Linear Representations of Finite Groups
Generalities on Linear Representations

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Let V be a vector space over $\mathbb{C}$ of dimension n with basis $\{e_i\}_{i=1}^n$. Denote the general linear group over V by

$$\begin{aligned} GL(V) =& \{a : V \to V | a \text{ is an isomorphism}\} \\ =& \left\{ (a_{ij})_{n\times n} \in \mathbb{C}^{n\times n} : a(e_j) = \sum_{i=1}^n a_{ij} e_i, \det(a_{ij}) \neq 0 \right\}. \end{aligned}$$

Let G be a group of finite order with identity e and multiplication $(s,t) \mapsto st$. A linear representation of G in V is a homomorphism

$$\begin{aligned} \rho :& G \to GL(V) \\ & s \mapsto \rho(s). \end{aligned}$$

We call V a representation (space) of G, and $\deg \rho = \dim V = n$. It must hold that

1. $$\det \rho(s) \neq 0.$$

2. $$\rho(st) = \rho(s)\rho(t), \text{ i.e., } \rho(st)(i,j) = \sum_{k=1}^n \rho(s)(i,k)\rho(t)(k,j).$$

Let $\rho : G \to GL(V), \rho' : G \to GL(V')$ be two representations of same degree. ρ is isomorphic to ρ' if $\exists \tau : V \to V'$ a linear isomorphism, such that $\forall s \in G, \tau \circ \rho(s) = \rho'(s) \circ \tau$, i.e.

$$\begin{array}{ccc} V & \xrightarrow{\rho(s)} & V \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau} \\ V' & \xrightarrow{\rho'(s)} & V' \end{array}$$

is commutative, or $\exists T \in GL_n(\mathbb{C})$, such that $\forall s \in G, T\rho(s) = \rho'(s)T$, or $\rho(s) = T^{-1}\rho'(s)T$.

Example 1. *Let $\rho : G \to \mathbb{C}^\times$ be a representation of G of degree 1. $\forall s \in G$, it has finite order m,*

$$1 = \rho(1) = \rho(s^m) = \rho(s)^m.$$

Thus $\forall s \in G, |\rho(s)| = 1$. In particular,

$$\begin{aligned} \mathbb{1} : & G \to \mathbb{C}^\times \\ & s \mapsto 1 \end{aligned}$$

is called the unit (trivial) representation.

Example 2. *Let G be a group of order g and V a vector space of dimension g with basis $\{e_t\}_{t\in G}$ indexed by elements in G. A representation*

$$\begin{aligned} \rho : & G \to GL(V) \\ & s \mapsto \rho(s) \\ \rho(s) : & V \to V \\ & e_t \mapsto e_{st} \end{aligned}$$

is called a regular representation. Thus $\{\rho(s)(e_1) = e_s : s \in G\}$ forms a basis of V.

Conversely, for a representation

$$\begin{aligned} \sigma : & G \to GL(W) \\ & s \mapsto \sigma(s), \end{aligned}$$

if $\{\sigma(s)(w) : s \in G\}$ forms a basis of W, then σ is isomorphic to ρ

Example 3. *Let G act on a finite set X, i.e. $\forall s \in G, x \mapsto sx$, such that*

1. $$1x = x.$$

2. $$s(tx) = (st)x, \forall s, t \in G, \forall x \in X.$$

Let V be a vector space of dimension X with basis $\{e_x\}_{x\in X}$. The following representation is called a permutation representation

$$\begin{aligned} \rho : & G \to GL(V) \\ & s \mapsto \rho(s) \\ \rho(s) : & V \to V \\ & e_x \mapsto e_{sx}. \end{aligned}$$