IV. HIGHER-ORDER LINEAR DIFFERENTIAL EQUATIONS

We now consider linear differential equations where derivatives of the dependent variable of higher order appear, e.g., d^2y/dx^2. In particular, we shall focus on second-order equations but most of the methods in this topic apply also to higher-order equations.

1 Second-order equations with constant coefficients

We begin by considering linear, second-order differential equations with constant coefficients. These take the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)\,, \tag{1}$$

where a, b and c are constants.

The differential operator on the left of Eq. (1) is *linear*.

Definition (Linear differential operator). A differential operator $\mathcal{D}$ is *linear* if for any $y_1(x)$ and $y_2(x)$, and constants α and β,

$$\mathcal{D}(\alpha y_1 + \beta y_2) = \alpha\mathcal{D}(y_1) + \beta\mathcal{D}(y_2)\,.$$

We can exploit linearity of $\mathcal{D}$ to solve Eq. (1) in two steps:

1. find the *complementary functions* that satisfy the homogeneous (unforced) equation, i.e.,

$$a\frac{d^2y_c}{dx^2} + b\frac{dy_c}{dx} + cy_c = 0\,;$$

2. find a *particular integral* y_p that satisfies the full equation.

A solution to the full equation can be found by adding the complementary function and particular integral since

$$\mathcal{D}(y_c + y_p) = \mathcal{D}(y_c) + \mathcal{D}(y_p) = 0 + f(x)\,.$$

If y_{c1} and y_{c2} are *linearly independent* complementary functions, then $y_{c1} + y_p$ and $y_{c2} + y_p$ are linearly independent solutions of the full equation.

Definition (Linear dependence of functions). A set of N functions $\{f_i(x)\}$ is *linearly dependent* if

$$\sum_{i=1}^{N} c_i f_i(x) = 0\,, \quad \forall x \in \text{range we are interested in}$$

for N constants $\{c_i\}$, where at least one of the c_i is nonzero. Otherwise, the functions are *linearly independent.*

Equivalently, the N functions are linearly dependent if any of them can be written as a linear combination of the others.

1.1 Complementary functions

Recall that $e^{\lambda x}$ is an eigenfunction of d/dx, i.e.,

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}\,.$$

It follows that $e^{\lambda x}$ is also an eigenfunction of d^2/dx^2 and, indeed, of any linear differential operator with constant coefficients. Taking

$$\mathcal{D} = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c\,,$$

we have

$$\mathcal{D}\left(e^{\lambda x}\right) = \left(a\lambda^2 + b\lambda + c\right) e^{\lambda x}\,.$$

Complementary functions of Eq. (1) satisfy $\mathcal{D}(y_c) = 0$ and so are eigenfunctions with eigenvalue zero. It follows that

$$y_c = Ae^{\lambda x}$$

is a complementary function provided that the *characteristic equation* is satisfied.

Definition (Characteristic equation). The *characteristic equation* of the (second-order) differential equation $ay'' + by' + cy = 0$ is

$$a\lambda^2 + b\lambda + c = 0\,.$$

Since the characteristic equation of a second-order differential equation is quadratic, there are *two* solutions, λ_1, λ_2, leading to two complementary functions

$$y_{c1} \propto e^{\lambda_1 x} \quad \text{and} \quad y_{c2} \propto e^{\lambda_2 x}\,.$$

If $\lambda_1 \neq \lambda_2$, then y_{c1} and y_{c2} are linearly independent. Furthermore, any other solution of the homogenous differential equation $\mathcal{D}(y) = 0$ can then be written as a linear combination of y_{c1} and y_{c2}, i.e.,

$$y(x) = c_1 y_{c1}(x) + c_2 y_{c2}(x)\,.$$

This is the most general complementary function for Eq. (1), with y_{c1} and y_{c2} forming a *basis* for the space of solutions of the homogeneous equation.

You should be aware that the roots of the characteristic equation may be complex, in which case the complementary functions have oscillatory character. Moreover, the roots may be degenerate, $\lambda_1 = \lambda_2$. In this case, we only have one linearly independent complementary function of the form $e^{\lambda_1 x}$. We shall explore how to deal with this case in an example below.

Example (Non-degenerate, real roots of the characteristic equation). Consider the equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0\,.$$

The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0 \quad \Rightarrow \quad (\lambda - 2)(\lambda - 3) = 0\,,$$

which is solved by $\lambda = 2$ or 3. It follows that the general complementary function is

$$y_c(x) = Ae^{2x} + Be^{3x} ,$$

for arbitrary constants A and B.

Example (Complex roots). Consider the equation

$$\frac{d^2y}{dx^2} + 4y = 0 .$$

The characteristic equation is

$$\lambda^2 + 4 = 0 \quad \Rightarrow \quad \lambda = \pm 2i .$$

The roots are non-degenerate but complex. The general complementary function is

$$y_c(x) = Ae^{2ix} + Be^{-2ix} ,$$

for arbitrary (complex) constants A and B. We can express this in terms of sine and cosine to emphasise the oscillatory character:

$$\begin{aligned} y_c &= A(\cos 2x + i \sin 2x) + B(\cos 2x - i \sin 2x) \\ &= \alpha \cos 2x + \beta \sin 2x , \end{aligned}$$

where $\alpha = A + B$ and $\beta = i(A - B)$ are two further arbitrary constants.

Example (Degeneracy and detuning). Consider the equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0 . \tag{2}$$

The characteristic equation is

$$\lambda^2 - 4\lambda + 4 = 0 \quad \Rightarrow \quad (\lambda - 2)^2 = 0 .$$

The roots are now degenerate, $\lambda = 2$, and we only generate one linearly independent solution $y_c \propto e^{2x}$. This does not form a basis for the solution space of Eq. (2) since the the solution space of any second-order differential equation is two-dimensional.

We can construct a second, linearly independent solution using a technique known as *detuning*. The idea is to modify Eq. (2) slightly to remove the degeneracy. In particular, consider the "detuned" equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + \left(4 - \epsilon^2\right) y = 0\,. \tag{3}$$

In the limit $\epsilon \to 0$, this reduces to Eq. (2), the equation that we really want to solve. The characteristic equation for Eq. (3) is

$$\lambda^2 - 4\lambda + 4 - \epsilon^2 = 0\,,$$

which has roots $\lambda = 2 \pm \epsilon$. The general solution of the detuned equation (3) is then

$$\begin{aligned} y &= Ae^{(2+\epsilon)x} + Be^{(2-\epsilon)x} \\ &= e^{2x}\left(Ae^{\epsilon x} + Be^{-\epsilon x}\right)\,. \end{aligned}$$

To take the limit as $\epsilon \to 0$, we use the series expansion of the exponential function to obtain

$$y = e^{2x}\left[(A+B) + \epsilon(A-B)x + O(A\epsilon^2) + O(B\epsilon^2)\right]\,.$$

Suppose now that we choose to solve Eq. (2) with the initial conditions $y(0) = C$ and $y'(0) = D$. Adopting the same initial conditions for the detuned equation, we have

$$C = A + B \quad \text{and} \quad D = 2(A+B) + \epsilon(A-B)\,.$$

It follows that

$$A + B = C \quad \text{and} \quad \epsilon(A-B) = D - 2C\,.$$

Moreover, terms of $O(A\epsilon^2)$, for example, become $O(\epsilon)$ since

$$A = \frac{1}{2}\left(C + \frac{D-2C}{\epsilon}\right)\,. \doteq \frac{1}{2}\left(\left(C\varepsilon^2 + (D-2C)\right)\varepsilon\right)$$

This term dominates

Taking the limit as $\epsilon \to 0$, we have

$$y \to e^{2x}\left[C + (D-2C)x\right]\,.$$

It follows that the general solution of Eq. (2) is

$$y_c = e^{2x}\left(\alpha + \beta x\right) ,$$

for arbitrary constants α and β.

We see that we have constructed a second, linearly independent complementary function of the degenerate equation (2) of the form xe^{2x}. This is reminiscent of the solutions we found in the radioactivity example back in Topic II.

This example illustrates a general rule. For linear equations with constant coefficients where the characteristic equation has a repeated root, if $y_{c1}(x)$ is a degenerate complementary function, then $y_{c2}(x) = xy_{c1}(x)$ is a linearly independent complementary function.

2 Homogeneous second-order equations with non-constant coefficients

Having seen how to solve homogeneous second-order equations with constant coefficients, let us consider the more general case of equations with non-constant coefficients. In this section, we shall discuss ways to find a second, linearly independent complementary function assuming that we have been able to find a first solution. We shall also look at some general properties of these solutions.

We shall consider equations of the form

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \,, \tag{4}$$

for general functions $p(x)$ and $q(x)$.

2.1 Second complementary function: reduction of order

Suppose that we know a first complementary function $y_1(x)$ that solves Eq. (4).

Let us assume that the second complementary function $y_2(x) = v(x)y_1(x)$ for some as-yet-undetermined function $v(x)$. Applying the product rule, we have

$$y_2' = vy_1' + v'y_1 \quad \text{and} \quad y_2'' = vy_1'' + 2v'y_1' + v''y_1 \,.$$

Given that y_2 is supposed to satisfy Eq. (4), substituting and collecting terms we find

$$v''y_1 + v'\left(2y_1' + py_1\right) + v\left(y_1'' + py_1' + qy_1\right) = 0 \,.$$

Since y_1 also satisfies Eq. (4), the bracket multiplying $v(x)$ is zero and so

$$v''y_1 + v'\left(2y_1' + py_1\right) = 0 \,.$$

This is a *first-order* equation for the variable $u \equiv v'$:

$$u'y_1 + u\left(2y_1' + py_1\right) = 0 \,. \tag{5}$$

Equation (5) is a separable, first-order equation and so can be integrated to find $u = v'$ up to an integration constant that appears as an overall normalisation factor, A:

$$\begin{aligned} \frac{u'}{u} &= -\frac{2y_1'}{y_1} - p \\ \Rightarrow \qquad \ln u &= -2\ln y_1 - \int^x p(u)\,du + \ln A \\ \Rightarrow \qquad u(x) &= \frac{A}{y_1^2(x)} \exp\left(-\int^x p(w)\,dw\right) . \end{aligned} \tag{6}$$

This equation can be further integrated to determine $v(x)$ up to addition of a further integration constant. This latter constant will just lead to a y_2 containing arbitrary amounts of y_1.

