Part IA — Groups

Based on lectures by R. Camina

Notes taken by Author

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Syllabus

Examples of groups

Axioms for groups. Examples from geometry: symmetry groups of regular polygons, cube, tetrahedron. Permutations on a set; the symmetric group. Subgroups and homomorphisms. Symmetry groups as subgroups of general permutation groups. The Möbius group; cross-ratios, preservation of circles, the point at infinity. Conjugation. Fixed points of Möbius maps and iteration. [4]

Lagrange's theorem

Cosets. Lagrange's theorem. Groups of small order (up to order 8). Quaternions. Fermat-Euler theorem from the group-theoretic point of view. [5]

Group actions

Group actions; orbits and stabilizers. Orbit-stabilizer theorem. Cayley's theorem (every group is isomorphic to a subgroup of a permutation group). Conjugacy classes. Cauchy's theorem. [4]

Quotient groups

Normal subgroups, quotient groups and the isomorphism theorem. [4]

Matrix groups

The general and special linear groups; relation with the Möbius group. The orthogonal and special orthogonal groups. Proof (in $\mathbb{R}^3$) that every element of the orthogonal group is the product of reflections and every rotation in $\mathbb{R}^3$ has an axis. Basis change as an example of conjugation. [3]

Permutations

Permutations, cycles and transpositions. The sign of a permutation. Conjugacy in S_n and in A_n. Simple groups; simplicity of A_5. [4]

Contents

Introduction

Dr Rachel Camina

Recommended books: Algebra & Geometry, Alan Beardon.

Notation: $\lightning$- contradiction

§1 Groups and homomorphisms

§1.1 Motivation

Groups are the abstractions of symmetries, a unified way to investigate symmetries.

§1.2 Basic Definitions and Examples

Definition 1.1 (Binary operation)
A binary operation $*$ on a set X is a way of combining two elements of X to unambiguously give another element of X, i.e. $* : X \times X \to X$.

Definition 1.2 (Group)
If G is a set and $*$ is a binary operation on G then $(G, *)$ is a *group* if the following four axioms hold:

1. $x, y \in G \implies x * y \in G$ (closure)
2. $\exists$ an element $e \in G$ satisfying $x * e = x = e * x$ (existence of an identity)
3. for every $x \in G$ there is a $y \in G$ s.t. $x * y = e = y * x$ (existence of inverses)
4. for every $x, y, z \in G$, $x * (y * z) = (x * y) * z$ (the associative law)

Remark 1. e is called the identity of G - see Lemma 1.1 for why it is unique. We will show in Lemma 1.1 that inverses are unique and we will write x^{-1} for the inverse of x.

Example 1.1
$(\mathbb{Z}, +)$, $e = 0$, $x^{-1} = -x$

Example 1.2
$(\mathbb{Q}, +)$, $(\mathbb{R}, +)$

Example 1.3
$(\mathbb{Q} \setminus \{0\}, *)$

Non Example 1.1 $(\mathbb{Z}, -)$ - associativity fails

Non Example 1.2 $(\mathbb{Z}, *)$ - no inverses

Non Example 1.3 $(\mathbb{Q}, *)$ - 0^{-1} does not exist

These have all had an infinite number of elements, so onto some finite groups.

Example 1.4 (The trivial group)

$$(e, *)$$

Example 1.5

$(\{\pm 1\}, \times)$. A nice way to look at a group is to look at a multiplication table.

$\times$	1	-1
1	1	-1
-1	-1	1

We can see $e = 1$ and $(-1)^{-1} = -1$

Example 1.6

$(\{0, 1, 2\}, +_3)$. $+_3$ is addition modulo 3.

$+_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

We can see $e = 0$ and $1^{-1} = 2$

Example 1.7

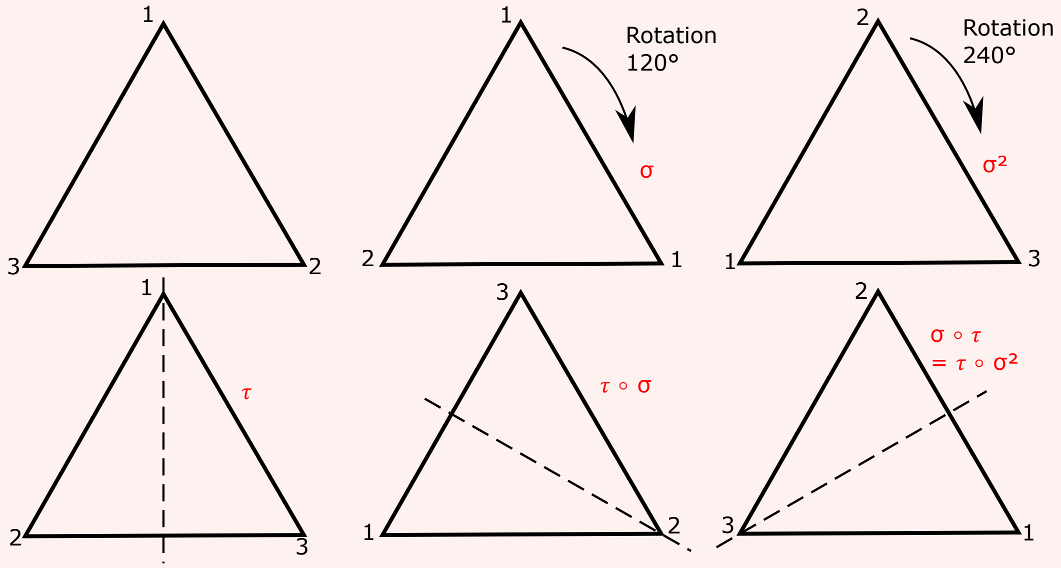
$$(\{e, a, b, c\}, *)$$

$*$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

You may notice that in any row no element is repeated, this is due to the cancellation law, Remark 6.

Example 1.8

Rotations and reflections of an equilateral triangle.



operation $\circ$ = do right transformation then left transformation

Claim: This defines a group with 6 elements

Example 1.9

$$\begin{aligned} M_2(\mathbb{R}) &= \{2 \times 2 \text{ matrices with entries in } \mathbb{R}\} \\ &= \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right] \end{aligned}$$

under addition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix}$$

A more interesting example is

Example 1.10

$$GL_2(\mathbb{R}) = \{\text{invertible } 2 \times 2 \text{ matrices with entries in } \mathbb{R}\}.$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ \det A = ad - bc \neq 0$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & -a \end{pmatrix}$$

Under multiplication this is a group. GL stands for general linear group.

Lemma 1.1

Let $(G, *)$ be a group.

i. The identity element is unique.

ii. Inverses are unique.

Proof.

i. Suppose e and $\hat{e}$ are both identities, so

$$a * e = a = e * a, \quad a * \hat{e} = a = \hat{e} * a \quad \forall a \in G.$$

In particular

$$e = e * \hat{e} = \hat{e}.$$

ii. Suppose both y and z are inverses for x, so

$$\begin{aligned} x * y = e = y * x&, \quad x * z = e = z * x \\ \text{Then, } y &= y * e \\ &= y * (x * z) \\ &= (y * x) * z && \text{(associative law)} \\ &= e * z \\ &= z. \end{aligned}$$

□

Remark 2. Associativity means we don't need brackets, $x * y * z$ is unambiguous. Furthermore, by induction, $x_1 * x_2 * \ldots * x_n$ is unambiguous.

We know the statement is true for the case $n = 3$.

$$\begin{aligned} x_1 * (x_2 * \ldots * x_n) &= (x_1 * x_2) * (x_3 * \ldots * x_n) \\ &= (x_1 * x_2 * x_3) * (x_4 * \ldots * x_n) \end{aligned}$$

$$
\begin{aligned}
&= \dots \\
&= (x_1 * x_2 * \dots * x_{n-1}) * x_n
\end{aligned}
$$

Remark 3. We often omit '$*$' and write xy for $x * y$ and G for $(G, *)$.

Remark 4. $(xy)^{-1} = y^{-1}x^{-1}$. Since it works:

$$
\begin{aligned}
(xy)y^{-1}x^{-1} &= x(yy^{-1})x^{-1} \\
&= xex^{-1} \\
&= xx^{-1} \\
&= e
\end{aligned}
$$

Note, inverses are unique

Remark 5. $(x^{-1})^{-1} = x$

Remark 6.

$$
\begin{aligned}
& & x, y, z \in G \text{ and } xy = xz & \\
&\implies & x^{-1}xy = x^{-1}xz & \\
&\implies & y = z & \qquad \text{(cancellation law)}
\end{aligned}
$$

Definition 1.3 (Abelian group)

A group G is *abelian* (or commutative) if $xy = yx$ for all $x, y \in G$.

Note all our examples above are abelian except 1.8 and 1.10.

Definition 1.4 (Order of group)

Let G be a group. If the number of elements in the set G is finite, then G is called a *finite group*. Otherwise, G is called an *infinite group*. If G is a finite group denote the number of elements in the set G by $|G|$ and call this the *order* of G.

Definition 1.5 (Subgroup)

Let $(G, *)$ be a group and H a subset of G ($H \subseteq G$ i.e. $h \in H \implies h \in G$). Then $(H, *)$ is a *subgroup* of $(G, *)$ if $(H, *)$ is a group (with same operation) i.e. if

a) $h, k \in H \implies h * k \in H$

b) $e_G \in H$

c) $h \in H \implies h^{-1} \in H$

(Note associativity is inherited)

i.e restricting operation to H still gives a group. We write $H \leq G$.

Example 1.11

$$(\mathbb{Z},+) \leq (\mathbb{Q},+) \leq (\mathbb{R},+)$$

Example 1.12

$$(\{\pm 1\},\times) \leq (\mathbb{Q}\setminus\{0\},\times)$$

Example 1.13

In Example 1.8 if we just take the rotations we get a subgroup, $\{1,\sigma,\sigma^2\}$ is a subgroup.

Example 1.14

In Example 1.10, we can take the matrices with determinant 1 which is $\mathrm{SL}_2(\mathbb{R})$ (SL stands for the special linear group).

$$\begin{aligned}\mathrm{SL}_2(\mathbb{R}) &= \{A \in GL_2(\mathbb{R}) : \det A = 1\} \\ &\leq GL_2(\mathbb{R})\end{aligned}$$

We always have identity and whole thing as subgroups.

Example 1.15

If G is a group then $\{e\} \leq G$ is the trivial subgroup.

Example 1.16

If G is a group then $G \leq G$ is the improper subgroup.

Proposition 1.1

Subgroups of $\mathbb{Z}$ are exactly $n\mathbb{Z} = nk : k \in \mathbb{Z}$ where $n \in \mathbb{Z}_{\geq 0}$ (under addition).

Proof. First note $n\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

a. If $a, b \in n\mathbb{Z}$ then $a = na'$ and $b = nb'$ for some $a', b' \in \mathbb{Z}$. Then $a + b = na' + nb' = n(a' + b') \in n\mathbb{Z}$

b. $0 \in n\mathbb{Z}$.

c. If we have $a = na' \in n\mathbb{Z}$ then $a^{-1} = -a = n(-a') \in n\mathbb{Z}$.

Conversely assume $H \leq \mathbb{Z}$.

If $H = \{0\} = 0\mathbb{Z}$.

Otherwise choose $0 < n \in H$ with n minimal (smallest positive element of H). Then $n\mathbb{Z} \in H$ by closure and inverses. We show $H = n\mathbb{Z}$. Suppose $\exists\, h \in H \setminus n\mathbb{Z}$, then we can write $h = nk + h'$ with $h' \in \{1, 2, \ldots, n-1\}$. But $h' = h - nk \in H$, contradicting minimality of n. Thus $H = n\mathbb{Z}$. □

Aside: Functions

We need the notion of functions.

Definition 1.6 (Function)

f is a *function* between sets A and B if it assigns each element of A a unique element of B.

$$\begin{aligned} f : A &\to B \\ a &\mapsto f(a) \end{aligned}$$

Example 1.17

$$\text{eg:}\quad \begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto x + 1 \end{aligned} \qquad\qquad \begin{aligned} g : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto 2x. \end{aligned}$$

Definition 1.7 (Composition of functions)

Suppose $g : A \to B$ and $f : B \to C$, define

$$\begin{aligned} f \circ g : A &\to C \\ a &\mapsto (f \circ g)(a) = f(g(a)) \end{aligned}$$

Example 1.18

$$(f \circ g)(x) = 2x + 1$$
$$(g \circ f)(x) = 2x + 2.$$

Suppose $f_1 : A \to B$ and $f_2 : A \to B$ then $f_1 = f_2$ if $f_1(a) = f_2(a) \quad \forall\, a \in A$.

Definition 1.8 (Bijective functions)
$f : A \to B$ is a *bijection* if it defines a paring between elements of A and elements of B. That is given $b \in B$ $\exists$ unique $a \in A$ s.t. $f(a) = b$.

Example 1.19

$$\begin{aligned} f : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto x + 1 \end{aligned}$$

Definition 1.9 (Inverse function)
If we have a bijection we can define

$$\begin{aligned} f^{-1} : B &\to A \\ b &\mapsto a \text{ where } f(a) = b. \\ \text{Then } f \circ f^{-1} &= id_B \qquad\qquad f^{-1} \circ f = id_A \\ id_B(b) &= b \end{aligned}$$

Lemma 1.2
If $g : A \to B$ and $f : B \to C$ are bijections then so is $f \circ g : A \to C$.

Proof. See Numbers and Sets $\square$

Definition 1.10 (Group homomorphism)
Let $(G, *_G)$ and $(H, *_H)$ be groups. Then the function

$$\theta : G \to H$$

is a *homomorphism* if

$$\theta(x *_G y) = \theta(x) *_H \theta(y)\ \forall\, x, y \in G.$$

'A map which respects the group operation'.

Example 1.20

$G = (\{0,1,2,3\}, +_4)$, $H = (\{1, e^{\pi i/2}, e^{\pi i}, e^{3\pi i/2}, \times)$ (the 4th roots of unity).

$$\begin{aligned}\theta : G &\to H\\ n &\mapsto e^{n\pi i/2}\\ \theta(n +_4 m) &= e^{(n+_4 m)\pi i/2}\\ &= e^{(n+_4 m)\pi i/2} \text{ since } n+m = n +_4 m + 4k \text{ and } e^{4k\pi i/2} = 1\\ &= e^{n\pi i/2} \times e^{m\pi i/2}\\ &= \theta(n) \times \theta(m)\end{aligned}$$

Lemma 1.3

Let G and H be groups and suppose we have a homomorphism $\theta : G \to H$. Then $\theta(G) = \{\theta(g) : g \in G\}$, the *image* of θ, is a subgroup of H, written $\theta(G) \leq H$.

Proof. $\theta(G) \subseteq H$ by definition of θ.

Closure: Let $x, y \in \theta(G)$. Then $x = \theta(g)$ and $y = \theta(h)$ for some $h, g \in G$.

$$\begin{aligned}x *_H y &= \theta(g) *_H \theta(h)\\ &= \theta(g *_G h) \text{ as } \theta \text{ is a homomorphism}\\ &= \theta(G).\end{aligned}$$

Identity:

$$\begin{aligned}\theta(e_G) &= \theta(e_G *_G e_G)\\ &= \theta(e_G) *_H \theta(e_G)\\ \text{premultiplying by } \theta(e_G)^{-1} \in H\\ \boldsymbol{e_H} &\boldsymbol{= \theta(e_G)} \in \theta(G)\end{aligned}$$

Inverses:

$$\begin{aligned}\text{Let } x &= \theta(g) \in \theta(G)\\ e_H = \theta(e_G) &= \theta(g *_G g^{-1})\\ &= \theta(g) *_H \theta(g^{-1})\\ &= x *_H \theta(g^{-1})\end{aligned}$$

$$\begin{aligned}
\text{also} \quad &= \theta(g^{-1} *_G g) \\
&= \theta(g^{-1}) *_H x
\end{aligned}$$

Inverses are unique

$$\implies \boldsymbol{\theta(g)^{-1} = \theta(g^{-1})} \in \theta(G)$$

Associativity is inherited. $\square$

Definition 1.11 (Isomorphism)

A bijective homomorphism is called an isomorphism. If G and H are groups and $\theta : G \to H$ is an isomorphism we say G and H are isomorphic and write $G \cong H$. The bijective part tells us the sets are the same and the homomorphism tells us the group operation is the same so an isomorphism tells us the groups are the "essentially the same".

See Example 1.20

$$\begin{aligned}
G = (\{0, 1, 2, 3\}, +_4) &\cong \{1, e^{\pi i/2}, e^{\pi i}, e^{3\pi i/2}, \times) = H \\
\theta &: G \to H \\
n &\mapsto e^{n\pi i/2}
\end{aligned}$$

θ is an isomorphism.

Lemma 1.4

i. The composition of two homomorphisms is a homomorphism, similarly for isomorphisms, thus if $G_1 \cong G_2$ and $G_2 \cong G_3$ then $G_1 \cong G_3$.

ii. If $\theta : G_1 \to G_2$ is an isomorphism then so is its inverse $\theta^{-1} : G_2 \to G_1$. So $G_1 \cong G_2 \implies G_2 \cong G_1$.

Proof.

i. Suppose

$$\begin{aligned}
\theta_1 &: (G_1, *_1) \to (G_2, *_2) \\
\theta_2 &: (G_2, *_2) \to (G_3, *_3)
\end{aligned}$$

are isomorphisms. Thus $\theta_2 \circ \theta_1$ is a function from G_1 to G_3, we need to check if its a homomorphism.

$$\begin{aligned}
&\text{Let } x, y \in G_1 \\
\theta_2 \circ \theta_1(x *_1 y) &= \theta_2(\theta_1(x) *_2 \theta_1(y)) \qquad \text{since } \theta_1 \text{ is a homomorphism}
\end{aligned}$$

$$
\begin{aligned}
&= \theta_2(\theta_1(x)) *_3 \theta_2(\theta_1(y)) && \text{since } \theta_2 \text{ is a homomorphism}\\
&= (\theta_2 \circ \theta_1)(x) *_3 (\theta_2 \circ \theta_1)(y)
\end{aligned}
$$

ii. θ is a bijection so θ^{-1} exists, we need to show its a homomorphism.

$$
\begin{aligned}
&\text{Let } y, z \in G_2.\\
&\text{Then } \exists\, x, k \in G_1, \text{ s.t.}\\
&\theta^{-1}(y) = x,\ \theta^{-1}(z) = k\\
&\text{Note, } \theta(x *_1 k) = \theta(x) *_2 \theta(k)\\
&\qquad\qquad\qquad\; = y *_2 z\\
&\implies \theta^{-1}(y *_2 z) = x *_1 k\\
&\qquad\qquad\qquad\; = \theta^{-1}(y) *_1 \theta^{-1}(z)
\end{aligned}
$$

$\square$

Notation. If $x \in (G, *)$, $n \in \mathbb{Z}$. Then

$$
x^n = \begin{cases} x * x * \ldots * x & n > 0 \\ e & n = 0 \\ x^{-1} * x^{-1} * \ldots * x^{-1} & n < 0 \end{cases}
$$

Definition 1.12 (Cyclic group)

A group H is *cyclic* if $\exists\, h \in H$ such that each element of H is a power of h, i.e. for each $x \in H\ \exists\, m \in \mathbb{Z}$ s.t. $x = h^m$. Then h is called a *generator* of H and we write $H = \langle h \rangle$.

Example 1.21

$(\mathbb{Z}, +) = \langle 1 \rangle = \langle -1 \rangle$, the infinite cyclic group. We showed all subgroups of $(\mathbb{Z}, +)$ are cyclic in Proposition 1.1.

Example 1.22

$(\{\pm 1\}, +) = \langle -1 \rangle$

Example 1.23

$(\{0, 1, 2, 3\}, +_4) = \langle 1 \rangle = \langle 3 \rangle$

Note a cyclic group is abelian.

Definition 1.13 (Order of element)
Let G be a group and $g \in G$. The *order* of g, written as $o(g)$, is the least positive integer n such that $g^n = e$, if it exists. Otherwise g has infinite order.

Lemma 1.5
Suppose G is a group, $g \in G$ and $o(g) = m$. Let $n \in \mathbb{N}_{>0}$. Then

$$g^n = e \iff m \mid n$$

Proof. ($\Longleftarrow$) Suppose $m \mid n$, then $n = qm$ for some $q \in \mathbb{N}$.

$$\implies g^n = g^{qm} = (g^m)^q = e^q = e$$

($\implies$) Suppose $g^n = e$. Write $n = qm + r$ with $0 \leq r < m$, $q \in \mathbb{N}$.

$$\begin{aligned}
\implies \qquad e = g^n &= g^{qm+r} \\
&= (g^m)^q + g^r \\
&= e^q g^r \\
&= e g^r \\
&= g^r \\
\implies \qquad & r = 0 \text{ by minimality of m} \implies n = qm \text{ as required.}
\end{aligned}$$

□

Remark 7.

i. Suppose $g \in G$. Then $\{g^n : n \in \mathbb{Z}\}$ is a subgroup of G, in fact it is the smallest subgroup of G containing g. We call it the subgroup of G generated by g and write $\langle g \rangle = \{g^n : n \in \mathbb{Z}\}$. Also $|\langle g \rangle| = o(g)$ if finite. Since if $o(g) = m$, $\langle g \rangle = \{e, g, g^2, \ldots, g^{m-1} = g^{-1}\}$. Otherwise both infinite.

ii. We can define the abstract cyclic group of order n

$$C_n = \langle x \rangle, \quad \text{where } o(x) = n.$$

Then $(\{0, 1, \ldots, n-1\}, +_n)$ and $(\{n^{\text{th}} \text{ roots of unity}\}, \times)$ are realisations of this group, they are all isomorphic.

iii. Let G be a group and $g_1, \ldots, g_k \in G$. Then the subgroup of G generated by $g_1, \ldots, g_k$ denoted $\langle g_1, \ldots, g_k \rangle$ is the smallest subgroup of G containing all the g_i's. It is the intersection of all the subgroups of G containing all the g_i's.

§2 The Dihedral and Symmetric Groups

First note composition of functions is associative

$$\begin{aligned} f, g, h : X \to X,& \text{ let } x \in X \\ (f \circ (g \circ h))(x) &= f((g \circ h)(x)) \\ &= f(g(h(x))) \\ &= (f \circ g)(h(x)) \\ &= ((f \circ g) \circ h)(x) \\ \implies f \circ (g \circ h) &= (f \circ g) \circ h. \end{aligned}$$

§2.1 Dihedral Groups

Definition 2.1 (Dihedral groups D_{2n})
Let P be a regular polygon with n sides and V its set of vertices. We can assume

$$V = \{e^{2\pi i k/n} : 0 \leq k < n\}$$

(nth roots of unity in $\mathbb{C}$). Then the symmetries of P are the isometries (i.e. distance preserving maps of $\mathbb{C}$ that map V to V).

We will show that: for $n \geq 3$ the set of symmetries of P, under composition, form a nonabelian group of order $2n$. This group is called the *dihedral group* of order $2n$ and denoted D_{2n}.

Warning - sometimes D_{2n} is denoted D_n

We have already met D_6 in Example 1.8.

