

# Graph Theory

Lectured by I. B. Leader, Michaelmas Term 2007

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# Course schedule

## GRAPH THEORY (D)

24 lectures, Michaelmas term

*No specific prerequisites.*

### Introduction

Basic definitions. Trees and spanning trees. Bipartite graphs. Euler circuits. Elementary properties of planar graphs. Statement of Kuratowski's theorem. [3]

### Connectivity and matchings

Matchings in bipartite graphs; Hall's theorem and its variants. Connectivity and Menger's theorem. [3]

### Extremal graph theory

Long paths, long cycles and Hamilton cycles. Complete subgraphs and Turan's theorem. Bipartite subgraphs and the problem of Zarankiewicz. The Erdős-Stone theorem; \*sketch of proof\*. [5]

### Eigenvalue methods

The adjacency matrix and the Laplacian. Strongly regular graphs. [2]

### Graph colouring

Vertex and edge colourings; simple bounds. The chromatic polynomial. The theorems of Brooks and Vizing. Equivalent forms of the four colour theorem; the five colour theorem. Heawood's theorem for surfaces; the torus and the Klein bottle. [5]

### Ramsey theory

Ramsey's theorem (finite and infinite forms). Upper bounds for Ramsey numbers. [3]

### Probabilistic methods

Basic notions; lower bounds for Ramsey numbers. The model  $G(n, p)$ ; graphs of large girth and large chromatic number. The clique number. [3]

## Appropriate books

B.Bollobás *Modern Graph Theory*. Springer 1998 (£26.00 paperback).

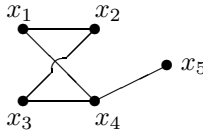
J.A.Bondy and U.S.R.Murty *Graph Theory with Applications*. Elsevier 1976  
(available online via <http://www.ecp6.jussieu.fr/pageperso/bondy/bondy.html>).




R.Diestel *Graph Theory*. Springer 2000 (£38.50 paperback).

D.West *Introduction to Graph Theory*. Prentice Hall 1999 (£45.99 hardback).

# Chapter 1 : Introduction

A **graph** is a pair  $(V, E)$ , where  $V$  is a set and  $E$  is a subset of  $V^{(2)} = \{\{x, y\} : x, y \in V, x \neq y\}$ , the set of unordered pairs from  $V$ .

E.g.,  has  $V = \{x_1, x_2, x_3, x_4, x_5\}$ ,  $E = \{x_1x_2, x_1x_4, x_2x_3, x_3x_4, x_4x_5\}$ .

- Notes.** 1. No loops:   
 2. No multiple edges:   
 3. No directed edges: 

Unless otherwise stated,  $V$  is *finite*.

We call  $V = V(G)$  the **vertex set** of  $G$ , and  $E = E(G)$  the **edge set** of  $G$ .

The **order** of  $G$  is  $|G| = |V(G)|$ , and the **size** is  $e(G) = |E(G)|$ .

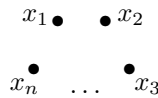
Often write, say,  $x \in G$  to mean  $x \in V(G)$ .

**Examples.**

1. The **empty graph**  $E_n$ .

$$V = \{x_1, \dots, x_n\}, E = \emptyset.$$

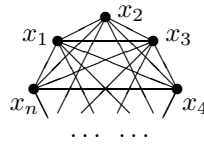
$$\text{So } |G| = n, e(G) = 0.$$



2. The **complete graph**  $K_n$ .

$$V = \{x_1, \dots, x_n\}, E = V^{(2)}.$$

$$\text{So } |G| = n, e(G) = \binom{n}{2}.$$

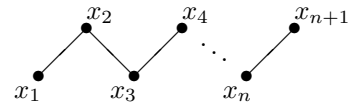


3. The **path**  $P_n$  of length  $n$

$$V = \{x_1, \dots, x_{n+1}\},$$

$$E = \{x_1x_2, \dots, x_nx_{n+1}\} = \{x_i, x_{i+1} : 1 \leq i \leq n\}.$$

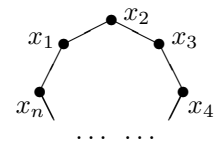
$$\text{So } |G| = n + 1, e(G) = n.$$



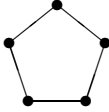
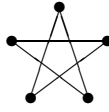
4. The **cycle**  $C_n$  of length  $n$ .  $n > 2$  to prevent loops and multiple edges

$$V = \{x_1, \dots, x_n\}, E = \{x_i, x_{i+1} : 1 \leq i \leq n-1\} \cup \{x_nx_1\}.$$

$$\text{So } |G| = n, e(G) = n.$$



Graphs  $G = (V, E)$  and  $H = (V', E')$  are **isomorphic** if there is a bijection  $f : V \rightarrow V'$  such that  $xy \in E \iff f(x)f(y) \in E'$ .

E.g.,  and  are isomorphic.

Say that  $H = (V', E')$  is a **subgraph** of  $G = (V, E)$  if  $V' \subset V$  and  $E' \subset E$ .

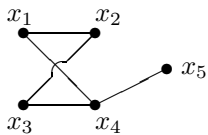
E.g.,  $C_n$  is a subgraph of  $K_n$ .

For  $xy \in E$ , write  $G - xy$  for the graph  $(V, E \setminus \{xy\})$ .

For  $xy \notin E$ , write  $G + xy$  for the graph  $(V, E \cup \{xy\})$ .

If  $xy \in E$ , say  $x, y$  are **adjacent** or **neighbours**.

The **neighbourhood** of  $x$  is  $\Gamma(x) = \{y \in V : xy \in E\}$ , and the **degree** of  $x$  is  $d(x) = |\Gamma(x)|$ .

E.g., in  we have  $\Gamma(x_4) = \{x_1, x_3, x_5\}$ , so  $d(x_4) = 3$ .

If  $V(G) = \{x_1, \dots, x_n\}$ , the **degree sequence** of  $G$  is  $d(x_1), \dots, d(x_n)$ .

The **maximum degree** of  $G$  is  $\Delta(G) = \max_{1 \leq i \leq n} d(x_i)$ .

The **minimum degree** of  $G$  is  $\delta(G) = \min_{1 \leq i \leq n} d(x_i)$ .

E.g., the above example has degree sequence: 2, 2, 2, 3, 1. So  $\delta(G) = 1$  and  $\Delta(G) = 3$ .




Say  $G$  is **regular of degree  $k$** , or  **$k$ -regular**, if  $d(x) = k$  for all  $x \in V(G)$ .

E.g.,  $C_n$  is regular of degree 2, and  $K_n$  is  $(n-1)$ -regular.

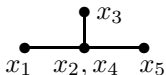
Unless otherwise stated,  $V$  is finite

In a graph  $G$ , an  $x$ - $y$  **path** is a sequence  $x_1, \dots, x_k$  ( $k \geq 1$ ) of distinct vertices of  $G$  with  $x_1 = x$ ,  $x_k = y$  and  $x_i x_{i+1} \in E$  for all  $1 \leq i \leq k-1$ . It has **length**  $k-1$ .

Say  $G$  is **connected** if for all  $x, y \in V$ , there is an  $x$ - $y$  path in  $G$ .

E.g.,  is connected, but   is not connected.



Write  $x \sim y$  if there is an  $x$ - $y$  path. Note,  $\sim$  is an equivalence relation.

If  $x_1 \dots x_k$  and  $x_k \dots x_l$  are paths, then  $x_1, \dots, x_l$  need not be a path, e.g., 

– but it *contains* a path from  $x_1$  to  $x_l$ . (E.g., choose minimal  $1 \leq i \leq l$  for which there is some  $k \leq j \leq l$  with  $x_i = x_j$ , and then  $x_1, \dots, x_i, x_{j+1}, \dots, x_l$  is a path.)

The equivalence classes of  $\sim$  are the **components** of  $G$ . So a connected graph has one component (or no vertices)

So the component of  $x$  is  $\{y : \exists x\text{-}y \text{ path in } G\}$ .

E.g.,   has two components.

A **walk** is a sequence  $x_1, \dots, x_k$  such that  $x_i x_{i+1} \in E(G)$  for all  $1 \leq i \leq k-1$ .

Thus  $G$  has an  $x$ - $y$  walk  $\iff G$  has an  $x$ - $y$  path.

## Trees

A graph is **acyclic** if it has no cycle.

A **tree** is a connected acyclic graph.

In a tree  $T$ , a vertex  $x$  with  $d(x) = 1$  is called a **leaf** or **endvertex**.

**Proposition 1.** Let  $G$  be a graph. The following are equivalent.

- (a)  $G$  is a tree.
- (b)  $G$  is **minimal connected** (i.e.,  $G$  connected,  $G - xy$  disconnected for all  $xy \in E$ ).
- (c)  $G$  is **maximal acyclic** (i.e.,  $G$  acyclic,  $G + xy$  has a cycle for all  $xy \notin E$ ).

**Proof.** (a)  $\Rightarrow$  (b). If  $G - xy$  is connected, then it has an  $x$ - $y$  path  $P$ , but then  $Pyx$  is a cycle in  $G$ .  $\times$

(b)  $\Rightarrow$  (a). If  $G$  has a cycle  $C$ , choose  $xy \in E(C)$ . Then  $G - xy$  is connected, because for all  $a, b \in V$ , we have an  $a$ - $b$  path in  $G$ , and if it used edge  $xy$  then replace  $xy$  with  $C - xy$  to obtain an  $a$ - $b$  walk in  $G - xy$ .  $\times$

For  $xy$  not in  $E$ ,

(a)  $\Rightarrow$  (c). We have  $x$ - $y$  path  $P$  in  $G$ , so  $G + xy$  contains the cycle  $Pyx$ .

(c)  $\Rightarrow$  (a). If  $G$  is not connected, choose  $x, y$  in different components of  $G$ , then  $G + xy$  is acyclic.  $\times$   $\square$

**Proposition 2.** Let  $T$  be a tree ( $|T| \geq 2$ ). Then  $T$  has a leaf.

**Proof.** Let  $P = x_1 \dots x_k$  be a longest path in  $T$  (which exists as we have a finite graph). Then  $\Gamma(x_k) \subset P$  (by maximality of  $P$ ), and  $\Gamma(x_k) \cap P = \{x_{k-1}\}$ , since  $T$  is acyclic. So  $d(x_k) = 1$ .  $\square$

**Remark.** The proof actually shows there are  $\geq 2$  leaves, as the same proof works for  $x_1$ .

**Alternatively,** if  $T$  has no leaf, “go for a walk”. Choose some  $x_1, x_2 \in E$  and then choose  $x_3, x_4, x_5, \dots$  as follows. Having chosen  $x_1, \dots, x_k$ , choose  $x_{k+1} \in \Gamma(x_k) \setminus \{x_{k-1}\}$ . This must repeat (as  $T$  finite), giving a cycle.  $\times$   $\square$

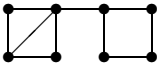
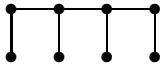
**Proposition 3.** Let  $T$  be a tree on  $n$  vertices ( $n \geq 1$ ). Then  $e(T) = n - 1$ .

For  $G$  a graph,  $W \subset V$ , write  $G[W]$  for the **subgraph spanned by**  $W$ . That is,  $G[W]$  has vertex set  $W$ , edge set  $E \cap W^{(2)}$ . For  $x \in V$ , write  $G - x$  for  $G[V \setminus \{x\}]$ .

**Proof of 3.** Induction on  $n$ . Done if  $n = 1$ .

Given  $T$  on  $n$  vertices ( $n \geq 2$ ), choose a leaf  $x$ . Then  $G - x$  is a tree, with  $|G - x| = n - 1$ , so  $e(G - x) = n - 2$ , by induction. Thus  $e(G) = e(G - x) + 1 = n - 1$ .  $\square$

For  $G$  a connected graph, a **spanning tree** of  $G$  is a subgraph  $T$  of  $G$ , with  $V(T) = V(G)$ , that is a tree.

E.g.,  has  as a spanning tree.

Clearly every connected  $G$  does have a spanning tree: just remove edges until we get a minimal connected graph.

For a non-inductive proof of Proposition 3 we'll show that any connected  $G$  has a spanning tree  $T$  on  $n - 1$  edges (then done as if  $G$  is a tree, then the only spanning tree of  $G$  is  $G$  itself, e.g., by minimal connectedness).

For  $x, y \in G$ , the **distance** from  $x$  to  $y$ ,  $d(x, y)$  is the shortest length of any  $x$ - $y$  path.

We construct our spanning tree in  $G$ , as follows.

Fix  $x_0 \in G$ , then for each  $x \in V \setminus \{x_0\}$  choose a shortest  $x$ - $x_0$  path, say  $xx' \dots x_0$ . (So  $d(x', x_0) = d(x, x_0) - 1$ .)

Let  $T$  consist of all the  $xx'$  for  $x \in V \setminus \{x_0\}$ . Then  $e(T) = n - 1$ .

$T$  is connected, as for all  $x$ ,  $xx'x'' \dots$  forms a path to  $x_0$ .

$T$  is acyclic. Suppose  $T$  has a cycle  $C$ . On  $C$  choose  $x$  at maximum distance from  $x_0$ , say  $d(x, x_0) = k$ . Then **both** neighbours of  $x$  are at distance  $\leq k$  from  $x_0$ ,  $\llcorner$  of construction.  $\square$

**Notes.** 1. A **forest** is an acyclic graph.


Thus  $G$  is a forest  $\iff$  every component of  $G$  is a tree.

2. For  $G$  a connected graph,  $xy \in E$ , say  $xy$  is a **bridge** if  $G - xy$  is disconnected.

Thus  $G$  is a tree  $\implies$  every edge is a bridge. And, e.g., in   $xy$  is a bridge.

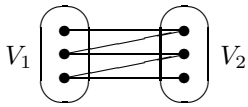
3. For  $G$  connected,  $x \in G$ , say  $x$  is a **cutvertex** if  $G - x$  is disconnected.

Clearly, if  $G$  has a bridge then it has a cutvertex (for  $|G| > 2$ ).

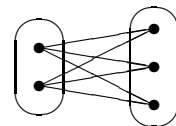
The converse is *false*, e.g., 

## Bipartite Graphs

A graph  $G$  is **bipartite** on vertex classes  $V_1$  and  $V_2$  if  $V_1, V_2$  **partition**  $V$  (i.e.,  $V_1 \cup V_2 = V$ ,  $V_1 \cap V_2 = \emptyset$ ) and  $E(G) \subset \{xy : x \in V_1, y \in V_2\}$  (i.e., no edges inside  $V_1, V_2$ ).

E.g., a path: 

The **complete bipartite graph**  $K_{n,m}$  has  $|V_1| = n, |V_2| = m$ , and  $E = \{xy : x \in V_1, y \in V_2\}$ . So  $e(K_{n,m}) = nm$ .

E.g., so  $K_{2,3}$  is: 

**Proposition 4.**  $G$  bipartite  $\iff G$  has no odd cycle.

A **circuit** in a graph  $G$  is a closed walk (i.e., a walk of the form  $x_1 \dots x_k$  where  $x_1 = x_k$ ).

Note that if  $G$  has an odd circuit, then it has an odd cycle. Indeed, if  $x_1 \dots x_k x_1$  is an odd circuit and  $x_i = x_j$  (some  $1 \leq i \leq j \leq k$ ), then one of  $x_i x_{i+1} \dots x_j$  and  $x_j x_{j+1} \dots x_k x_1 \dots x_i$  is an odd circuit. Then done by induction on  $k$ .


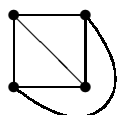
As  $k < \text{length of original circuit}$  and as we assume every odd circuit with length less than original circuit has an odd cycle (SPI). For base case we can consider odd circuit of length -3.

**Proof of 4.** ( $\Rightarrow$ ) The vertices in a cycle must alternate between  $V_1$  and  $V_2$ .


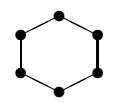

( $\Leftarrow$ ) Wlog  $G$  is connected (as if each component of  $G$  is bipartite, then so is  $G$ ). Fix  $x_1 \in G$  and put  $V_1 = \{x \in G : d(x, x_1) \text{ even}\}$ ,  $V_2 = \{x \in G : d(x, x_1) \text{ odd}\}$ . (The only possible choices.) If we had  $x, y \in V_1$  or  $x, y \in V_2$  with  $xy \in E$ , then  $xy$  together with shortest paths from  $x$  to  $x_1$  and  $y$  to  $x_1$  gives an odd circuit.  $\times \square$

## Planar Graphs

A graph  $G$  is **planar** if it can be drawn in the plane without crossing edges. A **plane graph** is such a drawing.

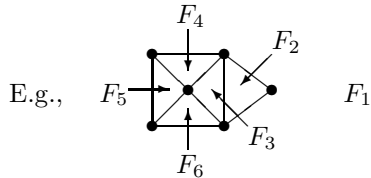
E.g.,  is not a plane graph, but  is. So  $K_4$  is planar.

**Examples.**

1. Any path is planar: 
2. Any cycle is planar: 
3. The empty graph is planar: 
4.  $K_4$  is planar (as shown above).

Which graphs are planar? How do we check if a given graph is not planar?

Given a graph  $G$ ,  $\mathbb{R}^2 - G$  (i.e., the plane with  $G$  removed) splits up into connected regions called **faces**. The **boundary** of a face consists of the vertices and edges that touch it.



Here, the boundaries of  $F_2, F_3, F_4, F_5, F_6$  consist of 3-cycles, and of  $F_1$  consists of a 5-cycle.

**Warning.** 1. The boundary of a face need not be a cycle:



2. The boundary of a face need not even be connected:



3. The two faces on either side of an edge may be the same:



**Formal bit:**  is officially 

Let  $x, y \in \mathbb{R}^2$ ,  $x \neq y$ . A **polygonal arc** from  $x$  to  $y$  is a finite union of (closed) straight line segments,  $\overline{x_1x_2} \cup \overline{x_2x_3} \cup \dots \cup \overline{x_{n-1}x_n}$ , with  $x = x_1, y = x_n$ , that are disjoint except for  $\overline{x_{i-1}x_i} \cap \overline{x_ix_{i+1}} = \{x_i\}$ .

For a graph  $G$  with vertex set  $\{v_1, \dots, v_n\}$ , a **drawing** of  $G$  consists of distinct points  $x_1, \dots, x_n \in \mathbb{R}^2$  together with, for each  $v_iv_j \in E$ , a polygonal arc  $p_{ij}$  from  $x_i$  to  $x_j$  such that  $p_{ij} \cap p_{kl} = \emptyset$  if  $i, j, k, l$  distinct, and  $p_{ij} \cap p_{jk} = \{x_j\}$ .