This technique is called *reduction of order*, because we have reduced a second-order differential equation to a first-order equation (for v') that we can solve since it is separable. This is actually a very general technique for higher-order differential equations.

Example. Consider the degenerate equation (2):

$$y'' - 4y' + 4y = 0\,,$$

which we know has a (degenerate) solution $y_1 = e^{2x}$. We look for a second solution in the form $y_2(x) = v(x)y_1(x)$, so that

$$y_2' = (v' + 2v)e^{2x} \quad \text{and} \quad y_2'' = (v'' + 4v' + 4v)\,e^{2x}\,.$$

Substituting into the differential equation, and dividing through by e^{2x} (which is guaranteed to be non-zero for all x) we obtain

$$v'' + 4v' + 4v - 4(v' + 2v) + 4v = 0\,.$$

Unsurprisingly there is lots of cancellation, and so

$$v'' = 0 \quad \Rightarrow \quad u = v' = A\,,$$

consistent with Eq. (6) since $p(x) = -4$. Finally, we integrate again to find $v = Ax + B$ for arbitrary constants A and B. It follows that

$$y_2(x) = (Ax + B)e^{2x}\,.$$

We see that the integration constant B just adds an arbitrary amount of $y_1(x)$ to $y_2(x)$, and we can take the second, linearly independent solution to be $y_2 \propto xe^{2x}$ in agreement with that we found previously by detuning.

2.2 Phase space

Quite generally, an nth-order differential equation determines the nth derivative, $y^{(n)}(x)$ in terms of $y(x)$ and its derivatives up to $y^{(n-1)}(x)$. Moreover, by differentiating the equation, these same derivatives determine all higher-order derivatives too.

If we think of specifying $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$ at some initial point x_0, then we can construct all derivatives there and hence the Taylor series of $y(x)$ about x_0.

Hence, the solution of a nth-order differential equation is uniquely determined by these initial conditions.[1]

We can therefore think of the state of the system governed by the differential equation as being fully specified at any value of the independent variable by the *solution vector* $\mathbf{Y}(x)$:

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix} . \tag{7}$$

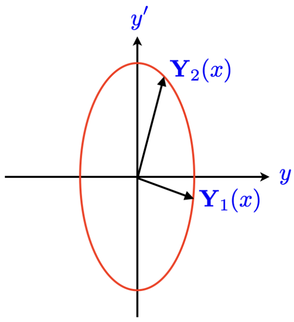
For every x, this vector (note the convention of writing it in bold font) defines a point in a n-dimensional *phase space*. As x varies $\mathbf{Y}(x)$ traces out a *trajectory* in phase space.

Example. Consider the undamped oscillator equation

$$y'' + 4y = 0 \,.$$

Two linearly independent solutions are $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Therefore, the associated solution vectors are

$$\mathbf{Y}_1(x) = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} = \begin{pmatrix} \cos 2x \\ -2\sin 2x \end{pmatrix} ,$$
$$\mathbf{Y}_2(x) = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = \begin{pmatrix} \sin 2x \\ 2\cos 2x \end{pmatrix} .$$



The two solution vectors thus trace out an elliptical trajectory in phase space as x varies, as shown in the figure to the right.

Note how in this example the two solution vectors are linearly independent (i.e., non-colinear in 2D) for all x. (We shall see shortly how this idea generalises.) Therefore, any point in phase space can be reached by a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$ at any x and the solution vectors form a *basis* for the 2D phase space.

[1] It takes rather more work to prove this statement rigorously!

Generally, for an nth-order linear equation, it is often convenient to use the solution vectors constructed from n linearly independent complementary functions as basis vectors for the phase space.

2.3 Wronskian and linear independence

Recall that n functions, $y_i(x)$ $(i = 1, \dots, n)$, are linearly *dependent* if

$$\sum_{i=1}^{n} c_i y_i(x) = 0$$

for some n constants c_i not all of which are zero. Since this has to hold for all x (within some domain of interest), we can differentiate $n-1$ times and collect the n constraints as

$$\sum_{i=1}^{n} c_i \mathbf{Y}_i(x) = 0\,, \tag{8}$$

where the $\mathbf{Y}_i(x)$ are the n vectors constructed from the y_i and their derivatives. This vector equation is the statement that the n vectors $\mathbf{Y}_i(x)$ are linearly dependent for all x.

Equation (8) implies that the determinant of the *fundamental matrix*, constructed with the ith column being $\mathbf{Y}_i(x)$, vanishes if the functions $y_i(x)$ $(i = 1, \dots, n)$ are linearly dependent.[2] We call this determinant the *Wronskian*.

Definition (Wronskian). The *Wronskian* $W(x)$ of n functions $y_i(x)$ $(i = 1, \dots, n)$ is the determinant of the fundamental matrix whose columns are the vectors $\mathbf{Y}_i$:

$$W(x) \equiv \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}\,.$$

[2]If the vectors are linearly dependent, some column of the fundamental matrix is a linear combination of other columns and so the determinant vanishes.

We have seen that

$$\text{linear dependence of the } y_i(x) \quad \Rightarrow \quad W(x) = 0\,.$$

It follows that[3]

$$W(x) \neq 0 \quad \Rightarrow \quad \text{the } y_i(x) \text{ are linearly independent.}$$

This is very useful in the context of solutions of nth-order linear differential equations as we can test for linear independence by calculating the Wronskian of n putative solutions.

For a second-order differential equation, as considered here, the Wronskian $W(x)$ takes the particularly simple form

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'. \tag{9}$$

Example. For the equation $y'' + 4y = 0$, we have $y_1 = \cos 2x$ and $y_2 = \sin 2x$. Therefore

$$W(x) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2\,,$$

so these solutions are linearly independent (as expected).

Example. For the equation $y'' - 4y' + 4y = 0$, we have $y_1 = e^{2x}$ and the independent solution $y_2 = xe^{2x}$. In this case,

$$W(x) = \begin{vmatrix} e^{2x} & xe^{2x} \\ e^{2x} & (1+2x)e^{2x} \end{vmatrix} = e^{4x}(1 + 2x - 2x) = e^{4x}\,,$$

establishing linear independence again as e^{4x} is never zero.

[3]Note that $W = 0$ does *not* necessarily imply linear *dependence*. For example, in two dimensions sufficient conditions for linear dependence are that $W = 0$ *and* that one of the functions, say y_1, is non-zero over the domain of interest. In this case, we can write the Wronskian in the form

$$W(x) = -y_1^2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right)$$

and then $W = 0$ implies $y_2 \propto y_1$.

2.4 Abel's theorem

In these examples, the Wronskian is non-zero for all x. Is it possible that it is zero for some x while non-zero for others? The answer is no, as a consequence of *Abel's theorem.*

Theorem (Abel's theorem). Given a linear, second-order, homogenous differential equation,

$$y'' + p(x)y' + q(x) = 0\,, \tag{10}$$

where $p(x)$ and $q(x)$ are continuous on an interval I, then, for any solutions of the differential equation, either $W = 0$ for all $x \in I$ or $W \neq 0$ for all $x \in I$.

Proof (sketch). From Eq. (9), the derivative of the Wronskian of two solutions of Eq. (10) is

$$W' = y_1 y_2'' - y_2 y_1''\,.$$

Since y_1 and y_2 satisfy the differential equation (10), we have

$$\begin{aligned} W' &= y_2\left(py_1' + qy_1\right) - y_1\left(py_2' + qy_2\right) \\ &= -p\left(y_1 y_2' - y_2 y_1'\right) \\ &= -pW\,. \end{aligned}$$

This is a separable, first-order equation for the Wronskian with solution (*Abel's identity*)

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p(u)\,du\right)\,, \tag{11}$$

for some arbitrary x_0. Since the exponential is never zero, either $W(x_0) = 0$, in which case $W = 0$ for all x, or $W(x_0) \neq 0$, in which case $W \neq 0$ for any x.

As a corollary of Abel's identity, note that if $p(x) = 0$, i.e., the differential equation has no y' term, the Wronskian is constant.

Example (Bessel's equation). Consider

$$x^2 y'' + xy' + \left(x^2 - n^2\right) y = 0\,.$$

This equation has no closed-form solutions for most values of n ((half-integer values are an exception). Dividing through by x^2, we have

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right) y = 0\,,$$

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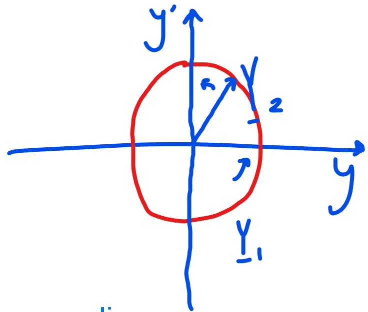
so that $p(x) = 1/x$. Abel's identity (11) gives

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x}\frac{du}{u}\right)$$

$$\Rightarrow \quad W(x) = W(x_0)\frac{x_0}{x}\,.$$

Phase space



Note how Abel's identity determines the form of the Wronskian without having to solve the differential equation directly.

Solution vectors always colinear or never colinear as x varies, basis at some x $\Rightarrow$ basis $\forall x$.

2.4.1 Application of Abel's theorem

Abel's identity (11) can be written as

$$y_1y_2' - y_2y_1' = W(x_0)\exp\left(-\int_{x_0}^{x} p(u)\,du\right)\,.$$

If y_1 is known, we can think of this is a first-order equation for y_2, which we can write as

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{W(x_0)}{y_1^2(x)}\exp\left(-\int_{x_0}^{x} p(u)\,du\right)\,. \qquad (12)$$

Integrating both sides, we obtain $y_2(x)$ up to the addition of a constant multiple of $y_1(x)$ (which arises from the integration constant). The $y_2(x)$ that we obtain is ensured to have the correct Wronskian with $y_1(x)$, and in particular the value $W(x_0)$ at $x = x_0$. (Note that adding any multiple of $y_1(x)$ to $y_2(x)$ does not change the Wronskian.)

This method of finding a second solution to a homogeneous equation is equivalent to the method of reduction of order (Sec. 2.1). Indeed, Eq. (12) is exactly Eq. (6) on recalling that $y_2 = v(x)y_1$ and $u = v'$ there.

2.4.2 Generalisation

Abel's theorem generalises to the Wronskian of solutions of higher-order, linear, homogeneous equations (see Question 7 on Examples Sheet 3).

