K-theory, Chern-Connes Character and Algebraic Novikov Conjecture

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1 K-theory

Recall history of K-theory.

1. Grothendieck, Riemann-Roch Theorem(for algebraic variety).
2. Atiyah, Hirzebruch, topological K-theory.
3. Quillen, Milnor, Bass, algebraic K-theory.

Applications: topology, operator algebra, algebra, number theory, etc.

R, a unital ring. $M_n(R) = \left\{ \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & \cdots & \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} : r_{ij} \in R \right\}$. $M_\infty(R) = \bigcup_{n=1}^{\infty} M_n(R)$.

$M_n(R) \hookrightarrow M_{n+1}(R)$, $\begin{pmatrix} r_{11} & \cdots & r_{1n} \\ & \cdots & \\ r_{n1} & \cdots & r_{nn} \end{pmatrix} \to \begin{pmatrix} r_{11} & \cdots & r_{1n} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r_{n1} & \cdots & r_{nn} & 0 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$. Idempotent $p \in M_\infty(R)$, $p^2 = p$.

Example 1. *X, a compact space. $R = C(X)$. Idempotent $p \in M_\infty(C(X)) \iff$ vector bundle over X. $p : X \to M_n(\mathbb{C}), x \mapsto p(x)$.*

$Idempotent(M_\infty(R))/\sim$, abelian semigroup. Two idempotents p, q are said to be equivalent if $\exists$ an invertible $w \in M_n(R)$ s.t. $w^{-1}pw = q$. $[p]+[q] = \left[\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}\right]$. Notice that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix}$.

S(abelian semigroup)$\xrightarrow{\text{Grothendieck process}}G(S)$(abelian group). $G(s) = \{(s,t) : s, t \in S\}/\sim$. $(s,t) \sim (s',t')$ iff $\exists x \in S$ s.t. $s + t' + x = s' + t + x$. $-[(s,t)] = [(t,s)]$.

Example 2. *$S = \mathbb{N}, G(s) = \mathbb{Z}$.*

Example 3. *$S = \mathbb{N} \cup \{+\infty\}, G(S) = \{0\}$. Notice that $(s,t) \sim (s',t'), s + t' + \infty = s' + t + \infty$.*

Definition 1. *$K_0(R) = G(Idempotent(M_\infty(R))/\sim)$.*

Γ, a group. Group ring, $\mathbb{C}\Gamma = \left\{\sum_{g\in\Gamma} c_g g : c_g \in \mathbb{C}\right\}$, where elements are finite sum. $U_g : \ell^2(\Gamma) \to \ell^2(\Gamma)$, $(U_g\xi)(x) = \xi(g^{-1}x)$. Then $\mathbb{C}\Gamma = \left\{\sum_{g\in\Gamma} c_g U_g\right\} \subset \mathcal{B}(\ell^2(\Gamma))$.

R, a ring. Group ring, $R\Gamma = \left\{\sum_{g\in\Gamma} r_g g : r_g \in R\right\}$. Question: what is $K_0(R\Gamma)$?

2 Chern-Connes Character

$\forall p \geq 1, T : H \to H$ is called a Schatten p-class operator if $tr(T^*T)^{\frac{p}{2}} < +\infty$. $T = diag(c_1, \cdots, c_n, \cdots)$, $tr(T^*T)^{\frac{p}{2}} = \sum_{n=1}^{\infty} |c_n|^p < +\infty$.

Definition 2. *S_p, the ring of all Schatten p-class operators. $S = \bigcup_{p=1}^{\infty} S_p$, the ring of all Schatten class operators.*

Γ, a group. S, the ring of Schatten class operators. $S\Gamma$, the group ring. Motivations: Connes-Moscovici's higher index theory(1990s, *Topology*). M^{2n}, a compact manifold. D, an elliptic differential operator on M^{2n}. Higher index, index $D \in K_0(S\Gamma)$, $\Gamma = \pi_1 M$.

Approximating $K_0(S\Gamma)$ using locally finite simplicial homology group of $P_F(\Gamma)$. Γ, a group. $\forall F \subset \Gamma$, a finite subset.

Definition 3. *The Rips' complex $P_F(\Gamma)$ is a simplicial complex whose set of vertices is Γ and $\{\gamma_0, \cdots, \gamma_n\}$ spans a simplex iff $\gamma_i^{-1}\gamma_j \in F$.*

Example 4. *$\Gamma = \mathbb{Z}$. $F = \{\pm 1\}$. $\{n_0, n_1\}$ spans a simplex. $n_1 - n_0 \in F = \{\pm 1\}$. $P_F(\Gamma)$ forms a line.*

Example 5. *$\Gamma = \mathbb{F}_2 =< a, b >$. $F = \{a^{\pm 1}, b^{\pm 1}\}$. $\gamma_0^{-1}\gamma_1 \in F$. $P_F(\Gamma)$ forms a tree.*

