

Analysis

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1 Limits and Convergence

1.1 Review of Numbers and Sets

Notation. Write sequences as: a_n , $(a_n)_{n=1}^{\infty}$, $a_n \in \mathbb{R}$

Definition. We say that $a_n \rightarrow a$ as $n \rightarrow \infty$ if given $\varepsilon > 0$, $\exists N$ s.t. $|a_n - a| < \varepsilon$ for all $n \geq N$

Note. $N = N(\varepsilon)$

Definition (increasing sequence). $a_n \leq a_{n+1}$

Definition (decreasing sequence). $a_n \geq a_{n+1}$

Definition (strictly increasing sequence). $a_n < a_{n+1}$

Definition (strictly decreasing sequence). $a_n > a_{n+1}$

Note. Say monotone if stays increasing or stays decreasing

1.2 Fundamental Axiom of the real numbers

Axiom. If $a_n \in \mathbb{R}, \forall n \geq 1, A \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3 \leq \dots$ with $a_n \leq A$ for all n , there exists $a \in \mathbb{R}$ s.t. $a_n \rightarrow a$ as $n \rightarrow \infty$
i.e. an increasing sequence of real numbers bounded above converges.

Note. Equivalently: a decreasing sequence of real numbers bounded below converges
Equivalent also to: every non-empty set of real numbers bounded above has a supremum

Notation. Say LUBA = Least Upper Bound Axiom.

Definition (supremum). For $S \subseteq \mathbb{R}, S \neq \emptyset$, $\sup S = K$ if

- (i) $x \leq K, \forall x \in S$
- (ii) given $\varepsilon > 0, \exists x \in S$, s.t. $x > K - \varepsilon$

Note. Supremum is unique (see N&S notes), infimum defined similarly.

Lemma 1.1.

- (i) The limit is unique. That is, if $a_n \rightarrow a$, and $a_n \rightarrow b$, then $a = b$
- (ii) If $a_n \rightarrow a$ as $n \rightarrow \infty$ and $n_1 < n_2 < n_3 < \dots$, then $a_{n_j} \rightarrow a$ as $j \rightarrow \infty$ (subsequences converge to the same limit)
- (iii) If $a_n = C \forall n$, then $a_n \rightarrow C$ as $n \rightarrow \infty$
- (iv) If $a_n \rightarrow a$ & $b_n \rightarrow b$, then

$$a_n + b_n \rightarrow a + b$$

- (v) If $a_n \rightarrow a$ & $b_n \rightarrow b$, then

$$a_n b_n \rightarrow ab$$

- (vi) If $a_n \rightarrow a$, $a_n \neq 0 \forall n$ & $a \neq 0$ then

$$\frac{1}{a_n} \rightarrow \frac{1}{a}$$

- (vii) If $a_n \leq A \forall n$ and $a_n \rightarrow a$, then $a \leq A$

Proof.

- (i) given $\varepsilon > 0$, $\exists n_1$ s.t. $|a_n - a| < \varepsilon \forall n \geq n_1$
and $\exists n_2$ s.t. $|a_n - b| < \varepsilon \forall n \geq n_2$
Let $N = \max\{n_1, n_2\}$. Then $\forall n \geq N$

$$|a - b| \leq |a_n - a| + |a_n - b| < 2\varepsilon \forall n \geq N$$

If $a \neq b$, take

$$\varepsilon = \frac{|a - b|}{3} \implies |a - b| < \frac{2}{3}|a - b| \times$$

- (ii) Given $\varepsilon > 0$, $\exists N$ s.t. $|a_n - a| < \varepsilon \forall n \geq N$. Since $n_j \geq j$ (induction),

$$|a_{n_j} - a| < \varepsilon \forall j \geq N$$

i.e. $a_{n_j} \rightarrow a$ as $j \rightarrow \infty$

- (iii) Exercise.
- (iv) Exercise.
- (v)

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

As $a_n \rightarrow a$, given $\varepsilon > 0$, $\exists N_1$ s.t. $|a_n - a| < \varepsilon \forall n \geq N_1$ (*)

As $b_n \rightarrow b$, given $\varepsilon > 0$, $\exists N_2$ s.t. $|b_n - b| < \varepsilon \forall n \geq N_2$

(*) \implies if $n \geq N_1(1)$, $|a_n - a| < 1$, so:

$$|a_n| \leq |a| + 1$$

$$\implies |a_n b_n - ab| \leq \varepsilon(|a| + 1 + |b|) \forall n \geq N_3 = \max\{N_1(1), N_1(\varepsilon), N_2(\varepsilon)\}$$

- (vi) Exercise.
- (vii) Exercise.

Lemma 1.2.

$$\frac{1}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Proof. $1/n$ is a decreasing sequence bounded below so by the fundamental Axiom it has limit a .

Claim. $a = 0$

Proof.

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \rightarrow \frac{a}{2}$$

by lemma 1.1(v)

But $\frac{1}{2n}$ is a subsequence, so by 1.1(ii) $\frac{1}{2n} \rightarrow a$. By uniqueness of limits, lemma 1.1(i), we have

$$a = \frac{a}{2} \implies a = 0 \quad \square$$

Remark. The definition of limit of a sequence makes perfect sense for $a_n \in \mathbb{C}$

Definition. $a_n \rightarrow a$ if given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.

First six parts of Lemma 1.1 are the same over \mathbb{C} .

The last one does not makes sense (over \mathbb{C}) since it uses the order of \mathbb{R} .

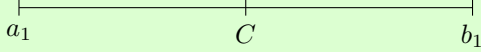
1.3 Bolzano-Weierstass Theorem

Theorem 1.3 (Bolzano-Weierstass). If $x_n \in \mathbb{R}$ and there exists K s.t. $|x_n| \leq K \forall n$, then we can find $n_1 < n_2 < n_3 < \dots$ and $x \in \mathbb{R}$ s.t. $x_{n_j} \rightarrow x$ as $j \rightarrow \infty$

In other words: every bounded sequence has a convergent subsequence.

Remark. We say nothing about uniqueness of limit, $x_n = (-1)^n$, $x_{2n+1} \rightarrow -1$, $x_{2n} \rightarrow 1$

Proof. set $[a_1, b_1] = [-K, K]$



C = mid point

Consider the following cases:

- (i) $x_n \in [a_1, c]$ for ∞ many values of n
- (ii) $x_n \in [c, b_1]$ for ∞ many values of n
- (i) & (ii) could both hold at the same time.

If (i) holds then we set $a_2 = a_1$ and $b_2 = C$. If (i) fails, we have that (ii) must hold and we set $a_2 = C$ & $b_2 = b_1$

Proceed inductively to construct sequences a_n, b_n s.t. $x_m \in [a_n, b_n]$ for infinitely many values of m .

$$\begin{aligned} a_{n-1} &\leq a_n \leq b_n \leq b_{n-1} \\ b_n - a_n &= \frac{b_{n-1} - a_{n-1}}{2} \end{aligned} \quad (*)$$

Note. Called ‘bijection method’ or “lion hunting”

a_n increasing sequence and bounded

b_n decreasing sequence and bounded

By the Fundamental Axiom,

$$a_n \rightarrow a \in [a_1, b_1]$$

$$b_n \rightarrow b \in [a_1, b_1]$$

Use (*),

$$\begin{aligned} b - a &= \frac{b - a}{2} \\ \implies b - a &= 0 \end{aligned}$$

Since $x_m \in [a_n, b_n]$ for ∞ many values of m , having chosen n_j s.t. $x_{n_j} \in [a_j, b_j]$, there is $n_{j+1} > n_j$ s.t. $x_{n_{j+1}} \in [a_{j+1}, b_{j+1}]$

(I have an “unlimited supply”!)

Hence

$$\begin{aligned} a_j &\leq x_{n_j} \leq b_j \\ \implies x_{n_j} &\rightarrow a \square \end{aligned}$$

1.4 Cauchy Sequences

Definition. $a_n \in \mathbb{R}$ is called a **Cauchy sequence** if given $\varepsilon > 0$, $\exists N > 0$ s.t. $|a_n - a_m| < \varepsilon \forall n, m \geq N$

Lemma 1.4. A convergent sequence is a Cauchy sequence.

Proof. if $a_n \rightarrow a$, given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$

Take $m, n \geq N$,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon \square$$

Theorem 1.5. Every Cauchy sequence is convergent.

Proof.

Claim. If a_n is Cauchy, then it is bounded.

Proof. Take $\varepsilon = 1$, $N = N(1)$, in the Cauchy property, then

$$|a_n - a_m| < 1, \forall n, m \geq N(1)$$

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \forall m \geq N$$

Let $K = \max\{1 + |a_N|, |a_n|, n = 1, 2, \dots, N-1\}$

Then $|a_n| \leq K \forall n \checkmark$

By the Bolzano-Weierstrass theorem,

$$a_{n_j} \rightarrow a$$

Claim. $a_n \rightarrow a$

Proof. Given $\varepsilon > 0$, $\exists j_0$ s.t. $\forall j \geq j_0$

$$|a_{n_j} - a| < \varepsilon$$

Also, $\exists N(\varepsilon)$ s.t. $|a_m - a_n| < \varepsilon \forall m, n \geq N(\varepsilon)$

Take j s.t. $n_j \geq \max\{N(\varepsilon), n_{j_0}\}$

Then if $n \geq N(\varepsilon)$,

$$|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < 2\varepsilon \square$$

Remark. Thus on \mathbb{R} a sequence is convergent iff it is Cauchy.

“Old-fashioned name”: “the general principle of convergence”

Note. This is a useful property since we do not need to know what the limit is.

1.5 Series

Definition. $a_n \in \mathbb{R}, \mathbb{C}$. We say that $\sum_{j=1}^{\infty} a_j$ **converges to** s if the sequence of partial sums

$$S_N = \sum_{j=1}^N a_j \rightarrow s$$

as $N \rightarrow \infty$
We write $\sum_{j=1}^{\infty} a_j = s$

If S_N does not converge, we say that $\sum_{j=1}^{\infty} a_j$ diverges.

Remark. Any problem on series can be turned into a problem on sequences just by considering the sequence of partial sums.

Lemma 1.6.

- (i) If $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ converge, then so does $\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j)$ where $\lambda, \mu \in \mathbb{C}$
- (ii) Suppose $\exists N$ s.t. $a_j = b_j \forall j \geq N$, then either $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ both converge or both diverge (initial terms do not matter)

Proof.

(i)

$$\begin{aligned} S_N &= \sum_{j=1}^N a(\lambda a_j + \mu b_j) \\ &= \lambda \sum_{j=1}^N a_j + \mu \sum_{j=1}^N b_j \\ &= \lambda c_N + \mu d_N \end{aligned}$$

- $c_N \rightarrow c$ & $d_N \rightarrow d$ so by lemma 1.1 (version \mathbb{C}), $s_N \rightarrow \lambda c + \mu d$
- (ii) $n \geq N$

$$\begin{aligned} s_n &= \sum_1^n a_j = \sum_1^{N-1} a_j + \sum_N^n a_j \\ d_n &= \sum_1^n b_j = \sum_1^{N-1} b_j + \sum_N^n b_j \\ \implies s_n - d_n &= \sum_1^{N-1} a_j - \sum_1^{N-1} b_j \end{aligned}$$

(as $a_j = b_j$ for $j \geq N$)
so s_n converges iff d_n does. \square

1.5.1 The Geometric Series

Claim. The geometric series converges iff $|x| < 1$

Proof. Set $a_n = x^n - 1$ $n \geq 1$

$$S_n = \sum_1^n a_g = 1 + x^2 + \cdots + x^{n-1}$$

Then

$$s_n = \begin{cases} \frac{1-x^n}{1-x} & \text{for } x \neq 1 \\ n & \text{for } x = 1 \end{cases}$$

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n$$

$$\implies S_n(1-x) = 1 - x^n$$

if $|x| < 1$, $x^n \rightarrow 0$ and $S_n \rightarrow \frac{1}{1-x}$

if $x > 1$, $x^n \rightarrow \infty$ & $S_n \rightarrow \infty$

if $x < -1$, S_n does not converge (oscillates)

if $x = -1$, $s = \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$

Note. Say $S_n \rightarrow \infty$ if given A , $\exists N$ s.t. $S_n > A$, $\forall n \geq N$

$S_n \rightarrow -\infty$, if given A , $\exists N$ s.t. $S_n < -A$ for all $n \geq N$

If S_n does not converge or tend to $\pm\infty$, we say that S_n oscillates.

Claim. $x^n \rightarrow 0$ if $|x| < 1$

Proof. Consider the case $0 < x < 1$ and we write $\frac{1}{x} = 1 + \delta$, $\delta > 0$

So:

$$x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+\delta n} \rightarrow 0$$

because $(1+\delta)^n \geq 1 + n\delta$ (from the binomial expansion)

Lemma 1.7. If $\sum_{j=1}^{\infty} a_j$ converges, then:

$$\lim_{j \rightarrow \infty} a_j = 0$$

Proof.

$$S_n = \sum_1^n a_j$$

$$a_n = S_n - S_{n-1}$$

So if $S_n \rightarrow a$ then $a_n \rightarrow 0$ (since $S_{n-1} \rightarrow a$ also)

Remark. The converse of 1.7 is false! Shown by example below:

Claim. $\sum_1^{\infty} \frac{1}{n}$ diverges (harmonic series)

Proof.

$$S_n = \sum_1^{\infty} \frac{1}{j}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} > S_n + \frac{1}{2}$$

Since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \dots, n$

So if $S_n \rightarrow a$, then $S_{2n} \rightarrow a$ also and thus

$$a \geq a + \frac{1}{2} \otimes$$

1.5.2 Series of Positive/ Non-negative terms

Theorem 1.8 (The Comparison Test). Suppose $0 \leq b_n \leq a_n \forall n$

Then if $\sum_1^{\infty} a_n$ converges, so does $\sum_1^{\infty} b_n$

Proof. Let $S_N = \sum_1^N a_n$

$$d_N = \sum_1^N b_n$$

$$b_n \leq a_n \implies d_N \leq S_N$$

But $S_N \rightarrow S$, then

$$d_N \leq S_N \leq S \forall N$$

and d_N is an increasing sequence bounded above $\implies d_N$ converges \square

An example using this below:

Claim. $\sum_1^n \frac{1}{n^2}$ converges

Proof.

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} = a_n$$

$$\begin{aligned}\sum_2^N a_n &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N} \\ &= 1 - \frac{1}{N} \rightarrow 1 \text{ as } N \rightarrow \infty\end{aligned}$$

By comparison, $\sum_1^n \frac{1}{n^2}$ converges

In fact, we get $\sum_1^n \frac{1}{n^2} \leq 1 + 1 = 2$

Note. Converges to $\frac{\pi^2}{6}$ but we do not prove that here.

Theorem 1.9 (Root test/ Cauchy's test for convergence). Assume $a_n \geq 0$ and $a_n^{1/n} \rightarrow a$ as $n \rightarrow \infty$. Then if $a < 1$, $\sum a_n$ converges; if $a > 1$, $\sum a_n$ diverges

Proof. If $a < 1$, choose $a < r < 1$.

By definition of limit,

$\exists N$ s.t. $\forall n \geq N$

$$a_n^{1/n} < r \implies a_n < r^n$$

But since $r < 1$, the geometric series $\sum r^n$ converges \implies by Theorem 1.8, $\sum a_n$ converges.

If $a > 1$, then for $n \geq N$,

$$a_n^{1/n} > 1 \implies a_n > 1$$

Thus $\sum a_n$ diverges (since a_n does not tend to zero). \square

Remark. Nothing can be said if $a = 1$, see examples later.

Theorem 1.10 (Ratio test/ D’Alanbert’s test). Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \rightarrow l$
 If $l < 1$, $\sum a_n$ converges.
 If $l > 1$, $\sum a_n$ diverges

Proof. Suppose $l < 1$ and choose r with $l < r < 1$
 Then $\exists N$ s.t. $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r$$

Therefore

$$\begin{aligned} a_n &= \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \quad n > N \\ \implies a_n &< K r^n \end{aligned}$$

with K independent of n

Since $\sum r^n$ converges, so does $\sum a_n$ by Theorem 1.8

If $l > 1$, choose $1 < r < l$

Then $\frac{a_{n+1}}{a_n} > r \quad \forall n \geq N$

And as before:

$$\begin{aligned} a_n &= \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \quad n > N \\ a_N r^{n-N} &\rightarrow \infty \text{ as } n \rightarrow \infty \end{aligned}$$

So $\sum a_n$ diverges. \square

Remark. Nothing can be said if $a = 1$.

