

# CS360 Machine Learning Homework 1

Due: Feb 22 23:55, 2024

100 Points

## Instructions

- **Collaboration policy:** Homework must be done individually, except where otherwise noted in the assignments. “Individually” means each student must hand in their own answers, and you must write and use your own code in the programming parts of the assignment. It is acceptable for you to collaborate in figuring out answers and to help each other solve the problems, and you must list the names of students you discussed this with. We will assume that, as participants in an undergraduate course, you will be taking the responsibility to make sure you personally understand the solution to any work arising from such collaboration.
- **Online submission:** You must submit your solutions online on the course Brightspace site. You need to submit (1) a PDF that contains the solutions to all questions (2) `x.py` or `x.ipynb` files for the programming questions. We recommend that you type the solution (e.g., using L<sup>A</sup>T<sub>E</sub>X or Word), but we will accept scanned/pictured solutions as well (clarity matters).
- **Generative AI Policy:** You are free to use any generative AI, but you are required to document the usage: which AI do you use, and what’s the query to the AI. You are responsible for checking the correctness.
- **Late Policy:** Due to the spring festival, we give extra time for this homework. **No late submission is allowed.**

## 1 Simpson’s Paradox [10 points]

Imagine you and a friend are playing the slot machine in a casino. Having played on two separate machines for a while, you decide to swap machines between yourselves, and measure for differences in “luck”. The wins/losses for you and your friend for each machine are tabulated below.

Machine 1	Wins	Losses
You	40	60
Friend	30	70
Machine 2	Wins	Losses
You	210	830
Friend	14	70

Assuming that the outcomes of playing the slot machine are independent of their history and that of the other machine, answer the following questions.

1. (*3 points*) Estimate the winning probability of you and your friend for each of the machines. Compare your winning probability with your friend’s on different machines. Who is more likely to win?
2. (*3 points*) Estimate the overall winning probability of you and your friend in the casino (assume that there are only two slot machines in the casino). Who is more likely to win?
3. (*4 points*) Compare your conclusions from (1) and (2). Can you explain them mathematically? I.e., write down the equations to compute the winning probabilities and compare the terms.

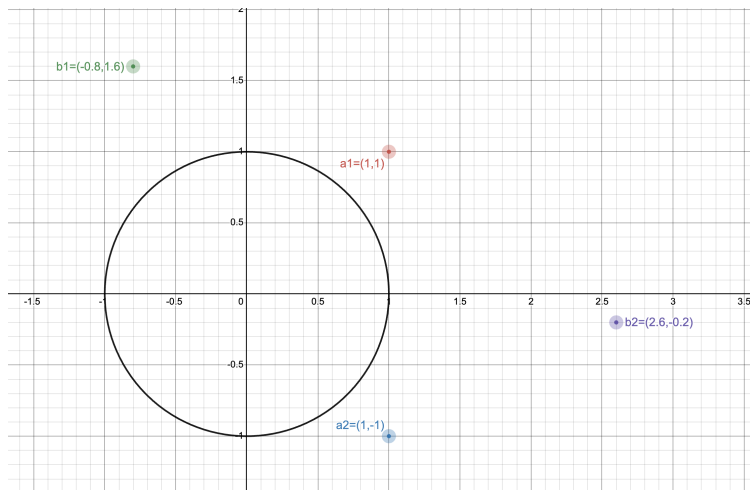


Figure 1: Problem 2

## 2 Matrix as Operations [25 points]

On the 2D plane, we have four vectors (all with the starting point on the origin)  $a_1 = (1, 1)$ ,  $a_2 = (1, -1)$ ,  $b_1 = (-0.8, 1.6)$  and  $b_2 = (2.6, -0.2)$ . See Fig. 1

1. [5 points] Write down one  $2 \times 2$  matrix  $W$  that transforms  $a_1$  to  $b_1$ , and  $a_2$  to  $b_2$ .
2. [5 points] Write down one rotation matrix  $V$ , which rotates clockwise by  $\alpha$  degrees such that  $\tan(\alpha) = 3$ . Then, write down one matrix  $\Sigma$ , which scales the y-axis by 2, while keeping the x-axis unchanged. Finally, write down one rotation matrix  $U$ , which rotates counter-clockwise by  $\beta$  degrees, such that  $\tan(\beta) = 1/3$ . Multiply three matrices together, namely  $U\Sigma V$ , what do you discover?
3. [10 points] Compute the eigenvalues and the corresponding eigenvectors of  $W^T W$ . Now consider the unit circle. Suppose every point on the unit circle gets transformed by  $W$ ; what shape do you get after the transformation (consider the break-up transformations in the previous question)? What relationship do you find between the eigenvalues/eigenvectors and the transformed shape, and why?
4. [5 points] Compute the determinant of  $W$ . What's the area of the shape transformed from the unit circle? What is the relationship between the determinant and the area of a transformed shape? Based on that, can you use one sentence to explain that the determinant of a product of two matrices is equal to the product of the determinants of the two matrices or, in other words,  $\det(AB) = \det(A)\det(B)$ ?

## 3 Some Practices [10 points]

1. [3 points] For a discrete random variable  $X$  of finite values, show that

$$(\mathbb{E}[X^3])^2 \leq \mathbb{E}[X^6]. \quad (1)$$

You only need the materials taught in class.

2. [3 points] Prove the above in a different way. Again, you only need the materials taught in class.
3. [4 points] For two PSD matrices  $A, B$  both of shape  $n \times n$ , show that  $\lambda A + (1 - \lambda)B$  is also PSD for  $0 \leq \lambda \leq 1$ .

