Part IB — Analysis and Topology

Based on lectures by Dr P. Russell

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Part I
Generalizing continuity and convergence

§2 Metric Spaces

§2.1 Definitions and Examples

Question

Can we think about convergence in a more general setting? Convergence seemed similar in our 3 settings.

What do we really need?

Answer

We need a notion of distance.

In $\mathbb{R}$: distance x to y is $|x-y|$.
In $\mathbb{R}^2$: its $\|x-y\|$.
For functions: distance f to g is $\sup_{x\in X}|f(x)-g(x)|$ (where this exists, i.e. if $f-g$ bounded).

The triangle inequality was often important (see the proof of uniqueness of limits).

Definition 2.1 (Metric)

A **metric** d is a function $d : X^2 \to \mathbb{R}$ satisfying:

- $d(x,y) \geq 0$ for all $x,y \in X$ with equality iff $x=y$;
- $d(x,y) = d(y,x)$ for all $x,y \in X$.
- $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Definition 2.2 (Metric Space)

A **metric space** is a set X endowed with a metric d.

We could also define a metric space as an ordered pair (X,d). If it is obvious what d is, we sometimes write 'The metric space X ...'.

Example 2.1

$X=\mathbb{R}$, $d(x,y)=|x-y|$ 'The usual metric on $\mathbb{R}$'.

Example 2.2

$X = \mathbb{R}^n$ with the Euclidean metric, $d(x, y) = \|x - y\| = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.

Uniform convergence of functions doesn't quite work: we want $d(f, g) = \sup|f - g|$ but this might not exist if $f - g$ is unbounded. However, we can do something with appropriate sets of functions.

Example 2.3

Let $Y \subset \mathbb{R}$. Take $X = B(Y) = \{f : Y \to \mathbb{R} \mid f \text{ bounded}\}$ with the uniform metric $d(f, g) = \sup_{x \in Y} |f - g|$.

Checking triangle inequality:

Proof. Let $f, g, h \in B(Y)$. Let $x \in Y$. Then

$$\begin{aligned} |f(x) - h(x)| &\le |f(x) - g(x)| + |g(x) - h(x)| \\ &\le d(f, g) + d(g, h) \end{aligned}$$

Taking sup over all $x \in Y$

$$d(f, h) \le d(f, g) + d(g, h).$$

□

Definition 2.3 (Subspace)

Suppose (X, d) a metric space and $Y \subset X$. Then $d\,|_{Y^2}$ is a metric on Y. We say Y with this metric is a **subspace** of X.

Example 2.4

Subspaces of $\mathbb{R}$: any of $\mathbb{Q}, \mathbb{Z}, \mathbb{N}, [0, 1], \dots$ with the usual metric $d(x, y) = |x - y|$.

Example 2.5

Recall that a cts function on a closed bounded interval is bounded. Define $C([a, b]) = \{f : [a, b] \to \mathbb{R} \mid f \text{ cts}\}$. This is a subspace of $B([a, b])$, example 2.3. That is $C([a, b])$ is a metric space with the uniform metric $\mathcal{L}(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$

Example 2.6

The empty metric space $X = \emptyset$ with the empty metric.

Could maybe define different metrics on the same set:

Example 2.7

The ℓ_1 metric on $\mathbb{R}^n$: $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$.

Example 2.8

The ℓ_∞ metric on $\mathbb{R}^n$: $d(x,y) = \max_i |x_i - y_i|$.[a]

[a] Proof of triangle inequality similar to example 2.3

Example 2.9

On $C([a,b])$ we can define the L_1 metric: $d(f,g) = \int_a^b |f - g|$.

Example 2.10

$X = \mathbb{C}$ with

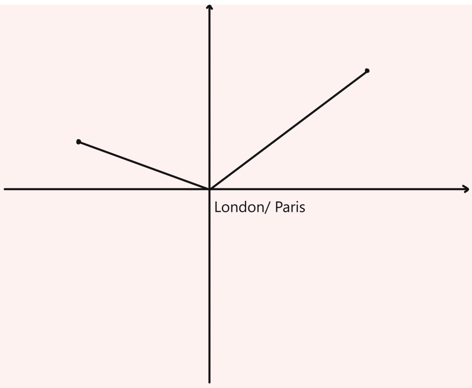
$$d(z,w) = \begin{cases} 0 & z = w \\ |z| + |w| & z \neq w. \end{cases}.$$

First two conditions of a metric hold obviously, for triangle inequality we need $d(u,w) \leq d(u,v) + d(v,w)$.

1. If $u = w$, LHS $= 0$ ✓
2. If $u = v$ or $v = w$ then LHS $=$ RHS ✓
3. If u, v, w distinct:

$$\begin{aligned} LHS &= |u| + |w| \\ RHS &= |u| + |w| + 2|v| \checkmark \end{aligned}$$

This metric is often called the British Rail metric or SNCF metric, you can think of it as for distinct points you have to travel through the origin.



