Stochastic Financial Models 2

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1 Mean-variance efficiency

To solve the mean-variance portfolio problem, we will use the following lemma

Lemma. *If* $\theta^\top a = b$ *then*

$$\theta^\top V \theta \geq \frac{b^2}{a^\top V^{-1} a}$$

with equality if and only if

$$\theta = \lambda \; V^{-1} a$$

where

$$\lambda = \frac{b}{a^\top V^{-1} a}.$$

Proof of lemma. Since V is non-negative definite we have

$$\begin{aligned}
\theta^\top V \theta =& \theta^\top V \theta + 2\lambda (b - \theta^\top a) \\
=& (\theta - \lambda \; V^{-1} a)^\top V (\theta - \lambda \; V^{-1} a) \\
&+ 2\lambda b - \lambda^2 \; a^\top V^{-1} a \\
\geq& 2\lambda b - \lambda^2 \; a^\top V^{-1} a = \frac{b^2}{a^\top V^{-1} a}
\end{aligned}$$

and since V is positive definite there is equality only if

$$\theta = \lambda \; V^{-1} a$$

□

Remark. This proof is secretly using the Lagrangian technique from IB Optimisation or Variational Principles. The constant λ could be thought of as a Lagrange multiplier.

Remark. The lemma is equivalent to

$$(\theta^\top a)^2 \leq (\theta^\top V \theta)(a^\top V^{-1} a).$$

This is just the Cauchy–Schwarz inequality applied to the vectors $V^{1/2}\theta$ and $V^{-1/2} a$.

By applying the lemma with $a = \mu - (1+r)S_0$ and $b = \mathbb{E}(X_1) - (1+r)X_0$, we see that

$$\mathrm{Var}(X_1) \geq \frac{(\mathbb{E}(X_1) - (1+r)X_0)^2}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]}$$

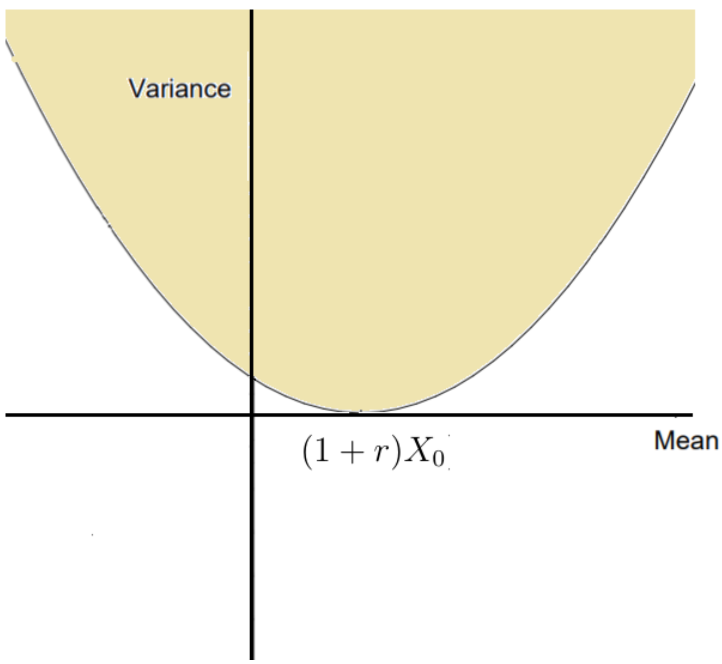
with equality if and only if

$$\theta = \lambda\, V^{-1}[\mu - (1+r)S_0]$$

where

$$\lambda = \frac{\mathbb{E}(X_1) - (1+r)X_0}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]}.$$

When the initial wealth X_0 is fixed, we can plot the set of all possible values of $(\mathbb{E}(X_1), \mathrm{Var}(X_1))$ as we vary the portfolio θ.



Definition. Given X_0, the *mean-variance efficient frontier* is the lower boundary of the set of possible values of $(\mathbb{E}(X_1), \mathrm{Var}(X_1))$; i.e. the set $\{m, (\min_{\mathbb{E}(X_1)=m} \mathrm{Var}(X_1)) : \ m \in \mathbb{R}\}$.

Remark. Note that we have shown that the mean-variance efficient frontier is a parabola.

Proof of mean-variance optimal portfolio. If $m > (1+r)X_0$, then it is optimal to take $\mathbb{E}(X_1) = m$ with portfolio $\theta = \lambda V^{-1}$, since minimised variance increases with $\mathbb{E}(X_1)$.

