

# Stochastic Financial Models 2

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## 1 Mean-variance efficiency

To solve the mean-variance portfolio problem, we will use the following lemma

**Lemma.** *If  $\theta^\top a = b$  then*

$$\theta^\top V \theta \geq \frac{b^2}{a^\top V^{-1} a}$$

*with equality if and only if*

$$\theta = \lambda V^{-1} a$$

*where*

$$\lambda = \frac{b}{a^\top V^{-1} a}.$$

*Proof of lemma.* Since  $V$  is non-negative definite we have

$$\begin{aligned} \theta^\top V \theta &= \theta^\top V \theta + 2\lambda(b - \theta^\top a) \\ &= (\theta - \lambda V^{-1} a)^\top V (\theta - \lambda V^{-1} a) \\ &\quad + 2\lambda b - \lambda^2 a^\top V^{-1} a \\ &\geq 2\lambda b - \lambda^2 a^\top V^{-1} a = \frac{b^2}{a^\top V^{-1} a} \end{aligned}$$

and since  $V$  is positive definite there is equality only if

$$\theta = \lambda V^{-1} a$$

□

**Remark.** This proof is secretly using the Lagrangian technique from IB Optimisation or Variational Principles. The constant  $\lambda$  could be thought of as a Lagrange multiplier.

**Remark.** The lemma is equivalent to

$$(\theta^\top a)^2 \leq (\theta^\top V \theta)(a^\top V^{-1} a).$$

This is just the Cauchy–Schwarz inequality applied to the vectors  $V^{1/2}\theta$  and  $V^{-1/2}a$ .

By applying the lemma with  $a = \mu - (1 + r)S_0$  and  $b = \mathbb{E}(X_1) - (1 + r)X_0$ , we see that

$$\text{Var}(X_1) \geq \frac{(\mathbb{E}(X_1) - (1 + r)X_0)^2}{[\mu - (1 + r)S_0]^\top V^{-1} [\mu - (1 + r)S_0]}$$

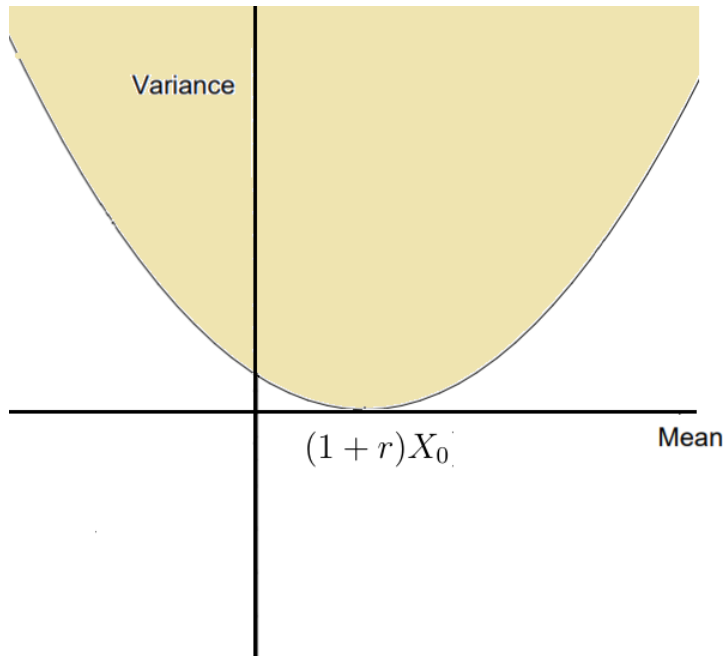
with equality if and only if

$$\theta = \lambda V^{-1} [\mu - (1 + r)S_0]$$

where

$$\lambda = \frac{\mathbb{E}(X_1) - (1 + r)X_0}{[\mu - (1 + r)S_0]^\top V^{-1} [\mu - (1 + r)S_0]}.$$

When the initial wealth  $X_0$  is fixed, we can plot the set of all possible values of  $(\mathbb{E}(X_1), \text{Var}(X_1))$  as we vary the portfolio  $\theta$ .



**Definition.** Given  $X_0$ , the *mean-variance efficient frontier* is the lower boundary of the set of possible values of  $(\mathbb{E}(X_1), \text{Var}(X_1))$ ; i.e. the set  $\{m, (\min_{\mathbb{E}(X_1)=m} \text{Var}(X_1)) : m \in \mathbb{R}\}$ .

**Remark.** Note that we have shown that the mean-variance efficient frontier is a parabola.

*Proof of mean-variance optimal portfolio.* If  $m > (1 + r)X_0$ , then it is optimal to take  $\mathbb{E}(X_1) = m$  with portfolio  $\theta = \lambda V^{-1}$ , since minimised variance increases with  $\mathbb{E}(X_1)$ .

However, if  $m \leq (1 + r)X_0$ , then the minimised variance decreases with  $\mathbb{E}(X_1)$  and hence it is optimal to take  $\mathbb{E}(X_1) = (1 + r)X_0 \geq m$ , with portfolio  $\theta = 0$ .  $\square$

**Definition.** Given  $X_0$ , we say that a portfolio is *mean-variance efficient* iff it is the optimal solution to a mean-variance portfolio problem for *some* target mean  $m$ .

