Introduction to von Neumann Algebra

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January 27, 2016

History of von Neumann algebra: von Neumann, Murray, *On rings of operators. I,II,III,IV*; von Neumann, *On rings of operators. Reduction theory.* It comes from ergodic theory and group algebra.

I_n	$M_n(\mathbb{C})$	$\{1, \cdots, n\}$
I_∞	$\mathcal{B}(H), \dim H < \infty$	$\{1, \cdots\}$
II_1		$[0, 1]$
II_∞	$II_\infty = II_1 \otimes I_\infty$	$\mathbb{R}$
III		$\{\infty\}$

G is a countable discrete group with identity e. Consider Hilbert space $\ell^2(G) = \{\sum_{g\in G} \alpha_g g : \alpha_g \in \mathbb{C}, \sum_{g\in G} |\alpha_g|^2 < \infty\}$ with inner product defined as $\left(\sum_{g\in G} \alpha_g g, \sum_{g'\in G} \beta_{g'} g'\right) = \sum_{g\in G} \alpha_g \overline{\beta_g}$. Observe that $\{\delta_g = g : g \in G\}$ is a family of orthonormal basis of $\ell^2(G)$. Consider left regular representation of G on $\ell^2(G)$. $\forall g \in G, L_g\delta_h = \delta_{gh}$. Thus L_g can be extended linearly to be a unitary operator on $\ell^2(G)$, s.t.

1. $g = 4, L_e = I$;
2. $L_{g_1g_2}\delta_h = \delta_{g_1g_2h} = L_{g_1}\delta_{g_2h} = L_{g_1}L_{g_2}\delta_h \Rightarrow L_{g_1g_2} = L_{g_1}L_{g_2}$;
3. $(L_g)^* = L_{g^{-1}}$.

$(\delta_h, (L_g)^*\delta_k) = (L_g\delta_h, \delta_k) = (\delta_{gh}, \delta_k) = \begin{cases} 1, & gh = k \\ 0, & gh \neq k \end{cases}$;

$(\delta_h, L_{g^{-1}}\delta_k) = (\delta_h, \delta_{g^{-1}k}) = \begin{cases} 1, & h = g^{-1}k \\ 0, & h \neq g^{-1}k \end{cases}$.

Thus $g \mapsto L_g$ is a unitary representation of G on $\ell^2(G)$, called a left regular representation of G.

Definition 1. *Group von Neumann algebra $\mathcal{L}(G)$ is the minimal von Neumann algebra containing $\{L_g : g \in G\}$ in $\mathcal{B}(\ell^2(G))$, i.e. $\mathcal{L}(G) = \overline{\mathrm{span}\{L_g : g \in G\}}^{SOT}$. The superscirpt SOT means strong operator topology.*

Remark: $\mathcal{C}_r^*(G) = \overline{\text{span}\{L_g : g \in G\}}^{\|\cdot\|}$.

We also have right regular representation, $\forall g \in G, R_g\delta_h = \delta_{hg^{-1}}$.

1. $R_e = I$;

2. $R_{g_1g_2} = R_{g_1}R_{g_2}$;

3. $(R_g)^* = R_{g^{-1}}$.

$\mathcal{R}(G) = \overline{\text{span}\{R_g : g \in G\}}^{SOT}$.

Observe $\forall g_1, g_2 \in G, L_{g_1}R_{g_2}\delta_h = L_{g_1}\delta_{hg_2^{-1}} = \delta_{g_1hg_2^{-1}} = R_{g_2}\delta_{g_1h} = R_{g_2}L_{g_1}\delta_h \Rightarrow L_{g_1}R_{g_2} = R_{g_2}L_{g_1}$. Thus $\mathcal{L}(G) \subset \mathcal{R}(G)', \mathcal{R}(G) \subset \mathcal{L}(G)'.(M \subset \mathcal{B}(H), M' = \{S \in \mathcal{B}(H) : ST = TS, \forall T \in M\}$. M' is called the commutator algebra of M).

