# Stochastic Financial Models 1

Michael Tehranchi

6 October 2023

**Remark.** The Part II course Probability & Measure is listed as desirable for this course. This is because we will be dealing with random variables, and being familiar with some probability theory will be handy. There are essentially three places where we use measure-theoretic probability:

- The convergence theorems will be used to justify statements such as $\lim_n \mathbb{E}(Z_n) = \mathbb{E}(\lim_n Z_n)$.
- The notions of measurability and sigma-algebra to model what information is available in a probabilistic setting
- The monotone class theorem, which says that in order to prove an identity involving expected values, it is usually sufficient check a special case.

However, this course is self-contained, so attending Probability & Measure is absolutely **not** necessary.

## 1 Standing assumptions and notation

Financial market consists of $d$ risky assets.

- No dividends.
- Infinitely divisibility.
- No bid-ask spread.
- No price impact.
- No transaction costs
- No short selling constraints

The price of asset $i$ at time $t$ will be denoted $S_t^i$. We will let $S_t = (S_t^1, \ldots, S_t^d)^\top$ be the column vector of prices. In addition, market participants can borrow or lend at a risk-free interest rate $r$, assumed constant.

## 2 The one-period set-up

Introduce an investor. Let $\theta^i$ be the number of shares of asset $i$ that the investor buys at time $t = 0$. (When $\theta^i < 0$ then the investor shorts $|\theta^i|$ shares of the asset.) Let $\theta = (\theta^1, \dots, \theta^d)^\top$ be the column vector of portfolio weights. In addition, let $\theta^0$ be the amount of money the investor puts in the bank. The investor's wealth at time $t$ is denoted $X_t$.

- Initial wealth $X_0 = \theta^0 + \theta^\top S_0$.
- Time-1 wealth $X_1 = \theta^0(1+r) + \theta^\top S_1$.
- $X_1 = (1+r)X_0 + \theta^\top [S_1 - (1+r)S_0]$

We think of the interest rate $r$ and the initial asset prices $S_0$ as known at time 0. We will model the time-1 asset prices $S_1$ as a random vector. Moreover, we make the (unrealistically) assumption that we are completely *certain* that we know the *distribution* of $S_1$. In particular, given the initial wealth $X_0$ and the portfolio $\theta$, we will model the time-1 wealth $X_1$ as a random variable with a known distribution.

## 3 The mean-variance portfolio problem

**Mean-variance portfolio problem** (Markowitz 1952) Given initial wealth $X_0$ and target mean $m$, find the portfolio $\theta$ to minimise $\mathrm{Var}(X_1)$ subject to $\mathbb{E}(X_1) \geq m$.

We will assume the random vector $S_1$ is square-integrable and adopt the notation

- $\mu = \mathbb{E}(S_1)$. We will assume $\mu \neq (1+r)S_0$.
- $V = \mathrm{Cov}(S_1) = \mathbb{E}[(S_1 - \mu)(S_1 - \mu)^\top]$. Recall that $V$ is automatically symmetric and non-negative definite. We will *assume* that $V$ is positive definite. In particular, the inverse $V^{-1}$ exists.

In this notation, we have

- $\mathbb{E}(X_1) = (1+r)X_0 + \theta^\top[\mu - (1+r)S_0]$ and
- $\mathrm{Var}(X_1) = \theta^\top V \theta$

so the mean-variance portfolio problem is to find $\theta$ such that

$$\text{minimise } \theta^\top V \theta \quad \text{subject to } \theta^\top[\mu - (1+r)S_0] \geq m - (1+r)X_0$$

**Theorem** (Mean-variance optimal portfolio)**.** *The unique optimal solution to the mean-variance portfolio problem is*

$$\theta = \lambda\, V^{-1}[\mu - (1+r)S_0]$$

*where*

$$\lambda = \frac{(m - (1+r)X_0)^+}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]}$$

**Notation.** Here and throughout the course we will use the common notation $x^+ = \max\{x, 0\}$ for a real number $x$.

*Proof.* Next lecture.

