正则方程组系数矩阵

如果向量组$\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1, ..., \boldsymbol{\varphi}_n$线性无关，则正则方程组的系数矩阵是对称正定矩阵。

证明

正则方程组系数矩阵:

$$A = \begin{pmatrix} (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_n) \\ (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_n) \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_n, \boldsymbol{\varphi}_n) \end{pmatrix}$$

下面对n归纳证明A对称正定，即$\forall \boldsymbol{x} \neq \mathbf{0}, \boldsymbol{x}^\top A \boldsymbol{x} > 0$

①$n = 1, \boldsymbol{x}_0 (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) \boldsymbol{x}_0 > 0$显然成立

②假设$n = k$时成立，即

$$\boldsymbol{x}_k^\top A_k \boldsymbol{x}_k > 0$$

其中

$$\boldsymbol{x}_k = \begin{pmatrix} x_1 & x_2 & \ldots & x_k \end{pmatrix}^\top$$

$$A_k = \begin{pmatrix} (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_k) \\ (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_k) \\ \vdots & \vdots & \ddots & \vdots \\ (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_k) \end{pmatrix}$$

$n = k + 1$时

$\boldsymbol{x}_{k+1}^\top A_{k+1} \boldsymbol{x}_{k+1} =$

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_k & x_{k+1} \end{pmatrix} \begin{pmatrix} (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_k) & (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_{k+1}) \\ (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_k) & (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_{k+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_k) & (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_{k+1}) \\ (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_0) & (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_1) & \cdots & (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_k) & (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_{k+1}) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ x_{k+1} \end{pmatrix}$$

记

$$\boldsymbol{\alpha}_k = \begin{pmatrix} (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_{k+1}) \\ (\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_{k+1}) \\ \vdots \\ (\boldsymbol{\varphi}_k, \boldsymbol{\varphi}_{k+1}) \end{pmatrix}$$

$$\boldsymbol{\beta}_k = ((\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_0)\ (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_1)\ \dots\ (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_k))$$

于是上式可写为

$$\begin{pmatrix} \boldsymbol{x}_k^\top & x_{k+1} \end{pmatrix} \begin{pmatrix} A_k & \boldsymbol{\alpha}_k \\ \boldsymbol{\beta}_k & (\boldsymbol{\varphi}_{k+1}, \boldsymbol{\varphi}_{k+1}) \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_k \\ x_{k+1} \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} \boldsymbol{x}^\top A_k + x_{k+1}\boldsymbol{\beta}_k & \boldsymbol{x}_k^\top \boldsymbol{\alpha}_k + x_{k+1} \begin{pmatrix} \boldsymbol{\varphi}_{k+1} & \boldsymbol{\varphi}_{k+1} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_k \\ x_{k+1} \end{pmatrix} \\
&= \boldsymbol{x}_k^\top A_k \boldsymbol{x}_k + x_{k+1}\boldsymbol{\beta}_k \boldsymbol{x}_k + \boldsymbol{x}_k^\top \boldsymbol{\alpha}_k x_{k+1} + x_{k+1} \begin{pmatrix} \boldsymbol{\varphi}_{k+1} & \boldsymbol{\varphi}_{k+1} \end{pmatrix} x_{k+1} \\
&= \boldsymbol{x}_k^\top A_k \boldsymbol{x}_k + 2x_{k+1}\boldsymbol{\alpha}_k^\top \boldsymbol{x}_k + x_{k+1} \begin{pmatrix} \boldsymbol{\varphi}_{k+1} & \boldsymbol{\varphi}_{k+1} \end{pmatrix}
\end{aligned}$$