But what does an element in $\mathcal{L}(G)$ look like? $\forall x = \sum_{g\in G} \alpha_g g \in \ell^2(G), L_x\delta_h = \sum_{g\in G} \alpha_g\delta_{gh} \in \ell^2(G)$. L_x is a densely defined unbounded operator. $\mathcal{D}(L_x) \supset \mathbb{C}(G) = \{\sum_{g\in G} \alpha_g\delta_g : \alpha_g\ has\ finitely\ many\ nonzero\ terms\}$. $(L_x)^* = L_{x^*}, x^* = \sum_{g\in G} \overline{\alpha_g}g^{-1} \in \ell^2(G)$.

Theorem 1. *$\mathcal{L}(G) = \{L_x : x \in \ell^2(G), L_x$ can be continued to be a bounded operator on $\ell^2(G)\}$*

Proof. 1. $\forall T \in \mathcal{L}(G)$, let $x = T\delta_e \in \ell^2(G)$. $\forall h, k \in G$,

$$\begin{aligned}&(T\delta_h, \delta_k)\\ =&(TR_{h^{-1}}\delta_e, \delta_k)\\ =&(R_{h^{-1}}T\delta_e, \delta_k)\\ =&(R_{h^{-1}}x, \delta_k)\\ =&\left(\sum_{g\in G}\alpha_g gh, \delta_k\right)\\ =&(L_x\delta_h, \delta_k).\end{aligned}$$

Thus $T\delta_h = L_x\delta_h, \forall h \in G$, that is to say, L_x can be continued to be a bounded operator and $T = L_x$.

2. $\forall S \in \mathcal{L}(G)'$,

$$\begin{aligned}&(L_xS\delta_h, \delta_k)\\ =&(S\delta_h, L_{x^*}\delta_k)\\ =&\left(S\delta_h, \sum_{g\in G}\overline{\alpha_g}\delta_{g^{-1}k}\right)\\ =&\left(\delta_h, S^*\left(\sum_{g\in G}\overline{\alpha_g}\delta_{g^{-1}k}\right)\right)\\ =&\sum_{g\in G}(\delta_h, \overline{\alpha_g}S^*\delta_{g^{-1}k})\end{aligned}$$

$$
\begin{aligned}
&= \sum_{g\in G} (\delta_h, \overline{\alpha_g} S^* L_{g^{-1}} \delta_k) \\
&= \sum_{g\in G} (\delta_h, \overline{\alpha_g} L_{g^{-1}} S^* \delta_k) \\
&= \sum_{g\in G} (\alpha_g L_g \delta_h, S^* \delta_k) \\
&= \sum_{g\in G} (S \alpha_g L_g \delta_h, \delta_k) \\
&= \left(S \left(\sum_{g\in G} \alpha_g L_g \right) \delta_h, \delta_k \right) \\
&= (S L_x \delta_h, \delta_k).
\end{aligned}
$$

Thus $L_x S = S L_x$. According to von Neumann bicommutant theorem, $L_x \in \mathcal{L}(G)$. □

Theorem 2 (von Neumann bicommutant)**.** *If $M \subset \mathcal{B}(H)$ is a *-sublgebra with an identity, then $M'' = \overline{M}^{\mathrm{SOT}}$.*

Theorem 3. *$(\mathcal{L}(G))' = \mathcal{R}(G), (\mathcal{R}(G))' = \mathcal{L}(G)$.*

Proof. We only need to prove $(\mathcal{R}(G))' \subset \mathcal{L}(G)$. $\forall T \in (\mathcal{R}(G))'$, let $x = T\delta_e = \sum_{g\in G} \alpha_g g \in \ell^2(G)$. $T\delta_h = TR_{h^{-1}}\delta_e = R_{h^{-1}}T\delta_e = R_{h^{-1}}x = \sum_{g\in G} \alpha_g gh = L_x\delta_h$. Thus $T = L_x \in \mathcal{L}(G)$. □

