Degenerate Elliptic Partial Differential Equations

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Contents

1 Introduction and preliminaries

We discuss regularity of solution of degenerate elliptic partial differential equations by energy estimates. Topics including hyperbolic equations, mixed equations, maximum principles will also be involved. In this note, we mainly care about H^k and L^p estimate.

We always assume that $a_{ij} = a_{ji}$.

Elliptic equations:

1. Non-divergence: $a_{ij}u_{ij} + b_i u_i + cu = f$

2. Divergence: $\partial_i(a_{ij}u_j) + cu = f$

Notation:

1. $\dfrac{\partial a_{ij}}{\partial x_k} = a_{ij,k}$

2. $\dfrac{\partial c}{\partial x_k} = c_{,k}$

3. Einstein summation convention: repeated index means summation.

2 Basic concepts and results

Elliptic partial differential equations:

1. (Strict) elliptic:

$$a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad \lambda > 0, \forall \xi = (\xi_1, \cdots, \xi_n) \in \mathbb{R}^n, \forall x \in \Omega.$$

   Or, all eigenvalues of $a_{ij}(x) \geq \lambda$.

2. Degenerate elliptic:

$$a_{ij}(x)\xi_i\xi_j \geq 0 \quad \lambda > 0, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega.$$

3. Uniformly elliptic:

$$\Lambda|\xi|^2 \geq a_{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \quad 0 < \lambda \leq \Lambda, \forall \xi \in \mathbb{R}^n, \forall x \in \Omega.$$

For uniformly elliptic PDEs, we have following results:

1. Perturbation results(assume $a_{ij} \in C$, and consider a_{ij}'s value on a given point, then equation has constant coefficients)

   (a) (Interior) Schauder Theory(1930s)
   If $\Omega' \subset\subset \Omega$, then

$$a_{ij}, b_i, c, f \in C^{\alpha}(\Omega) \Rightarrow u \in C^{2,\alpha}(\Omega),$$

   and

$$\|u\|_{C^{2,\alpha}(\Omega')} \leq C\left\{\|u\|_{L^\infty(\Omega)} + \|f\|_{C^\alpha(\Omega)}\right\},$$

   where C is a constant concerning with dimension of space, elliptic constant λ, Λ, and Hölder module of coefficients. More generally, we have

$$a_{ij}, b_i, c, f \in C^{k,\alpha}(\Omega) \Rightarrow u \in C^{k+2,\alpha}(\Omega).$$

   (b) $W^{2,p}$ Theory(1950s)

$$a_{ij} \in C, b_i, c \in L^\infty, f \in L^P(\Omega) \Rightarrow u \in W^{2,p}(\Omega),$$

   and

$$\|u\|_{W^{2,p}}(\Omega') \leq C\left\{\|u\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)}\right\},$$

   where C is a constant concerning with continuous module of a_{ij}.

   (c) H^k Theory(for divergence)

$$f \in H^k \Rightarrow u \in H^{k+2}.$$

   Due to some historical reason, here $H^k = W^{k,2}$.

2. Non-perturbation results(assume $a_{ij} \in L^\infty$)

    (a) DeGrorgi-Moser(for divergence, 1960s) $\Rightarrow$ Quasilinear

    (b) Krylov-Safanov(for non-divergence, 1970s) $\Rightarrow$ Fully nonlinear, geometry analysis, etc.

    (c) L^p estimate $\Rightarrow$ L^∞ estimate (for subsolution)

    (d) Weak Harnack (for supsolution)

    (e) (Combining 2c and 2d) Harnack $\Rightarrow$ Hölder continuity (for solution)

3 Basic estimates

3.1 Interior H^1-estimate for $\Delta u = f$

Consider $\Delta u = f$ in $B_1 \subset \mathbb{R}^n$. Our goal is to get regularity of u using regularity of f. Here technique integration by parts is often used to decrease order of derivative.

We will multiply both side of equation $\Delta u = f$ by $\varphi^2 u$, where cut-off function $\varphi \in C_0^\infty(B_1), \mathrm{supp}\varphi \subset\subset B_1, 0 \le \varphi \le 1$.

First, we try with $-\varphi u$ on $\Delta u = f$.

$$\begin{aligned}
\varphi u \Delta u &= \varphi u u_{ii} \\
&= (\varphi u u_i)_i - (\varphi u)_i u_i \\
&= (\varphi u u_i)_i - \varphi u_i^2 - \varphi_i u u_i
\end{aligned}$$

Then we have

$$-(\varphi u u_i)_i + \varphi \|\Delta u\|^2 + \varphi_i u u_i = -\varphi u f.$$

Integrate,

$$\begin{aligned}
\int \varphi |\Delta u|^2 + \int \varphi_i u u_i &= -\int \varphi u f \\
\Rightarrow \int \varphi |\Delta u|^2 &= -\int \varphi_i u u_i - \int \varphi u f.
\end{aligned}$$

Use Cauchy inequality($|2ab| \le a^2 + b^2$),

$$|\varphi_i u u_i| = \left| \frac{\varphi_i u}{\sqrt{\varphi}} \sqrt{\varphi} u_i \right| \le \frac{1}{2} \left(\varphi |\nabla u|^2 + \frac{|\nabla \varphi|^2}{\varphi} u^2 \right)$$

$$\begin{aligned}
\Rightarrow \int \varphi |\nabla u|^2 &\le \frac{1}{2} \int \left(\varphi |\nabla u|^2 + \frac{|\nabla \varphi|^2}{\varphi} u^2 \right) - \int \varphi u f \\
\Rightarrow \int \varphi |\nabla u|^2 &\le \int \frac{|\nabla \varphi|^2}{\varphi} u^2 - 2 \int \varphi u f.
\end{aligned}$$

Notice that when $\varphi = 0$, the first term on right side of inequality does not make sense. Replace φ by φ^2, notice

$$\frac{|\nabla \varphi^2|^2}{\varphi^2} = \frac{|2\varphi \nabla \varphi|^2}{\varphi^2} = 4|\nabla \varphi|^2,$$

then we have

$$\int \varphi^2 |\nabla u|^2 \le 4 \int |\nabla \varphi|^2 u^2 - 2 \int \varphi^2 u f.$$

Notice

$$2\varphi^2 u f = 2(\varphi u)(\varphi f) \le \varphi^2 u^2 + \varphi^2 f^2,$$

then

$$\int \varphi^2|\nabla u|^2 \leq \int \left(4|\nabla\varphi|^2+\varphi^2\right) u^2 + \int \varphi^2 f^2. \tag{$*$}$$

This is interior H^1-estimate, in other words,

$$u, f \in L^2 \Rightarrow \nabla u \in L^2.$$

3.2 Interior L^p-estimate for $\partial_i(a_{ij}u_j) = f$

Consider $\partial_i(a_{ij}u_j) = f, a_{ij} \in L^\infty$. And we have Poincaré inequality, if $u \in H_0^1(B_1)$, then

$$\int_{B_1} u^2 \leq c(n) \int_{B_1} |\nabla u|^2.$$

And Sobolev inequality, if $u \in H_0^1(B_1)$, then

$$\left(\int_{B_1} u^{2\chi}\right)^{\frac{1}{\chi}} \leq c(n) \int_{B_1} |\nabla u|^2,$$

where

$$\chi = \begin{cases} \dfrac{n}{n-2} & n \geq 3 \\ \text{arbitrary} & n = 2 \end{cases}.$$

For instance, take $n=3$, then $\chi = 3$, we have

$$\left(\int_{B_1} u^6\right)^{\frac{1}{3}} \leq c(3) \int_{B_1} |\nabla u|^2.$$

Notice

$$|\nabla(\varphi u)|^2 = |\varphi\nabla u + \nabla\varphi u|^2.$$

Use another form of Cauchy inequality($(a+b)^2 \leq 2\left(a^2+b^2\right)$), then

$$|\nabla(\varphi u)| \leq 2\left(\varphi^2|\nabla u|^2 + |\nabla\varphi|^2 u^2\right).$$

Thus we have

$$(*) \Rightarrow \int |\nabla(\varphi u)|^2 \leq \int \left(10|\nabla\varphi|^2+2\varphi^2\right) u^2 + 2\int \varphi^2 f^2,$$

which shows

$$\varphi u \in H_0^1(B_1).$$

Use Sobolev inequality and we have

$$\left(\int (\varphi u)^{2\chi}\right)^{\frac{1}{\chi}} \leq C\left\{\int \left(|\nabla\varphi|^2+\varphi^2\right) u^2 + \int \varphi^2 f^2\right\}.$$

Interior L^p- estimate

$$u \in L^2, f \in L^2 \Rightarrow u \in L^{2\chi}.$$

We easily ask whether such iteration hold to infinity,

$$f \in L^\infty, u \in L^2 \Rightarrow u \in L^{2\chi} \Rightarrow u \in L^{2\chi^2} \Rightarrow \cdots \Rightarrow u \in L^\infty.$$

Moser(1964) said yes and gave the so-called Moser iteration.

