

Stochastic Financial Models 21

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1 Properties of Brownian motion

Theorem (Wiener 1923). *Brownian motion exists.*

Remark. A Brownian motion is called a Wiener process in the US.

Theorem. *Brownian motion is a martingale in its filtration $\mathcal{F}_t = \sigma(W_s : 0 \leq s \leq t)$.*

Proof. Brownian motion is integrable, adapted and

$$\mathbb{E}[W_t - W_s | \mathcal{F}_s] = \mathbb{E}[W_t - W_s] = 0$$

for $0 \leq s \leq t$ by the independence of $W_t - W_s$ and \mathcal{F}_s . □

Theorem. *Brownian motion is a Markov process.*

Proof. Since W_s is \mathcal{F}_s measurable and $W_t - W_s$ is independent of \mathcal{F}_s for $0 \leq s \leq t$, we have

$$\begin{aligned}\mathbb{E}[g(W_t) | \mathcal{F}_s] &= \mathbb{E}[g(W_t - W_s + W_s) | \mathcal{F}_s] \\ &= \mathbb{E}[g(W_t - W_s + x)] \Big|_{x=W_s} \\ &= \mathbb{E}[g(W_t) | W_s]\end{aligned}$$

□

Definition. A process $(X_t)_{t \geq 0}$ is *Gaussian* iff the random variables X_{t_1}, \dots, X_{t_n} are jointly normal for all $0 \leq t_1 \leq \dots \leq t_n$, i.e. the random variable $\sum_{i=1}^n a_i X_{t_i}$ is normally distributed for all constants a_1, \dots, a_n .

Theorem. *The following are equivalent*

1. $(W_t)_{t \geq 0}$ is a Brownian motion
2. $(W_t)_{t \geq 0}$ is a Gaussian process such that
 - $t \mapsto W_t$ is continuous

- $\mathbb{E}[W_t] = 0$ for all $t \geq 0$
- $\mathbb{E}[W_s W_t] = s$ for all $0 \leq s \leq t$

Proof. Suppose $(W_t)_{t \geq 0}$ is a Brownian motion. Fix $0 = t_0 \leq t_1 \leq \dots \leq t_n$ and a_1, \dots, a_n . Note

$$\sum_{i=1}^n a_i W_{t_i} = \sum_{i=1}^n b_i (W_{t_i} - W_{t_{i-1}})$$

where $b_k = \sum_{i=k}^n a_i$. Since $W_{t_1} - W_{t_0}, \dots, W_{t_n} - W_{t_{n-1}}$ are independent normals, and the linear combination of independent normals is normal, we have that $(W_t)_{t \geq 0}$ is Gaussian with $\mathbb{E}[W_t] = \mathbb{E}[W_0] = 0$ and

$$\begin{aligned} \mathbb{E}[W_s W_t] &= \mathbb{E}[W_s^2] + \mathbb{E}[W_s(W_t - W_s)] \\ &= \text{Var}(W_s) + \mathbb{E}(W_s)\mathbb{E}(W_t - W_s) \\ &= 0. \end{aligned}$$

for $0 \leq s \leq t$, since W_s and $W_t - W_s$ are independent.

Conversely, suppose $(W_t)_{t \geq 0}$ is a continuous Gaussian process such that $\mathbb{E}[W_t] = 0$ and $\mathbb{E}[W_s W_t] = s$ for all $0 \leq s \leq t$. Then for $0 \leq u \leq s \leq t$ we have

$$\begin{aligned} \text{Cov}(W_u, W_t - W_s) &= \mathbb{E}[W_u W_t] - \mathbb{E}[W_u W_s] \\ &= u - u = 0 \end{aligned}$$

By normality, the increment $W_t - W_s$ is independent of W_u . By Gaussianity, the increment is independent of $(W_u)_{0 \leq u \leq s}$.

Theorem. *Let $(W_t)_{t \geq 0}$ be a Brownian motion. Then each of the following processes are also Brownian motions.*

1. $\tilde{W}_t = cW_{t/c^2}$, for any constant $c \neq 0$.
2. $\tilde{W}_t = W_{t+T} - W_T$ for any constant $T \geq 0$.
3. $\tilde{W}_0 = 0$ and $\tilde{W}_t = tW_{1/t}$ for $t > 0$.

Proof. Check that each process is a continuous mean-zero Gaussian process with the correct covariance. [For 3, we technically need the Brownian law of large number $\frac{W_s}{s} \rightarrow 0$ as $s \rightarrow \infty$ to prove continuity of \tilde{W} at $t = 0$.] \square

2 Reflection principle

Theorem. *Let $(W_t)_{t \geq 0}$ be a Brownian motion, and $T_a = \inf\{t \geq 0 : W_t = a\}$. Then $T_a < \infty$ almost surely.*

Proof. Consider the case $a > 0$. (The case $a < 0$ is similar.) We must show

$$\sup_{t \geq 0} W_t > a \text{ almost surely}$$

By Brownian scaling, for any $c > 0$ and $0 < a < b$, we have

$$\begin{aligned} \mathbb{P}(a < \sup_{t \geq 0} W_t < b) &= \mathbb{P}(a < \sup_{t \geq 0} cW_{t/c^2} < b) \\ &= \mathbb{P}(a/c < \sup_{s \geq 0} W_s < b/c) \quad \text{letting } t/c^2 = s \\ &\rightarrow 0 \end{aligned}$$

by sending $c \uparrow \infty$. Since $Z = \sup_{t \geq 0} W_t \geq W_0 = 0$, we have shown that $Z \in \{0, +\infty\}$ almost surely.