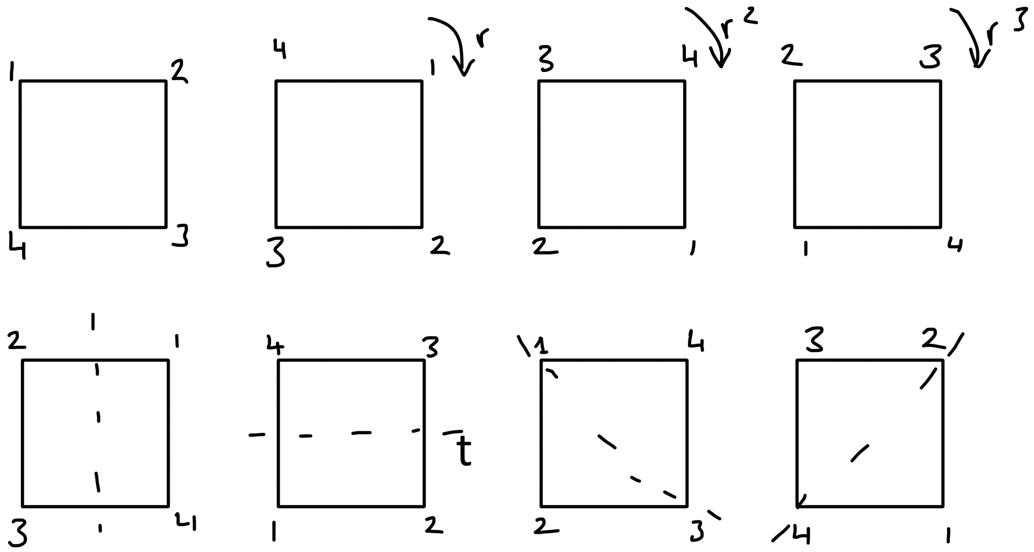
Consider D_8[1]

$$\begin{aligned} \text{Let } r : P &\to P \\ z &\mapsto e^{2\pi i/n} z \\ t : P &\to P \\ z &\mapsto \overline{z} \end{aligned}$$

These are both isometries

$$|r(z) - r(w)| = |e^{2\pi i/n} z - e^{2\pi i/n} w|$$

[1] The subgroup generated by r^2 and t includes the identity, as they are self inverses so no need for inverses, and by closure $r^2 t = t r^2 t$ and so it is a subgroup of order 4.



$$
\begin{aligned}
&= |e^{2\pi i/n}||z-w| \\
&= |z-w| \\
|t(z)-t(w)|^2 &= |\overline{z}-\overline{w}|^2 \\
&= (\overline{z}-\overline{w})(z-w) \\
&= |z-w|^2 \\
\implies |t(z)-t(w)| &= |z-w|
\end{aligned}
$$

$$
\begin{aligned}
\text{Note, } r^n &= \text{id} = \text{identity} \\
\implies r^{-1} &= r^{n-1} \\
t^2 &= \text{id} \\
\implies t^{-1} &= t \\
t * r * z &= e^{-2\pi i/n}\ \overline{z} = r^{-1} * t * z \\
\implies tr &= r^{-1}t
\end{aligned}
$$

We show that the set of symmetries of P is precisely $\{e, \underbrace{r, r^2, \ldots, r^{n-1}}_{\text{rotations}}, \underbrace{t, rt \ldots, r^{n-1}t}_{\text{reflections}}\}$.

Then this set under composition of functions gives the group D_{2n}.

Let f be a symmetry of P. Then $f(1) = e^{2\pi ik/n}$ for some k ($f(1)$ is mapping 1 to some vertex, so some element of V).

$$
\implies \qquad \underbrace{r^{-k} \circ f}_{g,\ \text{symmetry of } P \text{ fixing } 1} \qquad (1) = 1
$$

$$g = r^{-k} \circ f$$

So, $g(e^{2\pi i/n}) = e^{2\pi i/n}$ or $e^{-2\pi i/n}$ as the vertices next to 1 stay next to it.

If $g(e^{2\pi i/n}) = e^{2\pi i/n}$ then g fixes the points 1 and $e^{2\pi i/n}$. Also g interchanges the vertices of P so fixes P's centre of mass

$$\frac{1}{n} \sum_{k=0}^{n-1} e^{2\pi ik/n} = 0$$

So g fixes $0, 1$ and $e^{2\pi i/n} \implies g = \mathrm{id} \implies f = r^k$.

If $g(e^{2\pi i/n}) = e^{-2\pi i/n}$ then $t \circ g(e^{2\pi i/n}) = e^{2\pi i/n}$

$$\begin{aligned}
& & t \circ g(e^{2\pi i/n}) &= e^{2\pi i/n} \\
& & t \circ g(1) &= 1 \\
& & t \circ g(0) &= 0 \\
&\implies & t \circ g &= \mathrm{id} \\
&\implies & t \circ r^{-k} \circ f &= \mathrm{id} \\
&\implies & f &= r^k \circ t^{-1} \\
& & &= r^k \circ t
\end{aligned}$$

Algebraically we write,

$$D_{2n} = \left\langle \underbrace{r,\ t}_{\text{generators}} \,\middle|\, \underbrace{r^n = e,\ t^2 = e,\ trt = r^{-1}}_{\text{relations}} \right\rangle$$

Finally, $D_2 \cong C_2$ and D_4 is Example 1.7. Both are abelian. Also D_∞ exists.

§2.2 Symmetric Groups

Definition 2.2 (permutation)
Let X be a set. A bijection

$$f : X \to X$$

is called a *permutation* of X. Let $\mathrm{Sym}\, X$ denote the set of all permutations of X.

Proposition 2.1
$\mathrm{Sym}\, X$ is a group under composition of functions. It is called the symmetric group

on X.

Proof.

- closure - See Lemma 1.2 (Number's and Sets).
- identity, define $\iota(x) = x \ \forall \ x \in X$.
- Let $f \in \mathrm{Sym}(X)$. As f is a bijection, f^{-1} exists and is a bijection and satisfies $f \circ f^{-1} = \iota = f^{-1} \circ f$.
- composition of functions is associative.

$\square$

Notation. Suppose X is finite, $|X| = n$. Then we often take X to be the set $\{1, \dots, n\}$ and we write S_n for $\mathrm{Sym}\, X$. We call S_n the symmetric group of degree n.

Notation (Double row notation)**.** We'll use double row notation (for now).

If $\sigma \in S_n$ write

$$G = \begin{pmatrix} 1 & 2 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & \sigma(n) \end{pmatrix}$$

Example 2.1

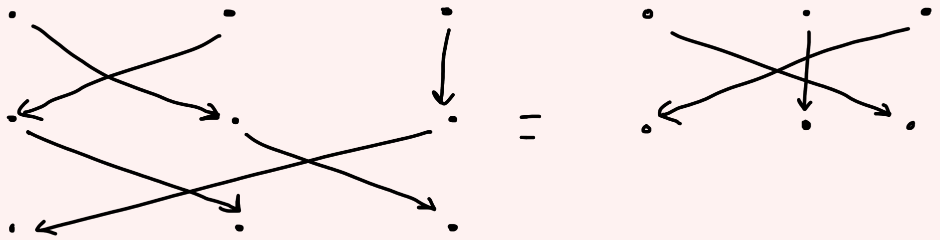
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \in S_5$$

Example 2.2

Composition:

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} &= \text{“} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \text{”} \\ &= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \end{aligned}$$

or:



§2.2.1 Small n

Example 2.3

$$
\begin{aligned}
S_1 &= \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} = \{\iota\} \quad \text{trivial group} \\
S_2 &= \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\} \\
&\cong (\{\pm 1\}, \times) \cong C_2 \\
S_3 &= \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \right\} \\
&\cong D_6
\end{aligned}
$$

Remark 8.

i. $|S_n| = n!$

ii. For $n \geq 3$ S_n is not abelian. Consider

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 3 & 1 & 4 & \dots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 1 & 3 & 2 & 4 & \dots & n \end{pmatrix}$$

They don't commute in S_3 so they won't in S_n.

iii. D_{2n} naturally embeds in S_n. e.g. $D_8 \lesssim S_4$ (isomorphic to a subgroup)

$$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \ t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

'Double row notation is cumbersome and hides what's going on, so we introduce cycle notation'.

Definition 2.3 (Cycle notation)

Let $a_1, \ldots, a_k$ be distinct integers in $\{1, \ldots, n\}$. Suppose $\sigma \in S_n$ and

$$\begin{aligned} \sigma(a_i) &= a_{i+1} \quad 1 \leq 1 \leq k-1 \\ \sigma(a_k) &= a_1 \end{aligned}$$

and $\sigma(x) = x \ \forall \ x \in \{1, \ldots, n\} \setminus \{a_1, \ldots, a_k\}$. Then σ is a *k-cycle* and we write $\sigma = (a_1, a_2, \ldots, a_k)$. e.g.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

is the 3-cycle $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ i.e. $1 \mapsto 2,\ 2 \mapsto 3,\ 3 \mapsto 1$ and all of numbers map to themselves.

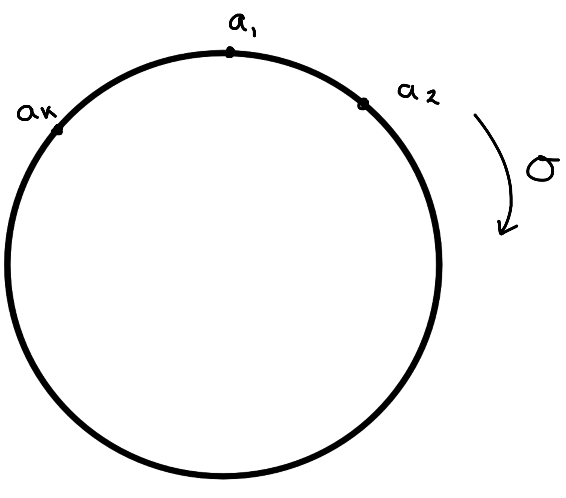
Remark 9.

i.
$$\begin{aligned} (a_1, a_2, \ldots, a_k) &= (a_k, a_1, a_2, \ldots, a_{k-1}) \\ &= \ldots \end{aligned}$$

We usually write the smallest a_i first.

ii.
$$(a_1, a_2, \ldots, a_k)^{-1} = (a_1, a_k, a_{k-1}, \ldots, a_2).$$

iii. $o(\sigma) = k$, σ is like the rotations of k points.



iv. a 2-cycle is called a *transposition*.

Definition 2.4 (Disjoint cycles)

Two cycles $\sigma = (a_1, \ldots, a_k)$ and $\tau = (b_1, \ldots, b_l)$ are *disjoint* if $\{a_1, \ldots, a_k\} \wedge \{b_1, \ldots, b_k\} = \emptyset$.

Lemma 2.1

If $\sigma, \tau \in S_n$ are disjoint then $\sigma\tau = \tau\sigma$

Proof. If $x \in 1, \ldots, n \setminus \{a_1, \ldots, a_k\} \vee \{b_1, \ldots, b_k\}$, $(\sigma \circ \tau)(x) = \sigma(\tau(x)) = (\tau \circ \sigma)(x)$.

Suppose $1 \leq i \leq k-1$

$$\begin{aligned}(\sigma \circ \tau)(a_i) &= \sigma(\tau(a_i)) \\ &= \sigma(a_i) = a_{i+1} \\ (\tau \circ \sigma)(a_i) &= \tau(\sigma(a_i)) \\ &= \tau(a_{i+1}) = a_{i+1}\end{aligned}$$

And $\sigma \circ \tau(a_k) = a_1$, $\tau \circ \sigma(a_k) = a_1$.

Similarly for b_j $\tau \circ \sigma(b_j) = \sigma \circ \tau(b_j)$

Thus $\sigma \circ \tau$ and $\tau \circ \sigma$ agree everywhere $\implies \sigma \circ \tau = \tau \circ \sigma$. $\square$

Example 2.4

$\begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4 & 5\end{pmatrix} = \begin{pmatrix}3 & 4 & 5\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix}$

However this is not necessarily true if two cycles are not disjoint.

Example 2.5

$$\begin{aligned}\sigma = \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \ \tau = \begin{pmatrix}2 & 4\end{pmatrix} \\ \sigma \circ \tau(1) = \sigma(1) = 2 \\ \sigma \circ \tau(2) = \sigma(4) = 4 \\ \sigma \circ \tau(3) = \sigma(3) = 1 \\ \sigma \circ \tau(4) = \sigma(2) = 3 \\ \sigma \circ \tau = \begin{pmatrix}1 & 2 & 4 & 3\end{pmatrix} \\ \text{But } \tau \circ \sigma = \begin{pmatrix}1 & 4 & 2 & 3\end{pmatrix}\end{aligned}$$

Example 2.6

$$\begin{aligned}\begin{pmatrix}1 & 2 & 3\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix} &= \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3\end{pmatrix}\\ &= \begin{pmatrix}1 & 2\end{pmatrix} \text{ suppress 1-cycles.}\\ \begin{pmatrix}2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix} &= \begin{pmatrix}1 & 3\end{pmatrix}\end{aligned}$$

Theorem 2.1

Every permutation can be written as a product of disjoint cycles (in an essentially unique way).

Example 2.7

$$\begin{aligned}\sigma &= \begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ 2 & 4 & 5 & 7 & 6 & 3 & 1 & 9 & 8\end{pmatrix}\\ &= \begin{pmatrix}1 & 2 & 4 & 7\end{pmatrix}\begin{pmatrix}3 & 5 & 6\end{pmatrix}\begin{pmatrix}8 & 9\end{pmatrix}\end{aligned}$$

Proof. Let $a_1 \in \{1, 2, \dots, n\} = X$. Consider $a_1, \sigma(a_1), \sigma^2(a_1), \dots$
Since X is finite $\exists$ minimal j such that $\sigma^j(a_1) \in \{a_1, \sigma(a_1), \sigma^2(a_1), \dots, \sigma^{j-1}(a_1)\}$.
We claim: $\sigma^j(a_1) = a_1$. Since if not

$$\begin{aligned}\sigma^j(a_1) &= \sigma^i(a_1),\ j > i \geq 1\\ \implies \sigma^{j-i}(a_1) &= a_1 \quad \text{of minimality of j.}\end{aligned}$$

So $\{a_1, \sigma(a_1), \sigma^2(a_1), \dots, \sigma^{j-1}(a_1)\}$ is a cycle in σ.
If $\exists\, b \in X \setminus \{a_1, \sigma(a_1), \sigma^2(a_1), \dots, \sigma^{j-1}(a_1)\}$. Consider $b, \sigma(b), \dots$
Note $(b, \sigma(b), \sigma^2(b), \dots, \sigma^{j-1}(b))$ is disjoint from $(a_1, \sigma(a_1), \sigma^2(a_1), \dots, \sigma^{j-1}(a_1))$ because σ is a bijection.[a]
Continue in this way until all elements of X are reached. □

[a] If $\sigma^i(b) = \sigma^j(a_1)$ then $b = \sigma^{j-i}(a_1)$ which contradicts $b \in X \setminus \{a_1, \sigma(a_1), \sigma^2(a_1), \dots, \sigma^{j-1}(a_1)\}$.

Lemma 2.2

Let σ, τ be disjoint cycles in S_n. Then $o(\sigma\tau) = \operatorname{lcm}\{o(\sigma), o(\tau)\}$

Proof. Let $k = \operatorname{lcm}\{o(\sigma), o(\tau)\}$ so $o(\sigma) \mid k$ and $o(\tau) \mid k$. Then

$$(\sigma\tau)^k = \sigma\tau\sigma\tau \dots \sigma\tau$$

$$
\begin{aligned}
&= \sigma^k \tau^k \textit{ Lemma } 2.1 \\
&= e \cdot e \textit{ Lemma } 1.5 \\
&= e \\
\implies & o(\sigma\tau) \mid k \textit{ Lemma } 1.5 \qquad (1)
\end{aligned}
$$

Now suppose $o(\sigma\tau) = n$ so $(\sigma\tau)^n = e \implies \sigma^n\tau^n = e$. But σ, τ move different elements of $X \implies \sigma^n = e,\ \tau^n = e$. By Lemma 1.5

$$
\begin{aligned}
&\implies o(\sigma) \mid n \text{ and } o(\tau) \mid n \\
&\implies k = \operatorname{lcm}\{o(\sigma), o(\tau)\} \\
&\qquad \mid n = o(\sigma\tau) \qquad (2) \\
\textit{eq. } (1), \textit{eq. } (2) &\implies o(\sigma\tau) = \operatorname{lcm}\{o(\sigma), o(\tau)\}
\end{aligned}
$$

□

Proposition 2.2

Any $\sigma \in S_n$ $(n \geq 2)$ can be written as a product of transpositions.

Proof. By Theorem 2.1 it is enough to show a k-cycle can be written as a product of transpositions.

$$
\begin{pmatrix} a_1 & a_2 & \dots & a_k \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \end{pmatrix} \begin{pmatrix} a_2 & a_3 \end{pmatrix} \dots \begin{pmatrix} a_{k-2} & a_{k-1} \end{pmatrix} \begin{pmatrix} a_{k-1} & a_k \end{pmatrix}
$$

□

Example 2.8

$$
\begin{aligned}
\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix} &= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix} \begin{pmatrix} 4 & 5 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix}
\end{aligned}
$$

not unique.

Definition 2.5 (Sign of permutation)

Let $\sigma \in S_n$ $(n \geq 2)$. Then the *sign* of σ, written $\operatorname{sgn}(\sigma)$, is $(-1)^k$ is the number of transpositions in some expression of σ as a product of transpositions.

Lemma 2.3

The function

$$\begin{aligned}\operatorname{sgn} : S_n &\to \{\pm 1\}\\ \sigma &\mapsto \operatorname{sgn}(\sigma)\end{aligned}$$

is well defined.
i.e. if $\sigma = \tau_1 \dots \tau_a = \tau_1' \dots \tau_b'$ with τ_i and τ_i' transpositions then $(-1)^a = (-1)^b$.

$$\begin{aligned}\operatorname{sgn}\left(\begin{pmatrix}1 & 2 & 3 & 4 & 5\end{pmatrix}\right) &= (-1)^4 = (-1)^6\\ &= 1\end{aligned}$$

Proof. Let $c(\sigma)$ denote the number of cycles in a disjoint cycle decomposition of σ including 1-cycles, so $c(\mathrm{id}) = n$.
Let τ be a transposition. Claim: $c(\sigma\tau) = c(\sigma) \pm 1 \equiv c(\sigma) + 1 \mod 2$.
Let $\tau = \begin{pmatrix}k & l\end{pmatrix}$.
2 cases:

i. k, l lie in different cycles of σ:

$$\begin{aligned}\begin{pmatrix}k & a_1 & \dots & a_r\end{pmatrix}\begin{pmatrix}l & b_1 & \dots & b_s\end{pmatrix}\begin{pmatrix}k & l\end{pmatrix} &= \begin{pmatrix}k & b_1 & b_2 & \dots & b_s & l & a_1 & \dots & a_r\end{pmatrix}\\ \implies c(\sigma\tau) &= c(\sigma) - 1\end{aligned}$$

ii. i. k, l lie in the same cycle in σ:

$$\begin{aligned}\begin{pmatrix}k & a_1 & \dots & a_r & l & b_1 & \dots & b_s\end{pmatrix}\begin{pmatrix}k & l\end{pmatrix} &= \begin{pmatrix}k & b_1 & b_2 & \dots & b_s\end{pmatrix}\begin{pmatrix}l & a_1 & \dots & a_r\end{pmatrix}\\ \implies c(\sigma\tau) &= c(\sigma) + 1\end{aligned}$$

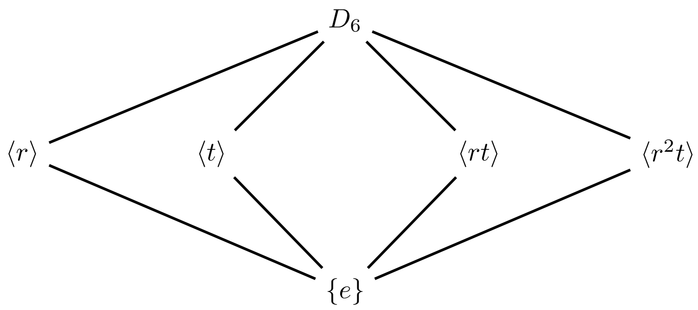
Assume

$$\begin{aligned}\sigma &= \mathrm{id}\tau_1 \dots \tau_a\\ &= \mathrm{id}\tau_1' \dots \tau_b'\\ \implies c(\sigma) &\equiv n + a \mod 2\\ &\equiv n + b \mod 2\\ \implies a &\equiv b \mod 2\\ \implies (-1)^a &= (-1)^b\end{aligned}$$

$\square$

Aside: Subgroup Lattices

Subgroup lattice of $D_6 = \{e, r, r^2, t, rt, r^2t\}$



We put the largest subgroups at the top and work our way down.

Theorem 2.2

Let $n \geq 2$. The map

$$\begin{aligned} \operatorname{sgn} : (S_n, \circ) &\to (\{\pm 1\}, \times) \\ \sigma * &\mapsto \operatorname{sgn}(\sigma) \end{aligned}$$

is a well-defined non-trivial (doesn't just map to identity) homomorphism.

Proof.

- We know its well-defined by Lemma 2.3
- $\operatorname{sgn}\left(\begin{pmatrix}1 & 2\end{pmatrix}\right) = -1$, so non-trivial
- homomorphism:

Let $\alpha, \beta \in S_n$ with $\operatorname{sgn}(\alpha) = (-1)^k$, $\operatorname{sgn}(\alpha) = (-1)$. So $\exists$ transpositions τ_i and τ_i' such that $\alpha = \tau_1 \dots \tau_k$ and $\beta = \tau_1' \dots \tau_l'$. This implies

$$\begin{aligned} \alpha\beta &= \tau_1 \dots \tau_k \tau_1' \dots \tau_l' \\ \implies \operatorname{sgn}(\alpha\beta) &= (-1)^{k+l} \\ &= (-1)^k (-1)^l \end{aligned}$$

$$= \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)$$

□

Definition 2.6

σ is an *even* permutation if $\operatorname{sgn}(\sigma) = 1$ and an odd permutation if $\operatorname{sgn}(\sigma) = -1$.

Corollary 2.1

The even permutations of S_n ($n \geq 2$) form a subgroup called the *alternating group* and we denote it by A_n.

Proof.

- $\mathrm{id} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} \in A_n$
- $$\begin{aligned} & \operatorname{sgn}(\sigma) = 1 = \operatorname{sgn}(\rho) \\ \implies & \operatorname{sgn}(\sigma\rho) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\rho) \\ & = 1 \end{aligned}$$

  by Theorem 2.2
- $\sigma = \tau_1 \dots \tau_k$ where τ_i are transpositions. Then $\sigma^{-1} = \tau_k \dots \tau_1 \implies \operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$
- associativity is inherited

□

Example 2.9

$A_4 = \{e, \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}3 & 4\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}2 & 4\end{pmatrix}, \begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix},$ $\begin{pmatrix}1 & 2 & 4\end{pmatrix}, \begin{pmatrix}1 & 4 & 2\end{pmatrix}, \begin{pmatrix}2 & 3 & 4\end{pmatrix}, \begin{pmatrix}2 & 4 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 4\end{pmatrix}, \begin{pmatrix}1 & 4 & 3\end{pmatrix}\}$.