As we discuss further in Topic V, any nth-order, linear, homogeneous differential equation for $y(x)$ can be written in the form

$$\mathbf{Y}' + \mathsf{A}(x)\mathbf{Y} = \mathbf{0}\,,$$

where $\mathsf{A}(x)$ is an $n \times n$ matrix, which may depend on x, and $\mathbf{Y}$ is the n-dimensional vector formed from $y(x)$ and its first $n-1$ derivatives (see Eq. 7). For example, for the equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0\,,$$

we have

$$\mathbf{Y}' + \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ r(x) & q(x) & p(x) \end{pmatrix} \mathbf{Y} = \mathbf{0}\,.$$

It can be shown that the Wronskian W of n solutions of the original nth-order differential equation satisfies

$$W' + \mathrm{Tr}[\mathsf{A}(x)]W = 0\,, \qquad \text{(qn7, sheet 3)}$$

where Tr denotes the trace. This equation is solved by

$$W = W(x_0)\exp\left(-\int_{x_0}^{x} \mathrm{Tr}[\mathsf{A}(u)]\,du\right)\,,$$

and so Abel's theorem still holds.

2.5 Linear equidimensional equations

Definition (Equidimensional equation). A linear, second-order equation is *equidimensional* if it has the form

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)\,, \tag{13}$$

where a, b and c are constants.

Such equations are called "equidimensional" (or sometimes "homogeneous", although this is confusing as we are using the latter term for equations with no forcing term, i.e., $f(x) = 0$) since solutions of the unforced equation remain solutions under scaling of x. Specifically, let $y(x)$ be a solution of Eq. (13) when $f(x) = 0$. Now consider the new function $\phi(x) = y(\alpha x)$, where α is an arbitrary scaling parameter. Since, by the chain rule,

$$x\frac{d\phi}{dx} = (\alpha x)y'(\alpha x)\,,$$

$x^2 \frac{d^2\phi}{dx^2} = x^2 \lambda^2 y''(\lambda x)$

we see that

$$ax^2\frac{d^2\phi}{dx^2} + bx\frac{d\phi}{dX} + c\phi = 0\,,$$

$= a(\lambda x)^2 \frac{d^2y}{dx^2}\Big|_{x=\lambda x} + b(x\lambda)\frac{dy}{dx}\Big|_{x=\lambda x} + cy(\lambda x)$

so $\phi(x)$ satisfies the same equation as $y(x)$.

Equivalently, if Eq. (13) describes some physical system, so that the variables have dimensions, the dimensions of each term on the left-hand side are the same (assuming that the constants a, b and c to be dimensionless). For example, in a dynamical system y might have dimensions of length (L) and x dimensions of time (T). Then y' has dimensions LT^{-1} and y'' has dimensions LT^{-2}, so that y, xy and xy'' all have dimensions L.

Let us now determine the complementary functions of Eq. (13).

2.5.1 Solving by eigenfunctions

Noting that $y = x^k$ is an eigenfunction of xd/dx (with eigenvalue k), we can look for complementary functions of this form. Substituting into

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0\,,$$

we require

$$ak(k-1) + bk + c = ak^2 + (b-a)k + c = 0\,. \qquad (14)$$

This equation has two roots k_1 and k_2, and so, provided they are distinct, the general complementary function is

$$y_c = Ax^{k_1} + Bx^{k_2} \,.$$

2.5.2 Solving by substitution

An alternative method for determing the complementary functions of equidimensional equations is to make the substitution $z \equiv \ln x$. This is useful since it turns Eq. (13) into an equation with constant coefficients.

To see this, note that for $y(e^z)$,

$$\begin{aligned} \frac{dy}{dz} &= e^z y'(e^z)\,, \qquad xy'(x) \\ \frac{d^2y}{dz^2} &= e^z y'(e^z) + e^{2z} y''(e^z)\,, \qquad x^2 y''(x) \end{aligned}$$

so that if $y(x)$ is a solution of Eq. (13), then $y(e^z)$ satisifes

$$a\frac{d^2y}{dz^2} + (b-a)\frac{dy}{dz} + cy = f(e^z)\,. \tag{15}$$

We can now use the techniques for equations with constant coefficients (Sec. 1) to solve Eq. (15). For example, we can look for complementary functions of the form $y = e^{\lambda z}$, in which case we require

$$a\lambda^2 + (b-a)\lambda + c = 0\,.$$

This is the same characteristic equation as Eq. (14), so the roots are k_1 and k_2. The general complementary function is then

$$y_c = Ae^{k_1 z} + Be^{k_2 z} = Ax^{k_1} + Bx^{k_2}\,,$$

as expected.

In the degenerate case, the roots of the characteristic equation are equal: $\lambda = k_1 = k_2 = k$. However, we know from our earlier example of "detuning" how to deal with such cases for equations with constant coefficients. The general complementary function is then

$$y_c = Ae^{kz} + Bze^{kz} = Ax^k + Bx^k \ln x\,.$$

$f(x)$	$y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx$ or $\cos kx$	$A\sin kx + B\cos kx$
Polynomial $p_n(x)$	$q_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

Table 1: Form of particular integrals $y_p(x)$ for linear, second-order equations with constant coefficients with some common forcing terms $f(x)$.

3 Inhomogeneous (forced) second-order differential equations

So far, we have focused on finding complementary functions. Let us now consider determining particular integrals of inhomogeneous equations, beginning with the case of equations with constant coefficients.

3.1 Particular integrals of equations with constant coefficients

We consider linear, second-order equations with constant coefficients, i.e.,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)\,.$$

For particularly simple forms of the *forcing function* $f(x)$ we can write down a particular integral $y_p(x)$ by inspection/guesswork. Table 1 lists some common cases.

The various arbitrary constants in the particular integrals are determined by substitution in the underlying differential equation. It is important to remember that this equation is *linear*, so we can superpose solutions and consider each forcing term separately.

Example. Consider the equation

$$y'' - 5y' + 6y = 2x + e^{4x}\,.$$

For the forcing term $2x$ we consider a particular integral $ax + b$, and for e^{4x} we consider ce^{4x}. Superposing, we try

$$y_p = ax + b + ce^{4x}\,,$$

so that

$$y_p' = a + 4ce^{4x}\,, \quad y_p'' = 16ce^{4x}\,.$$

Substituting these into the differential equation, and collecting terms multiplying the same function of x, we obtain

$$\underbrace{(16c - 20c + 6c)}_{\to 1} e^{4x} + \underbrace{(6a)}_{\to 2} x + \underbrace{(-5a + 6b)}_{\to 0} = 2x + e^{4x}\,.$$

Comparing coefficients, we find that $c = 1/2$, $a = 1/3$ and $b = 5/18$. Noting that the homogeneous equation is solved by e^{2x} and e^{3x}, we have the general solution

$$y = Ae^{3x} + Be^{2x} + \frac{1}{2}e^{4x} + \frac{1}{3}x + \frac{5}{18}\,.$$

Note that the boundary conditions used to determine the constant A and B must be applied to the entire solution, not just the complementary function.

3.1.1 Resonance

In the example above, the forcing term e^{4x} is not a complementary function of the differential equation. However, if the forcing term were e^{2x}, say, we would not have been able to construct a particular integral of the form $y_p(x) \propto e^{2x}$. We can deal with such cases by "detuning" the forcing term.

The process is best illustrated with a concrete example. Consider the forced equation

$$\ddot{y} + \omega_0^2 y = \sin \omega_0 t\,. \tag{16}$$

The complementary function is

$$y_c(t) = A \sin \omega_0 t + B \cos \omega_0 t\,,$$

where A and B are constants. Physically, this is an example of a simple harmonic oscillator, with natural frequency ω_0, being driven by a oscillatory force that is at the natural frequency. In such situations, the system is said to be driven *resonantly.* Since the forcing term is a complementary function at resonance, a linear combination of $\sin\omega_0 t$ and $\cos\omega_0 t$ is *not* a particular integral.

Instead, we proceed by "detuning" the forcing term away from the natural frequency by considering

$$\ddot{y} + \omega_0^2 y = \sin\omega t \quad (\omega \neq \omega_0)\,.$$

We try a particular integral C

$$y_p(t) = \sin\omega t\,,$$

where C is a constant to be determined. The form of the left-hand side of Eq. (3.1.1) means there is no $\cos\omega t$ term in y_p. Substituting into the differential equation we find

$$\ddot{y}_p + \omega_0^2 y_p = C\left(-\omega^2 + \omega_0^2\right)\sin\omega t \quad \Rightarrow \quad C = \frac{1}{\omega_0^2 - \omega^2}\,.$$

Now we recall that, ultimately, we are interested in the limit $\omega \to \omega_0$. The particular integral we have found does not have a finite limit as $C \to \infty$ there. However, we can try and fix this by adding in any solution of the homogeneous equation as this will still be a valid particular integral. If we take

C.F.

$$y_p(t) = \frac{1}{\omega_0^2 - \omega^2}\left(\sin\omega t - \sin\omega_0 t\right)\,, \tag{17}$$

evaluating the (now-indeterminate) limit with l'Hôpital's rule, we have

$$\lim_{\omega\to\omega_0} y_p(t) = -\frac{t}{2\omega_0}\cos\omega_0 t\,. \tag{18}$$

This is a valid particular integral of Eq. (16).

As a general rule, if the forcing term is a linear combination of complementary functions, then the particular integral is proportional to the independent variable (t

in the example above) times a non-resonant particular integral ($\cos \omega_0 t$ above). (just the guess from table 1)

Aside: Behaviour close to resonance

It is interesting to consider the particular integral in Eq. (17) in the case that the driving frequency ω is close to, but not equal to, the natural frequency ω_0.

