Stochastic Financial Models 24

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1 Black–Scholes PDE

Recall that the in the Black–Scholes model, the time-t price of a vanilla claim with time-T payout $g(S_T)$ is $\pi_t = V(t, S_t)$ where

$$V(t,s) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[g(se^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}Z})]$$

The Black–Scholes pricing function V solves the *Black–Scholes PDE*

$$\boxed{\partial_t V + rs\partial_s V + \frac{1}{2}\sigma^2 s^2 \partial_{ss} V = rV}$$

with boundary condition $V(T,s) = g(s)$.

One way to see this is to change variables and related V to a heat equation as described in Lecture 22. See example sheet 4.

Another derivation is to approximate the Black–Scholes model by a binomial model. Note that $S_t = S_{t-\delta}\xi_{t,\delta}$ where

$$\xi_{t,\delta} = e^{(r-\sigma^2/2)\delta+\sigma(W_t - W_{t-\delta})}$$

where W is a Brownian motion in the risk-neutral measure $\mathbb{Q}$. Approximating the Brownian motion by a random walk, we approximate $\xi_{t,\delta}$ by a random variable taking two values, $1+a$ with probability q and $1+a$ with probability $1-q$. Note

$$bq + a(1-q) \approx \mathbb{E}(\xi_{t,\delta} - 1) = e^{r\delta} - 1 \approx r\delta$$

and

$$b^2 q + a^2(1-q) \approx \mathbb{E}[(\xi_{t,\delta}-1)^2] = e^{(2r+\sigma^2)\delta} - 2e^{r\delta} + 1 \approx \sigma^2\delta$$

Now, in the binomial model we have

$$(1+r\delta)V(t-\delta, s) = q\ V(t, s(1+b)) + (1-q)V(t, s(1+a))$$

By Taylor expanding, the left-hand side is approximately

$$V + \delta\,(rV - \partial_t V)$$

where the functions are evaluated at the point (t, s). Similarly, the right-hand side is approximately,

$$q\left(V + sb\partial_s V + \frac{1}{2}s^2 b^2 \partial_{ss} V\right) + (1-q)\left(V + sa\partial_s V + \frac{1}{2}s^2 a^2 \partial_{ss} V\right)$$
$$= V + \delta\left(sr\partial_s V + \frac{1}{2}s^2\sigma^2\partial_{ss} V\right)$$

Equating the two sides and sending $\delta \to 0$ completes this (not rigorous) derivation.

2 Black–Scholes greeks

In the binomial model, in order to replicate the claim, at time t on the event $\{S_{t-\delta} = s\}$ you must hold

$$\frac{V(t, s(1+b)) - V(t, s(1+a))}{s(b-a)} \approx \frac{\partial V}{\partial s}(t, s)$$

shares of the underlying asset between times $t - \delta$ and t. In the Black–Scholes model, the quantity $\partial_s V$ is called the *delta* of the claim. The phrase *to delta-hedge a claim* just means to hold the delta (the partial derivative of the claim price with respect to the underlying asset price) in order to replicate the payout of the claim.

The quantity $\partial_{ss} V$ measures the sensitivity of the delta with respect to movements of the underlying asset price and is known as the *gamma* of the claim.

The quantity $-\partial_t V$ is known as the *theta* of the claim.

The partial derivatives of the Black–Scholes pricing function with respect to the various parameters are called the *greeks* of the claim. (There is a whole zoo of other greeks, including the *rho*, the *vega* and the *vanna*...)

Proposition. *If the payout function g is increasing, then the delta is always non-negative. If g is convex, then the gamma is always non-negative.*

Proof. From the formula

$$V(t, s) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[g(se^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}Z})]$$

it is clear (using the argument from example sheet 1) that $V(t, \cdot)$ is increasing when g is increasing, and that $V(t, \cdot)$ is convex when g is convex. □

3 Black–Scholes prices of barrier-type claims

Consider a market with a stock with price $(S_t)_{t\geq 0}$.

- Given a European contingent claim with payout Y and expiry T
- and given a level B

- A *down-and-in* version of the claim has payout $Y\mathbb{1}_{\{\min_{0\le t\le T} S_t \le B\}}$
- *down-and-out* has payout $Y\mathbb{1}_{\{\min_{0\le t\le T} S_t > B\}}$
- *up-and-in* has payout $Y\mathbb{1}_{\{\max_{0\le t\le T} S_t \ge B\}}$
- *up-and-out* has payout $Y\mathbb{1}_{\{\max_{0\le t\le T} S_t < B\}}$

Example. An up-and-in call option with strike K and barrier B, gives the owner of the option the right, but not the obligation, to buy the stock at time T for price K, provided that the price of the stock exceeds B at some time between time 0 and time T.

Proposition. *Within the Black–Scholes model, the initial price of an up-and-out claim with payout*

$$g(S_T)\mathbb{1}_{\{\max_{0\le t\le T} S_t < B\}}$$

is the same as that of a vanilla option with payout

$$g(S_T)\mathbb{1}_{\{S_T\le B\}} - (B/S_0)^{2r/\sigma^2-1} g(B^2 S_T/S_0^2)\mathbb{1}_{\{S_T\le S_0^2/B\}}$$

Proof. Since $\{\max_{0\le t\le T} S_t < B\} \subseteq \{S_T < B\}$, we have

$$\begin{aligned} g(S_T)\mathbb{1}_{\{\max_{0\le t\le T} S_t < B\}} =& g(S_T)\mathbb{1}_{\{S_T<B\}} \\ &- g(S_T)\mathbb{1}_{\{\max_{0\le t\le T} S_t\ge B, S_T<B\}} \end{aligned}$$

Looking at the second term on the right, letting $b = \log(B/S_0)/\sigma$ and $c = r/\sigma - \sigma/2$.

$$\begin{aligned} &g(S_T)\mathbb{1}_{\{\max_{0\le t\le T} S_t\ge B, S_T\le B\}} \\ &= g(S_0 e^{\sigma(W_T+cT)})\mathbb{1}_{\{\max_{0\le t\le T}(W_t+ct)\ge b, W_T+cT\le b\}} \end{aligned}$$

where $(W_t)_{0\le t\le T}$ is a Brownian motion under the risk-neutral measure.

By the Cameron–Martin theorem, the expected value is

$$\mathbb{E}[e^{cW_T - c^2T/2} g(S_0 e^{\sigma W_T})\mathbb{1}_{\{\max_{0\le t\le T} W_t\ge b, W_T\le b\}}]$$

by the reflection principle

$$= \mathbb{E}[e^{c(2b-W_T)-c^2T/2} g(S_0 e^{\sigma(2b-W_T)})\mathbb{1}_{\{W_T\ge b\}}]$$

by symmetry of W_T

$$= e^{2bc}\mathbb{E}[e^{cW_T-c^2T/2} g(S_0 e^{2b\sigma} e^{\sigma W_T})\mathbb{1}_{\{W_T\le -b\}}]$$

by Cameron–Martin again

$$= e^{2bc}\mathbb{E}[g(S_0 e^{2b\sigma} e^{\sigma(W_T+cT)})\mathbb{1}_{\{W_T+cT\le -b\}}]$$

Rewriting

$$\begin{aligned} &e^{2bc}\mathbb{E}[g(e^{2b\sigma} S_T)\mathbb{1}_{\{S_T\le S_0 e^{-b\sigma}\}}] \\ &= (B/S_0)^{2r/\sigma^2-1}\mathbb{E}[g(B^2 S_T/S_0^2)\mathbb{1}_{\{S_T\le S_0^2/B\}}] \end{aligned}$$

$\square$