1 $Jacobi$迭代法的收敛条件

若迭代格式系数矩阵为

$$A = \begin{pmatrix} 1 & \alpha & \dots & \alpha \\ \alpha & 1 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & 1 \end{pmatrix}$$

则其$Jacobi$方法的迭代矩阵为

$$B = \begin{pmatrix} 0 & \alpha & \dots & \alpha \\ \alpha & 0 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \alpha & \dots & 0 \end{pmatrix}$$

解其特征方程

$$\begin{aligned} \begin{vmatrix} \lambda & -\alpha & \dots & -\alpha \\ -\alpha & \lambda & \dots & -\alpha \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha & -\alpha & \dots & \lambda \end{vmatrix} &= \begin{vmatrix} \lambda-(n-1)\alpha & \lambda-(n-1)\alpha & \dots & \lambda-(n-1)\alpha \\ -\alpha & \lambda & \dots & -\alpha \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha & -\alpha & \dots & \lambda \end{vmatrix} \\ &= \begin{vmatrix} \lambda-(n-1)\alpha & 0 & \dots & 0 \\ -\alpha & \lambda+\alpha & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\alpha & 0 & \dots & \lambda+\alpha \end{vmatrix} \\ &= [\lambda-(n-1)\alpha](\lambda+\alpha)^{n-1} = 0 \end{aligned}$$

显然$-\alpha$是其$n-1$重特征值，还有一个特征值为$(n-1)\alpha$，显然矩阵谱半径为$|(n-1)\alpha|$，要想让$Jacobi$迭代法收敛，只需令

$$\rho(B) = |(n-1)\alpha| < 1$$

即可，解得

$$|\alpha| < \frac{1}{n-1}$$

这与令系数矩阵A严格对角占优，即

$$(n-1)|\alpha| < 1$$

得到的结果相同

2 条件数

证明$Cond_2(A^\top A) \neq [Cond_2(A)]^2$

$$\begin{aligned} Cond_2 &= ||A^\top A||_2||(A^\top A)^{-1}||_2 \\ &= ||A^\top A||_2||A^{-1}(A^\top)^{-1}||_2 \\ &= ||A^\top A||_2||A^{-1}(A^{-1})^\top||_2 \\ &= \sqrt{\rho((A^\top A)^2)}\sqrt{\rho((A^{-1}(A^{-1})^\top)^2)} \\ &= \sqrt{\rho^2(A^\top A)}\sqrt{\rho^2(A^{-1}(A^{-1})^\top)} \\ &= \rho(A^\top A)\rho(A^{-1}(A^{-1})^\top) \end{aligned}$$

$$\begin{aligned} [Cond_2(A)]^2 &= [||A||_2||A^{-1}||_2]^2 \\ &= \rho(A^\top A)\rho((A^{-1})^\top A^{-1}) \end{aligned}$$

由于$\rho(A^{-1}(A^{-1})^\top) \neq \rho(A^{-1}(A^{-1})^\top)$，故得证