Part IB — Analysis and Topology

Based on lectures by Dr P. Russell

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Contents

Part I
Generalizing continuity and convergence

§1 Three Examples of Convergence

§1.1 Convergence in $\mathbb{R}$

Let (x_n) be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. We say (x_n) *converges* to x and write $x_n \to x$ if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, n \geq N \quad |x_n - x| < \epsilon.$$

Useful fact: $\forall\, a, b \in \mathbb{R}$ $|a + b| \leq |a| + |b|$ (Triangle Inequality).

Bolzano-Weierstrass Theorem (BWT) A bounded sequence in $\mathbb{R}$ must have a convergent subsequence (Proof by interval bisection).

Recall: A sequence (x_n) in $\mathbb{R}$ is Cauchy if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, m, n \geq N \quad |x_m - x_n| < \epsilon.$$

Easy exercise Convergent $\implies$ Cauchy

General Principle of Convergence (GPC) Any Cauchy sequence in $\mathbb{R}$ converges.

> *Outline.* If (x_n) Cauchy then (x_n) bounded so by BWT has a convergent subsequence, say $x_{n_j} \to x$. But as (x_n) Cauchy, $x_n \to x$. □

§1.2 Convergence in $\mathbb{R}^2$

Remark 1. This all works in $\mathbb{R}^n$

Let (z_n) be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. What should $z_n \to z$ mean?

In $\mathbb{R}$: "As n gets large, z_n gets arbitrarily close to z."

What does 'close' mean in $\mathbb{R}^2$?

In $\mathbb{R}$: a, b close if $|a-b|$ small. In $\mathbb{R}^2$: Replace $|\cdot|$ by $\|\cdot\|$

Recall: If $z=(x,y)$ then $\|z\|=\sqrt{x^2+y^2}$.

Triangle Inequality If $a, b \in \mathbb{R}^2$ then $\|a+b\| \leq \|a\|+\|b\|$.

Definition 1.1

Let (z_n) be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. We say (z_n) **converges** to z and .. $z_n \to z$ if $\forall\, \epsilon>0\ \exists\, N\ \forall\, n \geq N\ \ \|z_n - z\| < \epsilon$.

Equivalently, $z_n \to z$ iff $\|z_n - z\| \to 0$ (convergence in $\mathbb{R}$).

Example 1.1

Let $(z_n), (w_n)$ be sequences in $\mathbb{R}^2$ with $z_n \to z, w_n \to w$. Then $z_n + w_n \to z + w$.

Proof.

$$\begin{aligned}\|(z_n+w_n)-(z+w)\| &\leq \|z_n - z\| + \|w_n - w\| \\ &\to 0+0=0 \text{ (by results from IA).}\end{aligned}$$

□

In fact, given convergence in $\mathbb{R}$, convergence in $\mathbb{R}^2$ is easy:

Proposition 1.1

Let (z_n) be a sequence in $\mathbb{R}^2$ and let $z \in \mathbb{R}^2$. Write $z_n = (x_n, y_n)$ and $z=(x,y)$. Then $z_n \to z$ iff $x_n \to x$ and $y_n \to y$.

Proof. ($\implies$): $|x_n - x|, |y_n - y| \leq \|z_n - z\|$. So if $\|z_n - z\| \to 0$ then $|x_n - x| \to 0$ and $|y_n - y| \to 0$.

($\Longleftarrow$): If $|x_n - x| \to 0$ and $|y_n - y| \to 0$ then $\|z_n - z\| = \sqrt{(x_n-x)^2 + (y_n - y)^2} \to 0$ by results in $\mathbb{R}$. □

Definition 1.2 (Bounded Sequence)

A sequence (z_n) in $\mathbb{R}^2$ is **bounded** if $\exists\, M \in \mathbb{R}$ s.t. $\forall\, n\ \|z_n\| \leq M$.

Theorem 1.1 (BWT in $\mathbb{R}^2$)

A bounded sequence in $\mathbb{R}^2$ must have a convergent subsequence.

Theorem 1.2 (GPC for $\mathbb{R}^2$)
Any Cauchy sequence in $\mathbb{R}^2$ converges.

Proof. Let (z_n) be a Cauchy sequence in $\mathbb{R}^2$. Write $z_n = (x_n, y_n)$. For all m, n, $|x_m - x_n| \le \|z_m - z_n\|$ so (x_n) is a Cauchy sequence in $\mathbb{R}$, so converges by GPC. Similarly, (y_n) converges in $\mathbb{R}$. So by 1.1, (z_n) converges. $\square$

Thought for the day What about continuity? Let $f : \mathbb{R}^2 \to \mathbb{R}$. What does it mean for f to be continuous? (Simple modification of defn for $\mathbb{R} \to \mathbb{R}$).

What can we do with it?

Big theorem in IA: If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on a closed bounded interval then f is bounded and attains its bounds.

Is there a similar theorem for $\mathbb{R}^2 \to \mathbb{R}$. What do we replace 'closed bounded interval' by? We proved the theorem using BWT. Why did it work? Why did we need a closed bounded interval to make it work? What can we do in $\mathbb{R}^2$?