Example 1. *$G = \mathbb{Z} = \{g^n : n \in \mathbb{Z}\}$. $e_n = \delta_{g^n}$. $L_g e_n = L_g \delta_{g^n} = \delta_{g^{n+1}} = e_{n+1}, \forall n \in \mathbb{Z}$. $\mathcal{L}(\mathbb{Z})$ is the von Neumann algebra generated by L_g. $\ell^2(\mathbb{Z}) \to \ell^2(S^1, m)$, where m is Haar measure and s.t. $e_n \overset{U}{\mapsto} z^n$. $UL_gU^*z^n = UL_ge_n = Ue_{n+1} = z^{n+1}$. $UL_gU^* = M_z$. $U\mathcal{L}(\mathbb{Z})U^* = \{M_{f(z)} : f(z) \in L^\infty(S^1, m)\}$. We can define a linear functional on $\mathcal{L}(G)$ s.t. $\forall T \in \mathcal{L}(G), \tau(T) = (T\delta_e, \delta_e)$.*

Theorem 4. *τ on $\mathcal{L}(G)$ is subject to the following,*

1. *$\tau(I) = 1$;*
2. *$\tau(ST) = \tau(TS), \forall S, T \in \mathcal{L}(G)$;*
3. *$\tau(T^*T) \geq 0$ and $\tau(T^*T) = 0 \Rightarrow T = 0$.*

Proof. Let $x = T\delta_e = \sum_{g\in G} \alpha_g g \in \ell^2(G), y = S\delta_e = \sum_{g\in G} \beta_g g \in \ell^2(G)$. $\tau(ST) = (ST\delta_e, \delta_e) = (T\delta_e, S^*\delta_e) = \left(\sum_{g\in G} \alpha_g g, \sum_{g\in G} \overline{\beta_g} g^{-1}\right) = \sum_{g\in G} \alpha_g \beta_{g^{-1}}$. $\tau(TS) = \sum_{g\in G} \alpha_{g^{-1}} \beta_g$. $\tau(T^*T) = 0 \Rightarrow (T\delta_e, T\delta_e) = 0$. Let $x = T\delta_e, (x, x) = 0 \Rightarrow x = 0$. $T = L_x = 0$. □

Definition 2. *Let M to be a von Neumann algebra, then the center of M, $Z(M) = \{S \in M : ST = TS, \forall T \in M\}$.*

Obviously, $Z(M) \supset \{\lambda I : \lambda \in \mathbb{C}\}$.

Definition 3. *M is called a factor if $Z(M) = \mathbb{C}I$.*

Definition 4. *M is called a II_1 factor if M is a infinite dimensional factor and there is a bounded linear functional τ on M s.t.*

1. *$\tau(T^*T) \geq 0, \forall T \in M$, $\tau(T^*T) = 0 \Rightarrow T = 0$;*
2. *$\tau(ST) = \tau(TS), \forall S, T \in M$;*
3. *τ is continuous under strong operator topology.*

Definition 5. *A discrete group G is called an i.c.c. (infinite conjugacy class) group if $\forall g \neq e$, $\{hgh^{-1} : h \in G\}$ is an infinite set.*

Theorem 5. *$\mathcal{L}(G)$ is called a II_1 factor if G is an i.c.c. group.*

Proof. $\forall T \in Z(\mathcal{L}(G))$, let $x = T\delta_e = \sum_{g\in G} \alpha_g g \in \ell^2(G)$. Notice that $\forall h \in G, L_h T = TL_h$. $L_h T L_{h^{-1}} = T \Rightarrow L_h T L_{h^{-1}}\delta_e = T\delta_e = \sum_{g\in G} \alpha_g g$. $L_h T \delta_{h^{-1}} = L_h \sum_{g\in G} \alpha_g g h^{-1} = \sum_{g\in G} \alpha_g hgh^{-1} \xlongequal{hgh^{-1}=g'} \sum_{g'\in G} \alpha_{h^{-1}g'h} g' \sum_{g\in G} \alpha_{h^{-1}gh} g$. $\alpha_{h^{-1}gh} = \alpha_g, \forall g, h \in G$. If $g \neq e$ and $\{h^{-1}gh : h \in G\}$ is infinite, then $\sum_{h\in G} |\alpha_{h^{-1}gh}|^2 \leq \sum_{k\in G} |\alpha_k|^2 < \infty$. Thus $\alpha_g = 0, \forall g \neq e$. $T\delta_e = \alpha_e e \Rightarrow T = L_{\alpha_e} e = \alpha_e I$. □