Example 2.11 (Discrete metric)

Let X be any set. Define a metric d on X by

$$d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases}$$

Easy to check this works. This is called the discrete metric on X.

Example 2.12 (p-adic metric)

Let $\mathbb{X} = \mathbb{Z}$. Let p be a prime. The **p-adic metric** on $\mathbb{Z}$ is the metric d defined by:

$$d(x, y) = \begin{cases} 0 & x = y \\ p^{-a} & \text{if } x \neq y \text{ and } x - y = p^a m \text{ with } p \nmid m. \end{cases}$$

'Two numbers are close if difference is divisible by a large power of p'.

Only thing we need to check is triangle inequality

Proof. STP: $d(x,z) \leq d(x,y) + d(y,z)$

1. If $x = z$, LHS $= 0$ ✓
2. If $x = y$ or $y = z$ then LHS $=$ RHS ✓

So easy if any two of x, y, z the same so assume x, y, z all distinct. Let $x - y = p^a m$ and $y - z = p^b n$ where $p \nmid m, p \nmid n$ and wlog $a \leq b$. So $d(x,y) = p^{-a}$ and $d(y,z) = p^{-b}$.

Now:

$$\begin{aligned} x - z &= (x - y) = (y - z) \\ &= p^a m + p^b n \\ &= p^a \underbrace{(m + p^{b-a} n)}_{\text{integer}} \text{ as } a \leq b. \end{aligned}$$

So $p^a \mid x - z$ so $d(x,z) \leq p^{-a}$. But $d(x,y) + d(y,z) \geq d(x,y) = p^{-a}$. □

[a] p^a is the largest a s.t. $p^a \mid x - y$

Definition 2.4 (Convergence)

Let (X,d) be a metric space, let (x_n) be a sequence in X and let $x \in X$. We say (x_n) **converges** to x and write '$x_n \to x$' or '$x_n \to x$ as $n \to \infty$' if

$$\forall\, \epsilon > 0\ \exists\, N\ \forall\, n \geq N\ d(x_n, x) < \epsilon.$$

Equivalently $x_n \to x$ iff $d(x_n, x) \to 0$ in $\mathbb{R}$.

Proposition 2.1

Limits are unique. That is, if (X,d) is a metric space, (x_n) a sequence in X, $x, y \in X$ with $x_n \to x$ and $x_n \to y$ then $x = y$.

Proof. For each n,

$$\begin{aligned} d(x,y) &\leq d(x, x_n) + d(x_n, y) \text{ by triangle inequality} \\ &\leq d(x_n, x) + d(x_n, y) \text{ by symmetry} \\ &\to 0 + 0 = 0 \text{ as } d(x_n, x), d(x_n, y) \to 0 \end{aligned}$$

So $d(x,y) \to 0$ as $n \to \infty$. But $d(x,y)$ is constant so $d(x,y) = 0$ so $x = y$. □

Remark 6. This justifies talking about the limit of a convergent sequence in a metric space, and writing $x = \lim_{n\to\infty} x_n$ if $x_n \to x$.

Remark 7 (Remarks on definition of convergence in a metric space).

1. Constant sequences obviously converge. More over, eventually constant sequences converge.
2. Suppose (X, d) is a metric space and Y is a subspace of X. Suppose (x_n) is a sequence in Y which converges in Y to x. Then also (x_n) converges in X to x.

   However, converse is false: e.g. in $\mathbb{R}$ with the usual metric then $\frac{1}{n} \to 0$ as $n \to \infty$. Consider the subspace $\mathbb{R} \setminus \{0\}$. Then $(\frac{1}{n})$ is a sequence in $\mathbb{R} \setminus \{0\}$ but it doesn't converge in $\mathbb{R} \setminus \{0\}$. (Why? Suppose $\frac{1}{n} \to x$ in $\mathbb{R} \setminus \{0\}$. Then also $\frac{1}{n} \to x$ in $\mathbb{R}$. But $\frac{1}{n} \to 0$ in $\mathbb{R}$ so by uniqueness of limits $x = 0$. But $x \in \mathbb{R} \setminus \{0\}$ and $0 \notin \mathbb{R} \setminus \{0\}$ $\lightning$.)

Example 2.13

Let d be the Euclidean metric on $\mathbb{R}^n$. Exactly as in $\mathbb{R}^2$, we have $x_n \to x$ iff the sequence converges in each coordinate in the usual way in $\mathbb{R}$.

What about other metrics on $\mathbb{R}^n$? E.g. let d_∞ be the uniform metric: $d_\infty(x, y) = \max_i |x_i - y_i|$. Which sequences converge in $(\mathbb{R}^n, d_\infty)$? $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2} \le \sqrt{\sum_{i=1}^n d_\infty(x, y)^2}$ so $d(x, y) \le \sqrt{n} d_\infty(x, y)$. But also $d_\infty(x, y) \le d(x, y)$ as one of the terms in $d(x, y)$ is d_∞^2.