Examples: Consider ratio test for series $\sum_{n=1}^{\infty} \frac{n}{2^n}$

$$\frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{2n} \rightarrow \frac{1}{2} < 1$$

So we have convergence by the ratio test.

The following examples show limit 1 inconclusive:

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges,

Since $n^{1/n} \rightarrow 1$ as $n \rightarrow \infty$, root test is also inconclusive when limit = 1.

To see this limit, write

$$n^{1/n} = 1 + \delta_n, \quad \delta > 0$$

$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2} \delta_n^2$$

(binomial expansion)

$$\implies \delta_n^2 < \frac{2}{n-1} \implies \delta_n \rightarrow 0$$

Another root test example:

$\sum_1^n \left[\frac{n+1}{3n+5} \right]^n$, root test gives:

$$\frac{n+1}{3n+5} \rightarrow \frac{1}{3} < 1$$

so converges.

Theorem 1.11 (Cauchy's Condensation Test). Let a_n be a decreasing sequence of positive terms.

Then $\sum_1^\infty a_n$ converges iff

$\sum_1^\infty 2^n a_{2^n}$ converges.

Proof. First we observe that if a_n is decreasing:

$$a_{2^k} \underset{(*_1)}{\leq} a_{2^{k-1}+i} \underset{(*_2)}{\leq} a_{2^{k-1}}, \quad 1 \leq i \leq 2^{k-1} \quad (\text{any } k \geq 1)$$

Assume now that $\sum_1^\infty a_n$ converges with sum let's say A

Then,

$$2^{n-1} a_{2^n} = \underbrace{a_{2^n} + \cdots + a_{2^n}}_{2^{n-1} \text{ times}} \underset{(*_1)}{\leq} a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} = \sum_{m=2^{n-1}+1}^{2^n} a_m$$

Thus

$$\begin{aligned} \sum_{n=1}^N 2^{n-1} a_{2^n} &\leq \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^n} a_m \\ \Rightarrow \sum_{n=1}^N 2^n a_{2^n} &\leq 2 \sum_{m=2}^{2^N} a_m \leq 2(A - a_1) \end{aligned}$$

Thus $\sum_{n=1}^N 2^n a_{2^n}$ increasing and bounded above, converges.

Conversely, assume $\sum 2^n a_{2^n}$ converges.

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^n} a_m \leq \sum_{n=1}^N 2^{n-1} a_{2^{n-1}} \leq B$$

$\Rightarrow \sum_{m=1}^N a_m$ is a bounded increasing sequence and thus it converges \square

Example/ Application

$\sum_1^{\infty} \underbrace{\frac{1}{n^k}}_{a_n}$ converges iff $k > 1$ (for $k > 0$)

Decreasing sequence of positive terms as:

$$\frac{1}{(n+1)^k} < \frac{1}{n^k} \iff \left(\frac{n}{n+1}\right)^k < 1 \iff \frac{n}{n+1} < 1$$

$$2^n a_{2^n} = 2^n \left[\frac{1}{2^n} \right]^k = 2^{n-nk} = \underbrace{(2^{1-k})}_r^n$$

And $\sum r^n$ converges iff $r < 1$.

$\implies \sum \frac{1}{n^k}$ converges iff $2^{1-k} < 1$ iff $k > 1$

1.5.3 Alternating Series

Theorem 1.12 (The alternating series test). If a_n decreases and tends to zero as $n \rightarrow \infty$, then the series $\sum_1^{\infty} (-1)^{n+1} a_n$ converges

Proof.

$$S_n = a_1 - a_2 + \cdots + (-1)^{n+1} a_n$$

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2n-1} - a_{2n}) \geq S_{2n-2}$$

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$$

So S_{2n} is increasing and bounded above $\implies S_{2n} \rightarrow S$

$$S_{2n+1} = S_{2n} + a_{2n+1} \rightarrow S + 0 = S$$

This implies that S_n converges to S as:

given $\varepsilon > 0$, $\exists N_1$ s.t. $\forall n \geq N_1$, $|S_{2n} - S| < \varepsilon$

$\exists N_3$ s.t. $\forall n \geq N_2$, $|S_{2n+1} - S| < \varepsilon$

Take $N = 2 \max\{N_1, N_2\} + 1$

Then if $k \geq N \implies$

$$|S_k - S| < \varepsilon, \text{ so } S_k \rightarrow S$$

Note. e.g. $\sum_1^{\infty} \frac{(-1)^{n+1}}{n}$ converges

1.5.4 Absolute Convergence

Definition. Take $a_n \in \mathbb{C}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series is **absolutely convergent**

Note. Since $|a_n| \geq 0$ we can use the previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

Theorem 1.13. IF $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Suppose first that $a_n \in \mathbb{R}$

Let

$$v_n = \begin{cases} a_n & \text{if } a_n \geq 0 \\ 0 & \text{if } a_n < 0 \end{cases}$$

$$w_n = \begin{cases} 0 & \text{if } a_n \geq 0 \\ -a_n & \text{if } a_n < 0 \end{cases}$$

$$v_n = \frac{|a_n| + a_n}{2}, \quad w_n = \frac{|a_n| - a_n}{2}$$

Clearly, $v_n, w_n \geq 0$,

$$a_n = v_n - w_n, \quad |a_n| = v_n + w_n \geq v_n, w_n$$

If $\sum |a_n|$ converges, by comparison, $\sum v_n, \sum w_n$ also converge

$$\implies \sum a_n \text{ converges}$$

If $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$

$$|x_n|, |y_n| \leq |a_n|$$

$\implies \sum x_n, \sum y_n$ are absolutely convergent, $\implies \sum x_n, \sum y_n$ converge, since $a_n = x_n + iy_n \implies \sum a_n$ converges as well \square

Examples.

- (i) $\sum \frac{(-1)^n}{n}$ converges, but not absolutely convergent (mod gives harmonic series).
- (ii)

$$\sum_{n=1}^{\infty} \frac{z^n}{2^n}, \quad \sum \left(\frac{|z|}{2} \right)^n \quad (*)$$

\implies if $|z| < 2$, convergence of (*) and hence absolute convergence.

if $|z| \geq 2$, then $|a_n| \geq 1$, so a_n does not tend to zero $\implies \sum \frac{z^n}{2^n}$ diverges

Definition. If $\sum a_n$ converges but $\sum |a_n|$ does not, it is said sometimes that $\sum a_n$ is **conditionally convergent**.

Note. “conditional”: because the sum to which the series converges is conditional on the order in which the elements of the sequence are taken.
If rearranged, the sum is altered.

Example.

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \quad (\text{I})$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots \quad (\text{II})$$

Let s_n be the partial sum of (I) and t_n be the sumpartial sum of (II)

$$s_n \rightarrow s > 0$$

$$t_n \rightarrow \frac{3s}{2}$$

Definition. Let σ be a bijection of the positive integers

$$a'_n = a_{\sigma(n)}$$

is a **rearrangement**.

Theorem 1.14. If $\sum_1^\infty a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

Proof. We do the proof first for $a_n \in \mathbb{R}$.
Let $\sum a'_n$ be a rearrangement of $\sum a_n$. Let

$$S_n = \sum_1^n a_n$$

$$t_n = \sum_1^n a'_n$$

Suppose first that $a_n \geq 0$

Given n , we can find q s.t. S_q contains every term of t_n

Since $a_n \geq 0$,

$$t_n \leq s_q \leq s$$

As $n \rightarrow \infty$, $t_n \rightarrow t$ (increasing sequence bounded above) $\implies t \leq s$. By symmetry,
 $s \leq t \implies s = t$

If a_n has any sign v_n and w_n from Theorem 1.13

$$v_n = \frac{|a_n| + a_n}{2}, \quad w_n = \frac{|a_n| - a_n}{2}$$

Consider, $\sum a'_n, \sum v'_n, \sum w'_n$

Since $\sum |a_n|$ converges, both $\sum v_n, \sum w_n$ converge, now use the case $v_n, w_n \geq 0$ to deduce that

$$\sum v'_n = \sum v_n, \quad \sum w'_n = \sum w_n$$

and the claim follows since $a_n = v_n - w_n$

For the case $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$

Since $|x_n|, |y_n| \leq |a_n| \implies \sum x_n, \sum y_n$ are absolutely convergent.

Then by the previous case $\sum x'_n = \sum x_n$ and $\sum y'_n = \sum y_n$. Since $a'_n = x'_n + iy'_n$, $\sum a_n = \sum a'_n$

□

2 Continuity

$E \subseteq \mathbb{C}$ non-empty, $f : E \rightarrow \mathbb{C}$ any function, $a \in E$
(includes case in which f is real valued and E is a subset of \mathbb{R})

Definition. f is **continuous at** $a \in E$ if for every sequence $z_n \in E$ with $z_n \rightarrow a$, we have $f(z_n) \rightarrow f(a)$
Equivalently below:

Definition. f is **continuous at** $a \in E$, if

$$\text{given } \varepsilon > 0, \exists \delta \text{ s.t. if } |z - a| < \delta, \text{ then } |f(z) - f(a)| < \varepsilon$$

(ε - f definition)

Claim. Two definitions equivalent

Proof. $2^{\text{nd}} \implies 1^{\text{st}}$:

We know that given $\varepsilon > 0$, $\exists \delta > 0$, s.t. $|z - a| < \delta$, $z \in E$, then $|f(z) - f(a)| < \varepsilon$.

Let $z_n \rightarrow a$.

Then $\exists n_0$ s.t. $\forall n \geq n_0$ we have

$$|z_n - a| < \delta \implies |f(z_n) - f(a)| < \varepsilon$$

$1^{\text{st}} \implies 2^{\text{nd}}$:

Assume $f(z_n) \rightarrow f(a)$ whenever $z_n \rightarrow a$ ($z_n \in E$). Suppose f is not continuous at a , according to 2^{nd} definition.

$$\exists \varepsilon > 0, \text{ s.t. } |z - a| < \delta \text{ and } |f(z) - f(a)| \geq \varepsilon \quad (*)$$

Let $\delta = \frac{1}{n}$, from (*) we get z_n s.t. $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \varepsilon$.

Clearly $z_n \rightarrow a$, but $f(z_n)$ does not tend to $f(a)$ because $|f(z_n) - f(a)| \geq \varepsilon$ $\forall n$.

Prop 2.1. $a \in E$, $g, f : E \rightarrow \mathbb{C}$ continuous at a . Then so are the functions $f(z) + g(z)$, $f(z)g(z)$ & $\lambda f(z)$ for any constant. In addition if $f(z) \neq 0 \forall z \in E$, then $\frac{1}{f}$ is continuous at a

Proof. Using 1^{st} definition, this is obvious using the analogous results for sequences (Lemma 1.1) e.g.

$$f(z_n) + g(z_n) \rightarrow f(a) + g(a) \text{ if } z_n \rightarrow a, f(z_n) \rightarrow f(a) \text{ \& } g(z_n) \rightarrow g(a) \text{ etc. } \square$$

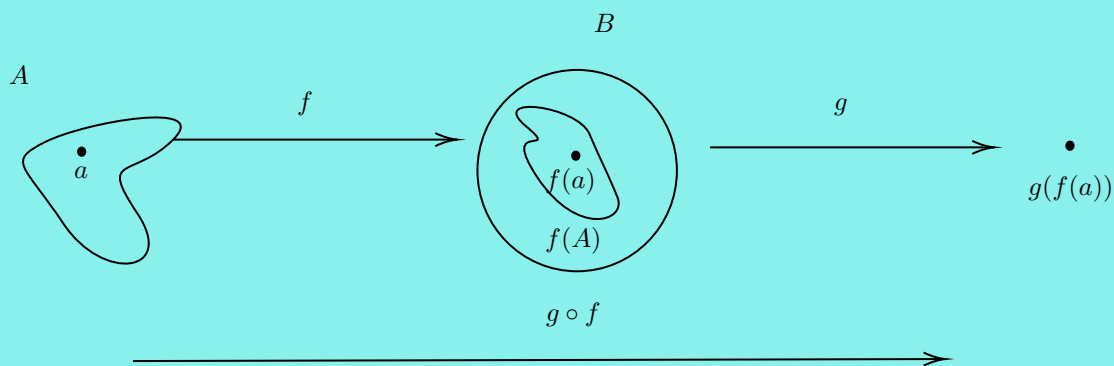
Example. The function $f(z) = z$ is continuous, so using the proposition we derive that every polynomial is continuous at every point in \mathbb{C}

Note. We say f is continuous on E if it is continuous at every $a \in E$.

Remark. Still it is instructive to prove above prop directly from the $\varepsilon - \delta$ definition

Next we look at compositions

Theorem 2.2. Let $f : A \rightarrow \mathbb{C}$ and $g : B \rightarrow \mathbb{C}$ be two functions s.t. $f(A) \subseteq B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$. Then $g \circ f : A \rightarrow \mathbb{C}$ is continuous at a .



Proof. Take any sequence $z_n \rightarrow a$. By assumption, $f(z_n) \rightarrow f(a)$. Set $w_n = f(z_n)$. then $w_n \in B$ and $w_n \rightarrow f(a)$; thus

$$g(w_n) \rightarrow g(f(a)) \square$$

Examples.

(i)

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(sin(x) continuous proved later)

if $x \neq 0$, then 2.1 and 2.2 imply that $f(x)$ is continuous at every $x \neq 0$.

Discontinuous at 0:

$$\frac{1}{x_n} = (2n + \frac{1}{2})\pi$$

$$f(x_n) = 1, \quad x_n \rightarrow 0 \text{ but } f(0) = 0$$

(ii)

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

 f is continuous at 0:take $x_n \rightarrow 0$, then

$$|f(x_n)| \leq |x_n| \text{ because } \left| \sin\left(\frac{1}{x}\right) \right| \leq 1$$

$$\implies f(x_n) \rightarrow 0 = f(0)$$

(iii)

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Discontinuous at every point:

if $x \in \mathbb{Q}$, take a sequence $x_n \rightarrow x$ with $x_n \notin \mathbb{Q}$, then

$$f(x_n) = 0 \not\rightarrow f(x) = 1$$

Similarly, if $x \notin \mathbb{Q}$, take a sequence $x_n \rightarrow x$ with $x_n \in \mathbb{Q}$, then

$$1 = f(x_n) \not\rightarrow f(x) = 0$$

2.1 Limit of a function

$$F : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$$

We wish to define what is meant by

$$\lim_{z \rightarrow a} f(z)$$

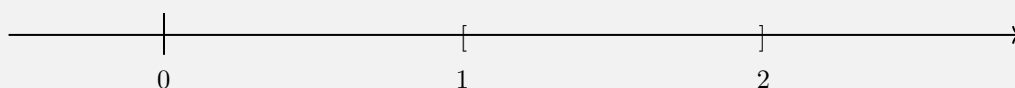
even when a might not be in E e.g.

$$\text{limit at } z \rightarrow 0 \frac{\sin z}{z} \quad E = \mathbb{C} \setminus \{0\} \quad a = 0$$

Also if

$$E \cup [1, 2]$$

it does not make sense to speak about $z \in E, z \neq 0, z \rightarrow 0$



Definition. $E \subseteq \mathbb{C}, a \in \mathbb{C}$. We say that a is a **limit point** of E if for any $\delta > 0, \exists z \in E$ s.t.

$$0 < |z - a| < \delta$$

Remark. a is a limit point iff \exists a sequence $z_n \in E$ s.t. $z_n \rightarrow a$ and $z_n \neq a$ for all n . (can check equivalence)

Definition. $f : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$, let $a \in \mathbb{C}$ be a limit point of E .

We say that

$$\lim_{z \rightarrow a} f(z) = l$$

(f tends to l as z tends to a)

If given $\varepsilon > 0, \exists \delta > 0$ s.t. whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \varepsilon$

Equivalently: $f(z_n) \rightarrow l$ for every sequence $z_n \in E, z_n \neq a$ and $z_n \rightarrow a$

(proved exactly the same as previously with 2 definitions of continuity).

Remark. Straight from the definition, we have if $a \in E$ is a limit point, then

$$\lim_{z \rightarrow a} f(z) = f(a) \iff f \text{ is continuous at } a$$

If $a \in E$ is isolated (i.e. $a \in E$ and is not a limit point), continuity of f at a always holds.