However, if $m \leq (1+r)X_0$, then the minimised variance decreases with $\mathbb{E}(X_1)$ and hence it is optimal to take $\mathbb{E}(X_1) = (1+r)X_0 \geq m$, with portfolio $\theta = 0$. $\square$

Definition. Given X_0, we say that a portfolio is *mean-variance efficient* iff it is the optimal solution to a mean-variance portfolio problem for *some* target mean m.

Theorem (Mutual fund theorem). *A portfolio θ is mean-variance efficient if and only there exists a scalar $\lambda \geq 0$ such that*

$$\theta = \lambda \, V^{-1}[\mu - (1+r)S_0]$$

Proof. We are given an initial wealth X_0.

Suppose we are given a target mean m. Then the optimal solution of the mean-variance portfolio problem if of the correct form with

$$\lambda = \frac{(m - (1+r)X_0)^+}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]} \geq 0$$

On the other hand, suppose that we are given $\lambda \geq 0$. Then the given portfolio is the optimal solution of the mean-variance portfolio problem for target mean

$$m = (1+r)X_0 + \lambda[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0].$$

□

2 Capital Asset Pricing Model

Theorem (Linear regression coefficients). *Let X and Y be two-$\overbrace{\textit{square integrable}}^{\text{have positive, finite variances}}$ random variables with* $\mathrm{Var}(X) > 0$. *The unique constants a and b such that*

$$Y = a + bX + Z$$

where $\mathbb{E}(Z) = 0$ *and* $\mathrm{Cov}(X, Z) = 0$ *are given by*

$$b = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} \text{ and } a = \mathbb{E}(Y) - b\mathbb{E}(X).$$

Proof. Let $Z = Y - a - bX$ and note

$$\mathbb{E}(Z) = \mathbb{E}(Y) - a - b\mathbb{E}(X)$$
$$\mathrm{Cov}(X,Z) = \mathrm{Cov}(X,Y) - b\mathrm{Var}(X)$$

The given a and b are the unique solution to the system of equations $\mathbb{E}(Z) = 0$ and $\mathrm{Cov}(X,Z) = 0$. □

Definition. The portfolio

$$\theta_{\text{Mar}} = V^{-1}[\mu - (1+r)S_0]$$

is called the *market portfolio.*

Remark. The name market portfolio is explained below.

Definition. Given initial wealth $X_0 > 0$, the excess return R^{ex} of a portfolio θ is defined by

$$R^{\text{ex}} = \frac{X_1}{X_0} - (1+r) = \frac{1}{X_0}\theta^\top [S_1 - (1+r)S_0]$$

Let $R^{\text{ex}}_{\text{Mar}}$ be the excess return of the market portfolio θ_{Mar}.

Theorem (Alpha is zero). *Fix $X_0 > 0$ and a portfolio θ. Suppose α and β are such that*

$$R^{\text{ex}} = \alpha + \beta R^{\text{ex}}_{\text{Mar}} + \varepsilon$$

where $\mathbb{E}(\varepsilon) = 0$ and $\text{Cov}(R^{\text{ex}}_{\text{Mar}}, \varepsilon) = 0$. Then $\alpha = 0$.

Proof. (next time) Note

$$\begin{aligned}
\text{Cov}(R^{\text{ex}}, R^{\text{ex}}_{\text{Mar}}) &= \frac{1}{X_0^2}\theta^\top \overbrace{\text{Cov}[S_1 - (1+r)S_0]}^{\text{Var}(S_1 - (1+r)S_0) = \text{Var}(S_1) = V} \underbrace{\theta_{\text{Mar}}}_{V^{-1}(\mu - (1+r)S_0)} \\
&= \frac{1}{X_0^2}\theta^\top [\mu - (1+r)S_0] \\
&= \frac{1}{X_0}\mathbb{E}(R^{\text{ex}})
\end{aligned}$$

and hence

$$\begin{aligned}
\text{Var}(R^{\text{ex}}_{\text{Mar}}) &= \text{Cov}(R^{\text{ex}}_{\text{Mar}}, R^{\text{ex}}_{\text{Mar}}) \\
&= \frac{1}{X_0}\mathbb{E}(R^{\text{ex}}_{\text{Mar}}).
\end{aligned}$$

By linear regression, we have

$$\begin{aligned}
\beta &= \frac{\text{Cov}(R^{\text{ex}}, R^{\text{ex}}_{\text{Mar}})}{\text{Var}(R^{\text{ex}}_{\text{Mar}})} \\
&= \frac{\mathbb{E}(R^{\text{ex}})}{\mathbb{E}(R^{\text{ex}}_{\text{Mar}})}
\end{aligned}$$

and

$$\alpha = \mathbb{E}(R^{\text{ex}}) - \beta\mathbb{E}(R^{\text{ex}}_{\text{Mar}}) = 0.$$

□