Now suppose (x_n) is a sequence in $\mathbb{R}^n$. Then $d(x_n, x) \to 0 \iff d(x_n, x) \to 0$. So exactly same sequences converge in $(\mathbb{R}^n, d)$ and $(\mathbb{R}^n, d_\infty)$

What about ℓ_1 metric d_1? $d_1(x, y) = \sum_{i=1}^n |x_i - y_i|$. Similarly, $d_\infty(x, y) \le d_1(x, y) \le n d_\infty(x, y)$. So again, exactly the same sequences converge in $(\mathbb{R}^n, d_1)$.

Example 2.14

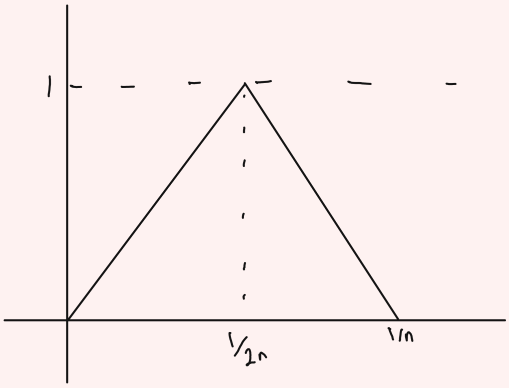
Let $X = C([0, 1]) = \{f : [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$. Let d_∞ be the uniform metric on X: $d_\infty(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$.

$$\begin{aligned} f_n \to f \text{ in } (X, d_\infty) &\iff d_\infty(f_n, f) \to 0 \\ &\iff \sup_{x \in [0,1]} |f_n(x) - f(x)| \to 0 \\ &\iff f_n \to f \text{ uniformly.} \end{aligned}$$

We also have L_1-metric d_1 on X: $d_1(f, g) = \int_0^1 |f - g|$. Now $d_1(f, g) = \int_0^1 |f - g| \le \int_0^1 d_\infty(f, g) = d_\infty(f, g)$. So similarly to previous example,

$$f_n \to f \text{ in } (X, d_\infty) \implies f_n \to f \text{ in } (X, d_1).$$

But converse does not hold, i.e. we can find a sequence (f_n) in X s.t. $f_n \to 0$ in d_1-metric but f_n doesn't converge in d_∞-metric, i.e. $\int_0^1 |f_n| \to 0$ as $n \to \infty$ but (f_n) does not converge uniformly.



$$f_n(x) = \begin{cases} 2nx & x \le \frac{1}{2n} \\ 2n(\frac{1}{n} - x) & \frac{1}{2n} < x \le \frac{1}{n} \\ 0 & x > \frac{1}{n}. \end{cases}$$

Then $d_1(f_n, 0) = \frac{1}{2} \times \frac{1}{n} \times 1 = \frac{1}{2n} \to 0$. So in (X, d_1) we have $f_n \to 0$. But f_n does not converge uniformly: indeed, $f_n \to 0$ pointwise; if we have uniform convergence then uniform limit is the same as pointwise limit; but $\forall\ n\ f_n(\frac{1}{2n}) = 1$ so $f_n \not\to 0$ uniformly.

Example 2.15

Let (X, d) be a discrete metric space; $d(x, y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$. When do we have $x_n \to x$ if (X, d)?

Suppose $x_n \to x$, i.e. $\forall\ \epsilon > 0\ \exists\ N\ \forall\ n \ge N\ d(x_n, x) < \epsilon$. Setting $\epsilon = 1$ in this, we can find N s.t. $\forall\ n \ge N\ d(x_n, x) < 1$, i.e. $\forall\ n \ge N\ d(x_n, x) = 0$ i.e. $\forall\ n \ge N\ x_n = x$. Thus (x_n) is eventually constant.
But we know in <u>any</u> metric space, eventually constant sequences converge.

So in this space, (x_n) converges iff (x_n) eventually constant.

Definition 2.5 (Continuity)

Let (X, d) and (Y, e) be metric spaces and let $f : X \to Y$.

1. Let $a \in X$ and $b \in Y$. We say $f(x) \to b$ as $x \to a$ if $\forall\, \epsilon > 0\, \exists\, \delta > 0\, \forall\, x \in X\ 0 < d(x,a) < \delta \implies e(f(x), b) < \epsilon$.
2. Let $a \in X$. We say f is **continuous** at a if $f(x) \to f(a)$ as $x \to a$.
   That is: $\forall\, \epsilon > 0\, \exists\, \delta > 0\, \forall\, x \in X\ d(x,a) < \delta \implies e(f(x), f(a)) < \epsilon$.
3. If $\forall\, a \in X$ f is continuous at a we say f is a **continuous** function or simply f is **continuous**.
4. We say f is **uniformly continuous** if $\forall\, \epsilon > 0\, \exists\, \delta > 0\, \forall\, x, y \in X\ d(x,y) < \delta \implies e(f(x), f(y)) < \epsilon$
5. Suppose $W \subset X$. We say f is **continuous on** W (respectively **uniformly continuous on** W) if the function $f|_W$ is continuous (resp. uniformly continuous), as a function from $W \to Y$ where we are now thinking of W as a subspace of X.

