# 1 三阶$R-K$方法

其他教材上的解法:

$$\begin{cases}\lambda_1+\lambda_2+\lambda_3=1\\ \alpha_2=\beta_{21}\\ \alpha_3=\beta_{31}+\beta_{32}\\ \lambda_2\alpha_2+\lambda_3\alpha_3=\frac{1}{2}\\ \lambda_2\alpha_2^2+\lambda_3\alpha_3^2=\frac{1}{3}\\ \lambda_3\alpha_2\beta_{32}=\frac{1}{6}\end{cases}$$

可取

$$\begin{cases}\lambda_1=\frac{1}{6}\\ \lambda_2=\frac{4}{6}\\ \lambda_3=\frac{1}{6}\\ \alpha_2=\frac{1}{2}\\ \alpha_3=1\\ \beta_{21}=\frac{1}{2}\\ \beta_{31}=-1\\ \beta_{32}=2\end{cases}$$

我的解法:

三阶$R-K$方法

$$\begin{cases}y_{n+1}=y_n+h(\lambda_1K_1+\lambda_2K_2+\lambda_3K_3)\\ K_1=f(x_n,y_n)\\ K_2=f(x_n+\alpha_2h,y_n+h\beta_{21}K_1)\\ K_3=f(x_n+\alpha_3h,y_n+h(\beta_{31}K_1+\beta_{32}K_2))\end{cases}$$

其中

$$K_1=f$$

$$K_2=f+\alpha_2hf_x+h\beta_{21}ff_y+\frac{1}{2}\left[(\alpha_2h)^2f_{xx}+2\alpha_2h\beta_{21}ff_y+(h\beta_{22}f)^2f_{yy}\right]+O(h^3)$$

$$\begin{aligned}K_3=&f+\alpha_3hf_x+h(\beta_{31}f+\beta_{32}K_2)f_y+\\ &\frac{1}{2}\left[(\alpha_3h)^2f_{xx}+2\alpha_3h^2(\beta_{31}f+\beta_{32}K_2)f_{xy}+h^2(\beta_{31}K_1+\beta_{32}K_2)^2f_{yy}\right]+O(h^3)\\ &=f+\alpha_3hf_x+h(\beta_{31}f+\beta_{32}(f+\alpha_2hf_x+h\beta_{21}ff_y))f_y+\\ &\frac{1}{2}\left[(\alpha_3h)^2f_{xx}+2\alpha_3h^2(\beta_{31}f+\beta_{32}f)f_{xy}+h^2(\beta_{31}f+\beta_{32}f)^2f_yy\right]+O(h^3)\end{aligned}$$

$$
\begin{aligned}
y(x_{n+1}) &= y(x_n + h) \\
&= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + O(h^4)
\end{aligned}
$$

而

$$
\begin{aligned}
y' &= f \\
y'' &= f_x + ff_y \\
y''' &= f_{xx} + f_{xy}f + f_y(f_x + f_yf) + f(f_{yx} + f_{yy}f) \\
&= f_{xx} + 2f_{xy}f + f_xf_y + ff_y^2 + f^2f_{yy}
\end{aligned}
$$

比较三阶$R-K$方法第一个等式左右两边，得到

$$
\begin{aligned}
&f: && \lambda_1 + \lambda_2 + \lambda_3 = 1 \\
&f_x: && \lambda_2\alpha_2 + \lambda_3\alpha_3 = \frac{1}{2} \\
&ff_y: && \lambda_2\beta_{21} + \lambda_3\beta_{31} + \lambda_3\beta_{32} = \frac{1}{2} \\
&f_{xx}: && \lambda_2\frac{\alpha_2^2}{2} + \lambda_3\frac{\alpha_3^2}{2} = \frac{1}{6} \\
&ff_{xy}: && \lambda_2\alpha_2\beta_{21} + \lambda_3\alpha_3(\beta_{31} + \beta_{32}) = \frac{1}{3} \\
&f_xf_y: && \lambda_3\alpha_2\beta_{32} = \frac{1}{6} \\
&ff_y^2: && \lambda_3\beta_{21}\beta_{32} = \frac{1}{6} \\
&f^2f_{yy}: && \lambda_2\frac{\beta_{21}^2}{2} + \lambda_3\frac{(\beta_{31} + \beta_{32})^2}{2} = \frac{1}{6}
\end{aligned}
$$