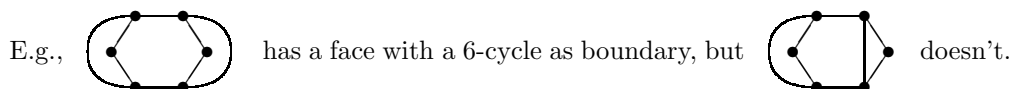
For  $x, y \in \mathbb{R}^2 - G$ , write  $x \sim y$  if there exists a polygonal arc in  $\mathbb{R}^2 - G$  from  $x$  to  $y$ . The components (equivalence classes) of  $\sim$  are called the **faces**. The **boundary** of a face consists of  $G$  intersect ~~its closure.~~  
the closure of the face

We'll assume various facts about  $\mathbb{R}^2$ , like "a cycle has two faces", or "a boundary of a face consists of vertices and (whole) edges". (Proved by induction on the total number of straight line segments in the drawing.)

**End of formal bit!**

**Remarks.** 1. Every tree is planar, with exactly one face. (Proof by induction, via removing a leaf.)

2. A planar graph may have genuinely different drawings.





Is the *number* of faces fixed? Yes:

**Theorem 5 (Euler's Formula).** Let  $G$  be a connected plane graph with  $n$  vertices,  $m$  edges and  $f$  faces. Then  $n - m + f = 2$ .

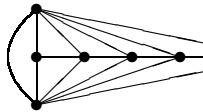
**Note.** We need  $G$  to be connected. E.g.,  $E_n$  has  $n$  vertices, 0 edges, 1 face.

**Proof of 5.** If  $G$  has no cycles, then  $G$  is a tree (acyclic and connected), so  $m = n - 1$ ,  $f = 1$ , so  $n - m + f = 2$ .

If  $G$  has a cycle, choose an edge  $e$  that is on a cycle. Then  $G - e$  is connected (as  $e$  was on a cycle), and has  $n$  vertices,  $m - 1$  edges, and  $f - 1$  faces (as  $e$  was on a cycle). But by induction,  $n - (m - 1) + (f - 1) = 2$ , so  $n - m + f = 2$ .  $\square$

**Theorem 6.** Let  $G$  be a plane graph on  $n$  vertices ( $n \geq 3$ ) with  $m$  edges. Then  $m \leq 3n - 6$ .

**Notes.** 1. This is a linear bound on  $m$  – whereas in general a graph could have anything up to  $\binom{n}{2} = \frac{n^2 - n}{2}$  edges.

2. This bound is best possible, e.g.:  (line of  $n - 2$  vertices)

Here, the number of edges is  $(n - 3) + 2(n - 2) + 1 = 3n - 6$ .

**Proof of 6.** Wlog,  $G$  connected. (If not, add edges to make it so.) Thus  $n - m + f = 2$ .

If we sum, for each face  $f$ , the number of edges in the boundary of  $f$ , we obtain  $\geq 3f$ , because each face has  $\geq 3$  edges in its boundary (for  $n \geq 3$  – theorem trivial for  $n \leq 2$ ). But we also obtain  $\leq 2m$  (since each edge counted  $\leq 2$ ).

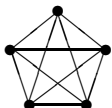
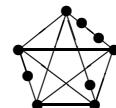
Thus  $3f \leq 2m$ , i.e.  $f \leq \frac{2m}{3}$ . So  $n - m + \frac{2m}{3} \geq 2$ , so  $\frac{m}{3} \leq n - 2$ .  $\square$

**Corollary 7.**  $K_5$  is not planar.

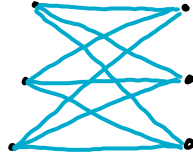
**Proof.**  $n = 5$ ,  $m = 10$ , so  $10 \leq 15 - 6$ .  $\times$   $\square$

Hence any graph containing  $K_5$  is not planar. E.g.,  $K_n$ ,  $n \geq 5$ .

A **subdivision** of a graph  $G$  is obtained by replacing edges of  $G$  with disjoint paths.

E.g.,  $K_5$ :  Subdivided  $K_5$ : 

So any subdivision of  $K_5$  is not planar.



**Proposition 8.**  $K_{3,3}$  is not planar.

**Remarks.** 1. We cannot use Theorem 6, as  $n = 6, m = 9$  holds.

2. So we cannot, e.g., connect up 3 houses to 3 utilities without pipes crossing.

**Proof of 8.**  $K_{3,3}$  is triangle-free, so if drawn in the plane, we must have  $\geq 4$  edges on the boundary of each face. So  $4f \leq 2m$ , i.e.  $f \leq m/2$ .

So  $n - m + \frac{m}{2} \geq 2$ , i.e.  $m \leq 2(n - 2)$ . But  $n = 6, m = 9$ .  $\times$  □

**Remark.** The **girth** of a graph is the length of a shortest cycle (with girth  $\infty$  if the graph has no cycle).

The proof above yields: if  $G$  is planar, girth  $\geq g$ , then  $m \leq \max\left(\frac{g}{g-2}(n-2), n-1\right)$ .

$gf \leq 2m$  thus  $n - m + \frac{2}{g}m \geq 2$  so ... ↑  
silly bit

**Corollary 9.** If  $G$  contains a subdivision of  $K_5$  or  $K_{3,3}$ , then  $G$  is not planar. □

**Kuratowski's Theorem** says that these are the *only* obstruction to being planar:

$G$  planar  $\iff G \not\supset$  subdivided  $K_5$  or  $K_{3,3}$ .

So, to show  $G$  planar: draw it.

To show  $G$  not planar: find a subdivided  $K_5$  or  $K_{3,3}$ .

## Chapter 2 : Connectivity and Matchings

Let  $G$  be a bipartite graph, with vertex classes  $X$  and  $Y$ . A **matching** from  $X$  to  $Y$  is a set  $\{xx' : x \in X\}$  of edges of  $G$ , such that  $x \mapsto x'$  is injective. In other words, it consists of  $|X|$  *independent* edges (i.e., no vertices in common).

When does  $G$  have a matching?

“Matchmaker” terminology: let  $X = \{\text{boys}\}$ ,  $Y = \{\text{girls}\}$ , with  $x$  joined to  $y$  if  $x$  knows  $y$ . Can we pair up each boy with a girl he knows?

Clearly impossible if  $d(x) = 0$  for some  $x \in X$ , or if there are (distinct)  $x_1, x_2 \in X$  with  $\Gamma(x_1) = \Gamma(x_2) = \{y\}$ , some  $y \in Y$ .

For  $A \subset X$ , write  $\Gamma(A) = \bigcup_{x \in A} \Gamma(x)$ , then we clearly need  $|\Gamma(A)| \geq |A|$  for all  $A \subset X$ .

Are there any other obstructions to a matching?

**Theorem 1 (Hall’s “Marriage” Theorem).** Let  $G$  be a bipartite graph, with vertex classes  $X, Y$ . Then  $G$  has a matching from  $X$  to  $Y \iff |\Gamma(A)| \geq |A|$  for all  $A \subset X$ . (This is “Hall’s condition”).

**Proof (1).**  $(\Rightarrow)$  Trivial.

$(\Leftarrow)$  Induction on  $|X|$ .  $|X| = 1$  is trivial.

We have  $G$ , vertex classes  $X, Y$ , with  $|X| > 1$ , such that  $|\Gamma(A)| \geq |A|$  for all  $A \subset X$ .

**Question:** do we have  $|\Gamma(A)| > |A|$  for all  $A \subset X$  ( $A \neq \emptyset, X$ )?

**If yes,** choose any  $x \in X$  and any  $y \in \Gamma(x)$ . Let  $G' = G - x - y$ .

**Claim.**  $G'$  has a matching from  $X \setminus \{x\}$  to  $Y \setminus \{y\}$ .

**Proof of claim.** We need  $|\Gamma_{G'}(A)| \geq |A|$  for all  $A \subset X \setminus \{x\}$  ( $A \neq \emptyset$ ). But  $|\Gamma_G(A)| \geq |A| + 1$ , so  $|\Gamma_{G'}(A)| \geq |A| + 1 - 1$ .

**If no,** then there is some  $A \subset X$  with  $|\Gamma(A)| = |A|$ . Let  $G' = G[A \cup \Gamma(A)]$  and  $G'' = G[(X \setminus A) \cup (Y \setminus \Gamma(A))]$ .

**Claim 1.**  $G'$  has a matching from  $A$  to  $\Gamma(A)$ .

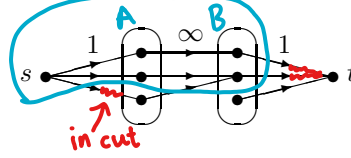
**Proof of claim.** For  $B \subset A$ , have  $|\Gamma_{G'}(B)| \geq |B|$ , because  $\Gamma_{G'}(B) = \Gamma_G(B)$ . So by induction  $G'$  has a matching from  $A$  to  $B$ .

**Claim 2.**  $G''$  has a matching from  $X \setminus A$  to  $Y \setminus \Gamma(A)$ .

**Proof of claim.** For  $B \subset X \setminus A$ , consider  $A \cup B$ . Have  $|\Gamma_G(A \cup B)| \geq |A| + |B|$ , so  $|\Gamma_G(A \cup B) \setminus \Gamma_G(A)| \geq |B|$ , so  $|\Gamma_{G''}(B)| \geq |B|$  so by induction  $G''$  has a matching  $\square$

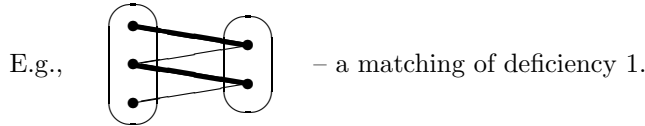
$$\text{as } |\Gamma(A)| = |A|.$$

**Proof (2).** Form a directed network as follows: add a source  $s$  joined to each  $x \in X$  by an edge of capacity 1, and a sink  $t$  joined to each  $y \in Y$  by an edge of capacity 1, and also direct each  $xy \in E(G)$  from  $x$  to  $y$ , capacity  $\infty$  (i.e., some huge number).



Then an integer-valued flow of size  $|X|$  is precisely a matching from  $X$  to  $Y$ . So, by integrality theorem of max-flow min-cut, we just need to show that every cut has size  $\geq |X|$ . So given a cut  $\{s\} \cup A \cup B$  ( $A \subset X$ ,  $B \subset Y$ ), wlog  $\Gamma(A) \subset B$  (else capacity  $\infty$ ), so capacity =  $|X| - |A| + |B| \geq |X|$  as  $|B| \geq |A|$ , since  $B \supset \Gamma(A)$ .  $\square$

A **matching of deficiency  $d$**  in a bipartite graph, vertex classes  $X$  and  $Y$ , consists of  $|X| - d$  independent edges.



**Corollary 2 (Defect Hall).** Let  $G$  be a bipartite graph, vertex classes  $X, Y$ . Then  $G$  has a matching of deficiency  $d$  from  $X$  to  $Y \iff |\Gamma(A)| \geq |A| - d$  for all  $A \subset X$ .

**Proof.** ( $\Rightarrow$ ) Trivial.

( $\Leftarrow$ ) Form  $G'$  by adding  $d$  new points to  $Y$ , each joined to all of  $X$ . Then  $|\Gamma_{G'}(A)| \geq |A|$  for all  $A \subset X$ , so by Hall there is a matching in  $G'$ , giving a matching of deficiency  $d$  in  $G$ .  $\square$

(In terms of boys and girls: add  $d$  imaginary girls to  $Y$ , known to all boys. Hall gives a matching, and at most  $d$  boys are paired with imaginary girls, so at least  $|X| - d$  are paired with real girls.)

Let  $S_1, \dots, S_n$  be sets. A **transversal** for  $S_1, \dots, S_n$  consists of distinct points  $x_1, \dots, x_n$  with  $x_i \in S_i$  for all  $i$ .

E.g.,  $\{1, 2, 3\}$ ,  $\{3, 4\}$ ,  $\{4, 5\}$ ,  $\{5\}$  has a transversal 1, 3, 4, 5.

When is there a transversal?

**Corollary 3.** Sets  $S_1, \dots, S_n$  have a transversal  $\iff \left| \bigcup_{i \in A} S_i \right| \geq |A|$  for all  $A \subset \{1, \dots, n\}$ .

**Proof.** ( $\Rightarrow$ ) Trivial.

( $\Leftarrow$ ) Wlog all  $S_i$  are finite. Form a bipartite graph as follows:  $X = \{1, \dots, n\}$ ,  $Y = S_1 \cup \dots \cup S_n$ , with  $i \in X$  joined to  $j \in Y$  if  $j \in S_i$ . Thus a transversal is precisely a matching from  $X$  to  $Y$ . But for  $A \subset X$ , have  $\left| \bigcup_{i \in A} S_i \right| \geq |A|$ , i.e.  $|\Gamma(A)| \geq |A|$ , so done by Hall.  $\square$

**Remarks.** 1. Actually, corollary 3 is equivalent to Hall, since a matching in a bipartite graph  $G$ , vertex classes  $\{x_1, \dots, x_n\}$  and  $Y$  is exactly a transversal for  $\Gamma(x_1), \dots, \Gamma(x_n)$ .

2. There is also a defect form: there exists a transversal for all but  $d$  of  $S_1, \dots, S_n \iff |\bigcup_{i \in A} S_i| \geq |A| - d$  for all  $A \subset \{1, \dots, n\}$ .

### A typical application of Hall

Let  $G$  be a finite group,  $H$  a subgroup of  $G$ . Have the left cosets  $L_1, \dots, L_k$  ( $k = |G|/|H|$ ), say  $g_1H, \dots, g_kH$ , and have the right cosets  $R_1, \dots, R_k$ , say  $Hg'_1, \dots, Hg'_k$ .

Can we choose representatives of the left cosets that are also representatives of the right cosets – that is,  $g_1, \dots, g_k$  such that  $g_1H, \dots, g_kH$  are the left cosets and  $Hg_1, \dots, Hg_k$  are the right cosets? In other words, we seek a permutation  $\pi$  of  $\{1, \dots, k\}$  such that  $L_i \cap R_{\pi(i)} \neq \emptyset \forall i$ . (“Pair up each left coset with a right coset that it meets.”)

Thus we seek a matching from  $X$  to  $Y$  in  $G$ , where  $X = \{L_1, \dots, L_k\}$ ,  $Y = \{R_1, \dots, R_k\}$ , and  $L_i$  is joined to  $R_j$  if  $L_i \cap R_j \neq \emptyset$ . Thus, by Hall, need to check that  $|\Gamma(A)| \geq |A| \forall A \subset X$ . But  $|\bigcup_{i \in A} L_i| = |A||H|$ , so  $\bigcup_{i \in A} L_i$  must meet at least  $|A|$  of the  $R_j$ , as  $|R_j| = |H|$  for all  $j$ . So done.  $\square$

### Connectivity

**Idea.** How “connected” is a graph?

A tree is connected, but we can remove a point to disconnect it.

A cycle  $G$  is connected, and  $G - x$  is connected for all  $x \in G$  (although we can remove two points to disconnect it).



—  $G - x - y$  still connected  $\forall x, y \in G$ .

For  $G$  connected,  $|G| > 1$ , the **connectivity** of  $G$ ,  $\kappa(G)$ , is the smallest  $|S|$  such that  $S \subset V(G)$  and  $G - S$  is disconnected, or a single point (because of complete graphs).

Say  $G$  is  **$k$ -connected** if  $\kappa(G) \geq k$ .

Thus  $G$  is  $k$ -connected  $\iff$  no set of size  $< k$  disconnects  $G$  (or makes it a single point).

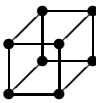
Equivalently,  $G$  is  $k$ -connected  $\iff |G| > k$  and no set of size  $< k$  disconnects  $G$ .

Thus:  $G$  1-connected  $\iff G$  is connected ( $|G| > 1$ )

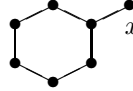
$G$  2-connected  $\iff G$  has no cutvertex ( $|G| > 2$ )

$G$  3-connected  $\iff G$  cannot be disconnected by removing 2 vertices ( $|G| > 3$ )

### Examples.

1. Any tree  $T$  is *not* 2-connected ( $|T| > 1$ )
2. A cycle  $C_n$  is 2-connected, but not 3-connected.
3.  is 3-connected.
4.  $K_n$  is  $(n - 1)$ -connected.

**Warning.** We can have  $\kappa(G - x) > \kappa(G)$ , e.g.

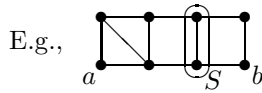


**Remark.** Always have  $\kappa(G) \leq \delta(G)$  = minimum degree in  $G$ : choose  $x$  with  $d(x) = \delta(x)$  and remove  $S = \Gamma(x)$ . Then  $|S| = \delta(x)$  and  $G$  is disconnected (or a single point).

We know:  $G$  connected  $\implies \forall a, b$  there is an  $a$ - $b$  path in  $G$ .

It would be nice if:  $G$   $k$ -connected  $\implies \forall a, b$  there are  $k$  *independent* (i.e., disjoint vertices apart from  $a, b$ )  $a$ - $b$  paths.

For  $G$  connected,  $a, b$  distinct vertices of  $G$ , say  $S \subset V(G) \setminus \{a, b\}$  **separates**  $a$  from  $b$ , or is an  $a$ - $b$  **separator**, if  $a, b$  are in different components of  $G - S$  (i.e., every  $a$ - $b$  path meets  $S$ ).



**Theorem 4 (Menger's Theorem).** Let  $G$  be a connected graph, and  $a, b$  distinct non-adjacent vertices of  $G$ . If all  $a$ - $b$  separators have size  $\geq k$  then there exists a family of  $k$  independent  $a$ - $b$  paths.

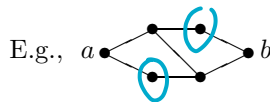
**Remarks.** 1. Converse trivial: any separator contains  $\geq 1$  point from each of the  $k$  paths.

2. Equivalent form: minimum size of an  $a$ - $b$  separator = maximum number of independent  $a$ - $b$  paths.

3. Need “non-adjacent”, else no separators!

4. Menger generalises Hall. Given bipartite  $G$  on  $X, Y$ , with  $|\Gamma(A)| \geq |A| \forall A \subset X$ , form  $G'$  by adding  $s$  (joined to all of  $X$ ) and  $t$  (joined to all of  $Y$ ). Then a matching is precisely a family of  $X$  independent  $s$ - $t$  paths, so by Menger, it's enough to check that every separator has size  $\geq |X|$ . So let  $S = A \cup B$  be a separator, where  $A \subset X, B \subset Y$ . Then  $\Gamma(X \setminus A) \subset B$ , so  $|A| + |B| \geq |\Gamma(X \setminus A)| + |A| \geq |A| + |X \setminus A| = |X|$ .  $\square$

5. *Cannot* prove Menger by choosing one point on each path in a maximum-sized “family of independent  $a$ - $b$  paths”.