3.3 Estimate for $-\partial_i(a_{ij}u_j) + b_iu_i + cu = 0$

Consider

$$-\partial_i(a_{ij}u_j) + b_iu_i + cu = 0.$$

Strict ellipticity:

$$a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2, \lambda > 0.$$

Degenerate ellipticity:

$$a_{ij}\xi_i\xi_j \geq 0.$$

Multiply both sides of equation by $\varphi^2 u$ and consider the first term of left side,

$$\begin{aligned}-\varphi^2 u\partial_i(a_{ij}u_j) &= -\partial_i\left(\varphi^2 u a_{ij}u_j\right) + \left(\varphi^2 u\right)_i a_{ij}u_j \\ &= -\partial_i\left(\varphi^2 u a_{ij}u_j\right) + \varphi^2 a_{ij}u_iu_j + 2\varphi u\varphi_i a_{ij}u_j.\end{aligned}$$

Notice

$$2\varphi u\varphi_i a_{ij}u_j = 2\varphi u\varphi_j a_{ij}u_i.$$

For uniform ellipticity,

$$\Rightarrow -\partial_i\left(\varphi^2 u a_{ij}u_j\right) + \varphi^2 a_{ij}u_iu_j + \left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j\right)uu_i + \varphi^2 cu^2 = \varphi^2 uf.$$

For degenerate ellipticity, notice

$$uu_i = \left(\frac{u^2}{2}\right)_i,$$

thus we have

$$\Rightarrow \partial_i\left(-\varphi^2 u a_{ij}u_j + \frac{1}{2}\left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j\right)u^2\right) + \varphi^2 a_{ij}u_iu_j$$
$$+ \left(\varphi^2 c - \frac{1}{2}\left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j\right)_i\right)u^2 = \varphi^2 uf.$$

3.3.1 Uniform ellipticity

Assume uniform ellipticity and integrate, we have

$$\int_{B_1} \varphi^2 a_{ij} u_i u_j = \int_{B_1} \left[-\left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j \right) u u_i - \varphi^2 c u^2 + \varphi^2 u f \right].$$

We already know

$$\int_{B_1} \varphi^2 a_{ij} u_i u_j \geq \lambda \int_{B_1} \varphi^2 |\nabla u|^2,$$

and apply Cauchy inequality to second term on right side of the equation,

$$\begin{aligned} -\left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j \right) u u_i &= -\left(\sqrt{\lambda}\varphi u_i \right) \left(\frac{1}{\sqrt{\lambda}} \left(\varphi b_i + 2 a_{ij}\varphi_j \right) u \right) \\ &\leq \frac{1}{2} \left(\lambda \varphi^2 |\nabla u|^2 + \frac{1}{\lambda} |\varphi \nabla b + 2 a_{ij}\varphi_j|^2 u^2 \right), \end{aligned}$$

thus we have

$$\lambda \int \varphi^2 |\nabla u|^2 \leq \int \left\{ \left(-2\varphi^2 c + \frac{1}{\lambda} |\varphi \nabla b + 2 a_{ij}\varphi_j|^2 \right) u^2 + 2\varphi^2 u f \right\}.$$

Notice

$$2\varphi^2 u f \leq \varphi^2 u^2 + \varphi^2 f^2,$$

thus

$$\lambda \int \varphi^2 |\nabla u|^2 \leq \int \left\{ \left(-2\varphi^2 c + \frac{1}{\lambda} |\varphi \nabla b + 2 a_{ij}\varphi_j|^2 + \varphi^2 \right) u^2 + \varphi^2 f^2 \right\},$$

which shows

$$u \in L^2, f \in L^2 \Rightarrow \nabla u \in L^2.$$

3.3.2 Degenerate ellipticity

Assume degenerate ellipticity and integrate, we have

$$\int_{B_1} \varphi^2 a_{ij} u_i u_j = \int_{B_1} \left(-\left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j \right) u u_i - \varphi^2 c u^2 + \varphi^2 u f \right)$$

$$\Rightarrow \int_{B_1} \varphi^2 a_{ij} u_i u_j + \int_{B_1} \left(\varphi^2 c - \frac{1}{2} \left(\varphi^2 b_i + 2\varphi a_{ij}\varphi_j \right)_i \right) u^2 = \int_{B_1} \varphi^2 u f.$$

We only know

$$\int_{B_1} \varphi^2 a_{ij} u_i u_j \geq 0,$$

while our goal is to obtain$\int u^2, \int |\nabla u|^2, \int a_{ij} u_i u_j$.

Global estimate($\varphi \equiv 1$)
At the very beginning, we have

$$\partial_i \left(-a_{ij} u u_j + \frac{1}{2} b_i u^2 \right) + a_{ij} u_i u_j + \left(c - \frac{1}{2} b_{i,i} \right) u^2 = uf.$$

Integrate by parts,

$$\int_{\partial B_1} \left(-a_{ij} u u_j \nu_i + \frac{1}{2} b_i \nu_i u^2 \right) + \int_{B_1} a_{ij} u_i u_j + \int_{B_1} \left(c - \frac{1}{2} b_{i,i} \right) u^2 = \int_{B_1} uf,$$

where ν_i is the ith component of normal of ∂B_1.
Consider Dirichlet problem

$$\begin{cases} -(a_{ij} u_j)_j + b_i u_i + cu = f, & \text{in } B_1 \\ u = 0, & \text{on } \partial B_1 \end{cases}.$$

$$\Rightarrow \int_{B_1} a_{ij} u_i u_j + \int_{B_1} \left(c - \frac{1}{2} b_{i,i} \right) u^2 = \int_{B_1} uf$$

(Kohn-Nirenberg) Assume $c - \frac{1}{2} b_{i,i} \geq 1$ in B_1,

$$\Rightarrow \int_{B_1} u^2 \leq \int_{B_1} uf \leq \frac{1}{2} \int_{B_1} \left(u^2 + f^2 \right)$$

$$\Rightarrow \int_{B_1} u^2 \leq \int_{B_1} f^2,$$

which shows

$$f \in L^2 \Rightarrow u \in L^2.$$

In fact,

$$c - \frac{1}{2} b_{i,i} \geq k \Rightarrow \text{estimates of derivative of } k\text{th oder}$$

$$\not\Rightarrow u \text{ smooth.}$$

4 Difficulties

Given (a_{ij}) in $B_1 \subset \mathbb{R}^n$,

$$(a_{ij}) \geq 0 \text{ (all eigenvalues } \geq 0).$$

Non-degenerate:

$$\det(a_{ij}) > 0 \text{ (all eigenvalues} > 0).$$

Degenerate:

$$\det(a_{ij}) = 0 \text{ (some eigenvalue} = 0).$$

Characteristic direction($a_{ij}u_{ij} + b_i u_i + cu = 0$):

$$\xi \text{ is a characteristic direction at } x_0 \text{ if } a_{ij}(x_0)\xi_i\xi_j = 0.$$

Under $(a_{ij}(x_0)) \geq 0$.

$$a_{ij}(x_0)\xi_i\xi_j = 0 \iff a_{ij}(x_0)\xi_j = \begin{pmatrix} a_{1j}(x_0)\xi_j \\ \vdots \\ a_{nj}(x_0)\xi_j \end{pmatrix} = 0, \forall i.$$

Consider

$$D_i = \left\{(x, y) \in \mathrm{R}^2 : |x| < 1, 0 < y < \varepsilon\right\}.$$

$$(\partial_x, \partial_y) = (\partial_1, \partial_2).\ \nu = (0, -1).$$

Example 1. $u_{yy} + y^2 u_{xx} = f$.

$$A = \begin{pmatrix} y^2 & 0 \\ 0 & 1 \end{pmatrix} \geq 0.$$

$$\det A = y^2 = 0 \iff y = 0.$$

$$A\nu = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Degenerate on non-characteristics.