The limit of functions has very similar properties to the limit of sequences

(i) it is unique $f(z) \rightarrow A, f(z) \rightarrow B$ as $z \rightarrow a$

$$|A - B| \leq |A - f(z)| + |f(z) - B|$$

if $z \in E$ is s.t. $0 < |z - a| < \delta_1, \delta_2$, then

$$|A - B| < 2\varepsilon \implies A = B$$

(the existence of such z is a consequence of the condition that a is a limit point of E)

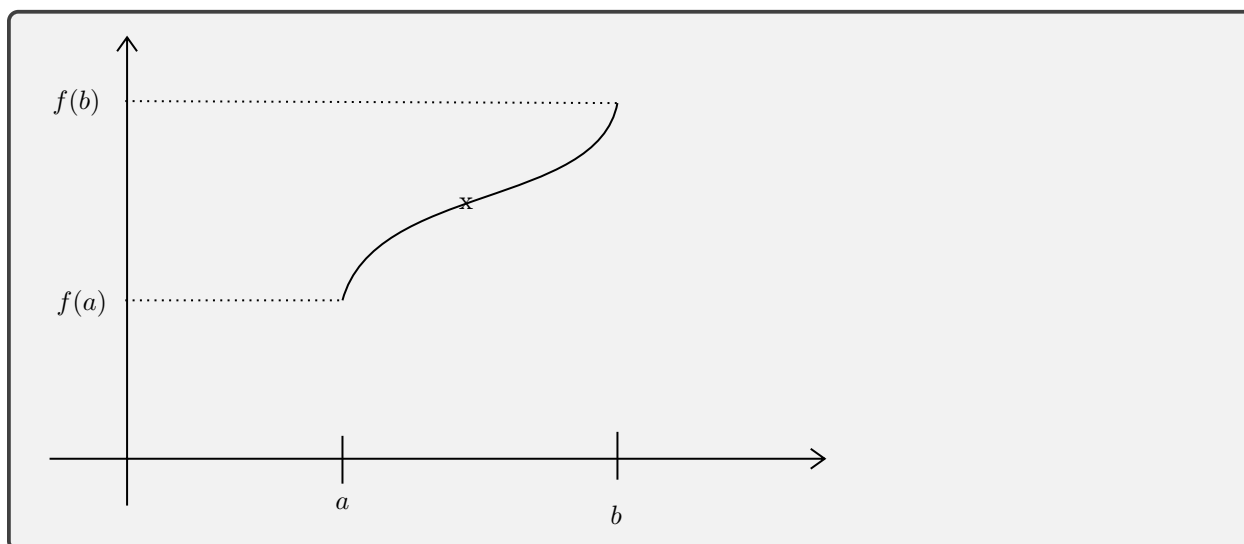
(ii) $f(z) + g(z) \rightarrow A + B$ if $f(z) \rightarrow A, g(z) \rightarrow B$ as $z \rightarrow a$

(iii) $f(z)g(z) \rightarrow AB$

(iv) if $B \neq 0, \frac{f(z)}{g(z)} \rightarrow \frac{A}{B}$

all proved in the same way as before.

2.2 The Intermediate Value Theorem



Theorem 2.3. $f : [a, b] \rightarrow \mathbb{R}$ continuous and $f(a) \neq f(b)$. Then f takes every value which lies between $f(a)$ and $f(b)$.

Proof. Without loss of generality, we may suppose $f(a) < f(b)$.

Take

$$f(a) < \eta < f(b)$$

Let

$$S = \{x \in [a, b] : f(x) < \eta\}$$

$a \in S$, so $S \neq \emptyset$. Clearly S is bounded above by b .

Then there is a supremum C where $C \leq b$. By definition of the supremum, given n , there exists $x_n \in S$ s.t.

$$C - \frac{1}{n} < x_n \leq C$$

So, $x_n \rightarrow C$. Since $x_n \in S$,

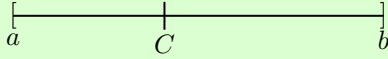
$$f(x_n) < \eta$$

By continuity of f , $f(x_n) \rightarrow f(C)$.

Thus

$$f(C) \leq \eta \tag{*}$$

Now observe that $C \neq b$, for if $C = b$, then $f(b) \leq \eta$ by (*) which is false.



Then for n large

$$C + \frac{1}{n} \in [a, b] \text{ and } C + \frac{1}{n} \rightarrow C$$

Again by continuity $f(C + \frac{1}{n}) \rightarrow f(C)$. But since

$$C + \frac{1}{n} > C, f(C + \frac{1}{n}) \geq \eta$$

Thus

$$f(C) \geq \eta \implies f(C) = \eta \square$$

Remark. The theorem is very useful for finding zeros of fixed points.

Example. Existence of the N -th root of a positive real number

$$f(x) = x^N, x \geq 0$$

Let y be a positive number.

f is continuous on $[0, 1 + y]$

$$0 = f(0) < y < (1 + y)^N = f(1 + y)$$

By the IVT, $\exists C \in (0, 1 + y)$ s.t. $f(C) = y$ i.e. $C^N = y$

C is a positive N -root of y .

Uniqueness: if $d^N = y$ with $d > 0$ and $d \neq C$, wlog suppose $d < c$

$$\implies d^N < c^N \implies y < y \times$$

2.3 Bounds of a Continuous Function

Theorem 2.4. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then there exists K s.t.

$$|f(x)| \leq K \quad \forall x \in [a, b]$$

Proof. We argue by contradiction.

Suppose statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a, b]$ s.t. $|f(x_n)| > n$.

By Bolzano-Weierstrauss, x_n has a convergent subsequence $x_{n_j} \rightarrow x$.

Since $a \leq x_{n_j} \leq b$, we must have $x \in [a, b]$. By continuity of f ,

$$f(x_{n_j}) \rightarrow f(x)$$

But

$$|f(x_{n_j})| > n_j \rightarrow \infty \quad \square$$

Theorem 2.5. $f : [a, b] \rightarrow \mathbb{R}$ continuous. Then $\exists x_1, x_2 \in [a, b]$ s.t.

$$f(x_1) \leq f(x) \leq f(x_2) \quad \forall x \in [a, b]$$

“A continuous function on a closed, bounded interval is bounded and attains its bounds.”

Proof (1st). Let

$$A = \{f(x) : x \in [a, b]\} = f([a, b])$$

By Theorem 2.4, A is bounded. Since it is clearly non-empty, it has supremum, M .

By definition of supremum,

$$\text{given integer } n \geq 1, \exists x_n \in [a, b] \text{ s.t. } M - \frac{1}{n} < f(x_n) \leq M \quad (*)$$

By Bolzano-Weierstrass,

$$\exists x_{n_j} \rightarrow x \in [a, b]$$

Since $f(x_{n_j}) \rightarrow M$ (because $*$) and f is continuous, we deduce that $f(x) = M$ so $x_2 = x$.

Reason similarly for the minimum \square

Proof (2nd).

$$A = f([a, b]), \quad M = \sup A$$

as before. Suppose $\nexists x_2$ s.t. $f(x_2) = M$.

Let

$$g(x) = \frac{1}{M - f(x)}, \quad x \in [a, b]$$

is defined and continuous. By Theorem 2.4 applied to g ,

$$\exists K > 0 \text{ s.t. } g(x) \leq K \quad \forall x \in [a, b]$$

This means that $f(x) \leq M - \frac{1}{K}$ on $[a, b]$. This is absurd since it contradicts that M is the supremum \square

Note. Theorems 2.4, 2.5 are false if the interval is not closed e.g.

$$x \in (0, 1], f(x) = \frac{1}{x}$$

2.4 Inverse functions

Definition. f is **increasing** for $x \in [a, b]$ if $f(x_1) \leq f(x_2)$ for all x_1, x_2 s.t. $a \leq x_1 \leq x_2 \leq b$.
If $f(x_1) < f(x_2)$ we say that f is **strictly increasing**.
Similarly for **decreasing** and **strictly decreasing**.

Theorem 2.6. $f : [a, b] \rightarrow \mathbb{R}$ continuous and strictly increasing for $x \in [a, b]$.

Let $c = f(a)$ and $d = f(b)$.

Then $f : [a, b] \rightarrow [c, d]$ is bijective and the inverse

$$g = f^{-1} : [c, d] \rightarrow [a, b]$$

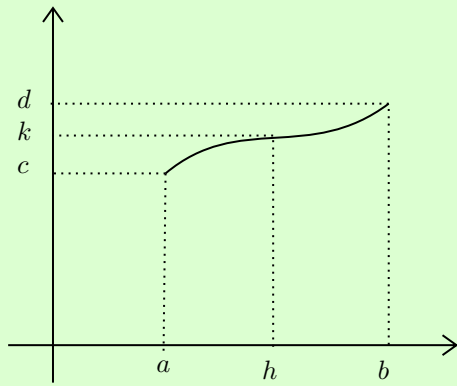
is continuous and strictly increasing

Remark. A similar theorem holds for strictly decreasing functions.

Proof. Take $c < k < d$.

From the intermediate value theorem

$$\exists h \text{ s.t. } f(h) = k$$



Since f is strictly increasing, h is unique.

Define $g(k) = h$ and this gives an inverse $g : [c, d] \rightarrow [a, b]$ for f . g is strictly increasing: $y_1 < y_2$

$$y_1 = f(x_1), y_2 = f(x_2)$$

If $x_2 \leq x_1$, since f is increasing

$$\implies f(x_2) \leq f(x_1) \implies y_2 \leq y_1 \text{ ✗}$$

g is continuous:

Given $\varepsilon > 0$, let

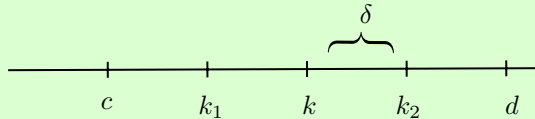
$$k_1 = f(h - \varepsilon), k_2 = f(h + \varepsilon)$$

f strictly increasing \implies

$$k_1 < k < k_2$$

If $k_1 < y < k_2$ then

$$h - \varepsilon < g(y) < h + \varepsilon$$



$$\delta = \min\{k_2 - k, k - k_1\}$$

(here $k \in (c, d)$ but a similar argument establishes continuity at the end points (can check))

3 Differentiability

Let $f : E \subseteq \mathbb{C} \rightarrow \mathbb{C}$, ost of the time $E = \text{interval} \subseteq \mathbb{R}$

Definition. Let $x \in E$ be a point s.t. $\exists x_n \in E$ with $x_n \neq x$ and $x_n \rightarrow x$ (i.e. a limit point) f is said to be **differentiable** at x with derviative $f'(x)$ if

$$\lim_{y \rightarrow x} \frac{f(y) - f(x)}{y - x} = f'(x)$$

If f is differentiable at each $x \in E$, we say f is differentiable on E

Note. Think of E as an interval or disc in the case of \mathbb{C}

Remark.

(i) Other common notations:

$$\frac{dy}{dx}, \frac{df}{dx}$$

(ii)

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

($y = x + h$)

(iii) “Another important look at the definition:”

Let

$$\varepsilon(h) = f(x+h) - f(x) - hf'(x)$$

then

$$\lim_{h \rightarrow 0} \frac{\varepsilon(h)}{h} = 0$$

$$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear}} + \varepsilon(h)$$

linear as $h \mapsto hf'(x)$

Definition (alternative). f is **differentiable** at x if $\exists A$ and ε s.t.

$$f(x+h) = f(x) + hA + \varepsilon(h)$$

where

$$\lim_{h \rightarrow 0} \frac{\varepsilon(h)}{h} = 0$$

If such an A exists, then it is unique, since

$$A = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Remark.

(iv) If f is differentiable at x then f is continuous at x as since $\varepsilon(h) \rightarrow 0$,

$$f(x+h) \rightarrow f(x) \text{ as } h \rightarrow 0$$

(v) Another alternative way of writing things:

$$f(x+h) = f(x) + hf'(x) + h\varepsilon_f(h)$$

with $\varepsilon_f(h) \rightarrow 0$ as $h \rightarrow 0$

or

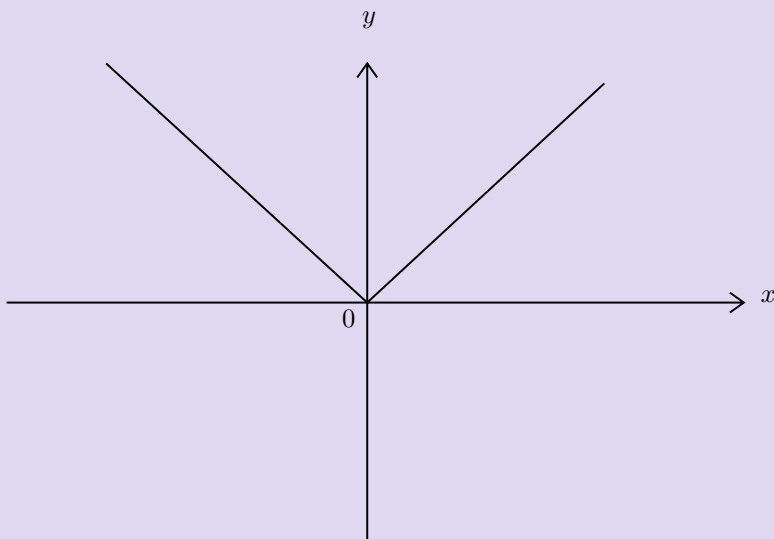
$$f(x) = f(a) + (x-a)f'(a) + (x-a)\varepsilon_f(x)$$

with

$$\lim_{x \rightarrow a} \varepsilon_f(x) \rightarrow 0$$

Example.

$$f(x) = |x|, f: \mathbb{R} \rightarrow \mathbb{R}$$



$$f'(x) = 1 \text{ if } x > 0$$

$$f'(x) = -1 \text{ if } x < 0$$

Take $h_n \rightarrow 0$ from above:

$$\lim_{n \rightarrow \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \rightarrow \infty} \frac{h_n}{h_n} = 1$$

Take $h_n \rightarrow 0$ from below:

$$\lim_{n \rightarrow \infty} \frac{f(h_n) - f(0)}{h_n} = \lim_{n \rightarrow \infty} \frac{-h_n}{h_n} = -1$$

So not differentiable at $x = 0$

3.1 Differentiation of Sums, Products, etc.

Prop 3.1.

- (i) IF $f(x) = c \forall x$ in E , then f is differentiable with $f'(x) = 0$
- (ii) f, g differentiable at x , then so is $f + g$ and

$$(f + g)'(x) = f'(x) + g'(x)$$

- (iii) f, g differentiable at x , then so is fg and

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x)$$

- (iv) If f is differentiable at x and $f(x) \neq 0 \forall x \in E$, then $1/f$ is differentiable at x and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$$

Proof.

- (i)

$$\lim_{h \rightarrow 0} \frac{C - C}{h} = 0$$

- (ii)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

- (iii)

$$\phi(x) = f(x)g(x)$$

$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= f(x+h) \left[\frac{g(x+h) - g(x)}{h} \right] + g(x) \left[\frac{f(x+h) - f(x)}{h} \right] \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

using standard properties of limits and the fact that f is continuous at x

- (iv)

$$\phi(x) = 1/f(x)$$

$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{1/f(x+h) - 1/f(x)}{h} \\ &= \frac{f(x) - f(x+h)}{hf(x)f(x+h)} \rightarrow -\frac{f'(x)}{[f(x)]^2} \square \end{aligned}$$

Remark. From (iii) and (iv) we immediately get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Example.

$$f(x) = x^n, \quad n \in \mathbb{Z}, \quad n > 0$$

$$n = 1$$

Clearly $f(x) = x$, $f'(x) = 1$

Claim.

$$f'(x) = nx^{n-1}$$

Proof. Induction:

$$f(x) = x \cdot x^n$$

$$f'(x) = x^n + x(nc^{n-1}) = (n+1)x^n$$

Using prop 3.1

$$f(x) = x^{-n} = \frac{1}{x^n} \quad n \in \mathbb{Z}, \quad n > 0$$

If $x \neq 0$, use prop 3.1 (iv) to derive

$$f'(x) = -\frac{(x^n)'}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

So can differentiate polynomials, rational functions ✓

Theorem 3.2 (Chain rule).

$$f : U \rightarrow \mathbb{C}$$

is s.t.

$$f(x) \in V \quad \forall x \in U$$

If f is differentiable at $a \in U$ and $g : V \rightarrow \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = f'(a)g'(f(a))$$

Proof. We know:

$$f(x) = f(a) + (x - a)f'(a) + \varepsilon_f(x)(x - a)$$

where

$$\lim_{x \rightarrow a} \varepsilon_f(x) = 0$$

$$g(y) = g(b) + (y - b)g'(b) + \varepsilon_g(y)(y - b)$$

where

$$\lim_{y \rightarrow b} \varepsilon_g(y) = 0$$

$$b = f(a)$$

Set

$$\varepsilon_f(a) = 0 \text{ \& } \varepsilon_g(b) = 0$$

to make them continuous at $x = a$ and $y = b$.