以下是利用$Mathematica$解得的结果
其中

$$
\begin{aligned}
&\lambda_1 = l1 \quad \lambda_2 = l3 \quad \lambda_3 = l3 \\
&\alpha_2 = a2 \quad \alpha_3 = a3 \\
&\beta_{21} = b21\ \beta_{31} = b31\ \beta_{32} = b32
\end{aligned}
$$

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Solve[{l1 + l2 + l3 == 1,
  l2 * a2 + l3 * a3 == 1 / 2,
  l2 * b21 + l3 * b31 + l3 * b32 == 1 / 2,
  (a2^2) / 2 * l2 + (a3^2) / 2 * l3 == 1 / 6,
  l2 * a2 * b21 + l3 * a3 * (b31 + b32) == 1 / 3,
  l3 * a2 * b32 == 1 / 6,
  l3 * b21 * b32 == 1 / 6,
  l2 * b21^2 / 2 + l3 * (b31 + b32)^2 / 2 == 1 / 6},
 {l1, l2, l3, a2, a3, b21, b31, b32}]
```

$$
\begin{aligned}
&\Big\{\Big\{l1 \to \frac{1}{2}\left(2 - l3 - 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right),\\
&\quad l2 \to \frac{1}{2}\left(-l3 + 6\,b32\,l3 - l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right),\\
&\quad a2 \to \frac{-1 - \frac{2\,l3}{-l3 + l3\sqrt{1-12\,b32+48\,b32^2\,l3}} + \frac{8\,b32\,l3^2}{-l3 + l3\sqrt{1-12\,b32+48\,b32^2\,l3}}}{l3 - 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}},\\
&\quad a3 \to \frac{-1 + 4\,b32\,l3}{-l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}},\ b21 \to -\frac{1}{l3 - 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}} -\\
&\qquad (2\,l3)\Big/\left(\left(-l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\left(l3 - 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\right) +\\
&\qquad \left(8\,b32\,l3^2\right)\Big/\\
&\qquad \left(\left(-l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\left(l3 - 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\right),\\
&\quad b31 \to -b32 - \frac{1}{-l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}} + \frac{4\,b32\,l3}{-l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}}\Big\},\\
&\Big\{l1 \to \frac{1}{2}\left(2 - l3 - 6\,b32\,l3 - l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right),\\
&\quad l2 \to \frac{1}{2}\left(-l3 + 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right),\\
&\quad a2 \to \frac{1 - \frac{2\,l3}{l3 + l3\sqrt{1-12\,b32+48\,b32^2\,l3}} + \frac{8\,b32\,l3^2}{l3 + l3\sqrt{1-12\,b32+48\,b32^2\,l3}}}{-l3 + 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}},\\
&\quad a3 \to \frac{1 - 4\,b32\,l3}{l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}},\ b21 \to \frac{1}{-l3 + 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}} -\\
&\qquad (2\,l3)\Big/\left(\left(l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\left(-l3 + 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\right) +\\
