# Part IB — Analysis and Topology

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# Part I
# Generalizing continuity and convergence

## §1 Three Examples of Convergence

### §1.1 Convergence in $\mathbb{R}$

Let $(x_n)$ be a sequence in $\mathbb{R}$ and $x \in \mathbb{R}$. We say $(x_n)$ *converges* to $x$ and write $x_n \to x$ if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, n \geq N \quad |x_n - x| < \epsilon.$$

Useful fact: $\forall\, a, b \in \mathbb{R}$ $|a + b| \leq |a| + |b|$ (Triangle Inequality).

Bolzano-Weierstrass Theorem (BWT) A bounded sequence in $\mathbb{R}$ must have a convergent subsequence (Proof by interval bisection).

Recall: A sequence $(x_n)$ in $\mathbb{R}$ is Cauchy if

$$\forall\, \epsilon > 0 \quad \exists\, N \quad \forall\, m, n \geq N \quad |x_m - x_n| < \epsilon.$$

Easy exercise Convergent $\implies$ Cauchy

General Principle of Convergence (GPC) Any Cauchy sequence in $\mathbb{R}$ converges.

> *Outline.* If $(x_n)$ Cauchy then $(x_n)$ bounded so by BWT has a convergent subsequence, say $x_{n_j} \to x$. But as $(x_n)$ Cauchy, $x_n \to x$. $\square$

### §1.2 Convergence in $\mathbb{R}^2$

*Remark* 1. This all works in $\mathbb{R}^n$

Let $(z_n)$ be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. What should $z_n \to z$ mean?

In $\mathbb{R}$: "As $n$ gets large, $z_n$ gets arbitrarily close to $z$."

What does 'close' mean in $\mathbb{R}^2$?

In $\mathbb{R}$: $a, b$ close if $|a-b|$ small. In $\mathbb{R}^2$: Replace $|\cdot|$ by $\|\cdot\|$

Recall: If $z=(x,y)$ then $\|z\|=\sqrt{x^2+y^2}$.

Triangle Inequality If $a, b \in \mathbb{R}^2$ then $\|a+b\| \le \|a\|+\|b\|$.

**Definition 1.1**

Let $(z_n)$ be a sequence in $\mathbb{R}^2$ and $z \in \mathbb{R}^2$. We say $(z_n)$ **converges** to $z$ and .. $z_n \to z$ if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, n \ge N\ \ \|z_n - z\| < \epsilon$.

Equivalently, $z_n \to z$ iff $\|z_n - z\| \to 0$ (convergence in $\mathbb{R}$).

**Example 1.1**

Let $(z_n), (w_n)$ be sequences in $\mathbb{R}^2$ with $z_n \to z, w_n \to w$. Then $z_n + w_n \to z + w$.

*Proof.*

$$\begin{aligned}\|(z_n+w_n)-(z+w)\| &\le \|z_n - z\| + \|w_n - w\| \\ &\to 0+0=0 \text{ (by results from IA).}\end{aligned}$$

□

In fact, given convergence in $\mathbb{R}$, convergence in $\mathbb{R}^2$ is easy:

**Proposition 1.1**

Let $(z_n)$ be a sequence in $\mathbb{R}^2$ and let $z \in \mathbb{R}^2$. Write $z_n = (x_n, y_n)$ and $z = (x, y)$. Then $z_n \to z$ iff $x_n \to x$ and $y_n \to y$.

*Proof.* ($\implies$): $|x_n - x|, |y_n - y| \le \|z_n - z\|$. So if $\|z_n - z\| \to 0$ then $|x_n - x| \to 0$ and $|y_n - y| \to 0$.

($\impliedby$): If $|x_n - x| \to 0$ and $|y_n - y| \to 0$ then $\|z_n - z\| = \sqrt{(x_n-x)^2 + (y_n-y)^2} \to 0$ by results in $\mathbb{R}$. □

**Definition 1.2** (Bounded Sequence)

A sequence $(z_n)$ in $\mathbb{R}^2$ is **bounded** if $\exists\, M \in \mathbb{R}$ s.t. $\forall\, n\ \ \|z_n\| \le M$.

