Stochastic Financial Models 17

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1 Example of optimal investment

- C_n consumed and $\eta_n = \theta_n S_{n-1}$ total held in stock between time $n-1$ and time n
- $S_n = S_{n-1}\xi_n$ where $(\xi_n)_n$ are IID
- wealth evolves as $X_n = (1+r)(X_{n-1} - C_n) + \eta_n[\xi_n - (1+r)]$
- Given a time horizon N, the goal is to

  $$\text{maximise } \mathbb{E}\left[\sum_{k=1}^{N} U(C_k) + U(X_N)\right]$$

  where U is the investor's utility function

The Bellman equation is

$$\begin{aligned} V(N,x) &= U(x) \\ V(n-1,x) &= \max_{c,\eta} \mathbb{E}\left[U(c) + V(n,(1+r)(x-c) + \eta(\xi - (1+r)))\right] \end{aligned}$$

- Generally, intractable
- but suppose the utility is CRRA: $U(x) = \frac{1}{1-R}x^{1-R}$ for $x > 0$, where $R > 0, R \neq 1$.
- Guess: $V(n,x) = U(x)A_n$
- Check: Correct for $n = N$ with $A_N = 1$. Assume correct for $n = k$ for some $k \leq N$. The

  $$\begin{aligned} V(k-1,x) & \max_{c,\eta}\{U(c) + \mathbb{E}[V(k,(1+r)(x-c) + \eta(\xi - (1+r))]\} \\ &= x^{1-R} \max_{c,\eta} \Big\{U(c/x) \\ &+ A_k(1-c/x)^{1-R}\mathbb{E}\left[U\left((1+r) + \frac{\eta}{x-c}(\xi - (1+r))\right)\right]\Big\} \\ &= x^{1-R}\max_{c}\left\{U(c/x) + A_k U(1-c/x)\alpha\right\} \end{aligned}$$

  where $\alpha = (1-R)\max_t \mathbb{E}\left[U\left((1+r) + t(\xi - (1+r))\right)\right]$

- Now optimise over $s = c/x$: differentiate and set equal to zero to get

$$s_k^{-R} = (1-s_k)^{-R} A_k \alpha \Rightarrow s_k = \frac{1}{1+(A_k\alpha)^{1/R}}$$

- plug this back in

$$V(k-1,x) = U(x)(1+(A_k\alpha)^{1/R})^R$$

- So induction would work if

$$A_{k-1} = (1+(A_k\alpha)^{1/R})^R \text{ for all } k \le N$$

- Solve this recursion: $A_{k-1}^{1/R} = 1 + \alpha^{1/R} A_k^{1/R}$ with $A_N = 1$ implying $A_k^{1/R} = 1 + \alpha^{1/R} + \cdots + \alpha^{(N-k)/R}$ yielding

$$A_n = \left(\frac{1-\alpha^{(N-n+1)/R}}{1-\alpha^{1/R}} \right)^R \text{ for } 0 \le n \le N$$

- Optimal strategy

$$C_n^* = X_{n-1}^* s_n = \frac{X_{n-1}^*}{1+(A_n\alpha)^{1/R}} = \frac{X_{n-1}^*(1-\alpha^{1/R})}{1-\alpha^{(N-n+2)/R}}$$
$$\theta_n^* = \frac{\eta_n^*}{S_{n-1}} = \frac{t^*(X_{n-1}^* - C_n^*)}{S_{n-1}}$$

where $t^* = \text{argmax}_t \mathbb{E}\left[U\left((1+r) + t(\xi - (1+r))\right)\right]$

- Optimised wealth process evolves as

$$X_n^* = X_{n-1}^* \alpha^{1/R} \left(\frac{1-\alpha^{(N-n+1)/R}}{1-\alpha^{(N-n+2)/R}} \right) \left((1+r)(1-t^*) + t^*\xi_n\right)$$

so

$$X_n^* = X_0 \left(\frac{\alpha^{n/R} - \alpha^{(N+1)/R}}{1-\alpha^{(N+1)/R}} \right) \prod_{k=1}^{n} \left((1+r)(1-t^*) + t^*\xi_k\right)$$

2 Infinite-horizon problems

- Consider a controlled Markov process

$$X_n = G(X_{n-1}, U_n, \xi_n)$$

where $(U_n)_{n\ge1}$ is the previsible control where $(\xi_n)_n$ is IID. Note note explicit time dependence in the function G.