# Stochastic Financial Models 2

Michael Tehranchi

9 October 2023

## 1 Mean-variance efficiency

To solve the mean-variance portfolio problem, we will use the following lemma

**Lemma.** *If* $\theta^\top a = b$ *then*

$$\theta^\top V \theta \geq \frac{b^2}{a^\top V^{-1} a}$$

*with equality if and only if*

$$\theta = \lambda \, V^{-1} a$$

*where*

$$\lambda = \frac{b}{a^\top V^{-1} a}.$$

*Proof of lemma.* Since $V$ is non-negative definite we have

$$\begin{aligned}
\theta^\top V \theta =& \theta^\top V \theta + 2\lambda(b - \theta^\top a) \\
=& (\theta - \lambda \, V^{-1} a)^\top V (\theta - \lambda \, V^{-1} a) \\
& + 2\lambda b - \lambda^2 \, a^\top V^{-1} a \\
\geq & 2\lambda b - \lambda^2 \, a^\top V^{-1} a = \frac{b^2}{a^\top V^{-1} a}
\end{aligned}$$

and since $V$ is positive definite there is equality only if

$$\theta = \lambda \, V^{-1} a$$

□

**Remark.** This proof is secretly using the Lagrangian technique from IB Optimisation or Variational Principles. The constant $\lambda$ could be thought of as a Lagrange multiplier.

**Remark.** The lemma is equivalent to

$$(\theta^\top a)^2 \leq (\theta^\top V \theta)(a^\top V^{-1} a).$$

This is just the Cauchy–Schwarz inequality applied to the vectors $V^{1/2}\theta$ and $V^{-1/2}a$.

By applying the lemma with $a = \mu - (1+r)S_0$ and $b = \mathbb{E}(X_1) - (1+r)X_0$, we see that

$$\mathrm{Var}(X_1) \geq \frac{(\mathbb{E}(X_1) - (1+r)X_0)^2}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]}$$

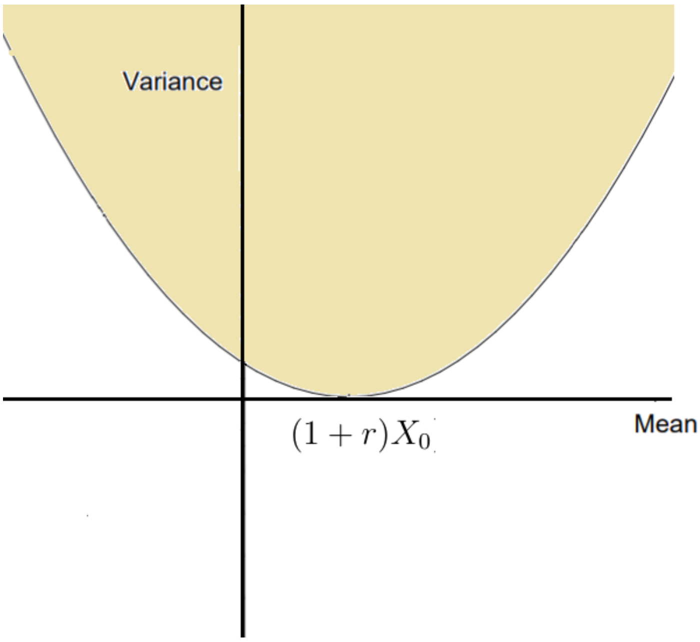
with equality if and only if

$$\theta = \lambda\, V^{-1}[\mu - (1+r)S_0]$$

where

$$\lambda = \frac{\mathbb{E}(X_1) - (1+r)X_0}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]}.$$

When the initial wealth $X_0$ is fixed, we can plot the set of all possible values of $(\mathbb{E}(X_1), \mathrm{Var}(X_1))$ as we vary the portfolio $\theta$.




**Definition.** Given $X_0$, the *mean-variance efficient frontier* is the lower boundary of the set of possible values of $(\mathbb{E}(X_1), \mathrm{Var}(X_1))$; i.e. the set $\{m, (\min_{\mathbb{E}(X_1)=m} \mathrm{Var}(X_1)) : \ m \in \mathbb{R}\}$.

**Remark.** Note that we have shown that the mean-variance efficient frontier is a parabola.

*Proof of mean-variance optimal portfolio.* If $m > (1+r)X_0$, then it is optimal to take $\mathbb{E}(X_1) = m$ with portfolio $\theta = \lambda V^{-1}$, since minimised variance increases with $\mathbb{E}(X_1)$.

However, if $m \leq (1+r)X_0$, then the minimised variance decreases with $\mathbb{E}(X_1)$ and hence it is optimal to take $\mathbb{E}(X_1) = (1+r)X_0 \geq m$, with portfolio $\theta = 0$. $\square$

**Definition.** Given $X_0$, we say that a portfolio is *mean-variance efficient* iff it is the optimal solution to a mean-variance portfolio problem for *some* target mean $m$.

