Stochastic Financial Models 22

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1 Cameron–Martin theorem

Motivation. Example sheet 1

- Let $Z \sim N(0,1)$.
- $\mathbb{E}[g(a+Z)] = \mathbb{E}[e^{aZ-a^2/2}g(Z)]$ for any $a \in \mathbb{R}$ and suitable g.
- Proof: Change of variables formula for integration.

Generalisation

- Let $Z \sim N_n(0,I)$ multi-variate normal.
- $\mathbb{E}[g(a+Z)] = \mathbb{E}[e^{a^\top Z-\|a\|^2/2}g(Z)]$ for any $a \in \mathbb{R}^n$ and suitable g.
- Essentially the same proof.

Theorem (Cameron–Martin theorem)**.** *Let $(W_t)_{t\geq 0}$ be a Brownian motion. For fixed $t \geq 0$ and $c \in \mathbb{R}$ we have*

$$\mathbb{E}[g((W_s+cs)_{0\leq s\leq t})] = \mathbb{E}[e^{cW_t-c^2t/2}g((W_s)_{0\leq s\leq t})]$$

for suitable functions g from the space of continuous functions on $[0,t]$ to the real line.

Sketch of proof. By measure theory, it is enough to consider functions g of the form

$$g(w) = G(w(t_1),\dots,w(t_n))$$

for a function G on $\mathbb{R}^n$, where $0 = t_0 < t_1 < \cdots < t_n = t$.

$$\begin{aligned}
\mathbb{E}[g((W_s+cs)_{0\leq s\leq t}) &= \mathbb{E}[G(W_{t_1}+ct_1,\dots,W_{t_n}+ct_n)] \\
&= \mathbb{E}[G((\sum_{i=1}^k \sqrt{t_i-t_{i-1}}(Z_i+a_i))_{k=1}^n)]] \\
&= \mathbb{E}[e^{a^\top Z-\|a\|^2/2}G((\sum_{i=1}^k \sqrt{t_i-t_{i-1}}Z_i)_{k=1}^n)]] \\
&= \mathbb{E}[e^{cW_t-c^2t/2}g((W_s)_{0\leq s\leq t})]
\end{aligned}$$

where $Z_i = \frac{W_{t_i} - W_{t_{i-1}}}{\sqrt{t_i - t_{i-1}}}$ are iid $N(0,1)$ for $1 \le i \le n$ and $a_i = c\sqrt{t_i - t_{i-1}}$ so that

$$a^\top Z = \sum_{i=1}^n a_i Z_i = W_t \cdot c$$

and

$$\|a\|^2 = \sum_{i=1}^n a_i^2 = c^2 t$$

□

2 An application of Cameron–Martin

Proposition. *Let $(W_t)_{t\ge0}$ be a Brownian motion. For $a \ge 0$ we have*

$$\begin{aligned}\mathbb{P}(\max_{0\le s\le t}(W_s + cs) \le a) &= \mathbb{P}(W_t \le a - ct) - e^{2ca}\mathbb{P}(W_t \ge a + ct) \\ &= \Phi\left(\frac{a - ct}{\sqrt{t}}\right) - e^{2ca}\Phi\left(\frac{-a - ct}{\sqrt{t}}\right)\end{aligned}$$

Proof.

$$\begin{aligned}\mathbb{P}(\max_{0\le s\le t}(W_s + cs) \le a) =& \mathbb{E}[\mathbb{1}_{\{\max_{0\le s\le t}(W_s+cs)\le a\}}] \\ =& \mathbb{E}[e^{cW_t - c^2t/2}\underbrace{\mathbb{1}_{\{\max_{0\le s\le t} W_s \le a\}}}_{= \mathbb{1}_{\{\max W_s \le a,\, W_t \le a\}}}] \\ =& \mathbb{E}[e^{cW_t - c^2t/2}\mathbb{1}_{\{W_t\le a\}}] \\ & - \mathbb{E}[e^{c(2a - W_t) - c^2t/2}\mathbb{1}_{\{W_t \ge a\}}] \\ =& \mathbb{E}[\mathbb{1}_{\{W_t + ct \le a\}}] - e^{2ac}\mathbb{E}[\mathbb{1}_{\{W_t - ct \ge a\}}]\end{aligned}$$

$= \mathbb{1}_{\{W_t \le a\}} - \underbrace{\mathbb{1}_{\{\max W_s \ge a,\, W_t \le a\}}}_{\text{Use reflection}}.$

□

To discuss risk-neutral measures, we need

Theorem (Cameron–Martin reformulation)**.** *Let $(W_t)_{t\ge0}$ be a Brownian motion under a given measure $\mathbb{P}$. Fix $T > 0$ and $c \in \mathbb{R}$, and define an equivalent measure $\mathbb{Q}$ by*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{cW_T - c^2T/2}$$

Then the process $(W_t - ct)_{0\le t\le T}$ is a Brownian motion under $\mathbb{Q}$.

Proof. Fix a function g on $C[0,T]$. Then

$$\begin{aligned}\mathbb{E}^\mathbb{Q}[g((W_t - ct)_{0\le t\le T})] &= \mathbb{E}^\mathbb{P}[e^{cW_T - c^2T/2}g((W_t - ct)_{0\le t\le T})] \\ &= \mathbb{E}^\mathbb{P}[g((W_t)_{0\le t\le T})]\end{aligned}$$

by the first formulation of Cameron–Martin. So the process $(W_t - ct)_{0\le t\le T}$ has the same law under $\mathbb{Q}$ as the process $(W_t)_{0\le t\le T}$ has under $\mathbb{P}$. □

3 Heat equation

Proposition. *Fix a suitable g and let*

$$u(t,x) = \mathbb{E}[f(x+\sqrt{\tau}Z)]$$

where $Z \sim N(0,1)$. Then u solves the heat equation

$$\partial_\tau u = \frac{1}{2}\partial_{xx}u$$

with boundary condition $u(0,x) = f(x)$.

Proof when g is well-behaved by example sheet 1,

$$\begin{aligned}\partial_\tau u &= \frac{1}{2\sqrt{\tau}}\mathbb{E}[Zg'(x+\sqrt{\tau}Z)] \\ &= \frac{1}{2}\mathbb{E}[g''(x+\sqrt{t}Z)] \\ &= \frac{1}{2}\partial_{xx}u\end{aligned}$$

If g is less well-behaved, then write

$$u(\tau,x) = \int f(y)p(\tau;x,y)dy$$

where

$$p(\tau;x,y) = \frac{1}{\sqrt{2\pi\tau}}\exp\left(-\frac{(y-x)^2}{2\tau}\right)$$

is the transition density of the Brownian motion (also called the *heat kernel* or *Green's function*) and use the fact that $p(\cdot;\cdot,y)$ satisfies the heat equation.

Since p is very well-behaved, interchange of derivatives and integrals is allowed by the dominated convergence theorem, provided that f has exponential growth. $\square$