Stochastic Financial Models 10

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27 October 2023

1 Examples of attainable claims

Example 1: Forward contract. A forward contract is the right *and the obligation* to buy a given asset at fixed price K (the strike) at time 1. When $d = 1$, the payout of a forward on the risky asset is given by $Y = S_1 - K$. Note that this is attainable by holding 1 share and borrowing $K/(1+r)$ from the bank. Hence the unique no-arbitrage initial price of the forward is $\pi = S_0 - K/(1+r)$

[The strike of a forward contract is usually chosen such that the initial price of the forward is zero. That is $K = (1+r)S_0$. This is called the forward price of the asset.]

Example 2: one-period binomial model. Suppose $d = 1$ as before and that S_1 can take exactly two values with $\mathbb{P}(S_1 = S_0(1+b)) = p = 1 - \mathbb{P}(S_1 = S_0(1+a))$, for constants $-1 < a < b$, where $0 < p < 1$.

First we find the risk-neutral measures. Let $\mathbb{Q}(S_1 = S_0(1+b)) = q = 1 - \mathbb{Q}(S_1 = S_0(1+a))$. Then

$$S_0 = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(S_1) = \frac{1}{1+r}S_0(1+b)q + \frac{1}{1+r}S_0(1+a)(1-q)$$

so

$$q = \frac{r-a}{b-a} \text{ and } 1 - q = \frac{b-r}{b-a}$$

Thus we learn that there exists a risk-neutral measure iff $0 < q < 1$ as $0 < p < 1$

$$\iff a < r < b$$

in which case the risk-neutral measure is unique. This means that every contingent claim is attainable! Consider a claim with payout $Y = g(S_1)$. We need only check that the unique solution (x, θ) to

$$(1+r)x + \theta[S_1 - (1+r)S_0] = g(S_1)$$

that is, the system of equations

$$(1+r)x + \theta S_0(b-r) = g(S_0(1+b))$$
$$(1+r)x + \theta S_0(a-r) = g(S_0(1+a))$$

is

$$\theta = \frac{g(S_0(1+b)) - g(S_0(1+a))}{S_0(b-a)}$$
$$x = \frac{1}{(1+r)(b-a)}[(r-a)g(S_0(1+b)) + (b-r)g(S_0(1+a))] = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[g(S_1)]$$

2 Multi-period models

Motivating discussion

- In a one period model, we think of S_0 as constant but S_1 as random
- In a two period model, S_0 is constant, but S_1 and S_2 are random, at least as observed at time 0.
- But at time 1, we can think of both S_0 and S_1 as constant, and only S_2 is random

flow of information

- Initially, an agent has information $\mathcal{F}_0$
- at time 1, has information $\mathcal{F}_1$
- and at time 2, has information $\mathcal{F}_2$.
- Naturally, we should have $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$
- We also want, for instance, S_0 and S_1 (but not S_2) to be $\mathcal{F}_1$-'measurable'.
- But what is information?

Given $(\Omega, \mathcal{F}, \mathbb{P})$, and a 'set of information' $\mathcal{G}$, an event $A \in \mathcal{F}$ is $\mathcal{G}$-measurable intuitively iff

$$\mathbb{P}(A|\mathcal{G}) \text{ is always either 0 or 1}$$

Example.

- Imagine flipping a coin two times.
- Let $\mathcal{G}$ be knowledge of the result of the first flip.
- $\mathbb{P}(\{HH, HT\}|\mathcal{G}) = 1$ if the first flip is heads and 0 otherwise. So $\{HH, HT\}$ is $\mathcal{G}$-measurable. That is to say, knowing $\mathcal{G}$, you can always measure whether the outcome is in $\{HH, HT\}$ or not.
- $\mathbb{P}(\{TT\}|\mathcal{G}) = 1/2$ if the first flip is tails, so $\{TT\}$ is not $\mathcal{G}$ measurable. That is, even knowing $\mathcal{G}$, sometimes you cannot perfectly measure whether the outcome is TT or not.

3 Measurability

Idea: Identify the information $\mathcal{G}$ with the collection of all $\mathcal{G}$-measurable events.
What kind of collection of events should it be?

Definition. Given a set Ω, a non-empty collection $\mathcal{G}$ of subsets of Ω is called a *sigma-algebra* iff

- $A \in \mathcal{G}$ implies $A^c \in \mathcal{G}$
- $A_1, A_2, \ldots \in \mathcal{G}$ implies $\cup_n A_n \in \mathcal{G}$.

Example. Consider tossing a coin twice. Let $\Omega = \{HH, HT, TH, TT\}$. The information measurable after the first coin toss is $\{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}, \}$

Definition. Given a sigma-algebra $\mathcal{G}$, a random variable X is $\mathcal{G}$-measurable iff the event $\{X \leq x\}$ is in $\mathcal{G}$ for all $x \in \mathbb{R}$.

Remark. Intuitively, knowing the information in $\mathcal{G}$ allows you measure the value of X.
Remark. If X is $\mathcal{G}$-measurable, then the event $\{X \in B\}$ is in $\mathcal{G}$ for all 'nice' (for the measure theory specialists: Borel) subsets $B \subseteq \mathbb{R}$.
Remark. If X takes values in the countable set $\{x_1, x_2, \ldots\}$ then X is $\mathcal{G}$-measurable iff $\{X = x_i\} \in \mathcal{G}$ for all i.
Exercise. Show that if X is measurable with respect to the trivial sigma-algebra $\{\emptyset, \Omega\}$ then X is equal to a constant.

Definition. The sigma-algebra *generated* by a random variable X is the sigma-algebra $\mathcal{G}$ containing all events of the form $\{X \in B\}$ where for 'nice' subsets $B \subseteq \mathbb{R}$. Notation: $\mathcal{G} = \sigma(X)$

Theorem (Sometimes called factorisation lemma)**.** *A random variable Y is measurable respect to $\sigma(X)$ if and only if there is a 'nice' function f such that $Y = f(X)$.*