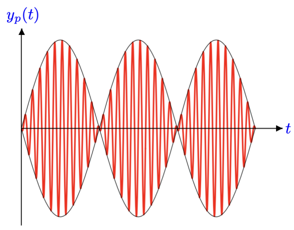
In this case, it is convenient to factorise $y_p(t)$ using

$$\begin{aligned}\sin \omega t - \sin \omega_0 t &= \mathrm{Im}\left(e^{i\omega t} - e^{i\omega_0 t}\right)\\ &= \mathrm{Im}\left[e^{i(\omega+\omega_0)t/2}\left(e^{i(\omega-\omega_0)t/2} - e^{-i(\omega-\omega_0)t/2}\right)\right]\\ &= 2\cos\left[\left(\frac{\omega+\omega_0}{2}\right)t\right]\sin\left[\left(\frac{\omega-\omega_0}{2}\right)t\right].\end{aligned}$$

This factors out an oscillation at the average frequency, $(\omega+\omega_0)/2$, and a slow oscillation at (half) the difference of the frequencies, $(\omega-\omega_0)/2$. If we write $\omega_0 - \omega = \Delta\omega$, we have

$$y_p(t) = \frac{-2}{(2\omega+\Delta\omega)\Delta\omega}\cos\left[\left(\omega+\frac{\Delta\omega}{2}\right)t\right]\sin\left(\frac{\Delta\omega}{2}t\right). \tag{19}$$

For $\Delta\omega \ll \omega$, the slow oscillation at frequency $\Delta\omega/2$ is much less rapid than that at the average frequency. In this limit, we observe the phenomenon of *beating*: the slow oscillation acts as a long-period modulation of the amplitude of the rapid oscillation. An example is shown in the figure to the right.



As $\Delta\omega \to 0$, the period of the modulation tends to infinity and we have linear growth of the amplitude of the oscillation at $\omega = \omega_0$. It is straightforward to show that in this limit, Eq. (19) reduces to our earlier result (18).

3.1.2 Resonance in equidimensional equations

The discussion above of resonance in equations with constant coefficients carries over to the case of equidimensional equations (Sec. 2.5). Recall that such equations have complementary functions of the $y_c \propto x^{k_1}$ or x^{k_2} (in the non-degenerate case). In the case that the forcing term is proportional to x^{k_1} (or x^{k_2}), there is a particular integral of the form $x^{k_1}\ln x$.

This result follows from transforming the equidimensional equation to one with constant coefficients by the substitution $z = \ln x$. For the transformed equation, a forcing term proportional to $e^{k_1 z}$ (or $e^{k_2 z}$) gives rise to a complementary function of the form $ze^{k_1 z}$ or, expressed in terms of x, $y_p(x) \propto x^{k_1}\ln x$.

3.2 Variation of parameters

So far, we have been determining particular integrals by making an educated guess. The method of *variation of parameters* provides a systematic way to find a particular integral given linearly independent complementary functions $y_1(x)$ and $y_2(x)$.

Consider the forced (inhomogeneous) second-order differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = f(x)\,, \tag{20}$$

with linearly independent complementary functions y_1 and y_2.

Recall that for any solution, $y(x)$, of Eq. (20), we define the solution vector

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \end{pmatrix}. \tag{21}$$

It will prove convenient to use the vectors

$$\mathbf{Y}_1(x) = \begin{pmatrix} y_1(x) \\ y_1'(x) \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(x) = \begin{pmatrix} y_2(x) \\ y_2'(x) \end{pmatrix}$$

as a basis for the solution space, so that at any x we write a particular integral as

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x)\,, \tag{22}$$

where $u(x)$ and $v(x)$ are functions to be determined.

Note that linear independence of the functions y_1 and y_2 ensures that the vectors $\mathbf{Y}_1$ and $\mathbf{Y}_2(x)$ are also linearly independent for *all* x.

The components of Eq. (22) give

$$y_p = uy_1 + vy_2 \,, \tag{23}$$

$$y_p' = uy_1' + vy_2' \,. \tag{24}$$

Differentiating the second equation, we have

$$y_p'' = uy_1'' + u'y_1' + vy_2'' + v'y_2' \,. \tag{25}$$

Adding this to $p(x)$ times Eq. (24) and $q(x)$ times Eq. (23), and demanding that y_p satisfies the differential equation (20), we have

$$u'y_1' + v'y_2' = f \,, \tag{26}$$

where we have used that y_1 and y_2 are complementary functions.

Now note that we derived Eq. (24) from the second row of the vector equation (22). However, this expression for y_p' has to be consistent with what we get by differentiating y_p in Eq. (23) directly. This requires that

$$u'y_1 + v'y_2 = 0 \,.$$

Along with Eq. (26), this gives us two simultaneous equations for u' and v', which we should be able to solve.

Writing these simultaneous equations in matrix form, we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix} .$$

Inverting the matrix on the left, we have

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} ,$$

where $W(x)$ is the (non-zero) Wronskian of y_1 and y_2. It follows that

$$u' = -\frac{y_2}{W} f \quad \text{and} \quad v' = \frac{y_1}{W} f \,. \tag{27}$$

Integrating these, and then substituting in Eq. (23), we obtain a particular integral

$$y_p(x) = y_2(x) \int^x \frac{y_1(\xi) f(\xi)}{W(\xi)}\, d\xi - y_1(x) \int^x \frac{y_2(\xi) f(\xi)}{W(\xi)}\, d\xi \,.$$

Note that we have not specified the lower limits on the integrals here. Changing these just adds constant multiples of y_1 and y_2 (i.e., a complementary function) to the particular integral.

Example. Consider the differential equation

$$y'' + 4y = \sin 2x \,.$$

Since the complementary functions are

$$y_1 = \sin 2x \quad \text{and} \quad y_2 = \cos 2x \,,$$

this is an example of an oscillator being driven resonantly (the forcing term is a complementary function). The Wronskian of y_1 and y_2 is $W = -2$, and so Eq. (27) gives

$$\begin{aligned} u' &= \frac{1}{2} \cos 2x \sin 2x = \frac{1}{4} \sin 4x \,, \\ v' &= -\frac{1}{2} \sin^2 2x = \frac{1}{4} \left(\cos 4x - 1 \right) \,. \end{aligned}$$

Integrating gives

$$u = -\frac{1}{16} \cos 4x \quad \text{and} \quad v = \frac{1}{16} \sin 4x - \frac{x}{4} \,.$$

It follows that

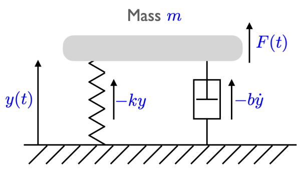
$$\begin{aligned} y_p &= \frac{1}{16} \left(-\cos 4x \sin 2x + \sin 4x \cos 2x \right) - \frac{1}{4} x \cos 2x \,, \\ &= \frac{1}{16} \sin 2x \left(-\cos 4x + 2 \cos^2 2x \right) - \frac{1}{4} x \cos 2x \,, \\ &= \frac{1}{16} \sin 2x - \frac{1}{4} x \cos 2x \,. \end{aligned}$$

The first term is clearly a multiple of one of the complementary functions. The second term is of the form x times a complementary function, as expected from our earlier discussion of resonance. Indeed, identifying $\omega_0 = 2$ and t with x in Eq. (16), we see that the term $-(x \cos 2x)/4$ above is exactly the same as the particular integral (18) that we found using detuning.

4 Forced oscillating systems: transients and damping

In this section we consider linear systems where there is a *restoring force* that tends to make the system oscillate and some *damping force* (e.g., friction in a mechanical system) that tends to oppose motion. There may also be some driving force.

Consider the set-up in the figure to the right, which might describe a car suspension system, for example. A mass m is acted on by a linear restoring force $-ky$, where k is a (positive) spring constant and $y(t)$ is the vertical displacement of the mass from its equilibrium position. There is also a damping force $-b\dot{y}$, where b is a positive damping constant, that resists the motion. In addition, the system is driven by an external force $F(t)$.



The equation of motion for the displacement of the mass is given by Newton's second law: $m\ddot{y} =$ total force. We have

$$\begin{aligned} m\ddot{y} &= -ky - b\dot{y} + F(t) \\ \Rightarrow \quad m\ddot{y} + b\dot{y} + ky &= F(t) \,. \end{aligned} \tag{28}$$

In the absence of damping and the external driving force, the system undergoes simple-harmonic motion with angular frequency $\omega_0 \equiv \sqrt{k/m}$. It is convenient to introduce a dimensionless time coordinate $\tau \equiv \omega_0 t$. Dividing Eq. (28) through by $m\omega_0^2$ (i.e., k), we can put the equation in dimensionless form:

tau is the period of oscillation up to a factor of 2pi.

$$y'' + 2\kappa y' + y = f(\tau)\,, \tag{29}$$

where

$$y' \equiv \frac{dy}{d\tau}\,, \quad \kappa = \frac{b}{m\omega_0} \quad f \equiv \frac{F}{k}\,.$$

In the form of Eq. (29), the unforced system is described by a single dimensionless parameter κ.

4.1 Free (unforced or natural) response

With $f = 0$, Eq. (29) reduces to

$$y'' + 2\kappa y' + y = 0\,. \tag{30}$$

We look for solutions of the form $y \propto e^{\lambda\tau}$, which gives the characteristic equation

$$\lambda^2 + 2\kappa\lambda + 1 = 0\,.$$

There are (generally) two roots given by

$$\lambda_1\,, \lambda_2 = -\kappa \pm \sqrt{\kappa^2 - 1}\,.$$

There are three cases to consider: $\kappa < 1$, $\kappa = 1$ and $\kappa > 1$.

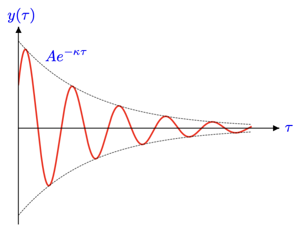
4.1.1 Light damping (underdamping): $\kappa < 1$

If $\kappa < 1$, both roots λ_1 and λ_2 are complex. We can write them as

$$\lambda_1\,, \lambda_2 = -\kappa \pm i\sqrt{1 - \kappa^2}\,.$$

The general solution of Eq. (30) is then

$$y(\tau) = e^{-\kappa\tau}\left[A\sin\left(\sqrt{1-\kappa^2}\tau\right) + B\cos\left(\sqrt{1-\kappa^2}\tau\right)\right]\,,$$



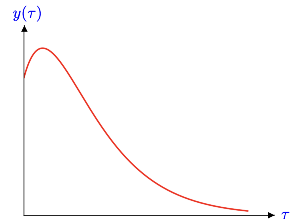
where A and B are constants. This solution describes damped oscillations at angular frequency

$$\omega_{\text{free}} = \sqrt{1 - \kappa^2}\omega_0\,, \tag{31}$$

which is lowered by damping from ω_0. The amplitude of the oscillation decays in time with a characteristic (dimensionless) decay time of $1/\kappa$. This behaviour is illustrated in the figure to the right.

