^2$ such that $\overline{D}(p) \cap \left(\overline{D}(p) + (n,m)\right) = \varnothing$ for all nonzero $(n,m) \in \mathbb{Z}^2$. Hence $e|_{\overline{D}(p)}$ and $q|_{\overline{D}(p)}$ are injective. Now, restricting to the open disc as before, we can find an open disc neighbourhood of $[p] \in {\mathbb{R}^2}/{\mathbb{Z}^2}$. Since $[p]$ was chosen arbitrarily, $S^1 \times S^1$ is a topological surface. $\square$

Another viewpoint:
${\mathbb{R}^2}/{\mathbb{Z}^2}$ is also given by imposing on $[0,1]^2$ the equivalence relation

$$\begin{aligned}
(x,0) &\sim (x,1) \quad \forall\, 0 \leq x \leq 1 \\
(0,y) &\sim (1,y) \quad \forall\, 0 \leq y \leq 1.
\end{aligned}$$

Example 1.6

Let P be a planar Euclidean polygon (including interior), with oriented edges. We will pair the edges, and without loss of generality we will assume that paired edges have the same Euclidean length.



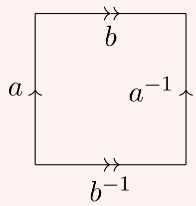
We can assign letter names to each edge pair, and denote a polygon by the sequence of edges found when traversing in a clockwise orientation. The edge pair name is inverted if the edge is traversed in the reverse direction. Note the difference between the annotations on the first two shapes above, due to the reversed direction of the edge.

If two edges $\{e, \hat{e}\}$ are paired, this defines a unique Euclidean isometry from e to $\hat{e}$ respecting the orientation, which will be written $f_{e\hat{e}} : e \to \hat{e}$.
The set of all such functions generate an equivalence relation on the polygon P, where we identify $x \in \partial P$ (a point on the boundary) with $f_{e\hat{e}}(x)$ whenever $x \in e$.

Lemma 1.4
${P}/{\sim}$, with the quotient topology, is a topological surface.

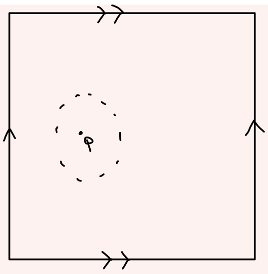
Example 1.7

Consider the torus, defined here as $T^2 = {[0,1]^2}/{\sim}$.

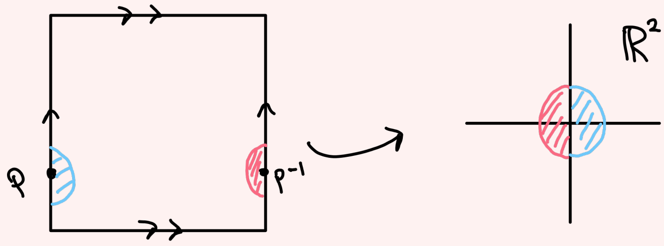


Let P be the polygon $[0,1]^2$.

If p is in the interior of P, then we pick $\delta > 0$ small s.t. $\overline{B_\delta(p)}$[a] lies in the interior of P. Arguing as before in $\mathbb{RP}^2$, the quotient map is injective on $\overline{B_\delta(p)}$ and is a homeomorphism on its interior.



Let p be on an edge, but not a vertex.



Let us say without loss of generality that $p = (0, y_0) \sim (1, y_0) = p'$. Let δ be sufficiently small that the closed half-discs U, V centred on p, p^{-1} with radius δ do not intersect any vertices.
Then we define a map from the union of the two half-discs to the disc $B(0, \delta) \subseteq \mathbb{R}^2$ via

$$U : (x, y) \underset{f_u}{\mapsto} (x, y - y_0)$$
$$V : (x, y) \underset{f_v}{\mapsto} (x - 1, y - y_0)$$

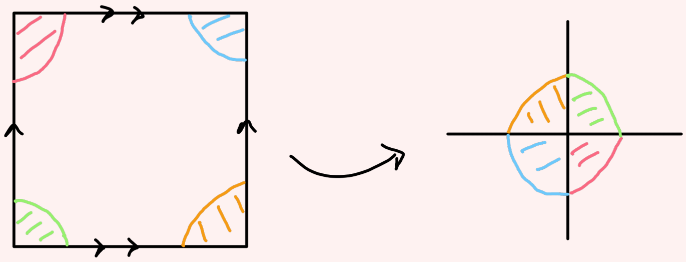
which will be a bijective map.

