Stochastic Financial Models 4

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13 October 2023

1 Properties of concave functions

We will nearly always assume our agent's utility function U is strictly increasing and strictly concave. If U is differentiable (always assumed), the gradient U' is called the *marginal utility*.

- $U'(x) > 0$ measures how much the utility increases at x
- $U''(x) < 0$ measures the concavity of the utility at x

Definition. The (Arrow–Pratt) *coefficient of absolute risk aversion* is

$$-\frac{U''(x)}{U'(x)}$$

The (Arrow–Pratt) *coefficient of relative risk aversion* for $x > 0$ is

$$-x\frac{U''(x)}{U'(x)}$$

Examples

- *exponential or CARA.* $U(x) = -e^{-\gamma x}$ with $\gamma > 0$ the constant coefficient of absolute risk aversion
- *power or CRRA.* $U(x) = \frac{1}{1-R}x^{1-R}$, $x > 0$, with $R > 0, R \neq 1$, modelling the constant coefficient of relative risk aversion
- *logarithmic.* $U(x) = \log x$, $x > 0$ with constant coefficient of relative risk aversion $R = 1$.
- *risk-neutral.* $U(x) = x$ so the coefficient of risk aversion is zero. Note that this function is concave, but not strictly concave, so we won't use it as a utility function!

Remark. To be really technically accurate, we should talk about the *domain* of a concave function, i.e. the set where the function is finite-valued.

Theorem (Concave functions are continuous, and their graphs lie ~~above~~ below their tangents)**.** *Let U be concave. Then U is continuous. If U is differentiable, then for any x, y we have*

$$U(y) \leq U(x) + U'(x)(y - x).$$

Proof. Fix x and $0 < \varepsilon < \ell$. We have

$$\begin{aligned}\frac{\varepsilon}{\ell}(U(x) - U(x - \ell)) &\geq U(x) - U(x - \varepsilon) \\ &\geq U(x + \varepsilon) - U(x) \\ &\geq \frac{\varepsilon}{\ell}(U(x + \ell) - U(x))\end{aligned}$$

This is proven by looking each inequality one at a time, and rearranging the definition of concavity. For instance, note

$$x - \varepsilon = \frac{\varepsilon}{\ell}(x - \varepsilon) + (1 - \frac{\varepsilon}{\ell})x$$

so by concavity

$$U(x - \varepsilon) \geq \frac{\varepsilon}{\ell}U(x - \ell) + (1 - \frac{\varepsilon}{\ell})U(x)$$

This is equivalent to the first inequality.

Sending $\varepsilon \to 0$ shows continuity. Now assuming differentiability, dividing by ε and taking the limit yields

$$U(x) - U(x - \ell) \geq \ell\; U'(x) \geq U(x + \ell) - U(x)$$

as claimed by letting $y = x + \ell$ or $x - \ell$. $\square$

Theorem (strictly Increasing concave functions are unbounded on the left)**.** *Suppose U is increasing and concave, but not constant. Then $U(x) \to -\infty$ as $x \to \infty$.*

Proof. Let $x < a < b$, where $U(a) < U(b)$. Then using $a = (\frac{b-a}{b-x})x + (\frac{a-x}{b-x})b$ in the definition of concavity yields

$$U(x) \leq U(a) + \frac{x - a}{b - a}(U(b) - U(a))$$

from which the conclusion follows. $\square$

2 Optimal investment and marginal utility

In this section we assume that U is strictly increasing, concave and differentiable.

Theorem (Marginal utility pricing)**.** *Suppose U is suitably nice*[1]*, and let θ^* maximise the expected utility $\mathbb{E}[U(X_1)]$ where $X_1 = (1 + r)X_0 + \theta^\top[S_1 - (1 + r)S_0]$. Then*

$$S_0 = \frac{\mathbb{E}[U'(X_1^*)S_1]}{(1 + r)\mathbb{E}[U'(X_1^*)]}$$

[1]That is, it satisfies a technical condition that allows the formal calculation to go through, but the condition is uninteresting for the main focus of this course. In this case, we assume $U(X_1)$ is integrable for all portfolios θ then the formal calculation is justified by the dominated convergence theorem of Probability & Measure.

where $X_1^* = (1+r)X_0 + (\theta^*)^\top[S_1 - (1+r)S_0]$ *is the optimal time-1 wealth.*

Proof. Let
$$f(\theta) = \mathbb{E}\{U\big((1+r)X_0 + \theta^\top[S_1 - (1+r)S_0]\big)\}$$
We can differentiate inside the expectation yielding

$$Df(\theta) = \mathbb{E}\{U'\big(X_1\big)[S_1 - (1+r)S_0]\}$$

where $X_1 = (1+r)X_0 + \theta^\top[S_1 - (1+r)S_0]$. Since by calculus, at the maximising portfolio θ^* the gradient vanishes $Df(\theta^*) = 0$, the conclusion follows upon rearrangement. $\square$