4.1.2 Critical damping: $\kappa = 1$

For $\kappa = 1$, the characteristic equation is degenerate with a repeated root



$$\lambda_1 = \lambda_2 = -\kappa\,.$$

The general solution of Eq. (30) is then

$$y(\tau) = e^{-\kappa\tau}\left(A + B\tau\right)\,,$$

where A and B are constants. An example behaviour is shown in the figure to the right.

4.1.3 Heavy damping (overdamping): $\kappa > 1$

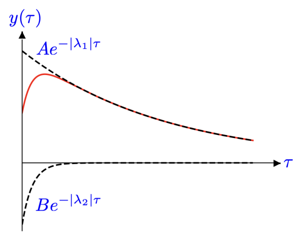
If $\kappa > 1$, both roots of the characteristic equation are real and negative. If we take $|\lambda_1| < |\lambda_2|$, then

$$\lambda_1 = -\kappa + \sqrt{\kappa^2 - 1} \quad \text{and} \quad \lambda_2 = -\kappa - \sqrt{\kappa^2 - 1}\,,$$

and the general solution of Eq. (30) is

$$y(\tau) = Ae^{-|\lambda_1|\tau} + Be^{-|\lambda_2|\tau}\,,$$

where A and B are constants.



The solution initially varies on the more rapid timescale $1/|\lambda_2|$, but once this has decayed the solution approaches $y(\tau) = Ae^{-|\lambda_1|\tau}$, which has decay time $1/|\lambda_1|$. The typical behaviour is shown in the figure to the right.

Note that in all cases, the unforced response decays eventually.

4.2 Forced response

In a forced system described by Eq. (29), the complementary function (i.e., unforced response) decays in time. The long-term behaviour is therefore determined by the driving force (through the particular integral), while the initial transient response is determined by *both* the complementary function and particular integral.

Example. Consider sinusoidal forcing so that Eq. (28) can be written as

$$\ddot{y} + \mu\dot{y} + \omega_0^2 y = \frac{F_0}{m}\sin\omega t\,, \tag{32}$$

where $\mu \equiv b/m$. We shall assume light damping, $\mu < 2\omega_0$, in which case the complementary function is

$$y_c(t) = e^{-\mu t/2}\left(A\sin\omega_{\text{free}}t + B\cos\omega_{\text{free}}t\right)\,,$$

where $\omega_{\text{free}} = \sqrt{\omega_0^2 - \mu^2/4}$.

For the particular integral, we try

$$y_p(t) = \frac{F_0}{m}\left(C\sin\omega t + D\cos\omega t\right)\,.$$

Substituting in Eq. (32) and comparing coefficients of the $\sin\omega t$ and $\cos\omega t$ terms, we have

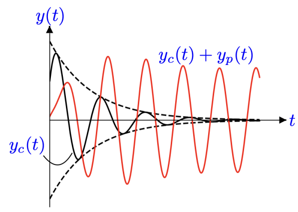
$$\begin{aligned}-\omega^2 C - \mu\omega D + \omega_0^2 C &= 1\\ -\omega^2 D + \mu\omega C + \omega_0^2 D &= 0\,.\end{aligned}$$

These are solved by

$$\begin{aligned}C &= \frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + (\mu\omega)^2}\\ D &= \frac{-\mu\omega}{\left(\omega_0^2 - \omega^2\right)^2 + (\mu\omega)^2}\end{aligned}$$

The full solution is $y(t) = y_c(t) + y_p(t)$, with

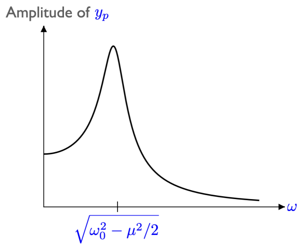
$$y_p(t) = \frac{F_0/m}{\left(\omega_0^2 - \omega^2\right)^2 + (\mu\omega)^2} \times \left[\left(\omega_0^2 - \omega^2\right)\sin\omega t - \mu\omega\cos\omega t\right]\,.$$



The solution is shown to the right. Note that the complementary function decays leaving only the particular integral asymptotically. This means that the damped oscillator has no long-term memory of its initial conditions since these only affect the constants A and B in the complementary function.

The particular integral determines the steady-state response to the driving force. This is sinusoidal with amplitude given by $(F_0/m)\sqrt{C^2+D^2}$:

$$\text{Amplitude of } y_p = \frac{F_0/m}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+(\mu\omega)^2}}\,.$$



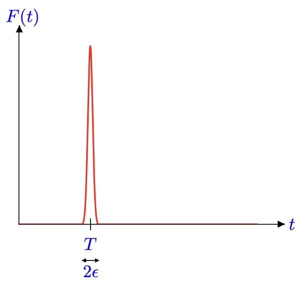
For light damping, the amplitude is sharply peaked at the *amplitude resonant frequency*

$$\omega_{\text{res}} = \sqrt{\omega_0^2-\mu^2/2}\,,$$

as illustrated in the figure to the right. The amplitude at ω_{res} increases as the damping is reduced and the peak becomes sharper.

5 Impulses and point forces

Consider a system that experiences a sudden force, extending from time $t = T-\epsilon$ to $t = T+\epsilon$, where ϵ is small compared to any other natural timescale (e.g., oscillation period or decay time) in the system. For example, the damped oscillator discussed in Sec. 4 might be struck from being at rest at its equilibrium position at time T (see figure to the right).



For small enough ϵ, the subsequent behaviour of the system does not depend on ϵ or the detailed form of the force when non-zero – all that matters is the *impulse* of the force,

$$I \equiv \int_{T-\epsilon}^{T+\epsilon} F(t)\,dt\,.$$

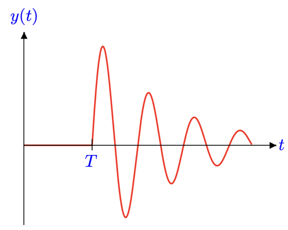
Mathematically, it is then convenient to consider the limit of a sudden, impulsive force with $\epsilon \to 0$, while preserving the impulse. (This means that the magnitude of the force at its peak must grow without limit.)

If we consider the damped, driven oscillator of Sec. 4, described by

$$m\ddot{y} + b\dot{y} + ky = F(t)\,,$$

we can integrate the equation of motion from $T - \epsilon$ to $T + \epsilon$ to find

$$\lim_{\epsilon\to 0}\left(m\,[\dot{y}]_{T-\epsilon}^{T+\epsilon} + b\,[y]_{T-\epsilon}^{T+\epsilon} + k\int_{T-\epsilon}^{T+\epsilon} y\,dt\right) = I\,.$$



The second term on the left is zero if y is continuous and the third is zero if y is finite in the interval. If we assume these to be true (they certainly will be in any physical system), we see that the velocity $\dot{y}$ undergoes a sudden change (it is discontinuous) that depends on the impulse of the force:

$$\lim_{\epsilon\to 0}\left(m\,[\dot{y}]_{T-\epsilon}^{T+\epsilon}\right) = I\,.$$

A typical behaviour is shown in the figure to the right.

5.1 The Dirac delta function

We can formalise the idea of an impulsive force by introducing the *Dirac delta function*.

We consider a family of functions $D(t;\epsilon)$ that have two key properties:

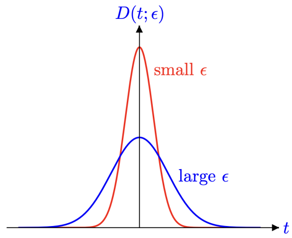
1. $\lim_{\epsilon\to 0} D(t;\epsilon) = 0\ \forall\ t \neq 0\,;$
2. $\int_{-\infty}^{\infty} D(t;\epsilon)\,dt = 1\,.$

The sudden force in the example above can then be represented by $F(t) = I\,D(t - T;\epsilon)$.

Multiply by I, to get correct impulse, and shift by T

An example of such a family is

$$D(t;\epsilon) = \frac{1}{\epsilon\sqrt{\pi}} e^{-t^2/\epsilon^2} \,,$$



which is illustrated in the figure to the right. (These functions do integrate to unity; see Question 14 on Examples Sheet 1.) Note that as ϵ decreases, the standard deviation or "width" of the Gaussian gets narrower while the peak value gets larger, preserving the integral. Therefore, $\lim_{\epsilon\to 0} D(0;\epsilon)$ does not exist.

Of course, the family of $D(t;\epsilon)$ having these two defining characteristics is not unique. However, for any such family the limit as $\epsilon \to 0$ yields a function (more carefully, a *distribution*), which we call the *Dirac delta function*.

Definition (Dirac delta function). The *Dirac delta function* is defined by

$$\delta(x) \equiv \lim_{\epsilon\to 0} D(x;\epsilon) \,,$$

on the understanding that we can only use its integral properties, i.e., when the delta function is multiplied by some suitably well-behaved "test function" and integrated over an appropriate interval (see below).

The delta function satisfies three key properties:

1. $\delta(x) = 0 \;\forall\; x \neq 0\,;$
2. $\int_{-\infty}^{\infty} \delta(x)\, dx = 1;$
3. for all functions $g(x)$ that are continuous at $x = 0$,
$$\int_{-\infty}^{\infty} g(x)\delta(x)\, dx = \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} g(x) D(x;\epsilon)\, dx = g(0)\,.$$

This last property is known as the *sampling property*. For any test function continuous at $x = 0$, the delta function samples its value there. The generalisation to functions $g(x)$ that are continuous at $x = x_0$ is (for $b > a$)

$$\int_a^b g(x)\delta(x - x_0)\, dx = \begin{cases} g(x_0) \text{ if } a < x_0 < b \\ 0 \text{ otherwise}\,. \end{cases}$$

5.2 Delta-function forcing

The delta function is a mathematically convenient way of expressing impulsive forcing terms.

Consider

$$y'' + p(x)y' + q(x)y = \delta(x)\,, \tag{33}$$

where $p(x)$ and $q(x)$ are continuous functions. For $x < 0$ and $x > 0$, $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0\,.$$

However, at $x = 0$ there will be a discontinuity in $y'(x)$.

We can see this by integrating Eq. (33) around a small interval $-\epsilon < x < \epsilon$ and taking the limit as $\epsilon \to 0$:

$$\lim_{\epsilon\to 0}\,[y']_{-\epsilon}^{\epsilon} + p(0)\lim_{\epsilon\to 0}\,[y]_{-\epsilon}^{\epsilon} + \lim_{\epsilon\to 0}\int_{-\epsilon}^{\epsilon} qy\,dx = 1\,.$$

If we assume that y is continuous at $x = 0$, the second and third term on the left vanish leaving the *jump condition*

$$\lim_{\epsilon\to 0}\,[y']_{-\epsilon}^{\epsilon} = 1\,.$$

Our assumption that $y(x)$ has to be continuous at $x = 0$ can be established by contradiction. If it were discontinuous there, then the first derivative would be like a delta function and the second derivative even worse behaved making it impossible to satisfy the original differential equation.

