Stochastic Financial Models 1

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Remark. The Part II course Probability & Measure is listed as desirable for this course. This is because we will be dealing with random variables, and being familiar with some probability theory will be handy. There are essentially three places where we use measure-theoretic probability:

- The convergence theorems will be used to justify statements such as $\lim_n \mathbb{E}(Z_n) = \mathbb{E}(\lim_n Z_n)$.

- The notions of measurability and sigma-algebra to model what information is available in a probabilistic setting

- The monotone class theorem, which says that in order to prove an identity involving expected values, it is usually sufficient check a special case.

However, this course is self-contained, so attending Probability & Measure is absolutely **not** necessary.

1 Standing assumptions and notation

Financial market consists of d risky assets.

- No dividends.

- Infinitely divisibility.

- No bid-ask spread.

- No price impact.

- No transaction costs

- No short selling constraints

The price of asset i at time t will be denoted S_t^i. We will let $S_t = (S_t^1, \ldots, S_t^d)^\top$ be the column vector of prices. In addition, market participants can borrow or lend at a risk-free interest rate r, assumed constant.

2 The one-period set-up

Introduce an investor. Let θ^i be the number of shares of asset i that the investor buys at time $t = 0$. (When $\theta^i < 0$ then the investor shorts $|\theta^i|$ shares of the asset.) Let $\theta = (\theta^1, \ldots, \theta^d)^\top$ be the column vector of portfolio weights. In addition, let θ^0 be the amount of money the investor puts in the bank. The investor's wealth at time t is denoted X_t.

- Initial wealth $X_0 = \theta^0 + \theta^\top S_0$.

- Time-1 wealth $X_1 = \theta^0(1 + r) + \theta^\top S_1$.

- $X_1 = (1 + r)X_0 + \theta^\top[S_1 - (1 + r)S_0]$

We think of the interest rate r and the initial asset prices S_0 as known at time 0. We will model the time-1 asset prices S_1 as a random vector. Moreover, we make the (unrealistically) assumption that we are completely *certain* that we know the *distribution* of S_1. In particular, given the initial wealth X_0 and the portfolio θ, we will model the time-1 wealth X_1 as a random variable with a known distribution.

3 The mean-variance portfolio problem

Mean-variance portfolio problem (Markowitz 1952) Given initial wealth X_0 and target mean m, find the portfolio θ to minimise $\mathrm{Var}(X_1)$ subject to $\mathbb{E}(X_1) \geq m$.

We will assume the random vector S_1 is square-integrable and adopt the notation

- $\mu = \mathbb{E}(S_1)$. We will assume $\mu \neq (1 + r)S_0$.

- $V = \mathrm{Cov}(S_1) = \mathbb{E}[(S_1 - \mu)(S_1 - \mu)^\top]$. Recall that V is automatically symmetric and non-negative definite. We will *assume* that V is positive definite. In particular, the inverse V^{-1} exists.

In this notation, we have

- $\mathbb{E}(X_1) = (1 + r)X_0 + \theta^\top[\mu - (1 + r)S_0]$ and

- $\mathrm{Var}(X_1) = \theta^\top V \theta$

so the mean-variance portfolio problem is to find θ such that

$$\text{minimise } \theta^\top V \theta \quad \text{subject to } \theta^\top[\mu - (1 + r)S_0] \geq m - (1 + r)X_0$$

Theorem (Mean-variance optimal portfolio). *The unique optimal solution to the mean-variance portfolio problem is*

$$\theta = \lambda \, V^{-1}[\mu - (1 + r)S_0]$$

where

$$\lambda = \frac{(m - (1 + r)X_0)^+}{[\mu - (1 + r)S_0]^\top V^{-1}[\mu - (1 + r)S_0]}$$

Notation. Here and throughout the course we will use the common notation $x^+ = \max\{x, 0\}$ for a real number x.

Proof. Next lecture.