**Theorem 1.1** (BWT in $\mathbb{R}^2$)

A bounded sequence in $\mathbb{R}^2$ must have a convergent subsequence.

**Theorem 1.2** (GPC for $\mathbb{R}^2$)
Any Cauchy sequence in $\mathbb{R}^2$ converges.

*Proof.* Let $(z_n)$ be a Cauchy sequence in $\mathbb{R}^2$. Write $z_n = (x_n, y_n)$. For all $m, n$, $|x_m - x_n| \le \|z_m - z_n\|$ so $(x_n)$ is a Cauchy sequence in $\mathbb{R}$, so converges by GPC. Similarly, $(y_n)$ converges in $\mathbb{R}$. So by 1.1, $(z_n)$ converges. $\square$

Thought for the day What about continuity? Let $f : \mathbb{R}^2 \to \mathbb{R}$. What does it mean for $f$ to be continuous? (Simple modification of defn for $\mathbb{R} \to \mathbb{R}$).

What can we do with it?

Big theorem in IA: If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function on a closed bounded interval then $f$ is bounded and attains its bounds.

Is there a similar theorem for $\mathbb{R}^2 \to \mathbb{R}$. What do we replace 'closed bounded interval' by? We proved the theorem using BWT. Why did it work? Why did we need a closed bounded interval to make it work? What can we do in $\mathbb{R}^2$?

## §1.3 Convergence of Functions

Let $X \subset \mathbb{R}$[1], let $f_n : X \to \mathbb{R}$ $(n \ge 1)$ and let $f : X \to \mathbb{R}$. What does it mean for $f_n$ to converge to $f$.

Obvious idea:

**Definition 1.3** (Pointwise convergence)
Say $(f_n)$ **converges pointwise** to $f$ and write $f_n \to f$ pointwise if $\forall\, x \in X\ f_n(x) \to f(x)$ as $n \to \infty$.

Pros

- Simple
- Easy to check
- Defined in terms of convergence in $\mathbb{R}$

Cons

- Doesn't preserve 'nice' properties.
- 'Doesn't feel right'.

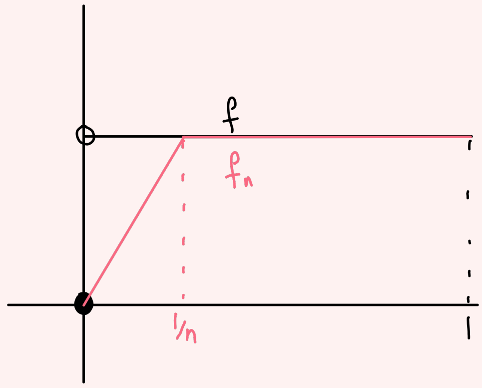
In all three examples, have $X = [0, 1]$, $f_n \to f$ pointwise.

[1] Mostly can think of $X = \mathbb{R}$ or some interval

**Example 1.2** (Every $f_n$ continuous but $f$ not)

$$f_n(x) = \begin{cases} nx & x \leq \frac{1}{n} \\ 1 & x \geq \frac{1}{n} \end{cases}$$

$$f(x) = \begin{cases} 0 & x = 0 \\ 1 & x > 0 \end{cases}$$




Clearly $f_n$ continuous for all $n$ but $f$ not. If $x = 0$, $\forall\, n$ $f_n(0) = 0 = f(0)$. If $x > 0$, for sufficiently large $n$ $f_n(x) = 1 = f(x)$ so $f_n(x) \to f(x)$.