Now $y = f(x)$ gives

$$\begin{aligned} g(f(x)) &= g(b) + (f(x) - b)g'(b) + \varepsilon_g(f(x))(f(x) - b) \\ &= g(f(a)) + [(x - a)f'(a) + \varepsilon_f(x)(x - a)][g'(b) + \varepsilon_g(f(x))] \\ &= g(f(a)) + (x - a)f'(a)g'(b) + (x - a) \underbrace{[\varepsilon_f(x)g'(b) + \varepsilon_g(f(x))(f'(a) + \varepsilon_f(x))]}_{\sigma(x)} \end{aligned}$$

$$\sigma(x) = \underbrace{\varepsilon_f(x)g'(b)}_0 + \underbrace{\varepsilon_g(f(x))}_{0 \text{ as continuous comp.}} \underbrace{(f'(a) + \varepsilon_f(x))}_{f'(a)}$$

so

$$\lim_{x \rightarrow a} \sigma(x) = 0$$

Examples.

(i)

$$f(x) = \sin(x^2)$$

$$(\sin x)' = \cos x$$

(to be seen later)

$$f'(x) = 2x \cos(x^2)$$

(ii)

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(this is continuous at every x)
differentiable at every $x \neq 0$ by the previous theorem.

At $x = 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x \sin(1/x)}{x} = \sin(1/x)$$

$$\implies \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist $\implies f$ is not differentiable at $x = 0$.

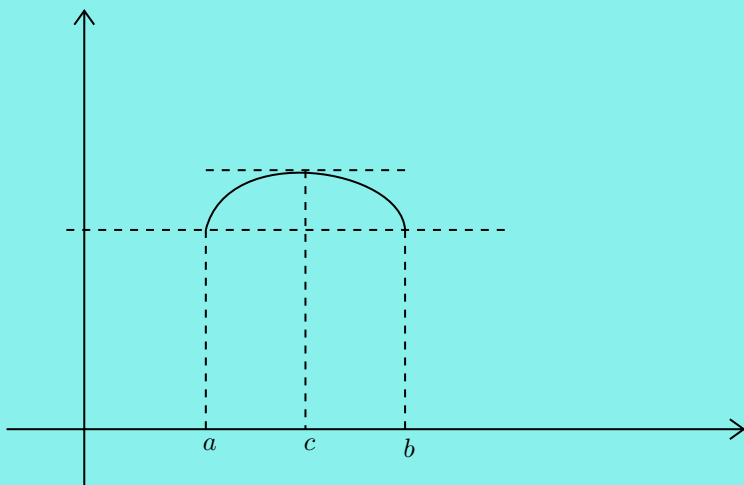
3.2 The Mean Value Theorem

Theorem 3.3 (Rolle's Theorem).

$$f : [a, b] \rightarrow \mathbb{R}$$

continuous on $[a, b]$ and differentiable on (a, b) . If $f(a) = f(b)$,

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$



Proof. Let

$$M = \max_{x \in [a, b]} f(x), \quad m = \min_{x \in [a, b]} f(x)$$

Recall (Theorem 2.5) that these values are achieved.

Let $k = f(a)$. If $M = m = k$, then f is constant and $f'(c) = 0 \quad \forall c \in (a, b)$

Then $M > k$ or $m < k$. Suppose $M > k$

By Theorem 2.5,

$$\exists c \text{ s.t. } f(c) = M$$

If $f'(c) > 0$, then there are values to the right of c for which $f(x) > f(c)$ since

$$f(x+h) - f(x) = h(f'(c) + \varepsilon(h)) > 0$$

Since $\varepsilon(h) \rightarrow 0$ as $h \rightarrow 0$ and thus

$$f'(x) + \varepsilon(h) > 0 \text{ if } h \text{ small}$$

This contradicts that M is the maximum.

Similarly, if $f'(c) < 0$, $\exists x$ to the left of c for which $f(x) > f(c)$

$$\implies f'(c) = 0 \square$$

Note. A simple tweak gives below:

Theorem 3.4 (The Mean Value Theorem). Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function which is differentiable on (a, b) . Then $\exists c \in (a, b)$ st.

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. Write

$$\phi(x) = f(x) - kx$$

Choose k s.t. $\phi(a) = \phi(b)$

$$\implies f(b) - bk = f(a) - bk \implies k = \frac{f(b) - f(a)}{b - a}$$

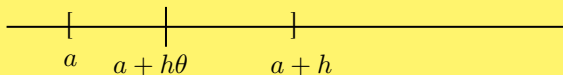
By Rolle's theorem applied to ϕ

$$\exists c \in (a, b) \text{ s.t. } \phi'(c) = 0$$

i.e. $f'(x) = k \square$

Remark. We will often write

$$f(a + h) = f(A) + hf'(a + \theta h)$$



$$\theta \in (0, 1)$$

$$(b = a + h)$$

Warning.

$$\theta = \theta(h)$$

Corollary 3.5. $f : [a, b] \rightarrow \mathbb{R}$ continuous and differentiable on (a, b) . Then we have

- (i) If $f'(x) > 0 \forall x \in (a, b)$, then f is strictly increasing on $[a, b]$
(i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)
- (ii) If $f'(x) \geq 0 \forall x \in (a, b)$, then f is increasing (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)
- (iii) If $f'(x) = 0 \forall x \in (a, b)$, then f is constant on $[a, b]$

Proof.

- (i) Have

$$f(y) - f(x) = f'(c)(y - x) \quad c \in (x, y)$$

from MVT

so

$$f'(c) > 0 \implies f(y) > f(x)$$

- (ii) same: but $f'(c) \geq 0 \implies f(y) \geq f(x)$

- (iii) Take $x \in [a, b]$. Then use MVT in $[a, x]$ to get $x \in (a, x)$ s.t.

$$f(x) - f(a) = f'(x)(x - a) = 0$$

$$\implies f(x) = f(a) \implies f \text{ is constant} \square$$

Remark. We have similar statements for decreasing functions

3.3 Inverse Rule/ Inverse Function Theorem

Theorem 3.6. $f : [a, b] \rightarrow \mathbb{R}$ continuous and differentiable on (a, b) with

$$f'(x) > 0 \quad \forall x \in (a, b)$$

Let $f(a) = c$ and $f(b) = d$. Then the function $f : [a, b] \rightarrow [c, d]$ is bijective and f^{-1} is differentiable on (c, d) with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. By corollary 3.5, f is strictly increasing on $[a, b]$. By Theorem 2.6

$$\exists g : [c, d] \rightarrow [a, b]$$

which is continuous, strictly increasing inverse of f .

RTP: g is differentiable and $g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$, $x \in (a, b)$

If $k \neq 0$ is given, let h be given by

$$y + k = f(x + h)$$

That is, $g(y + k) = x + h$, $h \neq 0$

Then

$$\frac{g(y + k) - g(y)}{k} = \frac{x + h - x}{f(x + h) - f(x)} \rightarrow \frac{1}{f'(x)}$$

Let $k \rightarrow 0$, then $h \rightarrow 0$ (g is continuous)

$$g'(y) = \lim_{h \rightarrow 0} \frac{g(y + k) - g(y)}{k} = \frac{1}{f'(x)}$$

Example.

$$g(x) = x^{1/q}$$

($x > 0$, q positive integer)

$$f(x) = x^q \quad (g(f(x)) = x)$$

$$f'(x) = qx^{q-1}$$

Since f is differentiable, so if g and by the inverse rule

$$g'(x) = \frac{1}{q(x^{1/q})^{1-q}} = \frac{1}{q}x^{1/q-1}$$

Now if $g(x) = x^{p/q}$ (p integer, q positive integer)

We can find $g'(x)$ by using the chain rule

$$g(x) = (x^p)^{1/q} = (x^{1/q})^p$$

We find (can check)

$$g'(x) = \frac{p}{q}x^{\frac{p}{q}-1}$$

So, if $g(x) = x^r$ $r \in \mathbb{Q}$
then $g'(x) = rx^{r-1}$

Remark. Suppose $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous, differentiable on (a, b) and $g(a) \neq g(b)$. Then the MVT gives us $s, t \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b - a)f'(s)}{(b - a)g'(t)} = \frac{f'(s)}{g'(t)}$$

Cauchy showed that one can take $s = t$

Theorem 3.7 (Cauchy's mean value theorem). Let $f, g : [a, b] \rightarrow \mathbb{R}$ be continuous functions and differentiable on (a, b) .

Then $\exists t \in (a, b)$ s.t.

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a))$$

Proof. Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$$

ϕ is continuous on $[a, b]$ and differentiable on (a, b)

Also,

$$\phi(a) = \phi(b) = 0$$

By Rolle's theorem, $\exists t \in (a, b)$ s.t. $\phi'(t) = 0$

If we expand the determinant, we get the desired result:

$$\begin{aligned} \phi'(x) &= f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x) \\ &= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)] \end{aligned}$$

$\phi'(t) = 0$ gives the result \square

Note. We recover the MVT if we take $g(x) = x$

Example. "L'Hopital's rule"

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin x} = \frac{e^x - e^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

as $x \rightarrow 0, t \rightarrow 0$, so

$$\frac{e^t}{\cos t} \rightarrow 1$$

Note. We want to extend the MVT to include higher order derivatives

Theorem 3.8 (Taylor's theorem with Lagrange's remainder). Suppose f and its derivatives up to order $n - 1$ are continuous in $[a, a + h]$ and $f^{(n)}$ exist for $x \in (a, a + h)$. Then

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

Where $\theta \in (0, 1)$

Proof. Define for $0 \leq t \leq h$

$$\phi(t) = f(a + t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}\beta$$

where we choose β s.t. $\phi(h) = 0$

(recall in the proof of the MVT we used $f(x) - kx$ and we picked k s.t. we could use Rolle's theorem)

We see that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0$$

We use Rolle's Theorem n -times:

$$\phi(0) = \phi(h) = 0 \implies \phi'(h_1) = 0 \quad 0 < h_1 < h$$

$$\phi'(0) = \phi(h_1) = 0 \implies \phi''(h_2) = 0 \quad 0 < h_2 < h_1$$

Finally

$$\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0 \implies \phi^{(n)}(h_n) = 0$$

$$0 < h_n < h_{n-1} < \cdots < h$$

So $h_n = \theta h$ for $\theta \in (0, 1)$

Now

$$\begin{aligned} \phi^{(n)}(t) &= f^{(n)}(a + t) - \beta \\ \implies \beta &= f^{(n)}(a + \theta h) \end{aligned}$$

Set $t = h$, $\phi(h) = 0$ and put this value of β in the second line in the proof \square

Note.

- (i) For $n = 1$, we get back the MVT, so this is a “ n -th order mean value theorem”
- (ii)

$$R_n = \frac{h^n}{n!}f^{(n)}(a + \theta h)$$

is known as Lagrange's form of the remainder

Theorem 3.9 (Taylor's theorem with Cauchy's form of remainder). With the same hypothesis as in Theorem 3.8 and $a = 0$ (to simplify), we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{h^n(1-\theta)^{n-1}f^{(n)}(\theta h)}{(n-1)!}, \quad \theta \in (0, 1)$$

Proof. Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1}f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \frac{(h-t)^2}{2}f''(t) + \cdots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

$$\implies F'(t) = -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

Set

$$\phi(t) = F(t) - \left[\frac{h-t}{h}\right]^p F(0)$$

where $p \in \mathbb{Z}, 1 \leq p \leq n$

Then $\phi(0) = \phi(h) = 0$ so by Rolle's theorem,

$$\exists \theta \in (0, 1) \text{ s.t. } \phi'(\theta h) = 0$$

But

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h} F(0) = 0$$

Thus

$$0 = -h^{n-1} \frac{(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h} \left[f(h) - f(0) - hf'(0) - \cdots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) \right]$$

$$\implies f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{h^n(1-\theta)^{n-1}f^{(n)}(\theta h)}{(n-1)! \cdot p \cdot (1-\theta)^{p-1}}, \quad \theta \in (0, 1)$$

If $p = n$ we get Lagrange's remainder

If $p = 1$ we get Cauchy's remainder

Method. To get a Taylor Series for f , one needs to show that $R_n \rightarrow 0$ as $n \rightarrow \infty$. This requires "estimates" and "effort"

Remark. Theorems 3.8 and 3.9 work equally well in an interval $[a+h, a]$ with $h < 0$

Example (The Binomial Series).

$$f(x) = (1+x)^r, \quad r \in \mathbb{Q}$$

Claim. if $|x| < 1$ then

$$(1+x)^r = 1 + \binom{r}{1}x + \cdots + \binom{r}{n}x^n + \cdots$$

where

$$\binom{r}{n} = \frac{r(r-1)\cdots(r-n+1)}{n!}$$

Proof. Clearly

$$f^{(n)}(x) = r(r-1)\cdots(r-n+1)(1+x)^{r-n}$$

If $r \in \mathbb{Z}$, $r \geq 0$, then $f^{(r+1)} \equiv 0$, we have a polynomial of degree r .

In general (Lagrange),

$$\begin{aligned} R_n &= \frac{x^n}{n!} f^{(n)}(\theta x) \\ &= \binom{r}{n} \frac{x^n}{(1+\theta x)^{n-r}} \end{aligned}$$

$\theta \in (0, 1)$ so have interval $[0, x]$ Note: in principle, θ depends on both x and n .

For $0 < x < 1$

$$(1+\theta x)^{n-r} > 1 \text{ for } n > r$$

Now observe that the series

$$\sum \binom{r}{n} x^n$$

is absolutely convergent for $|x| < 1$.

Indeed by the ratio test

$$a_n = \binom{r}{n} x^n$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{r(r-1)\cdots(r-n+1)(r-n)x^{n+1}}{(n+1)!} \right| \left| \frac{n!}{r(r-1)\cdots(r-n+1)x^n} \right| \quad (1)$$

$$= \left| \frac{(r-n)x}{n+1} \right| \rightarrow |x| \text{ as } n \rightarrow \infty \quad (2)$$

In particular, $a_n \rightarrow 0$, so $\binom{r}{n} x^n \rightarrow 0$ for $|x| < 1$

Hence for $n > r$ and $0 < x < 1$, we have

$$|R_n| \leq \left| \binom{r}{n} x^n \right| = |a_n| \rightarrow 0 \text{ as } n \rightarrow \infty$$

So the claim is proved in the range $0 \leq x < 1$

Example (continued).

Proof (continued). If $-1 < x < 0$ the argument above breaks down, but Cauchy's form of R_n works:

$$\begin{aligned} R_n &= \frac{(1-\theta)^{n-1} r(r-1) \dots (r-n+1) (1+\theta x)^{r-n} x^n}{(n-1)!} \\ &= \underbrace{\frac{r(r-1) \dots (r-n+1)}{(n-1)!}}_{r \binom{r-1}{n-1}} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^r \\ &= r \binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \underbrace{\left(\frac{1-\theta}{1+\theta x} \right)^{n-1}}_{< 1 \text{ for } x \in (-1,1)} \end{aligned}$$

$$|R_n| \leq \left| r \binom{r-1}{n-1} x^n \right| (1+\theta x)^{n-1}$$

Can check:

$$\begin{aligned} (1+\theta x)^{r-1} &< \max\{1, (1+x)^{r-1}\} \\ K_r &= r \max\{1, (1+x)^{r-1}\} \end{aligned}$$

which is independent of n

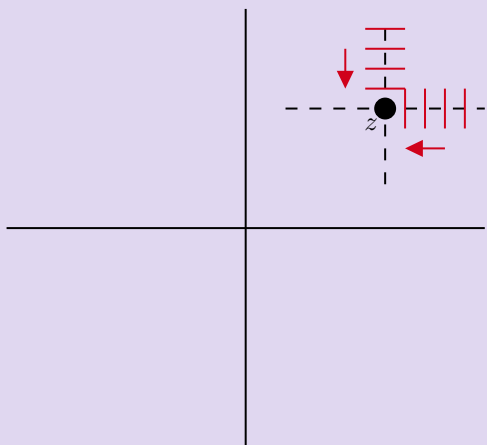
$$|R_n| \leq K_r \left| \binom{r-1}{n-1} x^n \right| \rightarrow 0$$

because $a_n \rightarrow 0$. Thus $R_n \rightarrow 0$ \square

3.4 Remarks on Complex Differentiation

Remark. Formally, we have regarding sums, products, chain rule etc. but it is much more restrictive than differentiability of functions on the real line.

