Stochastic Financial Models 7

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1 Proof of the convergence of indifference to marginal utility price

Fix Y and let
$$\pi_t = \frac{\pi(tY)}{t}$$
and $p = \sup_{t>0} \pi_t$. Example sheet: $t \mapsto \pi_t$ decreasing. [Hint: use $\pi(0) = 0$ and concavity] Hence $\pi_t \uparrow p$ as $t \downarrow 0$. We must show $p = \pi_0(Y)$.

From last time $\pi_t \leq \pi_0(Y)$ for all $t > 0$ so $p \leq \pi_0(Y)$. It remains to show the reverse inequality.

Now by definition of $X^* \in \mathcal{X}$ as maximiser of $\mathbb{E}[U(X)]$ we have

$$\begin{aligned}
0 &= \frac{1}{t}[V(tY - (1+r)t\pi_t) - V(0)] \\
&\geq \mathbb{E}\left[\frac{U(X^* + tY - (1+r)t\pi_t) - U(X^*)}{t}\right] \quad \text{as } V = \max_{x} \mathbb{E}[U(x)] \\
&\geq \mathbb{E}\left[\frac{U(X^* + tY - (1+r)tp) - U(X^*)}{t}\right] \text{ since } p \geq \pi_t \\
&\to \mathbb{E}\{U'(X^*)[Y - (1+r)p]\}
\end{aligned}$$

(by the dominated convergence theorem from Probability & Measure) Rearranging yields $p \geq \pi_0(Y)$. as $\pi_0 = \dfrac{\mathbb{E}[U'(X)Y]}{(1+r)\mathbb{E}[U'(X)]}$ □

2 Risk neutral measures

- Given an probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- Let Z be a strictly positive random variable such that $\mathbb{E}^{\mathbb{P}}(Z) = 1$.
- We can define a probability new measure $\mathbb{Q}$ by the formula
$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(Z\mathbb{1}_A)$$
for any event A.

- By measure theory, $\mathbb{E}^{\mathbb{Q}}(X) = \mathbb{E}^{\mathbb{P}}(ZX)$ for any $\mathbb{Q}$-integrable random variable X.
- Notation $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$
- $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is called the *density* or *likelihood ratio* of $\mathbb{Q}$ with respect to $\mathbb{P}$.

$\mathbb{P}(A)=0 \Rightarrow \mathbb{P}(\mathbb{1}_A=0)=1$ ✓

- Important point: $\mathbb{Q}(A) = 0$ if and only if $\mathbb{P}(A) = 0$ by the pigeon-hole principle.

$\mathbb{Q}(A)=0 \Rightarrow \mathbb{P}(Z\mathbb{1}_A=0)=1$ As $\mathbb{P}(Z\mathbb{1}_A \geq 0)=1$ and $\mathbb{P}(Z>0)=1$ ✓

Definition. Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures defined on the same measurable space $(\Omega, \mathcal{F})$. The measures are said to be *equivalent* if they have the property that $\mathbb{Q}(A) = 0$ if and only if $\mathbb{P}(A) = 0$.

Theorem (Radon–Nikodym theorem)**.** *Let $\mathbb{P}$ and $\mathbb{Q}$ be probability measures defined on the same measurable space $(\Omega, \mathcal{F})$. There exists a $\mathbb{P}$-a.s. positive random variable Z such that*

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}(Z\mathbb{1}_A)$$

for any event A if and only if $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

Remark. We don't need this theorem, but is only stated for mathematical context.

Example

- Let $\Omega = \{\omega_1, \omega_2, \dots\}$
- $\mathbb{P}\{\omega_i\} = p_i > 0$ for all i
- $\mathbb{Q}\{\omega_i\} = q_i > 0$ for all i
- $Z(\omega_i) = q_i/p_i$ for all i.
- Then $Z = \frac{d\mathbb{Q}}{d\mathbb{P}}$.

Example

- Let X be defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and μ, λ positive constants.
- $X \sim \exp(\lambda)$ under $\mathbb{P}$.
- Let $Z = \frac{\mu}{\lambda} e^{(\lambda-\mu)X}$. Note Z is positive and

$$\mathbb{E}^{\mathbb{P}}(Z) = \int_0^\infty \frac{\mu}{\lambda} e^{(\lambda-\mu)x} \lambda e^{-\lambda x} dx = \int_0^\infty \mu e^{-\mu x} dx = 1.$$

- Let $\mathbb{Q}$ have density Z with respect to $\mathbb{P}$. Then for any bounded function f we have

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[f(X)] &= \mathbb{E}^{\mathbb{P}}[Zf(X)] \\ &= \int_0^\infty \frac{\mu}{\lambda} e^{(\lambda-\mu)x} f(x) \lambda e^{-\lambda x} dx \\ &= \int_0^\infty f(x) \mu e^{-\mu x} dx \end{aligned}$$

- That is, the distribution of X under $\mathbb{Q}$ is $\exp(\mu)$

Now consider the one-period model set-up defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- interest rate r
- d risky assets with time t price vector S_t.

Definition. A *risk-neutral measure* is any probability measure $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that

$$S_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(S_1)$$

The probability measure $\mathbb{P}$ is called the *objective* or *statistical* measure.

Theorem (Marginal utility pricing 2)**.** *Consider the problem of maximising* $\mathbb{E}^{\mathbb{P}}[U(X)]$ *over*

$$X \in \mathcal{X} = \{(1+r)X_0 + \theta^\top(S_1 - (1+r)S_0) : \theta \in \mathbb{R}^d\}$$

where U is strictly increasing, and assume there exists a maximiser $X^* \in \mathcal{X}$. Define the equivalent probability measure $\mathbb{Q}$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}} \propto U'(X^*)$. Then $\mathbb{Q}$ is risk-neutral.

Proof. Let

$$Z = \frac{U'(X^*)}{\mathbb{E}^{\mathbb{P}}[U'(X^*)]}$$

Note that $Z > 0$ and $\mathbb{E}^{\mathbb{P}}(Z) = 1$. By assumption $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z$. But we already know from the first marginal utility pricing theorem (Lecture 4) that

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(S_1) = \frac{\mathbb{E}^{\mathbb{P}}[U'(X^*)S_1]}{(1+r)\mathbb{E}[U'(X^*)]} = S_0.$$

□