**Example 1.3** (Every $f_n$ integrable but $f$ not)

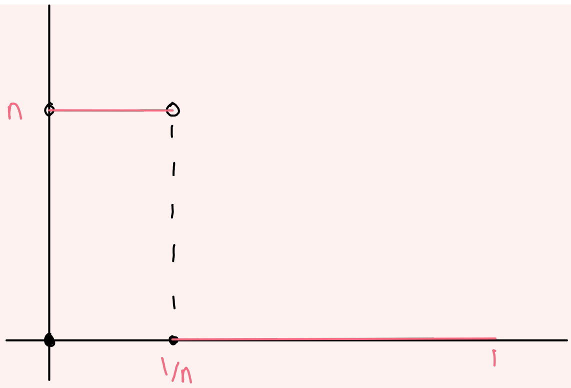
$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}.$$

This is a non integrable[a] function so now we want to find $f_n$ such that they converge pointwise to this. Enumerate the rationals in $[0,1]$ as $q_1, q_2, \ldots$ For $n \geq 1$, set $f_n(x) = \mathbb{1}_{q_1,\ldots,q_n}$. $f_n$ integrable as it is nonzero at finitely many points.

[a] N.B. As in IA 'integrable' means 'Riemann integrable'

**Example 1.4** (Every $f_n$ and $f$ integrable but $\int_0^1 f_n \not\to \int_0^1 f$)

Let $f(x) = 0$ for all $x$, so $\int_0^1 f = 0$. Define $f_n$ s.t. $\int_0^1 f_n = 1$ for all $n$.



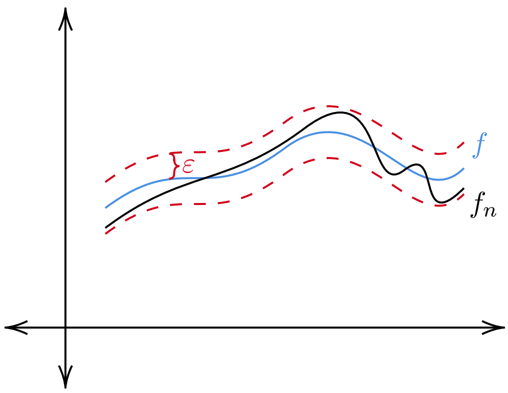

$$f_n(x) = \begin{cases} n & 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}.$$

Better definition:

**Definition 1.4** (Uniform convergence)

Let $X \subset \mathbb{R}$, $f_n : X \to \mathbb{R}$ ($n \geq 1$), $f : X \to \mathbb{R}$. We say $(f_n)$ **converges uniformly** to $f$ and write $f_n \to f$ uniformly if $\forall\ \epsilon > 0\ \exists\ N\ \forall\ x \in X\ \forall\ n \geq N\ |f_n(x) - f(x)| < \epsilon$.

cf $f_n \to f$ pointwise: $\forall\ \epsilon > 0\ \forall\ x \in X\ \exists\ N\ \forall\ n \geq N\ |f_n(x) - f(x)| < \epsilon$. (We have swapped the $\forall\ x \in x$ and $\exists\ N$). Pointwise convergence allows for $N$ to be a function of $x$ whilst uniform convergence requires $N$ to work for all $x$ even the worst case. In particular, $f_n \to f$ uniformly $\implies$ $f_n \to f$ pointwise.




Equivalently, $f_n \to f$ uniformly if for sufficiently large $n$ $f_n - f$ is bounded and $\sup_{x \in X} |f_n - f| \to 0$.

**Theorem 1.3** (A uniform limit of cts functions is cts)

Let $X \subset \mathbb{R}$, let $f_n : X \to \mathbb{R}$ be continuous $(n \geq 1)$ and let $f_n \to f : X \to \mathbb{R}$ uniformly. Then $f$ is cts.