**Proof (1).** We may assume that  $k \geq 2$ , as the theorem is trivial for  $k = 1$ .

Let  $k$  be the minimum size of any  $a$ - $b$  separator. We need  $k$  independent paths from  $a$  to  $b$ . If not possible, take a minimal counterexample (say minimal  $k$ , then minimal  $e(G)$  for  $k$ ). Let  $S$  be an  $a$ - $b$  separator,  $|S| = k$ .

Suppose first that  $S \not\subset \Gamma(a)$ ,  $S \not\subset \Gamma(b)$ .

Form  $G'$  from  $G$  by replacing the component of  $G - S$  containing  $a$  by a single point  $a'$  joined to all of  $S$ . Then  $e(G') < e(G)$ , as  $S \not\subset \Gamma(a)$ . In  $G'$ , there is no  $a'$ - $b$  separator of size  $< k$  (else the same set separates  $a$  and  $b$  in  $G$ ), so by minimality of  $e(G)$  we have  $k$  independent  $a'$ - $b$  paths in  $G'$ . I.e., we have  $k$  paths  $B_1, \dots, B_k$  from  $b$  to  $S$ , disjoint except at  $b$ .

Similarly, we have  $k$  paths  $A_1, \dots, A_k$  from  $a$  to  $S$ , disjoint except at  $a$ . No  $A_i$  can meet a  $B_j$  except on  $S$  (else  $S$  is not a separator), so put the  $A_i$  and  $B_j$  together to form  $k$  independent  $a$ - $b$  paths.  $\times$

Now suppose that every  $a$ - $b$  separator  $S$  of size  $k$  has  $S \subset \Gamma(a)$  or  $S \subset \Gamma(b)$ .

We cannot have any  $x \in \Gamma(a) \cap \Gamma(b)$ , because if so then consider  $G - x$ . All  $a$ - $b$  separators in  $G - x$  have size  $\geq k - 1$  (as with  $x$ , they separate  $a$  from  $b$  in  $G$ ), so by minimality of  $k$  there exist  $k - 1$  independent paths in  $G - x$ . Now add  $axb$ , and obtain  $k$  independent paths in  $G$ .  $\times$

Choose a shortest  $a$ - $b$  path, say  $ax_1 \dots x_r b$  ( $r \geq 2$ ) and consider  $G - x_1 x_2$ . In  $G - x_1 x_2$  have a separator  $S$  of size  $k - 1$  (by minimality), so  $S \cup \{x_1\}$  and  $S \cup \{x_2\}$  are separators in  $G$ . Since  $x_1 \notin \Gamma(b)$ , we must have  $S \cup \{x_1\} \subset \Gamma(a)$ . And since  $x_2 \notin \Gamma(a)$ , we must have  $S \cup \{x_2\} \subset \Gamma(b)$ . So  $S \subset \Gamma(a) \cap \Gamma(b)$ , contradicting  $\Gamma(a) \cap \Gamma(b) = \emptyset$ . (If  $S = \emptyset$  then  $k = 1$ .)  $\times$   $\square$

**Proof (2).** We'll apply vertex capacity form of max-flow min-cut. Form a directed network by replacing each edge  $xy$  by directed edges  $\vec{xy}$  and  $\vec{yx}$  and give each capacity 1. Then an integer-valued flow of size  $k$  is exactly a family of  $k$  independent  $a$ - $b$  paths. So, by integrality form of max-flow min-cut, just need that every vertex cut set has size  $\geq k$ , i.e. every  $a$ - $b$  separator has size  $\geq k$ .  $\square$

**Corollary 5.** Let  $G$  be a graph,  $|G| > 1$ . Then  $G$  is  $k$ -connected  $\iff$  for all  $a, b \in G$ , there exist  $k$  independent  $a$ - $b$  paths.

**Remark.** Sometimes also called "Menger's Theorem".

**Proof.** ( $\Leftarrow$ ) Certainly  $G$  is connected, with  $|G| > k$ . Also, no set of size  $< k$  can disconnect  $a$  from  $b$  (else choose  $a, b$  in different components).

( $\Rightarrow$ ) If  $a, b$  are non-adjacent, we are done by Menger (as  $G$  is  $k$ -connected, so no set of size  $< k$  can separate  $a, b$ ).

If  $a, b$  are adjacent, let  $G' = G - ab$ . Then  $G'$  is  $(k - 1)$ -connected (as  $G$  is  $k$ -connected), so by Menger there exist  $k - 1$  independent  $a$ - $b$  paths in  $G'$ . Now add edge  $ab$  as a  $k^{\text{th}}$  path.  $\square$


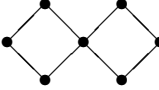
For  $G$  connected,  $|G| > 1$ , the **edge connectivity**  $\lambda(G)$  of  $G$  is the smallest size of a set  $W \subset E(G)$  such that  $G - W$  is disconnected.

Say  $G$  is  **$k$ -edge-connected** if  $\lambda(G) \geq k$ .

Thus:  $G$  1-edge-connected  $\iff G$  connected ( $|G| > 1$ )

$G$  2-edge-connected  $\iff G$  connected and has no bridge ( $|G| > 1$ )

(Note, different from 2-connected: while bridge  $\Rightarrow$  cutvertex, the converse is false.)

E.g.,  has  $\lambda(G) = 2$ , and  has  $\lambda(G) = 2$ ,  $\kappa(G) = 1$ .

We always have  $\lambda(G) \leq \delta(G)$ .

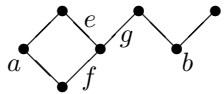
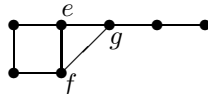
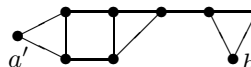
**Theorem 6 (Edge version of Menger).** Let  $G$  be a connected graph, and  $a, b$  distinct vertices of  $G$ . Then the minimum size of  $W \subset E(G)$  separating  $a$  from  $b$  equals the maximum number of edge-disjoint  $a$ - $b$  paths.

**Idea.** “View edges of  $G$  as vertices, and apply vertex Menger.”

The **line graph**  $L(G)$  of a graph  $G$  has vertex set  $E(G)$ , with  $e$  joined to  $f$  if they meet (share a vertex) in  $G$ .

**Proof of 6.** Form  $G'$  from  $L(G)$  by adding new vertices  $a', b'$  to  $L(G)$  with edges  $a'e$  for each  $e \in E(G) = V(L(G))$  with  $a \in e$ , and edges  $b'e$  for each  $e \in E(G)$  with  $b \in e$ .

Then  $\exists a$ - $b$  path in  $G \iff \exists a'$ - $b'$  path in  $G'$ , and deleting edges in  $G$  corresponds to deleting vertices (not  $a', b'$ ) in  $G'$ . So done by vertex Menger applied to  $G'$ .  $\square$

E.g., if  $G$  is  then  $L(G)$  is  and  $G'$  is 

**Corollary 7.** Let  $G$  be a graph,  $|G| > 1$ . Then  $G$  is  $k$ -edge-connected  $\iff$  for all distinct  $a, b \in G$  there exist  $k$  edge-disjoint  $a$ - $b$  paths.

**Proof.** ( $\Leftarrow$ ) Obvious.

( $\Rightarrow$ ) Edge form of Menger.  $\square$


**Remark.** Or prove Theorem 6 and Corollary 7 by max-flow min-cut (usual edge capacity version).



## Chapter 3 : Extremal Problems

An **Euler circuit**, or **Eulerian circuit**, in a graph  $G$  is a circuit passing through each edge exactly once. I.e.,  $x_1, \dots, x_k$  ( $x_k = x_1$ ) such that if  $xy \in E$  then there is a unique  $1 \leq i \leq k-1$  with  $xy = x_i x_{i+1}$ .

Say  $G$  is **Eulerian** if it has an Euler circuit.

E.g.,  is Eulerian. But any graph with a bridge is not Eulerian.

Which graphs are Eulerian?

**Proposition 1.** Let  $G$  be a connected graph. Then  $G$  is Eulerian  $\iff d(x)$  is even for all  $x \in G$ .

**Remark.** Hence  $G$  is Eulerian  $\iff$  all degrees are even and at most one component contains an edge.

**Proof of 1.** ( $\Rightarrow$ ) If an Eulerian circuit passes through  $x$   $k$  times, then  $d(x) = 2k$ .


( $\Leftarrow$ ) Use induction on  $e(G)$ . Done if  $e(G) = 0$ .

Given  $G$  connected, with  $e(G) > 0$  and  $d(x)$  even for all  $x \in G$ , suppose that  $G$  is not Eulerian and let  $C$  be a largest circuit in  $G$  with no edges repeated. Note that  $e(C) > 0$  as  $G$  has a cycle (since  $d(x) \geq 2$  for all  $x \in G$ , so  $G$  is not a tree).

Let  $H$  be a component of  $G - E(C)$  with  $e(H) > 0$ . Then  $H$  is connected and  $d_H(x)$  is even for all  $x \in H$  (as  $d_G(x)$  and  $d_C(x)$  are even for all  $x$ ). So, by the induction hypothesis,  $H$  has an Euler circuit  $C'$ . But now  $C$  and  $C'$  are edge disjoint circuits that share a vertex (we do have  $V(H) \cap V(C) \neq \emptyset$  as  $G$  is connected), so we can combine them to form a circuit longer than  $C$ .  $\times$   $\square$

Let  $G$  be a graph of order  $n$ . Say  $G$  is **Hamiltonian** if it has a cycle of length  $n$  (i.e., a cycle through all vertices). Such a cycle is called a **Hamilton cycle**.

There is no nice “ $\Leftrightarrow$ ” condition known for Hamiltonicity. We can’t have a parity type of condition, as  $G$  Hamiltonian  $\Rightarrow G + xy$  Hamiltonian.

E.g.,  is Hamiltonian (use the perimeter).

But any graph with a cutvertex is not Hamiltonian.

How “large” does a graph on  $n$  vertices have to be to ensure it is Hamiltonian?

**Silly question.** How many edges do we need to ensure that  $G$  is Hamiltonian?

This is silly because any  $x$  with  $d(x) = 1$  stops  $G$  from being Hamiltonian, so take  $G$  to be the **complement** (i.e.,  $\overline{G} = (V, V^{(2)} \setminus E)$ , for  $G = (V, E)$ ) of  $\{x_1 x_2, x_1 x_3, \dots, x_1 x_{n-1}\}$ . Then  $e(G) = \binom{n}{2} - (n-2)$ , but  $G$  is not Hamiltonian.

**Better question.** What  $\delta(G)$  ensures that  $G$  is Hamiltonian?

E.g.,  $n$  even, take two disjoint copies of  $K_{n/2}$ . Then  $\delta(G) = \frac{n}{2} - 1$ , but  $G$  is not Hamiltonian.

E.g.,  $n$  odd, take two copies of  $K_{(n+1)/2}$ , meeting at a single point. Then  $\delta(G) = \frac{n-1}{2}$ , but there is no Hamilton cycle.

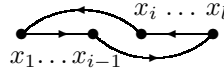
**Theorem 2.** Let  $G$  be a graph of order  $n$  ( $n \geq 3$ ) Then  $\delta(G) \geq \frac{n}{2} \implies G$  is Hamiltonian.

**Proof.**  $G$  is connected, since if  $x, y$  are non-adjacent vertices then  $\Gamma(x), \Gamma(y) \subset V \setminus \{x, y\}$ , so  $\Gamma(x) \cap \Gamma(y) \neq \emptyset$ , as  $|\Gamma(x)|, |\Gamma(y)| \geq n/2$  but  $|V \setminus \{x, y\}| < n$ .

Let  $x_1 x_2 \dots x_l$  be a longest path in  $G$ . (Note  $l \geq 3$  since  $G$  is connected and  $n \geq 3$ .)

Wlog  $G$  has no cycle of length  $l$ , because: if  $l = n$ , we're done; and if  $l < n$  then as  $G$  is connected, there exists  $x \notin \text{cycle}$ , adjacent to some  $y \in \text{cycle}$ , which yields a path on  $l + 1$  points.

Thus  $x_l x_1 \notin E$ . Moreover we cannot have  $2 \leq i \leq l$  with  $x_1 x_i, x_{i-1} x_l \in E$ , or else we have a cycle:



Now,  $\Gamma(x_1) \subset \{x_2, \dots, x_l\}$  and  $\Gamma(x_l) \subset \{x_1, \dots, x_{l-1}\}$  (by maximality of the path), and, by above,  $\Gamma(x_1)$  is disjoint from  $\Gamma_+(x_l) = \{x_i : 2 \leq i \leq l : x_{i-1} \in \Gamma(x_l)\}$ . But  $|\Gamma(x_1)|, |\Gamma_+(x_l)| \geq n/2$  and  $\Gamma(x_1), \Gamma_+(x_l) \subset \{x_2, \dots, x_l\}$ .  $\times$   $\square$

**Remark.** We didn't use the full strength of  $\delta(G) \geq n/2$ . We used only that  $x, y$  non-adjacent  $\implies d(x) + d(y) \geq n$ .

Similarly,

**Proposition 3.** Let  $G$  be a graph of order  $n$  ( $n \geq 3$ ). Then  $G$  connected and  $\delta(G) \geq \frac{k}{2}$ , where  $k < n \implies G$  has a path of length  $k$ .

**Note.** We do need  $k < n$ : e.g.,  $G = K_k$ . And we do need  $G$  to be connected: e.g., two disjoint copies of  $K_k$ .

**Proof of 3.** Let  $x_1 \dots x_l$  be a longest path in  $G$ . ( $l \geq 3$  since  $G$  is connected and  $n \geq 3$ .) Suppose that  $l < k$ . Then, as in the proof of Theorem 2: wlog  $G$  has no  $l$ -cycle, and thus  $\Gamma(x_1)$  and  $\Gamma_+(x_l)$  are disjoint subsets of  $\{x_2, \dots, x_l\}$ , each of size  $\geq k/2$ .  $\times$   $\square$

**Theorem 4.** Let  $G$  be a graph of order  $n$  with  $e(G) > \frac{n(k-1)}{2}$ . Then  $G$  contains a path of length  $k$ .

**Remarks.** 1. Equivalently,  $G \not\supset P_k \implies e(G) \leq \frac{n(k-1)}{2}$ .

2. This cannot be improved, if  $k$  divides  $n$ : e.g.,  $n/k$  disjoint  $K_k$ .

Each vertex has  $k-1$  edges going out of it,  $n$  vertices so  $\frac{n(k-1)}{2}$  edges  $\leftarrow$  double counted edges.

**Proof of 4.** Use induction on  $n$ . Done if  $n \leq 2$ .

Given  $G$ , with  $|G| = n \geq 3$  and  $G \not\supset P_k$ , we want  $e(G) \leq \frac{1}{2}n(k-1)$ .

Wlog,  $G$  is connected – if not, then components  $G_1, \dots, G_r$  of orders  $n_1, \dots, n_r$  have  $e(G_i) \leq \frac{1}{2}n_i(k-1)$  by induction, whence  $e(G) \leq \sum \frac{1}{2}n_i(k-1) = \frac{1}{2}n(k-1)$ .

Thus  $G$  has a vertex  $x$  of degree  $\leq \frac{1}{2}(k-1)$ , by Proposition 3.  
(Note: we may assume that  $k < n$ , as if  $n \leq k$  then  $e(G) \leq \frac{1}{2}n(k-1)$  trivially).

But then  $G - x$  is on  $n-1$  vertices and has no  $P_k$ , so  $e(G-x) \leq \frac{1}{2}(n-1)(k-1)$  by induction, whence  $e(G) \leq \frac{1}{2}(n-1)(k-1) + d(x) \leq \frac{1}{2}n(k-1)$ .  $\square$

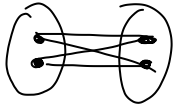
**Remark.** Both Theorem 2 and Theorem 4 are *extremal* results. They answer “how large can a graph be with a certain property?” Often this property is non-containment of a given graph. E.g., how big can  $e(G)$  be if  $G$  is triangle-free?

## Turán’s Theorem

How many edges guarantee a  $K_r$ ?

Equivalently, how many edges can a graph on  $n$  vertices have if it does not have  $K_r$  as a subgraph?

For  $r = 3$ , we would try  $G = K_{a,b}$  where  $a + b = n$ . Take  $a = b = n/2$  if  $n$  is even, and  $a = (n+1)/2$ ,  $b = (n-1)/2$  if  $n$  is odd.



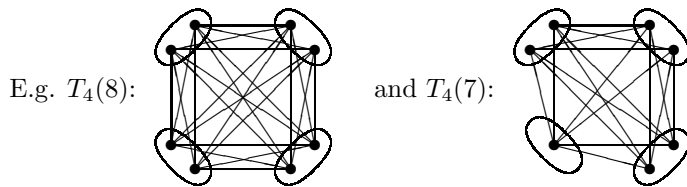
Say  $G$  is  **$k$ -partite** on classes  $V_1, \dots, V_k$  if  $V_1, \dots, V_k$  partition  $V$  and  $G[V_i] = \emptyset$  for all  $i$ .

So  $G$   $(r-1)$ -partite  $\Rightarrow G \not\supset K_r$  (else  $\geq 2$  vertices of  $K_r$  in some  $V_i$ ).

If in addition  $e(G) = \{xy : x \in V_i, y \in V_j, \text{ some } i \neq j\}$ , then say  $G$  is **complete  $k$ -partite**.

The **Turán graph**  $T_k(n)$  is the complete  $k$ -partite graph on  $n$  vertices, with vertex classes  $V_1, \dots, V_k$ , where  $|V_1|, \dots, |V_k|$  are as equal as possible.

(Integers are “as equal as possible” if  $|a_i - a_j| \leq 1 \ \forall i, j$ .)



Certainly  $T_{r-1}(n) \not\supset K_r$  (as  $T_{r-1}(n)$  is  $(r-1)$ -partite), and  $T_{r-1}(n)$  is maximal  $K_r$ -free: if we add any edge then we make a  $K_r$ .

If  $k$  divides  $n$  then all classes have size  $n/k$ , so  $d(x) = n - n/k = n(1 - 1/k)$  for all  $x$ . In general, the classes have size  $\lceil n/k \rceil$  or  $\lfloor n/k \rfloor$ , so all degrees are  $n - \lceil n/k \rceil$  or  $n - \lfloor n/k \rfloor$ .

**Notes.** 1. To obtain  $T_k(n-1)$  from  $T_k(n)$ , remove a point from a largest class, i.e. a point of minimum degree.

2. And to obtain  $T_k(n)$  from  $T_k(n-1)$ , add a point to a smallest class.

**Theorem 5 (Turán's Theorem).** Let  $G$  be a graph on  $n$  vertices. Then  $e(G) > e(T_{r-1}(n)) \implies G \supset K_r$ .

*Further  $T_{r-1}(n)$  is the unique graph without  $K_r$  of size  $e(T_{r-1}(n))$ .*

**Remarks.** 1. This is best possible, as  $T_{r-1}(n) \not\supset K_r$ .

2. If we knew  $G$  was  $(r-1)$ -partite, done by some kind of AM-GM inequality. But there's no reason why  $G$  should be  $(r-1)$ -partite: e.g.  $C_5$  is  $K_3$ -free, but not bipartite.