Example. $f(z) = \bar{z}$ is no-where \mathbb{C} -differentiable



$$z_n = z + \frac{1}{n} \rightarrow z$$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1$$

$$z_n = z + \frac{i}{n} \rightarrow z$$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1$$

so

$$\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z} \text{ does not exist}$$

On the other hand $f(x, y) = (x, -y)$ is differentiable

$$z = x + iy$$

Note. IB Complex Analysis explores the consequences of \mathbb{C} -differentiability

4 Power Series

We want to look at $\sum_{n=0}^{\infty} a_n z^n$ with $z_n \in \mathbb{C}$, $a_n \in \mathbb{C}$.

(The case $\sum_{n=0}^{\infty} a_n (z - z_0)^n$, z_0 fixed follows this one by translation)

Lemma 4.1. If $\sum_0^\infty a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum_0^\infty a_n z^n$ converges absolutely

Proof. Since $\sum_0^\infty a_n z_1^n$ converges, $a_n z_1^n \rightarrow 0$. Thus $\exists K > 0$ s.t.

$$|a_n z_1^n| < K \quad \forall n$$

Then

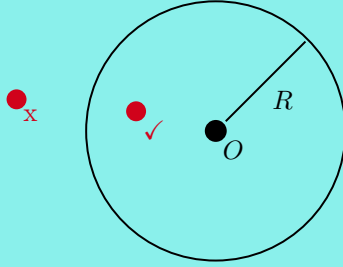
$$|a_n z^n| \leq K \left| \frac{z}{z_1} \right|^n$$

Since the geometric series $\sum_0^\infty \left| \frac{z}{z_1} \right|^n$ converges, the lemma follows by comparison \square

Using this lemma, we will prove that every power series has a radius of convergence

Theorem 4.2. A power series either

- (i) Converges absolutely for all z , or
- (ii) Converges absolutely for all z inside a circle $|z| = R$ and diverges for all z outside it, or
- (iii) Converges for $R = 0$ only



Proof. Let $S = \{x \in \mathbb{R}, x \geq 0 \text{ and } \sum a_n x^n \text{ converges}\}$. Clearly $0 \in S$. By Lemma 4.1, if $x_1 \in S$, then $[0, x_1] \in S$.

If $S = [0, \infty)$, we have case (i)

If not, there exists a finite supremum R ($R \geq 0$). For S , $R = \sup S < \infty$

If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely:
choose R_0 s.t. $|z_1| < R_0 < R$. Then $R_0 \in S$ and the series converges if $z = R_0$.
By Lemma 4.1, $\sum |a_n z_1^n|$ converges

Finally we show that if $|z_2| > R \geq 0$, then the series does not converge for z_2 . Now take R_0 s.t. $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converges, by Lemma 4.1, $\sum a_n R_0^n$ would be convergent, which contradicts that $R = \sup S$. \square

Definition. The circle $|z| = R$ is called the **circle of convergence** and R is the **radius of convergence**.

In (i), we agree that $R = \infty$ and in (iii) $R = 0$

The following lemma is useful for computing R

Lemma 4.3. If

$$\left| \frac{a_{n+1}}{a_n} \right| \rightarrow l$$

as $n \rightarrow \infty$, then $R = \frac{1}{l}$

Proof. By the ratio test, we have absolute convergence if

$$\lim \left| \frac{a_{n+1}}{a_n} \frac{z^{n+1}}{z^n} \right| < 1$$

so if $|z| < \frac{1}{l}$, we have absolute convergence. If $|z| > \frac{1}{l}$, the series diverges, again by the ratio test \square

Remark. One can also use the root test to get $|a_n|^{1/n} \rightarrow l$ then $R = \frac{1}{l}$

Examples.

(i) $\sum_0^\infty \frac{z^n}{n!}$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n!}{(n+1)!} = \frac{1}{n+1} \rightarrow 0 = l \implies R = \infty$$

(ii) Geometric series, $\sum_0^\infty z^n$

$R = 1$. Note that at $|z| = 1$, we have divergence

(iii) $\sum_0^\infty n!z^n$, has $R = 0$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)!z^{n+1}}{n!z^n} = (n+1)z \rightarrow \infty$$

Only converges at $z = 0$

(iv) $\sum_1^\infty \frac{z^n}{n}$ has $R = 1$, but diverges for $z = 1$ (harmonic series)

What happens for $|z| = 1$ and $z \neq 1$?

Consider

$$\sum_1^\infty \frac{z^n}{n} (1 - z)$$

$$\begin{aligned} S_N &= \sum_1^N \frac{z^n - z^{n+1}}{n} = \sum_1^N \frac{z^n}{n} - \sum_1^N \frac{z^{n+1}}{n} \\ &= \sum_1^N \frac{z^n}{n} - \sum_2^{N+1} \frac{z^n}{n-1} \\ &= z - \frac{z^{N+1}}{N} + \sum_2^{N+1} \frac{-z^n}{n(n-1)} \end{aligned}$$

if $|z| = 1$, $\frac{z^{N+1}}{N} \rightarrow 0$ as $N \rightarrow \infty$ and $\sum_2^\infty \frac{z^n}{n(n-1)}$ converges for all z with $|z| = 1$, $z \neq 1$

(v) $\sum_1^\infty \frac{z^n}{n^2}$, $R = 1$ and converges for all z with $|z| = 1$

(vi) $\sum_0^\infty n z^n$, $R = 1$ but diverges for all $|z| = 1$

Remark. In principle, nothing can be said about $|z| = R$ and each case has to be discussed separately.

Within the radius of convergence ‘life is great’. Power series will “behave as if they were polynomials”

Theorem 4.4. $f(z) = \sum_0^\infty a_n z^n$ has radius of convergence R . Then f is differentiable at all points with $|z| < R$ with

$$f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}$$

Proof. By Lemma 4.5, we may define

$$f'(z) = \sum_{n=1}^\infty n a_n z^{n-1}, \quad |z| < R$$

RTP:

$$\lim_{h \rightarrow 0} \frac{f(z+h) - f(z) - h f'(z)}{h} \rightarrow 0$$

Let

$$\begin{aligned} I &= \frac{f(z+h) - f(z) - h f'(z)}{h} \\ &= \frac{1}{h} \sum_0^\infty a_n ((z+h)^n - z^n - h n z^{n-1}) \\ |I| &= \frac{1}{|h|} \left| \lim_{N \rightarrow \infty} \sum_0^N a_n ((z+h)^n - z^n - h n z^{n-1}) \right| \\ &\leq \frac{1}{|h|} \sum_0^\infty |a_n| |(z+h)^n - z^n - h n z^{n-1}| \\ &\leq \frac{1}{|h|} \sum_2^\infty |a_n| n(n-1) (|z| + |h|)^{n-2} |h|^2 \end{aligned}$$

By Lemma 4.5, for $|h|$ small enough,

$$\sum_2^\infty |a_n| n(n-1) (|z| + |h|)^{n-2}$$

converges to $A(h)$, but $A(h) \leq A(r)$ for $h < r$ and $|z| + r < R$

$$\implies |I| \leq |h| A(h) \leq |h| A(r) \text{ as } h \rightarrow 0$$

Lemma 4.5. If $\sum_0^\infty a_n z^n$ has radius of convergence R , then so do

$$\sum_1^\infty n a_n z^{n-1} \text{ and } \sum_2^\infty n(n-1) a_n z^{n-2}$$

Proof. Take z and R_0 s.t. $0 < |z| < R_0 < R$. Since $a_n R_0^n \rightarrow 0$,

$$\exists K \text{ s.t. } |a_n R_0^n| \leq K \quad \forall n \geq 0$$

Thus

$$\begin{aligned} |a_n n z^{n-1}| &= \frac{n}{|z|} |a_n R_0^n| \left| \frac{z}{R_0} \right|^n \\ &\leq \frac{Kn}{|z|} \left| \frac{z}{R_0} \right|^n \end{aligned}$$

But $\sum n \left| \frac{z}{R_0} \right|^n$ converges by the ratio test

$$\frac{n+1}{n} \left| \frac{z}{R_0} \right|^{n+1} \left| \frac{R_0}{z} \right|^n = \frac{n+1}{n} \left| \frac{z}{R_0} \right| \rightarrow \left| \frac{z}{R_0} \right| < 1$$

if $|z| > R$, the series diverges since $|a_n z^n|$ is unbounded, hence so is $n|a_n z^n|$

Same proof applies to

$$\sum_2^\infty n(n-1) a_n z^{n-2} \quad \square$$

Lemma 4.6.

(i)

$$\binom{n}{r} \leq n(n-1) \binom{n-2}{r-2}$$

for all $2 \leq r \leq n$

(ii)

$$|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z|+|h|)^{n-2}|h|^2 \quad \forall z \in \mathbb{C}, h \in \mathbb{C}$$

Proof.

(i)

$$\begin{aligned} \frac{\binom{n}{r}}{\binom{n-2}{r-2}} &= \frac{n!}{r!(n-r)!} \frac{(r-2)!(n-r)!}{(n-2)!} \\ &= \frac{n(n-1)}{r(r-1)} \\ &\leq n(n-1) \quad \checkmark \end{aligned}$$

(ii)

$$\begin{aligned} (z+h)^n - z^n - nhz^{n-1} &= \sum_{r=2}^n \binom{n}{r} z^{n-r} h^r \quad \text{thus} \\ |(z+h)^n - z^n - nhz^{n-1}| &\leq \sum_{r=2}^n \binom{n}{r} |z|^{n-r} |h|^r \\ &\leq n(n-1) \underbrace{\left[\sum_{r=2}^n \binom{n-2}{r-2} |z|^{n-r} |h|^{r-2} \right]}_{(|z|+|h|)^{n-2}} |h|^2 \end{aligned}$$

4.1 The Standard Functions

We have already seen that

$$\sum_0^{\infty} \frac{z^n}{n!}$$

has $R = \infty$

Define $e : \mathbb{C} \rightarrow \mathbb{C}$

$$e(z) = \sum_0^{\infty} \frac{z^n}{n!}$$

Straight from Theorem 4.4, e is differentiable and $e'(z) = e(z)$

Claim. Observation: If $F : \mathbb{C} \rightarrow \mathbb{C}$ has $f'(z) = 0 \ \forall z \in \mathbb{C}$, then F is constant

Proof. Consider

$$\begin{aligned} g(t) &= F(tz) \\ &= u(t) + iv(t) \end{aligned}$$

By the chain rule:

$$\begin{aligned} g'(t) &= F'(tz)z = 0 = u'(t) + iv'(t) \\ \implies u' &= v' = 0 \end{aligned}$$

Now apply Corollary 3.5 \square

Now let $a, b \in \mathbb{C}$ and consider

$$\begin{aligned} F(z) &= e(a + b - z)e(z) \\ F'(z) &= -e(a + b - z)e(z) + e(a + b - z)^z = 0 \\ \implies F &\text{ is constant} \\ e(a + b - z)e(z) &= F(0) = e(a + b) \end{aligned}$$

Set $z = b$

$$e(a)e(b)e(a + b)$$

Now we restrict $e : \mathbb{R} \rightarrow \mathbb{R}$

Theorem 4.7.

- (i) $e : \mathbb{R} \rightarrow \mathbb{R}$ is everywhere differentiable and $e'(x) = e(x)$
- (ii) $e(x+y) = e(x)e(y)$
- (iii) $e(x) > 0 \ \forall x \in \mathbb{R}$
- (iv) e is strictly increasing
- (v) $e(x) \rightarrow \infty$ as $x \rightarrow \infty$, and $e(x) \rightarrow 0$ as $x \rightarrow -\infty$
- (vi) $e : \mathbb{R} \rightarrow (0, \infty)$ is a bijection

Proof.

- (i) done ✓
- (ii) done ✓
- (iii) Clearly $e(x) > 0 \ \forall x \geq 0$ and $e(0) = 1$
Also

$$e(0) = e(x-x) = e(x)e(-x) \\ \implies e(-x) > 0 \ \forall x > 0$$

- (iv) $e'(x) = e(x) > 0 \implies e$ is strictly increasing

- (v) $e(x) > 1+x$ for $x > 0$

So if $x \rightarrow \infty$, $e(x) \rightarrow \infty$
For $x > 0$ since

$$e(-x) = \frac{1}{e(x)}, \quad e(x) \rightarrow 0 \text{ as } x \rightarrow -\infty$$

- (vi) injectivity: follows right away from being strictly increasing
surjectivity: Take $y \in (0, \infty)$, since $e(x) \rightarrow \infty$ as $x \rightarrow \infty$ and $e(x) \rightarrow 0$ as $x \rightarrow -\infty$,

$$\exists a, b \in \mathbb{R} \text{ s.t. } e(a) < y < e(b)$$

By the intermediate value theorem, $\exists x \in \mathbb{R}$ s.t. $e(x) = y$

Remark.

$$e : (\mathbb{R}, +) \rightarrow ((0, \infty), \times)$$

is a group isomorphism.

Since e is a bijection, consider the inverse function

$$l : (0, \infty) \rightarrow \mathbb{R}$$

Theorem 4.8.

(i)

$$l : (0, \infty) \rightarrow \mathbb{R}$$

is a bijection and

$$l(e(x)) = x \quad \forall x \in \mathbb{R}$$

and

$$e(l(t)) = t \quad \forall t \in (0, \infty)$$

(ii) l is differentiable and

$$l'(t) = \frac{1}{t}$$

(iii)

$$l(xy) = l(x) + l(y) \quad \forall x, y \in (0, \infty)$$

Proof.

(i) obvious from the definition

(ii) Inverse rule (Theorem 3.6):
 l is differentiable and

$$l'(t) = \frac{1}{e(l(t))} = \frac{1}{t}$$

(iii) from IA Groups, if e is an isomorphism, so is its inverse \square Now define for $\alpha \in \mathbb{R}$ and $x > 0$,

$$r_\alpha(x) = e(\alpha l(x))$$

Theorem 4.9. Suppose $x, y > 0$ and $\alpha, \beta \in \mathbb{R}$. Then:

(i)

$$r_\alpha l(xy) = r_\alpha(x)r_\alpha(y)$$

(ii)

$$r_{\alpha+\beta}(x) = r_\alpha(x)r_\beta(x)$$

(iii)

$$r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$$

(iv)

$$r_1(x) = x, \quad r_0(x) = 1$$

Proof.

(i)

$$\begin{aligned} r_\alpha(xy) &= e(\alpha l(xy)) \\ &= e(\alpha l(x) + \alpha l(y)) \\ &= e(\alpha l(x))e(\alpha l(y)) \\ &= r_\alpha(x)r_\alpha(y) \end{aligned}$$

(ii)

$$\begin{aligned} r_{\alpha+\beta}(x) &= e((\alpha + \beta)l(x)) \\ &= e(\alpha l(x))e(\beta l(x)) \\ &= r_\alpha(x)r_\beta(x) \quad \checkmark \end{aligned}$$

(iii)

$$\begin{aligned} r_{\alpha\beta}(x) &= r_\alpha(e(\beta l(x))) \\ &= e(\alpha l(e(\beta l(x)))) \\ &= e(\alpha \beta l(x)) \\ r_{\alpha\beta}(x) &\quad \checkmark \end{aligned}$$

(iv)

$$\begin{aligned} r_1(x) &= e(l(x)) = x \quad \checkmark \\ r_0(x) &= e(0) = 1 \quad \checkmark \quad \square \end{aligned}$$

Equation.

$$r_n(x) = r_{1+\dots+1}(x) = x \cdot x \dots x = x^n$$

$$r_1(x)r_{-1}(x) = r_0(x) = 1$$

So

$$r_{-1}(x) = \frac{1}{x}$$

$$\implies r_{-n}(x) = \frac{1}{x^n}$$

$$(r_{1/q}(x))^q = r_1(x) = x \implies r_{1/q}(x) = x^{1/q}$$

$$r_{p/q} = (r_{1/q}(x))^p = x^{p/q}$$

Thus $r_\alpha(x)$ agrees with $\alpha \in \mathbb{Q}$ as previously defined.

Now we do a “baptism ceremony”

$$\exp(x) = e(x) \quad x \in \mathbb{R}$$

$$\log x = l(x) \quad x \in (0, \infty)$$

$$x^\alpha = r_\alpha(x) \quad \alpha \in \mathbb{R}, \quad x \in (0, \infty)$$

$$e(x) = e(x \log e) = r_x(e) = e^x$$

where

$$e = \sum_0^\infty \frac{1}{n!} = e(1)$$

so $\exp(x)$ is also a power, which we may as well denote e^x

Finally, we compute $(x^\alpha)'$

$$(x^\alpha)' = (e^{\alpha \log x})' = e^{\alpha \log x} \frac{\alpha}{x} = \alpha x^{\alpha-1} \quad \checkmark$$

Note. If we let $f(x) = a^x$, $a > 0$ then

$$f'(x) = (e^{x \log a})' = e^{x \log a} \log a = a^x \log a$$

Remark. “Exponentials beat polynomials”

$$\lim_{x \rightarrow \infty} \frac{e^x}{x^k} = \infty \quad \text{for } k > 0$$

$$e^x = \sum_0^\infty \frac{x^j}{j!} > \frac{x^n}{n!} \quad \text{for } x > 0$$

and pick $n > k$ so

$$\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \rightarrow \infty \quad \text{as } x \rightarrow \infty$$

4.2 Trigonometric Functions

Definition.