Remark 8.

1. Don't have a nice rephrasing of item 1 in terms of similar concepts in the reals. We would want to write '$e(f(x), b) \to 0$ as $d(x,a) \to 0$'. But this is meaningless, we haven't defined such a concept in the reals.
2. Item 1 says nothing about what happens at the point a itself. E.g. let $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$. Then $f(x) \to 0$ as $x \to 0$ (but $f(0) \neq 0$ so f is not continuous at 0).
   If we have f cts then $d(x,a) = 0 \implies x = a \implies f(x) = f(a) \implies e(f(x), f(a)) = 0$. So we can drop the '$0 <$' from definition of continuity.
3. We can rewrite item 5: f is continuous on W iff $f|_W$ is a continuous function $f|_W\colon W \to Y$ thinking of W as a subspace of X. That is: $\forall\, a \in W\, \forall\, \epsilon > 0\, \exists\, \delta > 0\, \forall\, x \in X\ d(x,a) < \delta \implies e(f(x), f(a)) < \epsilon$. In particular, note the subtlety that this <u>only</u> mentions points of W. So under this definition, e.g. $f : \mathbb{R} \to \mathbb{R}, f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$ then $f|_{[0,1]}$ is cts. But f is not cts at points $0, 1$.

Proposition 2.2

Let $(X,d), (Y,e)$ be metric spaces, $f : X \to Y$ and $a \in X$. Then f is continuous at a iff whenever (x_n) is a sequence in X with $x_n \to a$ then $f(x_n) \to f(a)$.

Proof. ($\implies$): Suppose f is cts at a. Let (x_n) be a sequence in X with $x_n \to a$. Let $\epsilon > 0$. As f cts at a we can find $\delta > 0$ s.t. $\forall\, x \in X$ s.t. $d(x,a) < \delta \implies e(f(x), f(a)) < \epsilon$. As $x_n \to x$ we can find N s.t. $n \geq N \implies d(x_n, a) < \delta$. Let $n \geq N$ then $d(x_n, a) < \delta$ so $e(f(x_n), f(a)) < \epsilon$. Hence $f(x_n) \to f(a)$.

($\Longleftarrow$): Suppose f is not cts at a. Then there is some $\epsilon > 0$ s.t. $\forall\, \delta > 0\ \exists\, x \in X$ with $d(x,a) < \delta$ but $e(f(x), f(a)) \geq \epsilon$. Now take $\delta = \frac{1}{n}$ we obtain a sequence (x_n) with, for each n $d(x_n, a) < \frac{1}{n}$ but $e(f(x_n), f(a)) > \epsilon$. Hence $x_n \to a$ but $f(x_n) \not\to f(a)$. □

Proposition 2.3

Let $(W,c), (X,d), (Y,e)$ be metric spaces, left $f : W \to X$, let $g : X \to Y$ and let $a \in W$. Suppose f is cts at a and g is cts at $f(a)$. Then $g \circ f$ is cts at a.

Proof. Let (x_n) be a sequence in W with $x_n \to a$. Then by proposition 2.2, $f(x_n) \to f(a)$ and so also $g(f(x_n)) \to g(f(a))$. So by proposition 2.2 $g \circ f$ cts at a.

What?airsntaeirsnt □

Example 2.16

In $\mathbb{R} \to \mathbb{R}$ with the usual metric, this is the same definition as when we defined continuity directly for $\mathbb{R}$ only. So we already have lots of cts fcns $\mathbb{R} \to \mathbb{R}$: polynomials, sin, e^x, ...

Example 2.17

Constant functions are continuous. Also if X is any metric space and $f : X \to X$ by $f(x) = x$ for all $x \in X$ (the indentity function) then that is continuous.

Example 2.18 (Projection Maps)

Consider $\mathbb{R}^n$ with the usual metric and $\mathbb{R}$ with the usual metric. The **projection maps** $\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x) = x_i$ are continuous.

(Why? We've seen convergence in $\mathbb{R}^n$ of sequences is the same as convergence in each coordinate. Let's denote a sequence in $\mathbb{R}^n$ by $(x^{(m)})_{m \geq 1}$. So e.g. $x_5^{(3)}$ is the 5th coord of the 3rd term. We know $x^{(m)} \to x$ iff for each i $x_i^{(m)} \to x_i$, i.e. for each i $\pi_i(x^{(m)}) \to \pi_i(x)$. Then we can use proposition 2.2)

Similarly, suppose $f_1, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ Let $f : \mathbb{R} \to \mathbb{R}^n$ defined by $f(x) = (f_1(x), \ldots, f_n(x))$. Then f is cts at a point iff all of $f_1, \ldots, f_n$ are. Using these facts example 2.16 and proposition 2.3, we have many cts fcns $\mathbb{R}^n \to \mathbb{R}^m$. E.g. $f : \mathbb{R}^3 \to \mathbb{R}^2, f(x,y,z) = (e^{-x} \sin y, 2x \cos z)$ is cts. (Why? write $w = (x,y,z) \in \mathbb{R}^3$, we have $f_1(w) = e^{-\pi_1(w)} \sin \pi_2(w)$ and $f_2(w) = 2\pi_1(w) \cos \pi_3(w)$. So f_1, f_2 cts so f cts.)