3. It looks like the proof has to be fiddly, as  $e(T_{r-1}(n))$  is a complicated function of  $n$  and  $r$ .

**Proof.** Key idea: we'll strengthen the theorem to make it easier to prove. We'll prove:  $|G| = n$ ,  $e(G) = e(T_{r-1}(n))$ ,  $G \not\supset K_r \implies G \cong T_{r-1}(n)$ . (This certainly implies Theorem 5, by maximality of  $T_{r-1}(n)$ .)

Use induction on  $n$ :  $n \leq r-1$  is trivial.

We have  $G$  with  $|G| = n$ ,  $e(G) = e(T_{r-1}(n))$ ,  $G \not\supset K_r$ .

**Claim.**  $\delta(G) \leq \delta(T_{r-1}(n))$ .

**Proof of claim.** We have  $e(G) = e(T_{r-1}(n))$  and so  $\sum_{y \in G} d_G(y) = \sum_{y \in T_{r-1}(n)} d_{T_{r-1}(n)}(y)$ .

But the  $d_{T_{r-1}(n)}(y)$  are as equal as possible, so  $\delta(G) \leq \delta(T_{r-1}(n))$ , as claimed.

Choose  $x \in G$  with  $d(x) = \delta(G)$  and let  $G' = G - x$ . Then  $|G'| = n-1$ ,  $G' \not\supset K_r$ , and  $e(G') = e(G) - \delta(G) \geq e(T_{r-1}(n)) - \delta(T_{r-1}(n)) = e(T_{r-1}(n-1))$  (by note 1 above). Thus  $\delta(G') = \delta(T_{r-1}(n))$  and  $G' \cong T_{r-1}(n-1)$  by the induction hypothesis.

Let  $G'$  have vertex classes  $V_1, \dots, V_{r-1}$ . We cannot have  $x$  joined to a point in every  $V_i$ , else  $G \supset K_r$ , so  $\Gamma(x) \cap V_i = \emptyset$  for some  $i$ . But  $d(x) = n-1 - \min |V_i|$  (by note 2 above). So  $\Gamma(x) = \bigcup_{j \neq i} V_j$ , for some  $i$  with  $|V_i|$  minimal. Thus  $G$  is complete  $(r-1)$ -partite, with vertex classes  $V_j$  ( $j \neq i$ ) and  $V_i \cup \{x\}$ . Thus  $G \cong T_{r-1}(n)$ .  $\square$

There are many other proofs of Turán.

## The Problem of Zarankiewicz

"Bipartite version of Turán's Theorem" : how many edges can a bipartite graph  $G$  (with  $n$  vertices in each class) have if  $G \not\supset K_{t,t}$ ?

Write  $Z(n, t)$  for this maximum. How large is  $Z(n, t)$ ?

**Theorem 6.** Let  $t \geq 2$ . Then  $Z(n, t) \leq n^{2-1/t} t^{1/t} + nt$ . In particular,  $Z(n, t) \leq 2n^{2-1/t}$  for  $n$  sufficiently large (i.e., for all  $n \geq n_0(t)$ ).

**Proof.** Let  $G$  be bipartite, vertex classes  $X, Y$ , where  $|X| = |Y| = n$ , with  $G \not\supset K_{t,t}$ . Let degrees in  $X$  be  $d_1, \dots, d_n$ . We'll see that the average degree  $d$  is  $\leq n^{1-1/t}t^{1/t} + t$ .

Wlog each  $d_i \geq t - 1$ . (If  $d_i \leq t - 2$ , add an edge.)

For each  $t$ -set  $A$  in  $Y$ , how many  $x \in X$  have  $A \subset \Gamma(x)$ ? At most  $t - 1$ , as  $G \not\supset K_{t,t}$ .

Thus, the number of  $(x, A)$  with  $x \in X$ ,  $A \subset \Gamma(x)$ ,  $|A| = t$  is  $\leq (t - 1) \binom{n}{t}$ .

But this number also equals  $\sum \binom{d_i}{t}$ , so  $\sum \binom{d_i}{t} \leq (t - 1) \binom{n}{t}$ .

Now, the function  $\binom{x}{t} = \frac{x(x-1)\dots(x-t+1)}{t!}$  is a convex function of  $x$  for  $x \geq t - 1$ . (E.g., put  $y = x - t + 1$ , then  $\binom{x}{t} = \frac{(y+t-1)\dots y}{t!}$ , which is a non-negative linear combination of powers of  $y$ .)

Thus  $\sum \binom{d_i}{t} \geq n \binom{d}{t}$ , so  $n \binom{d}{t} \leq (t - 1) \binom{n}{t}$ , and so  $\frac{n(d-t+1)^t}{t!} \leq \frac{(t-1)n^t}{t!}$ .

Thus  $d - t + 1 \leq n^{1-1/t}(t-1)^{1/t}$ , whence  $d \leq n^{1-1/t}t^{1/t} + t$ . □

Is this the right value? Does  $Z(n, t)$  actually grow as  $n^{2-1/t}$  ( $t$  fixed)?

$t = 2$ .  $G$  bipartite,  $G \not\supset K_{2,2} = C_4$ . Can  $e(G)$  be as large as  $cn^{3/2}$ ? Linear  $e(G)$  is easy, e.g. a  $2n$ -cycle. But  $n^{1.01}$ ? In fact, there are examples of  $G$  with  $e(G) = cn^{3/2}$  (coming from algebra – projective planes).

$t = 3$ . Here,  $cn^{5/3}$  is correct (but harder).

$t = 4$  is unknown!

**\*\* Non-examinable section \*\***

## The Erdős-Stone Theorem

For a fixed graph  $H$ , write  $\text{Ex}(n, H) = \max\{e(G) : |G| = n, G \not\supset H\}$ .

E.g., Turán's Theorem says:  $\text{Ex}(n, K_r) \sim \left(1 - \frac{1}{r-1}\right) \binom{n}{2}$ .

Or, more precisely, Turán says:  $\frac{\text{Ex}(n, K_r)}{\binom{n}{2}} \rightarrow 1 - \frac{1}{r-1}$  as  $r \rightarrow \infty$ .

**Note.**  $\frac{e(G)}{\binom{n}{2}}$  is called the **density** of  $G$ .

And Theorem 4 says:  $\text{Ex}(x, P_k) \sim \frac{n(k-1)}{2}$ .

How does  $\text{Ex}(n, H)$  behave for general  $H$  (as  $n \rightarrow \infty$ )?

Turán says:  $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} \implies G \supset K_r$ . But what if  $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + 0.001$ ?

Write  $K_r(t)$  for  $T_r(rt)$ . (“ $K_r$  blown up by  $t$ ”.)

Remarkably,  $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + 0.001 \implies G \supset K_r(t)$  for any  $t$  (for  $n$  large enough).

In general, we have the...

**Erdős-Stone Theorem.** For all  $r, \epsilon, t$ ,  $\frac{e(G)}{\binom{n}{2}} > 1 - \frac{1}{r-1} + \epsilon \implies G \supset K_r(t)$  for  $n > n_0(r, \epsilon, t)$ .

**Sketch proof.** We have  $G$ , average degree  $\geq \left(1 - \frac{1}{r-1} + \epsilon\right)n$ .

1. Get  $H \subset G$  ( $H$  large) with  $\delta(H) \geq \left(1 - \frac{1}{r-1} + \epsilon\right)n'$  ( $n' = |H|$ ).

(Similar to: average degree  $d \Rightarrow$  can get  $\delta \geq d/2$ .)

2. By induction on  $r$ ,  $H \supset K_{r-1}(t')$ , some  $t'$  large. Write  $K$  for the  $K_{r-1}(t')$ .

3. Have lots of points in  $H - K$ , each joined to  $\geq t$  of each class of  $K$  (by  $\delta(H)$ ).

4. Get  $\geq t$  of these points joined to the same  $t$ -set in each class of  $K$  (by pigeonhole principle).  $\square$

For given  $H$ , choose least  $r$  with  $H$   $r$ -partite. (E.g.,  $H$  = Petersen graph: not bipartite as it has a 5-cycle, but it is 3-partite.) So then  $H \subset K_r(t)$ , some  $t$ .

Then  $T_{r-1}(n) \not\supset H$  (as  $T_{r-1}(n)$  is  $(r-1)$ -partite), so  $\frac{\text{Ex}(n, H)}{\binom{n}{2}} \geq 1 - \frac{1}{r-1}$ .

But Erdős-Stone says  $\frac{\text{Ex}(n, H)}{\binom{n}{2}} \geq 1 - \frac{1}{r-1} + \epsilon \implies G \supset \text{any } K_r(t)$ , so  $G \supset H$ .

Conclusion:  $\frac{\text{Ex}(n, H)}{\binom{n}{2}} \rightarrow 1 - \frac{1}{r-1}$  (where  $r$  is least such that  $H$   $r$ -partite).

**Remark.** If  $H$  bipartite, this says  $\frac{\text{Ex}(n, H)}{\binom{n}{2}} \rightarrow 0$ . But what is the growth speed of  $\text{Ex}(n, H)$ ?

Unknown for most  $H$ . (E.g.,  $H = C_{2n}$ ,  $n \geq 6$ ).

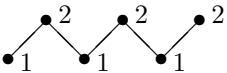
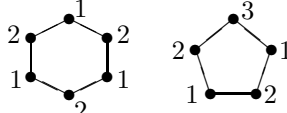
**\*\* End of non-examinable section \*\***

## Chapter 4 : Colourings

An  **$r$ -colouring** of a graph  $G$  is a function  $c : V(G) \rightarrow [r] = \{1, \dots, r\}$ , such that  $c(x) \neq c(y)$  whenever  $x, y$  are adjacent.

The **chromatic number**  $\chi(G)$  of  $G$  is the smallest  $r$  such that  $G$  has an  $r$ -colouring.

**Examples.**

1.  $\chi(P_n) = 2$  : 
2.  $\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$  : 
3.  $\chi(K_n) = n$  (as all vertices need a different colour).
4.  $\chi(E_n) = 1$ .
5.  $T$  a tree  $\Rightarrow \chi(T) = 2$ . (E.g., induction: remove a leaf.)
6.  $\chi(K_{m,n}) = 2$  (one colour for each class).

Clearly any bipartite graph is 2-colourable, and conversely if  $c$  is a 2-colouring of  $G$ , then  $X = \{x : c(x) = 1\}$ ,  $Y = \{y : c(y) = 2\}$  show that  $G$  is bipartite. Similarly,  $G$   $r$ -colourable  $\iff G$   $r$ -partite.

(So Erdős-Stone corollary says:  $\frac{\text{Ex}(n, H)}{\binom{n}{2}} \rightarrow 1 - \frac{1}{\chi(H) - 1}$  as  $n \rightarrow \infty$ .)

If  $|G| = n$  then trivially  $\chi(G) \leq n$ . We can improve this.

**Proposition 1.** Let  $G$  be a graph with maximum degree  $\Delta$ . Then  $\chi(G) \leq \Delta + 1$ .

**Note.** This is best possible: e.g.,  $G$  a complete graph or odd cycle.


**Proof.** Order  $V(G)$  as  $x_1, \dots, x_n$  and colour  $x_1, \dots, x_n$  in turn. When we come to colour  $x_i$ , we find  $\leq \Delta$  colours used already on neighbours of  $x_i$ , so there is  $\geq 1$  colour we can use for  $x_i$ .  $\square$

**Remarks.** 1. We can have  $\chi(G)$  much less than  $\Delta$ .

E.g.,  $K_{1,n}$  is 2-colourable:  (also called a **star**)

2. Could view proof of Proposition 1 as an application of the **greedy algorithm**: for a given ordering  $x_1, \dots, x_n$ , colour each in turn, always using the smallest colour available.

**Warning.** 1. Greedy algorithm may use *more* than  $\chi(G)$  colours.

E.g., in  greedy gives 3 colours.

2.  $G \supset K_r \Rightarrow \chi(G) \geq r$ , but  $G \not\supset K_r \not\Rightarrow \chi(G) < r$ . E.g.,  $C_5 \not\supset C_3$ , but not 2-colourable.

In fact, there is no simple formula for  $\chi(G)$ .

## Colouring Planar Graphs

How many colours do we need to colour a planar graph?

3 isn't enough, e.g.  $K_4$ .

**Proposition 2 (Six Colour Theorem).**  $G$  planar  $\implies G$  is 6-colourable.

**Proof.** Use induction on  $|G|$ . Done if  $|G| \leq 6$ .

We have planar  $G$ ,  $|G| > 6$ .

**Claim.**  $\delta(G) \leq 5$ .

**Proof of claim.** We have  $e(G) \leq 3n - 6$  (as  $G$  is planar), so  $\sum_{x \in G} d(x) \leq 6n - 12$ , so some  $x$  has  $d(x) \leq 5$ .

Choose  $x$  such that  $d(x) \leq 5$ . Then  $G - x$  is planar, so by induction it has a 6-colouring. Then  $\Gamma(x)$  receives  $\leq 5$  colours, so we can colour  $x$  with the remaining colour.  $\square$

**Theorem 3 (Five Colour Theorem).**  $G$  planar  $\implies G$  is 5-colourable.

**Proof.** Use induction on  $|G|$ . Done if  $|G| \leq 5$ .

Given planar  $G$ ,  $|G| > 5$ , we have  $\delta(G) \leq 5$  (as before), so choose  $x \in G$  with  $d(x) \leq 5$ . By the induction hypothesis, we have a 5-colouring of  $G - x$ , so done unless  $d(x) = 5$  and all 5 colours appear in  $\Gamma(x)$ .

Say  $\Gamma(x)$  is  $x_1, \dots, x_5$  (clockwise) with  $x_i$  having colour  $i$  for each  $i$ .

**Question:** is there a 1-3 path from  $x_1$  to  $x_3$ ? (A **1-3-path** is a path on which colours are alternately 1 and 3.)

**If no,** let  $H$  be the 1-3-component of  $x_1$  (i.e., all vertices we can reach from  $x_1$  by a 1-3-path). So  $x_3 \notin H$ . Swap colours 1 and 3 on  $H$ . This is still a colouring of  $G - x$ , but we now have  $x_1$  of colour 3, so we can use colour 1 for  $x$ .

**If yes,** we have a 1-3-path  $P$  from  $x_1$  to  $x_3$ , but then there is no 2-4-path from  $x_2$  to  $x_4$  (as it would have to meet  $P$ , as  $G$  is planar). So finish as above: swap 2, 4 on the 2-4-component of  $x_2$  and then use colour 2 for  $x$ .  $\square$

**Remark.** The  $i$ - $j$ -paths in the proof are called **Kempe chains**.



Suppose we wanted to colour the *faces* of a plane graph such that distinct faces sharing an edge get different colours – i.e., “colouring a plane map”.

Given a plane graph  $G$ , form the **dual graph**  $G'$  by taking a point for each face, and joining two points if they share an edge. So  $G'$  planar and a colouring  $G'$  corresponds exactly to colouring the faces of  $G$ . Thus Theorem 3 tells us that every planar map is 5-colourable.

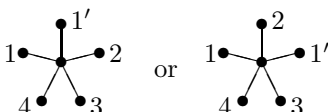
**\*\* Non-examinable section \*\***

**Theorem (Four Colour Theorem).**  $G$  planar  $\implies G$  is 4-colourable.

**“Proof”.** Use induction on  $|G|$ . Done if  $|G| \leq 4$ .

Given planar  $G$ ,  $|G| > 4$ , we have  $\delta(G) \leq 5$  (as usual), so choose  $x \in G$  with  $d(x) \leq 5$ . We can 4-colour  $G - x$  by induction, so done unless either  $d(x) = 4$  or 5, with all colours appearing in  $\Gamma(x)$ .

If  $d(x) = 4$ , proceed as in the Five Colour Theorem. If there is no 1-3-path from  $x_1$  to  $x_3$ , swap the colours 1 and 3 on the 1-3-component of  $x_1$  and use colour 1 for  $x$ . And if there is a 1-3-path from  $x_1$  to  $x_3$ , then there is no 2-4-path from  $x_2$  to  $x_4$ , so swap colours 2 and 4 on the 2-4-component of  $x_2$  and use colour 2 for  $x$ .

If  $d(x) = 5$ , we could have  or

First picture. If there is no 2-4-path from  $x_2$  to  $x_4$  then done (swap colours 2 and 4 on the 2-4-component of  $x_2$ ).

If there is a 2-4-path from  $x_2$  to  $x_4$  then there is no 1-3-path from  $x_3$  to  $x_1$  or  $x_{1'}$ , so done (swap colours 1 and 3 on the 1-3-component of  $x_3$ ).

Second picture. If there is no 2-4-path from  $x_2$  to  $x_4$  then done, as before.

So wlog there is a 2-4-path from  $x_2$  to  $x_4$  (so there is no 1-3-path from  $x_1$  to  $x_3$ ). Similarly, wlog there is a 2-3-path from  $x_2$  to  $x_3$  (so there is no 1-4-path from  $x_{1'}$  to  $x_4$ ). So swap colours 1 and 3 on the 1-3-component of  $x_1$ , and colours  $1'$  and 4 on the  $1'$ -4-component of  $x_{1'}$ . This leaves colour 1 for use at  $x$ .  $\square$

The above “proof” was given by Kempe in 1879.

In 1890, Heawood spotted a mistake – *but where is the mistake?*

The theorem was eventually proved in 1976 by Appel and Haken. In the proof of Theorem 3, we used the fact that “a vertex of degree 1, 2, 3, 4 or 5” forms an **unavoidable** (every planar graph contains  $\geq$  one) set of **reducible** (cannot be present in a minimal counterexample) configurations. Appel and Haken found an unavoidable set of about 1900 reducible configurations for the Four Colour Theorem (using computers a lot).

**\*\* End of non-examinable section \*\***

We have  $\chi(G) \leq \Delta + 1$  for all graphs  $G$ , and we can have equality – for example  $K_n$  and  $C_{\text{odd}}$ . We will show that in fact  $\chi(G) \leq \Delta$  for all connected  $G$  except  $K_n$  and  $C_{\text{odd}}$ .

**Remark.** If  $G$  is connected and not regular, we can certainly colour with  $\Delta$  colours.

Indeed, choose  $x_n$  with  $d(x_n) < \Delta$ . Then choose  $x_{n-1}$  adjacent to  $x_n$  (as  $G$  connected). Then choose  $x_{n-2} \in G - \{x_n, x_{n-1}\}$  adjacent to  $\{x_n, x_{n-1}\}$  (as  $G$  connected). And so on. Then run the greedy algorithm on the order  $x_1, \dots, x_n$ .

