Smith Normal Form of Powers of Matrices

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1 Introduction

Theorem 1. *For any σ^a-F-crystal (M, F), the Newton Polygon is above the Hodge Polygon. Both polygons start at $(0,0)$ and end at $(n, v_p(det(F)))$ where n is the number of columns of F.[1]*

Theorem 2. *Let R be a unique factorization domain (e.g., a PID), so that any two elements have a greatest common divisor (gcd). Suppose that the $m \times n$ matrix M over R satisfies $SNF(M) = diag(\alpha_1, \ldots, \alpha_m)$. Then for $1 \leq k \leq m$, we have that $\alpha_1\alpha_2 \cdots \alpha_k$ is equal to the gcd of all $k \times k$ minors of A.[2]*

For all ideas below, A_{ij} is the entry in the ith row and the jth column of A.

2 The 2×2 case

Lemma 3. *Let $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ where $a = T^i u_1$ and $d = T^j u_2$ with $i < j$ and b some power series in T for $a, b, d, u_1, u_2 \in \mathbb{F}_p[[T]]$ where u_1, u_2 are units. Let $SNF(A)$=$diag(s_1,\ s_2)$ be the Smith Normal Form of A. Then $HP(A^n)$, is the following points: $(0,0); (1, s_1 + v_T(a)(n-1)); (2, v_T(\det(A^n)))$ where the $HP(A^n)$ is the Hodge Polygon of A^n*

Proof. For the 2×2 case we know already know the start and end points from the Hodge Polygon of A^n due to **Theorem 1**, so we only need to consider the middle coordinate of $HP(A^n)$, call it y_1. Knowing this, we know y_1 is the gcd of the 1×1 minors of A^n due to **Theorem 2**.

When running computations, we see that A_{12} is of the form $b(a^{n-1} + a^{n-2}d + \cdots + d^{n-1})$. Since a and d share a common factor of at least $(T^i)^{n-1}$, we can factor this out meaning $(T^i)^{n-1}b$ is a divisor of A_{12}. Using **Theorem 2**, we know that $SNF(A^n)_{11}$ must be $(T^i)^{n-1}$ or $(T^i)^{n-1}b$ depending on whether b has $v_T(b) > 0$ and $v_T(b) > i$. We can infer the quantity $v_T(b)$ using $v_T(s_1)$, so we can keep $v_T(s_1)$ as a constant. Specifically, $v_T(b) = v_T(s_1)$ or $v_T(s_1) = i$ or both. Nonetheless, this means our coordinate in question is: $(1, v_T(s_1) + v_T(a)(n-1))$ giving us the desired conclusion. $\square$

Proposition 4. *Let* $A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}$ *where* $a = T^i u_1$ *and* $b = T^j u_2$ *with* $i < j$ *and* b *some power series in* T *for* $a, b, c, u_1, u_2 \in \mathbb{F}_p[[T]]$ *where* u_1, u_2 *are units. Let* $SNF(A)$*=diag(*$s_1 = T^{k_1}$, $s_2 = T^{k_2}$*) be the Smith Normal Form of* A. *Denote* $S_i = SNF(A^i)$. *Then, given* S_m, *we find*

$$S_{m+n} = \begin{bmatrix} T^{v_T(a)(n)} & 0 \\ 0 & T^{v_T(b)(n)} \end{bmatrix} \cdot S_m$$

Proof. Using **Theorem 2**, we find the second entry of A_i to be $m \cdot v_T(b) + v_T(a) - k_1$ Note by **Lemma 3**, we find

$$S_m = \begin{bmatrix} T^{k_1 + v_T(a)(m-1)} & 0 \\ 0 & T^{m \cdot v_T(b) - k_1 + v_T(a)} \end{bmatrix} \quad S_{m+n} = \begin{bmatrix} T^{k_1 + v_T(a)(m+n-1)} & 0 \\ 0 & T^{(m+n) \cdot v_T(b) - k_1 + v_T(a)} \end{bmatrix}$$

At the same time, note that we are only solving for D in $S_{m+n} = D \cdot S_m$, so we can rearrange this to $S_{m+n} \cdot S_m^{-1}$ by right multiplying by S_d^{-1}. From direct computations, we find

$$S_m^{-1} = \frac{1}{T^{k_1 + v_T(a)(m-1)} \cdot T^{m \cdot v_T(b) - k_1 + v_T(a)}} \begin{bmatrix} T^{m \cdot v_T(b) - k_1 + v_T(a)} & 0 \\ 0 & T^{k_1 + v_T(a)(m-1)} \end{bmatrix}$$