Example 2.19

Recall that if we have the Euclidean metric, the l_1 or l_∞ metric on $\mathbb{R}^n$ then the convergent sequences are the same. So by proposition 2.2, the ctf fcns $X \to \mathbb{R}^n$ or from $\mathbb{R}^n \to Y$ are the same with each of these three metrics.

Example 2.20

Let (x, d) be the discrete metric space, example 2.11, and let (Y, e) be any metric space. Which functions $f : X \to Y$ are cts? Suppose $a \in X$ and (x_n) a sequence in X with $x_n \to a$. Then (x_n) is eventually constant, i.e. for sufficiently large n $x_n = a$ and so $f(x_n) = f(a)$. So $f(x_n) \to f(a)$.

Hence every function on a discrete metric space is cts.

§2.2 Completeness

Question

In section 1 we saw a version of GPC held in each of the three examples we considered. Does GPC hold in a general metric space?

Definition 2.6 (Cauchy Sequences)

Let (X, d) be a metric space and let (x_n) be a sequence in X. We say (x_n) is **Cauchy** if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, m, n \geq N\ d(x_m, x_n) < \epsilon$.

Theorem 2.1

(x_n) convergent $\implies$ (x_n) Cauchy.

Proof. Left as an exercise. □

But converse is not true in general.

Example 2.21

Let $X = \mathbb{R} \setminus \{0\}$ with the usual metric and $x_n = \frac{1}{n}$. We say previously that (x_n) does not converge.
Note that X is a subspace of $\mathbb{R}$. In $\mathbb{R}$ (x_n) is convergent $(x_n \to 0)$ so (x_n) is Cauchy in $\mathbb{R}$ so (x_n) is Cauchy in X.

Example 2.22

$\mathbb{Q}$ with the usual metric. Let x_n be $\sqrt{2}$ to n decimal places. This converges in $\mathbb{R}$ so is Cauchy in $\mathbb{Q}$ but clearly doesn't converge in $\mathbb{Q}$.

Definition 2.7 (Completeness)

Let (X, d) be a metric space. We say X is **complete** if every Cauchy sequence in X converges.

Example 2.23

Example 2.21 says $\mathbb{R} \setminus \{0\}$ with the usual metric is not complete.
Similarly Q with usual metric is not complete.

Example 2.24

GPC says $\mathbb{R}$ with the usual metric is complete.

Example 2.25

GPC for $\mathbb{R}^n$ says $\mathbb{R}^n$ with Euclidean metric is complete.

Example 2.26

GPUC, theorem 1.8, (almost) says if $X \subset \mathbb{R}$ and $B(X) = \{f : X \to \mathbb{R} \mid f \text{ is bounded}\}$ with the uniform norm then $B(X)$ is complete.

Proof. Let (f_n) be a Cauchy sequence in $B(X)$. Then (f_n) is uniformly Cauchy so by GPUC is uniformly convergent. That is $f_n \to f$ uniformly for some $f : X \to \mathbb{R}$. As $f_n \to f$ uniformly we know $f_n - f$ is bounded for n suff. large. Take such an n, then $f_n - f$ and f_n are bounded so $f = f_n - (f_n - f)$ is bounded. That is, $f \in B(X)$. Finally, $f_n \to f$ uniformly and $d(f_n, f) \to 0$, i.e. $f_n \to f$ in $(B(X), d)$. $\square$

Remark 9. In many ways, this is typical of a proof that a given space (X, d) is complete:

1. Take (x_n) Cauchy in X;
2. Construct/ find a putative limit object x where it seems (x_n) converges to x in some sense;
3. Show $x \in X$,
4. Show $x_n \to x$ in metric space (X, d) i.e. that $d(x_n, x) \to 0$.

This is often tricky/ fiddly/ annoying/ repetitive/ boring. But we need to take care as for example, it's tempting to talk about $d(x_n, x)$ while doing (ii) or (iii); but makes no sense to write '$d(x_n, x)$' until we have completed (iii) as d is only defined on X^2 (if $x \notin X$ then can't use d).

Example 2.27

If $[a, b]$ is a closed interval then $C([a, b])$ with uniform norm d is complete.