Each  $x_i$  has a forward edge, i.e.  $x_i x_j \in E$ , some  $j > i$  (for all  $i \neq n$ ), so uses  $\leq \Delta$  colours. (The presence of a forward edge means that when choosing the colour of a vertex, we have already chosen colours for at most  $\Delta - 1$  of its neighbours.)  $\square$

**Proposition 4 (Brooks' Theorem).** Let  $G$  be connected, but not a complete graph or odd cycle. Then  $\chi(G) \leq \Delta$ .

**Proof.** Wlog  $G$  is regular (by the remark above), and  $\Delta \geq 3$  (as  $\Delta = 1$  is trivial, and  $\Delta = 2 \Rightarrow G$  is a cycle).

Let  $G$  be a minimal counterexample (i.e.,  $|G|$  minimal). Wlog  $G$  is 2-connected: as if  $x$  is a cutvertex, let  $G_1, \dots, G_k$  be the components of  $G - x$  together with  $x$ ; then each  $G_i$  is  $\Delta$ -colourable (by remark above, as  $d_{G_i}(x) < \Delta$  for all  $i$ ), so  $G$  itself is  $\Delta$ -colourable.

**Case 1.**  $G$  is 3-connected.

We want an ordering such that every vertex has a forward edge, and  $x_n$  has two neighbours of the same colour.

Choose any  $x_n$ . There must be some  $x_1, x_2 \in \Gamma(x_n)$  with  $x_1 x_2 \notin E$  (or else  $x_n \cup \Gamma(x_n)$  forms a  $K_{\Delta+1}$ , whence we must have a point of degree  $> \Delta$ , as  $G$  is connected but not itself a complete graph  $\mathbb{X}$ ).

Now,  $G - \{x_1, x_2\}$  is connected (as  $G$  is 3-connected) so order its vertices  $x_3, \dots, x_n$  (as in the remark above) such that for all  $3 \leq i \leq n - 1$ , there is  $j > i$  for which  $x_i x_j \in E(G)$ . Then run the greedy algorithm on the ordering  $x_1, \dots, x_n$ . This uses  $\leq \Delta$  colours.  $\square$

**Case 2.**  $G$  is not 3-connected.

Choose a separator  $\{x, y\}$  (i.e.,  $G - \{x, y\}$  is disconnected), and let  $G_1, \dots, G_k$  be the components of  $G - \{x, y\}$  together with  $x$  and  $y$ . Then each  $G_i$  has a  $\Delta$ -colouring (by the remark above, as  $d_{G_i}(x) \leq \Delta - 1$  for all  $i$ ).

If  $xy \in E$ , then  $x$  and  $y$  have different colours in the colouring of  $G_i$  for each  $i$ , so we can recolour and combine to form a  $\Delta$ -colouring of  $G$ .

So suppose  $xy \notin E$ . If each  $G_i$  has at least one of  $d_{G_i}(x), d_{G_i}(y)$  being  $\leq \Delta - 2$ , we can recolour to ensure  $x$  and  $y$  have different colours in  $G_i$ .

So done, unless some  $G_i$  has  $d_{G_i}(x) = d_{G_i}(y) = \Delta - 1$ , say  $i = 1$ . Then we must have  $k = 2$ , with  $d_{G_2}(x) = d_{G_2}(y) = 1$  (as  $d(x), d(y) \leq \Delta$ ). Let  $\Gamma_{G_2}(x) = \{u\}$ ,  $\Gamma_{G_2}(y) = \{v\}$ . Then  $\{x, v\}$  is a separator, *not* of this form.  $\square$

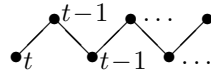
## The Chromatic Polynomial

We are carrying more information on  $G$  than the number  $\chi(G)$ . For any graph  $G$  and  $t = 1, 2, 3, \dots$ , let  $P_G(t)$  be the number of  $t$ -colourings of  $G$ . (So  $\chi(G)$  equals the least  $t$  such that  $P_G(t) > 0$ .)

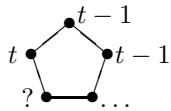
**Examples.**

$$P_{K_n}(t) = t(t-1)(t-2)\dots(t-n+1).$$

$$P_{E_n}(t) = t^n.$$

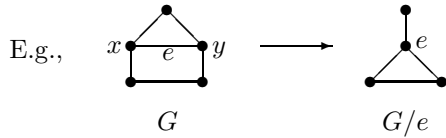
$$P_{P_n}(t) = t(t-1)^n :$$


And in general  $P_{T_n}(t) = t(t-1)^{n-1}$  for any tree  $T$  on  $n$  vertices (by induction)

$$P_{C_n}(t) = ..?$$


Is  $P_G(t)$  always a polynomial?

For a graph  $G$  and  $e = xy \in E(G)$ , the **contraction** of  $G$  by  $e$ , written  $G/e$ , is formed from  $G$  by replacing  $x$  and  $y$  by a new vertex  $e$ , joined to all points that were joined to  $x$  or  $y$ .



**Lemma 5.** Let  $G$  be a graph, and  $e \in E$ . Then  $P_G = P_{G-e} - P_{G/e}$ .

**Remark.** Called the **deletion-contraction relation** or **cut-fuse relation**.

**Proof.** The  $t$ -colourings of  $G - e$  in which  $x$  and  $y$  get different colours correspond exactly with the  $t$ -colourings of  $G$ .

And the  $t$ -colourings of  $G - e$  in which  $x$  and  $y$  get the same colour correspond exactly with the  $t$ -colourings of  $G/e$ .

$$\text{Thus } P_{G-e}(t) = P_G(t) + P_{G/e}(t). \quad \square$$

**Note.** We could not use Lemma 5 (and a base case of  $G = E_n$ ) to define  $P_G$ , as it's not clear that it gives a unique function  $P_G$  for all  $G$ .

**Proposition 6.**  $P_G(t)$  is a polynomial in  $t$ .

**Proof.** Use induction on  $e(G)$ .  $P_{E_n}(t) = t^n$ , so  $e(G) = 0$  is done.

Given  $G$ ,  $e(G) > 0$ , choose  $e \in E$ . Then  $P_{G-e}$  and  $P_{G/e}$  are polynomials, by induction. So  $P_G = P_{G-e} - P_{G/e}$  is also a polynomial.  $\square$

For a tree  $T$ , we had  $P_T(t) = t^n - (n-1)t^{n-1} + \dots$

**Proposition 7.** Let  $G$  be a graph on  $n$  vertices with  $m$  edges. Then the leading terms of  $P_G(t)$  are  $t^n - mt^{n-1} + \dots$

**Proof.** Use induction on  $e(G)$ .  $P_{E_n}(t) = t^n$ , so  $e(G) = 0$  is done.

Given  $G$ ,  $e(G) > 0$ , choose  $e \in E$ . Then, by induction,  $P_{G-e} = t^n - (m-1)t^{n-1} + \dots$  and  $P_{G/e} = t^{n-1} + \dots$ . So  $P_G(t) = t^n - mt^{n-1} + \dots$   $\square$

**Remarks.** 1. We can get other information about  $G$  from its chromatic polynomial. For example, it turns out that  $P_G(t) = t^n - mt^{n-1} + \left(\binom{m}{2} - \#\text{triangles of } G\right)t^{n-2} + \dots$

2. Since  $P_G$  is a polynomial, we can talk about  $P_G(t)$  for any real or complex  $t$ .

3. Four Colour Theorem says: planar  $G$  has  $P_G(4) > 0$ . I.e.,  $P_G$  has no root at 4. No polynomial-style direct proof is known that  $P_G(4) \neq 0$  for all planar  $G$ . It is known that  $P_G(\varphi + 2) \neq 0$  where  $\varphi = (1 + \sqrt{5})/2$ .

## Edge Colouring

A  **$k$ -edge-colouring** of  $G$  is a map  $c : E(G) \rightarrow \{1, \dots, k\}$  such that  $c(e) \neq c(f)$  whenever  $e, f$  share a vertex.

The smallest such  $k$  is called the **edge-chromatic number** or **chromatic index** of  $G$ , written  $\chi'(G)$ . (Thus  $\chi'(G) = \chi(L(G))$ .)

$$\text{E.g., } \chi'(C_n) = \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

But also,  $\chi'(G)$  can be far from  $\chi(G)$ . E.g.,  $K_{1,n}$  (a star) has  $\chi(G) = 2$ ,  $\chi'(G) = n$ .

Clearly  $\chi'(G) \geq \Delta(G)$  for all  $G$ . We can have  $\chi'(G) > \Delta(G)$ , for example  $C_{\text{odd}}$ .

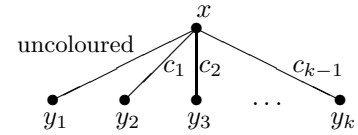
But very surprisingly:

**Theorem 8 (Vizing's Theorem).** For any  $G$ , we have  $\chi'(G) = \Delta(G)$  or  $\Delta(G) + 1$ .

**Proof.** Let  $\Delta = \Delta(G)$ . We must show that  $\chi'(G) \leq \Delta + 1$ , i.e. that  $G$  can be  $(\Delta + 1)$ -edge-coloured. Use induction on  $e(G)$ . Done if  $e(G) = 0$ .

Given  $G$ ,  $e(G) > 0$ , choose  $e \in E(G)$ . We have a  $(\Delta + 1)$ -edge-colouring of  $G - e$  (by induction). Let  $e$  be  $xy_1$ . At each vertex, some colour is not being used (as  $\Delta + 1 > d(y)$  for all  $y \in G$ ).

Choose a maximal sequence of distinct vertices  $y_1, \dots, y_k$  as follows. Given edge  $xy_i$ , choose a colour  $c_i$  missing at  $y_i$ . If there is a new edge from  $x$  with colour  $c_i$ , let  $xy_{i+1}$  be this edge.



This must stop, as  $G$  is finite. And it must stop because either  $c_k$  is not used at  $x$ , or  $c_k = c_j$  for some  $j < k$ .

If  $c_k$  is not used at  $x$ , then recolour by giving  $xy_i$  colour  $c_i$  for all  $1 \leq i \leq k$ . This is a  $(\Delta + 1)$ -edge-colouring of  $G$ .

If  $c_k = c_j$  for some  $j < k$ , wlog  $j = 1$ , because we can recolour  $xy_i$  with colour  $c_i$  for  $1 \leq i \leq j - 1$ , leaving  $xy_j$  as the uncoloured edge. Let  $c$  be a colour not used at  $x$ .

If there is no  $c - c_1$  path from  $y_1$  to  $x$ , then swap  $c$  and  $c_1$  on all edges of the  $c - c_1$  component of  $y_1$ . This leaves  $c$  missing at  $y_1$  (since  $c_1$  was missing at  $y_1$  previously) and at  $x$ , so we can give  $xy_1$  colour  $c$ , so done.

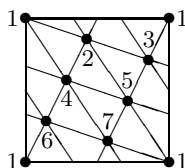
Similarly, if there is no  $c - c_1$  path from  $y_k$  to  $x$  (recall  $c_k = c_1$ ), then swap  $c$  and  $c_1$  on the  $c - c_1$  component of  $y_k$ . We can now use colour  $c$  for  $xy_k$  and colour  $c_i$  for  $xy_i$  ( $1 \leq i \leq k - 1$ ), so done.

Otherwise, the  $c - c_1$  component of  $x$  (call it  $H$ ) contains  $y_1$  and  $y_k$ , but  $H$  is connected and has  $\Delta(H) \leq 2$  (as  $H$  is 2-edge-coloured) and so is a path or cycle, but  $d_H(x) = d_H(y_1) = d_H(y_k) = 1$ .  $\nexists$   $\square$

## Graphs on Surfaces

We know that  $G$  drawn on the plane or a sphere has  $\chi(G) \leq 5$  (well, actually  $\leq 4$ ). What about  $G$  drawn on other surfaces?

E.g., we can draw  $K_7$  on a torus:



(with edges identified).

In general?

The **surface of genus  $g$**  (or the **compact orientable surface of genus  $g$** ) consists of a sphere with  $g$  handles attached.

For plane/sphere,  $n - m + f = 2$  for  $G$  connected, and  $n - m + f \geq 2$  for any planar  $G$  (add edges to make  $G$  connected).

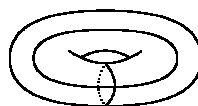
$G$  on a torus?



$$n - m + f = 2$$



$$n - m + f = 1$$



$$n - m + f = 0$$

**Fact.** For  $G$  on a surface of genus  $g$ , we have  $n - m + f \geq 2 - 2g$ . (This is the **Euler characteristic**,  $E = 2 - 2g$ .)

So  $n - m + f \geq E$ , and  $3f \leq 2m$  (for  $m \geq 3$ ) as usual.

Thus  $n - m + \frac{2m}{3} \geq E$ , and so  $m \leq 3(n - E)$ .

**Theorem 9 (Heawood's Theorem).** Let  $G$  be a graph drawn on a surface of Euler characteristic  $E \leq 0$ .

$$\text{Then } \chi(G) \leq H(E) = \left\lfloor \frac{7 + \sqrt{49 - 24E}}{2} \right\rfloor.$$

**Note.** Tantalizingly,  $H(2) = 4$ , so “almost” have the Four Colour Theorem (but need  $E = 2$ ).

**Proof.** Let  $G$  be a graph drawn on the surface, with  $\chi(G) = k$ . We want  $k \leq H(E)$ . Choose minimal  $G$  with  $\chi(G) = k$ . Then  $\delta(G) \geq k - 1$  (by minimality of  $G$ ) and  $n \geq k$ .

We know  $m \leq 3(n - E)$ , so the sum of all degrees  $= 2m \leq 6(n - E)$ , and so  $\delta(G) \leq 6(n - E)/n = 6 - 6E/n$ .

Thus  $k - 1 \leq \delta(G) \leq 6 - 6E/n \leq 6 - 6E/k$  (as  $n \geq k$  and  $E \leq 0$ , which is why this fails for the Four Colour Theorem).

So  $k^2 - k \leq 6k - 6E$ , i.e.  $k^2 - 7k + 6E \leq 0$ . Whence result.  $\square$

**Remark.** Equality holds. We can draw  $K_{H(E)}$  on a surface of characteristic  $E$  – this took 75 years to prove!

## Chapter 5 : Ramsey Theory

Can we find order in enough disorder?

Suppose we 2-edge-color  $K_6$ , e.g.,  $c : E(K_6) \rightarrow \{1, 2\}$ . Can we always find a monochromatic  $K_3$  (i.e., a  $K_3$  on which  $c$  is constant)?

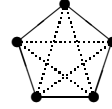
Choose  $x \in V(K_6)$ . We have  $d(x) = 5$ , so there are at least 3 edges from  $x$  with the same colour – say  $xy_1, xy_2, xy_3$  have colour 1. If any edge  $y_i y_j$  ( $i \neq j$ ) also has colour 1, then  $xy_i y_j$  is a colour-1 triangle. But if all edges  $y_i y_j$  are colour 2, then  $y_1 y_2 y_3$  is a colour-2 triangle.

What about  $K_4$ ? Is there an  $n$  such that  $K_n$  2-edge-coloured  $\Rightarrow \exists$  a monochromatic  $K_4$ ?

In general, write  $R(s)$  for the smallest  $n$  (if it exists) such that whenever  $K_n$  is 2-coloured (we now mean “edge-coloured”), there exists a monochromatic  $K_s$  (i.e., a  $K_s$  on which  $c : E(K_n) \rightarrow \{1, 2\}$  is constant).

**Aim.** To show that  $R(s)$  exists for all  $s$  (and find out roughly how fast  $R(s)$  grows).

E.g., the above shows that  $R(3) \leq 6$ . In fact,  $R(3) = 6$ , because of



**Idea.** “To go from monochromatic  $K_3$  to monochromatic  $K_4$ , first try to get a red  $K_3$  or blue  $K_4$ .”

For  $s, t \geq 2$ , write  $R(s, t)$  for the smallest  $n$  (if it exists) such that whenever  $K_n$  is 2-coloured, we have either a red  $K_s$  or a blue  $K_t$ .

So:  $R(s, s) = R(s)$ ,

$R(s, t) = R(t, s)$ ,

$R(s, 2) = s$ .

Equivalently,  $R(s, t)$  is the smallest  $n$  (if it exists) such that every graph  $G$  on  $n$  vertices has either  $K_s \subset G$  or  $K_t \subset \overline{G}$ .

**Theorem 1 (Ramsey’s Theorem).**  $R(s, t)$  exists for all  $s, t$ . And moreover, we have  $R(s, t) \leq R(s-1, t) + R(s, t-1)$ , for  $s, t \geq 3$ .

**Proof.** It is enough to show that if both  $R(s-1, t)$  and  $R(s, t-1)$  exist, then we have  $R(s, t) \leq R(s-1, t) + R(s, t-1)$ , as then  $R(s, t)$  exists for all  $s, t$  by induction on  $s+t$ .

So let  $a = R(s-1, t)$ ,  $b = R(s, t-1)$ . Choose a 2-colouring of  $K_{a+b}$ , and choose  $x \in K_{a+b}$ . Then  $d(x) = a+b-1$ , so we have either  $\geq a$  red edges or  $\geq b$  blue edges incident with  $x$ .

If  $\geq a$  are red, then consider the  $K_a$  spanned by the endpoints of the edges from  $x$ . By the definition of  $a$ , this  $K_a$  contains either a red  $K_{s-1}$  or a blue  $K_t$ .

If  $\geq b$  are blue, similarly. □

**Remarks.** 1. So any graph on  $n$  vertices has  $K_s \subset G$  or  $K_s \subset \overline{G}$  for  $n$  large enough.

2. Very few of the **Ramsey numbers**  $R(s, t)$  are known exactly. (See later.)

**Corollary 2.** Let  $s, t \geq 2$ . Then  $R(s, t) \leq \binom{s+t-2}{s-1}$ . In particular,  $R(s) \leq 2^{2s}$ .

**Proof.** Induction on  $s+t$ . Done if  $s=2$  or  $t=2$ . Given  $s, t \geq 3$ ,

$$R(s, t) \leq R(s-1, t) + R(s, t-1) \leq \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1}. \quad \square$$

**What about more colours?**

Write  $R_k(s_1, \dots, s_k)$  for the smallest  $n$  (if it exists) such that whenever  $K_n$  is  $k$ -coloured, there exists a  $K_{s_i}$  of colour  $i$ , for some  $1 \leq i \leq k$ .

**Corollary 3.**  $R_k(s_1, \dots, s_k)$  exists for all  $k \geq 1$  and  $s_1, \dots, s_k \geq 2$ .

**Proof (“Turquoise Spectacles”).** Induction on  $k$ , the number of colours. Done if  $k=1$ . (Or if  $k=2$ , by Ramsey’s Theorem.)

Given  $k \geq 2$  and  $s_1, \dots, s_k$ , let  $n = R(s_1, R_{k-1}(s_2, \dots, s_k))$ . Then any  $k$ -colouring of  $K_n$  may be viewed as a 2-colouring, with colours 1 and “2 or 3 or ... or  $k$ ”.