§1.3 Convergence of Functions

Let $X \subset \mathbb{R}$[1], let $f_n : X \to \mathbb{R}$ $(n \ge 1)$ and let $f : X \to \mathbb{R}$. What does it mean for f_n to converge to f.

Obvious idea:

Definition 1.3 (Pointwise convergence)
Say (f_n) **converges pointwise** to f and write $f_n \to f$ pointwise if $\forall\, x \in X$ $f_n(x) \to f(x)$ as $n \to \infty$.

Pros

- Simple
- Easy to check
- Defined in terms of convergence in $\mathbb{R}$

Cons

- Doesn't preserve 'nice' properties.
- 'Doesn't feel right'.

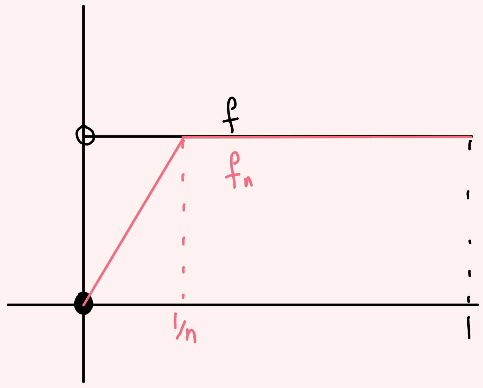
In all three examples, have $X = [0, 1]$, $f_n \to f$ pointwise.

[1] Mostly can think of $X = \mathbb{R}$ or some interval

Example 1.2 (Every f_n continuous but f not)

$$f_n(x) = \begin{cases} nx & x \le \frac{1}{n} \\ 1 & x \ge \frac{1}{n} \end{cases}$$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$



Clearly f_n continuous for all n but f not. If $x = 0$, $\forall\, n$ $f_n(0) = 0 = f(0)$. If $x > 0$, for sufficiently large n $f_n(x) = 1 = f(x)$ so $f_n(x) \to f(x)$.

Example 1.3 (Every f_n integrable but f not)

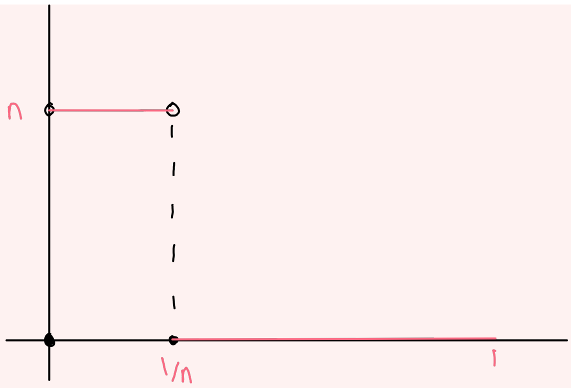
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

This is a non integrable[a] function so now we want to find f_n such that they converge pointwise to this. Enumerate the rationals in $[0, 1]$ as $q_1, q_2, \dots$ For $n \ge 1$, set $f_n(x) = \mathbb{1}_{q_1,\dots,q_n}$. f_n integrable as it is nonzero at finitely many points.

[a] N.B. As in IA 'integrable' means 'Riemann integrable'

Example 1.4 (Every f_n and f integrable but $\int_0^1 f_n \not\to \int_0^1 f$)

Let $f(x) = 0$ for all x, so $\int_0^1 f = 0$. Define f_n s.t. $\int_0^1 f_n = 1$ for all n.

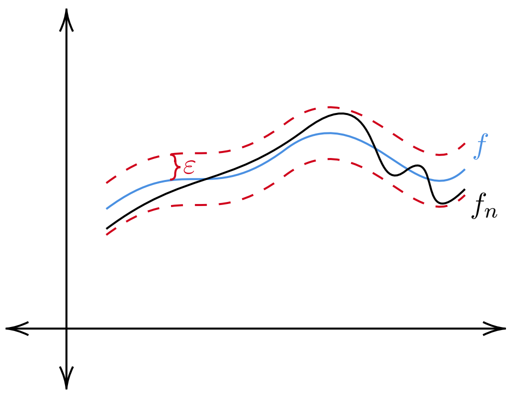


$$f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Better definition:

Definition 1.4 (Uniform convergence)
Let $X \subset \mathbb{R}$, $f_n : X \to \mathbb{R}$ $(n \geq 1)$, $f : X \to \mathbb{R}$. We say (f_n) **converges uniformly** to f and write $f_n \to f$ uniformly if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, x \in X\ \forall\, n \geq N\ |f_n(x) - f(x)| < \epsilon$.

cf $f_n \to f$ pointwise: $\forall\, \epsilon > 0\ \forall\, x \in X\ \exists\, N\ \forall\, n \geq N\ |f_n(x) - f(x)| < \epsilon$. (We have swapped the $\forall\, x \in x$ and $\exists\, N$). Pointwise convergence allows for N to be a function of x whilst uniform convergence requires N to work for all x even the worst case. In particular, $f_n \to f$ uniformly $\implies$ $f_n \to f$ pointwise.



Equivalently, $f_n \to f$ uniformly if for sufficiently large n $f_n - f$ is bounded and $\sup_{x \in X} |f_n - f| \to 0$.

