Stochastic Financial Models 18

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1 Optimal stopping problems

- Consider a Markov process of the form

$$X_n = G(n, X_{n-1}, \xi_n)$$

where $(\xi)_{n\geq 1}$ are independent.

- Fix a horizon N and consider the problem:

$$\text{maximise } \mathbb{E}\left[g(X_T)\right]$$

over stopping times $0 \leq T \leq N$.

- **The Bellman equation** is

$$\begin{aligned} V(N, x) &= g(x) \text{ for all } x \\ V(n-1, x) &= \max\{g(x), \mathbb{E}[V(n, G(n, x, \xi_n))]\} \text{ for all } x, 1 \leq n \leq N \end{aligned}$$

Theorem.

$$V(n, x) = \max\left\{\mathbb{E}\left[g(X_T)|X_n = x\right] : T \textit{ a stopping time, } n \leq T \leq N\right\}$$

The optimal stopping time is

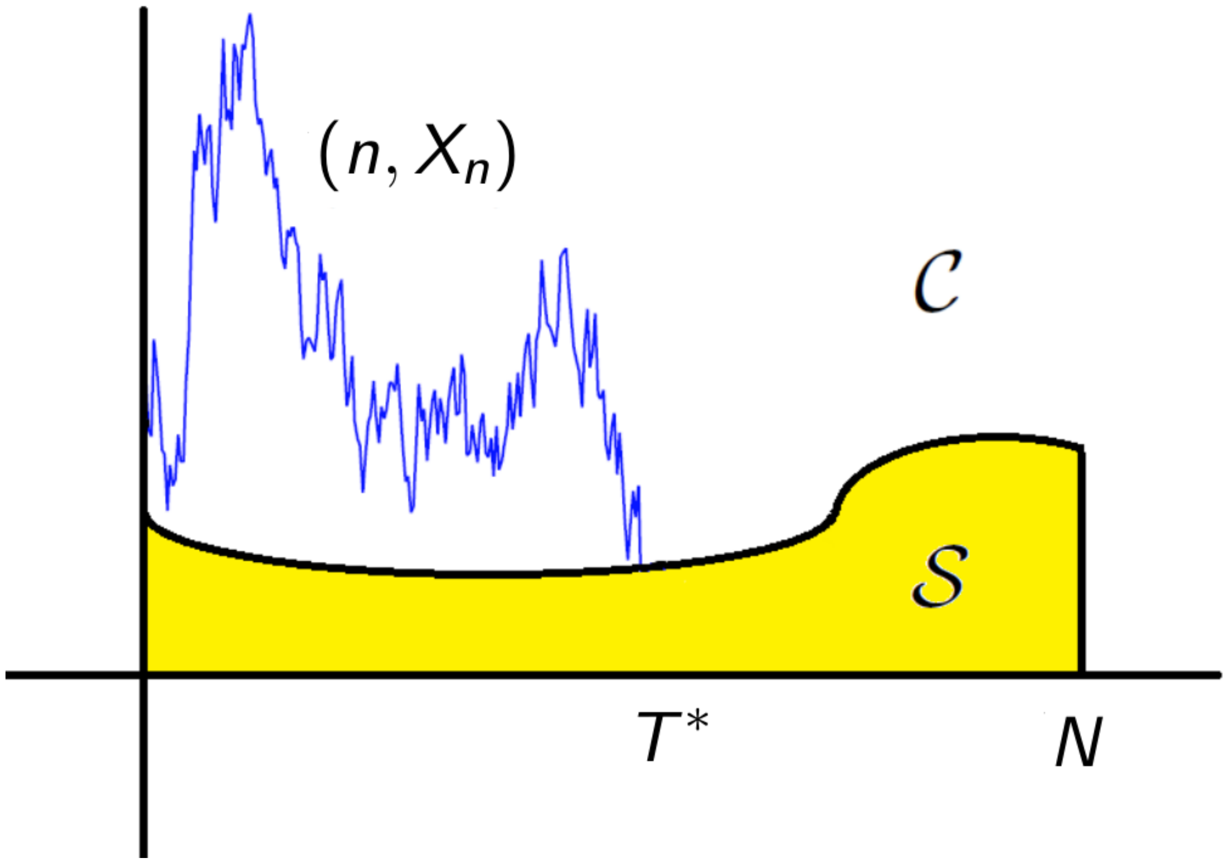
$$T^* = \inf\{n \geq 0 : V(n, X_n) = g(X_n)\}$$

It can be described graphically as follows Let

$$\begin{aligned} \mathcal{C} &= \{(n, x) : V(n, x) > g(x)\} = \text{ 'continuation region'} \\ \mathcal{S} &= \{(n, x) : V(n, x) = g(x)\} = \text{ 'stopping region'} \end{aligned}$$

Then

$$T^* = \inf\{n \geq 0 : (n, X_n) \in \mathcal{S}\}$$



2 Multi-period arbitrage

The set-up. Consider a market

- with a risk-free asset with interest rate r
- and d risky assets with time n prices $(S_n)_{n\geq 0}$.
- The investor holds the portfolio $\theta_n \in \mathbb{R}^d$ of risky assets during the time interval $(n-1, n]$, where θ_n is $\mathcal{F}_{n-1}$-measurable

The wealth of a self-financing investor evolves as

$$X_n = (1+r)X_{n-1} + \theta_n^\top [S_n - (1+r)S_{n-1}]$$

Hence

$$X_n = (1+r)^n X_0 + \sum_{k=1}^{n} (1+r)^{n-k} \theta_k^\top [S_k - (1+r)S_{k-1}]$$

The investor holds

$$\theta_n^0 = X_{n-1} - \theta_n^\top S_{n-1}$$

in the bank during the time interval $(n-1, n]$.