So, by the choice of  $n$ , we have either a  $K_{s_1}$  coloured 1 (so done), or a  $K_{R_{k-1}(s_2, \dots, s_k)}$  coloured with colours 2, 3, ...,  $k$  (i.e.,  $(k-1)$ -coloured), so done by the definition of  $R_{k-1}(s_2, \dots, s_k)$ .  $\square$

**What about  $r$ -sets?**

E.g.,  $r=3$ . Colour each *triangle* red/blue – do we get a 4-set all of whose triangles are the same colour? (We are asking for a much denser monochromatic structure.)

For  $X$  a set and  $r=1, 2, \dots$ , write  $X^{(r)} = \{A \subset X : |A| = r\}$ . Unless otherwise stated,  $X = [n] = \{1, \dots, n\}$ .

Write  $R^{(r)}(s, t)$  for the smallest  $n$  (if it exists) such that whenever  $X^{(r)}$  is 2-coloured (i.e., we have  $c : X^{(r)} \rightarrow \{1, 2\}$ ), there exists a red  $s$ -set (i.e.,  $A \subset X$ ,  $|A| = s$ , with  $c(B) = 1$  for all  $B \in A^{(r)}$ ) or a blue  $t$ -set.

So:  $R^{(2)}(s, t) = R(s, t)$ ,

$$R^{(1)}(s, t) = s + t - 1 = (s-1) + (t-1) + 1 \quad (\text{pigeonhole principle}),$$

$$R^{(r)}(s, t) = R^{(r)}(t, s),$$

$$R^{(r)}(s, r) = R^{(r)}(r, s) = s.$$

**Theorem 4 (Ramsey for  $r$ -sets).** Let  $r \geq 1$ ,  $s, t \geq r$ . Then  $R^{(r)}(s, t)$  exists.

**Idea.** In the proof of Theorem 1 ( $r=2$ ), we used the case  $r=1$ .



**Proof.** Induction on  $r$ . Done if  $r = 1$  (pigeonhole principle) or  $r = 2$  (Theorem 1).

Given  $r > 1$ , use induction on  $s + t$ . Done if  $s = r$  or  $t = r$ . So, suppose  $s, t > r$ .

**Claim.**  $R^{(r)}(s, t) \leq R^{(r-1)}(R^{(r)}(s-1, t), R^{(r)}(s, t-1)) + 1$ .

**Proof of claim.** Let  $a = R^{(r)}(s-1, t)$ ,  $b = R^{(r)}(s, t-1)$ , and  $n = R^{(r-1)}(a, b) + 1$ .

Given a 2-colouring  $c$  of  $X^{(r)}$ , choose  $x \in X$  and let  $Y = X \setminus \{x\}$ . Then  $c$  induces a 2-colouring  $c'$  of  $Y^{(r-1)}$  by  $c'(A) = c(A \cup \{x\})$ , for each  $A \in Y^{(r-1)}$ . So by the definition of  $R^{(r-1)}(a, b)$ , we have either a red  $a$ -set for  $c'$  or a blue  $b$ -set for  $c'$ .

By symmetry, wlog we have a red  $a$ -set  $Z$  for  $c'$ , i.e.  $A \cup \{x\}$  is red for all  $A \in Z^{(r-1)}$ . But by the definition of  $a$ ,  $Z$  contains either a red  $(s-1)$ -set for  $c$  or a blue  $t$ -set for  $c$ .

If a blue  $t$ -set, then we are done.

If a red  $(s-1)$ -set, then add  $x$  and obtain a red  $s$ -set. □

**Remarks.** 1. Similarly for  $k$  colours – e.g., by “turquoise spectacles”.

2. The bounds we get on  $R^{(s,t)}$  are quite large: “to get  $R^{(r)}$ , iterate  $R^{(r-1)}$  about  $s+t$  times”.

Define  $f_1, f_2, f_3, \dots : \mathbb{N} \rightarrow \mathbb{N}$  as follows:

$$f_1(x) = 2x, \text{ and for } n \geq 2, f_n(x) = \underbrace{f_{n-1}(f_{n-1}(\dots(f_{n-1}(x))\dots))}_{x \text{ times}}.$$

So  $f_2(x) = 2^x$ ,  $f_3(x) = 2^{2^{\cdot^{\cdot^2}}}$  }  $x$  times. And  $f_4(x) \dots$ ?

Well,  $f_4(1) = 2$ ,  $f_4(2) = 2^2 = 4$ ,  $f_3(2) = 2^{2^2} = 65536$ ,  $f_4(4) = 2^{2^{\cdot^{\cdot^2}}}$  } 65536 times

Then our bound on  $R^{(r)}(s, t)$  is of the form  $f_r(s+t)$ .

(These sorts of bounds are often a feature of such double induction proofs.)

## Infinite Ramsey Theory

Given a 2-colouring  $c$  of  $\mathbb{N}^{(2)}$ , can we always find an infinite monochromatic subset? (I.e.,  $M \subset \mathbb{N}$ ,  $M$  infinite, with  $c$  constant on  $M^{(2)}$ ?)

**Examples.** 1. Colour  $ij$  red if  $i+j$  is even, and blue if odd. Take, e.g.,  $M = \{n : n \text{ even}\}$ .

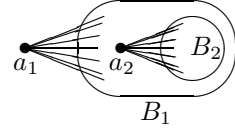
2. Colour  $ij$  red if  $\max\{n : 2^n \text{ divides } i+j\}$  is even, and blue otherwise. Take, e.g.,  $M = \{2^2, 2^4, 2^6, 2^8, \dots\}$ .

3. Colour  $ij$  red if  $i+j$  has an even number of (distinct) prime factors, and blue if odd.

**Note.** Asking for an infinite red set is more than asking for arbitrarily large finite red sets. For example, consider the colouring for which all edges within the sets  $\{1, 2\}$ ,  $\{3, 4, 5\}$ ,  $\{6, 7, 8, 9\}$ ,  $\{10, 11, 12, 13, 14\}$ ,  $\{15, 16, 17, 18, 19, 20\}$ , ... are coloured red, and all other edges are coloured blue. Then there are arbitrarily large finite red sets, but no infinite red set.

**Theorem 5 (Infinite Ramsey).** Let  $\mathbb{N}^{(2)}$  be 2-coloured. Then there exists an infinite monochromatic  $M \subset \mathbb{N}$ .

**Proof.** Choose  $a_1 \in \mathbb{N}$ . Then there exists infinite  $B_1 \subset \mathbb{N} \setminus \{a_1\}$  such that all edges from  $a_1$  to  $B_1$  have the same colour, say  $c_1$ . Choose  $a_2 \in B_1$ . Then there exists infinite  $B_2 \subset B_1 \setminus \{a_2\}$  such that all edges from  $a_2$  to  $B_2$  are the same colour, say  $c_2$ . Continue inductively.



We obtain points  $a_1, a_2, \dots$  in  $\mathbb{N}$  and colours  $c_1, c_2, \dots$  such that  $a_i a_j$  has colour  $c_i$  (for  $i < j$ ). We must have infinitely many of the  $c_i$  the same colour, say  $c_{i_1} = c_{i_2} = \dots$ . Then  $M = \{a_{i_1}, a_{i_2}, \dots\}$  is infinite monochromatic.  $\square$

**Remarks.** 1. Similarly for  $k$  colours. (E.g., by “turquoise spectacles”.)

2. This is called a **2-pass proof**.

3. In the third example above (prime factors), no explicit example is known.

**Example.** Any sequence in  $\mathbb{R}$  (or any totally ordered set) has a monotonic subsequence. Indeed, given sequence  $x_1, x_2, \dots$ , 2-colour  $\mathbb{N}^{(2)}$  by giving  $ij$  the colour “up” if  $x_i \leq x_j$  and “down” if  $x_i > x_j$ . Now apply infinite Ramsey.

**What about  $r$ -sets?**

E.g.,  $r = 3$ . 2-colour  $\mathbb{N}^{(3)}$  by giving  $ijk$  ( $i < j < k$ ) the colour red if  $i$  divides  $j + k$ , and blue if not. We can take  $M = \{1, 2, 4, 8, 16, \dots\}$ .

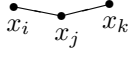
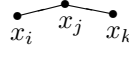
**Theorem 6 (Infinite Ramsey for  $r$ -sets).** Let  $r \geq 1$ , and let  $\mathbb{N}^{(r)}$  be 2-coloured. Then there exists an infinite monochromatic subset  $M \subset \mathbb{N}$ .

**Proof.** Induction on  $r$ . Done if  $r = 1$  (or if  $r = 2$ , by Theorem 5).

Given  $r > 1$ , and given a 2-colouring  $c$  of  $\mathbb{N}^{(r)}$ , choose  $a_1 \in \mathbb{N}$ . This induces a 2-colouring  $c'$  of  $(\mathbb{N} \setminus \{a_1\})^{(r-1)}$  by  $c'(F) = c(F \cup \{a_1\})$ . So there exists an infinite monochromatic  $B_1 \subset \mathbb{N} \setminus \{a_1\}$  for  $c'$ , by the induction hypothesis. I.e., we have some colour  $c_1$  such that every  $r$ -set of the form  $\{a_1\} \cup F$ , for  $F \subset B_1$ , has colour  $c_1$ .

Choose  $a_2 \in B_1$ . This induces a 2-colouring of  $(B_1 \setminus \{a_2\})^{(r-1)}$ , so we get an infinite  $B_2 \subset B_1 \setminus \{a_2\}$  and a colour  $c_2$  such that every  $r$ -set of the form  $\{a_2\} \cup F$ , for  $F \subset B_2$ , has colour  $c_2$ . Continue inductively.

We obtain points  $a_1, a_2, \dots$  in  $\mathbb{N}$  and colours  $c_1, c_2, \dots$  such that any  $r$ -set  $a_{i_1} \dots a_{i_r}$  (for  $i_1 < \dots < i_r$ ) has colour  $c_i$ . But infinitely many of the  $c_i$  agree, say  $c_{i_1} = c_{i_2} = \dots$ . Then  $M = \{a_{i_1}, a_{i_2}, \dots\}$  is monochromatic.  $\square$

**Example.** We saw that given points  $(1, x_1), (2, x_2), \dots$  in  $\mathbb{R}^2$ , we can find a subsequence such that the induced function (i.e., piecewise linear, through these dots) is monotonic. In fact, we can guarantee that the induced function is convex or concave. Colour  $\mathbb{N}^{(3)}$  by giving  $ijk$  the colour “convex” if  and “concave” if  and apply Theorem 6.

## Exact Ramsey Numbers

Very few non-trivial  $(s, t, \geq 3)$  of the  $R(s, t)$  are known exactly:

$$R(3, 3) = 6, R(3, 4) = 9, R(3, 5) = 14, R(3, 6) = 18, R(3, 7) = 23, R(3, 8) = 28, R(3, 9) = 36.$$

$$R(4, 4) = 18, R(4, 5) = 25. \text{ Known that } 43 \leq R(5, 5) \leq 49$$

For example, to show  $R(4, 4) > 17$ , 2-colour  $\mathbb{Z}_{17}^{(2)}$  by giving  $ij$  colour red if  $i - j$  is a square mod 17, and blue if not. (Have to check that there is no monochromatic  $K_4$ .)

For more than two colours, the only number is  $R_3(3, 3, 3) = 17$ .

For  $r$ -sets, the only known number is  $R^{(3)}(4, 4) = 13$ .

This is hard because we are asking “exactly how much disorder” guarantees a given amount of order – hard to analyse.

“Put it on a computer?”

To show, for example,  $R(5, 5) > 43$ , need to examine  $\binom{43}{5}$  5-sets, in each of  $2^{\binom{43}{2}}$  colourings. But  $2^{\binom{43}{2}} > 2^{800} > 10^{250}$ , so no chance.

## Chapter 6 : Random Graphs

How fast does  $R(s)$  grow?

We know  $R(s) \leq 4^s$ . What about a lower bound?

It's easy to see that  $R(s) > (s-1)^2$  : take  $s-1$  copies of  $K_{s-1}$ , then colour the edges within each  $K_{s-1}$  red, and those between different copies blue.

It was believed (in the 1940s) that perhaps  $R(s) \sim cs^2$ . However...

**Theorem 1 (Erdős, 1947).** Let  $s \geq 3$ . Then  $R(s) > 2^{s/2}$ .

**Proof.** Choose a colouring of  $K_n$  at random, by taking each edge to be red or blue with probability  $1/2$  each (independently). Then  $\mathbb{P}(\text{a fixed } s\text{-set is monochromatic}) = 2 \left(\frac{1}{2}\right)^{\binom{s}{2}}$ .

The number of  $s$ -sets is  $\binom{n}{s}$ , so  $\mathbb{P}(\exists \text{ a monochromatic } s\text{-set}) \leq \binom{n}{s} 2^{1-\binom{s}{2}}$ . ↖ two colours

Thus we must have  $R(s) > n$  if  $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$ , i.e. if  $\binom{n}{s} < 2^{\binom{s}{2}-1}$ .

But  $\binom{n}{s} \leq \frac{n^s}{s!}$  and  $s! \geq 2^{\frac{s}{2}+1}$  (for  $s \geq 3$ , by induction on  $s$ ).

So done if  $n^s \leq 2^{s^2/2}$ , i.e. if  $n \leq 2^{s/2}$  □

**Remarks.** 1. The above is a **random graphs** argument.

2. We could rewrite it as: there are  $2^{\binom{n}{2}}$  colourings, and a given  $s$ -set is monochromatic in  $2 \times 2^{\binom{n}{2}-\binom{s}{2}}$  of them, so done if  $\binom{n}{s} \times 2^{1+\binom{n}{2}-\binom{s}{2}} < 2^{\binom{n}{2}}$ .

(But this is a bad viewpoint for later, when we won't have all graphs equally likely.)

3. The proof gives no hint as to how to construct such a colouring.

4. No construction giving an exponential lower bound on  $R(s)$  is known!

So, to find an actual graph  $G$  on  $10^6$  points with no  $K_{40}$  in  $G$  or  $\overline{G}$ , the best thing to do is just toss a coin for each edge.

We have now  $\sqrt{2}^s \leq R(s) \leq 4^s$ . No better bounds ( $\sqrt{2} \rightarrow \sqrt{2} + \epsilon$  or  $4 \rightarrow 4 - \epsilon$ ) are known. (There exists a heuristic argument for each of  $\sqrt{2}^s$ ,  $2^s$ ,  $4^s$ .)

Proved in 2023, that  $R(s) \leq (4 - \epsilon)^s$  for small  $\epsilon$ .

The **probability space**  $G(n, p)$  is defined on the set of all graphs on  $\{1, \dots, n\}$  as follows. We choose each edge to be present with probability  $p$ , and absent with probability  $1 - p$ .

So, e.g., in the proof of Theorem 1, we worked inside  $G(n, \frac{1}{2})$ .

**Example.** In  $G(5, p)$ ,  $\mathbb{P}\left(\begin{array}{c} 2 \\ 1 \text{---} 3 \\ 5 \text{---} 4 \end{array}\right) = p^6(1-p)^4$ .

It can be useful to look at  $p$  other than  $1/2$ .

**Recall Zarankiewicz.** We had  $Z(n, t) \leq 2n^{2-1/t}$  (for  $t$  fixed,  $n$  large).

What about a lower bound? (Hopefully better than the trivial lower bound.)

**Could:** choose a random bipartite graph  $G$ , vertex classes  $X, Y$  (with  $|X| = |Y| = n$ ), choosing each edge independently with probability  $p$ , and choosing  $p$  to make the expected number of  $K_{t,t}$  less than 1.

Now, the number of  $K_{t,t}$  is  $\binom{n}{t}^2$ , and  $\mathbb{P}(\exists \text{ fixed } K_{t,t} \subset G) = p^{t^2}$ .

So the expected number of  $K_{t,t}$  in  $G$  is  $\binom{n}{t}^2 p^{t^2} < \frac{1}{4} n^{2t} p^{t^2}$ .

Take  $p = n^{-2/t}$  to get  $\mathbb{E}(\#K_{t,t} \text{ in } G) < 1/4$ . Whence  $\mathbb{P}(G \text{ has no } K_{t,t}) > 3/4$ .

Also,  $\mathbb{E}(\# \text{edges}) = pn^2$ , so we would get  $\mathbb{P}(e(G) > \frac{1}{2}pn^2) > 1/2$ , so there exists  $G$ , with no  $K_{t,t}$ , with  $e(G) > \frac{1}{2}pn^2 = \frac{1}{2}n^{2-2/t}$ .  $\square$

We can do better, however.

**Theorem 2.**  $Z(n, t) > \frac{1}{2}n^{2-2/(t+1)}$ .

**Idea.** If a graph  $G$  has  $m$  edges and  $r$  copies of  $K_{t,t}$ , remove an edge from each copy of  $K_{t,t}$  to obtain a graph with  $\geq m - r$  edges and no  $K_{t,t}$ , showing  $Z(n, t) > m - r$ .

**Proof.** Choose a random bipartite graph  $G$ , vertex classes  $X, Y$  ( $|X| = |Y| = n$ ), by taking each edge independently with probability  $p$ . Let  $M = e(G)$ , and  $R$  = the number of  $K_{t,t}$  in  $G$ .

As before,  $\mathbb{E}(M) = pn^2$  and  $\mathbb{E}(R) = \binom{n}{t}^2 p^{t^2}$ .

So, using linearity of expectation,  $\mathbb{E}(M - R) = pn^2 - \binom{n}{t}^2 p^{t^2} \geq pn^2 - \frac{1}{2}n^{2t} p^{t^2}$ .

Taking  $p = n^{-2/(t+1)}$ , we have  $\mathbb{E}(M - R) \geq n^{2-2/(t+1)} - \frac{1}{2}n^{2t-2t^2/(t+1)} = \frac{1}{2}n^{2-2/(t+1)}$ .

Thus there exists a graph with  $m - r \geq \frac{1}{2}n^{2-2/(t+1)}$ , so  $Z(n, t) > \frac{1}{2}n^{2-2/(t+1)}$ .  $\square$

**Remark.** The above proof is called **modifying a random graph**.

## Graphs with Large $\chi(G)$

To make  $\chi(G) \geq k$  – can ensure this by  $G \supset K_k$ .

To have  $\chi(G) \geq k$  – need not have  $G \supset K_k$ . E.g.,  $G = C_5$ .

In fact, we can have  $\chi(G)$  much larger than the **clique number**  $= \max\{k : G \supset K_k\}$ , because:

(a) the graph  $G$  in Theorem 1 (e.g., the red edges) is on  $n = 2^{s/2}$  vertices, with no  $K \subset G$  and no **independent set** (i.e., a set with no edges) of size  $s$ . But in any colouring, each colour class is an independent set, so  $\chi(G) \geq \frac{2^{s/2}}{s-1}$  – much more than the clique number, which is  $\leq s-1$ .

