# Stochastic Financial Models 13

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### 1 Discrete-time martingales

Example.

- Let  $X_1, X_2, ...$  be independent with  $\mathbb{E}(X_n) = 0$  for all n.
- Let  $S_0 = 0$  and  $S_n = X_1 + \ldots + X_n$ .

Then  $(S_n)_{n\geq 0}$  is a martingale in the filtration generated by  $(X_n)_{n\geq 1}$  since

- $S_n$  is integrable:  $\mathbb{E}(|S_n|) \leq \mathbb{E}(|X_1|) + \ldots + \mathbb{E}(|X_n|) < \infty$
- $S_n$  is clearly  $\mathcal{F}_n$  measurable (since it is a function of  $X_1, \ldots, X_n$ )
- $\mathbb{E}(S_n S_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}(X_n|\mathcal{F}_{n-1}) = \mathbb{E}(X_n) = 0$  by the independence of  $X_n$  and  $\mathcal{F}_{n-1}$ .

Note that in this example  $(S_n)_{n\geq 0}$  and  $(X_n)_{n\geq 1}$  generate the same filtration

**Definition.** A discrete-time process  $(H_n)_{n\geq 1}$  is *previsible* (or *predictable*) with respect to a filtration  $(\mathcal{F}_n)_{n\geq 0}$  iff  $H_n$  is  $\mathcal{F}_{n-1}$ -measurable for all  $n\geq 1$ .

Remark. The index set for a previsible process is usually  $\{1, 2, \ldots\}$ . Remark. Let  $X_n = H_{n+1}$ . Then  $(H_n)_{n \geq 1}$  is previsible if and only if  $(X_n)_{n \geq 0}$  is adapted.

**Definition.** The martingale transform of a previsible process  $(H_n)_{n\geq 1}$  with respect to an adapted process  $(X_n)_{n\geq 0}$  is the process defined by

$$Y_n = \sum_{k=0}^{n} H_k(X_k - X_{k-1})$$

**Theorem.** The martingale transform of a bounded previsible process with respect to a martingale is a martingale.

Proof. Let  $(H_n)_{n\geq 1}$  be bounded and previsible and  $(X_n)_{n\geq 0}$  a martingale, and let  $(Y_n)_{n\geq 0}$  be the martingale transform. Note that  $(Y_n)_{n\geq 0}$  is adapted since each term of the formula defining  $Y_n$  is  $\mathcal{F}_n$ -measurable by the adaptedness of  $(X_n)$  and the previsibility of  $(H_n)$ . Integrability follows from by the triangle inequality

$$\mathbb{E}(|Y_n|) \le \mathbb{E}\left(\sum_{k=1}^n |H_k||X_k - X_{k-1}|\right) \le C\sum_{k=1}^n \mathbb{E}(|X_k - X_{k-1}|) < \infty$$

and the integrability of  $(X_n)$  (from the definition of martingale), where C > 0 is the constant such that  $|H_k| \leq C$  a.s. for all k (from the assumption of boundedness of  $(H_n)$ )

Now

$$\mathbb{E}(Y_n - Y_{n-1}|\mathcal{F}_{n-1}) = \mathbb{E}[H_n(X_n - X_{n-1})|\mathcal{F}_{n-1}]$$

$$= H_n \mathbb{E}(X_n - X_{n-1}|\mathcal{F}_{n-1})$$

$$= 0$$

by taking out what is known, and the martingale property of  $(X_n)_{n>0}$ .

Important example from finance. Consider a market

- with a risk-free asset with interest rate r
- and d risky assets with time n prices  $(S_n)_{n>0}$ .

and investor who

- holds the portfolio  $\theta_n \in \mathbb{R}^d$  of risky assets during the time interval (n-1, n],
- and the rest of his wealth is held in the risk-free asset.
- Suppose the investor is *self-financing*: his changes in wealth are explained by the changes in asset prices (but not by consumption or non-market income)

$$X_n = (1+r)X_{n-1} + \theta_n^{\top} [S_n - (1+r)S_{n-1}]$$

**Definition.** The investor's discounted wealth at time n is  $\frac{X_n}{(1+r)^n}$ . The discounted asset prices at time n are  $\frac{S_n}{(1+r)^n}$ .

**Proposition.** A self-financing investor's discounted wealth is the initial wealth plus the martingale transform of the portfolio process with respect to the discounted risky asset prices.

*Proof.* It is easy to see by induction that

$$\frac{X_n}{(1+r)^n} = X_0 + \sum_{k=1}^n \theta_k^{\top} \left( \frac{S_k}{(1+r)^k} - \frac{S_{k-1}}{(1+r)^{k-1}} \right)$$

## 2 Stopping times

**Definition.** A stopping time for a filtration  $(\mathcal{F}_t)_{t\geq 0}$  is a random variable T valued in  $\{0,1,2,\ldots,+\infty\}$  (discrete-time) or  $[0,+\infty]$  (continuous time) such that

$$\{T \le t\} \in \mathcal{F}_t \text{ for all } t \ge 0$$

#### Example.

- Let  $(X_n)_{n\geq 0}$  be a discrete-time adapted process.
- Let  $T = \inf\{n \ge 0 : X_n > 0\}$
- Convention:  $\inf \emptyset = \infty$ .
- Then T is a stopping time.

Note  $\{T \leq n\} = \bigcup_{k=0}^n \{X_k > 0\} \in \mathcal{F}_n$  since  $\{X_k > 0\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$  for all  $k \leq n$ . (Recall that the sigma-algebra  $\mathcal{F}_n$  is closed under finite unions.)

#### Possible counter-example.

- Let  $(X_n)_{n\geq 0}$  be an adapted process.
- Let  $T = \sup\{n \ge 0 : X_n > 0\}$
- Then T is a *not* a stopping time in general.

Note  $\{T \leq n\} = \bigcap_{k=n+1}^{\infty} \{X_k \leq 0\}$  so the event  $\{T \leq n\}$  generally contains information about the future.