Stochastic Financial Models 22

Michael Tehranchi

24 November 2023

1 Cameron–Martin theorem

Motivation. Example sheet 1

- Let $Z \sim N(0,1)$.
- $\mathbb{E}[g(a+Z)] = \mathbb{E}[e^{aZ-a^2/2}g(Z)]$ for any $a \in \mathbb{R}$ and suitable g.
- Proof: Change of variables formula for integration.

Generalisation

- Let $Z \sim N_n(0, I)$ multi-variate normal.
- $\mathbb{E}[g(a+Z)] = \mathbb{E}[e^{a^{\top}Z ||a||^2/2}g(Z)]$ for any $a \in \mathbb{R}^n$ and suitable g.
- Essentially the same proof.

Theorem (Cameron–Martin theorem). Let $(W_t)_{t\geq 0}$ be a Brownian motion. For fixed $t\geq 0$ and $c\in \mathbb{R}$ we have

$$\mathbb{E}[g((W_s + cs)_{0 \le s \le t})] = \mathbb{E}[e^{cW_t - c^2 t/2}g((W_s)_{0 \le s \le t})]$$

for suitable functions g from the space of continuous functions on [0,t] to the real line.

Sketch of proof. By measure theory, it is enough to consider functions g of the form

$$g(w) = G(w(t_1), \dots, w(t_n))$$

for a function G on \mathbb{R}^n , where $0 = t_0 < t_1 < \cdots < t_n = t$.

$$\mathbb{E}[g((W_s + cs)_{0 \le s \le t}) = \mathbb{E}[G(W_{t_1} + ct_1, \dots, W_{t_n} + ct_n)]$$

$$= \mathbb{E}[G((\sum_{i=1}^k \sqrt{t_i - t_{i-1}} (Z_i + a_i))_{k=1}^n)]]$$

$$= \mathbb{E}[e^{a^\top Z - ||a||^2/2} G((\sum_{i=1}^k \sqrt{t_i - t_{i-1}} Z_i)_{k=1}^n)]]$$

$$= \mathbb{E}[e^{cW_t - c^2 t/2} g((W_s)_{0 \le s \le t})]$$

where $Z_i = \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}$ are iid N(0, 1) for $1 \le i \le n$ and $a_i = c\sqrt{t_i - t_{i-1}}$ so that

$$a^{\top}Z = \sum_{i=1}^{n} a_i Z_i = W_t$$

and

$$||a||^2 = \sum_{i=1}^n a_i^2 = c^2 t$$

2 An application of Cameron–Martin

Proposition. Let $(W_t)_{t>0}$ be a Brownian motion. For $a \geq 0$ we have

$$\mathbb{P}(\max_{0 \le s \le t} (W_s + cs) \le a) = \mathbb{P}(W_t \le a - ct) - e^{2ca} \mathbb{P}(W_t \ge a + ct)$$
$$= \Phi\left(\frac{a - ct}{\sqrt{t}}\right) - e^{2ca} \Phi\left(\frac{-a - ct}{\sqrt{t}}\right)$$

Proof.

$$\begin{split} \mathbb{P}(\max_{0 \leq s \leq t}(W_s + cs) \leq a) = & \mathbb{E}[\mathbb{1}_{\{\max_{0 \leq s \leq t}(W_s + cs) \leq a\}}] \\ = & \mathbb{E}[e^{cW_t - c^2 t/2} \mathbb{1}_{\{\max_{0 \leq s \leq t}W_s \leq a\}}] \\ = & \mathbb{E}[e^{cW_t - c^2 t/2} \mathbb{1}_{\{W_t \leq a\}}] \\ & - \mathbb{E}[e^{c(2a - W_t) - c^2 t/2} \mathbb{1}_{\{W_t \geq a\}}] \\ = & \mathbb{E}[\mathbb{1}_{\{W_t + ct \leq a\}}] - e^{2ac} \mathbb{E}[\mathbb{1}_{\{W_t - ct \geq a\}}] \end{split}$$

To discuss risk-neutral measures, we need

Theorem (Cameron–Martin reformulation). Let $(W_t)_{t\geq 0}$ be a Brownian motion under a given measure \mathbb{P} . Fix T>0 and $c\in\mathbb{R}$, and define an equivalent measure \mathbb{Q} by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{cW_T - c^2 T/2}$$

Then the process $(W_t - ct)_{0 \le t \le T}$ is a Brownian motion under \mathbb{Q} .

Proof. Fix a function g on C[0,T]. Then

$$\mathbb{E}^{\mathbb{Q}}[g((W_t - ct)_{0 \le t \le T})] = \mathbb{E}^{\mathbb{P}}[e^{cW_T - c^2T/2}g((W_t - ct)_{0 \le t \le T})]$$

= $\mathbb{E}^{\mathbb{P}}[g((W_t)_{0 \le t \le T})]$

by the first formulation of Cameron–Martin. So the process $(W_t - ct)_{0 \le t \le T}$ has the same law under \mathbb{Q} as the process $(W_t)_{0 \le t \le T}$ has under \mathbb{P} .

3 Heat equation

Proposition. Fix a suitable g and let

$$u(t,x) = \mathbb{E}[f(x+\sqrt{\tau}Z)]$$

where $Z \sim N(0,1)$. Then u solves the heat equation

$$\partial_{\tau} u = \frac{1}{2} \partial_{xx} u$$

with boundary condition u(0,x) = f(x).

Proof when g is well-behaved by example sheet 1,

$$\partial_{\tau} u = \frac{1}{2\sqrt{\tau}} \mathbb{E}[Zg'(x + \sqrt{\tau}Z)]$$
$$= \frac{1}{2} \mathbb{E}[g''(x + \sqrt{t}Z)]$$
$$= \frac{1}{2} \partial_{xx} u$$

If g is less well-behaved, then write

$$u(\tau, x) = \int f(y)p(\tau; x, y)dy$$

where

$$p(\tau; x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(y-x)^2}{2\tau}\right)$$

is the transition density of the Brownian motion (also called the *heat kernel* or *Green's function*) and use the fact that $p(\cdot;\cdot,y)$ satisfies the heat equation.

Since p is very well-behaved, interchange of derivatives and integrals is allowed by the dominated convergence theorem, provided that f has exponential growth.