Example 2. $y^2 u_{yy} + u_{xx} = f$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & y^2 \end{pmatrix} \geq 0.$$

Similarly,

$$A\nu = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Degenerate on characteristics, which is hard to deal with. We have to consider a more specific form

$$y^2 u_{yy} + u_{xx} + b_1 u_x + b_2 u_y + cu = f.$$

5 Framework

Consider

$$-\partial_i(a_{ij}u_j) + b_i u_i + cu = f \text{ in } D_\varepsilon \subset \mathbb{R}^2,$$

$$\text{where } D_\varepsilon = \{(x_1, x_2) : |x_1| < 1, 0 < x_2 < \varepsilon\}.$$

$$(a_{ij}) > 0 \text{ in } \overline{D_\varepsilon} \backslash \{x_2 = 0\}.$$

All functions are periodic in x_1. (Don't need cut-off functions when integrating by parts.)

Case 1. $\{x_2 = 0\}$ is not characteristic.

Case 2. $\{x_2 = 0\}$ is characteristic.

6 Degenerate non-characteristics

6.1 Estimate for u, u_2

$$D_\varepsilon = \{(x_1, x_2) : |x_1| < 1, |x_2| < \varepsilon\}, \varepsilon \in (0, 1].$$

All functions are 2-periodic in x_1. Consider

$$a_{ij}u_{ij} + b_i u_i + cu = f \text{ in } D_1.$$

Assume degenerate on $\{x_2 = 0\}$, which is non-characteristic.

$$\nu = (0, 1).$$

$$(0, 1)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = a_{22} \neq 0 \text{ on } \{x_2 = 0\}.$$

Assume

$$a_{ij}\xi_i\xi_j \geq \lambda_1\xi_1^2 + \lambda_2\xi_2^2,$$

$$(0 <)\lambda \leq \lambda_2 \leq \Lambda, 0 \leq \lambda_1 \leq \Lambda.$$

Energy estimate in D_ε.
Multiply both sides of $a_{ij}u_{ij} + b_i u_i + cu = f$ by $-u$,

$$-a_{ij}uu_{ij} = -(a_{ij}uu_j)_i + a_{ij}u_iu_j + a_{ij,i}uu_j$$

$$\Rightarrow -(a_{ij}uu_j)_i + a_{ij}u_iu_j + (a_{ij,j} - b_i)uu_i - cu^2 = -uf.$$

Notice

$$uu_i = \left(\frac{u^2}{2}\right)_i.$$

$$\Rightarrow \left(-a_{ij}uu_j + \frac{1}{2}(a_{ij,j} - b_i)u^2\right)_i + a_{ij}u_iu_j - (a_{ij,ij} - b_{i,i})\frac{u^2}{2} - cu^2 = -uf$$

$$\Rightarrow \left(-a_{ij}uu_j + \frac{1}{2}(a_{ij,j} - b_i)u^2\right)_i + a_{ij}u_iu_j - \left(c - \frac{1}{2}b_{i,i} + \frac{1}{2}a_{ij,ij}\right)u^2 = -uf$$

Integrate in D_ε,

$$\int_{\{x_2=\pm\varepsilon\}}\left(-a_{ij}uu_j + \frac{1}{2}(a_{ij,j} - b_i)u^2\right)\nu_i + \int_{D_\varepsilon} a_{ij}u_iu_j - \int_{D_\varepsilon}\left(c - \frac{1}{2}b_{i,i} + \frac{1}{2}a_{ij,ij}\right)u^2$$

$$= -\int_{D_\varepsilon} uf \leq \int_{D_\varepsilon}\left(\frac{u^2}{2} + \frac{f^2}{2}\right),$$

where

$$\nu_i = (0, 1) \text{ on } x_2 = \varepsilon, \nu_i = (0, -1) \text{ on } x_2 = -\varepsilon.$$

Recall

$$a_{ij}u_iu_j \geq \lambda u_2^2.$$

Set

$$M = \sup_{B_1} c + \left|\nabla^2 a_{ij}\right|_{L^\infty(B_1)} + |\nabla b|_{L^\infty(B_1)} + 1.$$

$$\int_{D_\varepsilon} a_{ij}u_iu_j + \int_{D_\varepsilon} \left(-c + \frac{1}{2}b_{i,i} - \frac{1}{2}a_{ij,ij} - \frac{1}{2}\right)u^2 \leq BI + \int_{D_\varepsilon} f^2,$$

where BI represents boundary integral concerning with u^2 and uu_2.

$$\int_{D_\varepsilon} a_{ij}u_iu_j \leq \int_{D_\varepsilon} \left(c - \frac{1}{2}b_{i,i} + \frac{1}{2}a_{ij,ij} + \frac{1}{2}\right)u^2 + BI + \int_{D_\varepsilon} f^2$$

$$\Rightarrow \int_{D_\varepsilon} a_{ij}u_iu_j \leq BI + M\int_{D_\varepsilon} u^2 + \int_{D_\varepsilon} f^2,$$

where

$$\int_{D_\varepsilon} a_{ij}u_iu_j \geq \lambda \int_{D_\varepsilon} u_2^2.$$

Recall Poincaré inequality. Assume $u \in C_0^1(\Omega)$, then

$$\int_\Omega u^2 \leq C(\Omega)\int_\Omega |\nabla u|^2.$$

Integrate from $(x_1, -\varepsilon)$ to (x_1, x_2),

$$u(x_1, x_2) = u(x_1, -\varepsilon) + \int_{-\varepsilon}^{x_2} u_2(x_1, s)\mathrm{d}s.$$

Square both sides of the equation,

$$\begin{aligned} u^2(x_1, x_2) &\leq 2u^2(x_1, -\varepsilon) + 2(x_2 + \varepsilon)\int_{-\varepsilon}^{x_2} u_2^2(x_1, s)\mathrm{d}s \\ &\leq 2u^2(x_1, -\varepsilon) + 4\varepsilon\int_{-\varepsilon}^{\varepsilon} u_2^2(x_1, s)\mathrm{d}s \end{aligned}$$

$$\Rightarrow \int_{D_\varepsilon} u^2(x_1, x_2) \leq 4\varepsilon\int_{x_2=-\varepsilon} u^2\mathrm{d}x_1 + 8\varepsilon^2\int_{D_\varepsilon} u_2^2,$$

which we call Poincaré inequality in narrow domain.

$$\Rightarrow \lambda\int_{D_\varepsilon} u_2^2 \leq BI + M\left\{4\varepsilon\int_{x_2=-\varepsilon} u^2 + 8\varepsilon^2\int_{D_\varepsilon} u_2^2\right\} + \int_{D_\varepsilon} f^2.$$

If $8\varepsilon^2 M \leq \frac{\lambda}{2}$, then

$$\frac{\lambda}{2}\int_{D_\varepsilon} u_2^2 \leq BI + 4\varepsilon M\int_{x_2=-\varepsilon} u^2 + \int_{D_\varepsilon} f^2.$$

$$\Rightarrow \int_{D_\varepsilon} u_2^2 \leq C \left\{ \int_{x_2=\pm\varepsilon} \left(u^2 + u_2^2\right) + \int_{D_\varepsilon} f^2 \right\}$$

Use Poincaré inequality,

$$\Rightarrow \int_{D_\varepsilon} (u^2 + u_2^2) \leq C \left\{ \int_{x_2=\pm\varepsilon} \left(u^2 + u_2^2\right) + \int_{D_\varepsilon} f^2 \right\}.$$

Lemma 1. *Set*

$$M = \sup_{B_1} c + \left|\nabla^2 a_{ij}\right|_{L^\infty(B_1)} + |\nabla b|_{L^\infty(B_1)} + 1.$$

If $4\varepsilon\sqrt{M} \leq 1$, *then*

$$\int_{D_\varepsilon} (u^2 + u_2^2) \leq C \left\{ \int_{x_2=\pm\varepsilon} (u^2 + u_2^2) + \int_{D_\varepsilon} f^2 \right\},$$

where $C = C(n, \lambda, \Lambda, |\nabla a_{ij}|, |b|)$.