Remark 10.

i. $|A_n| = \frac{|S_n|}{2} = \frac{n!}{2}$ (will be proved later on)

ii. cycles of even length are odd and cycles of odd length are even.

iii. $A_n = \ker(\operatorname{sgn})$, hence a subgroup (q9 - sheet 1).

§3 Cosets and Lagrange

Definition 3.1 (Cosets)

Let $H \leq G$ and $g \in G$. The *left coset* gH is defined to be $\{gh : h \in H\}$. Similarly the *right coset* $Hg = \{hg : h \in H\}$

Example 3.1

$$
\begin{aligned}
S_3 &= \{e, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}, \begin{pmatrix}2 & 3\end{pmatrix}\} \\
H &= \{\mathrm{id}, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}\} = A_3 \\
\begin{pmatrix}1 & 2\end{pmatrix} H &= \left\{\begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}1 & 3 & 2\end{pmatrix}\right\} \\
&= \left\{\begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}\right\} \\
\begin{pmatrix}1 & 2 & 3\end{pmatrix} H &= H \text{ (since H is a subgroup)}
\end{aligned}
$$

Note, $H \mathbin{\dot\cup} \begin{pmatrix}1 & 2\end{pmatrix} H = S_3$

Lemma 3.1

Let $H \leq G$ and $g \in G$. Then there is a bijection between H and gH. In particular if H is finite then $|H| = |gH|$.

Proof.

$$
\begin{aligned}
\text{Define } \theta_g : H &\to gH \\
h &\mapsto gh.
\end{aligned}
$$

We show θ_g is a bijection.

Surjectivity: if $gh \in gH$ then $\theta_g(h) = gh$.

Injectivity:

$$
\begin{aligned}
\theta_g(h_1) &= \theta_g(h_2) \\
gh_1 &= gh_2 \\
\implies h_1 &= h_2 \text{ (cancellation law)}
\end{aligned}
$$

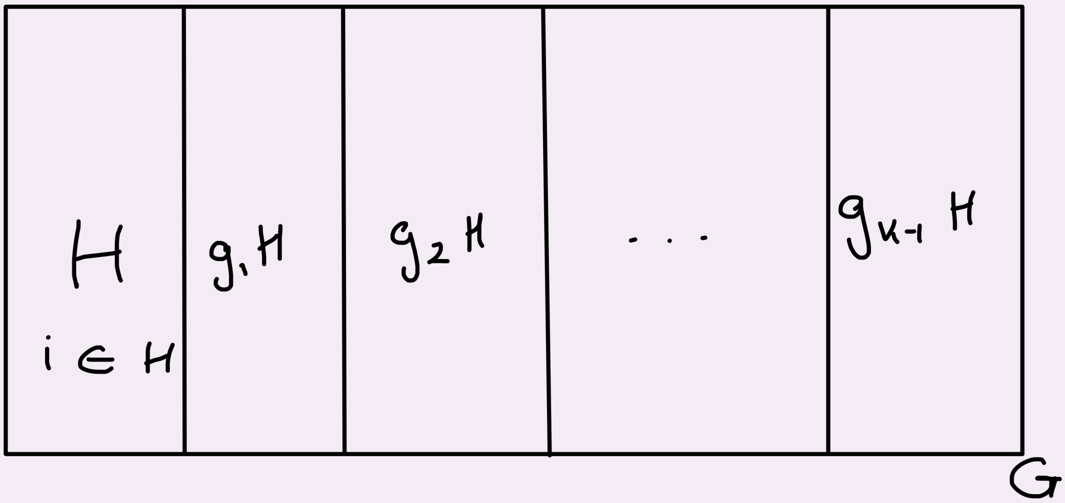
$\square$

Lemma 3.2

The left cosets of H in G form a partition of G i.e.

i. each $g \in G$ lies in some left coset of H in G.

ii. if $aH \cap bH \neq \emptyset$ (for some $a, b \in G$) $\implies aH = bH$.



Proof.

i. $g \in gH$ as $e \in H$

ii. Suppose $c \in aH \cap bH$.
Claim: $aH = cH = bH$.

Now $c \in aH$ so $c = ak$ for some $k \in H$.

$$\begin{aligned} \implies cH &= \{ch : h \in H\} \\ &= \{akh : h \in H\} \subseteq aH. \end{aligned}$$

Similarly, $a = ck^{-1} \in cH$

$$\implies aH \subseteq cH.$$

So $aH = cH$.
Similarly $cH = bH$ □

Example 3.2

$$S_n = \underbrace{A_n}_{\text{even elements}} \,\dot{\cup}\, \underbrace{\begin{pmatrix}1 & 2\end{pmatrix} A_n}_{\text{odd elements}}$$

Lemma 3.3

Let $H \leq G$, $a, b \in G$. Then $aH = bH \iff a^{-1}b \in H$.

Proof. ($\implies$):

$$\begin{aligned} & b \in bH = aH \\ \implies & b = ah \text{ for some } h \in H \\ \implies & a^{-1}b = h \in H \end{aligned}$$

($\impliedby$):

$$\begin{aligned} & \text{Suppose } a^{-1}b = k \in H \\ \implies & b = ak \in aH \\ & \text{also } b \in bH. \\ \implies & aH = bH \end{aligned}$$

by Lemma 3.2 □

Theorem 3.1 (Lagrange's Theorem)

Let H be a subgroup of the finite group G. Then the order of H divides the order of G (i.e. $|H| \Big| |G|$).

Proof. By Lemma 3.2 G is partitioned into distinct cosets of H, say $G = g_1 H \,\dot{\cup}\, g_2 H \,\dot{\cup} \ldots \dot{\cup}\, g_k H$ (say $g_1 = e$).
By Lemma 3.1[a]

$$\begin{aligned} & |g_i H| = |H| \quad 1 \leq i \leq k \\ \implies & |G| = |H| k \end{aligned}$$

□

[a] You would need to prove these lemma's in an exam q.

Definition 3.2 (Index of a subgroup)

Let $H \leq G$. The *index* of H in G is the number of left cosets of H in G, denoted $|G:H|$.

Remark 11.

i. If G is finite, $|G:H| = \frac{|G|}{|H|}$. But we can have $|G:H|$ finite, even if G and H are infinite, e.g. $\mathbb{Z}$ and $n\mathbb{Z}$ where $|G:H| = n$.

ii. We write $(G:H)$ for the set of left cosets of H in G.

From Dexter's Notes:

The k in Lagrange's Theorem is $|G:H|$, giving item 1. 'The hard part of this proof is to prove that the left cosets partition G and have the same size. If you are asked to prove Lagrange's theorem in exams, that is what you actually have to prove.'

Corollary 3.1 (Lagrange's Corollary)

G is a finite group, $g \in G$. Then $o(g) \Big| |G|$. In particular, $g^{|G|} = e$.

Proof.

$$\text{Note } \langle g \rangle = \{e, g, \ldots, g^{n-1}\} \text{ where } o(g) = n.$$

$$\text{Then } o(g) = |\langle g \rangle| \Big| |G| \text{ by Lagrange's Theorem}$$

$$\implies g^{|G|} = e \text{ by Lemma 1.5}$$

□

Corollary 3.2

If $|G| = p$ for some prime p, then G is cyclic.

Proof. Let $e \neq g \in G$. Then $\{e\} \neq \langle g \rangle \leq G$. By Lagrange's Corollary

$$1 \neq |\langle g \rangle| \Big| |G| = p$$

$$\implies |\langle g\rangle| = p = |G|$$
$$\implies \langle g\rangle = G$$

i.e. G cyclic. □

Definition 3.3 (Euler's totient function)
Let $n \in \mathbb{N}$ and $\phi(n) = |\{1 \leq a \leq n : \mathrm{hcf}(a,n) = 1\}|$.

Example 3.3
$\phi(12) = |\{1,5,7,11\}| = 4$.

Theorem 3.2 (Fermat-Euler Theorem)
Let $n \in \mathbb{N},\ a \in \mathbb{Z}$ and $\mathrm{hcf}(a,n) = 1$.
Then $a^{\phi(n)} \equiv 1 \pmod{n}$.

We can prove this by using Lagrange, but first we need to set it up.
Let $n \in \mathbb{N}$,

$$R_n = \{0,1,\dots,n-1\}$$
$$R_n^* = \{a \in R_n : \mathrm{hcf}(a,n) = 1\}$$

Notation. $n \in \mathbb{Z}$ then $\overline{u} \in R_n$ such that $u \equiv \overline{u} \pmod{n}$.

Define $\times_n$ to be multiplication mod n.
Claim: $(R_n^*, \times_n)$ is a group.

Closure:

$$\begin{aligned} &\mathrm{hcf}(a,n) = 1 = \mathrm{hcf}(b,n) \\ \implies &\mathrm{hcf}(ab,n) = 1 \\ \implies &\mathrm{hcf}(\overline{ab},n) = 1 \end{aligned}$$

Identity $= 1$

Associativity is fine.

Inverses: Let $a \in R_n^*$, $\mathrm{hcf}(a,n) = 1$

$$\implies \exists\, u,v \in \mathbb{Z} \text{ s.t. } au + vn = 1 \text{ (Bezout's Theorem)}$$
$$\implies au \equiv 1 \pmod{n}$$

Then $\overline{u} \in R_n^*$ is a^{-1}

Proof. Note $|R_n^*| = \phi(n)$.

$$a \equiv \overline{a} \pmod{n},\ \overline{a} \in R_n^*$$

By Corollary 3.1

$$\begin{aligned} &\overline{a}^{\phi(n)} = \overline{a}^{|R_n^*|} = 1 \in R_n^* \\ \implies &a^{\phi(n)} \equiv 1 \pmod{n} \end{aligned}$$

$\square$

§4 Normal Subgroups, Quotient groups and Homomorphisms

Given a group G, subgroup H of G and the set of left cosets of H in G, $(G : H)$. We would like to define a group operation on the cosets $\circ$, so that $((G : H), \circ)$ is a group. Would like:

$$(gH) \circ (kH) = gkH.$$

When does this work? Consider $gHkH$ if $Hk = kH$ then we get $gkHH = gkH$.
This motivates the following definition.

Definition 4.1 (Normal subgroup)
A subgroup K of G is called *normal* if $gK = Kg$ for all $g \in G$. We write $K \trianglelefteq G$.

Example 4.1

$$\begin{aligned}
K = \{\mathrm{id}, \begin{pmatrix}1 & 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}\} &\trianglelefteq S_3 \\
\begin{pmatrix}1 & 2\end{pmatrix} K = \{\begin{pmatrix}1 & 2\end{pmatrix}, \begin{pmatrix}2 & 3\end{pmatrix}, \begin{pmatrix}1 & 3\end{pmatrix}\} &= K \begin{pmatrix}1 & 2\end{pmatrix} \\
\begin{pmatrix}1 & 3\end{pmatrix} K &= K \begin{pmatrix}1 & 3\end{pmatrix} \\
\begin{pmatrix}2 & 3\end{pmatrix} K &= K \begin{pmatrix}2 & 3\end{pmatrix} \\
\begin{pmatrix}1 & 2 & 3\end{pmatrix} K = K &= K \begin{pmatrix}1 & 2 & 3\end{pmatrix}
\end{aligned}$$

But $H = \{1, \begin{pmatrix}1 & 2\end{pmatrix}\}$ is not normal in S_3.

$$\begin{aligned}
\begin{pmatrix}1 & 3\end{pmatrix} H &= \{\begin{pmatrix}1 & 3\end{pmatrix}, \begin{pmatrix}1 & 2 & 3\end{pmatrix}\} \\
H \begin{pmatrix}1 & 3\end{pmatrix} &= \{\begin{pmatrix}1 & 3\end{pmatrix}, \begin{pmatrix}1 & 3 & 2\end{pmatrix}\}
\end{aligned}$$

Proposition 4.1
Let $K \trianglelefteq G$. TFAE (The following are equivalent):

i. $gK = Kg \;\forall\, g \in G$
ii. $gKg^{-1} = K \;\forall\, g \in G$
iii. $gkg^{-1} \in K \;\forall\, k \in K,\; g \in G$

Proof.

$$\begin{aligned}(i) &\implies (ii)\\ gKg^{-1} &= \{gkg^{-1} : k \in K\}\\ &= (gK)g^{-1}\\ &= (Kg)g^{-1}\\ &= K.\end{aligned}$$

$(ii) \implies (iii)$

$(iii) \implies (i)$
For any $k \in K,\ g \in G\ \exists\, k' \in K$ s.t. $gkg^{-1} = k'$

$$\begin{aligned}&\implies gk = k'g \in Kg\\ &\implies gK \subseteq Kg\end{aligned}$$

Similarly $g^{-1}kg = k''$ for some $k'' \in K$

$$\begin{aligned}&\implies kg = gk''\\ &\implies Kg \subseteq gK\\ &\implies gK = Kg.\end{aligned}$$

□

Example 4.2

- $\{e\} \trianglelefteq G,\ G \trianglelefteq G$.
- If G is abelian, all subgroups are normal. Since if $k \in K,\ g \in G$ and $K \leq G$ then $gkg^{-1} = gg^{-1}k = k \in K$.
- Kernels of homomorphisms are normal subgroups (Sheet 1, q9) $\implies A_n \trianglelefteq S_n$ since $A_n = \ker sgn$
- $D_{2n} = \left\langle \underbrace{r,\ t}_{\text{generators}} \middle| \underbrace{r^n = e,\ t^2 = e,\ trt = r^{-1}}_{\text{relations}} \right\rangle$. Then $\langle r \rangle \trianglelefteq D_{2n}$.

$$\begin{aligned}\text{Clearly } r^i r^j r^{-i} &= r^j \in \langle r \rangle.\\ \text{Also, } (r^i t) r^j (r^i t)^{-1} &= r^i t r^j t r^{-i}\\ &= r^i r^{-j} t t r^{-i} \text{ as } tr^k = r^{-1}tr^{k-1} ... = r^{-k}t\\ &= r^{-j} \in \langle r \rangle.\end{aligned}$$

Or use the following lemma.

Lemma 4.1
If $K \leq G$ and the index, Definition 3.2, of K in G is 2, then $K \trianglelefteq G$.

Proof.

$$G = K \,\dot{\cup}\, gK \text{ for any } g \in G \setminus K$$

as g is in gK but not K so $gK \neq K$

$$= K \,\dot{\cup}\, Kg \text{ by Lemma 3.2}$$

as g is in Kg but not K so $Kg \neq K$

$$\implies gK = Kg \;\forall\, g \in G, \text{ as if } g \in K \text{ then we just get } K = K.$$

□

Theorem 4.1
If $K \trianglelefteq G$, the set $(G : K)$ of the left cosets of K in G is a group under coset multiplication i.e. $gK \circ hK = ghK$.
This group is called the *quotient group* (or factor group) of G by K and denoted G/K.

Proof. We need to check that coset multiplication is well defined.

$$\begin{gathered}\text{i.e. } gK = \hat{g}K \\ \text{and } hK = \hat{h}K \\ \text{then } ghK = \hat{g}\hat{h}K.\end{gathered}$$

By Lemma 3.3,

$$\begin{aligned} gK = \hat{g}K &\implies \hat{g}^{-1}g \in K \\ hK = \hat{h}K &\implies \hat{h}^{-1}h \in K \\ \text{Now } \hat{g}^{-1}g \in K & \\ \implies h^{-1}\hat{g}^{-1}gh &\in K \text{ since } K \trianglelefteq G \\ \implies (\hat{h}^{-1}h)(h^{-1}\hat{g}^{-1}gh) &\in K \\ \implies \hat{h}^{-1}\hat{g}^{-1}gh &\in K \end{aligned}$$

$$\implies ghK = \hat{g}\hat{h}K \text{ by Lemma 3.3.}$$

So coset multiplication is well-defined. Group axioms now follow easily.

- By construction coset multiplication is closed as $ghK \in (G : H)$ for $g, h \in G$.
- Identity given by $eK = K$
- $(gK)^{-1} = g^{-1}K$
- Associativity holds since it does in G, to check: $(gKhK)lK = (gh)lK = g(hl)K = gK(hKlK)$.

□

Example 4.3

i.

$$\begin{aligned} S_n/A_n &= \left(\left\{A_n, \begin{pmatrix}1 & 2\end{pmatrix} A_n\right\}, \circ\right) \\ &\cong C_2. \end{aligned}$$

ii.

$$D_8 = \langle a, b : a^4 = e = b^2,\ bab = a^{-1}\rangle$$

Let $K = \{1, a^2\}$.
Claim: $K \trianglelefteq D_8$.

$$\begin{aligned} (a^i b)a^2(a^i b)^{-1} &= a^i ba^2 ba^{-i} \\ &= a^i a^{-2} bba^{-i} \text{ as } ba^2 = a^{-1}ba = a^{-2}b \\ &= a^{-2}a^2 \in K \\ a^i a^2 a^{-i} &= a^2 \in K. \\ |D_8|/|K| &= 4 = |(D_8 : K)|. \end{aligned}$$

4 distinct left cosets:

$$\begin{aligned} K &= \{1, a^2\} \\ aK &= \{a, a^3\} \\ bK &= \{b, ba^2\} = \{b, a^2 b\} \\ abK &= \{ab, aba^2\} = \{ab, a^3 b\}. \end{aligned}$$

$\circ$	K	aK	bK	abK
K	K	aK	bK	abK
aK	aK	K	abK	bK
bK	bK	abK	K	aK
abK	abK	bK	aK	K

Note $aKaK = a^2K = K$ This is isomorphic to Example 1.7.

iii. Recall the subgroups of $(\mathbb{Z}, +)$ are precisely the groups $(n\mathbb{Z}, +)$ where $n \in N \cup \{0\}$, $n\mathbb{Z} = \{nk : k \in \mathbb{Z}\}$. Since $(\mathbb{Z}, +)$ is abelian, all subgroups are normal, $n\mathbb{Z} \trianglelefteq Z$.
Suppose $n = 5$, cosets given by,

$$\begin{aligned}
5\mathbb{Z} &= \{5k : k \in \mathbb{Z}\} \\
1 + 5\mathbb{Z} &= \{1 + 5k : k \in \mathbb{Z}\} \\
2 + 5\mathbb{Z} &= \{2 + 5k : k \in \mathbb{Z}\} \\
3 + 5\mathbb{Z} &= \{3 + 5k : k \in \mathbb{Z}\} \\
4 + 5\mathbb{Z} &= \{4 + 5k : k \in \mathbb{Z}\}. \\
(1 + 5\mathbb{Z}) + (2 + 5\mathbb{Z}) &= 3 + 5\mathbb{Z}. \\
(3 + 5\mathbb{Z}) + (4 + 5\mathbb{Z}) &= 2 + 5\mathbb{Z}.
\end{aligned}$$

Claim: $(\mathbb{Z}/5\mathbb{Z}, \underbrace{\circ}_{\text{coset`addition'}}) \cong (\{0, 1, 2, 3, 4\}, +_5)$[a]

We define a map $\theta : n + 5\mathbb{Z} \to \overline{n}$ where $n \equiv \overline{n} \mod 5$ and $\overline{n} \in \{0, 1, 2, 3, 4\}$.

Well-defined map:

$$\begin{aligned}
&\text{if } n + 5\mathbb{Z} = m + 5\mathbb{Z} \\
\implies &-m + n \in 5\mathbb{Z} \text{ by Lemma 3.3} \\
\implies &-m + n \equiv 0 \pmod 5 \\
&\implies n \equiv m \pmod 5 \\
&\implies \overline{n} \equiv \overline{m}
\end{aligned}$$

homomorphism:

$$\begin{aligned}
\theta\left((n + 5\mathbb{Z}) + (m + 5\mathbb{Z})\right) &= \theta(n + m + 5\mathbb{Z}) \\
&= \overline{n + m} \\
&= \overline{n} +_5 \overline{m} \\
&= \theta(n + 5\mathbb{Z}) + \theta(m + 5\mathbb{Z}).
\end{aligned}$$

In general $(\mathbb{Z}/n\mathbb{Z}, \circ) \cong (\{0, 1, \ldots, n-1\}, +_n)$.

[a]It is coset 'addition' as it is abelian

Recall the definition of a homomorphism, Definition 1.10,
$\mathrm{Im}(\theta) = \{\theta(g) : g \in G\} \leq H$ (Lemma 1.3). $\ker(\theta) = \{g \in G : \theta(g) = e_H\} \trianglelefteq G$ (Sheet 1).

Theorem 4.2 (1st Isomorphism Theorem)

Let G, H be groups and $\theta : G \to H$ a group homomorphism. Then

$$\mathrm{Im}\,\theta \leq H$$
$$\ker\theta \trianglelefteq G$$
$$\text{and } G/\ker\theta \cong \mathrm{Im}\,\theta.$$

Definition 4.2 (Simple group)

A group is called *simple* if its only normal subgroups are $\{e\}$ and G, e.g. C_p where p is prime.

Aside

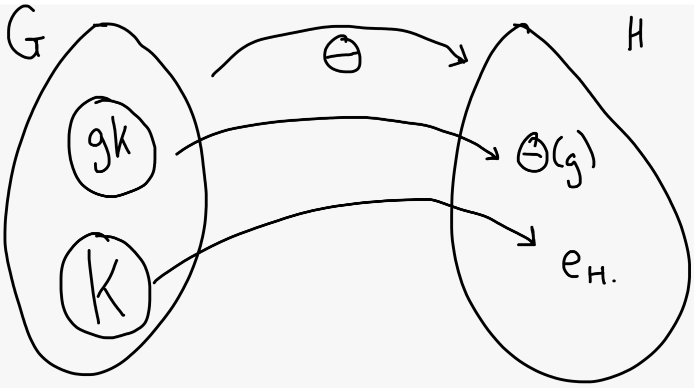
Definition 4.3

Suppose $f : A \to B$

i. f is *injective* (one-to-one) if $a_1, a_2 \in A$, $f(a_1) = f(a_2) \implies a_1 = a_2$ (each element of A maps to a different element of B).
ii. f is *surjective* (onto) if given $b \in B$ $\exists\, a \in A$ such that $f(a) = b$ (every element in B is 'hit').
iii. f is *bijective* if f is both injective and surjective.

Proof (1st Isomorphism Theorem). Let $K = \ker\theta$, we need to construct an isomorphism

$$\begin{aligned}\phi : G/\ker\theta &\to \mathrm{Im}\,\theta \\ gK &\mapsto \theta(g).\end{aligned}$$



Need ϕ well-defined:

$$\begin{aligned} &\text{Suppose } gK = hK \\ &\implies h^{-1}g \in K \text{ see Lemma 3.3} \\ &\implies \theta(h^{-1}g) = e_H \\ &\theta(h)^{-1}\theta(g) = e_H \text{ since } \theta \text{ is a homomorphism} \\ &\implies \theta(g) = \theta(h) \\ &\implies \phi(gK) = \phi(hK) \end{aligned}$$

Need ϕ to be a homomorphism:

$$\begin{aligned} \phi(gKhK) &= \phi(ghK) \\ &= \theta(gh) \\ &= \theta(g)\theta(h) \text{ since } \theta \text{ is a homomorphism} \\ &= \phi(gK)\phi(hK). \end{aligned}$$

ϕ surjective: If $\theta(g) \in \operatorname{Im}\theta$ then $\phi(gK) = \theta(g)$.

