

# Stochastic Financial Models 4

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## 1 Properties of concave functions

We will nearly always assume our agent's utility function  $U$  is strictly increasing and strictly concave. If  $U$  is differentiable (always assumed), the gradient  $U'$  is called the *marginal utility*.

- $U'(x) > 0$  measures how much the utility increases at  $x$
- $U''(x) < 0$  measures the concavity of the utility at  $x$

**Definition.** The (Arrow–Pratt) *coefficient of absolute risk aversion* is

$$-\frac{U''(x)}{U'(x)}$$

The (Arrow–Pratt) *coefficient of relative risk aversion* for  $x > 0$  is

$$-x \frac{U''(x)}{U'(x)}$$

### Examples

- *exponential or CARA.*  $U(x) = -e^{-\gamma x}$  with  $\gamma > 0$  the constant coefficient of absolute risk aversion
- *power or CRRA.*  $U(x) = \frac{1}{1-R}x^{1-R}$ ,  $x > 0$ , with  $R > 0, R \neq 1$ , modelling the constant coefficient of relative risk aversion
- *logarithmic.*  $U(x) = \log x$ ,  $x > 0$  with constant coefficient of relative risk aversion  $R = 1$ .
- *risk-neutral.*  $U(x) = x$  so the coefficient of risk aversion is zero. Note that this function is concave, but not strictly concave, so we won't use it as a utility function!

**Remark.** To be really technically accurate, we should talk about the *domain* of a concave function, i.e. the set where the function is finite-valued.

**Theorem** (Concave functions are continuous, and their graphs lie <sup>below</sup>~~above~~ their tangents).  
Let  $U$  be concave. Then  $U$  is continuous. If  $U$  is differentiable, then for any  $x, y$  we have

$$U(y) \leq U(x) + U'(x)(y - x).$$

*Proof.* Fix  $x$  and  $0 < \varepsilon < \ell$ . We have

$$\begin{aligned} \frac{\varepsilon}{\ell}(U(x) - U(x - \ell)) &\geq U(x) - U(x - \varepsilon) \\ &\geq U(x + \varepsilon) - U(x) \\ &\geq \frac{\varepsilon}{\ell}(U(x + \ell) - U(x)) \end{aligned}$$

This is proven by looking each inequality one at a time, and rearranging the definition of concavity. For instance, note

$$x - \varepsilon = \frac{\varepsilon}{\ell}(x - \ell) + (1 - \frac{\varepsilon}{\ell})x$$

so by concavity

$$U(x - \varepsilon) \geq \frac{\varepsilon}{\ell}U(x - \ell) + (1 - \frac{\varepsilon}{\ell})U(x)$$

This is equivalent to the first inequality.

Sending  $\varepsilon \rightarrow 0$  shows continuity. Now assuming differentiability, dividing by  $\varepsilon$  and taking the limit yields

$$U(x) - U(x - \ell) \geq \ell U'(x) \geq U(x + \ell) - U(x)$$

as claimed by letting  $y = x + \ell$  or  $x - \ell$ . □

**Theorem** (Increasing <sup>strictly</sup> concave functions are unbounded on the left). Suppose  $U$  is increasing and concave, but not constant. Then  $U(x) \rightarrow -\infty$  as  $x \rightarrow \infty$ .

*Proof.* Let  $x < a < b$ , where  $U(a) < U(b)$ . Then using  $a = (\frac{b-a}{b-x})x + (\frac{a-x}{b-x})b$  in the definition of concavity yields

$$U(x) \leq U(a) + \frac{x-a}{b-a}(U(b) - U(a))$$

from which the conclusion follows. □

## 2 Optimal investment and marginal utility

In this section we assume that  $U$  is strictly increasing, concave and differentiable.

**Theorem** (Marginal utility pricing). Suppose  $U$  is suitably nice<sup>1</sup>, and let  $\theta^*$  maximise the expected utility  $\mathbb{E}[U(X_1)]$  where  $X_1 = (1+r)X_0 + \theta^\top[S_1 - (1+r)S_0]$ . Then

$$S_0 = \frac{\mathbb{E}[U'(X_1^*)S_1]}{(1+r)\mathbb{E}[U'(X_1^*)]}$$

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<sup>1</sup>That is, it satisfies a technical condition that allows the formal calculation to go through, but the condition is uninteresting for the main focus of this course. In this case, we assume  $U(X_1)$  is integrable for all portfolios  $\theta$  then the formal calculation is justified by the dominated convergence theorem of Probability & Measure.

where  $X_1^* = (1 + r)X_0 + (\theta^*)^\top [S_1 - (1 + r)S_0]$  is the optimal time-1 wealth.

*Proof.* Let

$$f(\theta) = \mathbb{E}\{U((1 + r)X_0 + \theta^\top [S_1 - (1 + r)S_0])\}$$

We can differentiate inside the expectation yielding

$$Df(\theta) = \mathbb{E}\{U'(X_1)[S_1 - (1 + r)S_0]\}$$

where  $X_1 = (1 + r)X_0 + \theta^\top [S_1 - (1 + r)S_0]$ . Since by calculus, at the maximising portfolio  $\theta^*$  the gradient vanishes  $Df(\theta^*) = 0$ , the conclusion follows upon rearrangement.  $\square$