Comparing with uniformly elliptic equation,

$$\int |\nabla u|^2 \leq C \left\{ \int u^2 + \int f^2 \right\}.$$

6.2 Estimate for u_1

Next, how about $\int_{D_\varepsilon} u_1^2$?
Recall

$$a_{ij}u_{ij} + b_i u_i + cu = f.$$

Derive an equation for u_1.

$$a_{ij}(u_1)_{ij} + b_i(u_1)_i + cu_1 + a_{ij,1}u_{ij} + b_{i,1}u_i + c_{,1}u = f_1,$$

where

$$b_{i,1}u_i = b_{1,1}u_1 + b_{2,1}u_2.$$

And

$$a_{ij,1}u_{ij} = a_{11,1}u_{11} + 2a_{12,1}u_{12} + a_{22,1}u_{22}.$$

Recall

$$a_{11}u_{11} + 2a_{12}u_{12} + a_{22}u_{22} + b_i u_i + cu = f$$

$$\Rightarrow u_{22} = -\frac{1}{a_{22}}(a_{11}u_{11} + 2a_{12}u_{12} + b_i u_i + cu - f)$$

$$\Rightarrow a_{ij,1}u_{ij} = \left(a_{11,1} - \frac{a_{11}}{a_{22}}a_{22,1}\right)u_{11} + 2\left(a_{12,1} - \frac{a_{12}}{a_{22}}a_{22,1}\right)u_{12} - \frac{b_i}{a_{22}}u_i - \frac{c}{a_{22}}u + \frac{1}{a_{22}}f.$$

We can write

$$u_{11} = (u_1)_1, u_{12} = (u_1)_2.$$

Then

$$
\begin{aligned}
a_{ij}(u_1)_{ij} &+ \left(b_1 + a_{11,1} - \frac{a_{11}}{a_{22}}a_{22,1}\right)(u_1)_1 \\
&+ \left(b_2 + 2\left(a_{12,1} - \frac{a_{12}}{a_{22}}a_{22,1}\right)\right)(u_1)_2 \\
&+ \left(c + b_{1,1} - \frac{1}{a_{22}}b_1\right)u_1 \\
&= f_1 - (\cdots)u - (\cdots)u_2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow & a_{ij}(u_1)_{ij} + b_i^{(1)}(u_1)_i + c^{(1)}u_1 = f^{(1)} \\
& b_1^{(1)} = b_1 + a_{11,1} - \frac{a_{11}}{a_{22}}a_{22,1} \\
& b_2^{(1)} = b_2 + 2\left(a_{12,1} - \frac{a_{12}}{a_{22}}a_{22,1}\right) \\
& c^{(1)} = c + b_{1,1} - \frac{1}{a_{22}}b_1.
\end{aligned}
$$

Remark: $b_i^{(1)}$, $c^{(1)}(b_i^{(k)}$, $c^{(k)})$ differ from b_i, c by 1^{st} (and 2^{nd}) derivatives of a_{ij}, b_i.

Assume $a_{22} = 1$.

Lemma 2. *Assume $c_0 m\varepsilon\sqrt{M} \le 1$ where c_0 is universal, then*

$$
\begin{aligned}
\int_{D_\varepsilon} & \left\{u^2 + \left(\frac{\partial u}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial^m u}{\partial x_1^m}\right)^2\right. \\
& \left. + \left(\frac{\partial u}{\partial x_2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2 \partial x_1}\right)^2 + \cdots + \left(\frac{\partial^{m+1} U}{\partial x_2 \partial x_1^m}\right)^2\right\} \\
& \le C\left\{BI_m + \int_{D_\varepsilon}\left\{f^2 + \left(\frac{\partial f}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial^m f}{\partial x_1^m}\right)^2\right\}\right\}.
\end{aligned}
$$

Define $M^{(m)}$ for new $b_i^{(m)}$, $c^{(m)}$, then $M^{(m)} = M\mathcal{O}(m)$.

Examine the estimate

$$
\begin{array}{cccc}
u & u_1 & u_{11} & \cdots \\
u_2 & u_{12} & u_{112} & \cdots
\end{array}
$$

Recall

$$u_{22} = -(a_{11}u_{11} + 2a_{12}u_{12} + b_i u_i + cu + f).$$

Then we have

$$
\begin{array}{lllll}
0^{\text{th}} & u & & & \\
1^{\text{st}} & u_1 & u_2 & & \\
2^{\text{nd}} & u_{11} & u_{12} & (u_{22}) & \\
3^{\text{rd}} & u_{111} & u_{112} & (u_{122}) & (u_{222}) \\
\cdots & & & &
\end{array}
$$

Terms in bracket can be estimated by previous result.
Thus we can estimate $\partial_{x_1}^{k_1}\partial_{x_2}^{k_2}u, k_2 \geq 2$.

6.3 Conclusion

Theorem 1. *Assume* $c_0 m\varepsilon\sqrt{M} \leq 1$, *then*

$$\|u\|_{H^m(D_\varepsilon)} \leq C\left\{BI_m + \|f\|_{H^m(D_\varepsilon)}\right\}.$$

Remark: Fixed m, $\varepsilon \approx \frac{1}{m}$.

Theorem 2. *Consider*

$$u_{22} + au_{11} + b_i u_i + cu = f \text{ in } D_1,$$

where

$$a, b_i, c, f \in C^\infty\left(\overline{D_1}\right), u \in H^1.$$

Assume

$$a > 0 \text{ in } D_1 \backslash \{x_2 = 0\}.$$

Then

$$u \in C^\infty(D_1).$$

Remark: historically, $a = x_2^{2m} + hot$ is assumed in proof.

Proof. Assume $u \in C^\infty(D_1)$, attempt to derive $\|u\|_{H^m(D_1)}, \forall m$.
Fact: $\forall\varepsilon > 0, \Omega_\varepsilon = D_1\backslash D_\varepsilon, \lambda(\varepsilon) \leq a \leq A$.
Note: $\lambda(\varepsilon) > 0, \forall\varepsilon$.
Use standard interior H^k theory in Ω_ε,

$$\|u\|_{H^k\Omega_\varepsilon} \leq C(\varepsilon)\left\{\|u\|_{L^2(D_1)} + \|f\|_{H^{k-2}(D_1)}\right\}.$$

Fix m, take ε s.t. $c_0 m\varepsilon\sqrt{M} \leq 1$, then

$$\|u\|_{H^m(D_\varepsilon)} \leq C\left\{BI_m + \|f\|_{H^m(D_\varepsilon)}\right\}.$$

By combining,

$$\|u\|_{H^m(D_1)} \leq C\left\{\|u\|_{L^2(D_1)} + \|f\|_{H^m(D_1)}\right\}.$$

□

Theorem 3. *Consider*

$$u_{22} + au_{11} + b_i u_i + cu = f \text{ in } B_1,$$

where

$$a, b_i, c, f \in C^\infty, u \in H^1.$$

Assume $a \geq 0$. *And when looking at* $a^{-1}(0)$, *finite non-vertical curves intersecting at finitely many points. Then*

$$u \in C^\infty.$$

Question 1. *If we consider*

$$u_{11}u_{12} - u_{12}^2 = a,$$

where

$$u \in C^{1,1}.$$

Does conclusion of above theorem hold?

7 Degenerate on characteristics

$$D_\varepsilon = \{(x_1, x_2) : |x_1| < 1, x_2 \in (0, \varepsilon)\}.$$

All functions are 2-periodic in x_1.

$$\Sigma_\varepsilon = \{(x_1, \varepsilon) : |x_1| < 1\}.$$

$$\Sigma_0 = \{(x_1, 0) : |x_1| < 1\}.$$

$$a_{ij}u_{ij} + b_i u_i + cu = f \text{ in } D_1.$$

Assume

1. (a_{ij}) is positive definite in $\overline{D_1}\backslash\overline{\Sigma_0} = \{(x_1, x_2) : |x_1| \le 1, 0 < x_2 \le 1\}$, i.e. degeneracy at $\overline{\Sigma_0}$.

$$\Rightarrow \det a_{ij} = a_{11}a_{22} - a_{12}^2 = 0 \text{ on } \Sigma_0.$$
$$\Rightarrow a_{12} = 0 \text{ on } \Sigma_0.$$

2. Σ_0 is characteristic.

$$\Rightarrow \nu = (0, 1).\ \nu^{\mathrm{T}}(a_{ij})\nu = a_{22} = 0 \text{ on } \Sigma_0.$$

Assume $a_{11} \ne 0$ on $\overline{\Sigma_0}$, i.e. only one of two eigenvalues of (a_{ij}) is 0.
Assume

$$a_{ij}\xi_i\xi_j \ge \lambda_1\xi_1^2 + \lambda_2\xi_2^2,$$
$$\Lambda \ge \lambda_1 \ge \lambda, \Lambda \ge \lambda_2 \ge 0.$$

7.1 Energy estimate

Multiply both sides of

$$a_{ij}u_{ij} + b_i u_i + cu = f$$

by $-u$,

$$-a_{ij}uu_{ij} = (-a_{ij}uu_i)_j + a_{ij}u_iu_j + a_{ij,j}uu_i$$
$$\Rightarrow (-a_{ij}uu_i)_j + a_{ij}u_iu_j + (a_{ij,j} - b_i)uu_i - cu^2 = -uf.$$

