Stochastic Financial Models 23

Michael Tehranchi

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1 Black–Scholes model

- A risk-free asset with constant (instantaneously compounded) interest rate r.
- A risky stock with time t price $(S_t)_{t\geq 0}$ where
$$S_t = S_0 e^{\mu t + \sigma W_t}$$
and $(W_t)_{t\geq 0}$ is a Brownian motion.

A risk neutral measure in this context is an equivalent measure $\mathbb{Q}$ under which the discounted stock price $(e^{-rt}S_t)_{t\geq 0}$ process is a martingale.

Theorem (Risk-neutrality in Black–Scholes)**.** *Over any horizon $T \geq 0$, there is a risk-neutral measure $\mathbb{Q}$ with density*
$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{cW_T - c^2T/2}$$
where $c = \frac{r-\mu}{\sigma} - \frac{\sigma}{2}$.

Proof. By Cameron–Martin, the process $\hat{W}_t = W_t - ct$ is a Brownian motion under $\mathbb{Q}$. Notice that
$$\begin{aligned} e^{-rt}S_t &= S_0 e^{(\mu-r)t+\sigma W_t} \\ &= S_0 e^{(\mu-r+c\sigma)t+\sigma\hat{W}_t} \\ &= S_0 e^{-\sigma^2 t/2+\sigma\hat{W}_t} \end{aligned}$$
is a martingale under $\mathbb{Q}$ by example sheet 4. □

Black–Scholes pricing

Definition. Consider a European contingent claim with time T payout Y. Within the Black–Scholes model, the time t price is
$$\pi_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(Y|\mathcal{F}_t)$$
where $\mathbb{Q}$ is the risk-neutral measure.

Note $(e^{-rt}\pi_t)_{0\le t\le T}$ is a $\mathbb{Q}$-martingale.

For a vanilla European contingent claim with payout $Y = g(S_T)$ the price is

$$\begin{aligned}\pi_t &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}(g(S_T)|\mathcal{F}_t)\\ &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[g(S_t e^{(r-\sigma^2/2)(T-t)+\sigma(\hat{W}_T-\hat{W}_t)}|\mathcal{F}_t]\\ &= V(t,S_t)\end{aligned}$$

where

$$V(t,s) = e^{-r(T-t)}\mathbb{E}[g(s\ e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}Z})]$$

and $Z \sim N(0,1)$.

2 Black–Scholes formula

The Black–Scholes price of a European call

$$\begin{aligned}\pi_t &= e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[(S_T-K)^+|\mathcal{F}_t]\\ &= \boxed{S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)}\end{aligned}$$

where

$$d_1 = -\frac{\log(K/S_t)}{\sigma\sqrt{T-t}} + \left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)\sqrt{T-t}$$

and

$$d_2 = -\frac{\log(K/S_t)}{\sigma\sqrt{T-t}} + \left(\frac{r}{\sigma}-\frac{\sigma}{2}\right)\sqrt{T-t}$$

Derivation: Let $\delta = T-t$ and $\xi = e^{(r-\sigma^2/2)\delta+\sigma\sqrt{\delta}Z}$ where $Z \sim N(0,1)$.

$$\begin{aligned}V(t,s) &= e^{-r\delta}\mathbb{E}[(s\xi-K)^+]\\ &= e^{-r\delta}\mathbb{E}[(s\xi-K)\mathbb{1}_{\{\xi>K/s\}}]\\ &= s\mathbb{E}(e^{-r\delta}\xi\mathbb{1}_{\{\xi>K/s\}}) - -e^{-r\delta}K\mathbb{P}(\xi>K/s)\end{aligned}$$

Note that $\mathbb{P}(\xi > K/s) = 1-\Phi(-d_2) = \Phi(d_2)$. By the change of variables formula for normal random variables (see example sheet 1), the law of ξ under $\hat{\mathbb{P}}$ is the same as the law of $e^{\sigma^2\delta}\xi$ under $\mathbb{P}$, where $\frac{d\hat{\mathbb{P}}}{d\mathbb{P}} = e^{-r\delta}\xi$. Hence

$$\mathbb{E}(e^{-r\delta}\xi\mathbb{1}_{\{\xi>K/s\}}) = \mathbb{P}(\xi > Ke^{-\sigma^2\delta}/s) = 1-\Phi(-d_2-\sigma\sqrt{\delta}) = \Phi(d_1)$$

We can also calculate prices of European puts. Recall the payout is of the form $Y = (K - S_T)^+$. But by the identity

$$(K-S_T)^+ - (S_T-K)^+ = K - S_T$$

we see that the portfolio long one put and short one call of the same maturity T and strike K has the same payout as long K units of cash and short one share. Therefore, letting P_t and C_t be the time-t prices of the put and call, respectively, we have the *put-call parirty formula*

$$\boxed{P_t - C_t = Ke^{-r(T-t)} - S_t}$$

(This formula holds for all models as long as the interest rate is constant. However, in discrete time, the discount factor $e^{-(T-t)t}$ is replaced by $(1+r)^{-(N-n)}$.)

We can now apply this to the Black–Scholes model to calculate the price of a European put

$$\begin{aligned}
P_t &= C_t + Ke^{-r(T-t)} - S_t \\
&= S_t\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)Ke^{-r(T-t)} - S_t \\
&= \boxed{Ke^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1)}
\end{aligned}$$

using the identity $\Phi(x) = 1 - \Phi(-x)$.