Definition. An *arbitrage* is a previsible process $(\varphi_n)_{1 \le n \le N}$ such that

$$\sum_{k=1}^{N} (1+r)^{N-k} \varphi_k^\top [S_k - (1+r) S_{k-1}] \ge 0 \text{ almost surely}$$

and

$$\mathbb{P}\left(\sum_{k=1}^{N} (1+r)^{N-k} \varphi_k^\top [S_k - (1+r) S_{k-1}] > 0 \right) > 0$$

If φ is an arbitrage, then an investor would always prefer the investment strategy $\theta + \varphi$ to the strategy θ.

Definition. A risk-neutral measure is a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which the discounted asset price process

$$M_n = (1+r)^{-n} S_n$$

is a martingale, that is,

$$\frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}(S_n | \mathcal{F}_{n-1}) = S_{n-1}$$

for all $n \ge 1$.

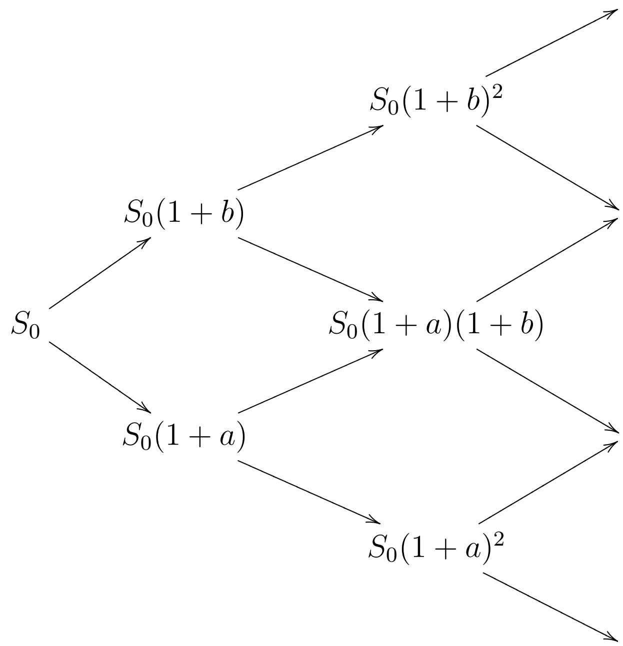
Theorem (Fundamental theorem of asset pricing)**.** *In a finite horizon multi-period model, there is no arbitrage if and only if there exists a risk-neutral measure.*

3 Introduction to the (Cox–Ross–Rubinstein) binomial model

- $d = 1$ and $S_n = S_{n-1} \xi_n$
- $(\xi_n)_{n \ge 1}$ generate a filtration $(\mathcal{F}_n)_n$ and such that

  $$0 < \mathbb{P}(\xi_n = 1 + b | \mathcal{F}_{n-1}) = 1 - \mathbb{P}(\xi_n = 1 + a | \mathcal{F}_{n-1}) < 1 \text{ a.s. for all } n$$

  That is, the stock price can follow any path along the tree with positive probability
- $S_0 > 0$ and $-1 < a < b$



Theorem. *Consider the N-step binomial model. There exists a risk-neutral measure if and only if $a < r < b$. When it exists it is the unique measure $\mathbb{Q}$ such that $(\xi_n)_{1\le n\le N}$ are IID under $\mathbb{Q}$ with*

$$\mathbb{Q}(\xi = 1+b) = q = \frac{r-a}{b-a} = 1 - \mathbb{Q}(\xi = 1+a).$$

Proof. Suppose such a risk-neutral measure $\mathbb{Q}$ exists. Then by definition

$$\begin{aligned}(1+r)S_{n-1} =& \mathbb{E}^{\mathbb{Q}}(S_n|\mathcal{F}_{n-1}) \\ =& S_{n-1}(1+b)\mathbb{Q}(\xi_n = 1+b|\mathcal{F}_{n-1}) \\ &+ S_{n-1}(1+a)\mathbb{Q}(\xi_n = 1+a|\mathcal{F}_{n-1})\end{aligned}$$

and hence

$$\mathbb{Q}(\xi_n = 1+b|\mathcal{F}_{n-1}) = q = 1 - \mathbb{Q}(\xi_n = 1+a|\mathcal{F}_{n-1}).$$

Note $0 < q < 1$ if and only if $a < r < b$. Also, under this condition, the conditional distribution of ξ_n is independent of n and $\mathcal{F}_{n-1}$, so the $(\xi_n)_{1\le n\le N}$ are IID. $\square$