Notice

$$uu_i = \left(\frac{u^2}{2}\right)_i.$$

$$\Rightarrow \left(-a_{ij}uu_j + \frac{1}{2}(a_{ij,j} - b_i)u^2\right)_i + a_{ij}u_iu_j - (a_{ij,ij} - b_{i,i})\frac{u^2}{2} - cu^2 = -uf$$
$$\Rightarrow \left(-a_{ij}uu_j + \frac{1}{2}(a_{ij,j} - b_i)u^2\right)_i + a_{ij}u_iu_j - \left(c - \frac{1}{2}b_{i,i} + \frac{1}{2}a_{ij,ij}\right)u^2 = -uf$$

Integrate in D_ε,

$$\int_{D_\varepsilon} a_{ij}u_iu_j - \int_{D_\varepsilon}\left(c-\frac{1}{2}b_{i,i}+\frac{1}{2}a_{ij,ij}\right)u^2 + \int_{\partial D_\varepsilon}\left(-a_{ij}uu_j+\frac{1}{2}(a_{ij,j}-b_i)u^2\right)\nu_i = -\int_{D_\varepsilon} uf,$$

where

$$a_{ij}u_iu_j \geq \lambda_1 u_1^2 + \lambda_2 u_2^2,\ \ \lambda_2 = 0 \text{ on } \Sigma_0.$$

Let

$$\begin{aligned} BI &= \int_{\{x_2=\varepsilon\}\cup\{x_2=0\}}\left(-a_{2j}uu_j+\frac{1}{2}(a_{2j,j}-b_2)u^2\right)\nu_2 \\ &= \int_{\{x_2=\varepsilon\}}\left(-a_{2j}uu_j+\frac{1}{2}(a_{2j,j}-b_2)u^2\right) - \int_{\{x_2=0\}}\left(-a_{2j}uu_j+\frac{1}{2}(a_{2j,j}-b_2)u^2\right). \end{aligned}$$

Notice on $\{x_2=0\}$,

$$a_{2j}=0,\ \ a_{2j,j}=a_{21,1}+a_{22,2},\ \ a_{21,1}=0.$$

Thus

$$a_{2j}uu_j+\frac{1}{2}(a_{2j,j}-b_2)u^2 = \frac{1}{2}(a_{22,2}-b_2)u^2.$$

Multiply both sides of

$$a_{ij}u_{ij}+b_iu_i+cu=f$$

by u_2,

$$a_{ij}u_2u_{ij} = (a_{ij}u_2u_i)_j - a_{ij}u_{2j}u_i - a_{ij,j}u_2u_i.$$

Notice

$$u_{2j}u_i = \left(\frac{u_iu_j}{2}\right)_2.$$

Then

$$a_{ij}u_2u_{ij} = (a_{ij}u_2u_i)_j - \left(\frac{1}{2}a_{ij}u_iu_j\right)_2 + \frac{1}{2}a_{ij,2}u_iu_j - a_{ij,j}u_2u_i.$$

Thus

$$(a_{ij}u_2u_i)_j - \left(\frac{1}{2}a_{ij}u_iu_j\right)_2 + \frac{1}{2}a_{ij,2}u_iu_j - a_{ij,j}u_2u_i + b_1u_1u_2 + b_2u_2^2 + cuu_2 = fu^2.$$

Notice

$$cuu_2 = c\left(\frac{u^2}{2}\right)_2 = \left(c\frac{u^2}{2}\right)_2 - c_{,2}\frac{u^2}{2}.$$

$$\begin{aligned} Q(\nabla u) &= \frac{1}{2}a_{ij,2}u_iu_j - a_{ij,j}u_2u_i + b_1u_1u_2 + b_2u_2^2 \\ &= \left(\frac{1}{2}a_{11,2}\right)u_1^2 + (a_{12,2}-a_{1j,j}+b_1)u_1u_2 + \left(\frac{1}{2}a_{22,2}-a_{2j,j}+b_2\right)u_2^2 \\ &= \frac{1}{2}a_{11,2}u_1^2 + (b_1-a_{11,1})u_1u_2 + \left(b_2-a_{12,1}-\frac{1}{2}a_{22,2}\right)u_2^2 \end{aligned}$$

$$(a_{ij}u_2u_j)_i + \left(-\frac{1}{2}a_{ij}u_iu_j + \frac{1}{2}cu^2\right)_2$$
$$+\left(\frac{1}{2}a_{11,2}u_1^2 + (b_1 - a_{11,1})u_1u_2 + \left(b_2 - a_{12,1} - \frac{1}{2}a_{22,2}\right)u_2^2\right) - \frac{c_{,2}}{2}u^2 = u_2 f$$

Coefficient for u_2^2
$$b_2 - a_{12,1} - \frac{1}{2}a_{22,2} = b_2 - \frac{1}{2}a_{22,2} \quad \text{on } \Sigma_0.$$

Define
$$Sgn(L,\Sigma_0) = b_2 - \frac{1}{2}a_{22,2} \quad \text{on } \Sigma_0.$$

Will derive estimates depending on the sign of $Sgn(L,\Sigma_0)$.
Integrate in D_ε,
$$\int_{D_\varepsilon} a_{ij}u_iu_j - \int_{D_\varepsilon}\left(c - \frac{1}{2}b_{i,i} + \frac{1}{2}a_{ij,ij}\right)u^2$$
$$+\int_{\Sigma_\varepsilon}\left(-a_{2j}uu_j + \frac{1}{2}(a_{2j,j} - b_2)u^2\right) - \int_{\Sigma_0}\frac{1}{2}(a_{22,2} - b_2)u^2 = -\int_{D_\varepsilon} uf, \tag{1}$$

where
$$a_{ij}u_iu_j = a_{11}u_1^2 + 2a_{12}u_1u_2 + a_{22}u_2^2.$$
$$\int_{D_\varepsilon}\left\{\frac{1}{2}a_{11,2}u_1^2 + (b_1 - a_{11,1})u_1u_2 + \left(b_2 - a_{12,1} - \frac{1}{2}a_{22,2}\right)u_2^2\right\} - \frac{1}{2}\int_{D_\varepsilon} c_{,2}u^2$$
$$+\int_{\Sigma_\varepsilon}\left(a_{2j}u_2u_j - \frac{1}{2}a_{ij}u_iu_j + \frac{1}{2}cu^2\right) - \int_{\Sigma_0}\left(a_{2j}u_2u_j - \frac{1}{2}a_{ij}u_iu_j + \frac{1}{2}cu^2\right) = \int_{D_\varepsilon} u_2 f, \tag{2}$$

where
$$a_{2j}u_2u_j - \frac{1}{2}a_{ij}u_iu_j + \frac{1}{2}cu^2 = -\frac{1}{2}a_{11}u_1^2 + \frac{1}{2}cu^2.$$

7.1.1 $Sgn < 0$

$Sgn(L,\Sigma_0) \le -2c_0 < 0$.

$$\exists \varepsilon > 0, \text{ s.t. } b_2 - a_{12,1} - \frac{1}{2}a_{22,2} \le -c_0 < 0.$$

$M(1) - (2)$. Check quadratic in ∇u first,

$$\begin{aligned} Q_1(\nabla u) = &\left(Ma_{11} - \frac{1}{2}a_{11,2}\right)u_1^2 \\ &+ 2\left(Ma_{12} - \frac{1}{2}(b_1 - a_{11,1})\right)u_1u_2 \\ &+ \left(Ma_{22} - \left(b_2 - a_{12,1} - \frac{1}{2}a_{22,2}\right)\right)u_2^2. \end{aligned}$$

On Σ_0,

$$
\begin{aligned}
Q_1(\nabla u) = & \left(Ma_{11} - \frac{1}{2}a_{11,2}\right) u_1^2 \\
& - 2\left(\frac{1}{2}(b_1 - a_{11,1})\right) u_1 u_2 \\
& - Sgn(L, \Sigma_0) u_2^2.
\end{aligned}
$$

By choosing ε small and M large,

$$C_1 |\nabla u|^2 \leq Q_1(\nabla u).$$

On Σ_0,

$$
\begin{aligned}
& -\frac{M}{2}(a_{22,2} - b_2)u^2 - \frac{1}{2}a_{11}u_1^2 + \frac{1}{2}cu^2 \\
= & \frac{1}{2}(c - Ma_{22,2} + Mb_2)u^2 - \frac{1}{2}a_{11}u_1^2.
\end{aligned}
$$

Lemma 3. *Assume $Sgn(L, \Sigma_0) < 0$ on $\overline{\Sigma_0}$. Then $\exists \varepsilon_0$, s.t. $\forall \varepsilon \in (0, \varepsilon_0)$,*

$$\int_{D_\varepsilon} \left(u^2 + |\nabla u|^2\right) \leq c_0 \left\{ \int_{D_\varepsilon} f^2 + \int_{\Sigma_\varepsilon} \left(u^2 + |\nabla u|^2\right) + \int_{\Sigma_0} \left(u^2 + u_1^2\right) \right\}.$$

Use Poincaré inequality in narrow domains, first term on the left side u^2 is controlled by u_2^2.