Notice that this expression simplifies to

$$S_m^{-1} = \begin{bmatrix} \frac{1}{T^{k_1 + v_T(a)(m-1)}} & 0 \\ 0 & \frac{1}{T^{m \cdot v_T(b) - k_1 + v_T(a)}} \end{bmatrix}$$

Now, we find

$$S_{m+n} \cdot S_m^{-} = \begin{bmatrix} T^{v_T(a)(n)} & 0 \\ 0 & T^{v_T(b)(n)} \end{bmatrix}$$

Rearranging, we have our result of

$$S_{m+n} = \begin{bmatrix} T^{v_T(a)(n)} & 0 \\ 0 & T^{v_T(b)(n)} \end{bmatrix} \cdot S_m$$

as desired. ∎

3 The $n \times n$ case

For the following proofs, let A be an $n \times n$ upper triangular matrix with entries from $\mathbb{F}_p[[T]]$. We will assume A satisfies the two following properties

$$A = \begin{bmatrix} T^{k_1} & & & & \\ & T^{k_2} & & \text{X} & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & T^{k_n} \end{bmatrix} \tag{1}$$

with $k_1 < k_2 < \cdots < k_n$. X represents the values in the upper triangle of A. We will assume all values in the matrix satisfy the following:

$$v_T(A_{ij}) \leq v_T(A_{i(j-1)}) \tag{2}$$

meaning every value in the matrix, excluding the main diagonal, has a valuation less than the value to its left. (I mainly put these assumptions in to eliminate counterexamples)

Lemma 5. *Let A be as defined above with M_i being the set of all $i \times i$ minors of A. There exists a unique $c_A = c \leq n$ such that such that $\forall I \in M_i$, $T^{k_i} \mid I$.*

Proof. We will split this into two cases: $c = n$ and $c < n$. Starting with $c = n$, we will use induction. Let $c = n = 2$. As we've seen earlier, we get

$$A^2 = \begin{bmatrix} T^{2k_1} & T^{k_1}A_{12}(T^{k_2-k_1}+1) \\ 0 & T^{2k_2} \end{bmatrix} \tag{1}$$

so trivially, our base case is satisfied. Now, suppose our proposition holds true for some m. Note that for an $m+1 \times m+1$ matrix A,

$$A^m = \begin{bmatrix} T^{mk_1} & & & & \\ & T^{mk_2} & & \text{X} & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & T^{mk_n} \end{bmatrix} \tag{2}$$

the only value not satifying our proposition currently is $A^m_{1(m+1)}$, the value in the upper right corner of A^m. We explain this by revealing the results of direct computations. From the matrix multiplications, we will see that the entries in the ith row have a common factor T^{k_i}. We see this by noting the general form of each diagonal.

Let d_2 be the diagonal to the right of the main diagonal and let $d_3, d_4, \ldots$ be defined in a similar fashion. By observing the 2×2 minors along the diagonal, we see that the values of d_2 will have the form $b(\sum_{i=1}^{m} a^{m-i}e^{i-1})$ where these minors are of the form $\begin{bmatrix} a = T^i & b \\ 0 & e = T^j \end{bmatrix}$. Looking at the upper triangular 3×3 minors along the diagonal, we see values in d_3 having the same form. We observe this idea up to d_m. $d_{m+1} = A^m_{1(m+1)}$ is the only wild card, so to speak. When we look at these upper triangular $i \times i$ minors in A^m, our two assumptions on A (maybe only $k_i < k_j$ for $i < j$ implies this) combined with the induction hypothesis implies that we must have T_{k_i} dividing every entry in the ith row for $i > 1$. Upon calculating A^{m+1}, we find that $A^{m+1}_{1(m+1)}$ has the form

$$A^{m+1}_{1(m+1)} = T^{k_1} A^m_{1(m+1)} + T^{k_1} R \tag{3}$$

where R is the remaining sum from the matrix multiplication. So it's clear, by induction, that the first case of our proposition holds. We need not check the rest of the minors since if T_{k_i} divides every entry in the ith row, it follows that T_{k_i} divides every $i \times i$ minor in A^{m+1}.