ϕ injective: Suppose $\phi(gK) = \phi(hK)$

$$\begin{aligned} &\text{Suppose } \phi(gK) = \phi(hK) \\ &\implies \theta(g) = \theta(h) \\ &\implies \theta(h)^{-1}\theta(g) = e_H \\ &\theta(h^{-1}g) = e_H \\ &\implies h^{-1}g \in K \\ &\implies gK = hK. \end{aligned}$$

□

Remark 12. By the 1st Isomorphism Theorem, $\operatorname{Im}\theta \cong G/\ker\theta$, a homomorphic image of G is isomorphic to a quotient of G.

§4.1 Examples

Example 4.4

$$\begin{aligned} \operatorname{sgn} : S_n &\to (\{\pm 1\}, \times) \\ \sigma &\mapsto \operatorname{sgn}(\sigma). \\ \operatorname{Im}(\operatorname{sgn}) &= (\{\pm 1\}, \times) \\ \ker(\operatorname{sgn}) &= A_n \\ \implies S_n/A_n &\cong (\{\pm 1\}, \times) \cong C_2 \\ \implies |A_n| &= \frac{|S_n|}{2} \end{aligned}$$

Example 4.5

$$\begin{aligned} \theta : (\mathbb{R}, +) &\to (\mathbb{C} \setminus \{0\}, \times) \\ r &\mapsto e^{2\pi i r} \\ \text{Note, } \theta(r+s) &= \theta(r)\theta(s) \\ \operatorname{Im}(\theta) &= S^1 = \{z \in \mathbb{C} : |z| = 1\} \text{ unit circle} \\ \ker(\theta) &= (\mathbb{Z}, +) \trianglelefteq (\mathbb{R}, +). \\ (\mathbb{R}, +)/(\mathbb{Z}, +) &\cong S^1. \end{aligned}$$

Example 4.6

Recall example 1.10 and 1.14.

$$GL_2(\mathbb{R}) = \{2 \times 2 \text{ matrices with entries in } \mathbb{R}, \det \neq 0\}.$$

$$\det : GL_2(\mathbb{R}) \to (\mathbb{R} \setminus, \{0\}, \times)$$

$$M \mapsto \det(M)$$

$$\det(AB) = \det A \det B \text{ so is a homomorphism.}$$

$$\operatorname{Im} \det = (\mathbb{R} \setminus \{0\}, \times)$$

$$\text{since } \det \begin{pmatrix} \alpha & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} = \alpha \in \mathbb{R} \setminus \{0\}$$

$$\ker \det = \mathrm{SL}_2(\mathbb{R}) = \{2 \times 2 \text{ matrices with entries in } \mathbb{R}, \det = 1\}.$$

$$\implies \mathrm{SL}_2(\mathbb{R}) \trianglelefteq GL_2(\mathbb{R})$$

$$\text{and } GL_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{R}) \cong (\mathbb{R} \setminus \{0\}, \times)$$

Example 4.7

$$\begin{aligned} \theta : (\mathbb{Z}, +) &\to (\{0, 1, \dots, n-1\}, +_n) \\ n &\mapsto \overline{n} \\ \ker \theta &= n\mathbb{Z}. \end{aligned}$$

Lemma 4.2

Given $K \trianglelefteq G$, the *quotient map* $q : G \to G/K$ with $g \mapsto gK$ is a surjective group homomorphism.

Proof. $q(ab) = (ab)K = aKbK = q(a)q(b)$. So q is a group homomorphism. Also for all $aK \in G/K$, $q(a) = aK$. So it is surjective. □

Note that the kernel of the quotient map is K itself.

Lemma 4.3

A homomorphism $\theta : G \to H$ is injective iff $\ker \theta = \{e_G\}$.

Proof. ($\implies$): Suppose $\theta(g) = e_H = \theta(e_G)$. So injectivity $\implies g = e_G$.

($\impliedby$):

$$\begin{aligned} \theta(g) &= \theta(h) \\ \implies \theta(h)^{-1}\theta(g) &= e_H \\ \implies \theta(h^{-1}g) &= e_H \\ \implies h^{-1}g &\in \ker \theta = \{e_G\} \end{aligned}$$

$$\begin{aligned} \implies h^{-1}g &= e_G \\ \implies h &= g. \end{aligned}$$

□

$$\begin{aligned} \text{Recall if } N \trianglelefteq G, g \in G, n \in N \\ \implies gng^{-1} \in N \\ \implies gng^{-1} = \hat{n} \text{ for some } \hat{n} \in N \\ \implies gn = \hat{n}g. \end{aligned}$$

This allows us to reorder our elements and helps in proving the following lemma.

Lemma 4.4

i. Let $N \trianglelefteq G, H \leq G$. Then $NH = \{nh : n \in N, h \in H\} \leq G$.

ii. Let $N \trianglelefteq G, M \trianglelefteq G$ then $NM \trianglelefteq G$.

Proof.

i. Closure, $nh, \overline{n}\overline{h} \in NH$

$$n \underbrace{h\overline{n}}_{\hat{n}h} \overline{h} = n\hat{n}h\overline{h} \in NH$$

id $= e = \exists \in NH$

inverse:

$$\begin{aligned} (nh)^{-1} &= h^{-1}n^{-1} \\ &= \hat{n}h^{-1} \text{ for some } \hat{n} \in N. \\ &\in NH \end{aligned}$$

ii. we need to check normality

$$g(nm)g^{-1} = \underbrace{gng^{-1}}_{\in N} \underbrace{gmg^{-1}}_{\in M} \in NM.$$

□

§5 Direct Products and Small Groups

§5.1 Direct Products

Definition 5.1 ((External) direct product of groups)

Let H and K be groups. We can construct the (external) *direct product*, $H \times K$, with a set $\{(h,k),\ h \in H, k \in K\}$ and an operation:

$$\begin{aligned}(h_1, k_1) * (h_2, k_2) &= (h_1 *_H h_2, k_1 *_K k_2)\\ &= (h_1 h_2, k_1 k_2)\end{aligned}$$

i.e. component wise multiplication. Then $(H \times K, *)$ is a group

Proof. closure: H is a group $\implies h_1 h_2 \in H$, K is a group $\implies k_1 k_2 \in K$

identity: (e_H, e_K)

inverse: $(h,k)^{-1} = (h^{-1}, k^{-1})$

associativity since group operation in both H and K are associative. □

Remark 13.

1. If H and K are both finite, $|H \times K| = |H||K|$.
2. $H \times K$ is abelian iff $(h_1,k_1)*(h_2,k_2) = (h_2,k_2)*(h_1,k_1)\ \forall\, h_1,h_2 \in H\ \forall\, k_1,k_2 \in K$
   iff $(h_1h_2, k_1k_2) = (h_2h_1, k_2k_1)$
   iff $h_1h_2 = h_2h_1$ and $k_1k_2 = k_2k_1$
   iff H is abelian and K is abelian.
3. $$\begin{aligned}H &\cong \{(h, e_K) : h \in H\} \le H \times K\\ K &\cong \{(e_H, k) : k \in K\} \le H \times K\end{aligned}$$

Example 5.1

1. $$\begin{aligned}C_2 \times C_2 &= \langle x \rangle \times \langle y \rangle\\ &= \{e, x\} \times \{e, y\}\\ \text{elements} &: (e,e),\ (x,e), (e,y),\ (x,y)\end{aligned}$$

$\circ$	(e,e)	(x,e)	(e,y)	(x,y)
(e,e)	(e,e)	(x,e)	(e,y)	(x,y)
(x,e)	(x,e)	(e,e)	(x,y)	(e,y)
(e,y)	(e,y)	(x,y)	(e,e)	(x,e)
(x,y)	(x,y)	(e,y)	(x,e)	(e,e)

This is the called the Klein 4-group and is $\cong$ to Example 1.7.
Note $o((x,e)) = o((e,y)) = o((x,y)) = 2$. So $C_2 \times C_2 \ncong C_4$, as there is no element of order 4.

2. However, $C_2 \times C_3 \cong C_6$ (Sheet 2, qn 10).

Lemma 5.1

Let $(h,k) \in H \times K$ where H, K are groups. Then $o((h,k)) = \operatorname{lcm}(o(h), o(k))$.

Proof. Let $n = o((h,k))$ and $m = \operatorname{lcm}(o(h), o(k))$.
Then $h^m = e_H, k^m = e_K$ by Lemma 1.5.

$$\begin{aligned} \text{So } (h,k)^m &= (h^m, k^m) \\ &= (e_H, e_K) \\ \implies n &\mid m \text{ by Lemma 1.5.} \\ \text{Also, } (h,k)^n &= (h^n, k^n) \\ &= (e_H, e_K) \\ \implies o(h) &\mid n \text{ and } o(k) \mid n \text{ by Lemma 1.5.} \\ \implies m &\mid n. \end{aligned}$$

Thus we know when $C_m \times C_n \cong C_{mn}$ (Sheet 2, qn 10). $\square$

Recognising when a group can be written as a direct product of subgroups is trickier.

Proposition 5.1

Let G be a group with subgroups H and K, if:

1. each element of G can be written as hk, for some $h \in H$, $k \in K$,
2. $H \cap K = \{e\}$
3. $hk = kh \; \forall \, h \in H, \; k \in K$

Then $G \cong H \times K$ and we call G the (internal) direct product of H and K.

Proof.

$$\text{Let } \theta : H \times K \to G$$
$$(h, k) \mapsto hk.$$

θ is a homomorphism:

$$\begin{aligned}\theta\left((h_1, k_1), (h_2, k_2)\right) &= \theta\left((h_1 h_2, k_1 k_2)\right) \\ &= h_1 h_2 k_1 k_2 \\ &= h_1 k_1 h_2 k_2 \text{ by } \textit{Item } 3 \\ &= \theta\left((h_1, k_1)\right) \theta\left((h_2, k_2)\right)\end{aligned}$$

θ is injective:

$$\begin{aligned}\theta\left((h_1, h_2)\right) &= \theta\left((h_2, k_2)\right) \\ h_1 k_1 &= h_2 k_2 \\ h_2^{-1} h_1 &= k_2 k_1^{-1} \in H \cap K = \{e\} \text{ by Item } 2 \\ \implies h_1 &= h_2 \text{ and } k_1 = k_2 \\ \text{So } (h_1, k_1) &= (h_2, k_2)\end{aligned}$$

θ is surjective by Item 1.
So θ is an isomorphism as required. □

Remark 14. There are alternative equivalent definitions of internal direct product. G is the internal direct product of subgroups H and K if:

1'. $H \trianglelefteq G, K \trianglelefteq G$

2'. $H \cap K = \{e\}$

3'. $HK = G$

Proof. We need to show Item 1, 2, 3 $\iff$ Item 1', 2', 3'.

($\implies$) we show $K \trianglelefteq G$.
Let $k \in K$, $g = h_1 k_1 \in G$ by Item 1. $gkg^{-1} = h_1 \underbrace{k_1 k k_1^{-1}}_{\overline{k} \in K} h_1^{-1}$

$$\begin{aligned}gkg^{-1} &= h_1 \underbrace{k_1 k k_1^{-1}}_{\overline{k} \in K} h_1^{-1} \\ &= \overline{k} \in K \text{ by } \textit{Item } 3\end{aligned}$$

Similarly $H \trianglelefteq G$.
And Item 1 $\implies$ Item 3'.

($\Longleftarrow$) need to show Item 3.

$$\begin{aligned}
h \in H, k \in K \text{ consider} \\
h^{-1}\underbrace{k^{-1}hk}_{\in H} &\in H, \text{ since } H \trianglelefteq G \\
&\in K, \text{ since } K \trianglelefteq G \\
\implies h^{-1}k^{-1}hk &= H \cap K = \{e\} \text{ by Item 2'} \\
\implies hk &= kh
\end{aligned}$$

□

Example 5.2

$G = \langle a \rangle = C_{15}$.

$$\begin{aligned}
C_5 &\cong \langle a^3 \rangle = H \trianglelefteq G \text{ (as } G \text{ is abelian).} \\
C_3 &\cong \langle a^5 \rangle = K \trianglelefteq G. \\
H \cap K &= a^{15n} = \{e\} \\
a^k &= (a^3)^{2k}(a^5)^{-k} \in HK \\
\implies C_{15} &\cong K \times H \cong C_3 \times C_5.
\end{aligned}$$

§5.2 Small Groups

Recall D_{2n}, the symmetries of a regular n-gon, generated by

$$\begin{aligned}
r &: z \mapsto e^{2\pi i/n} z \\
t &: z \mapsto \overline{z}
\end{aligned}$$

elements of

$$D_{2n} = \{e, \underbrace{r, r^2, \ldots, r^{n-1}}_{\text{rotations}}, \underbrace{t, rt \ldots, r^{n-1}t}_{\text{reflections}}\}.$$

Lemma 5.2

Now suppose G is a group, $n \geq 3$, with $|G| = 2n$, and $\exists\, b \in G$ with $o(b) = n$ and $a \in G$, $o(a) = 2$ and $aba = b^{-1}$. Then $G \cong D_{2n}$.

Proof. Note $\langle b \rangle \trianglelefteq G$ since of index two, Lemma 4.1.

Also $a \notin \langle b \rangle$, since $aba = aab = b$ as $a \in \langle b \rangle \implies ab = ba$.

$$\begin{aligned} \text{So } G &= \langle b \rangle \cup \langle b \rangle a \\ &= \{e, b, \dots, b^{n-1}, a, ba, \dots, b^{n-1}a\} \end{aligned}$$

Furthermore

$$\begin{aligned} ab &= b^{-1}a \\ \implies ab^k &= (ab)b^{k-1} \\ &= b^{-1}ab^{k-1} \\ &= b^{-2}ab^{k-2} \\ &\vdots \\ &= b^{-k}a \\ \text{So } (b^k a)(b^k a) &= b^k b^{-k} aa \\ &= e. \end{aligned}$$

We can check

$$\begin{aligned} \theta : D_{2n} &\to G \\ r &\mapsto b \\ t &\mapsto a \end{aligned}$$

is an isomorphism. □

§5.2.1 Classifying groups of small order

Example 5.3

- If $|G| = 1$, $G = \{e\}$
- If $|G| = 2 \implies G \cong C_2$, as we have identity and an another element which must be a self inverse or we can prove by Lagrange's Corollary.
- If $|G| = 3 \implies G \cong C_3$ by Corollary 3.2 (3 is prime).
- If $|G| = 5 \implies G \cong C_5$ by Corollary 3.2 (5 is prime).
- If $|G| = 7 \implies G \cong C_7$ by Corollary 3.2 (7 is prime).

Lemma 5.3

If $|G| = 4$, G is isomorphic to C_4 and $C_2 \times C_2$, both abelian.

Proof. By Lagrange's Theorem, if $1 \neq g \in G$ then $o(g) \mid 4$.

$$\begin{aligned} &\text{If } \exists\, g \in G \text{ s.t. } o(g) = 4 \\ \implies &\ G \cong C_4 \end{aligned}$$

Suppose not, then let

$$\begin{aligned} 1 \neq a \in G &\implies o(a) = 2 \\ &\implies G \text{ is abelian (qn 7, sheet 1)} \\ &\implies C_2 \cong \langle a \rangle \trianglelefteq G. \end{aligned}$$

Let $b \in G \setminus \langle a \rangle$, then $1 \neq b \in G$ so $C_2 \cong \langle b \rangle \trianglelefteq G$.
Also, $\langle a \rangle \cap \langle b \rangle = \{e\}$.
Consider the element ab:
if $ab = e \implies a = b^{-1} = b$ ↯.
if $ab = a \implies b = e$ ↯.
if $ab = b \implies a = e$ ↯.
So

$$\begin{aligned} G &= \{e, a, b, ab\} \\ &= \langle a \rangle \langle b \rangle \\ &\cong \langle a \rangle \times \langle b \rangle \text{ by Remark 14} \\ &\cong C_2 \times C_2. \end{aligned}$$

□

Lemma 5.4

If $|G| = 6$, G is isomorphic to C_6 and $D_6 \cong S_3$. Note $C_6 \not\cong D_6$ as C_6 is abelian and D_6 is not.

Proof. Let $1 \neq g \in G \implies o(g) = 2, 3$ or 6 by Corollary 3.1. If all non-identity elements have order 2 $\implies |G|$ is a 2-power ↯(qn 7, Sheet 1).
So $\exists\, b \in G, o(b) = 3$ (Note if $o(g) = 6$ then $o(g^2) = 3$).
$C_3 \cong \langle b \rangle \trianglelefteq G$, RHS by Lemma 4.1 (since of index 2).

Let $a \in G \setminus \langle b \rangle$,

$$\implies a^2 \in \langle b \rangle.$$

As $a\langle b\rangle \in G/\langle b\rangle$ and $|G : \langle b\rangle| = 2$, the group has order 2. So $(a\langle b\rangle)^2 = a^2\langle b\rangle = e = \langle b\rangle$ by Lagrange's Corollary. Then $a^2 \in \langle b\rangle$ by Lemma 3.3.

If $a^2 = b$ or b^2 then $o(a) = 6$, as if $a^2 = b$, $a^3 = ab$ and if $ab = e$, $a = b^{-1} \in \langle b\rangle$ ↯.
$o(a) = 6 \implies G \cong C_6$.

Suppose $a^2 = 1$.
Also, $aba^{-1} = aba \in \langle b\rangle$ as $\langle b\rangle \trianglelefteq G$.

$$\begin{aligned} \text{If } aba^{-1} &= e \implies b = e \text{ ↯} \\ &= b \implies ab = ba \\ & \implies o(ab) = 6 \\ & \implies G \cong C_6. \\ &= b^2 = b^{-1}. \end{aligned}$$

So $o(a) = 2$, $o(b) = 3$, $aba^{-1} = b^{-1} \implies G \cong D_6$. □

§5.2.2 Groups of order 8

Lemma 5.5

If G has order 8, then either G is abelian (i.e. $\cong C_8, C_4 \times C_2$ or $C_2 \times C_2 \times C_2$), or G is not abelian and isomorphic to D_8 or Q_8 (dihedral or quaternion).

Proof. Consider the different possible cases:

- If G contains an element of order 8, then $G \cong C_8$.
- If all non-identity elements have order 2 $\implies$ G is abelian (Sheet 1, Q7). If all non-identity elements have order 2 $\implies$ G abelian (qn7, Sheet 1). Let $1 \neq a \in G$, then $C_2 \cong \langle a\rangle \trianglelefteq G$, RHS as G is abelian. Choose $b \notin \langle a\rangle$, $\langle a, b\rangle = \{1, a, b, ab\}$

$$\begin{aligned} &\text{Choose } b \notin \langle a\rangle, \\ \langle a, b\rangle &= \{1, a, b, ab\} \\ &= \langle a\rangle\langle b\rangle \\ &\cong \langle a\rangle \times \langle b\rangle \text{ by Remark 14.} \end{aligned}$$

  Choose $c \in G \setminus \langle a, b\rangle$

$$\begin{aligned} G &= \langle a, b\rangle \cup \langle a, b\rangle c \\ &= \langle a, b\rangle\langle c\rangle \\ &\cong \langle a, b\rangle \times \langle c\rangle \text{ by Remark 14} \end{aligned}$$

$$\cong \langle a\rangle \times \langle b\rangle \times \langle c\rangle$$
$$\cong C_2 \times C_2 \times C_2$$

- Now suppose $\exists\, g \in G, o(g) > 2 \implies \exists\, a \in G, o(a) = 4$ (if we have $o(a) = 8$, then $o(a^2) = 4$). $C_4 \cong \langle a\rangle \trianglelefteq G$, RHS by Lemma 4.1 (since of index 2).

$$\begin{aligned} &\text{Let } b \in G \setminus \langle a\rangle \\ \implies &\ b^2 \in \langle a\rangle. \end{aligned}$$

(Same reasoning as in $n = 6$, alternative proof is $b^2 \in \langle a\rangle$ or $b^2 \in b\langle a\rangle$, if $b^2 = ba^n \implies b = a^n$ ↯ so $b^2 \in \langle a\rangle$).

$$\begin{aligned} &\text{If } b^2 = a \text{ or } a^3 \\ \implies &\ o(b) = 8 \implies G \cong C_8. \end{aligned}$$

$$\begin{aligned} \text{Else } b^2 &= e \text{ or } a^2 \\ \text{Now } bab^{-1} &\in \langle a\rangle \text{ since } \langle a\rangle \trianglelefteq G \\ &= a^i \text{ for some } i. \\ \implies b^2ab^{-2} &= b(bab^{-1})b^{-1} \\ &= ba^ib^{-1} \\ &= (bab^{-1})^i \text{ as } (bab^{-1})(bab^{-1}) = ba^2b \\ &= a^{i^2} \\ \text{But } b^2 \in \langle a\rangle &\implies b^2ab^{-2} = a \end{aligned}$$

as $b^2 = a^m$ so must commute with a.

$$\begin{aligned} \implies &\ i^2 \equiv 1 \pmod 4 \\ \implies &\ i \equiv \pm 1 \pmod 4. \end{aligned}$$

  - When $i \equiv 1 \pmod 4$, $bab^{-1} = a \implies ba = ab$. So $G = \langle a\rangle \cup b\langle a\rangle$ is abelian.
    * If $b^2 = e$, then $G = \langle a\rangle\langle b\rangle \cong \langle a\rangle \times \langle b\rangle \cong C_4 \times C_2$.
    * If $b^2 = a^2$, then $(ba^{-1})^2 = e$. So $G = \langle a, ba^{-1}\rangle \cong C_4 \times C_2$.
  - If $i \equiv -1 \pmod 4$, then $bab^{-1} = a^{-1}$.

* If $b^2 = e$, then $G = \langle a, b : a^4 = e = b^2, bab^{-1} = a^{-1}\rangle$. So $G \cong D_8$ by definition.
* If $b^2 = a^2$, then we have $G \cong Q_8$, a new group called the quaternion group.

To show all 5 groups are different, C_8 has an element of order 8, $C_4 \times C_2$ does not. $C_4 \times C_2$ has an element of order 4 whilst $C_2 \times C_2 \times C_2$ does not have elements of order 2 or 4.

D_8 and Q_8, Q_8 has 6 elements of order 4, but D_8 only has 2, so non-isomorphic. $\square$

§5.2.3 Realisations of Q_8

1.

$$\begin{aligned} Q_8 &= \{\pm 1, \pm i, \pm j, \pm k\} \\ \text{with } ij &= k, jk = i, ki = j \\ ji &= -k, kj = -i, ik = -j \\ i^2 &= j^2 = k^2 = -1 \\ \text{so } o(i) &= o(j) = o(k) = 4, o(-1) = 2. \end{aligned}$$

2.

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \right\} \le \mathrm{SL}_2(\mathbb{C})$$

3.

$$Q_8 = \langle a, b \mid a^4 = e, b^2 = a^2, bab^{-1} = a^{-1}\rangle$$

If $|G| = 9$, we will show later that groups of order p^2 (where p is prime) are abelian, so either $G \cong C_9$.
Or all non-identity elements have order 3 by Lagrange's Corollary Choose $e \neq a \in G, b \in G \setminus \langle a\rangle$

$$\begin{aligned} G &= \langle a\rangle \cup \langle a\rangle b \cup \langle a\langle b^2 \\ &= \langle a\rangle\langle b\rangle \\ &\cong \langle a\rangle \times \langle b\rangle \\ &\cong C_3 \times C_3. \end{aligned}$$

If $|G| = 10 \implies G \cong C_{10}$ or D_{10} (qn 12, Sheet 2, use proof similar to order 8 not 6).