X, a simplicial complex. $H_*^{lf}(X)$, the locally finite homology group of X. $C_n^{lf}(X)$, the n-dimensional locally finite simplicial chain group. $C_n^{lf}(X) = \left\{\sum c_{[v_0,\cdots,v_n]}[v_0, \cdots, v_n] : c_{[v_0,\cdots,v_n]} \in \mathbb{C}\right\}$. $[v_0, \cdots, v_n]$, oriented simplex with vertices $v_0, \cdots, v_n$. $\sum c_{[v_0,\cdots,v_n]}[v_0, \cdots, v_n]$ locally finite, i.e. $\forall$ a compact subset $K \subset X$, $\exists$ at most finitely many $[v_0, \cdots, v_n]$ s.t. $c_{[v_0,\cdots,v_n]} \neq 0$ and $[v_0, \cdots, v_n] \cap K \neq \phi$. $\partial_n : C_n^{lf}(X) \to C_{n-1}^{lf}(X)$, boundary map. $\partial_n[v_0, \cdots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \cdots, \hat{v_i}, \cdots, v_n]$.

Definition 4. $H_n^{lf}(X) = Ker\partial_n / Im\partial_{n+1}$.

Example 6. $X = \mathbb{R}$. $H_1^{lf}(\mathbb{R}) = \mathbb{C}$. $H_1(\mathbb{R}) = 0$. *Generator of* $H_1^{lf}(\mathbb{R})$, $\sum_{n=-\infty}^{\infty}[n, n+1] \in C_1^{lf}(\mathbb{R})$. $\partial_1 \left(\sum_{n=-\infty}^{\infty}[n, n+1]\right) = \sum_{n=-\infty}^{\infty}([n+1] - [n]) = 0$.

$(v_0, v_1, \cdots, v_n)$, an ordered simplex of $X(v_i$ may be the same as $v_j)$. $(v_0, v_1, \cdots, v_n) \to (v_n, v_0, \cdots, v_{n-1})$ is called a cyclic permutation. $(v_0, \cdots, v_n) \sim (v_0', \cdots, v_n')$ 1. if one can be obtained by any number of cyclic permutations if n is even; 2. one can be obtained from another by even number of cyclic permutation if n is odd. The cyclically oriented simplex $[v_0, \cdots, v_n]_\lambda$ is defined to be the equivalence class of ordered simplex.

Theorem 1. $H_{n,\lambda}^{lf}(X) \cong \bigoplus_{\substack{k \le n, \\ n-k \ even}} H_k^{lf}(X)$.

Proof. $H_{n-2}^{lf}(X) \hookrightarrow H_{n,\lambda}^{lf}(X)$. $\sum c_{[v_0,\cdots,v_{n-2}]}[v_0, \cdots, v_{n-2}] \to \sum c_{[v_0,\cdots,v_{n-2}]}[v_0, v_0, v_0, \cdots, v_{n-2}]_\lambda$. □

Γ, a group. X, a simplicial complex. Γ acts on X simplicially. Define $H_k^\Gamma(X), H_{k,\lambda}^\Gamma(X)$ by requiring all chains to be Γ-invariant.

Γ, a group(torsion free, i.e. if $g^n = 1$ for some $g \in \Gamma$, then $n = 0$ or $g = 1$.). S, the ring of Schatten class operators. $c : K_0(S\Gamma) \to \lim_{\substack{finite\ subsets\ F\ of\ \Gamma, \\ n \to \infty}} H_{2n,\lambda}^\Gamma(P_F(\Gamma)) = \lim_F \left(\bigoplus H_{2k}^\Gamma(P_F(\Gamma))\right)$. $[p] - [p_0] \in K_0(S_p\Gamma)$ for some p. $p \in M_\infty((S_p\Gamma)^+), p_0 \in M_\infty(\mathbb{C})$ are idempotents. $p = q + p_0, q \in M_\infty(S_p\Gamma)$. Let $q = \sum_{g\in\Gamma} k_g g, k_g \in M_\infty(S_p)$. $c([p] - [p_0]) = \sum tr(k_{x_0^{-1}x_1} k_{x_1^{-1}x_2} \cdots k_{x_n^{-1}x_b})[x_0, \cdots, x_n]_\lambda$, where $n \ge p$, n even. Let $F = \{g : k_g \ne 0\}$.

Proposition 1. $c([p] - [p_0]) \in H_n^\Gamma(P_F(\Gamma))$.

3 Algebraic K-theory and Novikov Conjecture

Conjecture 1 (Novikov)**.** $\lim_F H_n^\Gamma(P_F(\Gamma), \mathbb{K}(S)^{-\infty}) \xrightarrow{assembly\ map} K_n(S\Gamma)$ *is rationally injective.*

Theorem 2. *This conjecture is true!*

Proof.

$$\begin{tikzcd}
\lim_F H_n^\Gamma(P_F(\Gamma), \mathbb{K}(S)^{-\infty}) \arrow[r, "A"] \arrow[dr, dashed, "c \circ A"'] & K_n(S\Gamma) \arrow[d, "c"] \\
& \lim_F \left(\bigoplus H_{2k}^\Gamma(P_F(\Gamma))\right)
\end{tikzcd}$$

Claim: $c \circ A$ is rationally an isomorphism(Mayer-Vietoris sequence & Five Lemma). □