**Theorem** (Mutual fund theorem). *A portfolio $\theta$ is mean-variance efficient if and only there exists a scalar $\lambda \geq 0$ such that*

$$\theta = \lambda\, V^{-1}[\mu - (1+r)S_0]$$

*Proof.* We are given an initial wealth $X_0$.

Suppose we are given a target mean $m$. Then the optimal solution of the mean-variance portfolio problem if of the correct form with

$$\lambda = \frac{(m - (1+r)X_0)^+}{[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0]} \geq 0$$

On the other hand, suppose that we are given $\lambda \geq 0$. Then the given portfolio is the optimal solution of the mean-variance portfolio problem for target mean

$$m = (1+r)X_0 + \lambda[\mu - (1+r)S_0]^\top V^{-1}[\mu - (1+r)S_0].$$

□

# 2 Capital Asset Pricing Model

**Theorem** (Linear regression coefficients). *Let $X$ and $Y$ be two-square integrable random variables with $\mathrm{Var}(X) > 0$. The unique constants $a$ and $b$ such that*

$$Y = a + bX + Z$$

*where $\mathbb{E}(Z) = 0$ and $\mathrm{Cov}(X, Z) = 0$ are given by*

$$b = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)} \text{ and } a = \mathbb{E}(Y) - b\mathbb{E}(X).$$

*Proof.* Let $Z = Y - a - bX$ and note

$$\mathbb{E}(Z) = \mathbb{E}(Y) - a - b\mathbb{E}(X)$$
$$\mathrm{Cov}(X, Z) = \mathrm{Cov}(X, Y) - b\mathrm{Var}(X)$$

The given $a$ and $b$ are the unique solution to the system of equations $\mathbb{E}(Z) = 0$ and $\mathrm{Cov}(X, Z) = 0$. □

**Definition.** The portfolio

$$\theta_{\mathrm{Mar}} = V^{-1}[\mu - (1+r)S_0]$$

is called the *market portfolio.*

**Remark.** The name market portfolio is explained below.

**Definition.** Given initial wealth $X_0 > 0$, the excess return $R^{\text{ex}}$ of a portfolio $\theta$ is defined by

$$R^{\text{ex}} = \frac{X_1}{X_0} - (1+r) = \frac{1}{X_0}\theta^\top[S_1 - (1+r)S_0]$$

Let $R^{\text{ex}}_{\text{Mar}}$ be the excess return of the market portfolio $\theta_{\text{Mar}}$.

**Theorem** (Alpha is zero)**.** *Fix $X_0 > 0$ and a portfolio $\theta$. Suppose $\alpha$ and $\beta$ are such that*

$$R^{\text{ex}} = \alpha + \beta R^{\text{ex}}_{\text{Mar}} + \varepsilon$$

*where $\mathbb{E}(\varepsilon) = 0$ and $\text{Cov}(R^{\text{ex}}_{\text{Mar}}, \varepsilon) = 0$. Then $\alpha = 0$.*

*Proof.* (next time) Note

$$\begin{aligned}
\text{Cov}(R^{\text{ex}}, R^{\text{ex}}_{\text{Mar}}) &= \frac{1}{X_0^2}\theta^\top \text{Cov}[S_1 - (1+r)S_0]\theta_{\text{Mar}} \\
&= \frac{1}{X_0^2}\theta^\top[\mu - (1+r)S_0] \\
&= \frac{1}{X_0}\mathbb{E}(R^{\text{ex}})
\end{aligned}$$

and hence

$$\begin{aligned}
\text{Var}(R^{\text{ex}}_{\text{Mar}}) &= \text{Cov}(R^{\text{ex}}_{\text{Mar}}, R^{\text{ex}}_{\text{Mar}}) \\
&= \frac{1}{X_0}\mathbb{E}(R^{\text{ex}}_{\text{Mar}}).
\end{aligned}$$

By linear regression, we have

$$\begin{aligned}
\beta &= \frac{\text{Cov}(R^{\text{ex}}, R^{\text{ex}}_{\text{Mar}})}{\text{Var}(R^{\text{ex}}_{\text{Mar}})} \\
&= \frac{\mathbb{E}(R^{\text{ex}})}{\mathbb{E}(R^{\text{ex}}_{\text{Mar}})}
\end{aligned}$$

and

$$\alpha = \mathbb{E}(R^{\text{ex}}) - \beta\mathbb{E}(R^{\text{ex}}_{\text{Mar}}) = 0.$$