Proof. (i): Let (f_n) be a Cauchy sequence in $C([a, b])$.
(ii): We know $C([a, b])$ is a subspace of $B([a, b])$ with uniform metric. We know $B([a, b])$ is complete by example 2.26 and (f_n) is a Cauchy sequence in $B([a, b])$ so in $B([a, b])$, $f_n \to f$ for some f.
(iii) Each f_n is cts and $f_n \to f$ uniformly so f is cts, i.e. $f \in C([a, b])$.
(iv) Finally, each $f_n \in C([a, b])$, $f \in C([a, b])$ and $f_n \to f$ uniformly so $d(f_n, f) \to 0$. □

This generalises:

Definition 2.8 (Closed Metric Space)

Let (X, d) be a metric space and $Y \subset X$. We say Y is **closed** if whenever (x_n) a sequence in Y with $x_n \to x \in X$ then $x \in Y$.

Proposition 2.4

A closed subset of a complete metric space is complete.

Remark 10. This does make sense: if $Y \subset X$ then Y is itself a metric space or a subspace of X so we can say e.g. 'Y is complete' to mean the metric space Y (as a subspace of X) is complete.

We could do exactly the same with any other properties of metric spaces we define.

Proof. Let (X, d) be a metric space and $Y \subset X$ with X complete and Y closed. (i): Let (x_n) be a Cauchy sequence in Y.
(ii): Now (x_n) is a Cauchy sequence in X so by completeness $x_n \to x$ in X for some $x \in X$.
(iii) $Y \subset X$ is closed so $x \in Y$.
(iv) Finally we now have each $x_n \in Y, x \in Y$ and $x_n \to x$ in X, so $d(x_n, x) \to 0$ so $x_n \to x$ in Y. □

Example 2.28

Define $\ell_1 = \{(x_n)_{n\geq 1} \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=1}^{\infty} |x_n| \text{ converges}\}$. Define a metric d on ℓ_1 by $d((x_n),(y_n)) = \sum_{n=1}^{\infty} |x_n - y_n|$.

Note we have $\sum |x_n|, \sum |y_n|$ converge as we are in ℓ_1. For each n $|x_n - y_n| \leq |x_n| + |y_n|$ so by comparison test $\sum |x_n - y_n|$ converges. So d is well-defined. Easy to check d is a metric on ℓ_1. Then (ℓ_1, d) is complete.

Proof. (i): Let $(x^{(n)})_{n\geq 1}$ be a Cauchy sequence in ℓ_1, so for each n $(x_i^{(n)})_{i\geq 1}$ is a sequence in $\mathbb{R}$ with $\sum_{i=1}^{\infty} |x_i^{(n)}|$ convergent.
(ii) For each i, $(x_i^{(n)})_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, since if $y, z \in \ell_1$ then $|y_i - z_i| \leq d(y,z)$. But $\mathbb{R}$ is complete, so for each i we can find $x_i \in \mathbb{R}$ s.t. $x_i^{(n)} \to x_i$ as $n \to \infty$. Let $x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}}$.
(iii) We next show $x \in \ell_1$, i.e. that $\sum_{i=1}^{\infty} |x_i|$ converges.

Given $y \in \ell_1$, define $\sigma(y) = \sum_{i=1}^{\infty} |y_i|$, i.e. $\sigma(y) = d(y,z)$ where z is the constant zero sequence.
We now have, for any m, n

$$\begin{aligned}
\sigma(x^{(m)}) &= d(x^{(m)}, z) \\
&\leq d(x^{(m)}, x^{(n)}) + d(x^{(n)}, z) \\
&= d(x^{(m)}, x^{(n)}) + \sigma(x^{(n)})
\end{aligned}$$

So $\sigma(x^{(m)}) - \sigma(x^{(n)}) \leq d(x^{(m)}, x^{(n)})$. Similarly, for any m, n $\sigma(x^{(n)}) - \sigma(x^{(m)}) \leq d(x^{(m)}, x^{(n)})$ and so $|\sigma(x^{(m)}) - \sigma(x^{(n)})| \leq d(x^{(m)}, x^{(n)})$. Hence $(\sigma(x^{(m)}))_{m\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, and so by GPC converges, say $\sigma(x^{(m)}) \to K$ as $m \to \infty$.

Claim 2.1

For any $I \in \mathbb{N}$, $\sum_{i=1}^{I} |x_i| \leq K + 2$.

Proof. As $\sigma(x^{(n)}) \to K$ as $n \to \infty$ we can find N_1 s.t. $n \geq N_1 \implies \sum_{i=1}^{\infty} |x_i^{(n)}| \leq K + 1$. Also, $n \geq N_1 \implies \sum_{i=1}^{I} |x_i^{(n)}| \leq K + 1$ (as each term non-negative).

Next, for each $i \in \{1, 2, \dots, I\}$ we have $x_i^{(n)} \to x_i$ as $n \to \infty$. So we can find N_2 s.t. $n \geq N_2 \implies \forall\, i \in \{1, \dots, I\}$ $|x_i^{(n)} - x_i| < I^{-1}$.