For the second case, we make use of the constraint that $v_T(A_{ij}) \leq v_T(A_{i(j-1)})$. This structural aspect of A enforces a decreasing ordering on the entries in the rows of A. In particular, our c occurs one exponent after the second y coordinate of the Hodge Polygon is nonzero. Our result becomes trivial from there since if A^{c-1} has its nonzero entries all with $v_T(A_{ij}) > 0$ for all i, j, the valuations of the entries of A^{c+p} are incremented by their corresponding diagonal valuation. (I'm feeling more comfortable with my logic in the $c = n$ part, but now I'm unsure whether this paragraph of logic is complete.)

With these two explanations, we have proven our cases, and as a result, our proposition.

□

Lemma 6. *Let A be as defined above. Then,*

$$HP(A^m) = \{(0,0), (1, (m-c+1)k_1 + r_1), \ldots, (n, (m-c+1)\sum_{i=1}^{n} k_i + r_n\}$$

for $m \geq c_A = c$ and the r_i being the y-coordinate of the ith coordinate of $HP(A^{c-1})$ where $1 \leq i \leq n$.

Proof. This proof is rather short since this result directly follows from **Lemma 5**. It follows from **Lemma 5**, that $HP(A^c) = \{(0,0), (1, k_1 + r_1), \ldots, (n, \sum_{i=1}^{n} k_i + r_n\}$ since it follows from **Theorem 2** that the ith y coordinate of the Hodge Polygon is equal to the gcd of all $i \times i$ minors of a matrix. From there, proceeding matrix multiplication increments the values of the jth row by T^{k_j} giving us our conclusion. □

Theorem 7. *Let A be as defined above and $NP(A)$ given and $SNF(A)$=diag(T^{e_1}, T^{e_2}, …, T^{e_n}) being the Smith Normal Form of A. Denote $S_i = SNF(A^i)$. Then, given S_d for $d \geq c_A = c$, we find*

$$S_{d+n} = \begin{bmatrix} T^{nk_1} & & & & \\ & T^{nk_2} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & T^{nk_n} \end{bmatrix} \cdot S_d$$

Proof. Using **Theorem 2**, we find the ith entry of S_i to be $(i-c+1)\cdot k_i - \sum_{j=0}^{i} y_j + r_j$ where the y_j are the jth y coordinate of $HP(A^i)$ and r_j as defined before. Note by **Lemma 6**, we find

$$S_d = \begin{bmatrix} T^{(d-c+1)k_1} & & & & \\ & T^{(d-c+1)k_2-\sum_{j=0}^{2} y_j+r_j} & & & 0 \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & T^{(d-c+1)k_n-\sum_{j=0}^{n} y_j+r_j} \end{bmatrix}$$

$$S_{d+n} = \begin{bmatrix} T^{(d+n-c+1)k_1} & & & & \\ & T^{(d+n-c+1)k_2-\sum_{j=0}^{2} y_j+r_j} & & & 0 \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & T^{(d+n-c+1)k_n-\sum_{j=0}^{n} y_j+r_j} \end{bmatrix}$$

At the same time, note that we are only solving for D in $S_{d+n} = D \cdot S_d$, so we can rearrange this to $S_{d+n} \cdot S_d^{-1}$ by right multiplying by S_d^{-1}. Upon inspection, it is clear that

$$S_d^{-1} = \begin{bmatrix} \frac{1}{s_1} & & & & \\ & \frac{1}{s_2} & & & 0 \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \frac{1}{s_n} \end{bmatrix}$$

where the s_i are ith entry in S_d. Now, we find

$$S_{d+n} \cdot S_d^{-1} = \begin{bmatrix} T^{nk_1} & & & & \\ & T^{nk_2} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & T^{nk_n} \end{bmatrix}$$

Rearranging, we have our result of

$$S_{d+n} = \begin{bmatrix} T^{nk_1} & & & & \\ & T^{nk_2} & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & T^{nk_n} \end{bmatrix} \cdot S_d$$

as desired. $\square$

References

[1] N. KATZ, Slope Filtration of F-crystals. *Asterisque* ***63*** (1979), 113–164. `http://www.numdam.org/article/AST_1979__63__113_0.pdf`

[2] Stanley, Richard P. Smith normal form in combinatorics. *Journal of Combinatorial Theory, Series A* 144 (2016): 476-495.