Recall the gluing lemma from Analysis and Topology: that if $X = A \cup B$ is a union of closed subspaces, and $f : A \to Y$, $g : B \to Y$ are continuous and $f|_{A \cap B} = g|_{A \cap B}$, they define a continuous map on X.

f_U, f_V are continuous on $U, V \subset [0, 1]^2$. By the definition of the quotient topology, $q \circ f_U$ and $q \circ f_V$ are also continuous ($q : [0, 1]^2 \to {[0,1]^2}/{\sim}$).
In T^2, $1/2$-discs, $q \circ U, q \circ V$ overlap but our maps agree as they are compatible with the equivalence relation.
Hence, by the gluing lemma, f_U, f_V "glue" together to give a continuous map to an open neighbourhood of $[p] \in T^2$ to $\mathbb{R}^2$.

We can show that this is a homeomorphism using the usual process: pass to a closed disc, apply the topological inverse function theorem, then apply the result to the interior. If $[p] \in T^2$ lies in an edge on P, it has a neighbourhood homeomorphic to a disc.

Now it suffices to consider points p on a vertex. All four vertices of the square are identified to the same point in the torus as each vertex lies on two edges and so is identified to two other vertices.



and analogously we get that a vertex has a neighbourhood homeomorphic to a disc.

Thus, ${[0,1]^2}/{\sim}$ is a topological surface.

[a] The closure of $B_\delta(p)$

Example 1.8 (General Polygon)

We can generalise this proof to an arbitrary planar Euclidean polygon P, such as the hexagon above. The equivalence relation $x \sim f_{e\hat{e}}(x)$ induces an equivalence relation on the vertices of P, by considering the images of the vertices under all $f_{e\hat{e}}$. However, it is not necessarily the case that an equivalence class of vertices contains exactly four vertices, so quarter-discs are not necessarily applicable. Again, there are three types of point:

- interior points, for which a neighbourhood not intersecting the boundary is chosen;
- points on edges, for which a corresponding point exists and two half-discs can be glued to form the neighbourhood; and
- points on vertices. For this case, all vertices of the polygon have a neighbourhood which is a sector of a circle. Let there be r vertices in a given equivalence class. Let α be the sum of the angles of the sectors in a given class.
  Any sector can be identified with a given sector in the disc $B(0,\delta) \subseteq \mathbb{R}^2$, which we will choose to have angle α/r. Then, we can glue each sector together in $\mathbb{R}^2$, compatibly with the orientations of the edges and arrows, inducing a neighbourhood which is locally homeomorphic to a disc.

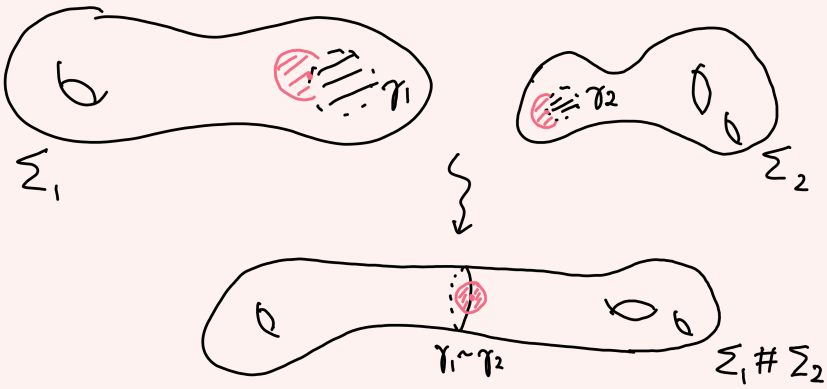
  If $r = 1$, we have an equivalence class comprising a single vertex, which gives a single sector. For r to be one, the two edges attached to this vertex must be paired and have the same direction (either both inwards or outwards from the vertex). This quotient space is simply a cone, which is homeomorphic to a disc as required.