Now let $n = \max(N_1, N_2)$ then $\sum_{i=1}^{I} |x_i| \leq \sum_{i=1}^{I} |x_i^{(n)}| + \sum_{i=1}^{I} |x_i^{(n)} - x_i| \leq K + 1 + I(I^{-1}) = K + 2$. $\square$

Now the partial sums of $\sum |x_i|$ are increasing and bounded above so $\sum |x_i|$ converges. That is $x \in \ell_1$.

(iv) Finally, need to check $x^{(n)} \to x$ as $n \to \infty$ in ℓ_1, i.e. that $d(x^{(n)}, x) \to 0$ as $n \to \infty$.
We have, for all n, I:

$$\begin{aligned} d(x^{(n)}, x) &= \sum_{i=1}^{\infty} |x_i^{(n)} - x_i| \\ &\le \sum_{i=1}^{I} |x_i^{(n)} - x_i| + \sum_{i=I+1}^{\infty} |x_i^{(n)}| + \sum_{i=I+1}^{\infty} |x_i|. \end{aligned}$$

Let $\epsilon > 0$. We know $\sum |x_i|$ convergent (as $x \in \ell_1$) so we can pick I_1 s.t. $\sum_{i=I_1+1}^{\infty} |x_i| < \epsilon$.
As $(x^{(n)})$ is Cauchy, we can find N_1 s.t. $m, n \ge N_1 \implies d(x^{(m)}, x^{(n)}) < \epsilon$. As $\sum_i |x_i^{(N_1)}|$ converges, we can find I_2 s.t. $\sum_{i=I_2+1}^{\infty} |x_i^{(N_1)}| < \epsilon$. Then

$$\begin{aligned} n \ge N_1 \implies \sum_{i=I_2+1}^{\infty} |x_i^{(n)}| &\le \sum_{i=I_2+1}^{\infty} |x_i^{(N_1)}| + \sum_{i=I_2+1}^{\infty} |x_i^{(n)} - x_i^{(N_1)}| \\ &< \epsilon + d(x^{(n)}, x^{(N_1)}) \\ &< 2\epsilon. \end{aligned}$$

Let $I = \max(I_1, I_2)$. For each $i = 1, 2, \ldots, I$ we have $|x_i^{(n)} - x_i| \to 0$ as $n \to \infty$, so $\sum_{i=1}^{I} |x_i^{(n)} - x_i| \to 0$ as $n \to \infty$. Hence we can find N_2 s.t $n \ge N_2 \implies \sum_{i=1}^{I} |x_i^{(n)} - x_i| < \epsilon$. Let $N = \max(N_1, N_2)$ and let $n \ge N$. Then

$$\begin{aligned} d(x^{(n)}, x) &\le \sum_{i=1}^{I} |x_i^{(n)} - x_i| + \sum_{i=I+1}^{\infty} |x_i^{(n)}| + \sum_{i=I+1}^{\infty} |x_i| \\ &\le \sum_{i=1}^{I} |x_i^{(n)} - x_i| + \sum_{i=I_2+1}^{\infty} |x_i^{(n)}| + \sum_{i=I_1+1}^{\infty} |x_i| \\ &< \epsilon + 2\epsilon + \epsilon = 4\epsilon \end{aligned}$$

Hence $d(x^{(n)}, x) \to 0$ as $n \to \infty$, i.e. $x^{(n)} \to x$ in ℓ_1.
Hence ℓ_1 is complete. □

Now we will move on to main theorem of completeness.

Definition 2.9 (Contraction mapping)

Let (X, d) be a metric space and $f : X \to X$. We say f is a **contraction** if $\exists\, \lambda \in [0, 1)$ s.t. $\forall\, x, y \in X$ $d(f(x), f(y)) \le \lambda d(x, y)$.

Theorem 2.2 (The Contraction Mapping Theorem)

Let (X, d) be a complete, non-empty metric space and $f : X \to X$ a contraction. Then f has a unique fixed point.

Proof. Let $\lambda \in [0, 1)$ satisfy $\forall\, x, y \in X\ d(f(x), f(y)) \le \lambda d(x, y)$.
Let $x_0 \in X$. Recursively define $x_n = f(x_{n-1})$ for $n \ge 1$. Let $\Delta = d(x_0, x_1)$. Then, by induction $d(x_n, x_{n+1}) \le \lambda^n \Delta$ for all n.
Now suppose $N \le m < n$. Then

$$\begin{aligned} d(x_m, x_n) &\le \sum_{i=m}^{n-1} d(x_i, x_{i+1}) \\ &\le \sum_{i=m}^{n-1} \lambda^i \Delta \\ &\le \sum_{i=N}^{\infty} \lambda^i \Delta \\ &= \frac{\lambda^N \Delta}{1-\lambda} \to 0 \text{ as } N \to \infty. \end{aligned}$$

So $\forall\ \epsilon > 0\ \exists\ N\ \forall\ m, n \ge N\ d(x_m, x_n) < \epsilon$ (i.e. we take N s.t. $\frac{\lambda^N \Delta}{1-\lambda} < \epsilon$). Thus (x_n) is Cauchy, so by completeness converges, say $x_n \to x \in X$. But also $x_n = f(x_{n-1}) \to f(x)$ as f continuous[a]. So by uniqueness of limits, $f(x) = x$.