*Proof.* Let $x \in X$. Let $\epsilon > 0$. As $f_n \to f$ uniformly, we can find $N$ s.t. $\forall\, n \geq N\ \forall\, y \in X\ |f_n(y) - f(y)| < \epsilon$. In particular, $\forall\, y \in X\ |f_N(y) - f(y)| < \epsilon$. As $f_N$ is cts, we can find $\delta > 0$ s.t. $\forall\, y \in X,\ |y - x| < \delta \implies |f_N(y) - f_N(x)| < \epsilon$. Now let $y \in X$ with $|y - x| < \delta$. Then

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)|^{a} \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Hence $f$ is cts. □

[a] The core of this proof is this inequality.

*Remark* 2. This is often called a '$3\epsilon$ proof' (or an $\frac{\epsilon}{3}$ proof).

**Theorem 1.4**

Let $f_n : [a, b] \to \mathbb{R}$ $(n \geq 1)$ be integrable and let $f_n \to f : [a, b] \to \mathbb{R}$ uniformly. Then $f$ is integrable and $\int_a^b f_n \to \int_a^b f$ as $n \to \infty$.

*Proof.* As $f_n \to f$ uniformly, we can pick $n$ suff. large s.t. $f_n - f$ is bounded. Also $f_n$ is bounded (as integrable). So by triangle inequality, $f = (f - f_n) + f_n$ is bounded. Let $\epsilon > 0$. As $f_n \to f$ uniformly there is some $N$ s.t. $\forall\, n \geq N\ \forall\, x \in [a, b]$ we have $|f_n(x) - f(x)| < \epsilon$.
In particular, $\forall\, x \in [a, b]\ |f_N(x) - f(x)| < \epsilon$.

By Riemann's criterion, there is some dissection $\mathcal{D}$ of $[a, b]$ for which $S(f_n, \mathcal{D}) - s(f_n, \mathcal{D}) < \epsilon$. Let $\mathcal{D} = \{x_0, x_1, x_2, \ldots, x_k\}$ where $a = x_0 < x_1 < \cdots < x_k = b$. Now

$$\begin{aligned} S(f, \mathcal{D}) &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f(x) \\ &\leq \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} (f_N(x) + \epsilon) \\ &= \sum_{i=1}^{k} (x_i - x_{i-1}) \left( \left( \sup_{x \in [x_{i-1}, x_i]} f_N(x) \right) + \epsilon \right) \\ &= \sum_{i=1}^{k} (x_i - x_{i-1}) \sup_{x \in [x_{i-1}, x_i]} f_N(x) + \sum_{i=1}^{k} (x_i - x_{i-1}) \epsilon \end{aligned}$$

$$= S(f_N, \mathcal{D}) + (b-a)\epsilon.$$

That is $S(f, \mathcal{D}) \leq S(f_N, \mathcal{D}) + (b-a)\epsilon$. Similarly $s(f, \mathcal{D}) \geq s(f_N, \mathcal{D}) - (b-a)\epsilon$. Hence

$$\begin{aligned} S(f, \mathcal{D}) - s(f, \mathcal{D}) &\leq S(f_N, \mathcal{D}) - s(f_N, \mathcal{D}) + 2(b-a)\epsilon \\ &< (2(b-a)+1)\epsilon \end{aligned}$$

But $2(b-a)+1$ is a constant so $(2(b-a)+1)\epsilon$ can be made arbitrarily small. Hence by Riemann's criterion, $f$ is integrable over $[a, b]$.

Now, for any $n$ suff. large that $f_n - f$ is bounded,

$$\begin{aligned} \left| \int_a^b f_n - \int_a^b f \right| &= \left| \int_a^b (f_n - f) \right| \\ &\leq \int_a^b |f_n - f| \\ &\leq (b-a) \sup_{x \in [a,b]} |f_n - f| \\ &\to 0 \text{ as } n \to \infty \text{ since } f_n \to f \text{ uniformly.}^{a} \end{aligned}$$

□

[a] Note we said that $f_n \to f$ uniformly if $\sup |f_n - f| \to 0$.

What about differentiation? Here even uniform convergence isn't enough.