(b) We can construct a graph  $G$  to be triangle-free (i.e.,  $K_3 \not\subset G$ ) but with  $\chi(G)$  large (although this is not easy).

Could we have large girth (length of shortest cycle) and still have  $\chi(G)$  large, for example, with girth  $\geq 10$ ,  $\chi(G) \geq 100$ ? This seems unlikely, however...

**Theorem 3.** For all  $k, g$ , there exists a graph  $G$  with  $\chi(G) \geq k$  and  $\text{girth}(G) \geq g$ .

**Idea.** Try to find  $G$  on  $n$  vertices such that the number of short cycles (length  $\leq g$ ) is  $\leq n/2$ , and every independent set has size  $\leq n/2k$ . Then done by removing a vertex in each short cycle to obtain a graph  $H$  with  $\text{girth}(H) \geq g$  and  $\chi(H) \geq (n/2)/(n/2k) = k$ .

**Proof.** Choose a random  $G \in G(n, p)$ , where  $p = n^{-1+1/g}$ .

Let  $x_i$  be the number of  $i$ -cycles in  $G$ ,  $3 \leq i \leq g-1$ , and let  $x$  be the number of cycles of length  $< g$ , so  $x = x_3 + \dots + x_{g-1}$ .

Then  $\mathbb{E}(x_i) = (\# \text{ possible } i\text{-cycles}) \times \mathbb{P}(\text{given } i\text{-cycle is present}) \leq n^i p^i$ .

So  $\mathbb{E}(x) \leq \sum_{i=3}^{g-1} (np)^i = \sum_{i=3}^{g-1} n^{i/g} \leq gn^{(g-1)/g} = n \frac{g}{n^{1/g}} < \frac{n}{4}$  for  $n$  large (as  $g/n^{1/g} \rightarrow 0$ ).

Thus  $\mathbb{P}(x \leq n/2) > 1/2$  (else  $\mathbb{P}(x \geq n/2) \geq 1/2$ , whence  $\mathbb{E}(x) \geq n/4$   $\times$ ).

Let  $t = n/2k$  (for  $n$  a multiple of  $2k$ ) and let  $y$  be the number of  $t$ -sets that are independent. Then

$$\begin{aligned} \mathbb{E}(y) &= (\#t\text{-sets}) \times \mathbb{P}(\text{given } t\text{-set independent}) = \binom{n}{t} (1-p)^{\binom{t}{2}} \\ &\leq n^t e^{-p \binom{t}{2}} \quad (\text{because } 1-x \leq e^{-x} \forall x) \\ &\leq \exp\left(\frac{n}{2k} \log n - \frac{n^2}{8k^2} n^{-1+1/g}\right) \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (\text{because } n \log n - nn^{1/g} \rightarrow -\infty). \end{aligned}$$

So  $\mathbb{E}(y) < 1/2$  (for  $n$  large). So  $\mathbb{P}(y = 0) > 1/2$ .

Thus there exists  $G \in G(n, p)$  with  $x \leq n/2$ ,  $y = 0$ , so done.  $\square$

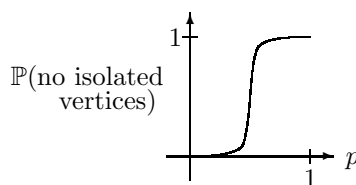
## The Structure of a Random Graph

What does a “typical” random graph  $G \in G(n, p)$  look like?

How do the properties vary as  $p$  varies?

For example, does  $G$  have no isolated vertices?

We would expect a gradual increase of  $\mathbb{P}(\text{no isolated vertices})$  as we increase  $p$ . But in fact we get a **threshold effect**:



This jump or threshold happens below  $p = \text{constant}$ , since if  $p = \text{constant}$  then  $\mathbb{P}(\text{a given vertex is isolated})$  is exponentially small, whence  $\mathbb{P}(\exists \text{ isolated vertex})$  is also small.

So, where does that jump happen?

### Probability digression/reminder

Let  $X$  be a random variable, taking values in  $\{0, 1, 2, \dots\}$ .

To show  $\mathbb{P}(X = 0)$  is large, it is enough to show that the mean  $\mu = \mathbb{E}(X)$  is small. Indeed, for any  $t$ , we have  $\mu \geq \mathbb{P}(X \geq t)t$  whence  $\mathbb{P}(X \geq t) \leq \mu/t$  (known as **Markov's inequality**), so  $\mathbb{P}(X \geq 1) \leq \mu$ , so  $\mathbb{P}(X = 0) \geq 1 - \mu$ .

To show  $\mathbb{P}(X = 0)$  is small, it is not enough to show that  $\mu$  is large. E.g., take  $X = 0$  with probability 999/1000, and  $X = 10^{10}$  with probability 1/1000.

So instead look at the variance  $V = \text{Var}(X) = \mathbb{E}((X - \mu)^2) = \mathbb{E}(X^2) - \mathbb{E}^2(X)$ . For any  $t$ ,  $\mathbb{P}(|X - \mu| \geq t) = \mathbb{P}(|X - \mu|^2 \geq t^2) \leq V/t^2$ , by Markov (known as **Chebyshev's inequality**).

Thus  $\mathbb{P}(|X - \mu| \geq \mu) \leq V/\mu^2$ . So in particular  $\mathbb{P}(X = 0) \leq V/\mu^2$ .

Conclusion: to show  $\mathbb{P}(X = 0)$  small, show  $V/\mu^2$  small.

Suppose  $X$  is the number of some events  $A$  that occur. Then  $\mu = \mathbb{E}(X) = \sum_n \mathbb{P}(A)$ .

Variance?  $\mathbb{E}^2(X) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B)$ , and  $\mathbb{E}(X^2) = \sum_{A,B} \mathbb{P}(A \cap B) = \sum_{A,B} \mathbb{P}(A)\mathbb{P}(B|A)$ .

(Using, e.g.,  $X = \sum_A I_A$ , so  $X^2 = \sum_{A,B} I_A I_B = \sum_{A,B} I_{A \cap B}$ .)

So variance  $V = \text{Var}(X) = \sum_{A,B} \mathbb{P}(A)(\mathbb{P}(B|A) - \mathbb{P}(B))$ .

(Note: no contribution to this from independent  $A, B$ .)

The phrase **almost surely** means “with probability  $\rightarrow 1$  as  $n \rightarrow \infty$ ”.

**Theorem 4.** Let  $\lambda$  be constant, and let  $G \in G(n, p)$  where  $p = \lambda \frac{\log n}{n}$ .

If  $\lambda < 1$  then almost surely  $G$  has an isolated vertex. While if  $\lambda > 1$ , then almost surely  $G$  has no isolated vertex.

**Remark.** Theorem 4 tells us that  $p = \log n/n$  is a threshold for the property of having an isolated vertex.

**Proof of 4.** Let  $X$  be the number of isolated vertices of  $G$ .

$$\text{Then } \mu = \mathbb{E}(X) = n(1-p)^{n-1} = \frac{n}{1-p}(1-p)^n.$$

For  $\lambda > 1$ , we have  $\mu \leq \frac{n}{1-p}e^{-pn} = \frac{n}{1-p}e^{-\lambda \log n} = \frac{n^{1-\lambda}}{1-p} \rightarrow 0$  as  $n \rightarrow \infty$ , so certainly  $\mathbb{P}(X=0) \rightarrow 1$ .

For  $\lambda < 1$ , we have  $1-p \geq e^{-(1+\delta)p}$ , for any  $\delta$  ( $p$  small enough).

$$\text{So } \mu \geq \frac{n}{1-p}e^{-(1+\delta)pn} = \frac{n}{1-p}e^{-(1+\delta)\lambda \log n} = \frac{n^{1-(1+\delta)\lambda}}{1-p}.$$

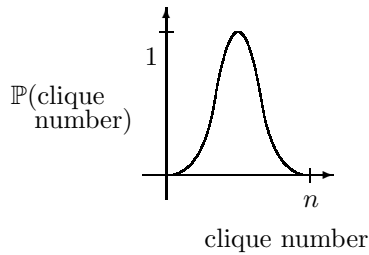
Choosing  $\delta$  such that  $(1+\delta)\lambda < 1$ , we have  $\mu \rightarrow \infty$  as  $n \rightarrow \infty$ .

$$\begin{aligned} \text{Var}(X) &= \underbrace{n(1-p)^{n-1}(1-(1-p)^{n-1})}_{n \text{ terms in which } A=B} + \underbrace{n(n-1)(1-p)^{n-1}((1-p)^{n-2} - (1-p)^{n-1})}_{n(n-1) \text{ terms in which } A \neq B} \\ &\leq \mu + n(n-1)(1-p)^{n-1}p(1-p)^{n-2} \\ &\leq \mu + \frac{p}{1-p}n^2(1-p)^{2n-2} = \mu + \frac{p}{1-p}\mu^2. \end{aligned}$$

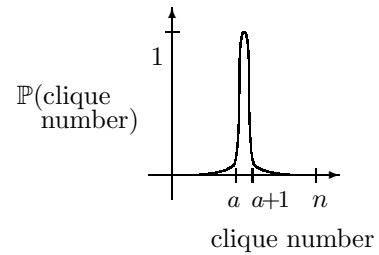
Thus  $\frac{V}{\mu^2} \leq \frac{1}{\mu} + \frac{p}{1-p} \rightarrow 0$  as  $n \rightarrow \infty$ . So  $\mathbb{P}(X=0) \rightarrow 0$  as  $n \rightarrow \infty$ .  $\square$

A different kind of threshold effect is the clique number of a random graph. Let  $0 < p < 1$  be fixed. What is the distribution of the clique number of  $G \in G(n, p)$ ?

We'd guess:



but in fact:



It turns out that, almost surely, the clique number is  $a$  or  $a+1$  (some  $a$ ).



**Theorem 5.** Let  $0 < p < 1$  be fixed, and let  $d$  be a real number with  $\binom{n}{d} p^{\binom{d}{2}} = 1$ .

Then, almost surely,  $G \in G(n, p)$  has clique number equal to  $\lceil d \rceil$ ,  $\lfloor d \rfloor$ , or  $\lfloor d \rfloor - 1$ .

**Remark.** With more work, we could check that only two values occur.

**Sketch proof of 5.** Fix an integer  $k$ , and let  $X$  be the number of  $K_k$  in  $G$ .

$$\text{So } \mu = \mathbb{E}(X) = \binom{n}{k} p^{\binom{k}{2}}.$$

Task: if  $k \leq d - 1$ , then almost surely  $X \neq 0$ , while if  $k \geq d + 1$  then almost surely  $X = 0$ .

If  $k \geq d + 1$ , then  $\mu \rightarrow 0$  (check), so  $\mathbb{P}(X = 0) \rightarrow 1$ .

If  $k \leq d - 1$ , then  $\mu \rightarrow \infty$  (check).

$$\text{Now, } V = \binom{n}{k} p^{\binom{k}{2}} \sum_{s=2}^k \underbrace{\binom{k}{s} \binom{n-k}{k-s}}_{\#B \text{ meeting } A \text{ in } s \text{ points}} \left( p^{\binom{k}{2} - \binom{s}{2}} - p^{\binom{k}{2}} \right).$$

$$\text{So, } \frac{V}{\mu^2} \leq \frac{1}{\mu} \sum_{s=2}^k \binom{k}{s} \binom{n-k}{k-s} p^{\binom{k}{2} - \binom{s}{2}}.$$

Check that the first and last terms dominate – i.e.,  $\text{sum} \leq (\text{first} + \text{last}) \times \text{constant}$ .

$$\text{So, } \frac{V}{\mu^2} \leq \text{constant} \times \left( \binom{k}{2} \binom{n-k}{k-2} p^{\binom{k}{2} - 1} + 1 \right), \text{ whence } V/\mu^2 \rightarrow 0.$$

Thus  $\mathbb{P}(X = 0) \rightarrow 0$ . □

## Chapter 7 : Algebraic Methods

The **diameter** of a graph  $G$  is  $\max\{d(x, y) : x, y, \in G\}$ .

So, e.g.,  $G$  has diameter 1  $\iff G$  is complete.

What about diameter 2? How many vertices can  $G$  have if  $G$  has diameter 2 and maximum degree  $\Delta$ ?

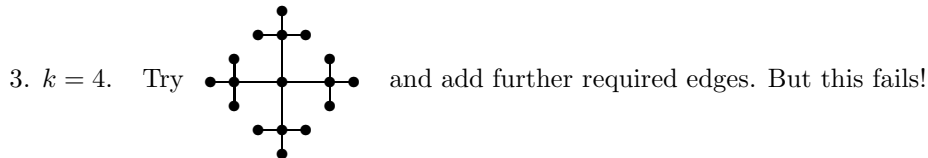
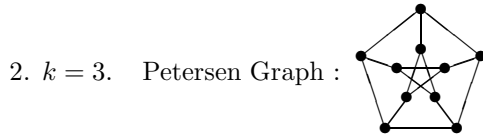
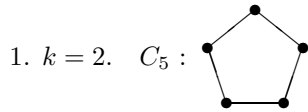
Expanding from a point  $x$ , we see that  $V(G) = \{x\} \cup \Gamma(x) \cup \Gamma(\Gamma(x))$ . And  $|\Gamma(x)| \leq \Delta$ , so  $|G| \leq 1 + \Delta + \Delta(\Delta - 1) = 1 + \Delta^2$ .

If  $|G| = 1 + \Delta^2$ , then  $G$  must be regular.

A  $k$ -regular graph of diameter 2 on  $n = 1 + k^2$  vertices is called a **Moore graph**, or **Moore graph of diameter 2**.

Equivalently, a  $k$ -regular graph is a Moore graph  $\iff$  for all  $x \neq y$  there exists a unique path of length  $\leq 2$  from  $x$  to  $y$  (for  $k \neq 1$ ).

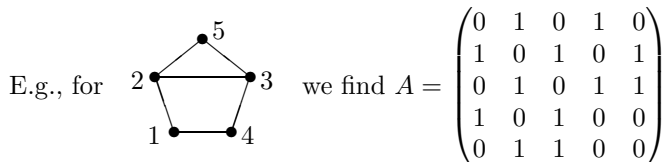
**Examples.**



4.  $k > 4$ . Ought to keep failing. Why?

Let  $G$  be a graph on vertex set  $[n] = \{1, \dots, n\}$ .

The **adjacency matrix** of  $G$  is the  $n \times n$  matrix  $A$  with  $A_{ij} = \begin{cases} 1 & \text{if } ij \in E(G) \\ 0 & \text{otherwise} \end{cases}$ .



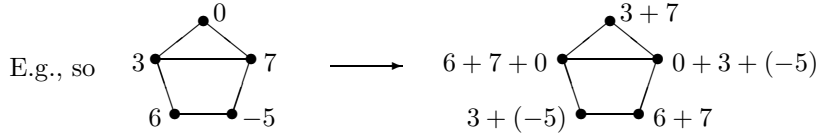
Note that  $A$  is always real and symmetric.

The matrix  $A$  contains all of the information of  $G$ .

E.g., look at  $A^2$ .  $(A^2)_{ij} = \sum_k A_{ik}A_{kj}$  = the number of walks of length 2 from  $i$  to  $j$ .

Similarly,  $(A^3)_{ij}$  = the number of walks of length 3 from  $i$  to  $j$ .

We have a linear map  $x \mapsto Ax$  from  $\mathbb{R}^n$  to  $\mathbb{R}^n$ . Thus  $(Ax)_i = \sum_j A_{ij}x_j$ .



I.e., add the values at neighbours. So if  $x = (6, 3, 7, -5, 0)$  then  $Ax = (-2, 13, -2, 13, 10)$ .

E.g., if  $A$  is  $k$ -regular, then  $(1, 1, \dots, 1) \mapsto (k, k, \dots, k)$ .

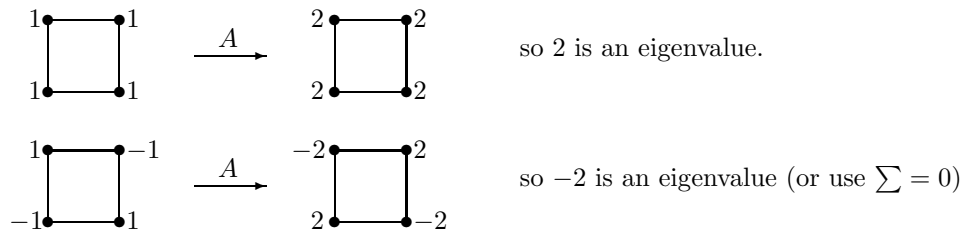
Since  $A$  is real symmetric, it is diagonalisable: it has a basis of eigenvectors, say  $e_1, e_2, \dots, e_n$  with corresponding eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . (We can take the  $e_1, \dots, e_n$  to be an orthonormal basis.)

Often write  $\lambda_{\max} = \lambda_1$  and  $\lambda_{\min} = \lambda_n$ . Note that  $\sum \lambda_i = 0$ , as  $\text{trace}(A) = 0$ . So  $\lambda_{\max} > 0$  and  $\lambda_{\min} < 0$  (unless  $G = E_n$ ).

To find eigenvalues, we often don't need lots of calculation.

**Example.**  $C_4$  has  $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ .

So  $\text{rank}(A) = 2$ , so we have 0 as an eigenvalue twice. The other two?



Thus the eigenvalues are 2, 0, 0, -2.

Let  $e_1, \dots, e_n$  be an orthonormal basis of eigenvectors. Take  $x \in \mathbb{R}^n$ , say  $x = \sum c_i e_i$ , with  $\|x\| = 1$ , i.e.,  $\sum c_i^2 = 1$ . Then  $Ax = \sum c_i \lambda_i e_i$ , so  $\langle x, Ax \rangle = \sum \lambda_i c_i^2$ .

So,

$$\left. \begin{array}{l} \min_{\|x\|=1} \langle x, Ax \rangle = \lambda_{\min} \quad (\text{take } c_n = 1, \text{ all other } c_i = 0) \\ \max_{\|x\|=1} \langle x, Ax \rangle = \lambda_{\max} \quad (\text{take } c_1 = 1, \text{ all other } c_i = 0) \end{array} \right\} (*)$$

**Proposition 1.** For any graph  $G$ ,

- (i)  $\lambda$  an eigenvalue  $\implies |\lambda| \leq \Delta$
- (ii) For  $G$  connected:  $\Delta$  is an eigenvalue  $\iff G$  is regular
- (iii) For  $G$  connected:  $-\Delta$  is an eigenvalue  $\iff G$  is regular and bipartite
- (iv)  $\lambda_{\max} \geq \delta$ .

**Proof.** (i) Choose an eigenvector  $x$  for  $\lambda$ , and choose  $i$  with  $|x_i|$  maximal. Wlog,  $x_i = 1$ . But then  $|(Ax)_i| = |\sum_{j \in \Gamma(i)} x_j| \leq \Delta \times 1$ . So  $|\lambda| \leq \Delta$ .

