一个简单而有♂趣的函数

定义函数

$$\Omega_{n+1}(t) = \prod_{i=0}^{n}(t-i)$$

1 极值点

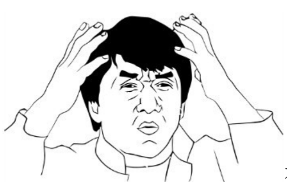
$$\Omega'_{n+1}(t) = \sum_{k=0}^{n} \frac{\Omega_{n+1}(t)}{t-k}$$

令$\Omega'_{n+1}(t) = 0$可求得极值点，但是该怎么解呢?

2 导数

利用$Mathematica$，我对$\Omega_{n+1}(t)$求了前12阶导，归纳出了下面的式子

$$\Omega_{n+1}^{(k)}(t) = \sum_{i=0}^{n}\sum_{j=1}^{k} \frac{(-1)^{j+1}(k-1)!\Omega_{n+1}^{k-j}(t)}{(k-j)!(t-i)^j}$$

看起来相当给力啊

3 积分

$$
\begin{aligned}
\int_0^n \Omega_{n+1}(t)\,\mathrm{d}t &= \int_0^n [t-0][t-1][t-2]\dots[t-(n-1)][t-n]\,\mathrm{d}t \\
&= t\Omega_{n+1}(t)\Big|_0^n - \int_0^n t\Omega'_{n+1}(t)\,\mathrm{d}t \\
&= t\Omega_{n+1}(t)\Big|_0^n - \left(\frac{t^2}{2}\Omega'_{n+1}(t)\Big|_0^n - \int_0^n \frac{t^2}{2}\Omega''_{n+1}(t)\,\mathrm{d}t\right) \\
&= t\Omega_{n+1}(t)\Big|_0^n - \frac{t^2}{2}\Omega'_{n+1}(t)\Big|_0^n + \int_0^n \frac{t^2}{2}\Omega''_{n+1}(t)\,\mathrm{d}t \\
&= t\Omega_{n+1}(t)\Big|_0^n - \frac{t^2}{2}\Omega'_{n+1}(t)\Big|_0^n + \frac{t^3}{6}\Omega''_{n+1}(t)\Big|_0^n - \int_0^n \frac{t^3}{6}\Omega'''_{n+1}(t)\mathrm{d}t \\
&= \dots \\
&= t\Omega_{n+1}(t)\Big|_0^n - \frac{t^2}{2!}\Omega'_{n+1}(t)\Big|_0^n + \frac{t^3}{3!}\Omega''_{n+1}(t)\Big|_0^n + \dots + (-1)^{n+1}\frac{t^{n+2}}{(n+2)!}\Omega^{n+1}_{n+1}(t)\Big|_0^n \\
&= \left[t\Omega_{n+1}(t) - \frac{t^2}{2!}\Omega'_{n+1}(t) + \frac{t^3}{3!}\Omega''_{n+1}(t) + \dots + (-1)^{n+1}\frac{t^{n+2}}{(n+2)!}\Omega^{n+1}_{n+1}(t)\right]\Bigg|_0^n
\end{aligned}
$$

当然，我们可以定义函数

$$
\begin{aligned}
f(t) &= t\Omega_{n+1}(t) - \frac{t^2}{2!}\Omega'_{n+1}(t) + \frac{t^3}{3!}\Omega''_{n+1}(t) + \dots + (-1)^{n+1}\frac{t^{n+2}}{(n+2)!}\Omega^{n+1}_{n+1}(t) \\
&= \sum_{i=1}^{n+2}\frac{(-1)^{i+1}t^i}{i!}\Omega^{(i-1)}_{n+1}(t)
\end{aligned}
$$

然后求出$f(n)-f(0)$即可，但事实上我连$\Omega^{(k)}_{n+1}(0)$和$\Omega^{(k)}_{n+1}(n)$都不会求
直观上看，$\Omega^{(k)}_{n+1}$是一个$n-k+1$次多项式，其各项均为一个累乘式，对它求导得到$\Omega^{(k+1)}_{n+1}$，它是$\Omega^{(k)}_{n+1}$中各累乘式中去掉一项的全组合。