

# Stochastic Financial Models 10

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## 1 Examples of attainable claims

*Example 1: Forward contract.* A forward contract is the right *and the obligation* to buy a given asset at fixed price  $K$  (the strike) at time 1. When  $d = 1$ , the payout of a forward on the risky asset is given by  $Y = S_1 - K$ . Note that this is attainable by holding 1 share and borrowing  $K/(1+r)$  from the bank. Hence the unique no-arbitrage initial price of the forward is  $\pi = S_0 - K/(1+r)$

[The strike of a forward contract is usually chosen such that the initial price of the forward is zero. That is  $K = (1+r)S_0$ . This is called the forward price of the asset.]

*Example 2: one-period binomial model.* Suppose  $d = 1$  as before and that  $S_1$  can take exactly two values with  $\mathbb{P}(S_1 = S_0(1+b)) = p = 1 - \mathbb{P}(S_1 = S_0(1+a))$ , for constants  $-1 < a < b$ , where  $0 < p < 1$ .

First we find the risk-neutral measures. Let  $\mathbb{Q}(S_1 = S_0(1+b)) = q = 1 - \mathbb{Q}(S_1 = S_0(1+a))$ . Then

$$S_0 = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}(S_1) = \frac{1}{1+r} S_0(1+b)q + \frac{1}{1+r} S_0(1+a)(1-q)$$

so

$$q = \frac{r-a}{b-a} \text{ and } 1-q = \frac{b-r}{b-a}$$

Thus we learn that there exists a risk-neutral measure iff  $0 < q < 1$   $\leadsto$   $0 < p < 1$

$$\Leftrightarrow a < r < b$$

in which case the risk-neutral measure is unique. This means that every contingent claim is attainable! Consider a claim with payout  $Y = g(S_1)$ . We need only check that the unique solution  $(x, \theta)$  to

$$(1+r)x + \theta[S_1 - (1+r)S_0] = g(S_1)$$

that is, the system of equations

$$\begin{aligned} (1+r)x + \theta S_0(b-r) &= g(S_0(1+b)) \\ (1+r)x + \theta S_0(a-r) &= g(S_0(1+a)) \end{aligned}$$

is

$$\theta = \frac{g(S_0(1+b)) - g(S_0(1+a))}{S_0(b-a)}$$

$$x = \frac{1}{(1+r)(b-a)}[(r-a)g(S_0(1+b)) + (b-r)g(S_0(1+a))] = \frac{1}{1+r} \mathbb{E}^{\mathbb{Q}}[g(S_1)]$$

## 2 Multi-period models

Motivating discussion

- In a one period model, we think of  $S_0$  as constant but  $S_1$  as random
- In a two period model,  $S_0$  is constant, but  $S_1$  and  $S_2$  are random, at least as observed at time 0.
- But at time 1, we can think of both  $S_0$  and  $S_1$  as constant, and only  $S_2$  is random

flow of information

- Initially, an agent has information  $\mathcal{F}_0$
- at time 1, has information  $\mathcal{F}_1$
- and at time 2, has information  $\mathcal{F}_2$ .
- Naturally, we should have  $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$
- We also want, for instance,  $S_0$  and  $S_1$  (but not  $S_2$ ) to be  $\mathcal{F}_1$ -‘measurable’.
- But what is information?

Given  $(\Omega, \mathcal{F}, \mathbb{P})$ , and a ‘set of information’  $\mathcal{G}$ , an event  $A \in \mathcal{F}$  is  $\mathcal{G}$ -measurable intuitively iff

$$\mathbb{P}(A|\mathcal{G}) \text{ is always either 0 or 1}$$

*Example.*

- Imagine flipping a coin two times.
- Let  $\mathcal{G}$  be knowledge of the result of the first flip.
- $\mathbb{P}(\{HH, HT\}|\mathcal{G}) = 1$  if the first flip is heads and 0 otherwise. So  $\{HH, HT\}$  is  $\mathcal{G}$ -measurable. That is to say, knowing  $\mathcal{G}$ , you can always measure whether the outcome is in  $\{HH, HT\}$  or not.
- $\mathbb{P}(\{TT\}|\mathcal{G}) = 1/2$  if the first flip is tails, so  $\{TT\}$  is not  $\mathcal{G}$  measurable. That is, even knowing  $\mathcal{G}$ , sometimes you cannot perfectly measure whether the outcome is  $TT$  or not.

### 3 Measurability

**Idea:** Identify the information  $\mathcal{G}$  with the collection of all  $\mathcal{G}$ -measurable events.

What kind of collection of events should it be?

**Definition.** Given a set  $\Omega$ , a non-empty collection  $\mathcal{G}$  of subsets of  $\Omega$  is called a *sigma-algebra* iff

- $A \in \mathcal{G}$  implies  $A^c \in \mathcal{G}$
- $A_1, A_2, \dots \in \mathcal{G}$  implies  $\cup_n A_n \in \mathcal{G}$ .

*Example.* Consider tossing a coin twice. Let  $\Omega = \{HH, HT, TH, TT\}$ . The information measurable after the first coin toss is  $\{\emptyset, \Omega, \{HH, HT\}, \{TH, TT\}, \}$

**Definition.** Given a sigma-algebra  $\mathcal{G}$ , a random variable  $X$  is  $\mathcal{G}$ -measurable iff the event  $\{X \leq x\}$  is in  $\mathcal{G}$  for all  $x \in \mathbb{R}$ .

**Remark.** Intuitively, knowing the information in  $\mathcal{G}$  allows you measure the value of  $X$ .

**Remark.** If  $X$  is  $\mathcal{G}$ -measurable, then the event  $\{X \in B\}$  is in  $\mathcal{G}$  for all 'nice' (for the measure theory specialists: Borel) subsets  $B \subseteq \mathbb{R}$ .

**Remark.** If  $X$  takes values in the countable set  $\{x_1, x_2, \dots\}$  then  $X$  is  $\mathcal{G}$ -measurable iff  $\{X = x_i\} \in \mathcal{G}$  for all  $i$ .

**Exercise.** Show that if  $X$  is measurable with respect to the trivial sigma-algebra  $\{\emptyset, \Omega\}$  then  $X$  is equal to a constant.

**Definition.** The sigma-algebra *generated* by a random variable  $X$  is the sigma-algebra  $\mathcal{G}$  containing all events of the form  $\{X \in B\}$  where for 'nice' subsets  $B \subseteq \mathbb{R}$ . Notation:  $\mathcal{G} = \sigma(X)$

**Theorem** (Sometimes called factorisation lemma). *A random variable  $Y$  is measurable respect to  $\sigma(X)$  if and only if there is a 'nice' function  $f$  such that  $Y = f(X)$ .*