Remark 15. There are lots of groups of order 2^k. e.g. 10 of order 16 and approximately 50,000,000,000 of order 2^{10}.

§6 Group Actions

It is often easier to understand a group if it's doing something, permuting elements, rotating a square etc.

Definition 6.1 (Group action)
Let G be a group and X a non-empty set. We say that G *acts* on X if there is a mapping

$$\begin{aligned} \rho : G \times X &\to X \\ (g, x) &\mapsto \rho(g, x) = g(x) \end{aligned}$$

such that

0. if $g \in G$, $x \in X$, then $\rho(g, x) = g(x) \in X$ (implied by notation, but something we should check).
1. $\rho(gh, x) = \rho(g, \rho(h, x))$. shorthand: $gh(x) = g(h(x))$.
2. $\rho(e, x) = x$, shorthand: $e(x) = x$.

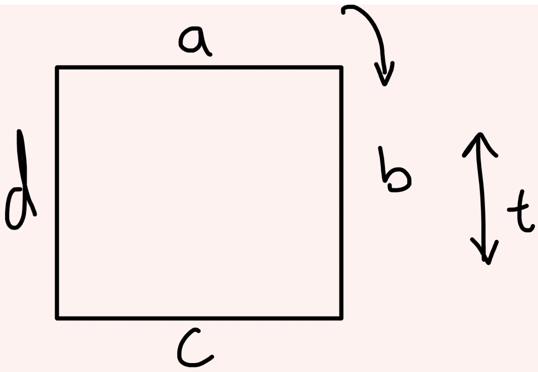
When G acts on a set it maps elements of X to X in a way that the multiplication of G is respected.

Example 6.1

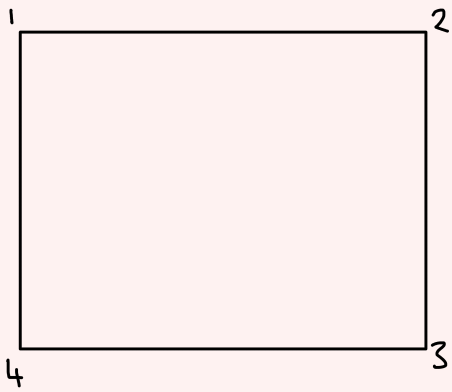
i. trivial action $\rho(g, x) = x \; \forall \; x \in X, g \in G$.

ii. S_n acts on $X = \{1, 2, \dots, n\}$ by permuting the elements of X. e.g. S_3 acts on $\{1, 2, 3\}$, $\sigma = \begin{pmatrix} 1 & 2 \end{pmatrix} \in S_3 : \sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3$. $\tau = \begin{pmatrix} 1 & 3 \end{pmatrix} \in S_3$, $\tau\sigma = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$
$(\tau\sigma)(1) = 2$, $\tau(\sigma(1)) = \tau(2) = 3$.
Similarly subgroups of S_n act on X.

iii. $D_8 = \{e, r, r^2, r^3, t, rt, r^2t, r^3t\}$ acts on the edges of a square.



$$\begin{aligned} t(a) &= c & t(c) &= a \\ t(b) &= b & t(d) &= d. \\ r(a) &= b. \end{aligned}$$



Also acts on the vertices of a square.

$$\begin{aligned} t(1) &= 4 & t(4) &= 1 \\ t(2) &= 3 & t(d) &= 2. \end{aligned}$$

iv. G acts on itself by left multiplication. This is called the *left regular action*

$$\begin{aligned} G \times G &\to G \\ (g, k) &\mapsto gk. \end{aligned}$$

Check:

0. $gk \in G$ (by closure)

1.

$$\rho(gh,k) = ghk$$
$$\rho(g,\rho(h,k)) = \rho(g,hk) = ghk$$

Or in shorthand:

$$gh(k) = ghk$$
$$g(h(k)) = g(hk) = ghk.$$

2. $\rho(e,k) = ek = k$.

v. We also have the G *right regular action*

$$G \times G \to G$$
$$(g,k) \mapsto kg^{-1}.$$

(we need inverse for Item 1 to hold.)

vi. G acts on itself by *conjugation*

$$G \times G \to G$$
$$(g,k) \mapsto gkg^{-1}.$$

Check:

0. $gkg^{-1} \in G$ (by closure)

1.

$$\rho(gh,k) = ghk(gh)^{-1}$$
$$= ghkh^{-1}g^{-1}\rho(g,\rho(h,k)) \quad = \rho(g,hkh^{-1}) = g(hkh^{-1})g^{-1}$$

2. $\rho(e,k) = eke^{-1} = k$.

vii. Let $N \trianglelefteq G$, then G acts on N by conjugation

$$G \times N \to N$$
$$(g,n) \mapsto gng^{-1}.$$

0. $gng^{-1} \in G$ since $N \trianglelefteq G$.
(1) and (2) as above.

viii. Let $H \leq G$, then G acts on the set of left cosets, $(G : H)$, if H in G. Called the *left coset action.*

$$\begin{aligned} G \times (G : H) &\to (G : H) \\ (g, kH) &\mapsto gkH. \end{aligned}$$

0. $gkH \in (G : H)$
1. $$\begin{aligned} \rho(gh, kH) &= (gh)kH = ghkH. \\ \rho(g, \rho(h, kH)) &= \rho(g, hkH) \\ &= ghkH \end{aligned}$$
2. $\rho(e, kH) = ekH = kH$.

Remark 16. Recall a permutation of a set X is a bijection of X, Definition 2.2. We have commented that a bijection $f : X \to X$ has a 2-sided inverse, i.e. $\exists\, g : X \to X$ s.t.

$$\begin{aligned} f \circ g(x) &= x \;\forall\, x \in X \\ g \circ f(x) &= x \;\forall\, x \in X. \end{aligned}$$

Conversely if $f : X \to X$ is a map with a 2-sided inverse then f is a bijection.

$f \circ g(x) = x \;\forall\, x \in X \implies f$ is surjective, as f is mapping to all elements in X
$g \circ f(x) = x \;\forall\, x \in X \implies f$ is injective, as if f took two elements to the same place then g wouldn't be able to split them up.

Note 2-sided is necessary:

$$\begin{aligned} \phi : \mathbb{Z} &\to \mathbb{Z} & \psi : \mathbb{Z} &\to \mathbb{Z} \\ x &\mapsto 2x & 2x &\mapsto x \\ & & 2x+1 &\mapsto 0 \\ \phi\psi &= \mathrm{id}\,. \end{aligned}$$

Lemma 6.1

Suppose the group G acts on the non-empty set X. Fix $g \in G$, then the

$$\begin{aligned} \phi_g : X &\to X \\ x &\mapsto \rho(g, x) = g(x) \end{aligned}$$

is a permutation of X, i.e. $\phi_g = \mathrm{Sym}(X)$.

Proof. Clearly ϕ_g is a map from X to X. We need to show ϕ_g is a bijection, enough to show it has a 2-sided inverse.

$$\begin{aligned}\phi_{g^{-1}} \circ \phi_g(x) &= \phi_{g^{-1}}(\rho(g,x)) \\ &= \rho(g^{-1}, \rho(g,x)) \\ &= \rho(g^{-1}g, x) \text{ since } \rho \text{ is a group action, } 1 \\ &= \rho(e,x) \\ &= x \;\forall\; x.\end{aligned}$$

Similarly, $\phi_g \circ \phi_{g^{-1}}(x) = x \;\forall\; x \in X$. □

Proposition 6.1

Suppose G acts on the set X. Then the map

$$\begin{aligned}\Phi : G &\to \mathrm{Sym}(x) \\ g &\mapsto \phi_g\end{aligned}$$

as in Lemma 6.1, is a homomorphism.

Proof. We need to show Φ is a homomorphism i.e. need

$$\begin{aligned}\Phi(gh) &= \Phi(g) \circ \Phi(h) \\ \text{i.e. } \phi_{gh} &= \phi_g \circ \phi_h.\end{aligned}$$

Let $x \in X$

$$\begin{aligned}\phi_{gh}(x) &= \rho(gh, x) \\ &= \rho(g, \rho(h,x)) \\ &= \phi_g \circ \phi_h(x).\end{aligned}$$

This is true $\forall\; x \in X$. □

Remark 17.

1. Proposition 6.1 gives us an equivalent definition of a group action. If G is a group and X a set such that $\Phi : G \to \mathrm{Sym}(X)$ is a group homomorphism, then

$$\begin{aligned}\rho : G \times X &\to X \\ (g,x) &\mapsto \phi_g(x)\end{aligned}$$

where $\Phi(g) = \phi_g$, is group action.

2. Using notation of Proposition 6.1, by 1st Isomorphism Theorem

   $G/\ker\Phi \cong \operatorname{Im}\Phi \leq \operatorname{Sym}(X)$.

   Note

$$\begin{aligned}\ker\Phi &= \{g \in G : \Phi(g) = \mathrm{id}_X \in \operatorname{Sym}(X)\\ &= \{g \in G : \phi_g(x) = \rho(g,x) = x \;\forall\, x \in X\}\\ &\trianglelefteq G \text{ Kernels of homomorphisms are normal subgroups (Sheet 1, q9).}\end{aligned}$$

   I.e. all those elements that fix every elements of X, that act 'trivially'.

   We say the action is *faithful* if $\ker\Phi = \{e\}$.

   e.g. the kernels of Example 6.1

   i. trivial action - $\ker\Phi = G$.

   ii. S_n acts on $\{1, \dots, n\}$ - faithful.

   iii. D_8 acts on the edges of a square - faithful.

   iv. left regular action - faithful.

   v. conjugation - $\ker\Phi = \{g \in G : \underbrace{gkg^{-1} = k}_{gk=kg} \;\forall\, k \in G\} = Z(G)$, *the centre of* G are 'the elements that commute with everything'.

   vi. conjugation of $N \trianglelefteq G$. $\ker\Phi = \{g \in g : gng^{-1} = n \;\forall\, n \in N\} = C_G(N)$, the *centraliser of* N *in* G.

   vii. left coset action -

$$\begin{aligned}\ker\Phi &= \{g \in G : gkH = kH \;\forall\, k \in G\}\\ &= \{g \in G : k^{-1}gk \in H \;\forall\, k \in G\}\\ &= \{g \in G : g \in kHk^{-1} \;\forall\, k \in G\}\\ &= \bigcap_{k\in G} kHk^{-1}\\ &= \operatorname{Core}_G(H) \trianglelefteq G\end{aligned}$$

$$\text{and } \underbrace{\operatorname{Core}_G(H)}_{\text{core of } H} \leq H \text{ (we can set } k = e).$$

   Useful Note for exm sheet: If $\ker\Phi = \{e\}$, then G is isomorphic to a subgroup of $\operatorname{Sym}(X)$, we write $G \lesssim \operatorname{Sym}(X)$. So if $|G| \nmid |\operatorname{Sym}(X)|$ then $\ker\Phi \neq \{e\}$.

Theorem 6.1 (Cayley's Theorem)

Any group G is isomorphic to a subgroup of $\mathrm{Sym}(X)$ for some non-empty set X.

Proof. We take X to be G and consider the left regular action

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto gh. \end{aligned}$$

This is a faithful action as $gh = h \; \forall \; h \in G \implies g = e$. Thus we have an injective homomorphism

$$\Phi : G \to \mathrm{Sym}(G)$$

and $G \lesssim \mathrm{Sym}(G)$. □

Definition 6.2 (Orbit)
Let G act on a set X and $x \in X$. The *orbit* of $x \in X$ is given by

$$\mathrm{Orb}_G(x) = \{g(x) : g \in G\} \subseteq X.$$

I.e. the set of points in X which x can be mapped to.

Example 6.2
The orbits of Example 6.1.

1. trivial action, $\mathrm{Orb}_G(x) = \{x\}$
2. S_n acts on $\{1, \ldots, n\}$ - $\mathrm{Orb}_G(1) = X$ (we can get $(1\ a)$ which maps x to any a).
   If $H = \langle(1\ 2)(3\ 4\ 5)\rangle \leq S_n$ acting on $X = \{1, 2, 3, 4, 5\}$ then $\mathrm{Orb}_G(1) = \{1, 2\}$ and $\mathrm{Orb}_G(3) = \{3, 4, 5\}$.
3. D_8 acts on the edges of a square - $\mathrm{Orb}_{D_8}(a) = \{a, b, c, d\}$.
4. left regular action - $\mathrm{Orb}_G(k) = G$, since $g = g(k^{-1}k) = (gk^{-1})k$ for any $g \in G$.
5. conjugation - $\mathrm{Orb}_G(k)$

$$\begin{aligned} \mathrm{Orb}_G(k) &= \{g(k) : g \in G\} \\ &= \{gkg^{-1} : g \in G\} \\ &= \mathrm{ccl}_G(k) \end{aligned}$$

, the *conjugacy class of* k *in* G. If $h \in \mathrm{ccl}_G(k)$ we say h and k are *conjugate*.

Definition 6.3 (Transitive orbits)

We say G acts *transitively* on X if for any $x \in X$, $\mathrm{Orb}_G(x) = X$. Equivalently, if given any pair $x_1, x_2 \in X \; \exists \, g \in G$ s.t. $g(x_1) = x_2$.

So the left regular action is a transitive action.

Lemma 6.2

The distinct G-orbits form a partition of X

Proof. Let $x \in X$, then $x \in \mathrm{Orb}_G(x)$ since $x = ex$.
Suppose $z \in \mathrm{Orb}_G(x) \cap \mathrm{Orb}_G(y)$, we show $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(z) = \mathrm{Orb}_G(y)$.
$z \in \mathrm{Orb}_G(x) \implies \exists \, g \in G$ s.t. $g(x) = z$.

$$\begin{aligned}
\text{Suppose } t \in \mathrm{Orb}_G(z) &\implies \exists \, h \in G \text{ s.t. } h(z) = t \\
&\implies t = h(g(x)) = (hg)(x) \\
&\implies t \in \mathrm{Orb}_G(x) \implies \mathrm{Orb}_G(z) \subseteq \mathrm{Orb}_G(x) \\
\text{Similarly } g(x) = z & \\
x = e(x) = (g^{-1}g)(x) = g^{-1}(z) & \\
\implies \mathrm{Orb}_G(x) \subseteq \mathrm{Orb}_G(z). &
\end{aligned}$$

Thus $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(z)$. Similarly, $\mathrm{Orb}_G(z) = \mathrm{Orb}_G(y)$. □

Remark 18.

1. We could have proved Lemma 6.2 by noticing that $x_1 \sim x_2$ if $\exists \, g \in G$ s.t. $g(x_1) = x_2$ is an equivalence relation.
2. $\mathrm{Orb}_G(x)$ is G-invariant, i.e. $g\left(\mathrm{Orb}_G(x)\right) \subseteq \mathrm{Orb}_G(x)$. Since if $y \in \mathrm{Orb}_G(x), y = hx$ for some $h \in G \implies g(y) = g(h(x)) = (gh)(x) \in \mathrm{Orb}_G(x)$.
3. G is transitive on $\mathrm{Orb}_G(x)$.

$$\begin{aligned}
&\text{Let } y, z \in \mathrm{Orb}_G(x) \\
&\qquad y = g(x), z = h(x), \text{ for some } g, h \in G. \\
&\text{Then } z = h(g^{-1}(y)).
\end{aligned}$$

Definition 6.4 (Stabiliser)

Let G act on X and $x \in X$. The *stabiliser* of x in G is given by

$$\mathrm{Stab}_G(x) = \{g \in G : g(x) = x\}$$

$$\subseteq G.$$

i.e. all those elements in G that fix x.

Example 6.3

The stabilisers of Example 6.1.

1. trivial action - $\mathrm{Stab}_G(x) = G$.
2. S_n on $X = \{1, 2, \ldots, n\}$ - $\mathrm{Stab}_G(1) \cong S_{n-1}$.
3. $H = \langle \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \rangle$ on X - $\mathrm{Stab}_H(1) = \langle \begin{pmatrix} 3 & 4 & 5 \end{pmatrix} \rangle$.
4. D_8 - $\mathrm{Stab}_{D_8}(b) = \{e, t\}$.
5. left regular action - $\mathrm{Stab}_G(k) = \{e\}$, $gk = k \implies g = e$.
6. conjugation -

$$\begin{aligned}
\mathrm{Stab}_G(k) &= \{g \in G : g(k) = k\} \\
&= \{g \in G : gkg^{-1} = k\} \\
&= \{g \in G : gk = kg\} \\
&= C_G(k), \ \textit{centraliser of } k \textit{ in } G.
\end{aligned}$$

I.e. all those elements of G that commute with k.

Lemma 6.3

$\mathrm{Stab}_G(x)$ is a subgroup of G.

Proof.

- $e(x) = x \implies e \in \mathrm{Stab}_G(x)$.
-

$$\begin{aligned}
\text{if } g, h &\in \mathrm{Stab}_G(x) \\
(gh)(x) &= g(h(x)) \\
&= g(x) \\
&= x \implies gh \in \mathrm{Stab}_G(x).
\end{aligned}$$

-
$$
\begin{aligned}
g &\in \operatorname{Stab}_G(x) \\
g(x) &= x \\
x &= e(x) = (g^{-1}g)(x) = g^{-1}(gx) \\
&= g^{-1}(x) \\
\implies g^{-1} &\in \operatorname{Stab}_G(x).
\end{aligned}
$$

- Associativity is inherited from G.

□

Remark 19. Recall, Proposition 6.1

$$
\begin{aligned}
\Phi &: G \to \operatorname{Sym}(X) \\
\ker \Phi &= \{g \in G : g(x) = x \ \forall \ x \in X\} \\
&= \bigcap \operatorname{Stab}_G(x).
\end{aligned}
$$

Theorem 6.2 (Orbit-Stabiliser Theorem)
Let G be a finite group acting on a non-empty set X. Then $\operatorname{Stab}_G(x) \leq G$ and

$$|G| = |\operatorname{Stab}_G(x)||\operatorname{Orb}_G(x)|.$$

Remark 20. We actually prove that $|G : \operatorname{Stab}_G(x)|$, the number of left cosets of $\operatorname{Stab}_G(x)$ in G, is equal to $|\operatorname{Orb}_G(x)|$, a more general statement.

Proof. $(G : \operatorname{Stab}_G(x))$ is the set of left cosets of $\operatorname{Stab}_G(x)$ in G. Consider the map

$$
\begin{aligned}
\theta : \operatorname{Orb}_G(x) &\to (G : \operatorname{Stab}_G(x)) \\
g(x) &\mapsto g \operatorname{Stab}_G(x).
\end{aligned}
$$

θ is well-defined:

$$
\begin{aligned}
g(x) = h(x) &\implies h^{-1}g(x)(x) = x \\
&\implies h^{-1}g \in \operatorname{Stab}_G(x) \\
&\implies g \operatorname{Stab}_G(x) = h \operatorname{Stab}_G(x) \text{ by Lemma 3.3} \\
&\implies \theta(g(x)) = \theta(h(x)).
\end{aligned}
$$

θ is injective:

$$\theta(g(x)) = \theta(h(x))$$

$$
\begin{aligned}
&\implies g\,\mathrm{Stab}_G(x) = h\,\mathrm{Stab}_G(x) \\
&\implies h^{-1}g \in \mathrm{Stab}_G(x) \text{ by Lemma 3.3} \\
&\implies h^{-1}g(x) = x \\
&\implies g(x) = h(x).
\end{aligned}
$$

θ is surjective:

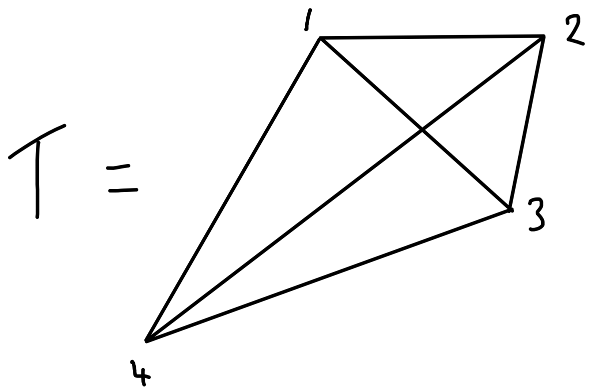
$$
\begin{gathered}
\text{Given } g\,\mathrm{Stab}_G(x) \in (G : \mathrm{Stab}_G(x)) \\
\text{then } g(x) \in \mathrm{Orb}_G(x) \\
\text{and } \theta(g(x)) = g\,\mathrm{Stab}_G(x).
\end{gathered}
$$

Thus θ is a well-defined bijection. □

§6.1 Application to Symmetry Groups of Regular Solids

Let S be a regular solid and V its vertices. The the symmetries of S are the isometries (distance preserving maps) of $\mathbb{R}^2$ or $\mathbb{R}^3$ that maps S to itself. The dual is the solid with vertices in the middle of each face of the input.

§6.1.1 Tetrahedron (self-dual)



faces are 4 equilateral triangles.

Let G be group of symmetries of T, and $X = \{\text{vertices of } T\} = \{1, 2, 3, 4\}$. Then $\exists$ a homomorphism

$$\Phi : G \to \mathrm{Sym}\, X \cong S_4 \text{ Proposition 6.1}$$

Note $\ker \Phi = \{e\}$, if all vertices are fixed, then T fixed.
Consider $G^+ \leq G$ be the subgroup of all rotations. The elements of G^+ are as follows:

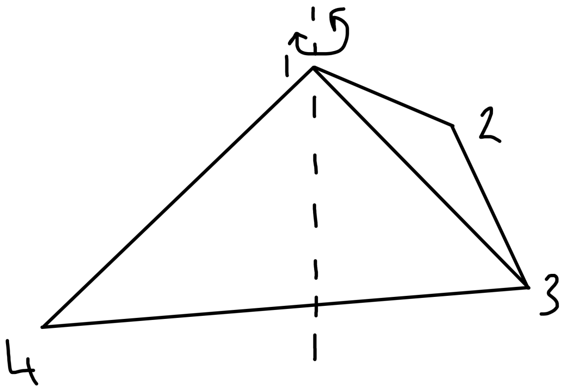


Figure 1: Rotation of $2\pi/3$, a 3-cycle $\begin{pmatrix}2 & 3 & 4\end{pmatrix}$, and $4\pi/3$ gives $\begin{pmatrix}2 & 4 & 3\end{pmatrix}$.

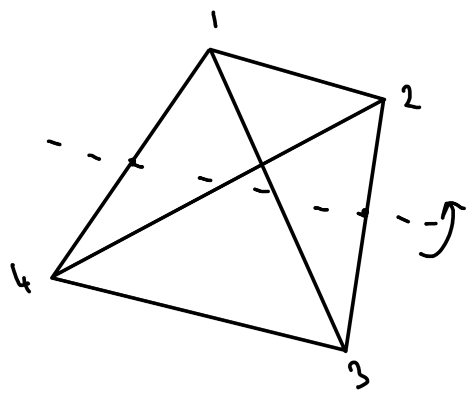


Figure 2: Rotation of π, a double transposition $\begin{pmatrix}1 & 4\end{pmatrix}\begin{pmatrix}2 & 3\end{pmatrix}$. We have one rotation for each pair of axes and we have 6 axes, so 3 such double transpositions.

