Stochastic Financial Models 9

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1 Harder direction of FTAP continued

Let $F(\theta) = \mathbb{E}[e^{-\theta^T \xi}]$ where $\xi = S_1 - (1+r)S_0$.

We will assume without loss that the random variables $\{\xi^1, \ldots, \xi^d\}$ are linearly independent. (Otherwise, we could consider a sub-market where the asset prices are linearly independent. Since there is no arbitrage in the given market, there is no arbitrage in the sub-market.)

We may assume $\|\theta_n\| \uparrow \infty$. Let

$$\varphi_n = \frac{\theta_n}{\|\theta_n\|}$$

Note $(\varphi_n)_n$ is bounded, so by the Bolzona–Weierstrass theorem, there exists a convergent subsequence. By passing to that subsequence, we assume $\varphi_n \to \varphi_0$. Note $\|\varphi_0\| = 1$.

We will show that $\varphi_0^\top \xi \geq 0$ almost surely. By no arbitrage, this will imply that $\varphi_0^\top \xi = 0$ almost surely. And by linear independence, this would show that $\varphi_0 = 0$, contradicting $\|\varphi_0\| = 1$.

Now to show $\varphi^\top \xi \geq 0$ almost surely, that is $\mathbb{P}(\varphi_0^\top \xi < 0) = 0$. By the continuity, it is enough to show $\mathbb{P}(\varphi_0^\top \xi < -\varepsilon, \|\xi\| < r) = 0$ for every $\varepsilon > 0, r > 0$. So fix ε, r. We can pick N such that $\|\varphi_n - \varphi_0\| \leq \frac{\varepsilon}{2r}$ for $n \geq N$. Note on the event $\{\varphi_0^\top \xi < -\varepsilon, \|\xi\| < r\}$ for $n \geq N$ we have

$$\begin{aligned}
\varphi_n^\top \xi &\leq \|\varphi_n - \varphi_0\| \|\xi\| + \varphi_0^\top \xi \\
&\leq -\frac{\varepsilon}{2}
\end{aligned}$$

by Cauchy–Schwarz.

Since $\theta = 0$ is not optimal we have for $n \geq N$ that

$$\begin{aligned}
1 = F(0) &\geq F(\theta) \\
&= \mathbb{E}[e^{-\theta_n^\top \xi}] \\
&\geq \mathbb{E}[(e^{-\varphi_n^\top \xi})^{\|\theta_n\|} \mathbb{1}_{\{\varphi_0^\top \xi < -\varepsilon, \|\xi\| < r\}}] \\
&\geq e^{\frac{1}{2}\|\theta_n\|\varepsilon} \mathbb{P}(\varphi_0^\top \xi < -\varepsilon, \|\xi\| < r)
\end{aligned}$$

so $\mathbb{P}(\varphi_0^\top \xi < -\varepsilon, \|\xi\| < r) \leq e^{-\frac{1}{2}\|\theta_n\|\varepsilon} \to 0$

$\square$

Remark on examining. The details of the above proof should individually be accessible to someone in Part II, and could be examined. However, the proof in its entirety is bit longer than usual bookwork questions for this course, so don't worry too much about memorising it.

2 No-arbitrage pricing

Given a market of tradable assets and a contingent claim with payout Y, how can you assign an initial price π? Possible solutions

- Given U and X_0, find the indifference price.
- Given U and X_0, find the marginal utility price.
- Pick π such that the augmented market (consisting of the original market and the contingent claim) has no arbitrage.

Theorem. *Suppose that the original market has no arbitrage. There is no arbitrage in the augmented market if and only if there exists a risk-neutral measure for the original market such that*

$$\pi = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(Y)$$

In particular, the set of no-arbitrage prices of the claim is an interval.