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_0^{\infty} \frac{(-1)^k z^{2k}}{(2k)!}$$
$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_0^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}$$

Both power series have infinite radius of convergence and by theorem 4.4., they are differentiable and

$$(\sin z)' = \cos z$$

$$(\cos z)' = -\sin z$$

Notation. Write

$$e^x = e(z)$$

Equation.

$$e^{iz} = \sum_0^{\infty} \frac{(-z)^n}{n!} = \sum_0^{\infty} \frac{(iz)^{2k}}{(2k)!} + \sum_0^{\infty} \frac{(iz)^{2k+1}}{(2k+1)!}$$

$$(iz)^{2k} = (-1)^k z^{2k}, \quad (iz)^{2k+1} = i(-1)^k z^{2k+1}$$

$$\implies e^{iz} = \cos z + i \sin z$$

Similarly,

$$e^{-iz} = \cos z - i \sin z$$

which gives:

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz})$$

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz})$$

From this we get many trigonometric identities:

$$\cos z = \cos(-z), \quad \sin(z) = -\sin z$$

$$\cos(0) = 1, \quad \sin(0) = 0$$

(i)

$$\sin(z + w) = \sin z \cos w + \cos z \sin w$$

(ii)

$$\cos(z + w) = \cos z \cos w - \sin z \sin w \quad z, w \in \mathbb{C}$$

Follows from

$$e^{a+b} = e^a \cdot e^b$$

to prove (ii) write:

$$\begin{aligned} \cos(z + w) &= \frac{1}{2} \left\{ e^{i(z+w)} + e^{-i(z+w)} \right\} \\ &= \frac{1}{2} \left\{ e^{iz} \cdot e^{iw} + e^{-iz} \cdot e^{-iw} \right\} \end{aligned}$$

$$\cos z \cos w - \sin z \sin w = \frac{1}{4} (e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) + \frac{1}{4} (e^{iz} - e^{-iz})(e^{iw} - e^{-iw}) \quad (*)$$

operate to get same result use (*) to get

$$\sin^2 z + \cos^2 z = 1 \quad \forall z \in \mathbb{C}$$

Now if $x \in \mathbb{R}$, then $\sin x, \cos x \in \mathbb{R}$

and (*) gives

$$|\sin x|, |\cos x| \leq 1$$

Warning.

$$\cos(iy) = \frac{1}{2} (e^{-y} + e^y) \quad (y \in \mathbb{R})$$

as $y \rightarrow \infty$, $\cos(iy) \rightarrow \infty$

4.2.1 Periodicity of the Trigonometric Functions

Prop 4.10. There is a smallest positive number ω (where $\sqrt{2} < \frac{\omega}{2} < \sqrt{3}$ s.t.

$$\cos\left(\frac{\omega}{2}\right) = 0$$

Proof. If $0 < x < 2$

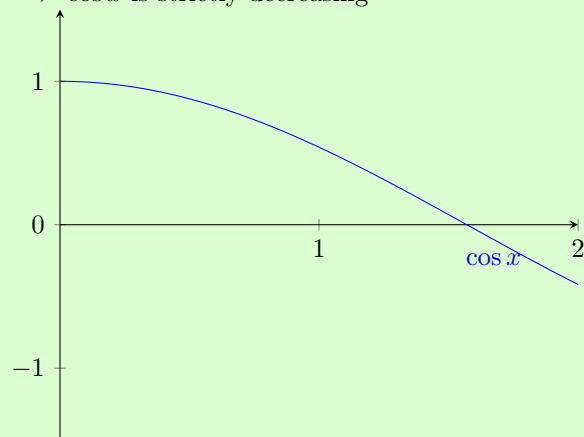
$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0$$

(if $0 < x < 2$ then $\frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)!}$)

So for $0 < x < 2$,

$$(\cos x)' = -\sin x < 0$$

$\Rightarrow \cos x$ is strictly decreasing



We'll show that $\cos \sqrt{2} > 0$ and $\cos \sqrt{3} < 0$. Then by the intermediate value theorem the existence of ω follows.

$$\cos \sqrt{2} = \left(\frac{(\sqrt{2})^4}{4!} - \frac{(\sqrt{2})^6}{6!}\right) + \underbrace{\left(\frac{(\sqrt{2})^8}{8!} - \frac{(\sqrt{2})^{10}}{10!}\right)}_{>0} + \cdots > 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \underbrace{\left(\frac{x^6}{6!} - \frac{x^8}{8!}\right)}_{>0} - \cdots$$

$x = \sqrt{3}$:

$$1 - \frac{3}{2} + \frac{9}{4 \times 3 \times 2} = 1 - \frac{3}{2} + \frac{3}{8} = -\frac{1}{8} < 0$$

$$\Rightarrow \cos \sqrt{3} < 0 \quad \square$$

Corollary 4.11.

$$\sin \frac{\omega}{2} = 1$$

Proof.

$$\sin^2 \frac{\omega}{2} + \cos^2 \frac{\omega}{2} = 1$$

and

$$\sin \frac{\omega}{2} > 0 \quad \square$$

Notation. Now define $\pi = \omega$

Theorem 4.12.

(i)

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \quad \cos\left(z + \frac{\pi}{2}\right) = -\sin z$$

(ii)

$$\sin(z + \pi) = -\sin z, \quad \cos(z + \pi) = -\cos z$$

(iii)

$$\sin(z + 2\pi) = \sin z, \quad \cos(z + 2\pi) = \cos z$$

Proof. immediate from addition formulas and

$$\cos \frac{\pi}{2}, \quad \sin \frac{\pi}{2} = 1 \quad \square$$

Note. This implies

$$\begin{aligned} e^{iz+2\pi i} &= \cos(z + 2\pi) + i \sin(z + 2\pi) \\ &= \cos(z) + i \sin z \\ &= e^{iz} \end{aligned}$$

$$\implies e^z \text{ is periodic with period } 2\pi i$$

Remark. We can “relate the trig functions with geometry”.

Given two vectors $x, y \in \mathbb{R}^2$, define $x \cdot y$ as in vector and matrices

$$x \cdot y = x_1 y_1 + x_2 y_2, \quad x = (x_1, x_2) \text{ and } y = (y_1, y_2)$$

By Cauchy-Swarz:

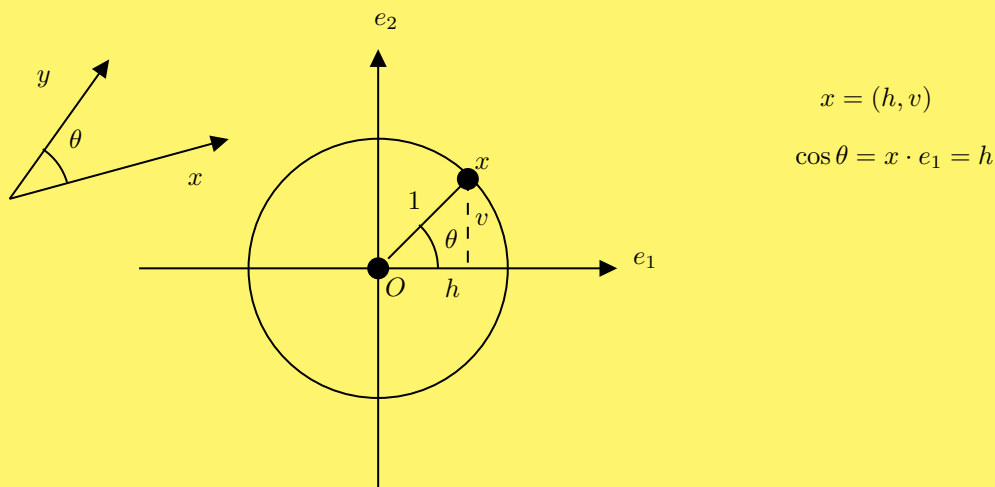
$$|x \cdot y| \leq \|x\| \|y\|$$

Thus if $x \neq 0, y \neq 0$

$$-1 \leq \frac{x \cdot y}{\|x\| \|y\|} \leq 1$$

So we define the angle between x and y as the unique $\theta \in [0, \pi]$ s.t.

$$\cos \theta = \frac{x \cdot y}{\|x\| \|y\|}$$



4.3 Hyperbolic Functions

Definition.

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$\implies \cosh z = \cos(iz), \quad \sinh z = -i \sin(iz)$$

Claim.

$$(\cosh z)' = \sinh z$$

$$(\sinh z)' = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1, \text{ etc.}$$

Proof. Exercise

Note. The rest of the trigonometric functions ($\tan, \cot, \sec, \operatorname{cosec}$) are defined in the usual way

5 Integration

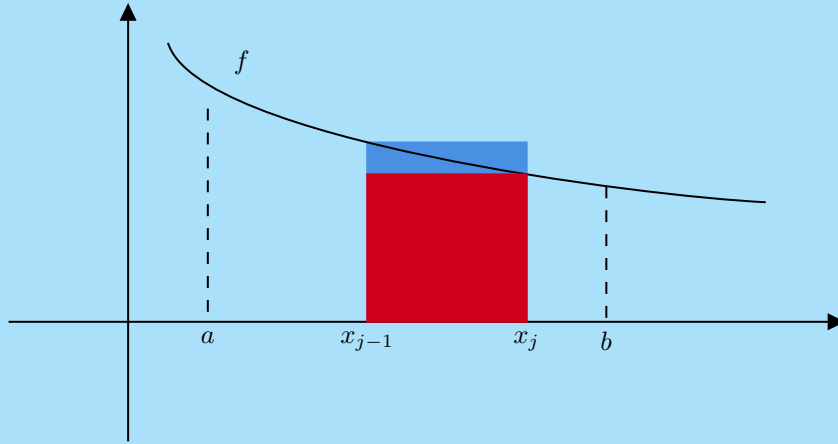
Note. $f : [a, b] \rightarrow \mathbb{R}$ bounded meand:

$$\exists K \text{ s.t. } |f(x)| \leq K, \forall x \in [a, b]$$

Definition. A **dissection** (or partition) \mathcal{D} of $[a, b]$ is a finite subset of $[a, b]$ containing the end points of a and b .

We write

$$\mathcal{D} = \{x_0, x_1, \dots, x_n\} \text{ with} \\ a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$



Definition. We define the **upper sum** and **lower sum** associated with \mathcal{D} by

$$S(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \text{ (upper)}$$

$$s(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \text{ (lower)}$$

Clearly

$$s(d, \mathcal{D}) \leq S(d, \mathcal{D}) \quad \forall \mathcal{D}$$

Lemma 5.1. If \mathcal{D} and \mathcal{D}' are dissections with $\mathcal{D} \subseteq \mathcal{D}'$, then

$$S(d, \mathcal{D}) \geq S(d, \mathcal{D}') \geq s(f, \mathcal{D}') \geq s(f, \mathcal{D})$$

Proof.

$$S(d, \mathcal{D}') \geq s(f, \mathcal{D}')$$

is obvious.

Suppose \mathcal{D}' contains an extra point than \mathcal{D} , let's say $y \in (x_{r-1}, x_r)$ clearly:

$$\begin{aligned} \sup_{x \in [x_{r-1}, y]} f(x), \quad \sup_{x \in [y, x_r]} f(x) &\leq \sup_{x \in [x_{r-1}, x_r]} f(x) \\ \implies (x_r - x_{r-1}) \sup_{x \in [x_{r-1}, x_r]} f(x) &\geq (y - x_{r-1}) \sup_{x \in [x_{r-1}, y]} f(x) + (x_r - y) \sup_{x \in [y, x_r]} f(x) \\ S(f, \mathcal{D}) &\geq s(f, \mathcal{D}') \end{aligned}$$

The same for s and the same if \mathcal{D}' has more extra points than \mathcal{D}

Lemma 5.2. $\mathcal{D}_1, \mathcal{D}_2$ two arbitrary dissections. Then

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_2)$$

So

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

Proof. Take

$$\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1 \mathcal{D}_2$$

ad apply the previous lemma. \square

Definition. The **upper integral** of f is

$$I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$$

(this always exists)

The **lower integral** of f is

$$I_*(f) = \sup_{\mathcal{D}} s(f, \mathcal{D})$$

(this always exists)

Claim. By lemma 5.2,

$$I^*(f) \geq I_*(f)$$

Proof.

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

$$I^*(f) = \inf_{\mathcal{D}_1} S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

$$I^*(f) \geq \sup_{\mathcal{D}_2} s(f, \mathcal{D}_2) \geq s(f, \mathcal{D}_2) = I_*(f)$$

Definition. A bounded function $f : [a, b] \rightarrow \mathbb{R}$ is said to be **Reimann integrable** (or first integrable) if

$$I^*(f) = I_*(f)$$

and we set

$$\int_a^b f(x) \, dx = I^*(f) = I_*(f) = \int_a^b f$$

Example.

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1] \\ 0 & x \notin \mathbb{Q} \cap [0, 1] \end{cases}$$

$f : [0, 1] \rightarrow \mathbb{R}$ is not Reimann integrable

$$\sup_{[x_{j-1}, x_j]} f = 1, \quad \inf_{[x_{j-1}, x_j]} f = 0 \quad \forall \mathcal{D}$$

$$\implies I^*(f) = 1, \text{ but } I_*(f) = 0$$

A useful criterion for integrability:

Theorem 5.3. A bounded function

$$f : [a, b] \rightarrow \mathbb{R}$$

is Riemann integrable iff given $\varepsilon > 0$, $\exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

Proof. For every dissection \mathcal{D} , we have

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - s(f, \mathcal{D})$$

If the given condition holds, then

$$\begin{aligned} 0 \leq I^*(f) - I_*(f) &\leq S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon \quad \forall \varepsilon > 0 \\ \implies I^*(f) &= I_*(f) \end{aligned}$$

Conversely, if f is integrable, by definition of sup, inf, there are partitions \mathcal{D}_1 and \mathcal{D}_2 s.t.

$$\int_a^b f \, dx - \frac{\varepsilon}{2} = I_*(f) - \frac{\varepsilon}{2} < s(f, \mathcal{D}_1)$$

$$S(f, \mathcal{D}_2) < I^*(f) + \frac{\varepsilon}{2} = \int_a^b f \, dx + \frac{\varepsilon}{2}$$

By lemma 5.1,

$$(\mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2)$$

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) - s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \leq S(f, \mathcal{D}_2) - s(f, \mathcal{D}_1) < \int_a^b f \, dx + \frac{\varepsilon}{2} - \int_a^b f \, dx + \frac{\varepsilon}{2} = \varepsilon \quad \square$$

We now use this condition to show that monotonic and continuous functions (separately) are integrable.

Remark. Monotonic and continuous are bounded (thm 2.6 for the case of continuous functions)

Theorem 5.4. $f : [a, b] \rightarrow \mathbb{R}$ monotonic. Then f is integrable

Proof. Suppose f is increasing (same proof for f decreasing)

Then

$$\sup_{x \in [x_{j-1}, x_j]} = f(x_j)$$

$$\inf_{x \in [x_{j-1}, x_j]} = f(x_{j-1})$$

Thus

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1}) [f(x_j) - f(x_{j-1})]$$

Now choose

$$\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \dots, b\}$$

$$x_j = a + \frac{(b-a)j}{n}, \quad 0 \leq j \leq n$$

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \frac{(b-a)}{n} (f(b) - f(a))$$

Take n large enough s.t.

$$\frac{b-a}{n} (f(b) - f(a)) < \varepsilon$$

and use Theorem 5.3 \square

5.0.1 Continuous Functions

Note. First we need an auxiliary lemma

Lemma 5.5. $f : [a, b] \rightarrow \mathbb{R}$ continuous. Then

$$\text{given } \varepsilon > 0, \text{ exists } \delta > 0 \text{ s.t. } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(uniform continuity)

Note. The point is δ works $\forall x, y$ as long as $|x - y| < \delta$
(in the definition of continuity of f at x , $\delta = \delta(x)$)

Proof. Suppose the claim is false. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, we can find $x, y \in [a, b]$ s.t.
 $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$
Take $\delta = \frac{1}{n}$, to get x_n, y_n with

$$|x_n - y_n| < \frac{1}{n}, \text{ but } |f(x_n) - f(y_n)| \geq \varepsilon$$

By Bolzano-Weierstrass, $\exists x_{n_k} > C$

$$|y_{n_k} - C| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - C| \rightarrow 0$$

(both parts of sum converge to 0)

But

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$$
$$0 \geq \varepsilon \times \square$$

Theorem 5.6. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then f is Riemann integrable.