We can also show that the quotient space is Hausdorff and second countable. By construction, two distinct points in the quotient space can be separated by open neighbourhoods by selecting a sufficiently small radius such that the discs considered in the derivation above are disjoint. For second countability, consider

- discs in the interior of P with rational centres and radii;
- for each edge of P, consider an isometry $e \to [0, \ell]$ where ℓ is the length of e, taking discs on e which are centred at rational values in $[0, \ell]$; and
- for each vertex, consider discs centred at these vertices with rational radii.

Example 1.9 (Connected Sums)

Given topological surfaces Σ_1, Σ_2 we can remove an open disc from each and glue the resulting boundary circles.



Explicitly, take $\Sigma_1 \setminus D_1 \amalg$ [a]$\Sigma_2 \setminus D_2$ and impose a quotient relation by identifying $\theta \in \partial D_1 \sim \theta \in \partial D_2$ where θ is an angle parametrising $S^1 = \partial D_i$, ∂D_i is the boundary of D_i. The result $\Sigma_1 \# \Sigma_2$ is called the **connected sum** of Σ_1, Σ_2.

In principle this depends on many choices and takes some effort to prove that it is well-defined.

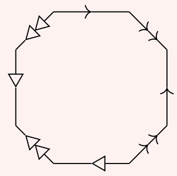
[a] Disjoint Union

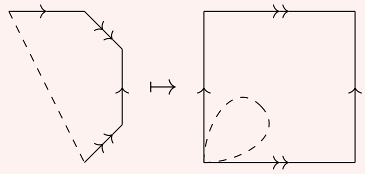
Lemma 1.5

The connected sum $\Sigma_1 \# \Sigma_2$ is a topological surface.

Proof. Not proved in this course, if you want to learn more try 'Introduction to topological manifolds' by Jack Lee. $\square$

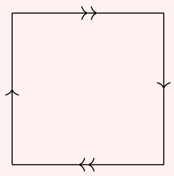
Example 1.10

Consider the following octagon.  The associated quotient space $P/\sim$ can be seen to be homeomorphic to a surface with two holes, known as a double torus. All vertices are identified as the same vertex in the quotient space. We can cut the octagon along a diagonal, leaving two topological surfaces which are homeomorphic to a torus.



Thus, the connected sum of the two half-octagons are the connected sum of two toruses.

Example 1.11

Consider the following square.  This is homeomorphic to the real projective plane $\mathbb{RP}^2$. This is because we identify points on the boundary with their antipodes, when interpreting the square as the closed disc $B(0,1)$. The real projective plane was constructed by identifying points on the unit sphere with their antipodes. Thus, we can construct a homeomorphism by considering only points in the upper hemisphere (taking antipodes as required), and then orthographically projecting onto the xy plane. Under this transformation, points on the boundary are identified with their antipodes as required.

§1.2 Subdivisions

Definition 1.5 (Subdivision)

A **subdivision** of a compact topological surface Σ comprises

1. a finite subset $V \subseteq \Sigma$ of vertices;

2. a finite subset of edges $E = \{e_i : [0,1] \to \Sigma\}$ s.t. 1) each e_i is a continuous injection on its interior and $e_i^{-1}V = \{0,1\}$, the endpoints. 2) e_i, e_j have disjoint images except perhaps at their endpoints.

3. we require that each connected component of $\Sigma \setminus (\cup_i e_i[0,1] \cup V)$ is homeomorphic to an open disc called a **face**. In particular, the closure of a face has boundary $\overline{F} \setminus F$ lying in $(\cup_i e_i[0,1] \cup V)$.

Definition 1.6 (Triangulation)

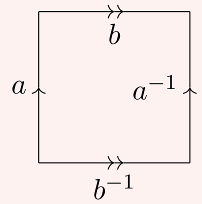
We say that a subdivision is a **triangulation** if each closed face (closure of a face) contains exactly three edges, and two closed faces are disjoint, meet at exactly one edge or just one vertex.

Example 1.12

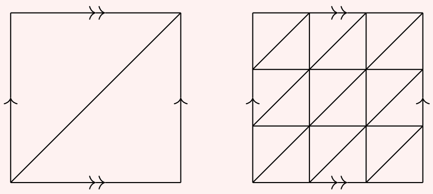
A cube displays a subdivision of S^2. A tetrahedron displays a triangulation of S^2.

Example 1.13

We can display subdivisions of surfaces constructed from polygons.