**Theorem** (Mutual fund theorem). *A portfolio  $\theta$  is mean-variance efficient if and only there exists a scalar  $\lambda \geq 0$  such that*

$$\theta = \lambda V^{-1}[\mu - (1 + r)S_0]$$

*Proof.* We are given an initial wealth  $X_0$ .

Suppose we are given a target mean  $m$ . Then the optimal solution of the mean-variance portfolio problem is of the correct form with

$$\lambda = \frac{(m - (1 + r)X_0)^+}{[\mu - (1 + r)S_0]^\top V^{-1}[\mu - (1 + r)S_0]} \geq 0$$

On the other hand, suppose that we are given  $\lambda \geq 0$ . Then the given portfolio is the optimal solution of the mean-variance portfolio problem for target mean

$$m = (1 + r)X_0 + \lambda[\mu - (1 + r)S_0]^\top V^{-1}[\mu - (1 + r)S_0].$$

□

## 2 Capital Asset Pricing Model

**Theorem** (Linear regression coefficients). *Let  $X$  and  $Y$  be two-square integrable random variables with  $\text{Var}(X) > 0$ . The unique constants  $a$  and  $b$  such that*

$$Y = a + bX + Z$$

*where  $\mathbb{E}(Z) = 0$  and  $\text{Cov}(X, Z) = 0$  are given by*

$$b = \frac{\text{Cov}(X, Y)}{\text{Var}(X)} \text{ and } a = \mathbb{E}(Y) - b\mathbb{E}(X).$$

*Proof.* Let  $Z = Y - a - bX$  and note

$$\begin{aligned}\mathbb{E}(Z) &= \mathbb{E}(Y) - a - b\mathbb{E}(X) \\ \text{Cov}(X, Z) &= \text{Cov}(X, Y) - b\text{Var}(X)\end{aligned}$$

The given  $a$  and  $b$  are the unique solution to the system of equations  $\mathbb{E}(Z) = 0$  and  $\text{Cov}(X, Z) = 0$ . □

**Definition.** The portfolio

$$\theta_{\text{Mar}} = V^{-1}[\mu - (1 + r)S_0]$$

is called the *market portfolio*.

**Remark.** The name market portfolio is explained below.

**Definition.** Given initial wealth  $X_0 > 0$ , the excess return  $R^{\text{ex}}$  of a portfolio  $\theta$  is defined by

$$R^{\text{ex}} = \frac{X_1}{X_0} - (1+r) = \frac{1}{X_0} \theta^\top [S_1 - (1+r)S_0]$$

Let  $R_{\text{Mar}}^{\text{ex}}$  be the excess return of the market portfolio  $\theta_{\text{Mar}}$ .

**Theorem** (Alpha is zero). Fix  $X_0 > 0$  and a portfolio  $\theta$ . Suppose  $\alpha$  and  $\beta$  are such that

$$R^{\text{ex}} = \alpha + \beta R_{\text{Mar}}^{\text{ex}} + \varepsilon$$

where  $\mathbb{E}(\varepsilon) = 0$  and  $\text{Cov}(R_{\text{Mar}}^{\text{ex}}, \varepsilon) = 0$ . Then  $\alpha = 0$ .

*Proof.* (next time) Note

$$\begin{aligned} \text{Cov}(R^{\text{ex}}, R_{\text{Mar}}^{\text{ex}}) &= \frac{1}{X_0^2} \theta^\top \overbrace{\text{Cov}[S_1 - (1+r)S_0]}^{\text{Var}(S_1 - (1+r)S_0) = \text{Var}(S_1) = V} \theta_{\text{Mar}} \\ &= \frac{1}{X_0^2} \theta^\top [\mu - (1+r)S_0] \quad \underbrace{V^{-1}(\mu - (1+r)S_0)} \\ &= \frac{1}{X_0} \mathbb{E}(R^{\text{ex}}) \end{aligned}$$

and hence

$$\begin{aligned} \text{Var}(R_{\text{Mar}}^{\text{ex}}) &= \text{Cov}(R_{\text{Mar}}^{\text{ex}}, R_{\text{Mar}}^{\text{ex}}) \\ &= \frac{1}{X_0} \mathbb{E}(R_{\text{Mar}}^{\text{ex}}). \end{aligned}$$

By linear regression, we have

$$\begin{aligned} \beta &= \frac{\text{Cov}(R^{\text{ex}}, R_{\text{Mar}}^{\text{ex}})}{\text{Var}(R_{\text{Mar}}^{\text{ex}})} \\ &= \frac{\mathbb{E}(R^{\text{ex}})}{\mathbb{E}(R_{\text{Mar}}^{\text{ex}})} \end{aligned}$$

and

$$\alpha = \mathbb{E}(R^{\text{ex}}) - \beta \mathbb{E}(R_{\text{Mar}}^{\text{ex}}) = 0.$$

□