Proof. given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta$

$$\implies |f(x) - f(y)| < \varepsilon$$

Let $\mathcal{D} = \{a + \frac{(b-a)j}{n}, j = 0, 1, \dots, n\}$

Choose n large enough s.t.

$$\frac{b-a}{n} < \delta$$

Then for $x, y \in [x_{j-1}, x_j]$

$$|f(x) - f(y)| < \varepsilon \quad (*)$$

since

$$|x - y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$$

This means that

$$\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) = f(p_j) - f(q_j) \quad p_j, q_j \in [x_{j-1}, x_j]$$

(max and min exist due to continuity)

$$\begin{aligned} \implies S(f, \mathcal{D}) - s(f, \mathcal{D}) &= \sum_{j=1}^n (x_j - x_{j-1}) \left[\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) \right] \\ &= \sum_{j=1}^n \frac{(b-a)}{n} \underbrace{(f(p_j) - f(q_j))}_{< \varepsilon \text{ by } (*)} \\ &< \varepsilon(b-a) \end{aligned}$$

Now use Theorem 5.3 \square

Remark. More complicated functions can be Riemann integrable

Example. $f : [0, 1] \rightarrow \mathbb{R}$

$$f(x) = \begin{cases} 1/q, & x = p/q \in (0, 1] \text{ in its lowest form} \\ 0, & \text{otherwise} \end{cases}$$

Clearly $s(f, \mathcal{D}) = 0 \forall \mathcal{D}$.

We will show that given $\varepsilon > 0, \exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) < \varepsilon$$

This implies f is integrable with

$$\int_0^1 f = 0$$

Take $N \in \mathbb{N}$ s.t.

$$\frac{1}{N} < \frac{\varepsilon}{2}$$

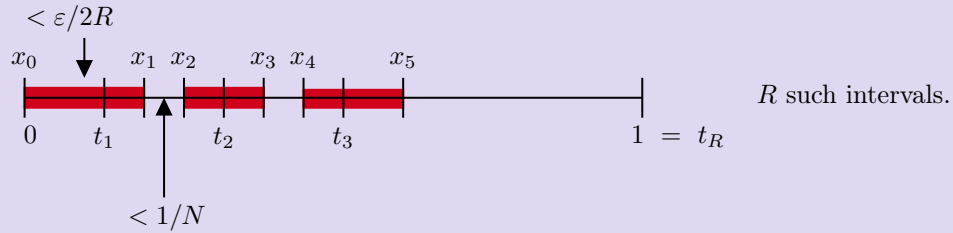
Consider the set

$$\{x \in [0, 1] : f(x) \geq 1/N\} = \{p/q : 1 \leq q \leq N \text{ and } 1 \leq p \leq q\}$$

This is a finite set $0 < t_1 < t_2 < \dots < t_R = 1$

Consider a dissection of $[a, b]$ s.t.

- (i) Each $t_k, 1 \leq k \leq R$ is in some $[x_{j-1}, x_j]$
- (ii) $\forall k$, the unique interval containing t_k has length at most $\varepsilon/2R$



Not: $f \leq 1$ everywhere

$$S(f, \mathcal{D}) \leq \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon$$

5.1 Elementary Properties of the Integral

Claim. For f, g bounded and integrable on $[a, b]$:

(i) If $f \leq g$ on $[a, b]$, then

$$\int_a^b f \leq \int_a^b g$$

(ii) $f + g$ is integrable on $[a, b]$ and

$$\int_a^b f + g = \int_a^b f + \int_a^b g$$

(iii) For any constant k , kf is integrable and

$$\int_a^b kg = k \int_a^b f$$

(iv) $|f|$ is integrable and

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

(v) The product fg is integrable

Proof.

(i) if $f \leq g$, then

$$\int_a^b f = I^*(f) \leq S(f, \mathcal{D}) \leq S(g, \mathcal{D})$$

hence

$$\int_a^b f = I^*(f) \leq I^*(g) = \int_a^b g$$

(ii)

$$\begin{aligned} \sup_{[x_{j-1}, x_j]} (f + g) &\leq \sup_{[x_{j-1}, x_j]} f + \sup_{[x_{j-1}, x_j]} g \\ \implies S(f + g, \mathcal{D}) &\leq S(f, \mathcal{D}) + S(g, \mathcal{D}) \end{aligned}$$

Now take two dissections \mathcal{D}_1 and \mathcal{D}_2

$$\begin{aligned} I^*(f + g) &\leq S(f + g, \mathcal{D}_1 \cup \mathcal{D}_2) \leq S(f, \mathcal{D}_1 \cup \mathcal{D}_2) + S(g, \mathcal{D}_1 \cup \mathcal{D}_2) \\ &\leq S(f, \mathcal{D}_1) + S(g, \mathcal{D}_2) \end{aligned}$$

last from lemma 5.1. Fix \mathcal{D}_1 and inf over \mathcal{D}_2 to get

$$I^*(f + g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g$$

Similarly

$$\int_a^b f + \int_a^b g \leq I_*(f + g)$$

$\implies f + g$ is integrable with integral equal to the sum of the integrals.

(iii) Exercise!

Claim (cont.).

Proof (cont.).

(iv) Consider

$$f_+(x) = \max(f(x), 0)$$

$$\sup_{[x_{j-1}, x_j]} f_+ - \inf_{[x_{j-1}, x_j]} f_+ \leq \sup_{[x_{j-1}, x_j]} f - \inf_{[x_{j-1}, x_j]} f$$

(can check)

and we know that given $\varepsilon > 0$, $\exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

$$\implies S(f_+, \mathcal{D}) - s(f_+, \mathcal{D}) < \varepsilon$$

$\implies f_+$ is integrable

But $|f| = 2f_+ - f$ By (ii) and (iii), $|f|$ is integrable.

Since $-|f| \leq f \leq |f|$, we use property (i) to see

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

(v) Take f integrable and ≥ 0

Then

$$\sup_{[x_{j-1}, x_j]} f^2 = \left(\overbrace{\sup_{[x_{j-1}, x_j]} f}^{M_j} \right)^2$$

$$\inf_{[x_{j-1}, x_j]} f^2 = \left(\underbrace{\inf_{[x_{j-1}, x_j]} f}_{m_j} \right)^2$$

Thus

$$S(f^2, \mathcal{D}) - s(f^2, \mathcal{D}) = \sum_{j=1}^n (x_j - x_{j-1})(M_j^2 - m_j^2)$$

$$= \sum_{j=1}^n (x_j - x_{j-1})(M_j + m_j)(M_j - m_j)$$

$$\leq 2K(S(f, \mathcal{D}) - s(f, \mathcal{D}))$$

using $|f(x)| \leq K \forall x \in [a, b]$

Using the criterion in Theorem 5.3, we deduce that f^2 is integrable.

Now take any f , then $|f| \geq 0$ and is integrable. Since $f^2 = |f|^2$.

We deduce that f^2 is integrable for any f

Finally for fg , note:

$$4fg = (f+g)^2 - (f-g)^2$$

$$\implies fg \text{ is integrable given what we proved}$$

Claim (6). f is integrable on $[ab]$. If $a < c < b$, then f is integrable over $[a, c]$ and $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Conversely if f is integrable over $[a, c]$ and $[c, b]$, then f is integrable over $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. We first make two observations:

if \mathcal{D}_1 is a dissection of $[a, c]$ and \mathcal{D}_2 is a dissection of $[b, c]$, then

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$$

is a dissection of $[a, b]$ and

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2) \quad (*_1)$$

Also if \mathcal{D} is a dissection of $[a, b]$, then

$$\begin{aligned} S(f, \mathcal{D}) &\geq S(f, \mathcal{D} \cup \{c\}) \\ &= S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2) \end{aligned} \quad (*_2)$$

where \mathcal{D}_1 dissects $[a, c]$ and \mathcal{D}_2 dissects $[a, b]$

$$(*_1) \implies I^*(f) \leq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$$

$$(*_2) \implies I^*(f) \geq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$$

Similarly

$$I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[c,b]})$$

Thus

$$0 \leq I^*(f) - I_*(f) = \underbrace{I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})}_{\geq 0} + \underbrace{I^*(f|_{[c,b]}) - I_*(f|_{[c,b]})}_{\geq 0}$$

From this, claim follows right away. \square

Notation. We have a convention that is if $a > b$, then

$$\int_a^b f = - \int_b^a f$$

if $a = b$, we agree that its value is zero.

With this convention, if $|f| \leq K$, then

$$\left| \int_c^b f \right| \leq K|b - a|$$

(from property (4) and convention)

5.2 The Fundamental Theorem of Calculus (FTC)

$f : [a, b] \rightarrow \mathbb{R}$ bounded and integrable. Write

$$F(x) = \int_a^x f(t) dt, \quad x \in [a, b]$$

Theorem 5.7. F is continuous

Proof.

$$F(x+h) - F(x) = \int_x^{x+h} f(t) dt$$

$$|F(x+h) - F(x)| = \left| \int_x^{x+h} f(t) dt \right| \leq K|h|$$

if $|f(t)| \leq K, \forall t \in [a, b]$. Now let $h \rightarrow 0$ and we are done. \square

Theorem 5.8 (FTC). If in addition f is continuous at x , then F is differentiable at x and

$$F'(x) = f(x)$$

Proof. We need to consider $(x+h) \in [a, b]$ & $h \neq 0$

$$\begin{aligned} \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &= \frac{1}{|h|} \left| \int_x^{x+h} f(t) dt - hf(x) \right| \\ &= \frac{1}{|h|} \left| \int_x^{x+h} [f(t) - f(x)] dt \right| \end{aligned}$$

f continuous at x , means that given $\varepsilon > 0$, $\exists \delta > 0$ s.t. if $|t - x| < \delta$ then

$$|f(t) - f(x)| < \varepsilon$$

IF $|h| < \delta$, we can write

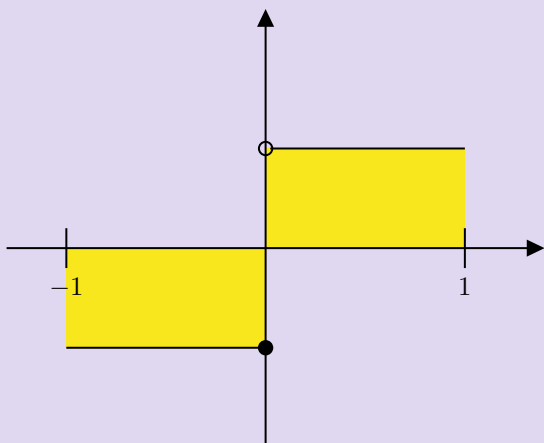
$$\begin{aligned} \frac{1}{|h|} \left| \int_x^{x+h} [f(t) - f(x)] dt \right| &\leq \frac{1}{|h|} \varepsilon |h| \\ &= \varepsilon \end{aligned}$$

This means

$$\lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h} = f(x) \quad \square$$

Example.

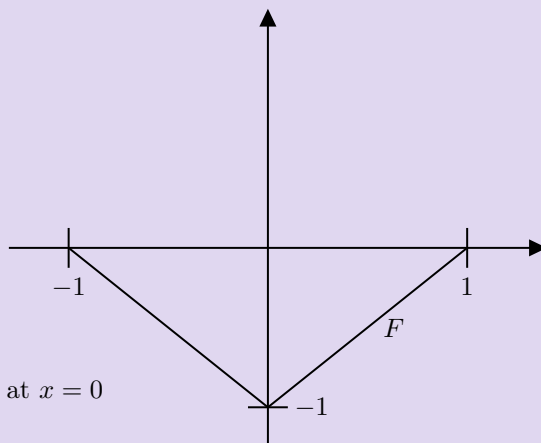
$$f(x) = \begin{cases} -1 & x \in [-1, 0) \\ 1 & x \in (0, 1] \end{cases}$$



monotonic \implies integrable

$$f(x) = \begin{cases} -x - 1 & x \leq 0 \\ x - 1 & x \geq 0 \end{cases}$$

$$F(x) = -1 + |x|$$



F not differentiable at $x = 0$

Corollary 5.9 (integration is the inverse of differentiation). If $f = g'$ is continuous on $[a, b]$, then

$$\int_a^x f(t) \, dt = g(x) - g(a) \quad \forall x \in [a, b]$$

Proof. From Theorem 5.8, $F - g$ has zero derivative in $[a, b] \implies F - g$ is constant and since $F(a) = 0$,

$$F(x) = g(x) - g(a) \quad \square$$

Notation. Every continuous function has an indefinite integral or anti-derivative written

$$\int f(x) \, dx$$

which is determined up to a constant.

Remark. We have solved the ODE:

$$\begin{cases} y'(x) = f(x) \\ y(a) = y_0 \end{cases}$$

Corollary 5.10 (integration by parts). Suppose f' and g' exist and are continuous on $[a, b]$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof. By the product rule,

$$(fg)' = f'g + fg'$$

By 5.9,

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg' \quad \square$$

Corollary 5.11 (integration by substitution). Let $g : [\alpha, \beta] \rightarrow [a, b]$ with $g(\alpha) = a$ and $g(\beta) = b$, g' exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous. Then

$$\int_a^b f(x) \, dx = \int_{\alpha}^{\beta} f(g(t))g'(t) \, dt$$

Proof. Set

$$F(x) = \int_a^x f(t) \, dt$$

as before. Let $h(t) = F(g(t))$ defined since g takes values in $[a, b]$. Then

$$\begin{aligned} \int_{\alpha}^{\beta} f(g(t))g'(t) \, dt &\stackrel{\text{FTC}}{=} \int_{\alpha}^{\beta} F'(g(t))g'(t) \, dt \\ &\stackrel{\text{chain rule}}{=} \int_{\alpha}^{\beta} h'(t) \, dt \\ &= h(\beta) - h(\alpha) \\ &= F(b) - F(a) \\ &= \int_a^b f(x) \, dx \quad \square \end{aligned}$$

Theorem 5.12 (Taylor's theorem with remainder an integral). Let $f^{(n)}(x)$ be continuous for $x \in [0, h]$. Then

$$f(h) = f(0) + \cdots + \frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) dt$$

Proof. By substituting $u = th$

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u) du$$

Integrating by parts now, we get:

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \underbrace{\frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u) du}_{R_{n-1}}$$

If we integrate by parts $n-1$ times, we arrive at:

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - hf'(0) + \underbrace{\int_0^h f'(u) du}_{f(h)-f(0)} \quad \square$$

Remark. Now we can get the Cauchy & Lagrange form of the remainder.

However, note that the proof above uses continuity of $f^{(n)}$ not just mere existence as in section 3. But first need to prove:

Theorem 5.13. $f, g : [a, b] \rightarrow \mathbb{R}$ continuous with $g(x) \neq 0 \forall x \in (a, b)$. Then $\exists c \in (a, b)$ s.t.

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx$$

Proof. We're going to use Cauchy's MVT (Theorem 3.7)

$$F(x) = \int_a^x fg, \quad G(x) = \int_a^x g$$

Theorem 3.7 $\implies \exists c \in (a, b)$ s.t.

$$(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a))$$

$$\left(\int_a^b fg \right) g(c) = f(c)g(c) \int_a^b g$$

if $g(c) \neq 0$, we simplify and we're done \square

Note. if we take $g(x) \equiv 1$, we get

$$\int_a^b f(x) dx = f(c)(b - a)$$

Claim. We can get the Cauchy & Lagrange form of the remainder from Taylor's theorem with remainder (given continuity of $f^{(n)}$)

Proof. Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th) dt$$

First we use Theorem 5.13 with $g \equiv 1$, to get

$$R_n = \frac{h^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(\theta h) \theta, \theta \in (0, 1)$$

Which is Cauchy's form of the remainder!

To get Lagrange, we use Theorem 5.13 with $g(t) = (1-t)^{n-1}$ which is > 0 for $t \in (0, 1)$

$$\implies \exists \theta \in (0, 1) \text{ s.t. } R_n = \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \underbrace{\left[\int_0^1 (1-t)^{n-1} dt \right]}_{=1/n}$$

$$\int_0^1 (1-t)^{n-1} dt = -\frac{(1-t)^n}{n} \Big|_0^1 = \frac{1}{n}$$

$$\implies R_n = \frac{h^n}{n!} f^{(n)}(\theta h), \quad \theta \in (0, 1)$$

which is Lagrange's form of the remainder!