There are 4 such axes, giving 8 rotations of order 3 (these are all the 3-cycles in S_4).

and identity,

$$\implies G^+ \cong A_4.$$

(only subgroup of order 12)

Now consider G (all symmetries). Clearly $\operatorname{Orb}_G(1) = \{1, 2, 3, 4\} = \operatorname{Orb}_{G^+}(1)$.
Consider $\operatorname{Stab}_G(1)$.

- Note if 3 vertices are fixed then T is fixed.
- Suppose vertices 1 and 2 are fixed

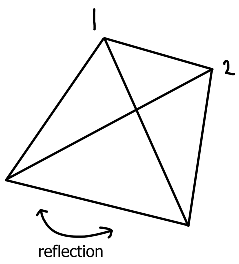


Figure 3: reflection through $1, 2$ giving $\begin{pmatrix}3 & 4\end{pmatrix} = \tau$. We have such a reflection for each of the 6 edges.

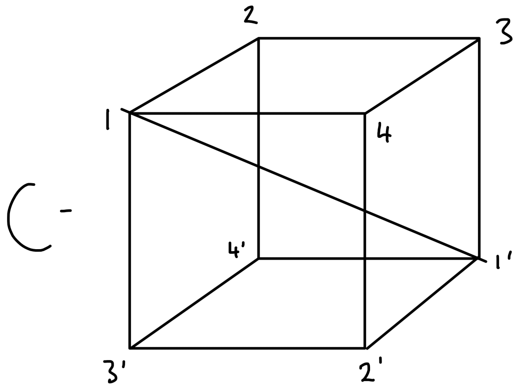
- If just 1 is fixed we have order 3 rotation from before, σ.

These are all elements in G

$$\begin{aligned}
\operatorname{Stab}_G(1) &= \langle \sigma, \tau \rangle \cong D_6 \\
\implies |G| &= |\operatorname{Orb}_G(1)||\operatorname{Stab}_G(1)| \text{ Orbit-Stabiliser Theorem} \\
&= 4 \times 6 = 24 \\
\implies G &\cong S_4.
\end{aligned}$$

(consider the effects of σ and τ on the bottom face to see why $\langle \sigma, \tau \rangle \cong D_6$).

Note $\operatorname{Stab}_{G^+}(1) = \langle G \rangle$. Also $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix} = \begin{pmatrix}1 & 2\end{pmatrix}\begin{pmatrix}2 & 3 & 4\end{pmatrix}$.



§6.1.2 Cube (dual to octahedron)

Let G^+ be the group of rotations of C. Then G^+ acts on set of diagonals $X = \{D_1, D_2, D_3, D_4\}$.

If a rotation, σ, that fixes all the diagonals, then $\sigma = \text{id}$. So we have an injective homomorphism

$$\Phi : G^+ \to \text{Sym}(X) \cong S_4.$$

Rotations:

- id

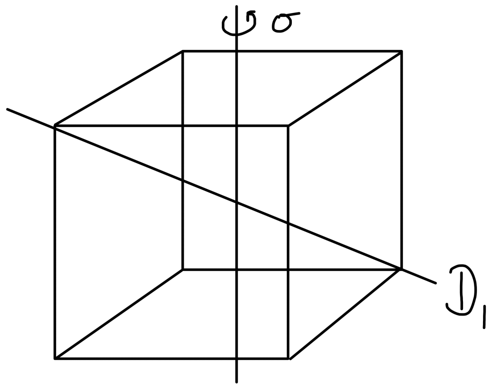


Figure 4: σ rotation of $\pi/2$ corresponds to $\begin{pmatrix}1 & 2 & 3 & 4\end{pmatrix}$ in action on diagonals. There are 3 such axes giving 6 elements of order 4 and 3 of order 2.

-
-
-

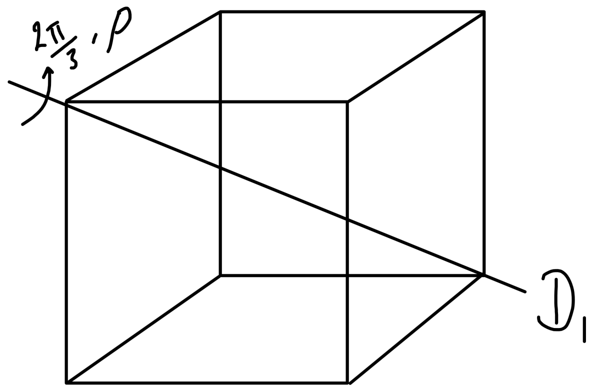


Figure 5: $o(\rho) = 3$ corresponds to $\begin{pmatrix} 2 & 3 & 4 \end{pmatrix}$, we have 4 such axes giving eight elements of order 3 (ρ, ρ^2).

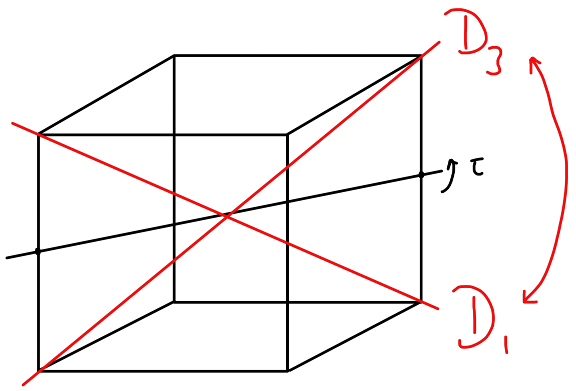


Figure 6: rotation of π, $o(\tau) = 2$ corresponds to $\begin{pmatrix} 1 & 3 \end{pmatrix}$. There are 6 such axes

I.e. $G^+ \cong S_4$. Note $\mathrm{Orb}_{G^+}(D_1) = \{D_1, D_2, D_3, D_4\}$. $\mathrm{Stab}_{G^+}(D_1) = \{\rho, \tau'\}$ or consider G^+ acting on vertex 1

$$\begin{aligned}
|\operatorname{Orb}_{G^+}(1)| &= 8 \\
|\operatorname{Stab}_{G^+}(1)| &= |\langle\rho\rangle| = 3. \\
\implies |G^+| &= 24.
\end{aligned}$$

Now consider full symmetry group of C, call it G.

Consider action on faces $F_1, \ldots, F_6$, this yields an injective (or faithful) homomorphism as fixing all the faces fixes everything.

$$\begin{aligned}
\Phi : G &\to \mathrm{Sym}(F_i) \cong S_6. \\
|\operatorname{Orb}(F_1)| &= 6. \\
\operatorname{Stab}(F_1) &\cong D_8. \text{ Consider the opposite face} \\
\implies |G| &= 6 \times 8 = 48.
\end{aligned}$$

So, action on diagonals is not faithful; $\exists\, g \in G$ $g(D_i) = D_i$ $1 \le i \le 4$ but $g \neq \mathrm{id}$. g can swap vertex i with i', the diagonals will stay unchanged however. Alternatively, label vertices of C as $\{(\pm 1, \pm 1, \pm 1)\}$, then

$$g : (x, y, z) \mapsto (-x, -y, -z).$$

If we label the faces of the cube like a dice (1 opposite 6, 2 opposite 5, 3 opposite 4) then $g = \begin{pmatrix} 1 & 6 \end{pmatrix} \begin{pmatrix} 2 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 \end{pmatrix}$.

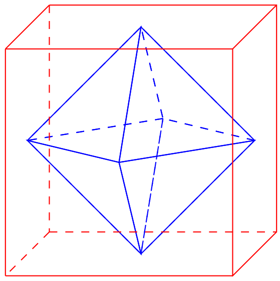
Then $G \cong G^+ \times \langle g \rangle$.

Proof.

$$\begin{aligned}
G^+ &\trianglelefteq G \text{ by Lemma 4.1 (since of index 2)} \\
\langle g \rangle &\trianglelefteq G \text{ commutes with all rotations} \\
G^+ \cap \langle g \rangle &= \{e\} \\
|G^+ \langle g \rangle| &= 48 = |G|.
\end{aligned}$$

□

In fact, we have also proved that the group of symmetries of an octahedron is $S_4 \times C_2$ since the octahedron is the dual of the cube. (if you join the centers of each face of the cube, you get an octahedron)



§6.1.3 Dodecahedron (dual to Icosahedron) - Non examinable

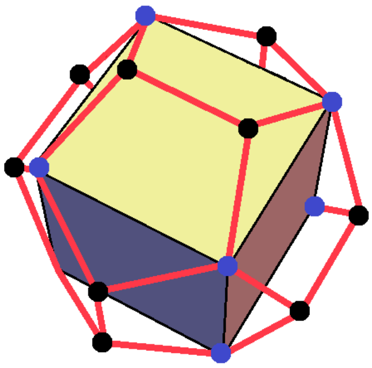
Let D be the dodecahedron.

- 12 regular pentagonal faces
- 30 edges
- 20 vertices

Let G^+ = group of rotations of D.
Let F be a face of D.

$$\begin{aligned} |\operatorname{Orb}_{G^+}(F)| &= 12 \\ |\operatorname{Stab}_{G^+}(F)| &= 5 \text{ just rotating about face} \\ \implies |G^+| &= 5 \times 12 = 60 \text{ by Orbit-Stabiliser Theorem.} \end{aligned}$$

There are five cubes embedded in D.



15 pairs of edges, 3 pairs per cube $\implies$ 5 cubes.

G^+ acts faithfully on cubes, giving us an injective map $\Phi : G^+ \to S_5$. And $|G^+| = 60 \implies G^+ \cong A_5$ (A_5 is the only subgroup of order 60 of S_5). The smallest non-abelian simple group comes up as the rotational group of a dodecahedron.

We can find the elements of A_5

- Rotations through opposite faces - 5 cycles (6 axes, 4 elements per axis giving 24 elements).
- Rotations through opposite vertices - 3 cycles.
- Rotations through opposite edges - double transpositions (15 such).

§6.1.4 Another Application of Orbit-Stabiliser Theorem

Theorem 6.3 (Cauchy's Theorem)
Let G be a finite group and p a prime that divides $|G|$. Then there exists an element in G of order p.

Proof. Let

$$X = \{(x_1, x_2, \dots, x_p) : x_1 x_2 \dots x_p = e, x_i \in G\}.$$

Let $H = \langle h : o(h) = p \rangle \cong C_p$ acts on X as follows:

$$\begin{aligned} H \times X &\to X \\ (h, (x_1, x_2, \dots, x_p)) &\mapsto (x_2, x_3, \dots, x_p, x_1) \end{aligned}$$

in general

$$(h^i, (x_1, x_2, \dots, x_p)) \mapsto (x_{1+i}, x_{2+i}, \dots, x_{p+i})$$
suffices are taken modulo p.

Check this is a group action

0.

$$\begin{aligned} x_1 x_2 \dots x_p &= e \\ x_{i+1} x_{2+i} \dots, x_{p+i} &= [(x_1 x_2 \dots x_i)^{-1} (x_1 x_2 \dots x_i)] x_{i+1} x_{2+i} \dots, x_p x_1 x_2 \dots x_i \\ &= (x_1 x_2 \dots x_i)^{-1} x_1 x_2 \dots x_p (x_1 x_2 \dots x_i) \\ &= (x_1 x_2 \dots x_i)^{-1} e (x_1 x_2 \dots x_i) \\ &= e. \end{aligned}$$

1.

$$\begin{aligned} h^{i+j}(x_1, \dots, x_p) &= (x_{1+i+j}, \dots, x_{p+i+j}) \\ &= h^i(h^j(x_1, \dots, x_p)). \end{aligned}$$

2.

$$\begin{aligned} e(x_1, \ldots, x_p) &= h^p(x_1, \ldots, x_p) \\ &= (x_1, \ldots, x_p). \end{aligned}$$

□

Let $\overline{x} = (x_1, x_2, \ldots, x_p) \in X$. As distinct orbits partition X (Lemma 6.2)

$$\implies \sum_{\text{distinct orbits}} |\operatorname{Orb}_H(\overline{x})| = |X|.$$

Note $|X| = |G|^{p-1}$ (choose $x_1, \ldots, x_{p-1}$ then x_p is determined, we have $|G|$ choices for each free variable).

$$\begin{aligned} p \mid |G| &\implies p \mid |X| \\ &\implies p \mid \sum_{\text{distinct orbits}} |\operatorname{Orb}_H(\overline{x})| \end{aligned} \tag{3}$$

But by Orbit-Stabiliser Theorem

$$\begin{aligned} |\operatorname{Orb}_H(\overline{x})| &\Big| |H| = p \\ \implies |\operatorname{Orb}_H(\overline{x})| &= 1 \text{ or } p. \end{aligned}$$

Now, $\overline{e} = (e, e, \ldots, e) \in X$ and $|\operatorname{Orb}_H(\overline{e})| = 1$. So $\exists$ at least $p-1$ other orbits of length 1 by (3). So $\exists\, \overline{x} \in X$ s.t $\operatorname{Orb}_H(\overline{x}) = 1 \implies \overline{x} = (x, x, \ldots, x)$ (has to look the same after permutation) with $x \neq e$ and $x^p = e$.

§6.2 Conjugacy Action

$$\begin{aligned} G \times G &\to G \\ (g, h) &\mapsto ghg^{-1}. \end{aligned}$$

Orbits are called conjugacy classes

$$\operatorname{ccl}_G(h) = \{ghg^{-1} : g \in G\}$$

stabilisers are called centralisers

$$C_G(h) = \{g \in G : ghg^{-1} = h\}.$$

Remark 21.

1. By Lemma 6.2 the conjugacy classes partition G.
2. By Orbit-Stabiliser Theorem, $h \in G$

$$|G| = |C_G(h)||\operatorname{ccl}_G(h)|.$$

In particular ($C_G(h)$ is a subgroup)

$$|\operatorname{ccl}_G(h)|\Big||G|.$$

3. If $k \in \operatorname{ccl}_G(h)$ then $o(k) = o(h)$.

Proof. Since $k = ghg^{-1}$ for some $g \in G$, so

$$\begin{aligned} k^{o(h)} &= (ghg^{-1})^{o(h)} \\ &= gh^{o(h)}g^{-1} \\ &= e \\ \implies o(k) &\mid o(h) \end{aligned}$$

Similarly, $h = g^{-1}kg \implies o(h) \mid o(h)$. □

4. Recall the centre of G is

$$\begin{aligned} Z(G) &= \{g \in G : gh = hg \; \forall\, h \in G\} \\ &\trianglelefteq G. \\ \text{And } Z(G) &= \bigcap_{h \in G} C_G(h) \end{aligned}$$

Note $z \in Z(G) \iff |\operatorname{ccl}_G(z)| = 1$.

Proof. If $z \in Z(G)$

$$\begin{aligned} \implies \operatorname{ccl}_G(z) &= \{\underbrace{gzg^{-1}}_{gg^{-1}z=z} : g \in G\} \\ &= \{z\} \end{aligned}$$

If $|\operatorname{ccl}_G(z)| = 1$, note $z = eze^{-1} \in \operatorname{ccl}_G(z)$. So $gzg^{-1} = z \; \forall\, g \in G$. □

5. Let $H \leq G$, then H is normal iff it is a union of conjugacy classes (Sheet 3, Q3).
6. G is abelian iff $G = Z(G)$.

Proposition 6.2

Let p be a prime and G a group of order p^n. Then $Z(G)$ is nontrivial, i.e. $Z(G) > \{e\}$ (not equal to).

Proof. Let G act on G by conjugation. Then the conjugacy classes of G partition G,

$$G = \dot{\bigcup}_{\text{distinct conj classes}} \operatorname{ccl}_G(x) \text{ by Lemma 6.2.}$$

By Orbit-Stabiliser Theorem

$$|\operatorname{ccl}_G(x)| \Big| |G| = p^n.$$

Either $|\operatorname{ccl}_G(x)| = 1$ or $p \Big| |\operatorname{ccl}_G(x)|$. By Item 4,

$$|G| = \sum_{x \in Z(G)} |\operatorname{ccl}_G(x)| + \sum_{\substack{\text{distinct} \\ \text{conj classes} \\ \text{with } p \big| |\operatorname{ccl}_G(x)|}} |\operatorname{ccl}_G(x)|$$

Now $p \mid |G|$ and $p \mid RHS$

$$\implies p \Bigg| \sum_{x \in Z(G)} |\operatorname{ccl}_G(x)| = |Z(G)|$$

□

Lemma 6.4

Let G be a finite group and $Z(G)$ the centre of G. If $G/Z(G)$ is cyclic then G is abelian (so $G = Z(G)$).

Proof. Let $Z = Z(G)$. G/Z is cyclic, so $G/Z = \langle yZ \rangle$ for some $y \in G$. Let $g, h \in G$.
Then $gZ = y^i Z$ for some $i \implies g = y^i z_1$ for some $z_1 \in Z$.
Similarly, $hZ = y^j Z$ for some $j \implies h = y^j z_2$ for some $z_2 \in Z$.
Now,

$$\begin{aligned} gh &= y^i z_1 y^j z_2 \\ &= y^i y^j z_1 z_2 \text{ as } z_1, z_2 \in Z \\ &= y^j y^i z_2 z_1 \end{aligned}$$

$$
\begin{aligned}
&= y^j z_2 y^i z_1 \\
&= hg \\
\implies & G \text{ is abelian.}
\end{aligned}
$$

□

Corollary 6.1

Suppose $|G| = p^2$ for some prime p. Then G is abelian and there are, up to isomorphism, just two groups of order p^2, namely C_{p^2} and $C_p \times C_p$.

Proof. (Q10, Sheet 3). □

Remark 22.

A group of order p^n for a prime p is called a finite p-group.

If all the elements have p-power order, G is called a p-group. E.g. C_{p^∞} is the Prüfer group.

§6.2.1 Conjugation in S_n

Definition 6.5 (Cycle type)

Let $\sigma \in S_n$ and write σ as a product of disjoint cycles including $1-$cycles. Then the *cycle type* of σ is $(n_1, n_2, \ldots, n_k)$ where $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$ and the cycles in σ have length n_i.
Note $n = n_1 + n_2 + \ldots + n_k$.

Example 6.4

$$
\begin{aligned}
\begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \end{pmatrix} &= \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 & 7 \end{pmatrix}\begin{pmatrix} 8 \end{pmatrix} \\
&= S_8
\end{aligned}
$$

has cycle type $(4, 3, 1)$.
$e \in S_5$ has cycle type $(1, 1, 1, 1, 1)$.

Theorem 6.4

The permutations π and σ in S_n are conjugate *in* S_n (i.e. $\exists\, g \in S_n$ s.t. $g\pi g^{-1} = \sigma$)

iff they have the cycle type.

Proof. Suppose σ has cycle type $(n_1, n_2, \dots, n_k)$. Write

$$\sigma = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n_1} \end{pmatrix} \begin{pmatrix} a_{21} & a_{22} & \dots & a_{2n_2} \end{pmatrix} \dots \begin{pmatrix} a_{k1} & a_{k2} & \dots & a_{kn_k} \end{pmatrix}.$$

Let $\tau \in S_n$.
Then

$$\begin{aligned} \tau\sigma\tau^{-1}(\tau(a_{ij})) &= \tau\sigma(a_{ij}) \\ &= \begin{cases} \tau(a_{ij+1}) & j < n_i \\ \tau(a_{i1}) & j = n_i \end{cases}. \end{aligned}$$

So

$$\begin{aligned} \tau\sigma\tau^{-1} = &\begin{pmatrix} \tau(a_{11}) & \tau(a_{12}) & \dots & \tau(a_{1n_1}) \end{pmatrix} \\ &\begin{pmatrix} \tau(a_{21}) & \tau(a_{22}) & \dots & \tau(a_{2n_2}) \end{pmatrix} \dots \begin{pmatrix} \tau(a_{k1}) & \tau(a_{k2}) & \dots & \tau(a_{kn_k}) \end{pmatrix}. \end{aligned}$$

So if two elements of S_n are conjugate they have the same cycle type.
Furthermore if π has the same cycle type as σ, we can write

$$\pi = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n_1} \end{pmatrix} \dots \begin{pmatrix} b_{k1} & b_{k2} & \dots & b_{kn_k} \end{pmatrix}.$$

Define $\tau(a_{ij}) = b_{ij}$ then $\pi = \tau\sigma\tau^{-1}$. Thus 2 permutations of the same cycle type are conjugate. □

Example 6.5

$$\begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix}$$
$$\begin{pmatrix} 1 & l \end{pmatrix} \begin{pmatrix} 1 & k \end{pmatrix} \begin{pmatrix} 1 & l \end{pmatrix} = \begin{pmatrix} l & k \end{pmatrix}$$

Consider S_4: Let $x \in S_4$, recall

$$24 = |S_4| = |\operatorname{ccl}_{S_4}(x)||C_{S_4}(x)| \text{ by Orbit-Stabiliser Theorem}.$$

example member, x	cycle type	no. of	sgn	$\lvert C_{S_4}(x)\rvert$	$C_{S_4}(x)$
e	$(1,1,1,1)$	1	1	24	S_4
$\begin{pmatrix}1 & 2\end{pmatrix}(3)(4)$	$(2,1,1)$	6	-1	4	$\langle(1\ 2)(3\ 4)\rangle \cong C_2 \times C_2$
$(1\ 2\ 3)(4)$	$(3,1)$	8	1	3	$\langle(1\ 2\ 3)\rangle \cong C_3$
$(1\ 2)(3\ 4)$	$(2,2)$	3	1	8	$\langle(1\ 3\ 2\ 4),(1\ 2)\rangle \cong D_8$
$(1\ 2\ 3\ 4)$	(4)	6	-1	4	$\langle(1,2,3,4)\rangle \cong C_4$.

Corollary 6.2

The number of distinct conjugacy classes of S_n is given by $p(n)$, the number of partitions of n into positive integers, i.e. $n = n_1 + \ldots + n_k$ with $n_1 \geq n_2 \geq \ldots \geq n_k \geq 1$.

However in A_n conjugation is less clear. Certainly

$$\begin{aligned}\mathrm{ccl}_{A_n}(x) &= \{gxg^{-1} : g \in A_n\}\\ &\subseteq \{gxg^{-1} : g \in S_n\} = \mathrm{ccl}_{S_n}(x)\end{aligned}$$

since $A_n \leq S_n$. So if two elements are conjugate in A_n they have the same cycle type. But having the same cycle type in A_n does not guarantee being conjugate.

E.g. $(1\ 2\ 3)$ is not conjugate to $(1\ 3\ 2)$ in A_4.
If $\tau(1\ 2\ 3)\tau^{-1} = (1\ 3\ 2)$ then $\tau = (1\ 2)$ or $(3\ 2)$ or $(1\ 3) \in S_4 \setminus A_4$.