7.1.2 $Sgn > 0$

$Sgn(L, \Sigma_0) \geq 2c_0 > 0$.
$\exists \varepsilon > 0$, s.t. $b_2 - a_{12,1} - \frac{1}{2}a_{22,2} > c_0 > 0$.
M(1)+(2). Check quadratic in ∇u on Σ_0 first,

$$
\begin{aligned}
Q_2(\nabla u) = & \left(Ma_{11} + \frac{1}{2}a_{11,2}\right) u_1^2 \\
& + 2\left(\frac{1}{2}(b_1 - a_{11,1})\right) u_1 u_2 \\
& + Sgn(L, \Sigma_0) u_2^2.
\end{aligned}
$$

By choosing ε small and M large,

$$C_2 |\nabla u|^2 \leq Q_2(\nabla u).$$

On Σ_0,

$$-\frac{M}{2}(a_{22,2} - b_2)u^2 + \frac{1}{2}a_{11}u_1^2 - \frac{1}{2}cu^2$$

$$= \frac{1}{2}(-c - Ma_{22,2} + Mb_2)u^2 + \frac{1}{2}a_{11}u_1^2.$$

Look at several terms,

$$C_2 \int_{D_\varepsilon} |\nabla u|^2 + \int_{\Sigma_0} a_{11}u_1^2$$

$$\leq \int_{D_\varepsilon} (\cdots)u^2 + \int_{\Sigma_\varepsilon} (\cdots)\left(u^2 + |\nabla u|^2\right) + \int_{\Sigma_0} (\cdots)u^2 + \int_{D_\varepsilon} f^2.$$

Note: $\int_{\Sigma_0} u^2$ is not free.

$$\begin{aligned}
&u(x_1, 0) = u(x_1, \varepsilon) - \int_0^\varepsilon u_2(x_1, t)\mathrm{d}t \\
\Rightarrow &u^2(x_1, 0) \leq 2u^2(x_1, \varepsilon) + 2\varepsilon \int_0^\varepsilon u_2^2(x_1, x_2)\mathrm{d}x_2 \\
\Rightarrow &\int_{\Sigma_0} u^2 \leq 2\int_{\Sigma_\varepsilon} u^2 + 2\varepsilon \int_{D_\varepsilon} u_2^2
\end{aligned}$$

Lemma 4. *Assume $Sgn(L, \Sigma_0) > 0$ on Σ_0. Then $\exists \varepsilon_0$, s.t. $\forall \varepsilon \in (0, \varepsilon_0)$*

$$\int_{D_\varepsilon} \left(u^2 + |\nabla u|^2\right) + \int_{\Sigma_0} u_1^2 \leq c_0 \left\{ \int_{D_\varepsilon} f^2 + \int_{\Sigma_\varepsilon} \left(u^2 + |\nabla u|^2\right) \right\}.$$

The above lemma shows a weird fact that uniqueness of solution is determined by equations itself rather than boundary values.

7.2 Higher regularity

Calculate derivatives of kth order of both sides of

$$\mathrm{L}u = a_{ij}u_{ij} + b_iu_i + cu = f.$$

$$a_{ij}u_{kij} + b_iu_{ki} + cu_k + a_{ij,k}u_{ij} + b_{i,k}u_i = f_k - c_{,k}u, \;\; k = 1, 2.$$

$$\mathrm{L}u_1 + a_{11,1}u_{11} + 2a_{12,1}u_{12} + a_{22,1}u_{22} + b_{1,1}u_1 + b_{2,1}u_2 = f_1 - c_{,1}u$$

$$\mathrm{L}u_2 + a_{11,2}u_{11} + 2a_{12,2}u_{12} + a_{22,2}u_{22} + b_{1,2}u_1 + b_{2,2}u_2 = f_2 - c_{,2}u$$

Let

$$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

$$a_{ij}U_{ij} + b_iU_i + cU + \tilde{B}_1U_1 + \tilde{B}_2U_2 + \tilde{B}_0U = F_2$$

$$\tilde{B}_1U_1 + \tilde{B}_2U_2 = \begin{pmatrix} a_{11,1} & b_{12} \\ a_{11,2} & b_{22} \end{pmatrix}\begin{pmatrix} u_{11} \\ u_{21} \end{pmatrix} + \begin{pmatrix} b_{11}^2 & a_{22,1} \\ b_{21}^2 & a_{22,2} \end{pmatrix}\begin{pmatrix} u_{12} \\ u_{22} \end{pmatrix}$$

$$b_{12}^1 + b_{11}^2 = 2a_{12,1}$$

$$b_{22}^1 + b_{21}^2 = 2a_{12,2}$$

$U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ satisfies

$$\mathrm{L}_1 u = a_{ij} U_{ij} + \left(b_i I + \tilde{B}_i\right) U_i + \left(cI + \tilde{B}_0\right) U = F.$$

$$\begin{aligned} Sgn(L, \Sigma_0) &= \frac{1}{2}\left(\left(b_2 I + \tilde{B}_2\right) + \left(b_2 I + \tilde{B}_2\right)^{\mathrm{T}}\right) - \frac{1}{2} a_{22,2} I \\ &= \left(b_2 - \frac{1}{2} a_{22,2}\right) I + \frac{1}{2}\left(\tilde{B}_2 + \tilde{B}_2^{\mathrm{T}}\right) \quad \text{on } \Sigma_0. \end{aligned}$$

Notice

$$b_2 - \frac{1}{2} a_{22,2} I = Sgn(L, \Sigma_0).$$

Take

$$\tilde{B}_2 = \begin{pmatrix} 0 & a_{22,1} \\ 0 & a_{22,2} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & a_{22,2} \end{pmatrix} \text{ on } \Sigma_0.$$

$$\Rightarrow Sgn(L, \Sigma_0) = \left(b_2 - \frac{1}{2} a_{22,2}\right) I + \begin{pmatrix} 0 & 0 \\ 0 & a_{22,2}|_{\Sigma_0} \end{pmatrix} = \left.\begin{pmatrix} b_2 - \frac{1}{2} a_{22,2} & 0 \\ 0 & b_2 + \frac{1}{2} a_{22,2} \end{pmatrix}\right|_{\Sigma_0},$$

where

$$Sgn(L, \Sigma_0) = b_2 - \frac{1}{2} a_{22,2}.$$

Lemma 5. *$Sgn(L, \Sigma_0) < 0$,*

$$\int_{D_\varepsilon} f^2, \ \int_{\Sigma_0} \left(u^2 + u_1^2\right) \xmapsto{\text{control}} \int_{D_\varepsilon} \left(u^2 + |\nabla u|^2\right).$$

Lemma 6. *$Sgn(L, \Sigma_0) < 0$,*

$$\int_{D_\varepsilon} f^2 \xmapsto{\text{control}} \int_{D_\varepsilon} \left(u^2 + |\nabla u|^2\right) + \int_{\Sigma_0} u_1^2.$$

For estimates on 2^{nd} derivative, need

$$b_2 - \frac{1}{2} a_{22,2} < 0 \ (\Leftarrow) \ b_2 + \frac{1}{2} a_{22,2} < 0;$$

or

$$b_2 - \frac{1}{2} a_{22,2} > 0 \ (\Rightarrow) \ b_2 + \frac{1}{2} a_{22,2} > 0$$

on Σ_0.