□

# Stochastic Financial Models 3

Michael Tehranchi

11 October 2023

## 1 CAPM, continued

Now let's model the entire market. Assumptions:

- There is a total of $n_i > 0$ shares of asset $i = 1, \ldots, d$, and let $n = (n_1, \ldots, n_d)^\top$.
- There are $K$ agents in the market, and agent $k$ holds portfolio $\theta_k$.
- Total supply equals total demand so that

$$\sum_k \theta_k = n.$$

- Each agent's portfolio is mean-variance efficient

By the mutual fund theorem, for each $k$ we have

$$\theta_k = \lambda_k \theta_{\text{Mar}}$$

where $\lambda_k \geq 0$. Hence,

$$n = \Lambda \ \theta_{\text{Mar}}$$

where $\Lambda = \sum_k \lambda_k$. Since $n \neq 0$, it follows $\Lambda > 0$. That is the say, in this model, the entire market is just some positive scalar multiple of the market portfolio (explaining the name).

A prediction of the CAPM is that when the excess returns of a portfolio are statistically regressed against the excess returns of a broad market index (such as the FTSE or S&P) then you should find $\alpha = 0$.

**Remark.** Markowitz and Sharpe shared the 1990 Nobel Prize in Economics for studying mean-variance efficiency and the CAPM.

## 2 Expected utility hypothesis

*Up to now*, given two random payouts $X$ and $Y$ we have implicitly assumed that an agent prefers $X$ over $Y$ if either

- $\mathbb{E}(X) > \mathbb{E}(Y)$ and $\mathrm{Var}(X) = \mathrm{Var}(Y)$, or
- $\mathbb{E}(X) = \mathbb{E}(Y)$ and $\mathrm{Var}(X) < \mathrm{Var}(Y)$

This is rather crude. Here is a historical example that illustrates one of the issues.
**Aside: historical origin of expected utility hypothesis** (not lectured). Consider the *St Petersburg paradox*: You and I play a game. I toss a coin repeatedly until it comes up heads. If toss the coin a total of $n$ times, I will pay you $2^n$ pounds. How much would you pay me to play this game? This problem was invented by Nicolaus Bernoulli in 1713. The issue is that according to N Bernoulli's intuition, the answer should be the expected value of the payout $\sum_n 2^n \times 2^{-n} = \infty$, but he thought no sensible person would pay more than 20 pounds. His cousin Daniel Bernoulli proposed in 1738 that people don't care about the expected payout *per se*, but instead the relevant quantity is the expected *utility* of the payout.

**Definition.** The *expected utility hypothesis* says that each agent has a function $U$ (called the *utility function*) such that the agent prefers random payout $X$ to $Y$ if and only if

$$\mathbb{E}[U(X)] > \mathbb{E}[U(Y)]$$

In the case $\mathbb{E}[U(X)] = \mathbb{E}[U(Y)]$ the agent is said to be *indifferent* between $X$ and $Y$.

**Remark.** If $\tilde{U}(x) = a + b\, U(x)$ with $b > 0$, then $\tilde{U}$ gives rise to the same expected utility preferences as $U$.

**Remark.** In 1947, von Neumann–Morgenstern axioms derived a short list of properties of an agent's preferences which are equivalent to the assumption that the agent's preferences are derived from expected utility.

## 3 Risk-aversion and concavity

Once we've assumed the expected utility hypothesis, there are two additional properties we will assume of the agent's utility function:

- *(Strictly) increasing.* $x > y$ implies $U(x) > U(y)$.
- *(Strictly) concave.*

  $$U(px + (1-p)y) > p\, U(x) + (1-p)U(y)$$

  for any $x \neq y$ and $0 < p < 1$.

**Remark.** Note that if $X \geq Y$ almost surely, then $X \succeq Y$. Furthermore, if $\mathbb{P}(X > Y) > 0$ then $X \succ Y$.

**Remark.** Recall Jensen's inequality:

$$U(\mathbb{E}[X]) \geq \mathbb{E}[U(X)]$$

whenever the expectations are defined. Hence $\mathbb{E}(X) \succeq X$ for any random payout $X$. If $X$ is not constant, then $\mathbb{E}(X) \succ X$.