

Stochastic Financial Models 16

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1 Dynamic programming principle

Given

- A sequence $(\xi_n)_{n \geq 1}$ of independent random variables generating a filtration $(\mathcal{F}_n)_{n \geq 0}$
- A function $G(\cdot, \cdot, \cdot, \cdot)$
- An initial condition X_0

for any previsible $(U_n)_{n \geq 1}$ construct the controlled Markov process by

$$X_n^U = \begin{cases} X_0 & \text{if } n = 0 \\ G(n, X_{n-1}^U, U_n, \xi_n) & \text{if } n \geq 1 \end{cases}$$

Now given

- A non-random time horizon $N > 0$
- Suitably integrable functions $f(\cdot, \cdot)$ and $g(\cdot)$

we seek to maximise

$$\mathbb{E} \left[\sum_{k=1}^N f(k, U_k) + g(X_N^U) \right]$$

Theorem (The dynamic programming principle). *Let V solve the Bellman equations:*

$$\begin{aligned} V(N, x) &= g(x) \quad \text{for all } x \\ V(n-1, x) &= \sup_u \{f(n, u) + \mathbb{E}[V(n, G(n, x, u, \xi_n))]\} \quad \text{for all } 1 \leq n \leq N, \text{ and } x \end{aligned}$$

and suppose for each n and x there is an optimal solution $u^(n, x)$ to the maximisation problem, so that*

$$V(n-1, x) = f(n, u^*(n, x)) + \mathbb{E}[V(n, G(n, x, u^*(n, x), \xi_n))] \quad \text{for all } 1 \leq n \leq N, \text{ and } x$$

Fix the initial condition $X_0^* = X_0$ and let

$$\begin{aligned} U_n^* &= u^*(n, X_{n-1}^*), \\ X_n^* &= G(n, X_{n-1}^*, U_n^*, \xi_n) \text{ for all } 1 \leq n \leq N \end{aligned}$$

so that $X_n^{U^*} = X_n^*$ for all $0 \leq n \leq N$. Then $(U_n^*)_{1 \leq n \leq N}$ is the optimal control and V is the value function.

Proof. Fix X_0 and let $(U_n)_{1 \leq n \leq N}$ be a previsible control, and consider the associated controlled process $(X_n^U)_{0 \leq n \leq N}$. Let

$$M_n^U = \sum_{k=1}^n f(k, U_k) + V(n, X_n^U)$$

Claim: $(M_n)_{0 \leq n \leq N}$ is a supermartingale.

Indeed, this process is adapted and integrable (by assumption). Now using the ‘fix known quantities and average over independent quantities’ property of conditional expectation, we have by the Bellman equation that

$$\begin{aligned} &\mathbb{E}[M_n^U - M_{n-1}^U | \mathcal{F}_{n-1}] \\ &= f(n, U_n) + \mathbb{E}[V(n, X_n^U) | \mathcal{F}_{n-1}] - V(n-1, X_{n-1}^U) \\ &= \{f(n, u) + \mathbb{E}[V(n, G(n, x, u, \xi_n))] - V(n-1, x)\} \Big|_{u=U_n, x=X_{n-1}^U} \\ &\leq 0 \end{aligned}$$

with equality if $U_n = u^*(n, X_{n-1}^U)$.

Hence, using $V(N, x) = g(x)$ for all x , we have by the tower property that

$$\begin{aligned} \mathbb{E} \left[\sum_{k=n+1}^N f(k, U_k) + g(X_N^U) \Big| X_n^U \right] &= \mathbb{E} \left[M_N^U - \sum_{k=1}^n f(k, U_k) \Big| X_n^U \right] \\ &= \mathbb{E} \left[\mathbb{E} \left(M_N^U - \sum_{k=1}^n f(k, U_k) \Big| \mathcal{F}_n \right) \Big| X_n^U \right] \\ &\leq \mathbb{E} \left[M_n^U - \sum_{k=1}^n f(k, U_k) \Big| X_n^U \right] \\ &= V(n, X_n^U) \end{aligned}$$

with equality if $U = U^*$. This shows

$$V(n, x) = \max_{(U_k)_{k+1 \leq n \leq N}} \mathbb{E} \left[\sum_{k=n+1}^N f(k, U_k) + g(X_N^U) \Big| X_n^U = x \right]$$

as claimed. □

Remark. The above proof uses an argument sometimes called the *martingale principle of optimal control*.

2 Optimal investment

Given a market with interest rate r and d risky assets with prices $(S_n)_{n \geq 0}$

- consider an investor who, between time $n-1$ and time n holds θ_n shares and consumes C_n cash, where $0 \leq C_n \leq X_{n-1}$.
- wealth evolves as $X_n = (1+r)(X_{n-1} - C_n) + \theta_n^\top [S_n - (1+r)S_{n-1}]$

To have a tractable problem, we make some simplifying assumptions:

- assume $d = 1$
- assume $S_n = S_{n-1}\xi_n$ where $(\xi_n)_n$ are independent
- Note $X_n = G(n, X_{n-1}, \binom{C_n}{\eta_n}, \xi_n)$ where $\eta_n = S_{n-1}\theta_n$ and

$$G(n, x, \binom{c}{\eta}, v) = (1+r)(x - c) + \eta(\xi - (1+r))$$

so the wealth is a controlled Markov process with two-dimensional controls $u = \binom{c}{\eta}$.

Given a time horizon N , a natural goal is to

$$\text{maximise } \mathbb{E} \left[\sum_{k=1}^N U(C_k) + U(X_N) \right]$$

where U is the investor's utility function

The Bellman equation is

$$\begin{aligned} V(N, x) &= U(x) \\ V(n-1, x) &= \max_{c, \eta} \mathbb{E} [U(c) + V(n, (1+r)(x - c) + \eta(\xi - (1+r)))] \end{aligned}$$

- Generally, intractable
- but suppose the utility is CRRA: $U(x) = \frac{1}{1-R}x^{1-R}$ for $x > 0$, where $R > 0, R \neq 1$.