Recall

$$\left.\begin{array}{l} a_{22} = 0 \text{ on } \Sigma_0 \\ a_{22} > 0 \text{ in } D_\varepsilon \end{array}\right\} \Rightarrow a_{22,2} \geq 0 \text{ on } \Sigma_0.$$

Theorem 4. *Assume*

$$\left(b_2 - \frac{1}{2}a_{22,2}\right) + (m-1)a_{22,2} < 0$$

on $\overline{\Sigma_0}$. *Then*

$$\|u\|_{H^m(D_\varepsilon)} \leq C\left\{\|f\|_{H^{m-1}(D_\varepsilon)} + \|u\|_{H^m(\Sigma_\varepsilon)} + \|u\|_{H^m_{\text{tangent}}(\Sigma_0)}\right\}.$$

Theorem 5. *Assume*

$$b_2 - \frac{1}{2}a_{22,2} > 0 \text{ on } \overline{\Sigma_0},$$

Then

$$\|u\|_{H^m(D_\varepsilon)} + \|u\|_{H^m_{\text{tangent}}(\Sigma_0)} \leq C\left\{\|f\|_{H^{m-1}(D_\varepsilon)} + \|u\|_{H^m(\Sigma_\varepsilon)}\right\}.$$

Question 2 (Isometric Embedding of Positive Disc). $B_1 \subset \mathbb{R}^2$, *g is a metric in* $\overline{B_1}$. *K is Gaussian curvature.* $K > 0$ *in* B_1. $K = 0$, $\nabla K \neq 0$ *on* ∂B_1. $\int_{B_1} K \mathrm{d}g = 4\pi$.

Question: $\left(\overline{B_1}, g\right) \xhookrightarrow{\text{isometric embedding}} \mathbb{R}^3$.

8 Hyperbolic equations

Question 3. $u_{xx}u_{yy} - u_{xy}^2 = K(x, y)$ *in* $B_1 \subset \mathbb{R}^2 = \{(x, y)\}$.
Notice $\det(D^2u) = \lambda_1\lambda_2$.
$K > 0 \iff \lambda_1, \lambda_2$ *have the same sign* $\Rightarrow$ *elliptic.*
$K < 0 \iff \lambda_1, \lambda_2$ *have different signs* $\Rightarrow$ *hyperbolic.*
Question(existence of local solution): for any smooth K *in* B_1*, does there exist a smooth solution* u*? (Possibly in smaller* B_r*).*

Analysis

1. $K(0) > 0$, elliptic(solved).
2. $K(0) < 0$, hyperbolic(solved).
3. $K(0) = 0$, unknown.

Results

1. $K \geq 0$, solved.
2. $K \leq 0$, solved with stability requirement.
3. K mixed sign.
   (a) $K(x, y) \approx y$, solved.
   (b) $K(x, y) \approx x^2 - y^2$, solved.

One-dimensional hyperbolic equation, Cauchy problem($\mathrm{R}^2 = \{(x, t)\}$)

$$\begin{cases} u_{tt} - a(x, t)u_{xx} + b_0u_t + b_1u_x + cu = f & \text{in } \mathbb{R} \times (0, T) \\ u|_{t=0} = u_0, \ u_t|_{t=0} = u_1 & \text{on } \mathbb{R} \end{cases}$$

Strict hyperbolic: $a \geq a_0 > 0 \Rightarrow$ well-posed.
Degenerate hyperbolic: $a \geq 0$.

Example 3 (1983)**.** $\exists a = a(t)$ *smooth,* $\exists u_0, u_1$ *smooth,*

$$\begin{cases} u_{tt} - a(t)u_{xx} = 0 & \text{in } \mathbb{R} \times (0, T) \\ u|_{t=0} = u_0, \ u_t|_{t=0} = u_1 & \text{on } \mathbb{R} \end{cases}$$

Conjecture 1. *If* $a = a(x, t)$ *analytic in* t*, and smooth in* x*, well-posed holds.*

A strategy:

$$a(x,t) = t^m + c_1(x)t^{m-1} + \cdots + c_m(x) \geq 0 \quad \text{in } \mathbb{R}\times(-T,T),$$

where $c_1, \cdots, c_m$ smooth in x, m even.

Known: $m = 2$ and $m = 4$.

Back to hyperbolic equation

$$u_{tt} - au_{xx} = f.$$

Recall elliptic equation. We multiplied both sides of

$$a_{ij}u_{ij} + b_i u_i + cu = f$$

by u.

Now we multiply both sides of hyperbolic equation by $u_t \mathrm{e}^{-\mu t}$.

$$\mathrm{e}^{-\mu t}u_t u_{tt} = \mathrm{e}^{-\mu t}\left(\frac{u_t^2}{2}\right)_t = \left(\mathrm{e}^{-\mu t}\frac{u_t^2}{2}\right)_t + \mu \mathrm{e}^{-\mu t}\frac{u_t^2}{2}$$

$$-\mathrm{e}^{-\mu t}au_t u_{xx} = -\left(\mathrm{e}^{-\mu t}au_t u_x\right)_x + \mathrm{e}^{-\mu t}a_x u_t u_x + \mathrm{e}^{-\mu t}au_{xt}u_x$$

Notice

$$u_{xt}u_x = \left(\frac{u_x^2}{2}\right)_t.$$

Then

$$-\mathrm{e}^{-\mu t}au_t u_{xx} = -\left(\mathrm{e}^{-\mu t}au_t u_x\right)_x + \left(\mathrm{e}^{-\mu t}a\frac{u_x^2}{2}\right)_t + \mu \mathrm{e}^{-\mu t}a\frac{u_x^2}{2} - \mathrm{e}^{-\mu t}a_t\frac{u_x^2}{2} + \mathrm{e}^{-\mu t}a_x u_x u_t$$

$$\Rightarrow \frac{1}{2}\left(\mathrm{e}^{-\mu t}u_t^2 + \mathrm{e}^{-\mu t}au_x^2\right)_t - \left(\mathrm{e}^{-\mu t}au_t u_x\right)_x + \frac{1}{2}\mu \mathrm{e}^{-\mu t}\left(u_t^2 + au_x^2\right) - \frac{1}{2}\mathrm{e}^{-\mu t}a_t u_x^2 + \mathrm{e}^{-\mu t}a_x u_t u_x = \mathrm{e}^{-\mu t}u_t f.$$

8.1 Strict hyperbolicity

Strict hyperbolicity: $a \geq a_0 > 0$.

Cauchy inequality

$$\left|\mathrm{e}^{-\mu t}a_x u_t u_x\right| \leq \mathrm{e}^{-\mu t}|a_x|\frac{1}{2}\left(u_x^2 + u_t^2\right).$$

$$\Rightarrow (\cdots)_t + (\cdots)_x + \frac{1}{2}\mu \mathrm{e}^{-\mu t}\left(u_t^2 + au_x^2\right) \leq \frac{1}{2}\mathrm{e}^{-\mu t}\left(|a_t|u_x^2 + |a_x|u_x^2 + |a_x|u_t^2 + u_t^2\right) + \frac{1}{2}\mathrm{e}^{-\mu t}f^2$$

Take

$$\mu = \frac{1}{a_0}|\nabla a|_{L^\infty} + |a_x|_{L^\infty} + 2.$$

$$\Rightarrow (\cdots)_t + (\cdots)_x + \frac{1}{2}\mathrm{e}^{-\mu t}\left(u_t^2 + a_0 u_x^2\right) \leq \frac{1}{2}\mathrm{e}^{-\mu t}f^2$$

Integrate in $\mathbb{R}\times(0,T)$,

$$\int_{\mathbb{R}\times\{T\}} \mathrm{e}^{-\mu t}\left(u_t^2 + au_x^2\right) + \int_{\mathbb{R}\times(0,T)} \mathrm{e}^{-\mu t}\left(u_t^2 + a_0 u_x^2\right) \leq \int_{\mathbb{R}\times\{0\}}\left(u_t^2 + au_x^2\right) + \int_{\mathbb{R}\times(0,T)} \mathrm{e}^{-\mu t}f^2.$$

Key: the positive lower bound for a.

8.2 Degenerate hyperbolicity

Degenerate hyperbolicity: $a \geq 0$.

$$(\cdots)_t - 2(\cdots)_x + \mu \mathrm{e}^{-\mu t}\left(u_t^2 + a u_x^2\right) = \mathrm{e}^{-\mu t} a_t u_x^2 - 2\mathrm{e}^{-\mu t} a_x u_t u_x + 2\mathrm{e}^{-\mu t} u_t f$$

$$\left|2\mathrm{e}^{-\mu t} a_x u_t u_x\right| \leq \mathrm{e}^{-\mu t}\left(u_t^2 + a_x^2 u_x^2\right) \leq \mathrm{e}^{-\mu t}\left(u_t^2 + C_* a u_x^2\right)$$

Lemma 7. *Suppose $h = h(x) \geq 0 \in C^2(\mathbb{R})$. Then*

$$\left|\left(\sqrt{h}\right)'(x)\right| = \frac{h'}{2\sqrt{h}} \leq C\left(\left|\nabla^2 h\right|_{L^\infty}\right).$$

Proof. Use Taylor expansion. $\square$

$$\Rightarrow (\cdots)_t - 2(\cdots)_x + (\mu - \mu_0)\mathrm{e}^{-\mu t}\left(u_t^2 + a u_x^2\right) \leq \mathrm{e}^{-\mu t} a_t u_x^2 + \mathrm{e}^{-\mu t} f^2.$$

If we have $a_t \leq C a$,

$$\Rightarrow (\cdots)_t - 2(\cdots)_x + (\mu - \mu_0^*)\mathrm{e}^{-\mu t}\left(u_t^2 + a u_x^2\right) \leq \mathrm{e}^{-\mu t} f^2.$$

However, it's impossible to satisfy $\frac{a_t}{a} \leq C$. See

Example 4.

$$a(t) = \prod_{i=1}^{m}(t - t_i)$$

$$\Rightarrow \frac{a_t}{a} = \sum_{i=1}^{m} \frac{1}{t - t_i},$$

which is ∞ for $t = t_i$.