Theorem 1.3 (A uniform limit of cts functions is cts)
Let $X \subset \mathbb{R}$, let $f_n : X \to \mathbb{R}$ be continuous ($n \geq 1$) and let $f_n \to f : X \to \mathbb{R}$ uniformly. Then f is cts.

Proof. Let $x \in X$. Let $\epsilon > 0$. As $f_n \to f$ uniformly, we can find N s.t. $\forall\, n \geq N\ \forall\, y \in X\ |f_n(y) - f(y)| < \epsilon$. In particular, $\forall\, y \in X\ |f_N(y) - f(y)| < \epsilon$. As f_N is cts, we can find $\delta > 0$ s.t. $\forall\, y \in X,\ |y - x| < \delta \implies |f_N(y) - f_N(x)| < \epsilon$. Now let $y \in X$ with $|y - x| < \delta$. Then

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)|^{a} \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Hence f is cts. □

[a]The core of this proof is this inequality.

Remark 2. This is often called a '3ϵ proof' (or an $\frac{\epsilon}{3}$ proof).

Theorem 1.4
Let $f_n : [a, b] \to \mathbb{R}$ ($n \geq 1$) be integrable and let $f_n \to f : [a, b] \to \mathbb{R}$ uniformly. Then f is integrable and $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$.

Proof. As $f_n \to f$ uniformly, we can pick n suff. large s.t. $f_n - f$ is bounded. Also f_n is bounded (as integrable). So by triangle inequality, $f = (f - f_n) + f_n$ is bounded. Let $\epsilon > 0$. As $f_n \to f$ uniformly there is some N s.t. $\forall\, n \geq N\ \forall\, x \in [a, b]$ we have $|f_n(x) - f(x)| < \epsilon$.
In particular, $\forall\, x \in [a, b]\ |f_N(x) - f(x)| < \epsilon$.

By Riemann's criterion, there is some dissection $\mathcal{D}$ of $[a, b]$ for which $S(f_n, \mathcal{D}) - s(f_n, \mathcal{D}) < \epsilon$. Let $\mathcal{D} = \{x_0, x_1, x_2, \ldots, x_k\}$ where $a = x_0 < x_1 < \cdots < x_k = b$. Now

$$\begin{aligned} S(f, \mathcal{D}) &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x) \\ &\leq \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} (f_N(x) + \epsilon) \\ &= \sum_{i=1}^{k} (x_i - x_{i-1}) \left(\left(\sup_{x \in [x_{i-1}, x_i]} f_N(x) \right) + \epsilon \right) \\ &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f_N(x) + \sum_{i=1}^{k} (x_i - x_{i-1}) \epsilon \end{aligned}$$

$$= S(f_N, \mathcal{D}) + (b-a)\epsilon.$$

That is $S(f, \mathcal{D}) \leq S(f_N, \mathcal{D}) + (b-a)\epsilon$. Similarly $s(f, \mathcal{D}) \geq s(f_N, \mathcal{D}) - (b-a)\epsilon$. Hence

$$\begin{aligned} S(f, \mathcal{D}) - s(f, \mathcal{D}) &\leq S(f_N, \mathcal{D}) - s(f_N, \mathcal{D}) + 2(b-a)\epsilon \\ &< (2(b-a)+1)\epsilon \end{aligned}$$

But $2(b-a)+1$ is a constant so $(2(b-a)+1)\epsilon$ can be made arbitrarily small. Hence by Riemann's criterion, f is integrable over $[a, b]$.

Now, for any n suff. large that $f_n - f$ is bounded,

$$\begin{aligned} \left| \int_a^b f_n - \int_a^b f \right| &= \left| \int_a^b (f_n - f) \right| \\ &\leq \int_a^b |f_n - f| \\ &\leq (b-a) \sup_{x \in [a,b]} |f_n - f| \\ &\to 0 \text{ as } n \to \infty \text{ since } f_n \to f \text{ uniformly.}^a \end{aligned}$$

□

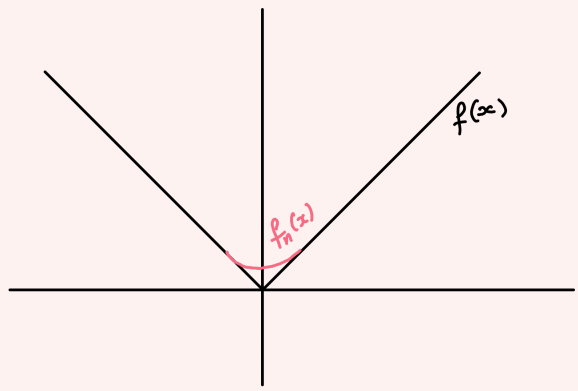
[a] Note we said that $f_n \to f$ uniformly if $\sup |f_n - f| \to 0$.

What about differentiation? Here even uniform convergence isn't enough.

Example 1.5

$f_n : (-1, 1) \to \mathbb{R}$, each f_n differentiable, $f_n \to f$ uniformly, f not diff.

Let $f(x) = |x|$ which is not differentiable at 0.



$$f_n = \begin{cases} |x| & |x| \geq \frac{1}{n} \\ ax^2 + bx + c & |x| < \frac{1}{n} \end{cases}.$$

We need $a(\frac{1}{n})^2 + \frac{b}{n} + c = \frac{1}{n}$ for continuity. Thus $b = 0$ and $c = \frac{1}{n} - \frac{a}{n^2}$.