5.3 Improper Integrals

Definition. Suppose $f : [a, \infty) \rightarrow \mathbb{R}$ integrable (and bounded) on every interval $[a, R]$ and that as $R \rightarrow \infty$

$$\int_a^R f(x) \, dx \rightarrow l$$

Then we say that $\int_a^\infty f(x) \, dx$ exists or converges and that its value is l . If $\int_a^R f(x) \, dx$ does not tend to a limit, we say that $\int_a^\infty f(x) \, dx$ diverges.

A similar definition applies to $\int_{-\infty}^a f(x) \, dx$. If

$$\int_a^\infty f(x) \, dx = l_1$$

and

$$\int_{-\infty}^a f(x) \, dx = l_2$$

we write

$$\int_{-\infty}^\infty f(x) \, dx = l_1 + l_2$$

(independent of the particular value of a)

Warning. This is not the same as saying that

$$\lim_{R \rightarrow \infty} \int_{-R}^R f(x) \, dx$$

exists. It is stronger: e.g.

$$\int_{-R}^R x \, dx = 0$$

Example.

$$\int_1^\infty \frac{dx}{x^k} \text{ converges iff } k > 1$$

Indeed, if $k \neq 1$,

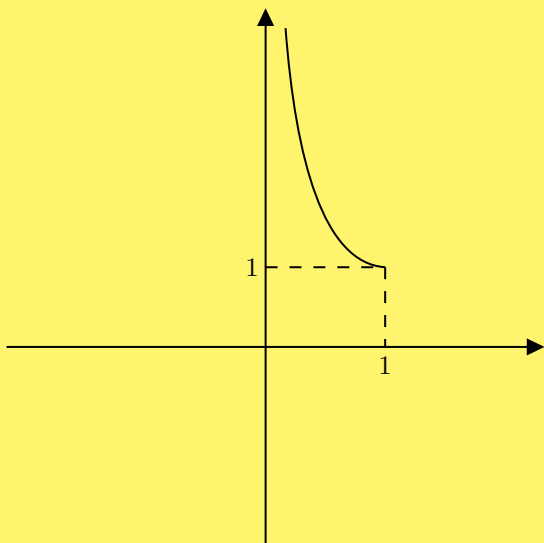
$$\int_1^R \frac{dx}{x^k} = \left. \frac{x^{1-k}}{1-k} \right|_1^R = \frac{R^{1-k}}{1-k}$$

and as $R \rightarrow \infty$, this limit is finite iff $k > 1$ (and equals $-\frac{1}{1-k}$) if $k = 1$,

$$\int_1^R \frac{dx}{x} = \log R \rightarrow \infty$$

Remark. $1/\sqrt{x}$ continuous on $[\delta, 1]$, for any $\delta > 0$. and

$$\int_{\delta}^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x} \Big|_{\delta}^1 = 2 - 2\sqrt{\delta} \rightarrow 2 \text{ as } \delta \rightarrow 0$$



$1/\sqrt{x}$ is unbounded on $[0, 1]$

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{\delta \rightarrow 0} \int_{\delta}^1 \frac{dx}{\sqrt{x}} = 2$$

Exercise: give a general definition

$$\int_0^1 \frac{dx}{x} = \lim_{\delta \rightarrow 0} \int_{\delta}^1 \frac{dx}{x} = \lim_{\delta \rightarrow 0} \log x \Big|_{\delta}^1 = \log 1 - \log \delta$$

limit does not exist as $\delta \rightarrow 0$

Remark. If $f \geq 0$ and $g \geq 0$ for $x \geq a$ and $f(x) \leq Kg(x)$, K constant $x \geq a$, then

$$\int_a^\infty g \text{ converges} \implies \int_a^\infty f \text{ converges}$$

and

$$\int_a^\infty f \leq K \int_a^\infty g$$

Just note that

$$\int_a^R f \leq K \int_a^R g$$

The function $R \rightarrow \int_a^R f$ is increasing ($f \geq 0$) and bounded above ($\int_a^\infty g$ converges)
Take

$$l = \sup_{R \geq a} \int_a^R f < \infty$$

Now check that

$$\lim_{R \rightarrow \infty} \int_a^R f = l$$

given $\varepsilon > 0, \exists R_0$ s.t.

$$\int_a^{R_0} f \geq l - \varepsilon$$

Thus

$$\begin{aligned} \forall R \geq R_0, \int_a^R f &\geq \int_a^{R_0} f \geq l - \varepsilon \\ \implies 0 &\leq l - \int_a^R f \leq \varepsilon \checkmark \end{aligned}$$

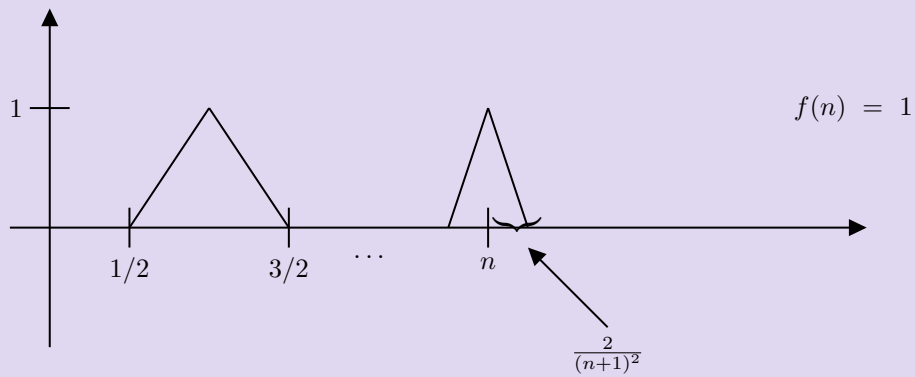
Example.

$$\begin{aligned} &\int_0^\infty e^{-x^2/2} dx \\ &e^{-x^2/2} \leq e^{-x/2}, x \geq 1 \\ &\int_1^R e^{-x/2} dx = \frac{1}{2}[e^{-1/2} - e^{-R/2}] \rightarrow \frac{e^{-1/2}}{2} \\ &\implies \int_0^\infty e^{-x^2/2} dx \text{ converges} \end{aligned}$$

Remark. We know that if $\sum a_n$ converges, then $a_n \rightarrow 0$. We have to be careful with improper integrals.

$\int_a^\infty f$ converges may not imply that $f \rightarrow 0$

Example.



$$\text{area}(\Delta) = \frac{2}{(n+1)^2}$$

5.4 The Integral Test

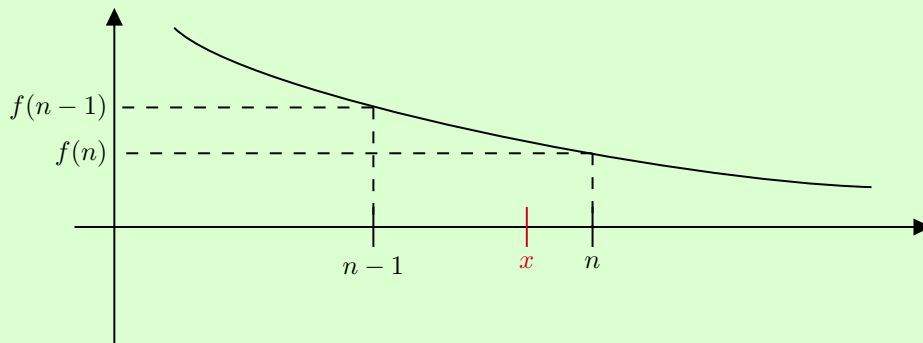
Theorem 5.14 (integral test). Let $f(x)$ be a positive decreasing function for $x \geq 1$. Then

- (i) The integral $\int_1^\infty f(x) dx$ and the series $\sum_1^\infty f(n)$ both converge or diverge.
- (ii) As $n \rightarrow \infty$,

$$\sum_{r=1}^n f(r) - \int_1^n f(x) dx$$

tends to a limit l s.t. $0 \leq l \leq f(1)$

Proof.



(f decreasing $\implies f$ integrable on every bounded subinterval by Theorem 5.4)

If $n-1 \leq x \leq n$, then

$$\begin{aligned} f(n-1) &\geq f(x) \geq f(n) \\ \implies f(n-1) &\geq \int_{n-1}^n f(x) dx \geq f(n) \end{aligned} \quad (*)$$

Adding:

$$\sum_{r=1}^{n-1} f(r) \geq \int_1^n f(x) dx \geq \sum_{r=2}^n f(r) \quad (**)$$

From this claim (i) is obvious.

For the proof of (ii) set

$$\phi(n) = \sum_1^n f(r) - \int_1^n f(x) dx$$

Then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^n f(x) dx \leq 0$$

using (*).

Also from (**),

$$0 \leq \phi(n) \leq f(1)$$

thus $\phi(n)$ is decreasing and tends to a limit l s.t.

$$0 \leq l \leq f(1) \quad \square$$

Examples.

(i)

$$\sum_1^{\infty} \frac{1}{n^k} \text{ converges iff } k > 1$$

We saw that

$$\int_1^{\infty} \frac{1}{x^k} \text{ converges iff } k > 1$$

so we just apply the integral test.

(ii)

$$\sum_1^{\infty} \frac{1}{n \log n}, \quad f(x) = \frac{1}{x \log x}, \quad x \geq 2$$

$$\int_2^R \frac{dx}{x \log x} = \log(\log x) \Big|_2^R$$

$$\log(\log R) - \log(\log 2) \rightarrow \infty \text{ as } R \rightarrow \infty$$

then by the integral test

$$\sum_2^{\infty} \frac{1}{n \log n} \text{ diverges}$$

Corollary 5.15 (Euler's constant). As $n \rightarrow \infty$,

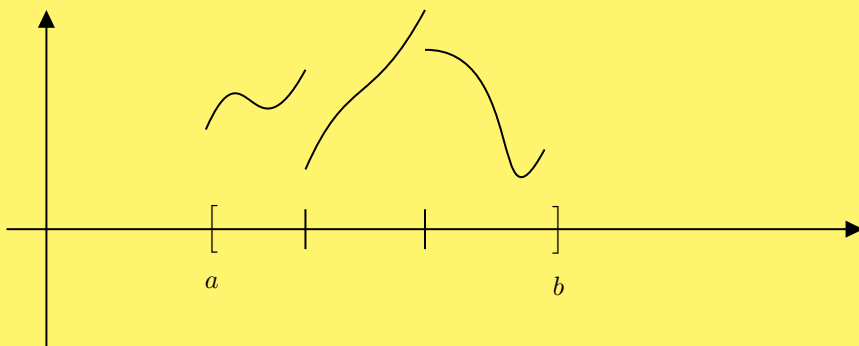
$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \rightarrow \gamma$$

with $0 \leq \gamma \leq 1$

Proof. Set $f(x) = 1/x$ and use Theorem 5.14 \square

Remark. We have an open problem: is γ irrational?
($\gamma \sim 0.577$)

Remark. We have seen: monotone functions and continuous functions are integrable
We can generalise this a bit and say that piece-wise continuous functions are integrable



Definition. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be **piece-wise continuous** if there is a dissection $\mathcal{D} = \{x_0 = a, x_1, \dots, x_n = b\}$ s.t.

- (i) f is continuous on $(x_{j-1}, x_j) \forall j$
- (ii) the one-sided limits

$$\lim_{x \rightarrow x_{j-1}^+} f(x), \quad \lim_{x \rightarrow x_j^-} f(x) \text{ exist}$$

5.5 Characterization for Riemann integrability (Non-Examinable)

Note. It is now an exercise to check that f is Riemann integrable: first check that $f|_{[x_{j-1}, x_j]}$ is integrable for each j (the values of f at the end points won't really matter) and use additivity of domain (property (6))

Note. Q: How large can the discontinuity of f be while f is still Riemann integrable? Recall the example

$$f(x) = \begin{cases} 1/q & x = p/q \\ 0 & \text{otherwise} \end{cases}$$

The question has been answered by Henri Lebesgue:

Characterization for Riemann integrability:

$f : [a, b] \rightarrow \mathbb{R}$ bounded. Then f is Riemann integrable iff the set of discontinuity points has measure zero.

Definition. Let $l(I)$ be the length of an interval I .

A subset $A \subseteq \mathbb{R}$ is said to have **measure zero** if for each $\varepsilon > 0 \exists$ a countable family of intervals st.

$$A \subseteq \bigcup_{j=1}^{\infty} I_j$$

and

$$\sum_j l(I_j) < \varepsilon$$

Lemma 5.16.

- (i) Every countable set has measure zero.
- (ii) if B has measure zero and $A \subseteq B$, the A has measure zero.
- (iii) if A_k has measure zero $\forall k \in \mathbb{N}$ then $\bigcup_{k \in \mathbb{N}} A_k$ also has measure zero.

Note. The proof of Lebesgue's criterion uses the concept of oscillation of f :
 I interval:

$$\omega_f(I) = \sup_I f - \inf_I f$$

Oscillation at a point

$$\omega_f(x) = \lim_{\varepsilon \rightarrow 0} \omega_f(x - \varepsilon, x + \varepsilon)$$

Proof (Sketch).

$$\begin{aligned} D &= \{x \in [a, b] : f \text{ discontinuous at } x\} \\ &= \{x : \omega_f(x) > 0\} \end{aligned}$$

\implies RTP: D has measure zero.

$$N(\alpha) = \{x : \omega_f(x) \geq \alpha\}$$

$$D = \bigcup_{k=1}^{\infty} N(1/k)$$

We'll show that for fixed α , $N(\alpha)$ has measure zero.

Let $\varepsilon > 0$, $\exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \frac{\varepsilon \alpha}{2}$$

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \sum_{j=1}^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1})$$

$$F = \{j : (x_{j-1}, x_j) \cap N(\alpha) \neq \emptyset\}$$

then for each $j \in F$,

$$\omega_f([x_{j-1}, x_j]) \geq \alpha$$

$$\alpha \sum_{j \in F} (x_j - x_{j-1}) \leq \sum_{j \in F} \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) < \frac{\varepsilon \alpha}{2}$$

$$\implies \sum_{j \in F} (x_j - x_{j-1}) < \frac{\varepsilon}{2}$$

These cover $N(\alpha)$ except perhaps for $\{x_0, x_1, x_n\}$. But these can be covered by intervals of total length $< \frac{\varepsilon}{2}$

$\implies N(\alpha)$ can be covered by intervals of total length $< \varepsilon$ ✓

Lemma 5.17 (cont.).

Proof (cont.). \Leftarrow : let $\varepsilon > 0$ be given

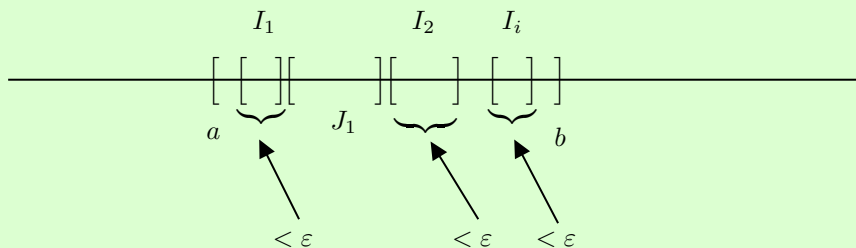
$$N(\varepsilon) \subseteq D$$

so $N(\varepsilon)$ has measure zero. It is closed and bounded, \Rightarrow it can be covered with finitely many open sets of total length $< \varepsilon$

$$N(\varepsilon) \subseteq \bigcup_{i=1}^m U_i$$

let $I_i = \overline{U_i}$ (closure = adding end points)

wlog, I_i do not overlap



The complement

$$K = [a, b] \setminus \bigcup_{i=1}^m U_i$$

is compact so it can be covered by finitely many disjoint closed intervals J_i s.t.

$$\omega_f(J_j) < \varepsilon$$

Now the I_i 's and J_j 's give a dissection for $[a, b]$ s.t.

$$\begin{aligned} \sum_1^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) &= \sum_{i=1}^m \underbrace{\omega_f(I_i)}_{\leq 2K} l(I_i) + \sum_{j=1}^k \underbrace{\omega_f(J_j)}_{< \varepsilon} l(J_j) \\ &\leq 2K \sum_1^m l(I_i) + \varepsilon(b - a) \\ &\leq 2K\varepsilon + \varepsilon(b - a) \quad \square \end{aligned}$$

(using $|f| \leq K$)

Lemma 5.18. f is continuous at x iff $\omega_f(x) = 0$

Proof. Exercise.