Generally, discontinuities arising from delta-function forcing are addressed by the highest-order derivative appearing in the differential equation.

Example. Consider the *boundary-value* problem

$$y'' - y = 3\delta\left(x - \frac{\pi}{2}\right) \quad \text{with} \quad y = 0 \text{ at } x = 0 \text{ and } \pi\,.$$

We solve this by considering the regions $0 \le x < \pi/2$ and $\pi/2 < x \le \pi$ separately, and then join these together using the appropriate jump condition.

Away from $x = \pi/2$, we have

$$y'' - y = 0\,, \tag{34}$$

with general solution $y = A\sinh x + B\cosh x$. (It is more convenient to use hyperbolic functions rather than exponentials since we require $y(x)$ to vanish at the boundary points.) For $0 \leq x < \pi/2$, the relevant solution is

$$y(x) = A\sinh x\,,$$

while for $\pi/2 < x \leq \pi$ we have

$$y(x) = C\sinh(\pi - x)\,.$$

We now have to join these solutions up at $x = \pi/2$ such that y is continuous there and

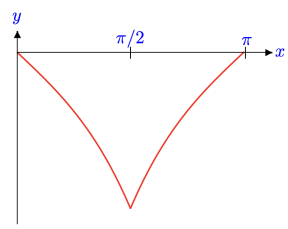
$$\lim_{\epsilon\to 0}\,[y']_{\pi/2-\epsilon}^{\pi/2+\epsilon} = 3\,.$$

Continuity of y implies that $A = C$, while the jump condition on y' gives

$$\begin{aligned}-C\cosh(\pi/2) - A\cosh(\pi/2) &= 3\\ \Rightarrow \quad A = C &= \frac{-3}{2\cosh(\pi/2)}\,.\end{aligned}$$

The full solution is therefore

$$y(x) = \begin{cases} -\frac{3}{2}\frac{\sinh x}{\cosh(\pi/2)} & 0 \leq x < \pi/2 \\ -\frac{3}{2}\frac{\sinh(\pi-x)}{\cosh(\pi/2)} & \pi/2 < x \leq \pi\,. \end{cases}$$



This is shown in the figure to the right. Note, in particular, the discontinuity in the gradient at $x = \pi/2$.

5.3 Heaviside step function $H(x)$

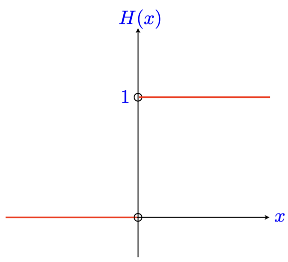
Definition (Heaviside step function). The *Heaviside step function* is defined as the integral of the delta function:

$$H(x) = \int_{-\infty}^{x} \delta(t)\,dt\,.$$

It follows that

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \\ \text{undefined} & \text{at } x = 0\,. \end{cases}$$

The Heaviside function is shown schematically in the figure to the right. Note that it is undefined at $x = 0$. From the fundamental theorem of calculus, we have that



$$\frac{dH}{dx} = \delta(x)\,,$$

but, recall, that such functions and relationships can only be used inside integrals.

Generally, we see the smoothing effect of integration and the sharpening effect of differentiation: the derivative of the Heaviside function is very rapidly varying around the origin. On the other hand, the integral of the Heaviside function (sometimes called the *ramp function*) is continuous at $x = 0$ and just has a discontinuous first derivative.

5.3.1 Forcing with the Heaviside step function

The Heaviside step function can be used to describe situations where the forcing changes discontinuously. For example, if a switch is closed in an electrical circuit, the electromotive force of the battery is suddenly now applied to the other circuit components.

Consider a system described by

$$y'' + p(x)y' + q(x)y = H(x)\,,$$

where $p(x)$ and $q(x)$ are continuous at $x = 0$. For $x < 0$, $y(x)$ satisfies

$$y'' + p(x)y' + q(x)y = 0\,,$$

while for $x > 0$,

$$y'' + p(x)y' + q(x)y = 1\,.$$

The solutions of these equations are joined by noting that

$$\lim_{\epsilon\to 0}\left([y'']_{-\epsilon}^{\epsilon} + p(0)\,[y']_{-\epsilon}^{\epsilon} + q(0)\,[y]_{-\epsilon}^{\epsilon}\right) = 1\,.$$

If y'' goes like $H(x)$ in the vicinity of $x = 0$, then y' and y are both continuous there and the above condition is satisfied. We thus have the jump conditions:

$$\lim_{\epsilon\to 0}\,[y']_{-\epsilon}^{\epsilon} = 0 \quad\text{and}\quad \lim_{\epsilon\to 0}\,[y]_{-\epsilon}^{\epsilon} = 0\,.$$

6 Higher-order discrete (difference) equations

Much of what we have learnt about the solutions of linear differential equations goes over to discrete (difference) equations. Recall that we first met these in Topic II as approximations to first-order differential equations.

Consider a discrete second-order equation of the form

$$ay_{n+2} + by_{n+1} + cy_n = f_n\,, \tag{35}$$

where a, b and c are constants. This now couples y_{n+2} to both y_{n+1} and y_n. Such couplings arise if we consider discretising a second-order differential equation at points $\{x_n\}$ spaced by h, since the second derivative at x_n can be approximated by

$$\left.\frac{d^2y}{dx^2}\right|_{x_n} \approx \frac{y_{n+1} + y_{n-1} - 2y_n}{h^2}\,.$$

We can solve Eq. (35) by exploiting linearity and eigenfunctions, as in the case of differential equations.

We first look for complementary functions that satisfy

$$ay_{n+2} + by_{n+1} + cy_n = 0\,. \tag{36}$$

For a linear second-order differential equation with constant coefficients, the complementary functions take the form $y_c \propto e^{\lambda x}$. The discrete version of this is $y_n^{(c)} \propto k^n$

f_n	$y_n^{(p)}$
k^n	Ak^n if $k \neq k_1$ or k_2
k_1^n	Ank_1^n
n^p (p a non-negative integer)	$An^p + Bn^{p-1} + \cdots + Cn + D$

Table 2: Form of particular "integrals" $y_n^{(p)}$ for discrete equations of the type in Eq. (35). Here, k_1 and k_2 are the roots of the characteristic equation for the homogeneous equation.

for some k to be determined. Trying this in Eq. (36), we have

$$\begin{aligned} ak^{n+2} + bk^{n+1} + ck^n &= 0 \\ \Rightarrow \quad ak^2 + bk + c &= 0\,. \end{aligned}$$

This characteristic equation has two roots in general, $k = k_1$ and $k = k_2$. The general complementary function is then

$$y_n^{(c)} = \begin{cases} Ak_1^n + Bk_2^n & \text{if } k_1 \neq k_2\,, \\ (A + Bn)k_1^n & \text{if } k_1 = k_2 = k \end{cases}\,.$$

The degenerate case, $k_1 = k_2 = k$, follows by analogy with degenerate differential equations: $xe^{\lambda x} \to nk^n$.

We can guess particular "integrals" of Eq. (35) for simple forcing sequences f_n; see Table 2.

Example (Fibonacci sequence). The Fibonacci sequence is defined by

$$y_n = y_{n-1} + y_{n-2}\,, \quad y_0 = y_1 = 1\,. \tag{37}$$

The sequence arises in all sorts of unexpected contexts. For example, in biological systems it arises in the arrangements of leaves on a stem or spikes on a pineapple. The first few elements in the sequence for $n = 0, 1, 2, 3, 4, 5$ are, of course, $y_n = 1, 1, 2, 3, 5, 8$.

We can rewrite Eq. (37) as

$$y_{n+2} - y_{n+1} - y_n = 0\,.$$

Trying $y_n = k^n$, we find

$$k^2 - k - 1 = 0 \quad \Rightarrow \quad k = \frac{1 \pm \sqrt{5}}{2}\,,$$

which you may recognise as the "golden ratio" or "golden mean" and (the negative of) its inverse:

$$\varphi_1 = \frac{1+\sqrt{5}}{2}\,, \quad \varphi_2 = \frac{1-\sqrt{5}}{2} = \frac{-1}{\varphi_1}\,.$$

The solution of Eq. (37) is therefore of the form

$$y_n = A\varphi_1^n + B\varphi_2^n\,,$$

where A and B are given by the initial conditions

$$y_0 = 1 = A + B \quad \text{and} \quad y_1 = 1 = A\varphi_1 + B\varphi_2\,.$$

These are solved by

$$A = \frac{\varphi_1}{\sqrt{5}} \quad \text{and} \quad B = -\frac{\varphi_2}{\sqrt{5}}\,,$$

so that

$$y_n = \frac{\varphi_1^{n+1} - \varphi_2^{n+1}}{\sqrt{5}} = \frac{\varphi_1^{n+1} - (-1/\varphi_1)^{n+1}}{\sqrt{5}}\,.$$

This result is remarkable! It expresses a sequence of *integers* in terms of the difference of powers of the irrational golden ratio. Noting that $\varphi_1 > 1$, we have

$$\lim_{n\to\infty} y_{n+1}/y_n = \varphi_1\,,$$

so the ratio of adjacent terms of the Fibonacci sequence tends to the golden ratio.

7 Series solutions

In this section, we develop techniques to find series solutions to linear, homogeneous second-order differential equations when we (perhaps) cannot find simple closed forms. This builds on the brief discussion of series solutions for first-order differential equations in Topic II.

We consider equations of the form

$$p(x)y'' + q(x)y' + r(x)y = 0\,. \tag{38}$$

The feasibility of constructing a series solution in the vicinity of the point $x = x_0$ depends on the nature of the functions $p(x)$, $q(x)$ and $r(x)$ there.

7.1 Classification of singular points

Definition (Ordinary and singular points). The point $x = x_0$ is an *ordinary point* of the differential equation (38) if both $q(x)/p(x)$ and $r(x)/p(x)$ have Taylor series around $x = x_0$ (i.e., they are *analytic* there). Otherwise, $x = x_0$ is a *singular point*.