Also need $2a\frac{1}{n} + b = 1$ and $2a(-\frac{1}{n}) = -1$ for differentiability so take $a = \frac{n}{2}$, $c = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$.

If $|x| \geq \frac{1}{n}$ then $|f_n(x) - f(x)| = 0$. If $|x| < \frac{1}{n}$:

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{n}{2}x^2 + \frac{1}{2n} - |x| \right| \\ &\leq \frac{n}{2}x^2 + \frac{1}{2n} + |x| \\ &\leq \frac{n}{2}(\frac{1}{n})^2 + \frac{1}{2n} + \frac{1}{n} \\ &= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{n} \\ &= \frac{2}{n} \end{aligned}$$

So $\sup_{x \in (-1,1)} |f_n(x) - f(x)| \leq \frac{2}{n} \to 0$ as $n \to \infty$. So $f_n \to f$ uniformly.

If fact we need uniform convergence of the derivatives.

Theorem 1.5

Let $f_n : (u, v) \to \mathbb{R}$ $(n \geq 1)$ with $f_n \to f : (u, v) \to \mathbb{R}$ pointwise. Suppose further each f_n is continuously differentiable and that $f'_n \to g : (u, v) \to \mathbb{R}$ uniformly. Then f is differentiable with $f' = g$.

Proof. Fix $a \in (u, v)$. Let $x \in (u, v)$, by FTC we have each f'_n is integrable over $[a, x]$ and $\int_a^x f'_n = f_n(x) - f_n(a)$. But $f'_n \to g$ uniformly so by theorem 1.4 g is integrable over $[a, x]$ and $\int_a^x g = \lim_{n\to\infty} \int_a^x f'_n = f(x) - f(a)$. So we have shown that for all $x \in (u, v)$

$$f(x) = f(a) + \int_a^x g.$$

By theorem 1.3, g is cts so by FTC, f is differentiable with $f' = g$. □

Remark 3. It would have sufficed to assume $f_n(x) \to f(x)$ for a single value of x rather than $f_n \to f$ pointwise.

GPC?

Definition 1.5 (Uniform Cauchy)

Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say $f_n)$ is **uniformly Cauchy** if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, m, n \geq N\ \forall\, x \in X\ |f_m(x) - f_n(x)| < \epsilon$

exercise: uniformly convergence $\implies$ uniformly Cauchy.

Theorem 1.6 (General principle of Uniform Convergence (GPUC))

Let (f_n) be a uniformly Cauchy sequence of functions $X \to \mathbb{R}$ $(X \subset \mathbb{R})$. Then (f_n) is uniformly convergent.

Proof. Let $x \in X$. Let $\epsilon > 0$. Then $\exists\, N\ \forall\, m, n \geq N\ \forall\, y \in X\ |f_m(y) - f_n(y)| < \epsilon$. In particular, $\forall\, m, n \geq N\ |f_m(x) - f_n(x)| < \epsilon$. So $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ so by GPC it converges, say $f_n(x) \to f(x)$ as $n \to \infty$.

We have now constructed $f : X \to \mathbb{R}$ s.t. $f_n \to f$ pointwise.
Let $\epsilon > 0$. Then we can find a N s.t. $\forall\, m, n \geq N\ \forall\, y \in X\ |f_m(y) - f_n(y)| < \epsilon$. Fix $y \in X$, keep $m \geq N$ fixed and let $n \to \infty$: $|f_m(y) - f(y)| \leq \epsilon$. So we have shown that $\forall\, m \geq N$, $|f_m(y) - f(y)| < \epsilon$.

But y was arbitrary so $\forall\, x \in X\ \forall\, m \geq N\ |f_m(x) - f(x)| \leq \epsilon$. That is $f_n \to f$ uniformly. $\square$

BW?

Definition 1.6 (Pointwise bounded)

Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say (f_n) is **pointwise bounded** if $\forall\, x\ \exists\, M\ \forall\, n\ |f_n(x)| \leq M$.

Definition 1.7 (Uniformly bounded)

Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say (f_n) is **uniformly bounded** if $\exists\, M\ \forall\, x\ \forall\, n\ |f_n(x)| \leq M$.[a]

[a] Again we have just swapped ... as in convergence.

What would uniform BW say? ‘If (f_n) is a uniformly bounded sequence of functions that it has a uniformly convergent subsequence’. But this is <u>not</u> true.

Example 1.6 (Counterexample of BW)

$$f_n : \mathbb{R} \to \mathbb{R}$$

$$f_n(x) = \begin{cases} 1 & x = n \\ 0 & x \neq n. \end{cases}$$

Obviously (f_n) uniformly bounded (by 1). However, if $m \neq n$ then $f_m(m) = 1$ and $f_n(m) = 0$ so $|f_m(m) - f_n(m)| = 1$ so no subsequence can be uniformly Cauchy so no subsequence can be uniformly convergent.

Application to power series Recall that if $\sum a_n x^n$ is a real power series with r.o.c $R > 0$ then we can differentiate/ integrate it term-by-term within $(-R, R)$.