Or consider

$$\begin{aligned}C_{A_4}((1\ 2\ 3)) &= C_{S_4}((1\ 2\ 3)) \cap A_4\\ C_{S_4}((1\ 2\ 3)) &= \langle(1\ 2\ 3)\rangle \leq A_4\\ \text{So, } C_{A_4}((1\ 2\ 3)) &= C_{S_4}((1\ 2\ 3))\\ \implies \lvert\mathrm{ccl}_{A_4}((1\ 2\ 3))\rvert &= \frac{\lvert A_4\rvert}{\lvert C_{A_4}((1\ 2\ 3))\rvert} \text{ by Orbit-Stabiliser Theorem}\\ &= \frac{\lvert S_4\rvert/2}{\lvert C_{S_4}((1\ 2\ 3))\rvert}\\ &= \frac{\lvert\mathrm{ccl}_{S_4}((1\ 2\ 3))\rvert}{2}.\end{aligned}$$

So the conjugacy class of 8 3-cycles in S_4 splits into 2 conjugacy classes in A_4.

Key point Let $x \in A_n$. If $C_{A_n}(x) = C_{S_n}(x) \implies \mathrm{ccl}_{A_n}(x) = \frac{\lvert\mathrm{ccl}_{S_n}(x)\rvert}{2}$ by Orbit-Stabiliser Theorem.
If $C_{A_n}(x) \lneq C_{S_n}(x)$, then C_{S_n} contains an odd permutation and $\lvert C_{A_n}(x)\rvert = \lvert C_{S_n} \cap A_n\rvert = \frac{\lvert C_{S_n}\rvert}{2}$ (Q4, Sheet 2) $\implies \lvert\,\mathrm{ccl}_{A_n}(x)\rvert = \lvert\,\mathrm{ccl}_{S_n}(x)\rvert$.

A_4:

example member, x	cycle type	$C_{A_4}(x)$	size of ccl
e	$(1,1,1,1)$	A_4	1
$(1\ 2\ 3)$	$(3,1)$	$\langle(1\ 2\ 3)\rangle$	4
$(1\ 3\ 2)$	$(3,1)$	$\langle(1\ 3\ 2)\rangle$	4
$(1\ 2)(3\ 4)$	$(2,2)$	$\{e,(1\ 2)(2\ 4),(1\ 4)(2\ 3)\} \cong C_2 \times C_2$	3

Remark 23. The number of elements in S_n with k_l cycles of length l is given by

$$\frac{n!}{\Pi_l k_l!\, l^{k_l}}.$$

Think of cycles as trays, put in elements of $X = \{1,2,\dots,n\}$. This gives $n!$ options, but we've overcounted. Eah cycle of length l can be written l ways, this given l^{k_l} factor. Also k_l cycle of length l can be permuted in $k_l!$ ways.

Example 6.6

E.g. Number of cycles in S_5 of type $(\cdot\ \cdot)(\cdot\ \cdot)(\cdot)$, so $k_2 = 2$ and $k_1 = 1$.

$$\text{no. of} = \frac{5!}{2!\,2^2 1!\,1} = 15$$

Or $(\cdot\ \ \cdot\ \ \cdot)(\cdot\cdot)$, $k_3 = 1, k_2 = 1$

$$\text{no. of} = \frac{5!}{1!\,3^1 1!\,2^1} = 20.$$

Let us consider S_5, $|S_5| = 120$.

example member, x	cycle type	no. of	sgn	$\lvert C_{S_5}(x)\rvert$	$C_{S_5}(x)$
e	$(1,1,1,1,1)$	1	1	120	S_5
$(1\ 2)$	$(2,1,1,1)$	10	-1	12	$\langle(1\ 2)\rangle \times \text{Sym}\{3,4,5\} \cong C_2 \times S_3$
$(1\ 2)(3\ 4)$	$(2,2,1)$	15	1	8	$\langle(1\ 3\ 2\ 4),(1\ 2)\rangle \cong D_8$
$(1\ 2\ 3)$	$(3,1,1)$	20	1	6	$\langle(1\ 2\ 3),(4\ 5)\rangle \cong C_6$
$(1\ 2\ 3)(4\ 5)$	$(3,2)$	20	-1	6	"
$(1\ 2\ 3\ 4)$	$(4,1)$	30	-1	4	$\langle(1\ 2\ 3\ 4)\rangle$
$(1\ 2\ 3\ 4\ 5)$	(5)	24	1	5	$\langle(1\ 2\ 3\ 4\ 5)\rangle$

Now consider A_5, $|A_5| = 60$.

example member, x	cycle type	$C_{A_5}(x)$	$\lvert \text{ccl}_{A_5}(x)\rvert$
e	$(1,1,1,1,1)$	A_5	1
$(1\ 2)(3\ 4)$	$(2,2,1)$	$\langle(1\ 2)(3\ 4),(1\ 3)(2\ 4)\rangle$	15
$(1\ 2\ 3)$	$(3,1,1)$	$\langle(1\ 2\ 3)\rangle$	20
$(1\ 2\ 3\ 4\ 5)$	(5)	$\langle(1\ 2\ 3\ 4\ 5)\rangle$	12
$(2\ 1\ 3\ 4\ 5)$	(5)	$\langle(2\ 1\ 3\ 4\ 5)\rangle$	12

Recall Definition 4.2, a group is *simple* if it has no non-trivial proper normal subgroups, i.e. if the only normal subgroups are $\{e\}$ and the group itself.

Theorem 6.5

A_5 is a simple group.

Proof. Suppose $N \trianglelefteq A_5$. Then N is a union of conjugacy classes (Sheet 3, Q3(a)). $\implies |N| = 1 + 15a + 20b + 12c + 12d$ where $a, b, c, d \in \{0, 1\}$. But by Lagrange's Theorem $|N| \Big| |A_5| = 60 \implies |N| = 1$ or 60. $\square$

Comments

1. A_5 is the smallest non-abelian simple group.
2. A_n is simple $\forall\, n \geq 5$ (GRM), but A_4 is not simple.
3. Classification of finite simple groups exists, includes ∞ families
   - C_p where p is prime (only abelian simple groups)
   - A_n for $n \geq 5$
   - groups of 'Lie type', matrix groups
   - 26 sporadic groups, include Monster and Baby Monster

§7 Matrix Groups

Let $M_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with entries in $\mathbb{R}$. Define

$$\mathrm{GL}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det A \neq 0\}.$$

Proposition 7.1

$\mathrm{GL}_n(\mathbb{R})$ is a group under matrix multiplication. It is called the *general linear group*.

Proof. closure: $A, B \in \mathrm{GL}_n(\mathbb{R})$, clearly $AB \in \mathrm{M_n}(\mathbb{R})$ and $\det(AB) = \det A \det B \neq 0$ so $AB \in \mathrm{GL}_n(\mathbb{R})$.

identity: $I_n = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$

inverse: $\det A \neq 0 \implies A^{-1}$ exists, $\det A^{-1} = \frac{1}{\det A} \neq 0$.

matrix multiplication is associative:

$$\begin{aligned} (A(BC))_{ij} &= A_{ik}(BC)_{kj} \\ &= A_{ik}B_{kt}C_{tj} \\ ((AB)C)_{ij} &= (AB)_{ik}C_{kj} \\ &= A_{it}B_{tk}C_{kj}. \end{aligned}$$

□

Example 7.1

$$\mathrm{GL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc \neq 0 \right\}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & -a \end{pmatrix}.$$

Proposition 7.2

$$\mathrm{Det} : \mathrm{GL}_n(\mathbb{R}) \to (\mathbb{R} \setminus \{0\}, \times)$$

$$A \mapsto \det A.$$

is a surjective group homomorphism.

Proof. Note $(\mathbb{R} \setminus \{0\}, \times)$ is a group. Det is clearly a map to $(\mathbb{R} \setminus \{0\}, \times)$, we need to check it's a group homomorphism,

$$\begin{aligned} \operatorname{Det}(AB) &= \det(AB) \quad (AB \text{ is multiplication in } \operatorname{GL}_n) \\ &= \det A \cdot \det B \quad (\text{multiplication in } (\mathbb{R} \setminus \{0\}, \times)) \\ &= \operatorname{Det} A \operatorname{Det} B \end{aligned}$$

and that Det is surjective. Let $r \in (\mathbb{R} \setminus \{0\}, \times)$ then

$$A = \begin{pmatrix} r & & & & 0 \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \in \operatorname{GL}_n(\mathbb{R}) \mapsto \det(A) = r.$$

□

By 1st Isomorphism Theorem $\ker(\operatorname{Det}) \trianglelefteq \operatorname{GL}_n(\mathbb{R})$.

$$\begin{aligned} \ker(\operatorname{Det}) &= \{A \in \operatorname{GL}_n(\mathbb{R}) : \det A = 1\} \\ &= \operatorname{SL}_n(\mathbb{R}) \text{ the special linear group.} \end{aligned}$$

Furthermore, by 1st Isomorphism Theorem

$$\operatorname{GL}_n(\mathbb{R}) / \operatorname{SL}_n(\mathbb{R}) \cong (\mathbb{R} \setminus \{0\}, \times).$$

Remark 24. More generally we can define the general linear group and special linear group over any field.
Examples of fields:
$\mathbb{R}, \mathbb{C}, \mathbb{Q}, \mathbb{F}_p$ where $\mathbb{F}_p = (\{0, 1, 2, \ldots, p-1\}, +_p, \times_p)$ and p is prime. Note $\operatorname{GL}_n(\mathbb{F}_p)$ and $\operatorname{SL}_n(\mathbb{F}_p)$ are finite groups.

What is $|GL_3(\mathbb{F}_p)|$?
Non-zero determinant means we have linearly independent columns. The no. of choices for the first column is $p^3 - 1$ (can't be $\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^T$). The second column is not a multiple of first so $p^3 - p$ (there are p multiples of first column). Third column not in space spanned by first two columns, this space has size p^2 (consider $\alpha c_1 + \beta c_2$ with $\alpha, \beta \in \mathbb{F}_p$), so $p^3 - p^2$.
$\implies |GL_3(\mathbb{F}_p)| = (p^3 - 1)(p^3 - p)(p^3 - p^2)$.

We can still consider

$$\begin{aligned} \text{Det} : \text{GL}_3(\mathbb{F}_p) &\to (\mathbb{F}_p \setminus \{0\}, \times) \\ A &\mapsto \det(A). \end{aligned}$$

Note $(\mathbb{F}_p \setminus \{0\}, \times)$ is a group: we have closure, identity $= 1$ and associativity. Let $a \in \mathbb{F}_p \setminus \{0\}$, by Bezout's Thm $\exists\, x, y$ s.t.

$$\begin{aligned} ax + py &= 1 \\ \therefore ax &\equiv 1 \pmod p \\ \text{Choose } \bar{x} &\equiv x \pmod p, \\ 1 \le \bar{x} &\le p - 1, \ a^{-1} = x \end{aligned}$$

Det is a surjective homomorphism to $(\mathbb{F}_p \setminus \{0\}, \times)$ so by 1st Isomorphism Theorem

$$\begin{aligned} |\,\text{GL}_3(\mathbb{F}_p)|/|\,\text{SL}_3(\mathbb{F}_p)| &= p - 1 \\ \implies |\,\text{SL}_3(\mathbb{F}_p)| &= \frac{(p^3-1)(p^3-p)(p^3-p^2)}{p-1}. \end{aligned}$$

§7.1 Actions of $\text{GL}_n(\mathbb{C})$

1. Let $\mathbb{C}^n$ denote vectors of length n with entries in $\mathbb{C}$:

$$\begin{aligned} \text{GL}_n(\mathbb{C}) \times \mathbb{C}^n &\to \mathbb{C}^n \\ (A, v) &\mapsto Av. \end{aligned}$$

   Note $Iv = v$, $(AB)v = A(B(v))$. This action is faithful: $Av = v \forall\ v \in \mathbb{C}^n \implies A = I_n$ (consider $v = e_i$). The action has two orbits

$$\begin{aligned} \text{Orb}_{\text{GL}_n(\mathbb{C})}(\underline{0}) &= \{\underline{0}\} \\ \text{Orb}_{\text{GL}_n(\mathbb{C})}(v) &= \mathbb{C}^n \setminus \{0\} \text{ for } v \neq \underline{0}, \end{aligned}$$

   i.e. given $w \neq 0\ \exists\ A \in \text{GL}_n(\mathbb{C})$ s.t. $Av = w$.

2. Conjugation action of $\text{GL}_n(\mathbb{C})$ on $M_n(\mathbb{C})$ (set of all matrices)

$$\begin{aligned} \text{GL}_n(\mathbb{C}) \times M_n(\mathbb{C}) &\to M_n(\mathbb{C}) \\ (P, A) &\mapsto PAP^{-1}. \\ \text{Note: } PQ(A) &= PQA(PQ)^{-1} \\ &= PQAQ^{-1}P^{-1} \\ &= P(Q(A)). \end{aligned}$$

Remark 25. Matrices A and B are conjugate if they represent the same linear map. If $PAP^{-1} = B$, then P represents a change of basis matrix. (See LA next year)

Example 7.2

$$\begin{aligned} A : e_1 &\mapsto 2e_1 \\ e_2 &\mapsto 3e_2 \\ A &= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \\ \text{Let } P : e_1 &\mapsto e_2 \\ e_2 &\mapsto e_1 \\ P &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= P^{-1} \end{aligned}$$

P is a change of basis

$$\begin{aligned} PAP^{-1} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \\ \text{i.e. } e_2 &\mapsto 3e_2 \\ e_1 &\mapsto 2e_1. \end{aligned}$$

We will use the following result from V&M when investigating Möbius groups.

Result Let $A \in M_2(\mathbb{C})$ and consider conjugation action of $\mathrm{GL}_2(\mathbb{R})$ on $M_2(\mathbb{C})$. Then precisely one of the following occurs

1. The orbit of A contains a diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ with $\lambda \neq \mu$.
2. The orbit of A is $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I$ for some λ.
3. The orbit of A contains a matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ for some λ.

Proof. See V&M but essentially

1. In this case A has 2 distinct eigenvalues $\lambda \neq u$, take a basis consisting of an eigenvector for λ and an eigenvector for μ. Distinct pairs given distinct orbits.
2. $PAP^{-1} = \lambda I \implies A = P\lambda I P^{-1} = \lambda I$, eigenvalues λ, λ, 2 linearly indepen-

dent eigenvectors.

3. In this case A has a repeated eigenvalue, but just one linearly independent eigenvector.

□

---

§7.2 Orthogonal Group

Aside - Recalling transpose properties

Recall if $A \in M_n(\mathbb{R})$, A^T is defined by $(A^T)_{ij} = A_{ji}$, i.e. the ij-th entry of A^T is the ji-th entry of A

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$$
$$A^T = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Note

1.

$$\begin{aligned} (AB)^T &= B^T A^T \\ [(AB)^T]_{ij} &= (AB)_{ji} \\ &= A_{jk} B_{ki} \\ [B^T A^T]_{ij} &= B^T_{ik} A^T_{kj} \\ &= B_{ki} A_{jk} \checkmark \end{aligned}$$

2.

$$\begin{aligned} AA^T &= I \\ \implies 1 &= \det(A^T A) \\ &= \det A^T \det A \\ &= (\det A)^2 \\ \implies \det A &\neq 0 \end{aligned}$$

3.

$$
\begin{aligned}
&AA^T = I \iff A^T A = I. \\
\implies &A^T A = A^{-1} \underbrace{AA^T}_{I} A \\
&\qquad = A^{-1} A \\
&\qquad = I.
\end{aligned}
$$

4.

$$
\begin{aligned}
(A^T)^{-1} &= (A^{-1})^T \\
\text{Since } I_n &= (AA^{-1})^T \\
&= (A^{-1})^T A^T
\end{aligned}
$$

5.

$$
\det A^T = \det A.
$$

Definition 7.1 (Orthogonal group)

$$
O_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^T A = I\}
$$

(so columns of A form an orthonormal basis for $\mathbb{R}^n$).

Proposition 7.3

$O_n(\mathbb{R})$ is a subgroup of $\mathrm{GL}_n(\mathbb{R})$ called the *orthogonal group*.

Proof.

- $\det A \neq 0 \implies O_n(\mathbb{R}) \subseteq \mathrm{GL}_n(\mathbb{R})$.
- closure:

$$
\begin{aligned}
A, B &\in O_n(\mathbb{R}) \\
(AB)^T (AB) &= B^T A^T AB \\
&= B^T B = I \\
\implies AB &\in O_n(\mathbb{R})
\end{aligned}
$$

- $I_n \in O_n(\mathbb{R})$
- Associativity is inherited.
- inverse: $A^T A = I_n \implies A^T = A^{-1}$ and $A^T \in O_n(\mathbb{R})$ since $(A^T)^T = A$ and $AA^T = I$.

□

Note $1 = (\det A)^2 \implies \det A = \pm 1$ if $A \in O_n(\mathbb{R})$.
So,

$$\begin{aligned} \text{Det} : O_n(\mathbb{R}) &\to (\{\pm 1\}, \times) \\ A &\mapsto \det A \end{aligned}$$

is a surjective homomorphism as, $\begin{pmatrix} -1 & & & & \\ & 1 & & \Large 0 & \\ & & 1 & & \\ & \Large 0 & & \ddots & \\ & & & & 1 \end{pmatrix} \in O_n(\mathbb{R})$.

$$\begin{aligned} \text{So } \ker(\text{Det}) &= \{A \in O_n(\mathbb{R}) : \det A = 1\} \\ &= SO_n(\mathbb{R}) \trianglelefteq O_n(\mathbb{R}). \end{aligned}$$

By 1st Isomorphism Theorem:

$$O_n(\mathbb{R})/SO_n(\mathbb{R}) \cong C_2.$$

Lemma 7.1

Let $A \in O_n(\mathbb{R})$ and $\underline{x}, \underline{y} \in \mathbb{R}^n$. Then

1. $A\underline{x} \cdot A\underline{y} = \underline{x} \cdot \underline{y}$
2. $|A\underline{x}| = |\underline{x}|$.

So A is an isometry (distance preserving map) of Euclidean space $\mathbb{R}^n$.

Proof. 1.

$$\begin{aligned} A\underline{x} \cdot A\underline{y} &= (A\underline{x})^T (A\underline{y}) \\ &= \underline{x}^T A^T A \underline{y} \\ &= \underline{x}^T \underline{y} \\ &= \underline{x} \cdot \underline{y}. \end{aligned}$$

2.

$$\begin{aligned}|A\underline{x}|^2 &= A\underline{x} \cdot A\underline{x} \\ &= \underline{x} \cdot \underline{x} \\ &= |\underline{x}|^2\end{aligned}$$

□

Note by Item 2 if λ is an eigenvalue of A, $A\underline{x} = \lambda\underline{x} \implies |\lambda\underline{x}| = |\underline{x}|$ i.e. $|\lambda| = 1$.

§7.2.1 In 2 dimensions

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ and

$$\begin{aligned} I &= AA^T \\ &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} \\ \implies 1 &= a^2 + b^2 = c^2 + d^2 \\ 0 &= ac + bd. \\ I &= A^TA \\ &= \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \implies 1 &= a^2 + c^2 = b^2 + d^2 \\ 0 &= ab + cd. \end{aligned}$$

For $0 \le \theta < 2\pi$ let

$$\begin{aligned} \begin{pmatrix} a \\ c \end{pmatrix} &= \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \\ \text{so } \begin{pmatrix} b \\ d \end{pmatrix} &= \begin{pmatrix} \mp\sin\theta \\ \pm\cos\theta \end{pmatrix}. \\ A &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

$\det A = 1$ so a rotation and all elements of $SO_2(\mathbb{R})$ are of this form.

Or $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$

$$\det A = -1$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = e^{i\theta}\overline{x+iy} = e^{i\theta}\bar{z}$$

What are the fixed points?

$$\begin{aligned} z = e^{i\theta}\bar{z} &\iff e^{-i\theta/2}z = e^{i\theta/2}\bar{z} \\ &\iff e^{-i\theta/2}z = t \in \mathbb{R} \\ &\iff z = e^{i\theta/2}t. \\ &\implies \text{ a reflection in line } e^{i\theta/2} \end{aligned}$$

All elmenents of $O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$ of this form.

$$\text{So, } O_2(\mathbb{R}) = \underbrace{SO_2(\mathbb{R})}_{\text{rotations}} \dot\cup \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SO_2(\mathbb{R})}_{\text{reflections}}$$

Note any element of $O_2(\mathbb{R})$ is a product of at most two reflections. Since if $A \in SO_2(\mathbb{R})$ then $A = \underbrace{\left(A\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)}_{\text{reflection}}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{reflection}}$.

§7.2.2 In 3 dimensions

Proposition 7.4
Let $A \in SO_3(\mathbb{R})$. Then A has an eigenvector with eigenvalue 1.

Proof.

$$\begin{aligned} \det(A-I) &= \det(A - AA^T) \\ &= \det A \det(I - A^T) \\ &= 1 \cdot \det(I-A)^t \\ &= \det(I-A) \\ &= (-1)^3 \det(A-I) \\ &= -\det(A-I) \\ \implies \det(A-I) &= 0 \text{ and } A \text{ has an eigenvalue} = 1. \end{aligned}$$

□

Alternative Proof. Consider $\chi_A(x)$ (characteristic poly of A), it is a cubic in $\mathbb{R}$. Thus it has a real root, and $|\lambda| = 1 \implies \lambda = \pm 1$. But the other eigenvalues are either a complex conjugate pair, then $\lambda = 1$ as product of eigenvalues give $\det A = 1$, or all are real so either $1, -1, -1$ or $1, 1, 1$. $\square$

Theorem 7.1

Let $A \in SO_3(\mathbb{R})$ then A is conjugate (i.e. there is a change of basis) to a matrix of the form $\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for some $\theta \in [0, 2\pi]$. In particular, A is a rotation around an axis through the origin.

Proof. By Proposition 7.4 $\exists\ \underline{v} \in \mathbb{R}^3$ with $A\underline{v} = \underline{v}$, we can assume $|\underline{v}| = 1$. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for $\mathbb{R}^3$. There exists $P \in SO_3(\mathbb{R})$ s.t. $P\underline{v} = e_3$. So $PAP^{-1}(e_3) = e_3$ and for Π plane perpendicular to e_3 then $PAP^{-1}(\Pi)$ is perpendicular to e_3. So,

$$\begin{aligned} PAP^{-1} &= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\text{action}} & 0 \\ \multicolumn{2}{c|}{\text{on } \Pi} & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \\ &= \left(\begin{array}{cc|c} \multicolumn{2}{c|}{\multirow{2}{*}{Q}} & 0 \\ & & 0 \\ \hline 0 & 0 & 0 \end{array}\right) \end{aligned}$$

$$\det(PAP^{-1}) = \det A = 1$$

$$\implies \det Q = 1$$

$$(PAP^{-1})(PAP^{-1})^T = I \implies QQ^T = I.$$

$$\text{So, } Q \in SO_2(\mathbb{R}).$$

$\square$

Suppose $\underline{r}$ is a reflection in a plane Π through 0. Let $\underline{n}$ be a unit vector perpendicular to Π. Then $r(\underline{x}) = \underline{x} - 2(\underline{x} \cdot \underline{n})\underline{n}$, $\underline{n} \mapsto -\underline{n}$, Π is fixed. So $\underline{r}$ is conjugate to $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in O_3(\mathbb{R})$ by taking basis $\underline{n}$ and two orthogonal unit vectors in Π.

$$O_3(\mathbb{R}) = \underbrace{SO_3(\mathbb{R})}_{\text{rotations, det} = 1} \ \dot{\cup}\ \underbrace{\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} SO_3(\mathbb{R})}_{\det = -1}$$

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} SO_3(\mathbb{R})$ includes

- reflections
- inversion in origin, $-I_3$
- combinations of rotations and reflections

Theorem 7.2

Any element of $O_3(\mathbb{R})$ is a product of at most 3 reflections.