as arb in augmented market implies arb in original market

Proof. The first part is just the fundamental theorem of asset pricing. The second part. Fix two risk neutral measures $\mathbb{Q}_0$ and $\mathbb{Q}_1$ and let $\mathbb{Q}_p$ have density

$$\frac{d\mathbb{Q}_p}{d\mathbb{P}} = p\frac{d\mathbb{Q}_1}{d\mathbb{P}} + (1-p)\frac{d\mathbb{Q}_0}{d\mathbb{P}}$$

where $0 \le p \le 1$. Note that $\frac{d\mathbb{Q}_p}{d\mathbb{P}}$ is strictly positive, so $\mathbb{Q}_p$ is equivalent to $\mathbb{P}$. Also

$$\mathbb{E}^{\mathbb{Q}_p}(S_1) = p\mathbb{E}^{\mathbb{Q}_1}(S_1) + (1-p)\mathbb{E}^{\mathbb{Q}_0}(S_1) = (1+r)S_0$$

and hence $\mathbb{Q}_p$ is a risk-neutral measure. Hence for any $0 \le p \le 1$ the expression

$$\frac{1}{1+r}\mathbb{E}^{\mathbb{Q}_p}(Y) = p\pi_1 + (1-p)\pi_0$$

is a no-arbitrage price of the claim. This shows that the set of no-arbitrage prices is an interval. $\square$

Remark. Note that the *marginal utility price* of a claim

$$\pi_0(Y) = \frac{\mathbb{E}[U'(X_1^*)Y]}{(1+r)\mathbb{E}[U'(X_1^*)]}$$

is also a no-arbitrage price since $U'(X_1^*)$ is proportional to the density of a risk-neutral measure. However, in general we cannot say that an *indifference price* is a no-arbitrage prices, but since $\pi(Y) \le \pi_0(Y)$, we can say it is bounded from above by a no-arbitrage price.

3 Attainable claims

Definition. A contingent claim with payout Y is *attainable* iff $Y = a + b^\top S_1$ for some $a \in \mathbb{R}$ and $b \in \mathbb{R}^d$.

Remark. We can equivalently write

$$a + b^\top S_1 = (1+r)x + b^\top[S_1 - (1+r)S_0]$$

with

$$x = \frac{a}{1+r} + b^\top S_0.$$

- Attainable claims have indifference prices independent of U and X_0 (example sheet). It is x.
- Attainable claims have marginal utility prices independent of U and X_0
- Attainable claims have unique no-arbitrage prices (today)

Theorem (Attainable claims have unique no-arbitrage prices)**.** *Suppose that our given market of tradable assets has no arbitrage. If a contingent claim is attainable then there is unique initial price such that the augmented market has no arbitrage.*

Proof. Suppose

$$Y = (1+r)x + b^\top[S_1 - (1+r)S_0]$$

To show: the unique no arbitrage price is $\pi = x$.
Method 1. Use the FTAP (in lecture) The only possible no arbitrage prices of the claim are of the form

$$\pi = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}(Y) = x + \frac{b^\top}{1+r}\mathbb{E}^{\mathbb{Q}}[S_1 - (1+r)S_0] = x$$

where $\mathbb{Q}$ is a risk-neutral measure. Since the answer is always x, $\forall \mathbb{Q}$ the no-arbitrage price is unique.
Method 2. Use the definition of arbitrage (not lectured) First, suppose $\pi = x$. Let $(\varphi^\top, \phi)^\top$ be a candidate arbitrage:

$$\varphi^\top[S_1 - (1+r)S_0] + \phi[Y - (1+r)x] \geq 0 \text{ almost surely}$$

This means

$$(\varphi + \phi b)^\top[S_1 - (1+r)S_0] \geq 0 \text{ almost surely}$$

Since the original market has no arbitrage, the almost sure inequalities are almost sure equalities. So there is no arbitrage in the augmented market. So $\pi = x$ is a no-arbitrage price.

Now suppose $\pi > x$. Note

$$b^\top[S_1 - (1+r)S_0] - [Y - (1+r)\pi] = (1+r)(\pi - x) > 0$$

so the portfolio $(b^\top, -1)^\top \in \mathbb{R}^{d+1}$ is an arbitrage in the augmented market. Otherwise, if $\pi < x$ the portfolio $(-b^\top, +1)^\top$ is an arbitrage. Hence there is exactly one price such that the augmented market has no arbitrage. $\square$

Theorem (Claims with unique no-arbitrage prices are attainable)**.** *Suppose that our given market of tradable assets has no arbitrage. A contingent claim is attainable if there is unique initial price such that the augmented market has no arbitrage.*

Proof. Use the FTAP. Details are on the example sheet. □