Definition 1.8

Let $f_n : X \to \mathbb{R}$ ($X \subset \mathbb{R}$) for each $n \geq 0$. We say the series $\sum_{n=0}^{\infty} f_n$ **uniformly converges** if the sequence of partial sums (F_n) does, where $F_n = \sum_{m=0}^{n} f_m$.

We can apply theorems 1.3 to 1.5 to get e.g. if conditions hold with f_n cts diff and uniform convergence then $\sum f_n$ has derivative $\sum f_n'$.

Hope Prove $\sum a_n x^n$ converges uniformly on $(-R, R)$ then hit it with earlier theorems.

Not quite true:

Example 1.7

$\sum_{n=0}^{\infty} x^n$ r.o.c 1. This does not converge uniformly on $(-1, 1)$. Let $f(x) = \sum_{n=0}^{\infty} x^n$ and $F_n(x) = \sum_{m=0}^{n} x^m$. Note $f(x) = \frac{1}{1-x} \to \infty$ as $x \to 1$. However, $\forall\ x \in (-1, 1)$ $|F_n(x)| \leq n + 1$.

Fix any n. We can find a point $x \in (-1, 1)$ where $f(x) \geq n+2$ and so $|f(x) - F_n(x)| \geq 1$. So we don't have uniform convergence.

Back-up plan: It does work if we look at a smaller interval.

New plan: show if $0 < r < R$ then we do have uniform convergence on $(-r, r)$.
Given $x \in (-R, R)$ there's some r with $|x| < r < R$: use uniform convergence on $(-r, r)$ to check everything is nice at x. 'Local uniform convergence of power series'.

Aside

In $\mathbb{R}$ $x_n \to 0$ if

1. $\forall\ \epsilon > 0\ \exists\ N\ \forall\ n \geq N\ |x_n| < \epsilon$.
2. Equivalently: $\forall\ \epsilon > 0\ \exists\ N\ \forall\ n \geq N\ |x_n| \leq \epsilon$.

Proof. i $\implies$ ii: obvious
ii $\implies$ ii: Let $\epsilon > 0$. Pick N s.t. $\forall\, n \geq N$ $|x_n| \leq \frac{1}{2}\epsilon$. Then $\forall\, n \geq N$ $|x_n| < \epsilon$. □

Also: $f_n, f : X \to \mathbb{R}$, $f_n \to f$ uniformly.

1. $\forall\, \epsilon > 0\, \exists\, N\, \forall\, x \in X\, \forall\, n \geq N$ $|f_n(x) - f(x)| < \epsilon$.
2. For n suff large $f_n - f$ is bounded and $\forall\, \epsilon > 0\, \exists\, N\, \forall\, n \geq N$ $\sup_{x\in X} |f_n(x) - f(x)| < \epsilon$.

Proof. ii $\implies$ i: obvious
i $\implies$ ii: if i holds then $\sup_{x\in X} |f_n(x) - f(x)| \leq \epsilon$. But OK by same argument as previously. □

Lemma 1.1

Let $\sum a_n x^n$ be a real power series with r.o.c $R > 0$. Let $0 < r < R$. Then $\sum a_n x^n$ converges uniformly on $(-r, r)$.

Proof. Define $f, f_n : (-r, r) \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $f_m(x) = \sum_{n=0}^{m} a_n x^n$. Recall that $\sum a_n x^n$ converges absolutely for all x with $|x| < R$.

Let $x \in (-r, r)$. Then f

$$\begin{aligned} |f(x) - f_m(x)| &= \left| \sum_{n=m+1}^{\infty} a_n x^n \right| \\ &\leq \sum_{n=m+1}^{\infty} |a_n||x|^n \\ &\leq \sum_{n=m+1}^{\infty} |a_n| r^n \end{aligned}$$

which converges by absolute convergence at r. Hence if m suff large, $f - f_m$ is bounded and

$$\sup_{x\in(-r,r)} |f(x) - f_m(x)| \leq \sum_{n=m+1}^{\infty} |a_n| r^n \to 0$$

as $m \to \infty$ by absolute convergence of r. □

Theorem 1.7

Let $\sum a_n x^n$ be a real power series with r.o.c $R > 0$. Define $f : (-R, R) \to \mathbb{R}$ by

$f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

1. f is continuous;
2. for any $x \in (-R, R)$ f is integrable over $[0, x]$ with

$$\int_0^x f = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Proof. Let $x \in (-R, R)$. Pick r s.t. $|x| < r < R$. By lemma 1.1, $\sum a_n y^n$ converges uniformly on $(-r, r)$. But the partial sum functions $y \mapsto \sum_{n=0}^{m} a_n y^n$ $(m \geq 0)$ are all cts functions on $(-r, r)$ (as they are polynomials). Hence by theorem 1.3, $f\,|_{(-r,r)}$ [a] is cts. Hence f is cts at x, but x was arbitrary so f is a cts fcn on $(-R, R)$.