Example 2. *$F_2 =< a, b >$ is i.c.c.*

Example 3. *$\pi(\mathbb{Z}) = \{permutations\ of\ \mathbb{Z}\ that\ change\ at\ most\ finitely\ many\ positions\}$.*

Theorem 6 (Murray-von Neumann). *$\mathcal{L}(F_2) \ncong \mathcal{L}(\pi(\mathcal{Z}))$.*

Definition 6. *Let (M, τ) to be a II_1 factor. M is said to have property Γ if $\forall \varepsilon > 0, \forall x_1, \cdots, x_n \in M, \exists U \in M, \tau(U) = 0$, and $\|x_i U - U x_i\|_2 < \varepsilon, i = 1, 2, \cdots, n$, where $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$. (Notice that for any unitary operator v, $\|vx\|_2 = \|xv\|_2 = \|x\|_2$.)*

Theorem 7. *$\mathcal{L}(\pi(\mathcal{Z}))$ has property Γ.*

Theorem 8. *$\mathcal{L}(F_2)$ doesn't have property Γ.*

$F_2 =< a, b >$. For L_a, L_b, take $\varepsilon > 0$ small enough($\frac{1}{24}$), $\exists U, \tau(U) = 0$ and ① $\|L_a U - UL_a\|_2 < \varepsilon$, ② $\|L_b U - UL_b\|_2 < \varepsilon$. Let $x = U\delta_e = \sum_{g\in F_2\backslash\{e\}} \alpha_g g \in \ell^2(G)(\alpha_e = 0)$. As U is a unitary operator, $\sum_{g\in F_2\backslash\{e\}} |\alpha_g|^2 = 1(\|U\|_2 = 1)$. ① $\Rightarrow \|L_a U L_{a^{-1}} - U\|_2^2 < \varepsilon^2 \Rightarrow \sum_{g\in F_2\backslash\{e\}} |\alpha_{a^{-1}ga} - \alpha_g|^2 < \varepsilon^2 \Rightarrow \sum_{g\in S} |\alpha_g|^2 - \sum_{g\in S} |\alpha_{a^{-1}ga}|^2 < 2\varepsilon$, and $\|L_{a^{-1}} U L_a - U\|_2^2 < \varepsilon^2 \Rightarrow \sum_{g\in F_2\backslash\{e\}} |\alpha_{aga^{-1}} - \alpha_g|^2 < \varepsilon^2 \Rightarrow \sum_{g\in S} |\alpha_g|^2 - \sum_{g\in S} |\alpha_{aga^{-1}}|^2 < 2\varepsilon$. ② $\Rightarrow \|L_b U L_{b^{-1}} - U\|_2^2 < \varepsilon^2 \Rightarrow \sum_{g\in F_2\backslash\{e\}} |\alpha_{b^{-1}gb} - \alpha_g|^2 < \varepsilon^2$. Consider $S = \{reduced\ words\ starting\ with\ b^{\pm 1}\} \subset F_2$. Observe $aSa^{-1} \cup a^{-1}Sa \cap S \subset F_2\backslash\{e\}$, $b^{-1}Sb \cup S \supset F_2\backslash\{e\}$, which is disjoint pairwise. $\sum_{g\in S} |\alpha_g|^2 + \sum_{g\in S} |\alpha_{aga^{-1}}|^2 + \sum_{g\in S} |\alpha_{a^{-1}ga}|^2 \leq 1 \approx 3\sum_{g\in S} |\alpha_g|^2 + 4\varepsilon \leq 1$, $\sum_{g\in S} |\alpha_g|^2 + \sum_{g\in S} |\alpha_{b^{-1}gb}|^2 \approx 2\sum_{g\in S} |\alpha_g|^2 - 2\varepsilon \geq 1$. Thus $\frac{1}{2} \leq \sum_{g\in S} |\alpha_g|^2 \leq \frac{1}{3}$, which is contradicted.

Question 1. *Is it true that $\mathcal{L}(F_2) \ncong \mathcal{L}(F_3)$?*

Recent result is shown below.

Theorem 9. *It is alternative that $\mathcal{L}(F_n) \cong \mathcal{L}(F_m), \forall n, m$ or $\mathcal{L}(F_n) \ncong \mathcal{L}(F_m), \forall n, m$.*