

# Stochastic Financial Models 6

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## 1 Proofs of indifference pricing properties

To prove the properties listed last time, it is convenient to define for a any suitable random variable  $Z$  the *indirect utility*

$$V(Z) = \max_{X \in \mathcal{X}} \mathbb{E}[U(X + Z)]$$

In this notation,  $\pi$  is an indifference price for the claim with payout  $Y$  iff

$$V(Y - (1 + r)\pi) = V(0).$$

We prove two lemmata:

**Lemma** (Indirect utility is strictly increasing). *If  $Z_0 \leq Z_1$  almost surely with  $\mathbb{P}(Z_0 < Z_1) > 0$  then*

$$V(Z_1) > V(Z_0)$$

*Proof of lemma.* Let  $X^i$  be the maximiser for the two problems, i.e.

$$V(Z_i) = \mathbb{E}[U(X^i + Z_i)]$$

for  $i = 0, 1$ . Then

$$\begin{aligned} V(Z_1) &= \mathbb{E}[U(X^1 + Z_1)] \\ &\geq \mathbb{E}[U(X^0 + Z_1)] \\ &> \mathbb{E}[U(X^0 + Z_0)] \\ &= V(Z_0) \end{aligned}$$

□

**Lemma** (Indirect utility is concave). *Given random variable  $Z_0, Z_1$  and  $0 < p < 1$ . Then*

$$V(pZ_1 + (1 - p)Z_0) \geq pV(Z_1) + (1 - p)V(Z_0)$$

*Proof of lemma.* Let  $X^i$  be the maximiser for the two problems for  $i = 0, 1$ .

Now noting that  $pX^1 + (1-p)X^0 \in \mathcal{X}$  yields (expand  $X_1, X_0$  to see this)

$$\begin{aligned} pV(Z_1) + (1-p)V(Z_0) &= \mathbb{E}[pU(X^1 + Z_1) + (1-p)U(X^0 + Z_0)] \\ &\leq \mathbb{E}[U(pX^1 + (1-p)X^0 + pZ_1 + (1-p)Z_0)] \text{ as } U \text{ concave} \\ &\leq \max_{X \in \mathcal{X}} \mathbb{E}[U(X + pZ_1 + (1-p)Z_0)] \\ &= V(pZ_1 + (1-p)Z_0) \end{aligned}$$

□

*Proof of existence and uniqueness of indifference prices.* By our assumption of the existence of a maximiser, we have  $V(0) = \mathbb{E}[U(X^*)]$  for some  $X^* \in \mathcal{X}$ . In particular we have that  $U(-\infty) < V(0) < U(\infty)$ .

For fixed  $Y$ , we will show that the function  $x \mapsto V(Y + x)$  is a bijection from  $(-\infty, \infty)$  to  $(U(-\infty), U(\infty))$ . This would imply that there is a unique solution  $x$  to  $V(Y + x) = V(0)$ . The indifference price is uniquely defined by  $\pi(Y) = -\frac{1}{1+r}x$ .

Note the function  $x \mapsto V(Y + x)$  is strictly increasing, and hence an injection. To complete the proof, we need only show its range is the interval  $(U(-\infty), U(\infty))$ .

The function is concave, hence continuous, so its range is an interval. Since strictly increasing concave functions are unbounded from the left, we have

$$V(Y + x) \downarrow -\infty = U(-\infty) \text{ as } x \downarrow -\infty.$$

Also

$$V(Y + x) \geq \mathbb{E}[U(X^* + Y + x)] \uparrow U(+\infty) \text{ as } x \uparrow +\infty$$

by a form of the monotone convergence theorem from Probability & Measure (this step is not examinable). This shows  $x \mapsto V(Y + x)$  is a bijection. □

*Proof that indifference prices are increasing.* Suppose  $Y_0 \leq Y_1$  a.s. and  $\mathbb{P}(Y_0 < Y_1) > 0$ . Note

$$\begin{aligned} V(Y_1 - (1+r)\pi(Y_1)) &= V(0) \\ &= V(Y_0 - (1+r)\pi(Y_0)) \\ &< V(Y_1 - (1+r)\pi(Y_0)). \end{aligned}$$

Since  $x \mapsto V(Y_1 + x)$  is strictly increasing, we have  $-(1+r)\pi(Y_1) < -(1+r)\pi(Y_0)$  as desired. □

*Proof of concavity of indifference prices.* Given  $Y_0, Y_1$  and  $0 < p < 1$ , let  $Y_p = pY_1 + (1-p)Y_0$  and  $\pi_i = \pi(Y_i)$  for  $i = 0, p, 1$ . By definition of indifference price and concavity of  $V$  we have

$$\begin{aligned} V(Y_p - (1+r)\pi_p) &= V(0) \\ &= V(Y_1 - (1+r)\pi_1) \\ &= V(Y_0 - (1+r)\pi_0) \\ &= pV(Y_1 - (1+r)\pi_1) + (1-p)V(Y_0 - (1+r)\pi_0) \\ &\leq V(Y_p - (1+r)(p\pi_1 + (1-p)\pi_0)) \end{aligned}$$

Since  $x \mapsto V(Y_p + x)$  is strictly increasing, we have  $-(1+r)\pi_p \leq -(1+r)(p\pi_1 + (1-p)\pi_0)$ .  $\square$

*Proof that marginal utility price is larger than indifference price.* Let  $X^*$  be the optimiser without the claim, and  $X^1$  be the optimiser with the claim. Using the supporting line property of the concave function  $U$  we have

$$\begin{aligned} V(0) &= V(Y - (1+r)\pi(Y)) \\ &= \mathbb{E}[U(X^1 + Y - (1+r)\pi(Y))] \\ &\leq \mathbb{E}[U(X^*)] + \mathbb{E}[U'(X^*)(X^1 - X^*) + Y - (1+r)\pi(Y)] \\ &= V(0) + \mathbb{E}[U'(X^*)Y] - \mathbb{E}[U'(X^*)(1+r)\pi(Y)] \end{aligned}$$

as a concave fn  $U \leq \text{tangent.}$

where we have used the fact that

$$\mathbb{E}[U'(X^*)(X^1 - X^*)] = (\theta^1 - \theta^*)^\top \mathbb{E}[U'(X^*)(S_1 - (1+r)S_0)] = 0.$$

by first marginal utility pricing result.  $\square$

The conclusion follows upon rearranging.

$$\pi(y) \leq \frac{\mathbb{E}[U'(X^*)Y]}{\mathbb{E}[U'(X^*)](1+r)} \quad \text{So } = \frac{\mathbb{E}[U'(X^*)S_1]}{(1+r)\mathbb{E}[U'(X^*)]}$$