(ii) ( $\Leftarrow$ ) Let  $x = (1, \dots, 1)$ , then  $Ax = (\Delta, \dots, \Delta)$ .

( $\Rightarrow$ ) From (i) we must have  $d(i) = \Delta$  and that all  $j \in \Gamma(i)$  also have  $x_j = 1$ . So we can repeat for each  $j \in \Gamma(i)$ , then for each  $k \in \Gamma(j)$ , etc. We obtain  $d(k) = \Delta$  for all  $k$  (as  $G$  is connected).

(iii) ( $\Leftarrow$ ) Choose  $x = (\underbrace{1, 1, \dots}_{\text{on } X}, \underbrace{-1, -1, \dots}_{\text{on } Y})$ .

( $\Rightarrow$ ) From (i) we must have  $d(j) = \Delta$  and  $x_j = -1$  for all  $j \in \Gamma(i)$ . Repeat for each  $k \in \Gamma(j)$ , etc. We obtain  $d(j) = \Delta$  for all  $j \in G$ . and for all  $jk \in E$  we have  $x_j = 1, x_k = -1$  or  $x_j = -1, x_k = 1$ . Thus  $G$  is regular and has no odd cycle.

(iv) Let  $x = (1, \dots, 1)$ . Then  $(Ax)_i \geq \delta$  for all  $i$ , so  $\langle Ax, x \rangle \geq \delta \langle x, x \rangle = \delta n$ . Hence  $\lambda_{\max} \geq \delta$ , by (\*) (immediately before Proposition).  $\square$

**Remark.** In (ii), if  $\Delta$  is an eigenvalue, it has multiplicity 1 (as it has the unique eigenvector  $(1, \dots, 1)$ ).

Eigenvalues can link in to other graph parameters. We know that  $\chi(G) \leq \Delta + 1$ . We can strengthen this to

**Proposition 2.** For any graph  $G$ ,  $\chi(G) \leq \lambda_{\max} + 1$ .

**Proof.** Induction on  $|G|$ . Done if  $|G| = 1$ .

Given  $G$  with  $|G| > 1$ , choose  $v \in G$  with  $d(v) = \delta(G)$ .

**Claim.**  $\lambda_{\max}(G - v) \leq \lambda_{\max}$ .

Then done: colour  $G - v$  by induction, and we can colour  $v$  as  $d(v) \leq \lambda_{\max}$ .


**Proof of claim.** Let  $B$  be  $A$  with row and column  $v$  removed – wlog, the last row and column.

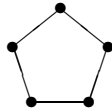
For any  $x = (x_1, \dots, x_{n-1})$ , let  $y = (x_1, \dots, x_{n-1}, 0)$ . Then  $(Ay).y = (Bx).x$ , and so  $\lambda_{\max}(G - v) \leq \lambda_{\max}$ , by (\*).  $\square$

## Towards Moore Graphs

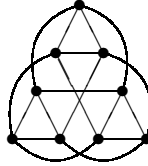
A graph  $G$  is **strongly regular** with parameters  $(k, a, b)$  if  $G$  is  $k$ -regular, with any two adjacent points having exactly  $a$  common neighbours, and any two non-adjacent points having exactly  $b$  common neighbours.

**Examples.**

$C_4$ :  has  $(2, 0, 2)$ .

$C_5$ :  has  $(2, 0, 1)$

– in general a Moore graph of degree  $k$  is strongly regular with  $(k, 0, 1)$ .

 has  $(4, 1, 2)$ .

A huge constraint on such  $G$  is given by...

**Theorem 3 (Rationality condition for strongly regular graph).** Let  $G$  be a graph on  $n$  vertices, strongly regular with parameters  $(k, a, b)$ , with  $b \geq 1$ .

Then the the numbers  $\frac{1}{2} \left( n - 1 \pm \frac{(n-1)(b-a) - 2k}{\sqrt{(a-b)^2 + 4(k-b)}} \right)$  are integers.

**Proof.**  $G$  is connected, as  $b \geq 1$ . So  $k$  is an eigenvalue with multiplicity 1. What are the other eigenvalues?

We have  $(A^2)_{ij} = \begin{cases} k & \text{if } i = j \\ a & \text{if } i \neq j, ij \in E \\ b & \text{if } i \neq j, ij \notin E \end{cases}$ .

Thus  $A^2 = kI + aA + b(J - I - A)$ , where  $J$  has all entries equal to 1.

So  $A^2 + (b-a)A + (b-k)I = bJ$ .

For  $\lambda$  an eigenvalue ( $\lambda \neq k$ ) with eigenvector  $x$ , we have  $(1, \dots, 1) \cdot x = 0$ , since the eigenvectors are orthogonal. I.e.,  $Jx = 0$ .

Thus  $(\lambda^2 + (b-a)\lambda + (b-k))x = 0$ , so  $\lambda^2 + (b-a)\lambda + (b-k) = 0$ .

So eigenvalues not equal to  $k$  are  $\lambda = \frac{1}{2} \left( a - b \pm \sqrt{(b-a)^2 + 4(k-b)} \right)$  – say  $\lambda, \mu$ , with multiplicities  $r, s$ . So  $r + s = n - 1$ , and  $\lambda r + \mu s = -k$ , as the eigenvalues sum to 0 (e.g., consider the trace). Solving for  $r, s$ , we get the numbers in the theorem.  $\square$

Finally,

**Corollary 4.** If there exists a Moore graph of degree  $k$ , then  $k \in \{2, 3, 7, 57\}$ .

**Remark.**  $k = 2 : C_5$

$k = 3 : \text{Petersen graph}$

$k = 7 : \text{exists}$

$k = 57 : \text{unknown}$

**Proof of 4.** By Theorem 3, with  $n = k^2 + 1$ ,  $a = 0$ ,  $b = 1$ , we must have

$$\frac{1}{2} \left( k^2 \pm \frac{k^2 - 2k}{\sqrt{1 + 4(k-1)}} \right) \in \mathbb{Z}.$$

So either  $k^2 - 2k = 0$  or  $1 + 4(k-1) = 4k - 3$  is a perfect square.

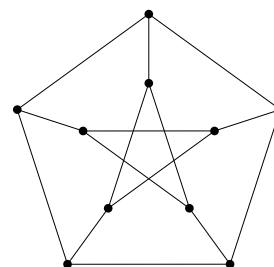
If  $k^2 - 2k = 0$ , then  $k = 2$ . Done.

If  $4k - 3 = t^2$ , then  $t$  divides  $k^2 - 2k = \left( \frac{t^2 + 3}{4} \right)^2 - 2 \left( \frac{t^2 + 3}{4} \right)$ .

So  $t$  divides  $(t^2 + 3)^2 - 8(t^2 + 3) = t^4 - 2t^2 + 15$ .

Thus  $t$  divides 15, so  $t = 1, 3, 5, 15$ , giving  $k = 1$  (not possible), 3, 7, 57. □

1. Construct a 3-regular graph on 8 vertices. Is there a 3-regular graph on 9 vertices?
2. How many spanning trees does  $K_4$  have?
3. Prove that every connected graph has a vertex that is not a cutvertex.
4. Let  $G$  be a graph on  $n$  vertices,  $G \neq K_n$ . Show that  $G$  is a tree if and only if the addition of any edge to  $G$  produces exactly 1 new cycle.
5. Let  $n \geq 2$ , and let  $d_1 \leq d_2 \leq \dots \leq d_n$  be a sequence of integers. Show that there is a tree with degree sequence  $d_1, \dots, d_n$  if and only if  $d_1 \geq 1$  and  $\sum d_i = 2n - 2$ .
6. Let  $T_1, \dots, T_k$  be subtrees of a tree  $T$ , any two of which have at least one vertex in common. Prove that there is a vertex in all the  $T_i$ .
7. Show that every graph of average degree  $d$  contains a subgraph of minimum degree at least  $d/2$ .
8. The *clique number* of a graph  $G$  is the maximum order of a complete subgraph of  $G$ . Show that the possible clique numbers for a regular graph on  $n$  vertices are  $1, 2, \dots, \lfloor n/2 \rfloor$  and  $n$ .
9. Let  $G$  be a graph on vertex set  $V$ . Show that there is a partition  $V_1 \cup V_2$  of  $V$  such that in each of  $G[V_1]$  and  $G[V_2]$  all vertices have even degree.
10. For which  $n$  and  $m$  is the complete bipartite graph  $K_{n,m}$  planar?
11. Prove that the Petersen graph (shown) is not planar.



12. The *square* of a graph  $G$  has vertex set that of  $G$  and edge set  $\{xy : d_G(x, y) \leq 2\}$ . For which  $n$  is the square of the  $n$ -cycle planar?
13. Prove that every planar graph has a drawing in the plane in which every edge is a straight-line segment.
- +14. The group of all isomorphisms from a graph  $G$  to itself is called the *automorphism group* of  $G$ . Show that every finite group is the automorphism group of some graph. Is every group the automorphism group of some (possibly infinite) graph?

1. For which  $n$  and  $m$  is the complete bipartite graph  $K_{n,m}$  Hamiltonian? Is the Petersen graph Hamiltonian?
2. Let  $G$  be a graph of order  $n$  with  $e(G) > \binom{n}{2} - (n - 2)$ . Prove that  $G$  is Hamiltonian.
3. Let  $G$  be a bipartite graph with vertex classes  $X, Y$ . Show that if  $G$  has a matching from  $X$  to  $Y$  then there exists  $x \in X$  such that every edge incident with  $x$  extends to a matching from  $X$  to  $Y$ .
4. Let  $G$  be a connected bipartite graph with vertex classes  $X, Y$ . Show that every edge of  $G$  extends to a matching from  $X$  to  $Y$  if and only if  $|\Gamma(A)| > |A|$  for every  $A \subset X$ ,  $A \neq \emptyset, X$ .
5. Let  $A$  be a matrix with each entry 0 or 1. Prove that the minimum number of rows and columns containing all the 1s of  $A$  equals the maximum number of 1s that can be found with no two in the same row or column.
6. An  $n \times n$  *Latin square* (resp.  $r \times n$  *Latin rectangle*) is an  $n \times n$  (resp.  $r \times n$ ) matrix, with each entry from  $\{1, \dots, n\}$ , such that no two entries in the same row or column are the same. Prove that every  $r \times n$  Latin rectangle may be extended to an  $n \times n$  Latin square.
- +7. Let  $G$  be a (possibly infinite) bipartite graph, with vertex classes  $X, Y$ , such that  $|\Gamma(A)| \geq |A|$  for every  $A \subset X$ . Give an example to show that  $G$  need not contain a matching from  $X$  to  $Y$ . Show however that if  $G$  is countable and  $d(x) < \infty$  for every  $x \in X$  then  $G$  does contain a matching from  $X$  to  $Y$ . Does this remain true if  $G$  is uncountable?
8. Show that we always have  $\kappa(G) \leq \lambda(G)$ . For any positive integers  $k \leq l$ , construct a graph  $G$  with  $\kappa(G) = k$  and  $\lambda(G) = l$ .
9. For a set  $B \subset V(G)$  and a vertex  $a$  not in  $B$ , an  $a$ - $B$  *fan* is a family of  $|B|$  paths from  $a$  to  $B$ , any two meeting only at  $a$ . Show that a graph  $G$  (with  $|G| > k$ ) is  $k$ -connected if and only if there is an  $a$ - $B$  fan for every  $B \subset V(G)$  with  $|B| = k$  and every vertex  $a$  not in  $B$ .
10. Let  $G$  be a  $k$ -connected graph ( $k \geq 2$ ), and let  $x_1, \dots, x_k$  be vertices of  $G$ . Show that there is a cycle in  $G$  containing all the  $x_i$ .
11. For each  $r \geq 3$ , construct a graph  $G$  such that  $G$  does not contain  $K_r$  but  $G$  is not  $(r - 1)$ -partite.
12. Let  $G$  be a graph of order  $n$  that does not contain an even cycle. Prove that each vertex  $x$  of  $G$  with  $d(x) \geq 3$  is a cutvertex, and deduce that  $G$  has at most  $\lfloor 3(n - 1)/2 \rfloor$  edges. Give (for each  $n$ ) a graph for which equality holds. How does this bound compare with the maximum number of edges of a graph of order  $n$  containing no *odd* cycles?
13. A *deleted*  $K_r$  consists of a  $K_r$  from which an edge has been removed. Show that if  $G$  is a graph of order  $n$  ( $n \geq r + 1$ ) with  $e(G) > e(T_{r-1}(n))$  then  $G$  contains a deleted  $K_{r+1}$ .
14. A *bowtie* consists of two triangles meeting in one vertex. Show that if  $G$  is a graph of order  $n$  ( $n \geq 5$ ) with  $e(G) > \lfloor n^2/4 \rfloor + 1$  then  $G$  contains a bowtie.
- +15. Let  $G$  be an  $r$ -regular graph on  $2r + 1$  vertices. Prove that  $G$  is Hamiltonian.



1. What is the chromatic number of the Petersen graph? What is its edge-chromatic number?
2. What is  $\chi'(K_{n,n})$ ? What is  $\chi'(K_n)$ ?
3. Let  $G$  be a graph with chromatic number  $k$ . Show that  $e(G) \geq \binom{k}{2}$ .
4. Show that, for any graph  $G$ , there is an ordering of the vertices of  $G$  for which the greedy algorithm uses only  $\chi(G)$  colours.
5. For each  $k \geq 3$ , find a bipartite graph  $G$ , with an ordering  $v_1, v_2, \dots, v_n$  of its vertices, for which the greedy algorithm uses  $k$  colours. Give an example with  $n = 2k - 2$ . Is there an example with  $n = 2k - 3$ ?
6. Let  $G$  be a bipartite graph of maximum degree  $\Delta$ . Must we have  $\chi'(G) = \Delta$ ?
7. Find the chromatic polynomial of the  $n$ -cycle.
8. Let  $G$  be a graph on  $n$  vertices, with  $p_G(t) = t^n + a_{n-1}t^{n-1} + a_{n-2}t^{n-2} + \dots + a_1t + a_0$ . Show that the  $a_i$  alternate in sign (in other words,  $a_i \leq 0$  if  $n - i$  is odd and  $a_i \geq 0$  if  $n - i$  is even). Show also that if  $G$  has  $m$  edges and  $c$  triangles then  $a_{n-2} = \binom{m}{2} - c$ .
9. An *acyclic orientation* of a graph  $G$  is an assignment of a direction to each edge of  $G$  in such a way that there is no directed cycle. Show that the number of acyclic orientations of  $G$  is precisely  $|p_G(-1)|$ .
10. Let  $G$  be a plane graph in which every face is a triangle. Show that the faces of  $G$  may be 3-coloured, unless  $G = K_4$ .
11. Can  $K_{4,4}$  be drawn on the torus? What about  $K_{5,5}$ ?
12. A *minor* of a graph  $G$  is any graph that may be obtained from a subgraph of  $G$  by successively contracting edges – equivalently, a graph  $H$  on vertex-set  $\{v_1, \dots, v_r\}$  is a minor of  $G$  if we can find disjoint connected subgraphs  $S_1, \dots, S_r$  of  $G$  such that whenever  $v_i v_j \in E(H)$  there is an edge from  $S_i$  to  $S_j$ . Show that for any  $k$  there is an  $n$  such that every graph  $G$  with  $\chi(G) \geq n$  has a  $K_k$  minor. Writing  $c(k)$  for the least such  $n$ , show that  $c(k+1) \leq 2c(k)$ . [Hint: choose  $x \in G$ , and look at the sets  $\{y \in G : d(x, y) = t\}$ .] Show that  $c(k) = k$  for  $1 \leq k \leq 4$ , and explain why  $c(5) = 5$  would imply the 4-Colour Theorem.
13. Let  $G$  be a countable graph in which every finite subgraph can be  $k$ -coloured. Show that  $G$  can be  $k$ -coloured.
- +14. Construct a triangle-free graph of chromatic number 1526.

1. Show that  $R(3, 4) \leq 9$ . By considering the graph on  $\mathbb{Z}_8$  (the integers modulo 8) in which  $x$  is joined to  $y$  if  $x - y = \pm 1$  or  $\pm 2$ , show that  $R(3, 4) = 9$ .
2. By considering the graph on  $\mathbb{Z}_{17}$  in which  $x$  is joined to  $y$  if  $x - y$  is a square modulo 17, show that  $R(4, 4) = 18$ .
3. Show that  $R_3(3, 3, 3) \leq 17$ .
4. Let  $A$  be a set of  $R^{(4)}(n, 5)$  points in the plane, with no three points of  $A$  collinear. Prove that  $A$  contains  $n$  points forming a convex  $n$ -gon.
5. Let  $A$  be an infinite set of points in the plane, with no three points of  $A$  collinear. Prove that  $A$  contains an infinite set  $B$  such that no point of  $B$  is a convex combination of other points of  $B$ .
6. Show that every graph  $G$  has a partition of its vertex-set as  $X \cup Y$  such that the number of edges from  $X$  to  $Y$  is at least  $\frac{1}{2}e(G)$ . Give three proofs: by induction, by choosing an optimal partition, and by choosing a random partition.
7. In a *tournament* on  $n$  players, each pair play a game, with one or other player winning (there are no draws). Prove that, for any  $k$ , there is a tournament in which, for any  $k$  players, there is a player who beats all of them. [Hint: consider a random tournament.] Exhibit such a tournament for  $k = 2$ .
8. Let  $X$  denote the number of copies of  $K_4$  in a random graph  $G$  chosen from  $G(n, p)$ . Find the mean and the variance of  $X$ . Deduce that  $p = n^{-2/3}$  is a threshold for the existence of a  $K_4$ , in the sense that if  $pn^{2/3} \rightarrow 0$  then almost surely  $G$  does not contain a  $K_4$ , while if  $pn^{2/3} \rightarrow \infty$  then almost surely  $G$  does contain a  $K_4$ .
9. Find the eigenvalues of  $K_n$ . Find the eigenvalues of  $K_{n,m}$ .
10. Prove that the matrix  $J$  (all of whose entries are 1) is a polynomial in the adjacency matrix of a graph  $G$  if and only if  $G$  is regular and connected.
11. Let  $G$  be a graph in which every edge is in a unique triangle and every non-edge is a diagonal of a unique 4-cycle. Show that  $G$  is  $k$ -regular, for some  $k$ , and that the number of vertices of  $G$  is  $1 + k^2/2$ . Show also that  $k$  must belong to the set  $\{2, 4, 14, 22, 112, 994\}$ .
12. Let the infinite subsets of  $\mathbb{N}$  be 2-coloured. Must there exist an infinite set  $M \subset \mathbb{N}$  all of whose infinite subsets have the same colour?
- +13. Let  $A$  be an uncountable set, and let  $A^{(2)}$  be 2-coloured. Must there exist an uncountable monochromatic set in  $A$ ?