Moreover, $[0, x] \subset (-r, r)$ so we also have $\sum a_n y^n$ converges uniformly on $[0, x]$. Each partial sum function on $[0, x]$ is a poly so can be integrated with $\int_0^x \sum_{n=0}^{m} a_n y^n \, dy = \sum_{n=0}^{m} \int_0^x a_n y^n \, dy = \sum_{n=0}^{m} \frac{a_n}{n+1} x^{n+1}$. Hence by theorem 1.4, f is integrable over $[0, x]$ with

$$\begin{aligned} \int_0^x f &= \lim_{m \to \infty} \int_0^x \sum_{n=0}^{m} a_n y^n \, dy \\ &= \lim_{m \to \infty} \sum_{n=0}^{m} \frac{a_n}{n+1} x^{n+1} \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}. \end{aligned}$$

□

[a] f restricted to domain $(-r, r)$

For differentiation, need technical lemma:

Lemma 1.2

Let $\sum a_n x^n$ be a real power series with r.o.c $R > 0$. Then the power series $\sum_{n \geq 1} n a_n x^{n-1}$ has r.o.c at least R.

Proof. Let $x \in \mathbb{R}$ with $0 < x < R$. Pick w with $x < w < R$. Then $\sum a_n w^n$ is absolutely convergent, so $a_n w^n \to 0$ (terms of a convergent series) so $\exists\ M$ s.t. $\forall\ n$, $|a_n w^n| \leq M$.

For each n,

$$|na_n x^{n-1}| = |a_n w^n| \left|\frac{x}{w}\right|^n \frac{1}{|x|} n.$$

Fix n. Let $\alpha = |\frac{x}{w}| < 1$. Let $c = \frac{M}{|x|}$, a constant. Then $|na_n x^{n-1}| \le cn\alpha^n$. By comparison test, ETS (enough to show) $\sum n\alpha^n$ converges.
Note $\left|\frac{(n+1)\alpha^{n+1}}{n\alpha^n}\right| = (1+\frac{1}{n})\alpha \to \alpha < 1$ as $n \to \infty$ so done by ratio test. □

Theorem 1.8

Let $\sum a_n x^n$ be a real power series with r.o.c. $R > 0$. Let $f : (-R, R) \to \mathbb{R}$ be defined by $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then f is differentiable and $\forall\ x \in (-R, R)$ $f'(x) = \sum_{n=1}^{\infty} na_n x^{n-1}$.

Proof. Let $x \in (-R, R)$. Pick r with $|x| < r < R$. Then $\sum a_n y^n$ converges uniformly on $(-r, r)$. Moreover, the power series $\sum_{n\ge1} na_n y^{n-1}$ has r.o.c at least R and so also converges uniformly on $(-r, r)$.

The partial sum functions $f_m(y) = \sum_{n=0}^{m} a_n y^n$ are polys so differentiable with $f'_m(y) = \sum_{n=1}^{m} na_n y^{n-1}$. We now have f'_m converging uniformly on $(-r, r)$ to the function $g(y) = \sum_{n=1}^{\infty} na_n y^{n-1}$.

Hence by theorem 1.5, $f\ |_{(-r,r)}$ is differentiable and $\forall\ y \in (-r, r)$ $f'(y) = g(y)$.

In particular, f is differentiable at x with $f'(x) = g(x)$. Hence f is a differentiable function on $(-R, R)$ with derivative g as desired. □

§1.4 Uniform Continuity

Let $X \subset \mathbb{R}$. Let $f : X \to \mathbb{R}$. (May as well think of $X = \mathbb{R}$ or $X = (a, b)$).

Definition 1.9 (Continuous function)

f is **continuous** if

$$\forall\ \epsilon > 0\ \forall\ x \in X\ \exists\ \delta > 0\ \forall\ y \in X\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Definition 1.10 (Unifomly Continuous function)

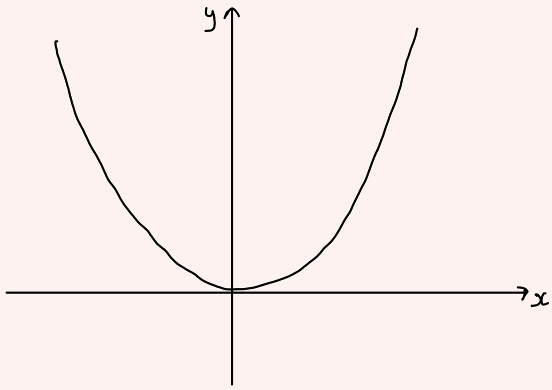
f is **uniformly continuous** if

$$\forall\ \epsilon > 0\ \exists\ \delta > 0\ \forall\ x \in X\ \forall\ y \in X\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Remark 4. Clearly if f is uniformly cts then f is cts. We would suspect that f cts doesn't imply f uniformly cts.

Example 1.8

A function $f : \mathbb{R} \to \mathbb{R}$ that is cts but not uniformly cts.