We can classify singular points further as follows. If $x = x_0$ is a singular point, but Eq. (38) can be rewritten as

$$P(x)(x-x_0)^2y'' + Q(x)(x-x_0)y' + R(x)y = 0\,,$$

where $Q(x)/P(x)$ and $R(x)/P(x)$ do have Taylor series around $x = x_0$, then $x = x_0$ is a *regular singular point*. Otherwise, $x = x_0$ is an *irregular singular point*.

Note that this condition for a regular singular point is equivalent to $(x-x_0)q(x)/p(x)$ and $(x-x_0)^2r(x)/p(x)$ in Eq. (38) having Taylor series around $x = x_0$.

Loosely, for a regular singular point, the equation is singular, but not *too* singular, due to the properties of the derivative (cf. equidimensional equations).

Example. Consider

$$(1-x^2)y'' - 2xy' + 2y = 0\,.$$

We have

$$\frac{q(x)}{p(x)} = \frac{-2x}{1-x^2} \quad \text{and} \quad \frac{r(x)}{p(x)} = \frac{2}{1-x^2}\,.$$

It follows that $x = \pm 1$ are singular points. However, as

$$(x-1)\frac{q(x)}{p(x)} = \frac{2x}{x+1} \quad \text{and} \quad (x-1)^2\frac{r(x)}{p(x)} = \frac{-2(x-1)}{x+1}\,,$$

for example, we see that these are both regular singular points.

Example. Consider

$$\sin x y'' + \cos x y' + 2y = 0\,.$$

Here,

$$\frac{q(x)}{p(x)} = \frac{\cos x}{\sin x} \quad \text{and} \quad \frac{r(x)}{p(x)} = \frac{2}{\sin x}\,.$$

Clearly, $x = n\pi$, for n an integer, are singular points while all others are ordinary points. Writing $x - n\pi = \epsilon$, we have

$$(x - n\pi)\frac{q(x)}{p(x)} = \epsilon\frac{\cos\epsilon}{\sin\epsilon}\,,$$

which is analytic at $\epsilon = 0$. Similarly, $(x-n\pi)^2 r(x)/p(x)$ is analytic at $x = n\pi$, so the points $x = n\pi$ are all regular singular points.

For this equation, all other points are ordinary points.

Example. Consider

$$\left(1 + \sqrt{x}\right) y'' - 2xy' + 2y = 0\,.$$

We have

$$\frac{q(x)}{p(x)} = \frac{-2x}{1 + \sqrt{x}}\,,$$

which does not have a Taylor series around $x = 0$ (the second derivative is undefined). Similarly, $xq(x)/p(x)$ does not have a Taylor series there either so the point $x = 0$ is an irregular singular point.

7.2 Method of Frobenius

We now develop the series-expansion method of obtaining at least one solution of the linear, homogeneous, second-order differential equation (38). We can always do this provided the expansion point, $x = x_0$ is no worse than a regular singular point.

Theorem (Fuchs' theorem). If $x = x_0$ is an ordinary point of Eq. (38), then there are two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n\,,$$

i.e., in the form of a Taylor series, convergent in some neighbourhood of x_0.

If, instead, $x = x_0$ is an irregular singular point of Eq. (38), then there is at least one solution of the form

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma} , \tag{39}$$

where σ is real and $a_0 \neq 0$ (so that σ is unique). This is an example of a *Frobenius series.* It can also be written as

$$y = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n ,$$

where the summation here is a Taylor series.

Note that when expanding about a regular singular point, there is no guarantee that one will obtain *two* linearly independent series solutions. We shall return to the construction of a second solution in such cases later.

Attempting a series solution about an irregular singular point may fail completely.

The method of Frobenius is best illustrated by example. Let us first consider an example of expanding about an ordinary point, and then a regular singular point.

Example (ordinary point). Let us consider again

$$(1 - x^2) y'' - 2xy' + 2y = 0 , \tag{40}$$

which we saw earlier has regular singular points at $x = \pm 1$ while all other points are ordinary. Expanding about $x = 0$ (an ordinary point), we try

$$y = \sum_{n=0}^{\infty} a_n x^n ,$$

so that

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} .$$

It is convenient to multiply the original equation through by x^2 to make it like an equidimensional equation but with polynomial coefficients:

$$(1-x^2)\left(x^2y''\right) - 2x^2\left(xy'\right) + 2x^2y = 0\,.$$

Substituting for $y(x)$ and its derivatives, we have

$$\sum_{n=2}^{\infty} a_n\left[(1-x^2)n(n-1)\right]x^n - 2\sum_{n=1}^{\infty} a_n(x^2n)x^n + 2\sum_{n=0}^{\infty} a_n(x^2)x^n = 0\,.$$

Equating coefficients of x^n, for $n \geq 2$ we have

$$a_n n(n-1) - a_{n-2}(n-2)(n-3) - 2a_{n-2}(n-2) + 2a_{n-2} = 0\,,$$

or

$$\begin{aligned} n(n-1)a_n &= n(n-3)a_{n-2} \qquad (n \geq 2) \\ \Rightarrow \quad a_n &= \frac{n-3}{n-1}a_{n-2}\,. \end{aligned} \tag{41}$$

This is a recurrence relation determining a_n in terms of a_{n-2}. Note that a_0 and a_1 are not fixed by this procedure – they are arbitrary constants set by the initial/boundary conditions for the differential equation.

From the recursion relation, we have $a_3 = 0$ and so $a_{2k+1} = 0$ for all $k \geq 1$. This gives one solution, $y = a_1x$.

On the other hand, for n even, we have

$$\begin{aligned} a_n &= \frac{(n-3)}{(n-1)}a_{n-2} = \frac{(n-3)}{(n-1)}\frac{(n-5)}{(n-3)}a_{n-4} = \cdots \\ &= -\frac{1}{n-1}a_0\,, \end{aligned}$$

as terms alternately cancel. Therefore

$$\begin{aligned} y &= a_0\left[1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots\right] \\ &= a_0\left[1 - \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right)\right]\,, \end{aligned}$$

noting that

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \cdots .$$

Finally, the general solution of the differential equation (40) is

$$y(x) = a_1 x + a_0 \left[1 - \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right)\right] .$$

Note the logarithmic behaviour of this solution at $x = \pm 1$, which, recall, are regular singular points of the differential equation. We shall return to this observation when discussing the construction of a second solution when expanding about a regular singular point.

Example (regular singular point). Consider

$$4xy'' + 2(1 - x^2)y' - xy = 0 .$$

For this equation, $x = 0$ is a regular singular point. Let us look for a series solution about this point. From Fuchs' theorem, we try

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} ,$$

with $a_0 \neq 0$. Again, it is convenient to multiply through the differential equation (this time by x) to write it as

$$4\left(x^2 y''\right) + 2(1 - x^2)\left(xy'\right) - x^2 y = 0 .$$

Substituting for y and its derivatives, we have

$$\sum_{n=0}^{\infty} a_n x^{n+\sigma} \left[4(n+\sigma)(n+\sigma-1)\right.$$
$$\left. +2(1-x^2)(n+\sigma) - x^2\right] = 0 .$$

As before, we equate coefficients of powers of x. The lowest power is x^σ with coefficient

$$a_0 \left[4\sigma(\sigma - 1) + 2\sigma\right] = 0$$
$$\Rightarrow \quad a_0 \sigma(2\sigma - 1) = 0 .$$

This is called the *indicial equation* for the *index* σ. Since $a_0 \neq 0$ by construction, we must have $\sigma = 0$ or $\sigma = 1/2$. Generally, the lowest power of x gives rise to the indicial equation, and its roots determine the index σ in the Frobenius series.

The next-lowest power is $x^{\sigma+1}$ with coefficient

$$\begin{aligned} a_1\left[4\sigma(\sigma+1)+2(\sigma+1)\right] &= 0 \\ \Rightarrow \quad a_1(\sigma+1)(2\sigma+1) &= 0\,. \end{aligned}$$

For both cases $\sigma = 0$ and $\sigma = 1/2$, this implies $a_1 = 0$.

Finally, we consider the power $x^{n+\sigma}$ for $n \geq 2$. This gives rise to the recursion relation

$$\begin{aligned} &\left[4(n+\sigma)(n+\sigma-1)+2(n+\sigma)\right]a_n \\ &\qquad\qquad + \left[-2(n+\sigma-2)-1\right]a_{n-2} = 0\,, \end{aligned}$$

which can be rearranged to obtain

$$2(n+\sigma)(2n+2\sigma-1)a_n = (2n+2\sigma-3)a_{n-2}\,. \quad (42)$$

We now consider the two roots of the indicial equation separately.

For $\sigma = 0$, the recursion relation (42) becomes

$$2n(2n-1)a_n = (2n-3)a_{n-2} \quad (n \geq 2)\,.$$

Since $a_1 = 0$, it follows that $a_{2k+1} = 0$ for all $k \geq 0$. For n even, we have

$$a_n = \frac{2n-3}{2n(2n-1)}a_{n-2}\,,$$

and so

$$a_2 = \frac{1}{4\times 3}a_0\,, \quad a_4 = \frac{5}{8\times 7}a_2 = \frac{5}{8\times 7}\times\frac{1}{4\times 3}a_0\,, \quad \text{etc.}$$

It follows that we have one solution

$$y = a_0\left(1 + \frac{1}{4\times 3}x^2 + \frac{5\times 1}{8\times 7\times 4\times 3}x^4 + \cdots\right)\,.$$

Note that this is a Taylor series.

We now consider the root $\sigma = 1/2$. In this case, the recursion relation (42) becomes

$$2n(2n+1)a_n = (2n-2)a_{n-2} \quad (n \geq 2)$$
$$\Rightarrow \quad a_n = \frac{n-1}{n(2n+1)}a_{n-2}\,.$$

This gives

$$a_2 = \frac{1}{2\times 5}a_0\,, \quad a_4 = \frac{3}{4\times 9}a_2 = \frac{3}{4\times 9}\times\frac{1}{2\times 5}a_0\,, \quad \text{etc.}$$

It follows that we have a second solution

$$y = b_0 x^{1/2}\left(1 + \frac{1}{2\times 5}x^2 + \frac{3\times 1}{4\times 9\times 2\times 5}x^4 + \cdots\right)\,,$$

where we have relabelled a_0 to b_0 to distinguish it from the constant in the other solution.

We see that for this example we have generated two linearly independent solutions with the series-expansion method. However, this is not generally the case, as we now discuss.