Proof. Let $\{e_1, e_2, e_3\}$ be the standard orthonormal basis for $\mathbb{R}^3$. Let $A \in O_3(\mathbb{R})$.

$$|Ae_3| = |e_3| = 1 \text{ since } A \text{ is an isometry.}$$

So $\exists$ a reflection r_1 s.t. $r_1 A(e_3) = e_3$. Let $\Pi = \langle e_1, e_2 \rangle \perp e_3$. So $r_1 A(\Pi) = \Pi$, as angles are preserved.
$\exists$ a reflection r_2 s.t. $r_2(e_3) = e_3$, $r_2(r_1 A(e_2)) = e_2$. So $r_2 r_1 A$ fixes e_2 and e_3.
So $r_2 r q A(e_1) = \pm e_1$, if $e_1 = e_1$, set $r_3 = \mathrm{id}$. Else $e_1 = -e_1$, let r_3 be the reflection in the plane $\perp$ to e_1.
So $r_3 r_2 r_1 A$ fixes e_1, e_2, e_3, so $r_3 r_2 r_1 A = \mathrm{id} \implies A = r_1^{-1} r_2^{-1} r_3^{-1} = r_1 r_2 r_3$. $\square$

Alternatively, any element in $SO_3(\mathbb{R})$ is a product of at most 2 reflections, via 2-dimensional case. Thus any element of $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} SO_3(\mathbb{R})$ is a product of at most 3 reflections. Note we do need 3, e.g. $-I_3$.

§8 Möbius Groups

Definition 8.1 (Möbius transformation)

A Möbius transformation (or map) is a function of a complex variable z that can be written in the form

$$f(z) = \frac{az+b}{cz+d}$$

for some $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$.

Why $ad - bc \neq 0$?

$$f(z) - f(w) = \frac{(ad-bc)(z-w)}{(cz+d)(cw+d)}.$$

So, $ad - bc = 0 \implies f$ is constant (not interesting). If, $ad - bc \neq 0 \implies f$ is injective.

When does $f(z) = g(z)$ ($g(z)$ is $f(z)$ with different a, b, c, d)? Suppose $\exists$ at least 3 values of z in $\mathbb{C}$ s.t.

$$\frac{az+b}{cz+d} = \frac{\alpha z+\beta}{\gamma z+\delta}$$
$$ad - bc \neq 0, \ \alpha\delta - \beta\gamma \neq 0.$$

Then $\exists\, \lambda \neq 0, \lambda \in \mathbb{C}$ s.t.

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Since, we have 3 distinct values of z for which

$$(az+b)(\gamma z+\delta) = (\alpha z+\beta)(cz+d)$$

so these quadratics are identical

$$\begin{aligned} &\implies \alpha\gamma = \alpha c, \ b\delta = \beta d \\ &a\delta + \beta\delta = \alpha d + \beta c \\ &\quad \text{Let } \mu = a\delta - \beta c = \alpha d - b\gamma \\ &\quad (\text{so } \mu^2 = (ad-bc)(\alpha\delta - \beta\gamma) \neq 0). \end{aligned}$$

$$\text{Then } \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \mu & 0 \\ 0 & \mu \end{pmatrix}$$
$$\implies \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \frac{\mu}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Problem: f is not defined at $z = -\frac{d}{c}$. We would like $f(-\frac{d}{c}) = \infty$. We consider f defined on $\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$ the extended complex plane. So if $f(z) = \frac{az+b}{cz+d}$, domain is now $\mathbb{C}_\infty$. If $c \neq 0$; $f(\infty) = \frac{a}{c}$; $f(-\frac{d}{c}) = \infty$ else $c = 0$; $f(\infty) = \infty$.

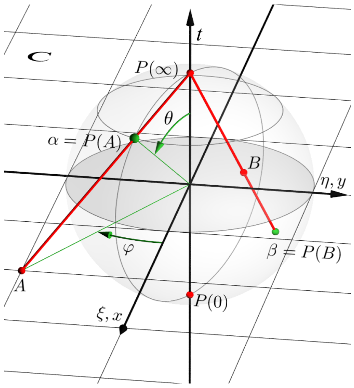


Figure 7: Stereographic projection of a complex number A onto a point α of the Riemann sphere

Theorem 8.1

The set $\mathcal{M}$ of all Möbius maps on $\mathbb{C}_\infty$ is a group under composition. It is a subgroup of $\mathrm{Sym}(\mathbb{C}_\infty)$.

Proof.

- Composition of maps is associative.
- $I(z) = z \in \mathcal{M}$ $(a = d = 1, c = d = 0)$
- closure:

$$\text{Let } f(z) = \frac{az+b}{cz+d},\ g(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$$

$$\text{Suppose } c \neq 0,\ \delta \neq 0$$

$$\text{First suppose } z \in \mathbb{C} \setminus \{-\delta/\gamma\}$$

$$\text{Then } f(g(z)) = \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right)+d} = \frac{(a\alpha+b\gamma)z+(a\beta+b\delta)}{(c\alpha+d\gamma)z+(c\beta+\delta d)} \in \mathcal{M}$$

$$\text{since}(a\alpha+b\gamma)(c\beta+\delta d)-(a\beta+b\delta)(c\alpha+d\gamma) = (ad-bc)(\alpha\delta-\beta\gamma) \neq 0$$

$$\text{Also, } f\left(g\left(-\frac{\delta}{\gamma}\right)\right) = f(\infty) = \frac{a}{c}$$

$$\text{and } \frac{(a\alpha+b\gamma)\left(-\frac{\delta}{\gamma}\right)+(a\beta+b\delta)}{(c\alpha+d\gamma)\left(-\frac{\delta}{\gamma}\right)+(c\beta+\delta d)} = \frac{a\alpha\left(-\frac{\delta}{\gamma}\right)+\alpha\beta}{c\alpha\left(-\frac{\delta}{\gamma}\right)+c\beta}$$

$$= \frac{a}{c} \checkmark$$

Need to check $c = 0$

- Inverses:

$$f(z) = \frac{az+b}{cz+d}, \; ad - bc \neq 0$$

$$\text{Let } f^*(z) = \frac{dz-b}{-cz+a}$$

$$\text{Then } f(f^*(z)) = z = f^*(f(z)) \text{ for } z \neq -\frac{d}{c}, -\frac{a}{c}, \infty$$

These cases are ok.

$$\text{If } c = 0$$

$$f(f^*(\infty)) = f(\infty) = \infty = f^*(f(\infty)).$$

□

Theorem 8.2

$GL_2(\mathbb{C})/Z \cong \mathcal{M}$ where $Z = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathbb{C} \setminus \{0\} \right\}$ (Z is the centre of $GL_2(\mathbb{C})$).

Proof. We construct a surjective homomorphism from $GL_2(\mathbb{C})$ onto $\mathcal{M}$ with kernel Z.

$$\text{Let } \Phi : GL_2(\mathbb{C}) \to \mathcal{M}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto f(z) = \frac{az+b}{cz+d}$$

Note Φ is a homomorphism

$$f(z) = \frac{az+b}{cz+d}, \; g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

$$\Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \Phi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)(z) = f \circ g(z)$$

$$= \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + \delta d)} \text{ from proof of Theorem 8.1}$$

$$= \Phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)(z).$$

Clearly Φ is surjective.

$$\begin{aligned}
&\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \ker \Phi \\
\iff &\frac{az+b}{cz+d} = z \; \forall \; z \in C_\infty \\
&\text{Let } z = 0 \implies c = 0 \\
&z = 0 \implies b = 0 \\
&z = 1 \implies a = d \\
\implies &\ker \Phi = Z
\end{aligned}$$

Finally apply 1st Isomorphism Theorem. □

Corollary 8.1

$$\frac{\mathrm{SL}_2(\mathbb{C})}{\{\pm I\}} = \mathcal{M}.$$

Proof. Restrict ϕ to $\mathrm{SL}_2(\mathbb{C})$

$$\begin{aligned}
\phi : \mathrm{SL}_2(\mathbb{C}) &\to \mathcal{M} \\
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \frac{az+b}{cz+d}
\end{aligned}$$

We require ϕ to be surjective:

$$\begin{aligned}
f(z) &= \frac{az+b}{cz+d} \\
&= \frac{\frac{a}{\sqrt{ad-bc}}z + \frac{b}{\sqrt{ad-bc}}}{\frac{c}{\sqrt{ad-bc}}z + \frac{d}{\sqrt{ad-bc}}}.
\end{aligned}$$

And $\ker \phi = \{\pm I\}$. □

Proposition 8.1

Every Möbius map can be written as a composition of maps of the following forms:

1. $z \mapsto az$, $a \neq 0$; represents a dilation or a rotation
2. $z \mapsto z + b$; a translation

3. $z \mapsto \frac{1}{z}$; inversion.

Proof. Let $f(z) = \frac{az+b}{cz+d}$.

$$\text{If } c = 0;\ z \mapsto \underbrace{\left(\frac{a}{d}\right) z}_{f_1,\ Item\ 1} \mapsto \underbrace{\left(\frac{a}{d}\right) z + \frac{b}{d}}_{f_2,\ Item\ 2}$$

$$f = f_2 \circ f_1$$

$$\text{If } c \neq 0$$

$$\begin{aligned} f(z) = \frac{az+b}{cz+d} &= \frac{\left(\frac{a}{c}\right) z + \left(\frac{b}{c}\right)}{z + \left(\frac{d}{c}\right)} \\ &= \left(\frac{a}{c}\right) + \frac{\frac{-ad+bc}{c^2}}{z + \frac{d}{c}} \\ &= A + \frac{B}{z + \frac{d}{c}},\quad B \neq 0 \end{aligned}$$

$$\begin{aligned} z &\underset{f_1,\ Item\ 2}{\mapsto} z + \frac{d}{c} \\ &\underset{f_2,\ Item\ 3}{\mapsto} \frac{1}{z + \frac{d}{c}} \\ &\underset{f_3,\ Item\ 1}{\mapsto} \frac{B}{z + \frac{d}{c}} \\ &\underset{f_4,\ Item\ 2}{\mapsto} A + \frac{B}{z + \frac{d}{c}} \end{aligned}$$

$$f = f_4 \circ f_3 \circ f_2 \circ f_1$$

□

Definition 8.2 (Triply transitive action)

A group G acts *triply transitively* on a set X if given $x_1, x_2, x_3 \in X$ all distinct and $y_1, y_2, y_3 \in X$ all distinct there exists $g \in G$ such that $g(x_i) = y_i,\ i = 1, 2, 3$.
A group G acts *sharply triply transitively* if such a g is unique.

Theorem 8.3

The action of $\mathcal{M}$ on $\mathbb{C}_\infty$ is sharply triply transitive.

Proof. Label first triple $\{z_0, z_1, z_\infty\}$ and second triple $\{w_0, w_1, w_\infty\}$. We construct $g \in \mathcal{M}$ s.t.

$$\begin{aligned} g : z_0 &\mapsto 0 \\ z_1 &\mapsto 1 \\ z_\infty &\mapsto \infty. \end{aligned}$$

First suppose $z_0, z_1, z_\infty \neq \infty$

$$g(z) = \frac{(z - z_0)(z_1 - z_\infty)}{(z - z_\infty)(z_1 - z_0)}$$
$$\text{Check: ``}ad - bc\text{''} = (z_0 - z_\infty)(z_1 - z_\infty)(z_1 - z_0) \neq 0.$$

If $z_\infty = \infty$

$$g(z) = \frac{(z - z_0)}{(z_1 - z_0)}$$

If $z_1 = \infty$

$$g(z) = \frac{(z - z_0)}{(z - z_\infty)}$$

If $z_0 = \infty$

$$g(z) = \frac{(z_1 - z_\infty)}{(z - z_\infty)}$$

Similarly find h s.t.

$$\begin{aligned} g : w_0 &\mapsto 0 \\ w_1 &\mapsto 1 \\ w_\infty &\mapsto \infty. \\ \text{Then } f = h^{-1}g &: z_i \to w_i \end{aligned}$$

Now to prove uniqueness.

$$\text{Suppose } f' : z_i \mapsto w_i$$
$$\text{Then } f^{-1} \circ f' : z_i \mapsto z_i$$

Let g be as above

$$\begin{aligned} gf^{-1}f'g^{-1} &: 0 \mapsto 0 \implies b = 0 \\ &: 1 \mapsto 1 \implies a = d \end{aligned}$$

$$\begin{aligned} :\infty \mapsto \infty &\implies c = 0 \\ \implies gf^{-1}f'g^{-1} &= \mathrm{id} \\ \implies f^{-1}f' &= \mathrm{id} \\ \implies f &= f'. \end{aligned}$$

So, the image of just three points determines the map □

§8.1 Conjugacy classes in $\mathcal{M}$

Recall $\Phi : \mathrm{GL}_2(\mathbb{C}) \twoheadrightarrow \mathcal{M}$ from proof of Theorem 8.2 ($\twoheadrightarrow$ means its a surjective homomorphism). Suppose A, B are conjugate in $\mathrm{GL}_2(\mathbb{C})$, i.e. $\exists\ P \in \mathrm{GL}_2(\mathbb{C})$ s.t. $PAP^{-1} = B$ then

$$\begin{aligned} \Phi(P)\Phi(A)\Phi(P)^{-1} &= \phi(PAP^{-1}) \\ &= \Phi(B) \in \mathcal{M} \end{aligned}$$

i.e. $\Phi(A)$ and $\Phi(B)$ are conjugate in $\mathcal{M}$.

Use knowledge of conjugacy classes in $\mathrm{GL}_2(\mathbb{C})$.

Theorem 8.4

Any non-identity Möbius map is conjugate to $f(z) = \nu z$ for some $\nu \neq 0, 1$ or to $f(z) = z + 1$.

Proof. 1.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ where } \lambda \neq \mu,\ \lambda \neq 0 \neq \mu.$$

$$\Phi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\right) = f,\ f(z) = \frac{\lambda}{\mu} z = \nu z,\ \nu \neq 0, 1.$$

2.

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ where } \lambda \neq 0$$

$$\Phi\left(\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \mathrm{id}.$$

3.

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \text{ where } \lambda \neq 0$$

$$\Phi\left(\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right) = f,\ f(z) = \frac{\lambda z + 1}{\lambda} = z + \frac{1}{\lambda}$$

$$\text{i.e. } f = \Phi\left(\begin{pmatrix} 1 & \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix}\right).$$

$$\text{And } \begin{pmatrix} 1 & \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix} \text{ is conjugate to } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\text{via } \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{\lambda} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

So f is conjugate to g where $g(z) = z + 1$.

□

Corollary 8.2

A non-identity Möbius map has either

1. 2 fixed points or
2. 1 fixed point.

Proof. Suppose $gfg^{-1} = h$. Then α is a fixed point of f (i.e. $f(\alpha) = \alpha$) $\iff$ $g(\alpha)$ is a fixed point of h (i.e. $h(g(\alpha)) = g(\alpha)$).[a]
So number of fixed points of f = number of fixed points of h as g is a bijection. By Theorem 8.4 either, f conjugate to $z \mapsto \nu z$ which has two fixed points: $0, \infty$; or f conjugate to $z \mapsto z + 1$ which has one fixed point: ∞. □

[a] $f(\alpha) = \alpha \iff gf(\alpha) = g(\alpha) \iff h(g(\alpha)) = gfg^{-1}(g(\alpha)) = g(\alpha)$ or $gf = hg \implies h(g(\alpha)) = g(\alpha)$ if $f(\alpha) = \alpha$ and conversely $g(f(\alpha)) = h(g(\alpha)) \implies g(f(\alpha)) = g(\alpha)$.

§8.2 Circles in $\mathbb{C}_\infty$

A Euclidean circle is the set of points in $\mathbb{C}$ given by some equation $|z - z_0| = r,\ r > 0$. A Euclidean line is the set of points in $\mathbb{C}$ given by some equation $|z - a| = |z - b|,\ a \neq b$.

Definition 8.3 (Circle in $\mathbb{C}_\infty$)

A *circle in* $\mathbb{C}_\infty$ is either a Euclidean circle of a set $L \cup \{\infty\}$ where L is a Euclidean line. Its general equation is of the form

$$Az\bar{z} + \bar{B}z + B\bar{z} + C = 0, \tag{4}$$

where $A, C \in \mathbb{R}$ and $|B|^2 > AC$.

Where $z = \infty$ is a solution iff $A = 0$.
$A = 0$: line
$C = 0$: goes through the origin.

There is a unique circle passing through any 3 distinct points in $\mathbb{C}_\infty$.

Theorem 8.5

Let $f \in \mathcal{M}$ and C a circle in $\mathbb{C}_\infty$, then $f(C)$ is a circle in C_∞.

Proof. By Proposition 8.1, just need to consider $f(z) = az,\ z + b$ or $\frac{1}{z}$. Let $S_{A,B,C}$ be the circle defined by Equation (4).

$$\begin{aligned}
&f(z) = az : S_{A,B,C} \mapsto S_{A/a\bar{a}, B/\bar{a}, C} \\
&f(z) = z + b : S_{A,B,C} \mapsto S_{A, B - Ab, C + Ab\bar{b} - \bar{B}b - B\bar{b}} \\
&f(z) = \frac{1}{z} = w : S_{A,B,C} \mapsto A + Bw + \bar{B}\bar{w} + Cw\bar{w} = 0 = S_{C,\bar{B},A}.
\end{aligned}$$

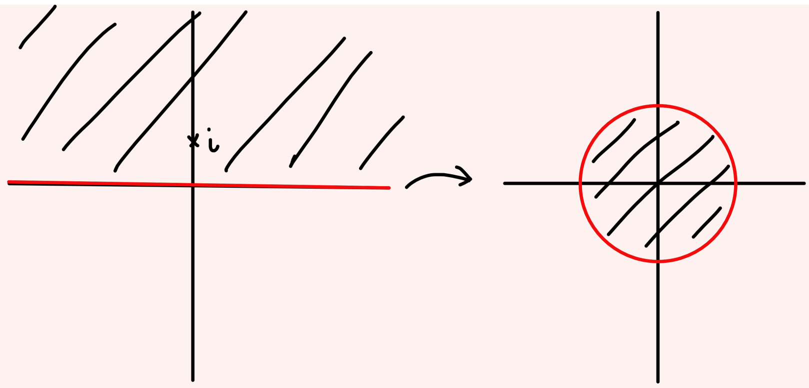
□

Example 8.1

Consider the image of $\mathbb{R} \cup \{\infty\}$ (a circle in $\mathbb{C}_\infty$) under

$$f(z) = \frac{z - i}{z + i}.$$

It is thus a circle in $\mathbb{C}_\infty$ containing $f(0) = -1$, $f(\infty) = 1$, $f(1) = -i$ so $f(\mathbb{R} \cup \{\infty\}) =$ unit circle. Furthermore, complimentary components are mapped to complementary



components.

§8.3 Cross-Ratios

Definition 8.4

The cross-ratio of distinct points $z_1, z_2, z_3, z_4 \in \mathbb{C}$.

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_2)(z_3 - z_4)}$$
$$[\infty, z_2, z_3, z_4] = \frac{(z_2 - z_4)}{(z_3 - z_4)}$$
$$[z_1, \infty, z_3, z_4] = -\frac{(z_1 - z_3)}{(z_3 - z_4)}$$
$$[z_1, z_2, \infty, z_4] = -\frac{(z_2 - z_4)}{(z_1 - z_2)}$$
$$[z_1, z_2, z_3, \infty] = \frac{(z_1 - z_3)}{(z_1 - z_2)}$$

Note $[0, 1, w, \infty] = \frac{-w}{-1} = w$

Warning: different authors use different permutations of $1, 2, 3, 4$ in the definition.

Theorem 8.6

Given $z_1, z_2, z_3, z_4 \in \mathbb{C}_\infty$ distinct $w_1, w_2, w_3, w_4 \in \mathbb{C}_\infty$ distinct then $\exists\ f \in \mathcal{M}$ s.t. $f(z_i) = w_i \iff [z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$. In particular, Möbius maps preserve cross-ratios $[z_1, z_2, z_3, z_4] = [f(z_1), f(z_2), f(z_3), f(z_4)]$

Proof. ($\implies$): Suppose $f(z_j) = w_j$ and $z_i, w_i \neq \infty\quad \forall\ i$ and $f(z) = \frac{az+b}{cz+d}$, then

$cz_j + d \neq 0 \quad \forall\, j.$

$$\begin{aligned}
\text{So, } w_j - w_k &= f(z_j) - f(z_k) \\
&= \frac{(ad-bc)(z_j - z_k)}{(cz_j + d)(cz_k + d)} \\
\implies [z_1, z_2, z_3, z_4] &= [w_1, w_2, w_3, w_4] \\
&= [f(z_1), f(z_2), f(z_3), f(z_4)]
\end{aligned}$$

Need to check other cases; $z_1 = \infty, w_1 = f(\infty) = \frac{a}{c}$ etc.

($\Longleftarrow$): Suppose $[z_1, z_2, z_3, z_4] = [w_1, w_2, w_3, w_4]$. Let $g \in \mathcal{M}$ s.t. $g(z_1) = 0, g(z_2) = 1, g(z_4) = \infty$ as triple transitive. Let $h \in \mathcal{M}$ s.t. $h(w_1) = 0, h(w_2) = 1, h(w_4) = \infty$.

$$\begin{aligned}
g(z_3) &= [0, 1, g(z_3), \infty] \\
&= [g(z_1), g(z_2), g(z_3), g(z_4)] \\
&= [z_1, z_2, z_3, z_4] \text{ by above} \\
&= [w_1, w_2, w_3, w_4] \\
&= [h(w_1), h(w_2), h(w_3), h(w_4)] \\
&= [0, 1, h(w_3), \infty] \\
&= h(w_3).
\end{aligned}$$

So $h^{-1}g$ is required map. □

So $[z_1, z_2, z_3, z_4] = f(z_3)$ where f is the unique Möbius map that sends $z_1 \mapsto 0,\ z_2 \mapsto 1,\ z_4 \mapsto \infty$.

Corollary 8.3

z_1, z_2, z_3, z_4 lie in some circle in $\mathbb{C}_\infty \iff [z_1, z_2, z_3, z_4] \in \mathbb{R}$.

Proof. Let C be a circle through z_1, z_2, z_4.
Let $g : C \to \mathbb{R} \cup \{\infty\}$ s.t. $g(z_1) = 0,\ g(z_2) = 1,\ g(z_4) = \infty$.

$$\begin{aligned}
g(z_3) &= [0, 1, g(z_3), \infty] \\
&= [g(z_1), g(z_2), g(z_3), g(z_4)] \\
&= [z_1, z_2, z_3, z_4] \text{ by Theorem 8.6} \\
\text{So, } [z_1, z_2, z_3, z_4] \in \mathbb{R} &\iff g(z_3) \in \mathbb{R} \\
&\iff z_3 \in C.
\end{aligned}$$

□