Stochastic Financial Models 6

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18 October

1 Proofs of indifference pricing properties

To prove the properties listed last time, it is convenient to define for a any suitable random variable Z the *indirect utility*

$$V(Z) = \max_{X \in \mathcal{X}} \mathbb{E}[U(X + Z)]$$

In this notation, π is an indifference price for the claim with payout Y iff

$$V(Y - (1+r)\pi) = V(0).$$

We prove two lemmata:

Lemma (Indirect utility is strictly increasing)**.** *If $Z_0 \leq Z_1$ almost surely with $\mathbb{P}(Z_0 < Z_1) > 0$ then*

$$V(Z_1) > V(Z_0)$$

Proof of lemma. Let X^i be the maximiser for the two problems, i.e.

$$V(Z_i) = \mathbb{E}[U(X^i + Z_i)]$$

for $i = 0, 1$. Then

$$\begin{aligned} V(Z_1) &= \mathbb{E}[U(X^1 + Z_1)] \\ &\geq \mathbb{E}[U(X^0 + Z_1)] \\ &> \mathbb{E}[U(X^0 + Z_0)] \\ &= V(Z_0) \end{aligned}$$

□

Lemma (Indirect utility is concave)**.** *Given random variable Z_0, Z_1 and $0 < p < 1$. Then*

$$V(pZ_1 + (1-p)Z_0) \geq pV(Z_1) + (1-p)V(Z_0)$$

Proof of lemma. Let X^i be the maximiser for the two problems for $i = 0, 1$.

Now noting that $pX^1 + (1-p)X^0 \in \mathcal{X}$ yields (expand X_1, X_0 to see this)

$$\begin{aligned} pV(Z_1) + (1-p)V(Z_0) &= \mathbb{E}[pU(X^1 + Z_1) + (1-p)U(X^0 + Z_0)] \\ &\leq \mathbb{E}[U(pX^1 + (1-p)X^0 + pZ_1 + (1-p)Z_0)] \quad \text{as } U \text{ concave} \\ &\leq \max_{X \in \mathcal{X}} \mathbb{E}[U(X + pZ_1 + (1-p)Z_0)] \\ &= V(pZ_1 + (1-p)Z_0) \end{aligned}$$

□

Proof of existence and uniqueness of indifference prices. By our assumption of the existence of a maximiser, we have $V(0) = \mathbb{E}[U(X^*)]$ for some $X^* \in \mathcal{X}$. In particular we have that $U(-\infty) < V(0) < U(\infty)$.

For fixed Y, we will show that the function $x \mapsto V(Y + x)$ is a bijection from $(-\infty, \infty)$ to $(U(-\infty), U(\infty))$. This would imply that there is a unique solution x to $V(Y + x) = V(0)$. The indifference price is uniquely defined by $\pi(Y) = -\frac{1}{1+r}x$.

Note the function $x \mapsto V(Y + x)$ is strictly increasing, and hence an injection. To complete the proof, we need only show its range is the interval $(U(-\infty), U(\infty))$.

The function is concave, hence continuous, so its range is an interval. Since strictly increasing concave functions are unbounded from the left, we have

$$V(Y + x) \downarrow -\infty = U(-\infty) \text{ as } x \downarrow -\infty.$$

Also

$$V(Y + x) \geq \mathbb{E}[U(X^* + Y + x)] \uparrow U(+\infty) \text{ as } x \uparrow +\infty$$

by a form of the monotone convergence theorem from Probability & Measure (this step is not examinable). This shows $x \mapsto V(Y + x)$ is a bijection. □

Proof that indifference prices are increasing. Suppose $Y_0 \leq Y_1$ a.s. and $\mathbb{P}(Y_0 < Y_1) > 0$. Note

$$\begin{aligned} V(Y_1 - (1+r)\pi(Y_1)) &= V(0) \\ &= V(Y_0 - (1+r)\pi(Y_0)) \\ &< V(Y_1 - (1+r)\pi(Y_0)). \end{aligned}$$

Since $x \mapsto V(Y_1 + x)$ is strictly increasing, we have $-(1+r)\pi(Y_1) < -(1+r)\pi(Y_0)$ as desired. □

Proof of concavity of indifference prices. Given Y_0, Y_1 and $0 < p < 1$, let $Y_p = pY_1 + (1-p)Y_0$ and $\pi_i = \pi(Y_i)$ for $i = 0, p, 1$. By definition of indifference price and concavity of V we have

$$\begin{aligned} V(Y_p - (1+r)\pi_p) =& V(0) \\ =& V(Y_1 - (1+r)\pi_1) \\ =& V(Y_0 - (1+r)\pi_0) \\ =& pV(Y_1 - (1+r)\pi_1) + (1-p)V(Y_0 - (1+r)\pi_0) \\ \leq& V(Y_p - (1+r)(p\pi_1 + (1-p)\pi_0)) \end{aligned}$$

Since $x \mapsto V(Y_p + x)$ is strictly increasing, we have $-(1+r)\pi_p \leq -(1+r)(p\pi_1 + (1-p)\pi_0)$. $\square$

Proof that marginal utility price is larger than indifference price. Let X^* be the optimiser without the claim, and X^1 be the optimiser with the claim. Using the supporting line property of the concave function U we have

$$\begin{aligned}
V(0) &= V(Y - (1+r)\pi(Y)) \\
&= \mathbb{E}[U(X^1 + Y - (1+r)\pi(Y))] \\
&\leq \mathbb{E}[U(X^*)] + \mathbb{E}[U'(X^*)(X^1 - X^* + Y - (1+r)\pi(Y))] \\
&= V(0) + \mathbb{E}[U'(X^*)Y] - \mathbb{E}[U'(X^*)](1+r)\pi(Y)
\end{aligned}$$

as a concave fcn $U \leq$ tangent.

where we have used the fact that

$$\mathbb{E}[U'(X^*)(X^1 - X^*)] = (\theta^1 - \theta^*)^\top \mathbb{E}[U'(X^*)(S_1 - (1+r)S_0)] = 0.$$

by first marginal utility pricing result.

The conclusion follows upon rearranging. $\square$

$$\pi(y) \leq \frac{\mathbb{E}[U'(X^*)Y]}{\mathbb{E}[U'(X^*)](1+r)}$$

$$S_0 = \frac{\mathbb{E}[U'(X^*)S_1]}{(1+r)\mathbb{E}[U'(X^*)]}$$