- Problem:
$$\text{maximise } \mathbb{E}\left[\sum_{k=1}^{\infty} \beta^{k-1} f(U_k)\right]$$
where the subjective rate of discounting $0 < \beta < 1$ is given

- The value function is
$$V(x) = \mathbb{E}\left[\sum_{k=1}^{\infty} \beta^{k-1} f(U_k) | X_0 = x\right]$$

- The Bellman equation is
$$V(x) = \max_u \{f(u) + \beta \mathbb{E}[V(G(x, u, \xi))]\}$$

- When is the solution of the Bellman equation the value function?

Theorem. *Suppose $f(u) \geq 0$ for all u and that V is a non-negative solution to the Bellman equation. Suppose $u^*(x)$ is the maximiser of*
$$f(u) + \beta \mathbb{E}[V(G(x, u, \xi))]$$
and let $X_0^* = X_0$ and $U_n^* = u^*(X_{n-1}^*)$ and $X_n^* = G(X_{n-1}^*, U_n^*, \xi_n)$ for $n \geq 1$. If
$$\beta^n \mathbb{E}[V(X_n^*)] \to 0$$
then V is the value function and U^* is the optimal control.

To fully prove this, we need an important result from measure theory:

Theorem (Monotone convergence theorem)**.** *Let $(Z_n)_n$ be an almost sure increasing sequence of non-negative random variables. Then* $\lim_n \mathbb{E}(Z_n) = \mathbb{E}(\lim_n Z_n)$

Proof of that the solution to the Bellman equation is the value function. Given a control $(U_n)_{n\geq 1}$ let
$$M_n = \sum_{k=1}^{n} \beta^{k-1} f(U_k) + \beta^n V(X_n)$$
Note $(M_n)_{n\geq 0}$ is a supermartingale
$$\begin{aligned} \mathbb{E}[M_n - M_{n-1} | \mathcal{F}_{n-1}] &= \beta^{n-1} \left(f(U_n) + \beta \mathbb{E}[V(X_n)|\mathcal{F}_{n-1}] - V(X_{n-1}) \right) \\ &\leq 0 \end{aligned}$$

with equality if $U = U^*$. Hence

$$\begin{aligned} V(x) &= M_0 \\ &\geq \mathbb{E}[M_n] \\ &= \mathbb{E}\left[\sum_{k=1}^{n} \beta^{k-1} f(U_k)\right] + \beta^n \mathbb{E}[V(X_n)] \end{aligned}$$

with equality if $U = U^*$.

Now since $V \geq 0$, we have

$$\begin{aligned} V(x) &\geq \mathbb{E}\left[\sum_{k=1}^{n} \beta^{k-1} f(U_k)\right] \\ &\to \mathbb{E}\left[\sum_{k=1}^{\infty} \beta^{k-1} f(U_k)\right] \end{aligned}$$

for any control, where we have used that $f \geq 0$ and the monotone convergence theorem.

And for $U = U^*$ we have

$$\begin{aligned} V(x) &= \mathbb{E}\left[\sum_{k=1}^{n} \beta^{k-1} f(U_k^*)\right] + \beta^n \mathbb{E}[V(X_n^*)] \\ &\to \mathbb{E}\left[\sum_{k=1}^{\infty} \beta^{k-1} f(U_k^*)\right] \end{aligned}$$

since $\beta^n \mathbb{E}[V(X_n^*)] \to 0$ by assumption. $\square$

Remark. The monotone convergence theorem is not technically examinable for this course. That is, if you need it for an exam question, then the text of the question will provide you with a statement of the monotone convergence to use without proof.