Let $\hat{Z} = \sup_{t \geq 1} (W_t - W_1)$. Note Z and \hat{Z} have the same distribution, so $\hat{Z} \in \{0, +\infty\}$ almost surely.

Note that $\{\hat{Z} = \infty\} = \{Z = \infty\}$ since $\sup_{0 \leq t \leq 1} W_t$ is finite by the continuity of Brownian motion. Hence

$$\begin{aligned} p = \mathbb{P}(Z = 0) &= \mathbb{P}(Z = 0, \hat{Z} = 0) \\ &\leq \mathbb{P}(W_1 \leq 0, \hat{Z} = 0) \\ &= \frac{1}{2} \mathbb{P}(\hat{Z} = 0) = \frac{1}{2} p \end{aligned}$$

so $p = 0$. Hence $\sup_{t \geq 0} W_t = \infty$ almost surely. \square

Theorem. Let $(W_t)_{t \geq 0}$ be a Brownian motion and T a finite stopping time. The process $W_{t+T} - W_T$ is also a Brownian motion independent of $(W_t)_{0 \leq t \leq T}$

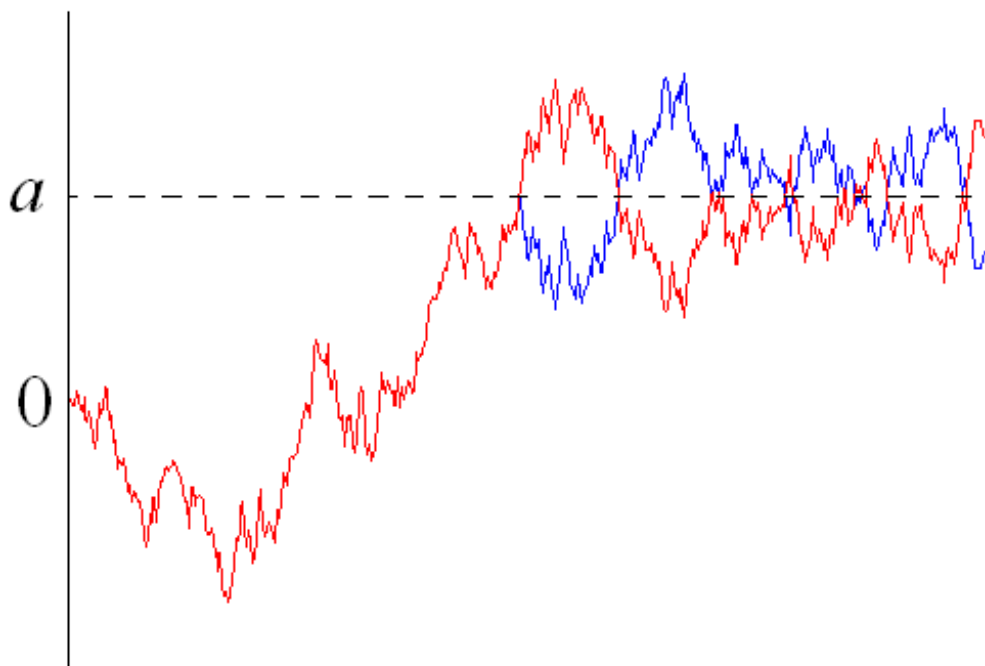
Proof. Omitted. The idea is Brownian motion is a *strong* Markov process. \square

Applying this with the finite stopping time T_a together with the symmetry of Brownian motion, we have

Theorem (Reflection principle). Let $(W_t)_{t \geq 0}$ be a Brownian motion and let

$$\tilde{W}_t = \begin{cases} W_t & \text{if } 0 \leq t < T_a \\ 2a - W_t & \text{if } t \geq T_a \end{cases}$$

Then $(\tilde{W}_t)_{t \geq 0}$ is a Brownian motion.



Reflection principle: Key formula

$$\mathbb{P}(\max_{0 \leq s \leq t} W_s \geq a, W_t \leq b) = \mathbb{P}(W_t \geq 2a - b) \text{ for } a \geq 0, b \leq a$$

Proof. We have

$$\begin{aligned} \mathbb{P}(\max_{0 \leq s \leq t} W_s \geq a, W_t \leq b) &= \mathbb{P}(\tilde{W}_t \geq 2a - b) \\ &= \mathbb{P}(W_t \geq 2a - b) \end{aligned}$$

□