**Example 1.5**

$f_n : (-1, 1) \to \mathbb{R}$, each $f_n$ differentiable, $f_n \to f$ uniformly, $f$ not diff.

Let $f(x) = |x|$ which is not differentiable at 0.

$$f_n = \begin{cases} |x| & |x| \geq \frac{1}{n} \\ ax^2 + bx + c & |x| < \frac{1}{n} \end{cases}.$$

We need $a(\frac{1}{n})^2 + \frac{b}{n} + c = \frac{1}{n}$ for continuity. Thus $b = 0$ and $c = \frac{1}{n} - \frac{a}{n^2}$.

Also need $2a\frac{1}{n} + b = 1$ and $2a(-\frac{1}{n}) = -1$ for differentiability so take $a = \frac{n}{2}$, $c = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$.

If $|x| \geq \frac{1}{n}$ then $|f_n(x) - f(x)| = 0$. If $|x| < \frac{1}{n}$:

$$\begin{aligned} |f_n(x) - f(x)| &= \left| \frac{n}{2}x^2 + \frac{1}{2n} - |x| \right| \\ &\leq \frac{n}{2}x^2 + \frac{1}{2n} + |x| \\ &\leq \frac{n}{2}(\frac{1}{n})^2 + \frac{1}{2n} + \frac{1}{n} \\ &= \frac{1}{2n} + \frac{1}{2n} + \frac{1}{n} \\ &= \frac{2}{n} \end{aligned}$$

So $\sup_{x \in (-1,1)} |f_n(x) - f(x)| \leq \frac{2}{n} \to 0$ as $n \to \infty$. So $f_n \to f$ uniformly.

If fact we need uniform convergence of the derivatives.

**Theorem 1.5**

Let $f_n : (u, v) \to \mathbb{R}$ $(n \geq 1)$ with $f_n \to f : (u, v) \to \mathbb{R}$ pointwise. Suppose further each $f_n$ is continuously differentiable and that $f_n' \to g : (u, v) \to \mathbb{R}$ uniformly. Then $f$ is differentiable with $f' = g$.

*Proof.* Fix $a \in (u, v)$. Let $x \in (u, v)$, by FTC we have each $f_n'$ is integrable over $[a, x]$ and $\int_a^x f_n' = f_n(x) - f_n(a)$. But $f_n' \to g$ uniformly so by thm 5 $g$ is integrable over $[a, x]$ and $\int_a^x g = \lim_{n\to\infty} \int_a^x f_n' = f(x) - f(a)$. So we have shown that for all $x \in (u, v)$

$$f(x) = f(a) + \int_a^x g.$$

By thm 4, $g$ is cts so by FTC, $f$ is differentiable with $f' = g$. □

*Remark* 3. It would have sufficed to assume $f_n(x) \to f(x)$ for a single value of $x$ rather than $f_n \to f$ pointwise.

GPC?

**Definition 1.5** (Uniform Cauchy)

Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say $f_n)$ is **uniformly Cauchy** if $\forall\, \epsilon > 0\ \exists\, N\ \forall\, m, n \geq N\ \forall\, x \in X\ |f_m(x) - f_n(x)| < \epsilon$

exercise: uniformly convergence $\implies$ uniformly Cauchy.

**Theorem 1.6** (General principle of Uniform Convergence (GPUC))

Let $(f_n)$ be a uniformly Cauchy sequence of functions $X \to \mathbb{R}$ ($X \subset \mathbb{R}$). Then $(f_n)$ is uniformly convergent.

*Proof.* Let $x \in X$. Let $\epsilon > 0$. Then $\exists\, N\ \forall\, m, n \geq N\ \forall\, y \in X\ |f_m(y) - f_n(y)| < \epsilon$. In particular, $\forall\, m, n \geq N\ |f_m(x) - f_n(x)| < \epsilon$. So $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$ so by GPC it converges, say $f_n(x) \to f(x)$ as $n \to \infty$.