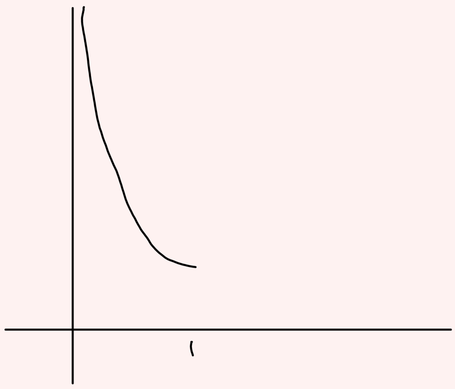
We want some function that looks like this, a continuous function which gets steeper as we go to infinity. So $f(x) = x^2$ ought to work. We know f is cts (as it's a poly). Suppose $\delta > 0$. Then

$$\begin{aligned} f(x+\delta) - f(x) &= (x+\delta)^2 - x^2 \\ &= 2\delta x + \delta^2 \to \infty \text{ as } x \to \infty. \end{aligned}$$

So in particular, $\forall\ \delta > 0\ \exists\ x, y \in \mathbb{R}$ s.t. $|x - y| < \delta$ but $|f(x) - f(y)| \geq 1$. So conditions for uniform cty fails for $\epsilon = 1$. So f not uniform cty.

Example 1.9

Make domain bounded. We can still fail, e.g. $f : (0, 1) \to \mathbb{R}$ cts but not uniform cts.



Let $f(x) = \frac{1}{x}$, clearly cts. Proof that its not uniform continuity is left as an exercise to the reader.

Theorem 1.9

A continuous real-valued function on a closed bounded interval is uniformly continuous.

Proof. Let $f : [a,b] \to \mathbb{R}$ and suppose f is cts but not uniformly cts. Then we can find $\epsilon > 0$ st. $\delta > 0 \; \exists \; x, y \in [a,b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon$.

In particular, taking $\delta = \frac{1}{n}$ we can find sequences $(x_n), (y_n) \in [a,b]$ with, for each n, $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon$. The sequence (x_n) is bounded so by BW[a] it has a convergent subsequence $x_{n_j} \to x$. And $[a,b]$ is a closed interval so $x \in [a,b]$. Then $x_{n_j} - y_{n_j} \to 0$ so $y_{n_j} \to x$.

But f is cts at x so $\exists \; \delta > 0$ s.t. $\forall \; y \in [a,b] \; |y - x| < \delta \implies |f(y) - f(x)| < \frac{\epsilon}{2}$. Take such a δ. As $x_{n_j} \to x$ we can find J_1 s.t. $j \geq J_1 \implies |x_{n_j} - x| < \delta$. Similarly we can find J_2 s.t. $j \geq J_2 \implies |y_{n_j} - x| < \delta$. Now let $j = \max(J_1, J_2)$ then $|x_{n_j} - x|, |y_{n_j} - x| < \delta$ so we have $|f(x_{n_j}) - f(x)|, |f(y_{n_j}) - f(x)| < \epsilon/2$. Then $|f(x_{n_j}) - f(y_{n_j})| \leq |f(x_{n_j}) - f(x)| + |f(y_{n_j}) - f(x)| < \epsilon$ ↯. $\square$

[a] Bolzano Weierstrass

Corollary 1.1

A continuous real-valued function on a closed bounded interval is bounded.

Proof. Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and so uniformly continuous by theorem 1.9. Then we can find $\delta > 0$ s.t. $\forall \; x, y \in [a,b] \; |x - y| < \delta \implies |f(x) - f(y)| < 1$.

Let $M = \lceil \frac{b-a}{\delta} \rceil$. Let $x \in [a,b]$. We can find $a = x_0 \leq x_1 \leq \cdots \leq x_M = x$ with $|x_i - x_{i-1}| < \delta$ for each i. Hence

$$\begin{aligned}
|f(x)| &= \left| f(a) + \sum_{i=1}^{M} f(x_i) - f(x_{i-1}) \right| \\
&\leq |f(a)| + \sum_{i=1}^{M} |f(x_i) - f(x_{i-1})| \\
&< |f(a)| + \sum_{i=1}^{M} 1 \\
&= |f(a)| + M.
\end{aligned}$$

$\square$

Remark 5. Referring back to example 1.9, starting at $x = 1$ and going towards $x = 0$ we can that δ gets smaller and smaller s.t. you require an infinite number of steps to get 0. So $M = \infty$ essentially.

Corollary 1.2

A continuous real-valued function on a closed bounded interval is integrable.

Proof. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and so uniformly continuous by theorem 1.9. Let $\epsilon > 0$. Then we can find $\delta > 0$ s.t. $\forall\, x, y \in [a, b]$ $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$. Let $\mathcal{D} = \{x_0 < x_1 < \cdots < x_n\}$ be a dissection s.t. for each i we have $x_i - x_{i-1} < \delta$.

Let $i \in \{1, \ldots, n\}$. Then for any $u, v \in [x_{i-1}, x_i]$ we have $|u-v| < \delta$ so $|f(u)-f(v)| < \epsilon$. Hence

$$\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \le \epsilon.$$

Hence:

$$\begin{aligned}
S(f, \mathcal{D}) - s(f, \mathcal{D}) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right) \\
&\le \sum_{i=1}^{n} (x_i - x_{i-1}) \epsilon \\
&= \epsilon \sum_{i=1}^{n} (x_i - x_{i-1}) \\
&= \epsilon (b - a).
\end{aligned}$$

But $\epsilon(b - a)$ can be made arbitrarily small by taking ϵ small. So by Riemann's criterion f is integrable over $[a, b]$. $\square$

§2 Metric Spaces

§2.1 Definitions and Examples

Question

Can we think about convergence in a more general setting? Convergence seemed similar in our 3 settings.

What do we really need?