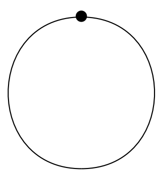


This is a subdivision of a torus with one vertex, two edges, and one face. We can construct additional subdivisions of a torus, for example:



The first of these examples is not a triangulation, since the two faces meet in more than one edge. The second is a triangulation.

Remark 4. The following is a very degenerate subdivision of S_2.



[1] This

[1]This is not a circle, its a 2-sphere.

has one vertex, no edges, and one face.

§1.3 Euler classification

Definition 1.7 (Euler Characteristic)

The **Euler characteristic** of a subdivision is

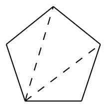
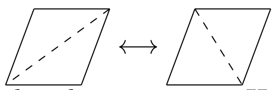
$\#$[a]$V - \#E + \#F$

[a]The number/size of the set

Theorem 1.1 1. Every compact topological surface has a subdivision (and indeed triangulations).

2. The Euler characteristic is invariant under choice of subdivision, and is topologically invariant of the surface (depends only on the homeomorphism type of Σ).

Hence, we might say that a surface has a particular Euler characteristic, without referring to subdivisions. We write this $\chi(\Sigma)$.

Remark 5. It is not trivial to prove part (i). For part (ii), note that subdivisions can be converted into triangulations by constructing triangle fans.  Triangulations can be related by local moves, such as  It is easy to check that both of these moves do not change the Euler characteristic. However, it is hard to make this argument rigorous, and it does not give much explanation for why the result is true. In Part II Algebraic Topology, a more advanced definition of the Euler characteristic is given, which admits a more elegant proof.

Proof. No proof will be given. □

Example 1.14

The Euler characteristic of S^2 is $\chi(S^2) = 2$.

Example 1.15

For the torus, $\chi(T^2) = 0$.

Example 1.16

If Σ_1, Σ_2 are compact surfaces, then the connected sum $\Sigma_1 \# \Sigma_2$ can be constructed by removing a face of a triangulation, then gluing together the boundary circles (three edges) in a way that matches the edges.

Then the connected sum inherits a subdivision, and we can find that it has Euler characteristic $\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$, where the remaining term corresponds to the two faces that were removed; the changes of three vertices and three edges cancel each other.

In particular, a surface Σ_g with g holes can be written $\#_{i=1}^{g} T^2$, so $\chi(\Sigma_g) = 2 - 2g$. We call g the **genus** of Σ.

§2 Abstract smooth surfaces

§2.1 Charts and atlases

Recall that if Σ is a topological surface, any point lies in an open neighbourhood homeomorphic to a disc.

Definition 2.1 (Chart)
A pair (U, φ), where U is an open set in Σ and $\varphi\colon U \to V$ is a homeomorphism to an open set $V \subseteq \mathbb{R}^2$, is called a **chart** for Σ. If $p \in U$, we might say that (U, φ) is a chart for Σ *at* p.

Definition 2.2 (Local parameterisation)
The inverse $\sigma = \varphi^{-1}\colon V \to U$ is known as a **local parametrisation** for the surface.

Definition 2.3 (Atlas)
A collection of charts $\{(U_i, \varphi_i)_{i\in I}\}$ whose domains cover Σ ($\cup_{i\in I} U_i = \Sigma$) is known as an **atlas** for Σ.

Example 2.1
If $Z \subseteq \mathbb{R}^2$ is closed, $\mathbb{R}^2 \setminus Z$ is a topological surface with an atlas containing one chart, $(\mathbb{R}^2 \setminus Z, \varphi = \mathrm{id})$.

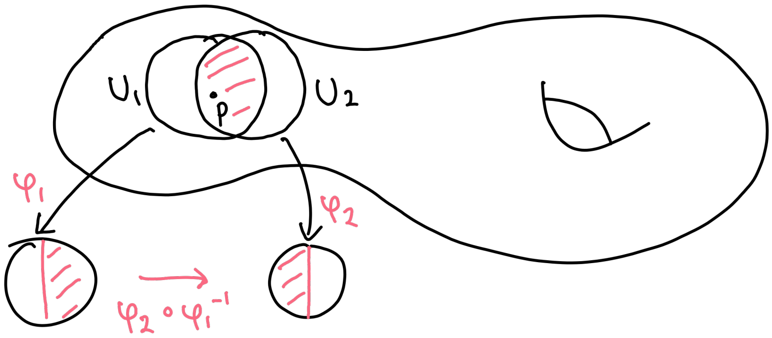
Example 2.2
For S^2, there is an atlas with two charts, which are the two stereographic projections from the poles.