7.3 Second solutions

When expanding around a regular singular point $x = x_0$, we are guaranteed to be able to construct one solution as a Frobenius series. Whether we can generate a second depends critically on the roots, σ_1 and σ_2, of the indicial equation. There are three cases to consider.

1. If $\sigma_2 - \sigma_1$ is not an integer, then there are always two linearly independent solutions:

$$y = (x-x_0)^{\sigma_1}\sum_{n=0}^{\infty} a_n (x-x_0)^n\,,$$
$$y = (x-x_0)^{\sigma_2}\sum_{n=0}^{\infty} b_n (x-x_0)^n\,.$$

   Note that as $x \to x_0$, a linear combination of these goes like $a_0(x-x_0)^{\sigma_1}$ if $\sigma_1 < \sigma_2$.

2. If $\sigma_2 - \sigma_1$ is a non-zero integer, there is one solution of the form
$$y_1 = (x - x_0)^{\sigma_2} \sum_{n=0}^{\infty} a_n (x - x_0)^n \,,$$
involving the larger root σ_2 of the indicial equation. The smaller root σ_1 will generally not generate a valid series solution (though in some special cases it may). Instead, the second solution is of the form
$$y_2 = (x - x_0)^{\sigma_1} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1 \ln(x - x_0) \,.$$
Here c is usually non-zero, i.e., the solution must generally include a part involving the solution y_1 multiplied by $\ln(x - x_0)$. The constant c is not arbitrary – it is fixed in terms of the constant a_0 in the first series and b_0, leaving the general solution with two arbitrary constants, as required.

3. If $\sigma_1 = \sigma_2 = \sigma$, then the log term is *always* required, i.e., $c \neq 0$. The solutions are then
$$y_1 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} a_n (x - x_0)^n \,,$$
$$y_2 = (x - x_0)^{\sigma} \sum_{n=0}^{\infty} b_n (x - x_0)^n + c y_1 \ln(x - x_0) \,.$$

Example (Case 2). Consider the differential equation
$$x^2 y'' - x y = 0 \,.$$
The point $x = 0$ is a regular singular point. As usual, we look for a solution
$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} \,,$$
with $a_0 \neq 0$. The differential equation then requires
$$\sum_{n=0}^{\infty} \left[a_n (n + \sigma)(n + \sigma - 1) x^{n+\sigma} - a_n x^{n+\sigma+1} \right] = 0 \,.$$

The coefficient of the lowest power of x (x^σ) determines the indicial equation:

$$a_0\sigma(\sigma - 1) = 0 \quad \Rightarrow \quad \sigma = 0 \quad \text{or} \quad \sigma = 1\,,$$

since $a_0 \neq 0$. The coefficients of the higher powers of x give

$$a_n(n+\sigma)(n+\sigma-1) = a_{n-1} \quad (n \geq 1)\,. \qquad (43)$$

We see that we have two roots of the indicial equation that differ by an integer (i.e., Case 2 above).

First consider the larger root, $\sigma = 1$. The recursion relation (43) gives (for $n \geq 1$)

$$a_n = \frac{a_{n-1}}{n(n+1)} \quad \Rightarrow \quad a_n = \frac{a_0}{n!(n+1)!}\,,$$

giving a solution

$$y_1 = a_0 x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots\right)\,.$$

Now consider the root $\sigma = 0$. The recursion relation gives

$$a_n n(n-1) = a_{n-1} \quad (n \geq 1)\,.$$

For $n = 1$, the left-hand side vanishes implying that $a_0 = 0$. However, this is a contradiction since we have required that $a_0 \neq 0$.

In this example, we see that the smaller root of the indicial equation does not generate a series solution to the differential equation. Instead, the solution will take the form

$$y_2 = cy_1 \ln x + \sum_{n=0}^{\infty} b_n x^n\,. \qquad (44)$$

Construction of the second solution (non-examinable)

There are several ways to construct the second solution. The most direct is simply to assume the trial solution (44) and substitute it into the differential equation.

Proceeding this way, we have

$$xy_2 = cy_1 x \ln x + \sum_{n=0}^{\infty} b_n x^{n+1} \, ,$$

$$x^2 y_2'' = c\left(y_1'' x^2 \ln x + 2xy_1' - y_1\right) + \sum_{n=0}^{\infty} b_n n(n-1) x^n \, ,$$

Recalling that y_1 is a solution of the differential equation, we require

$$2cxy_1' - cy_1 + \sum_{n=0}^{\infty} b_n \left[n(n-1)x^n - x^{n+1}\right] = 0 \, . \tag{45}$$

Note how the log terms have cancelled.

Recalling that

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1} \quad \text{with} \quad a_n = \frac{a_0}{n!(n+1)!} \, ,$$

so that

$$xy_1' = \sum_{n=0}^{\infty} a_n (n+1) x^{n+1} \, ,$$

Eq. (45) reduces to

$$c \sum_{n=0}^{\infty} a_n (2n+1) x^{n+1} + \sum_{n=0}^{\infty} b_n \left[n(n-1)x^n - x^{n+1}\right] = 0 \, .$$

The coefficient of the lowest power of x (x^0) vanishes identically, while for $n \geq 1$ we must have

$$ca_{n-1}(2n-1) + n(n-1)b_n - b_{n-1} = 0 \, . \tag{46}$$

For $n = 1$, Eq. (46) gives

$$ca_0 = b_0 \, ,$$

which determines the constant c in terms of a_0 and b_0, as noted earlier. Substituting $c = b_0/a_0$ and using the explicit form of the a_n, Eq. (46) then reduces to

$$\frac{(2n-1)}{n!(n-1)!} b_0 - b_{n-1} + n(n-1)b_n = 0 \quad (n \geq 1) \, .$$

In this recursion relation, we can choose b_0 and b_1 arbitrarily and then all b_n for $n > 1$ are determined linearly from b_0 and b_1.

By assumption, $b_0 \neq 0$. If we were to take $b_0 = 0$, then $c = 0$ and the log term vanishes in the trial solution (44). Moreover, the recursion relation would then give

$$b_n = \frac{b_1}{n!(n-1)!} \, ,$$

and the trial solution would reduce to

$$y_2 = b_1 \sum_{n=1}^{\infty} \frac{x^n}{n!(n-1)!} = b_1 \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} \, .$$

This is proportional to y_1 and so we have not generated a linearly independent second solution.

We therefore need only consider the case $b_1 = 0$, so that

$$b_2 = -\frac{3}{4}b_0\,, \quad b_3 = -\frac{7}{36}b_0\,, \quad \text{etc.}$$

The second solution is then

$$y_2 = b_0 \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + b_0\left(1 - \frac{3}{4}x^2 - \frac{7}{36}x^3 + \cdots\right)\,, \qquad (47)$$

where the first summation on the right is y_1/a_0 and we have used $c = b_0/a_0$.

Reduction of order

An alternative way to construct a second solution is to use the method of reduction of order (Sec. 2.1). This has the benefit of showing why the second solution has to have the form of the trial solution (44).

Recall that with this method, given a solution y_1, we look for a second solution in the form $y_2 = v(x)y_1$. Given the trial second solution (44), we expect that $v(x)$ will involve a log term. Substituting $y_2 = v(x)y_1$ into the differential equation

$$x^2 y_2'' - x y_2 = 0\,,$$

and using the fact that y_1 is also a solution, we must have

$$v'' y_1 + 2v' y_1' = 0\,.$$

This means that $u \equiv v'$ satisfies

$$u' y_1 + 2u y_1' = 0 \quad \Rightarrow \quad \frac{u'}{u} = -2\frac{y_1'}{y_1}\,.$$

The solution of this is

$$\ln u = -2 \ln y_1 + \ln B \quad \Rightarrow \quad u = v' = \frac{B}{y_1^2}\,,$$

where B is an arbitrary constant.

We now use the series solution for y_1, which we repeat here for convenience:

$$y_1 = a_0 x\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots\right)\,.$$

It follows that

$$\begin{aligned} v' &= \frac{B}{a_0^2 x^2}\left(1 + \frac{x}{2} + \frac{x^2}{12} + \frac{x^3}{144} + \cdots\right)^{-2} \\ &= \frac{B}{a_0^2 x^2}\left(1 - x + \frac{7}{12}x^2 - \frac{19}{72}x^3 + \cdots\right)\,, \end{aligned}$$

where we have performed a binomial expansion. Let us write this as

$$v' = \frac{B}{a_0^2}\left(\frac{1}{x^2} - \frac{1}{x} + \sum_{n=0}^{\infty} B_n x^n\right) ,$$

where $B_0 = 7/12$, $B_1 = -19/72$, etc. Integrating, we find

$$v = \frac{B}{a_0^2}\left(-\frac{1}{x} - \ln x + \sum_{n=0}^{\infty} \frac{B_n}{n+1} x^{n+1}\right) ,$$

where we have ignored any constant of integration since it would just add a multiple of the first solution, y_1, to y_2.

The final step is to multiply by y_1, where, recall,

$$y_1 = a_0 x \sum_{n=0}^{\infty} \frac{x^n}{n!(n+1)!} .$$

This gives

$$\begin{aligned} y_2 &= -\frac{B}{a_0} \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + \frac{B}{a_0}\left(1 + \frac{x}{2} + \frac{x^2}{12} + \cdots\right) \\ &\qquad\qquad\qquad \times \left(-1 + \sum_{n=0}^{\infty} \frac{B_n}{n+1} x^{n+2}\right) \\ &= -\frac{B}{a_0} \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} - \frac{B}{a_0}\left[1 + \frac{x}{2} + \left(\frac{1}{12} - B_0\right) x^2 + \cdots\right] , \end{aligned}$$

which is exactly of the form of the trial second solution (44).

We can make contact with the second solution (47), derived above by direct substitution, by identifying $-B/a_0$ with b_0 and using $B_0 = 7/12$:

$$y_2 = b_0 \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + b_0\left(1 + \frac{x}{2} - \frac{x^2}{2} + \cdots\right) .$$

The second series on the right differs from that in Eq. (47). However, they only differ by a series proportional to y_1. If we subtract $b_0(x + x^2/2 + \cdots)/2$, we get

$$y_2 = b_0 \ln x \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!(n+1)!} + b_0\left(1 - \frac{3}{4}x^2 + \cdots\right) ,$$

which is exactly the second solution obtained previously.