Answer

We need a notion of distance.

In $\mathbb{R}$: distance x to y is $|x-y|$.
In $\mathbb{R}^2$: its $\|x-y\|$.
For functions: distance f to g is $\sup_{x\in X}|f(x)-g(x)|$ (where this exists, i.e. if $f-g$ bounded).

The triangle inequality was often important (see the proof of uniqueness of limits).

Definition 2.1 (Metric)

A **metric** d is a function $d : X^2 \to \mathbb{R}$ satisfying:

- $d(x,y) \geq 0$ for all $x, y \in X$ with equality iff $x = y$;
- $d(x,y) = d(y,x)$ for all $x, y \in X$.
- $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.

Definition 2.2 (Metric Space)

A **metric space** is a set X endowed with a metric d.

We could also define a metric space as an ordered pair (X, d). If it is obvious what d is, we sometimes write 'The metric space X ...'.

Example 2.1

$X = \mathbb{R}$, $d(x,y) = |x-y|$ 'The usual metric on $\mathbb{R}$'.

Example 2.2

$X = \mathbb{R}^n$ with the Euclidean metric, $d(x,y) = \|x-y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

Uniform convergence of functions doesn't quite work: we want $d(f,g) = \sup|f-g|$ but this might not exist if $f-g$ is unbounded. However, we can do something with appropriate sets of functions.

Example 2.3

Let $Y \subset \mathbb{R}$. Take $X = B(Y) = \{f : Y \to \mathbb{R} \mid f \text{ bounded}\}$ with the uniform metric

$d(f,g) = \sup_{x \in Y} |f - g|$.

Checking triangle inequality:

Proof. Let $f, g, h \in B(Y)$. Let $x \in Y$. Then

$$\begin{aligned} |f(x) - h(x)| &\le |f(x) - g(x)| + |g(x) - h(x)| \\ &\le d(f,g) + d(g,h) \end{aligned}$$

Taking sup over all $x \in Y$

$$d(f,h) \le d(f,g) + d(g,h).$$

$\square$

Definition 2.3 (Subspace)

Suppose (X,d) a metric space and $Y \subset X$. Then $d\,|_{Y^2}$ is a metric on Y. We say Y with this metric is a **subspace** of X.

Example 2.4

Subspaces of $\mathbb{R}$: any of $\mathbb{Q}, \mathbb{Z}, \mathbb{N}, [0,1], \ldots$ with the usual metric $d(x,y) = |x - y|$.

Example 2.5

Recall that a cts function on a closed bounded interval is bounded. Define $C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ cts}\}$. This is a subspace of $B([a,b])$, example 2.3. That is $C([a,b])$ is a metric space with the uniform metric $\mathcal{L}(f,g) = \sup_{x \in [a,b]} |f(x) - g(x)|$

Example 2.6

The empty metric space $X = \emptyset$ with the empty metric.

Could maybe define different metrics on the same set:

Example 2.7

The ℓ_1 metric on $\mathbb{R}^n$: $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$.

Example 2.8

The ℓ_∞ metric on $\mathbb{R}^n$: $d(x,y) = \max_i |x_i - y_i|$.[a]

[a] Proof of triangle inequality similar to example 2.3

Example 2.9

On $C([a,b])$ we can define the L_1 metric: $d(f,g) = \int_a^b |f-g|$.

Example 2.10

$X = \mathbb{C}$ with

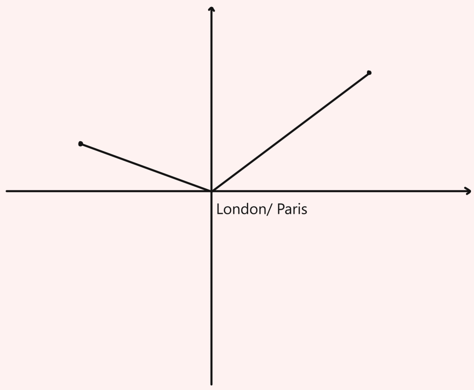
$$d(z,w) = \begin{cases} 0 & z = w \\ |z| + |w| & z \neq w. \end{cases}.$$

First two conditions of a metric hold obviously, for triangle inequality we need $d(u,w) \leq d(u,v) + d(v,w)$.

1. If $u = w$, LHS $= 0$ ✓
2. If $u = v$ or $v = w$ then LHS $=$ RHS ✓
3. If u, v, w distinct:

$$\begin{aligned} LHS &= |u| + |w| \\ RHS &= |u| + |w| + 2|v| ✓ \end{aligned}$$

This metric is often called the British Rail metric or SNCF metric, you can think of it as for distinct points you have to travel through the origin.



Example 2.11

Let X be any set. Define a metric d on X by

$$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Easy to check this works. This is called the discrete metric on X.

Example 2.12

Let $\mathbb{X} = \mathbb{Z}$. Let p be a prime. The p-adic metric on $\mathbb{Z}$ is the metric d defined by:

$$d(x,y) = \begin{cases} 0 & x = y \\ p^{-a} & \text{if } x \neq y \text{ and } x - y = p^a m \text{ with} \end{cases}$$

Two numbers are close if difference is divisible by a large power of p.

§3 Topological Spaces

Part II
Generalizing differentiation