Topological Space

1 Distance

Let x, y, z be elements of a set X and $d(x, y)$ be a real number that satisfies conditions below.

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \leq d(x, y) + d(y, z)$

Then, d is called adistance function of X and a set X with a distance function is called a distance space. $d(x, y)$ is called a distance between x and y.

Let X be a distance space, $x \in X$ and $\varepsilon > 0$. Then,

$$U(x, \varepsilon) = U_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\} \tag{1}$$

is called a ε-neighbourhood and

$$S(x, \varepsilon) = \{y \in X \mid d(x, y) = \varepsilon\} \tag{2}$$

is called a sphere.a

Example 1: Distance in R

The distance between x and y which are placed on a line R is denoted by

$$d(x, y) = |x - y| \tag{3}$$

Example 2: Distance in R^n

The distance between $x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ which are placed on a line R^n is denoted by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2} \tag{4}$$

This $d(x, y)$ is called Euclid distance and R^n with the distance is called n-dimension Euclid space.

Example 3: Hilbert Space

Hilbert space, which is an extension of an Euclidian space R^n to infinity dimention is defined as below.

$$R^\infty = \{x = (x_1, x_2, \ldots, x_n, \ldots) \mid \sum_{i=1}^{\infty} x_i^2 < \infty\} \tag{5}$$

The distance between two points $x = (x_1, \ldots, x_n, \ldots)$ and $y = (y_1, \ldots, y_n, \ldots)$ is denoted by

$$d(x, y) = \sqrt{\sum_{i=1}^{\infty} (x_i - y_i)^2} \tag{6}$$

This is finite since for arbitary n,

$$\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \geq \sum_{i=1}^{n} (x_i - y_i)^2 \tag{7}$$

Both, $\sum_{i=1}^{n} x_i^2$ and $\sum_{i=1}^{n} y_i^2$ converge from the definition, do right-hand side also converges.

Example 4: Discrete Distance Space

For 2 elements x, y of a set X define

$$d(x, x) = 0$$
$$d(x, y) = 1$$

This $d(x, y)$ is a distance function, X forms a distance space and this is called a discrete distance space. Its ε-neighbourhood is

$$U(x, \varepsilon) = \begin{cases} \{x\} \ (\varepsilon \leq 1) \\ X \ (\varepsilon > 1) \end{cases}$$

Its open ball is

$$S(x, \varepsilon) = \begin{cases} \emptyset \ (\varepsilon \neq 1) \\ X - \{x\} \ (\varepsilon = 1) \end{cases}$$

For a subset A of a distance space X,

$$\delta(A) = \sup\{d(x, y) \mid x \in A, y \in A\} \tag{8}$$

is called a diameter of A. If diameter of A is finite, then a set A is bounded.

Let A, B be subset of a distance space X. Then, the distance between A and B is denoted by

$$d(A, B) = \inf\{d(x, y) \mid x \in A, y \in B\} \tag{9}$$

Definition of topological space is noted below Its definition is

$$O \in P \tag{10}$$

hello this is zen thanks