### Stochastic Financial Models 18

#### Michael Tehranchi

#### 15 November 2023

## 1 Optimal stopping problems

• Consider a Markov process of the form

$$X_n = G(n, X_{n-1}, \xi_n)$$

where  $(\xi)_{n\geq 1}$  are independent.

 $\bullet$  Fix a horizon N and consider the problem:

maximise 
$$\mathbb{E}\left[g(X_T)\right]$$

over stopping times  $0 \le T \le N$ .

• The Bellman equation is

$$V(N,x)=g(x) \text{ for all } x$$
 
$$V(n-1,x)=\max\{g(x),\mathbb{E}[V(n,G(n,x,\xi_n))]\} \text{ for all } x,1\leq n\leq N$$

Theorem.

$$V(n,x) = \max \{ \mathbb{E} [g(X_T)|X_n = x] : T \text{ a stopping time, } n \le T \le N \}$$

The optimal stopping time is

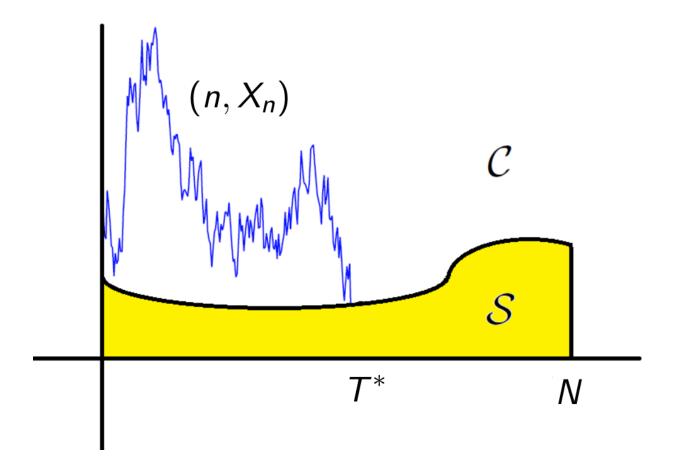
$$T^* = \inf\{n \ge 0 : V(n, X_n) = g(X_n)\}$$

It can be described graphically as follows Let

$$\mathcal{C} = \{(n,x): V(n,x) > g(x)\} = \text{`continuation region'}$$
 
$$\mathcal{S} = \{(n,x): V(n,x) = g(x)\} = \text{`stopping region'}$$

Then

$$T^* = \inf\{n \ge 0 : (n, X_n) \in \mathcal{S}\}$$



#### 2 Multi-period arbitrage

The set-up. Consider a market

- $\bullet$  with a risk-free asset with interest rate r
- and d risky assets with time n prices  $(S_n)_{n\geq 0}$ .
- The investor holds the portfolio  $\theta_n \in \mathbb{R}^d$  of risky assets during the time interval (n-1,n], where  $\theta_n$  is  $\mathcal{F}_{n-1}$ -measurable

The wealth of a self-financing investor evolves as

$$X_n = (1+r)X_{n-1} + \theta_n^{\top} [S_n - (1+r)S_{n-1}]$$

Hence

$$X_n = (1+r)^n X_0 + \sum_{k=1}^n (1+r)^{n-k} \theta_k^{\top} [S_k - (1+r)S_{k-1}]$$

The investor holds

$$\theta_n^0 = X_{n-1} - \theta_n^\top S_{n-1}$$

in the bank during the time interval (n-1, n].

**Definition.** An arbitrage is a previsible process  $(\varphi_n)_{1 \le n \le N}$  such that

$$\sum_{k=1}^{N} (1+r)^{N-k} \varphi_k^{\top} [S_k - (1+r)S_{k-1}] \ge 0 \text{ almost surely}$$

and

$$\mathbb{P}\left(\sum_{k=1}^{N} (1+r)^{N-k} \varphi_k^{\top} [S_k - (1+r)S_{k-1}] > 0\right) > 0$$

If  $\varphi$  is an arbitrage, then an investor would always prefer the investment strategy  $\theta + \varphi$  to the strategy  $\theta$ .

**Definition.** A risk-neutral measure is a measure  $\mathbb{Q}$  equivalent to  $\mathbb{P}$  under which the discounted asset price process

$$M_n = (1+r)^{-n} S_n$$

is a martingale, that is,

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(S_n|\mathcal{F}_{n-1}) = S_{n-1}$$

for all  $n \geq 1$ .

**Theorem** (Fundamental theorem of asset pricing). In a finite horizon multi-period model, there is no arbitrage if and only if there exists a risk-neutral measure.

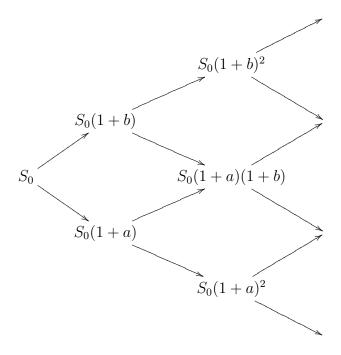
# 3 Introduction to the (Cox–Ross–Rubinstein) binomial model

- d = 1 and  $S_n = S_{n-1}\xi_n$
- $(\xi_n)_{n\geq 1}$  generate a filtration  $(\mathcal{F}_n)_n$  and such that

$$0 < \mathbb{P}(\xi_n = 1 + b | \mathcal{F}_{n-1}) = 1 - \mathbb{P}(\xi_n = 1 + a | \mathcal{F}_{n-1}) < 1 \text{ a.s. for all } n$$

That is, the stock price can follow any path along the tree with positive probability

•  $S_0 > 0$  and -1 < a < b



**Theorem.** Consider the N-step binomial model. There exists a risk-neutral measure if and only if a < r < b. When it exists it is the unique measure  $\mathbb{Q}$  such that  $(\xi_n)_{1 \le n \le N}$  are IID under  $\mathbb{Q}$  with

$$\mathbb{Q}(\xi = 1 + b) = q = \frac{r - a}{b - a} = 1 - \mathbb{Q}(\xi = 1 + a).$$

*Proof.* Suppose such a risk-neutral measure  $\mathbb Q$  exists. Then by definition

$$(1+r)S_{n-1} = \mathbb{E}^{\mathbb{Q}}(S_n|\mathcal{F}_{n-1})$$
  
=  $S_{n-1}(1+b)\mathbb{Q}(\xi_n = 1+b|\mathcal{F}_{n-1})$   
+  $S_{n-1}(1+a)\mathbb{Q}(\xi_n = 1+a|\mathcal{F}_{n-1})$ 

and hence

$$\mathbb{Q}(\xi_n = 1 + b | \mathcal{F}_{n-1}) = q = 1 - \mathbb{Q}(\xi_n = 1 + a | \mathcal{F}_{n-1}).$$

Note 0 < q < 1 if and only if a < r < b. Also, under this condition, the conditional distribution of  $\xi_n$  is independent of n and  $\mathcal{F}_{n-1}$ , so the  $(\xi_n)_{1 \le n \le N}$  are IID.