We have now constructed $f : X \to \mathbb{R}$ s.t. $f_n \to f$ pointwise.
Let $\epsilon > 0$. Then we can find a $N$ s.t. $\forall\, m, n \geq N\ \forall\, y \in X\ |f_m(y) - f_n(y)| < \epsilon$. Fix $y \in X$, keep $m \geq N$ fixed and let $n \to \infty$: $|f_m(y) - f(y)| \leq \epsilon$. So we have shown that $\forall\, m \geq N$, $|f_m(y) - f(y)| < \epsilon$.

But $y$ was arbitrary so $\forall\, x \in X\ \forall\, m \geq N\ |f_m(x) - f(x)| \leq \epsilon$. That is $f_n \to f$ uniformly. □

BW?

**Definition 1.6** (Pointwise bounded)

Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say $(f_n)$ is **pointwise bounded** if $\forall\, x\ \exists\, M\ \forall\, n\ |f_n(x)| \leq M$.

**Definition 1.7** (Uniformly bounded)

Let $X \subset \mathbb{R}$ and let $f_n : X \to \mathbb{R}$ for each $n \geq 1$. We say $(f_n)$ is **uniformly bounded** if $\exists\, M\ \forall\, x\ \forall\, n\ |f_n(x)| \leq M$.[a]

[a] Again we have just swapped ... as in convergence.

What would uniform BW say? 'If $(f_n)$ is a uniformly bounded sequence of functions that it has a uniformly convergent subsequence'. But this is <u>not</u> true.

**Example 1.6** (Counterexample of BW)

$$f_n : \mathbb{R} \to \mathbb{R}$$

$$f_n(x) = \begin{cases} 1 & x = n \\ 0 & x \neq n. \end{cases}$$

Obviously $(f_n)$ uniformly bounded (by 1). However, if $m \neq n$ then $f_m(m) = 1$ and $f_n(m) = 0$ so $|f_m(m) - f_n(m)| = 1$ so no subsequence can be uniformly Cauchy so no subsequence can be uniformly convergent.

Application to power series Recall that if $\sum a_n x^n$ is a real power series with r.o.c $R > 0$ then we can differentiate/ integrate it term-by-term within $(-R, R)$.

**Definition 1.8**

Let $f_n : X \to \mathbb{R}$ ($X \subset \mathbb{R}$) for each $n \geq 0$. We say the series $\sum_{n=0}^{\infty} f_n$ **uniformly converges** if the sequence of partial sums $(F_n)$ does, where $F_n = \sum_{m=0}^{n} f_m$.

We can apply thm 4 to 6 to get e.g. if conditions hold with $f_n$ cts diff and uniform convergence then $\sum f_n$ has derivative $\sum f_n'$.

Hope Prove $\sum a_n x^n$ converges uniformly on $(-R, R)$ then hit it with earlier theorems.

Not quite true:

**Example 1.7**

$\sum_{n=0}^{\infty} x^n$ r.o.c 1. This does not converge uniformly on $(-1, 1)$. Let $f(x) = \sum_{n=0}^{\infty} x^n$ and $F_n(x) = \sum_{m=0}^{n} x^m$. Note $f(x) = \frac{1}{1-x} \to \infty$ as $x \to 1$. However, $\forall\ x \in (-1, 1)$ $|F_n(x)| \leq n + 1$.

Fix any $n$. We can find a point $x \in (-1, 1)$ where $f(x) \geq n+2$ and so $|f(x) - F_n(x)| \geq 1$. So we don't have uniform convergence.

Back-up plan: It does work if we look at a smaller interval.

New plan: show if $0 < r < R$ then we do have uniform convergence on $(-r, r)$.
Given $x \in (-R, R)$ there's some $r$ with $|x| < r < R$: use uniform convergence on $(-r, r)$ to check everything is nice at $x$. 'Local uniform convergence of power series'.

## §2 Metric Spaces

## §3 Topological Spaces

# Part II
# Generalizing differentiation