

# Stochastic Financial Models 15

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8 November 2023

## 1 Submartingales and supermartingales

**Definition.** An integrable adapted process  $(X_t)_{t \geq 0}$  with respect to a filtration  $(\mathcal{F}_t)_{t \geq 0}$  (in either discrete- or continuous-time) is a submartingale if and only if

$$\mathbb{E}(X_t | \mathcal{F}_s) \geq X_s \text{ for all } 0 \leq s \leq t$$

An integrable adapted  $(X_t)_{t \geq 0}$  is called *supermartingale* with respect to a filtration iff  $(-X_t)_{t \geq 0}$  is a submartingale.

For the rest of the time, we work in **discrete-time**.

**Remark.** In discrete-time, a *submartingale* is an integrable adapted process  $(X_n)_{n \geq 0}$  such that

$$\mathbb{E}(X_n | \mathcal{F}_{n-1}) \geq X_{n-1} \text{ for all } n \geq 1$$

by the tower property and the positivity of conditional expectation.

**Theorem.** *The martingale transform of a non-negative bounded previsible process with respect to a submartingale is a submartingale.*

*Proof.* Let  $(H_n)_{n \geq 1}$  be non-negative, bounded and previsible, and  $(X_n)_{n \geq 0}$  a submartingale, and let  $(Y_n)_{n \geq 0}$  be the martingale transform. Integrability of  $(Y_n)_n$  follows from the boundedness of  $(H_n)_n$  and integrability of  $(X_n)_n$ . The adaptedness of  $(Y_n)_n$  follows from the adaptedness of both  $(H_n)_n$  and  $(X_n)_n$ .

Now

$$\begin{aligned} \mathbb{E}(Y_n - Y_{n-1} | \mathcal{F}_{n-1}) &= \mathbb{E}[H_n(X_n - X_{n-1}) | \mathcal{F}_{n-1}] \\ &= H_n \mathbb{E}(X_n - X_{n-1} | \mathcal{F}_{n-1}) \\ &\geq 0 \end{aligned}$$

by taking out what is known, and the submartingale property of  $(X_n)_{n \geq 0}$ . □

**Theorem.** Let  $(X_n)_{n \geq 0}$  be a submartingale and  $S \leq T$  are stopping times. Let

$$M_n = X_{n \wedge T} - X_{n \wedge S}.$$

Then  $(M_n)_{n \geq 0}$  is a submartingale.

*Proof.* Note

$$M_n = \sum_{k=1}^n \mathbb{1}_{\{S < k \leq T\}} (X_k - X_{k-1}).$$

Also  $H_n = \mathbb{1}_{\{S < k \leq T\}} = \mathbb{1}_{\{S \leq n-1\}} - \mathbb{1}_{\{T \leq n-1\}}$  is bounded and  $\mathcal{F}_{n-1}$ -measurable. Hence  $(M_n)_n$  is the martingale transform of a non-negative bounded previsible process with respect to a submartingale.  $\square$

**Theorem** (Optional sampling theorem). Let  $(X_n)_{n \geq 0}$  be a submartingale and  $S \leq T$  are bounded stopping times, then

$$\mathbb{E}(X_T) \geq \mathbb{E}(X_S)$$

*Proof.* Let  $M_n = X_{n \wedge T} - X_{n \wedge S}$ . Now pick a constant  $N$  such that  $T \leq N$  a.s. The conclusion follows from  $\mathbb{E}(M_N) \geq M_0 = 0$  since  $M_N = X_T - X_S$ .  $\square$

## 2 Controlled Markov processes

**Definition.** A Markov process  $(X_t)_{t \geq 0}$  with respect to a filtration  $(\mathcal{F}_t)_{t \geq 0}$  (in either discrete- or continuous-time) is an adapted process such that

$$\mathbb{P}(X_t \in A | \mathcal{F}_s) = \mathbb{P}(X_t \in A | X_s)$$

for all  $0 \leq s \leq t$ , (measurable) sets  $A$ , and where  $(\mathcal{F}_t)_{t \geq 0}$ .

We now work in discrete time. To check that a process  $(X_n)_n$  is a martingale, we need only check

$$\mathbb{P}(X_n \in A | \mathcal{F}_{n-1}) = \mathbb{P}(X_n \in A | X_{n-1})$$

for all  $n \geq 1$ .

A useful way of to think about a Markov process is as **random dynamical system**. A Markov process valued in  $\mathcal{X}$  can be constructed with

- Initial condition  $X_0 = x$
- A function  $G : \mathbb{N} \times \mathcal{X} \times \mathcal{V} \rightarrow \mathcal{X}$
- An sequence  $(\xi_n)_{n \geq 1}$  of independent  $\mathcal{V}$ -valued random variable
- Then we construct the process recursively

$$X_n = G(n, X_{n-1}, \xi_n)$$

for  $n \geq 1$ .

**Example.** A simple symmetric random walk on  $\mathbb{Z}$  starting at  $X_0 = 0$  can be constructed as follows

- Let  $\mathcal{V} = \{-1, 1\}$
- Let  $(\xi_n)_{n \geq 1}$  be an IID sequence such that  $\mathbb{P}(\xi_n = \pm 1) = 1/2$ .
- Let  $G(n, x, v) = x + v$  for all  $n$ .
- Then  $X_n = G(n, X_{n-1}, \xi_n)$  for  $n \geq 1$ .

A **controlled Markov process** is built from

- Initial condition  $X_0 = x$
- A previsible process  $(U_n)_{n \geq 1}$
- A function  $G : \mathbb{N} \times \mathcal{X} \times \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{X}$
- A sequence  $(\xi_n)_{n \geq 1}$  of independent  $\mathcal{V}$ -valued random variables
- Then we construct the process recursively

$$X_n^U = G(n, X_{n-1}^U, U_n, \xi_n)$$

for  $n \geq 1$ .

### 3 Stochastic optimal control

A typical problem that we will encounter is this. Given a controlled Markov process  $(X_n^U)_{n \geq 0}$  and a (non-random) time horizon  $N$  we wish to

$$\text{maximise } \mathbb{E} \left[ \sum_{k=1}^N f(k, U_k) + g(X_N^U) \middle| X_0 = x \right]$$

over previsible controls  $(U_k)_{1 \leq k \leq N}$ , where the controlled Markov process evolves as  $X_n = G(n, X_{n-1}, U_n, \xi_n)$  for  $n \geq 1$  for a given function  $G$  and independent sequence  $(\xi_n)_n$ .

**Definition.** The system of equations

$$\begin{aligned} V(N, x) &= g(x) \quad \text{for all } x \\ V(n-1, x) &= \sup_u \{f(n, u) + \mathbb{E}[V(n, G(n, x, u, \xi_n))]\} \quad \text{for all } x, 1 \leq n \leq N \end{aligned}$$

is called the *Bellman equation* for the problem.

**Definition.** The *value function* for the problem is

$$V(n, x) = \sup_{(U_k)_{n+1 \leq k \leq N}} \mathbb{E} \left[ \sum_{k=n+1}^N f(k, U_k) + g(X_N^U) \middle| X_n^U = x \right].$$

**The dynamic programming principle:** Under some assumptions, the solution to the Bellman equation is the value function. (details in next lecture)