Definition 2.4 (Transition Map)
Let (U_i, φ_i) be charts containing the point $p \in \Sigma$, for $i = 1, 2$. Then the map

$$\varphi_2 \circ \varphi_1^{-1}\Big|_{\varphi_1(U_1 \cap U_2)} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$$

converts between the corresponding charts, and is called a *transition map* between charts. This is a homeomorphism of open sets in $\mathbb{R}^2$.

Recall from Analysis and Topology that if $V \subseteq \mathbb{R}^n$ and $V' \subseteq \mathbb{R}^m$ are open, then a



continuous map $f : V \to V'$ is called *smooth* if it is infinitely differentiable. Equivalently, it is smooth if continuous partial derivatives of all orders in all variables exist at all points.

Definition 2.5 (Diffeomorphism)
A homeomorphism $f : V \to V'$ is called a **diffeomorphism** if it is smooth and it has a smooth inverse.

Definition 2.6 (Abstract Smooth Surface)
An **abstract smooth surface** Σ is a topological space with an atlas of charts $\{(U_i, \varphi_i)_{i\in I}\}$ s.t. all transition maps are diffeomorphisms.

Remark 6. We could not simply consider a smoothness condition for Σ itself without appealing to atlases, since Σ is an arbitrary topological space and could have almost any topology.

Example 2.3
The atlas of two charts with stereographic projections gives S^2 the structure of an abstract smooth surface.

Example 2.4
For the torus $T^2 = {}^{\mathbb{R}^2}\!/\!_{\mathbb{Z}^2}$, we can find charts of all points by choosing sufficiently small discs in $\mathbb{R}^2$ such that they do not intersect any of their non-trivial integer translates. The transition maps for this atlas are all translations of $\mathbb{R}^2$. Hence T^2 inherits the structure of an abstract smooth surface. Explicitly, let us define $e : \mathbb{R}^2 \to T^2$ by $(t, s) \mapsto (e^{2\pi it}, e^{2\pi is})$, then consider the atlas

$$\Big\{(e(D_\varepsilon(x, y)), e^{-1} \text{ on this image})\Big\}$$

for $\varepsilon < \frac{1}{3}$. These are charts on T^2, and the transition maps are (restricted to appropriate domains) translations in $\mathbb{R}^2$. Hence T^2, via this atlas, has the structure of an abstract smooth surface.

Remark 7. The definition of a topological surface is a notion of structure. One can observe a topological space and determine whether it is a topological surface. Conversely, to be an abstract smooth surface is to have a specific set of data; that is, we must provide charts for the surface in order to see that it is indeed an abstract smooth surface.

Definition 2.7

Let Σ be an abstract smooth surface, and $f\colon \Sigma \to \mathbb{R}^n$ be a continuous map. We say that f is *smooth* at $p \in \Sigma$ if, for all charts (U, φ) of p belonging to the smooth atlas for Σ, the map

$$f \circ \varphi^{-1}\colon \varphi(U) \to \mathbb{R}^n$$

is smooth at $\varphi(p) \in \mathbb{R}^2$.

Remark 8. Note that the choice of chart and atlas was arbitrary, but smoothness of f at p is independent of the choice of chart, since the transition maps between two such charts are diffeomorphisms.

Definition 2.8

Let Σ_1, Σ_2 be abstract smooth surfaces. Then a map $f\colon \Sigma_1 \to \Sigma_2$ is *smooth* if it is 'smooth in the local charts'. Given a chart (U, φ) at p and a chart (U', ψ) at $f(p)$, both mapping to open subsets of $\mathbb{R}^2$, the map $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(p)$. Smoothness of f does not depend on the choice of chart, provided that the charts all belong to the same atlas.

Definition 2.9

Two surfaces Σ_1, Σ_2 are *diffeomorphic* if there exists a homeomorphism $f\colon \Sigma_1 \to \Sigma_2$ which is smooth and has smooth inverse.

Remark 9. Often, we convert from a given smooth atlas for an abstract smooth surface Σ to the *maximal compatible* smooth atlas. That is, we consider the atlas with the maximal possible set of charts, all of which have transition maps that are diffeomorphisms. This can be accomplished formally by use of Zorn's lemma.