Suppose also $f(y) = y$ for some $y \in X$. Then $d(x, y) = d(f(x), f(y)) \le \lambda d(x, y)$ with $\lambda < 1$. So $d(x, y) = 0$ so $x = y$. □

[a] follows immediately from definition of contraction mapping (e.g. let $\delta = \epsilon$ in definition of continuity).

Remark 11.

1. Why is f cts? We have, for all $x, y \in X\ d(f(x), f(y)) \le d(x, y)$. So $\forall\ \epsilon > 0$, $d(x, y) < \epsilon \implies d(f(x), f(y)) < \epsilon$. (Indeed, this shows f is uniformly continuous.)

2. We have proved more than claimed. Not only does f have a unique fixed point, but starting from <u>any</u> point of the space and repeatedly apply f then the resulting sequence converges to the fixed point. In fact, the speed of convergence is exponential.

Example 2.29 (Application)

Suppose we want to numerically approximate the solution to $\cos x = x$. Any root must lie in $[-1, 1]$. Consider metric space $X = [-1, 1]$ with usual metric. X is a closed subset of a complete space $\mathbb{R}$ so is complete. Obviously X is non-empty.

Think of $\cos : [-1, 1] \to [-1, 1]$. Suppose $x, y \in [-1, 1]$.

$$\begin{aligned} |\cos x - \cos y| &= |x - y||\cos' z| \text{ for some } z \in [-1, 1] \text{ by MVT} \\ &|x - y|| - \sin' z| \\ &\leq |x - y| \sin 1 \end{aligned}$$

But $0 \leq \sin 1 < 1$ so cos is a contraction of $[-1, 1]$. So by The Contraction Mapping Theorem, cos has a unique fixed point in $[-1, 1]$. That is $\cos x = x$ has a unique solution.

How do we find it numerically? Use remark 2, we will have rapid convergence to the root.

We will see two major applications of CMT (The Contraction Mapping Theorem) later.

§2.3 Sequential Compactness

Recall BW for $\mathbb{R}^n$ says a bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Definition 2.10 (Bounded)

Let (X, d) be a metric space. We say X is **bounded** if

$$\exists\, M \in \mathbb{R}\; \forall\, x, y \in X\;\; d(x, y) \leq M.$$

Remark 12. Easy to check by triangle inequality that X bounded $\iff$ ($X = \emptyset$ or $\exists\, M \in \mathbb{R}\; \epsilon x \in X$ s.t. $\forall\, y \in X\;\; d(x, y) \leq M$). So definition agrees with earlier definition for subsets of $\mathbb{R}^n$.

Definition 2.11 (Closed subspace)

Let (X, d) be a metric space and $Y \subset X$. We say Y is **closed** in X if whenever (x_n) is a sequence in Y with, in X, $x_n \to x \in X$ then $x \in Y$.

Definition 2.12 (Sequentially Compact)

A metric space is **sequentially compact** if every sequence has a convergent subsequence.

BW for $\mathbb{R}^n$ is essentially the following:

Theorem 2.3

Let $X \subset \mathbb{R}^n$ with the Euclidean metric. Then X is sequentially compact iff X is closed and bounded.

Proof. ($\Longleftarrow$) Suppose X is closed and bounded. Let (x_n) be a sequence in X. Then (x_n) is a bounded sequence in $\mathbb{R}^n$ so by BW, in $\mathbb{R}^n$, $x_{n_j} \to x$ for some $x \in \mathbb{R}^n$ and some subsequence (x_{n_j}) of (x_n).
As X is closed, $x \in X$. Hence the subsequence (x_{n_j}) converges in X. So X is sequentially compact.

($\implies$) Suppose X is not closed. Then we can find a sequence (x_n) in X s.t. in $\mathbb{R}^n$ $x_n \to x \in \mathbb{R}^n$ with $x \notin X$. Now any subsequence $(x_{n_j}) \to x$ in $\mathbb{R}^n$. But $x \notin X$ so by uniqueness of limits (x_{n_j}) does not converge in X. So X is not sequentially compact.

Suppose instead X is not bounded. Then we can find a sequence (x_n) in X with $\forall\ n\ \|x_n\| \geq n$, i.e. $\|x_n\| \to \infty$ as $n \to \infty$. Suppose we have a subsequence $x_{n_j} \to x \in X$. Then $\left\|x_{n_j}\right\| \to \|x\|$ but $\left\|x_{n_j}\right\| \to \infty$ ↯. So, again, X is not sequentially compact. $\square$

Part II
Generalizing differentiation