Stochastic Financial Models 3

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1 CAPM, continued

Now let's model the entire market. Assumptions:

- There is a total of $n_i > 0$ shares of asset $i = 1, \ldots, d$, and let $n = (n_1, \ldots, n_d)^\top$.
- There are K agents in the market, and agent k holds portfolio θ_k.
- Total supply equals total demand so that

$$\sum_k \theta_k = n.$$

- Each agent's portfolio is mean-variance efficient and they agree on the mean and covariance of S_1.

By the mutual fund theorem, for each k we have

$$\theta_k = \lambda_k \theta_{\mathrm{Mar}}$$

where $\lambda_k \geq 0$. Hence,

$$n = \Lambda\, \theta_{\mathrm{Mar}}$$ as $n = \sum \theta_k = \theta_{mar} \sum \lambda_k$

where $\Lambda = \sum_k \lambda_k$. Since $n \neq 0$, it follows $\Lambda > 0$. That is the say, in this model, the entire market is just some positive scalar multiple of the market portfolio (explaining the name).

A prediction of the CAPM is that when the excess returns of a portfolio are statistically regressed against the excess returns of a broad market index (such as the FTSE or S&P) then you should find $\alpha = 0$.

Remark. Markowitz and Sharpe shared the 1990 Nobel Prize in Economics for studying mean-variance efficiency and the CAPM.

2 Expected utility hypothesis

Up to now, given two random payouts X and Y we have implicitly assumed that an agent prefers X over Y if either

- $\mathbb{E}(X) > \mathbb{E}(Y)$ and $\mathrm{Var}(X) \leq \mathrm{Var}(Y)$, or
- $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathrm{Var}(X) < \mathrm{Var}(Y)$

This is rather crude. Here is a historical example that illustrates one of the issues.
Aside: historical origin of expected utility hypothesis (not lectured). Consider the *St Petersburg paradox*: You and I play a game. I toss a coin repeatedly until it comes up heads. If toss the coin a total of n times, I will pay you 2^n pounds. How much would you pay me to play this game? This problem was invented by Nicolaus Bernoulli in 1713. The issue is that according to N Bernoulli's intuition, the answer should be the expected value of the payout $\sum_n 2^n \times 2^{-n} = \infty$, but he thought no sensible person would pay more than 20 pounds. His cousin Daniel Bernoulli proposed in 1738 that people don't care about the expected payout *per se*, but instead the relevant quantity is the expected *utility* of the payout.

Definition. The *expected utility hypothesis* says that each agent has a function U (called the *utility function*) such that the agent prefers random payout X to Y if and only if

$$\mathbb{E}[U(X)] > \mathbb{E}[U(Y)]$$

In the case $\mathbb{E}[U(X)] = \mathbb{E}[U(Y)]$ the agent is said to be *indifferent* between X and Y.

Remark. If $\tilde{U}(x) = a + b\, U(x)$ with $b > 0$, then $\tilde{U}$ gives rise to the same expected utility preferences as U.

Remark. In 1947, von Neumann–Morgenstern axioms derived a short list of properties of an agent's preferences which are equivalent to the assumption that the agent's preferences are derived from expected utility.

3 Risk-aversion and concavity

Once we've assumed the expected utility hypothesis, there are two additional properties we will assume of the agent's utility function:

- *(Strictly) increasing.* $x > y$ implies $U(x) > U(y)$.
- *(Strictly) concave.*

  $$U(px + (1-p)y) > p\, U(x) + (1-p)U(y)$$

  for any $x \neq y$ and $0 < p < 1$.

Remark. Note that if $X \geq Y$ almost surely, then $X \succeq Y$. Furthermore, if $\mathbb{P}(X > Y) > 0$ then $X \succ Y$.

Remark. Recall Jensen's inequality:

$$U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)]$$

whenever the expectations are defined. Hence $\mathbb{E}(X) \succeq X$ for any random payout X. If X is not constant, then $\mathbb{E}(X) \succ X$.