Introduce weight function $w = w(x,t)$ and multiply both sides of

$$u_{tt} - a u_{xx} = f$$

by $w u_t \mathrm{e}^{-\mu t}$.

$$\mathrm{e}^{-\mu t} w u_t u_{tt} = \mathrm{e}^{-\mu t} w \left(\frac{u_t^2}{2}\right)_t = \left(\mathrm{e}^{-\mu t} w \frac{u_t^2}{2}\right)_t + \mu \mathrm{e}^{-\mu t} w \frac{u_t^2}{2} - \mathrm{e}^{-\mu t} w_t \frac{u_t^2}{2}$$

$$-\mathrm{e}^{-\mu t} a w u_t u_{xx} = -\left(\mathrm{e}^{-\mu t} a w u_t u_x\right)_x + \mathrm{e}^{-\mu t} a w u_{xt} u_x + \mathrm{e}^{-\mu t}(aw)_x u_t u_x$$

Notice

$$u_{xt} u_x = \left(\frac{u_x^2}{2}\right)_t.$$

Then

$$-\mathrm{e}^{-\mu t}awu_tu_{xx}=-\left(\mathrm{e}^{-\mu t}awu_tu_x\right)_x+\left(\mathrm{e}^{-\mu t}aw\frac{u_x^2}{2}\right)_t+\mu\mathrm{e}^{-\mu t}aw\frac{u_x^2}{2}-\mathrm{e}^{-\mu t}(aw)_t\frac{u_x^2}{2}+\mathrm{e}^{-\mu t}(aw)_xu_tu_x.$$

$$\begin{aligned}&\left(\mathrm{e}^{-\mu t}\left(wu_t^2+wau_x^2\right)\right)_t-2\left(\mathrm{e}^{-\mu t}awu_tu_x\right)_x+\mu\mathrm{e}^{-\mu t}w\left(u_t^2+au_x^2\right)\\&=\mathrm{e}^{-\mu t}w_tu_t^2+\mathrm{e}^{-\mu t}(aw)_tu_x^2-2\mathrm{e}^{-\mu t}(aw)_xu_tu_x+2\mathrm{e}^{-\mu t}wu_tf=RHS\\&RHS\le\mathrm{e}^{-\mu t}\left(\frac{w_t}{w}wu_t^2+\frac{(wa)_t}{wa}wau_x^2+2(aw)_xu_tu_x+wu_t^2+wf^2\right)\end{aligned}$$

Notice

$$2(aw)_xu_tu_x=2\frac{(aw)_x}{w\sqrt{a}}\left(\sqrt{w}u_t\right)\left(\sqrt{wa}u_x\right)\le\frac{|(aw)_x|}{w\sqrt{a}}wu_t^2+\frac{|(aw)_x|}{w\sqrt{a}}wau_x^2,$$

and

$$\frac{(aw)_x}{w\sqrt{a}}=\frac{a_xw}{w\sqrt{a}}+\frac{aw_x}{w\sqrt{a}}=\frac{a_x}{\sqrt{a}}+\frac{w_x}{w}\sqrt{a},$$

where $\frac{a_x}{\sqrt{a}}$ is bounded. Then

$$\begin{aligned}RHS\le\mathrm{e}^{-\mu t}\Bigg(&\frac{w_t}{w}wu_t^2+\frac{(wa)_t}{wa}wau_x^2+\frac{|w_x|}{w}\sqrt{a}wu_t^2+\frac{|w_x|}{w}\sqrt{a}wau_x^2\\&+\frac{|a_x|}{\sqrt{a}}wu_t^2+\frac{|a_x|}{\sqrt{a}}wau_x^2+wu_t^2+wf^2\Bigg),\end{aligned}$$

where terms in first line in bracket are uncontrolled while terms in second line are controlled. Require

$$\begin{cases}\dfrac{w_t}{w}+\dfrac{|w_x|}{w}\sqrt{a}\le C_1\\ \dfrac{(wa)_t}{wa}+\dfrac{|w_x|}{w}\sqrt{a}\le C_2\end{cases}.\tag{$*$}$$

Lemma 8. *Assume (*) holds. Then*

$$\int_{\mathbb{R}\times\{T\}}\mathrm{e}^{-\mu t}w\left(u_t^2+au_x^2\right)+(\mu-\mu_0)\int_{\mathbb{R}\times(0,T)}\mathrm{e}^{-\mu t}\left(wu_t^2+wau_x^2\right)$$

$$\le\int_{\mathbb{R}\times\{0\}}w\left(u_t^2+au_x^2\right)+\int_{\mathbb{R}\times(0,T)}\mathrm{e}^{-\mu t}wf^2.$$

Example 5. $a\ge 0,\partial_ta\ge 0$. *(e.g.* $a(x,t)=t^m$ *in* $\mathbb{R}\times(0,T)$*) Take* $w=\frac{1}{a}$.

$$\Rightarrow\frac{w_x}{w}=\frac{-\frac{a_x}{a^2}}{\frac{1}{a}}=-\frac{a_x}{a}$$

$$\Rightarrow \frac{|w_x|}{w}\sqrt{a} = \frac{|a_x|}{\sqrt{a}},$$

which is bounded.

$$w_t = -\frac{a_t}{a^2} \le 0, wa = 1.$$

Hence (*) is verified.

$$\frac{(wa)_t}{wa} = \frac{w_t}{w} + \frac{a_t}{a}$$

$$a = \prod_{i=1}^{m}(t - t_i) \Rightarrow \frac{a_t}{a} = \sum_{i=1}^{m}\frac{1}{t - t_i}$$

Consider $a(x,t) = t^m + c_1(x)t^{m-1} + \cdots + c_m(x)$ *in* $\mathbb{R}\times(0,T)$. *Study the set* $\{a(x,t) = 0\}$. *Fix* x, $a(x,t) = 0$ *has* m *(complex) zeros. For* m *continuous functions* $t = t_i(x), i = 1, \cdots, m$, $a(x, t_i(x)) = 0$.

Question:

1. $t_i = t_i(x) \in C^{\frac{1}{m}}(\mathbb{R}) \Rightarrow$ *Sobolev embedding. (solved)*
2. $t_i = t_i(x) \in BV(\mathbb{R}) \Rightarrow$ *Integration by parts. (unsolved)*

Theorem 6. *If* a *has no complex roots, then* w *can be chosen to satisfy (*) and hence energy estimates hold.*

Theorem 7. *Same if* a *has at most one pair of complex roots.*

Theorem 8. *Same if* $\deg a = 4$.

8.3 Mixed type

$$u_{tt} + tu_{xx} = f \qquad (Tricomi)$$

Construct a smooth solution in B_1.

Step 1. $B_1^+ \subset \Omega_+$. Solve

$$\begin{cases} u_{tt} + tu_{xx} = f & \text{in } \Omega_+ \\ u = 0 & \text{on } \partial\Omega_+. \end{cases}$$

Degenerate elliptic on non-characteristic $x = 0$.

$$\Rightarrow u^+ \in C^\infty\left(\overline{\Omega_+}\right)$$

Step 2.

$$\begin{cases} u_{tt} + tu_{xx} = f & \text{in } \mathbb{R}\times(-T,0) \\ u|_{t=0} = 0, u_t|_{t=0} = \left.\frac{\partial u^+}{\partial t}\right|_{t=0} \end{cases}$$

Degenerate hyperbolic.

$$\Rightarrow u^- \in C^\infty(\mathbb{R}\times[-T,0])$$

$$\Rightarrow u \in C^\infty(B_1)$$

Same method applies $u_{tt} + (x^2 - t^2)\, u_{xx} = f$. Difficulty: smoothness at the origin.