## 4 Programming Problem: Density Estimation of Multivariate Gaussian [25 points]

General instructions for programming problems (if you are not familiar with python): please install anaconda python (python version 3.x, and 3.8 or above is recommended) by following the instructions on <https://www.anaconda.com/download/>, and then install sklearn, numpy, matplotlib, seaborn, pandas and jupyter notebook in anaconda, for example, by running command “conda install seaborn”. Note some of the above packages may have already been installed in anaconda, depending on which version of anaconda you just installed.

Please check the following tutorials if you are not familiar with numpy: <http://cs231n.github.io/python-numpy-tutorial/> and <https://docs.scipy.org/doc/numpy-1.15.0/user/quickstart.html>

In this problem, you will estimate a multivariate Gaussian distribution from 2D points sampled a multivariate Gaussian distribution, and learn how to use python and matplotlib to generate simple plots in ipython notebook. If you are not familiar with jupyter notebook, please check a quick tutorial on <https://jupyter-notebook-beginner-guide.readthedocs.io/en/latest/execute.html>.

We provide an ipython notebook “2d\_gaussian.ipynb” for you to complete. In your terminal, please go to the directory where this ipynb file is located, and run command “jupyter notebook”. A local webpage will be automatically opened in your web browser. Click the above file to open the notebook.

You need to complete everything below each of the “TODO”s that you find (please search for every “TODO”). Once you have done that, please submit the completed ipynb file as part of your included .zip file.

In your write-up, you also need to include the plots and answers to the questions required in this session. Please include the plots and answers in the pdf file in your solution.

1. [2 points] Run the first cell in the ipython notebook to generate 1000 points  $X$  (a  $1000 \times 2$  matrix, each point has two coordinates) sampled from a multivariate Gaussian distribution. Estimate the mean and covariance of the sampled points and report them in your write-up.
2. [3 points] Plot the histogram for the x-coordinates of  $X$  and y-coordinates of  $X$  respectively. You can use the `plt.hist()` function from matplotlib.
3. [5 points] From the histogram, can you tell whether the x-coordinates of the 1000 points (the first column of  $X$ ) follow some Gaussian distribution? If so, compute the mean and the variance. How about the y-coordinates? If so, compute the mean and the variance.
4. [5 points] Sample 1000 numbers from a 1D Gaussian distribution with the mean and variance of the x-coordinates you got from subproblem 3. Sample another 1000 numbers from a 1D Gaussian distribution with the mean and variance of the y-coordinates. You can use `np.random.normal` from numpy. Generate a new 2D scatter plot of 1000 points with the first 1000 numbers as x-coordinates and the second 1000 numbers as y-coordinates. Compared to the original 1000 points, what is the difference in their distributions? What causes the difference?
5. [5 points] Plot a line segment with  $x = [-10, 10]$  and  $y = 2x + 2$  on the 2D-Gaussian plot. The `np.linspace()` function may be helpful. Project  $X$  onto line  $y = 2x + 2$  and plot the projected points on the 2D space.
6. [5 points] Draw the histogram of the x-coordinates of the projected points. Are the x-coordinates of the projected points sampled from some Gaussian distribution? If so, compute the mean and the variance.

## 5 Programming Problem: KNN for Iris flowers classification [20 points]

For this problem, we want to use the K nearest neighbor algorithm to classify Iris flowers.

1. *[5 Points]* Load the Iris data using `sklearn.datasets`. Calculate how many elements there are for every class.
2. *[5 Points]* Build a `KNeighborsClassifier` with  $k = 1$  to predict the class. Train it on the whole dataset. For the classification problem, different goodness-of-fit metrics are used. For this exercise, you can use accuracy, which is defined in the formula given below. Calculate the accuracy of the KNN classifier on the iris dataset. Does this result give you meaningful information?

$$\text{Accuracy} = \frac{\text{\#correctly predicted}}{M} \quad (2)$$

3. *[5 Points]* Split the dataset into two parts using `sklearn's` train test split. Use the following arguments:
  - (a)  $X, y$  : the dataset
  - (b) test size = 0.5 (use 50% of the dataset for testing)
  - (c) shuffle = True (randomly shuffle the dataset before making a cut)
  - (d) random state = 0 (random seed, this ensures consistent results)

Use the split to find the optimal value of  $k$ . Please try different values of  $k$  from 1 to 50, fit the model on the training data, and calculate the model's accuracy on the training data and the testing data, respectively. Plot the training accuracy and testing accuracy against the value of  $k$ . Which  $k$  value is the best?

4. *[5 Points]* You observed a flower and measured the following characteristics:
  - (a) sepal width = 5.0
  - (b) petal width = 4.1
  - (c) sepal length = 3.8
  - (d) petal length = 1.2

Use your prediction model to classify this plant. What's the predicted class?

## 6 Programming Problem: K Means clustering [10 points]

For this question, use the data in `clust_data.csv`. We will attempt to cluster the data using k-means. But, what  $k$  should we use?

1. *[5 Points]* Apply k-means to this data 15 times, changing the number of centers from 1 to 15. Each time use `nstart = 10` and store the within-cluster sum-of-squares/inertia value from the resulting object. The inertia measures how variable the observations are within a cluster. This value will be lower with more centers, no matter how many clusters there truly are naturally in the given data. Plot this value against the number of centers. Look for an "elbow", the number of centers where the improvement suddenly drops off. Based on this plot, how many clusters do you think should be used for this data?
2. *[2 Points]* Re-apply k-means for your chosen number of centers. How many observations are placed in each cluster? What is the value of inertia?
3. *[3 Points]* Visualize this data. Plot the data using the first two variables and color the points according to the k-means clustering. Based on this plot, do you think you made a good choice for the number of centers? Briefly explain your conclusion.