&\qquad \left(8\,b32\,l3^2\right)\Big/\\
&\qquad \left(\left(l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\left(-l3 + 6\,b32\,l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}\right)\right),\\
&\quad b31 \to -b32 + \frac{1}{l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}} - \frac{4\,b32\,l3}{l3 + l3\sqrt{1 - 12\,b32 + 48\,b32^2\,l3}}\Big\},\\
&\Big\{l1 \to \frac{1}{4},\ l2 \to \frac{1}{4}(3 - 4\,l3),\ a2 \to \frac{2}{3},\ a3 \to \frac{2}{3},\ b21 \to \frac{2}{3},\ b31 \to \frac{-3 + 8\,l3}{12\,l3},\ b32 \to \frac{1}{4\,l3}\Big\},\\
&\Big\{l1 \to \frac{1}{4},\ l2 \to 0,\ l3 \to \frac{3}{4},\ a2 \to \frac{2}{9\,b32},\ a3 \to \frac{2}{3},\ b21 \to \frac{2}{9\,b32},\ b31 \to \frac{1}{3}(2 - 3\,b32)\Big\}\Big\}
\end{aligned}
$$

我的方程组与未知数一样多，而其他教材上的都有2个自由变量。

## 2 8 − 12 8 − 13

这两题很类似。

### 2.1 $8-12$

确定二步方法

$$y_{n+1}=\frac{1}{2}(y_n+y_{n-1})+\frac{h}{4}(4f_{n+1}-f_n+3f_{n-1})$$

的局部截断误差主项和阶

我的解法:
由于

$$y_{n-1}=y-hy'+\frac{h^2}{2}y''-\frac{h^3}{6}y'''+\frac{h^4}{24}y^{(4)}+O(h^5)$$

$$f_n=y'$$

$$\begin{aligned}f_{n+1}&=y'(x_n+h)\\&=y'+hy''+\frac{h^2}{2}y'''+\frac{h^3}{6}y_{(4)}+\frac{h^4}{24}y^{(5)}+O(h^5)\\f_{n-1}&=y'(x_n-h)\\&=y'-hy''+\frac{h^2}{2}y'''-\frac{h^3}{6}y_{(4)}+\frac{h^4}{24}y^{(5)}+O(h^5)\end{aligned}$$

故

$$\begin{aligned}y_{n+1}&=y_n+\left(-\frac{1}{4}+1+\frac{3}{4}-\frac{1}{2}\right)h^2y'+\left(1-\frac{3}{4}+\frac{1}{4}\right)h^3y''+\left(\frac{1}{2}+\frac{3}{4}\frac{1}{2}-\frac{1}{6}\frac{1}{2}\right)h^4y'''+O(h^4)\\&=y+hy'+\frac{h^2}{2}y''+\frac{19}{24}h^3y'''+O(h^4)\end{aligned}$$

而

$$\begin{aligned}y(x_{n+1})&=y(x_n+h)\\&=y+hy'+\frac{h^2}{2}y''+\frac{h^3}{6}y'''+O(h^4)\end{aligned}$$

故截断误差为

$$\begin{aligned}y(x_{n+1})-y_{n+1}&=\left(\frac{19}{24}-\frac{1}{6}\right)h^3y'''+O(h^4)\\&=\frac{5}{8}h^3y'''+O(h^4)\\&=O(h^3)\end{aligned}$$

## 2.2 $8-13$

试求系数$\alpha$，$\beta_0$，$\beta_1$，使二步方法

$$y_{n+1}=\alpha(y_n+y_{n+1})+h(\beta_0 f_n+\beta_1 f_{n-1})$$

的局部截断误差阶尽可能的高，并写出截断误差主项

我的解法（直接利用上题中算的）：

$$y_{n+1}=(\alpha+\alpha)y_n+(\beta_0+\beta_1-\alpha)hy'+\left(-\beta_1+\frac{\alpha}{2}\right)h^2y''+\left(\frac{\beta_1}{2}-\frac{\alpha}{6}\right)h^3y'''+O(h^4)$$

与$y(x_{n+1})$比较，有方程

$$\begin{cases}2\alpha=1\\ \beta_0+\beta_1+\alpha=1\\ -\beta_1+\dfrac{\alpha}{2}=1\end{cases}$$

解得

$$\begin{cases}\alpha=\dfrac{1}{2}\\ \beta_0=\dfrac{7}{4}\\ \beta_1=-\dfrac{1}{4}\end{cases}$$

同理，截断误差为

$$\begin{aligned}\left(\frac{\beta_1}{2}-\frac{\alpha}{6}\right)h^3y'''+O(h^4)&=\left\{\left[\frac{1}{2}\left(-\frac{1}{4}\right)-\frac{1}{6}\frac{1}{2}\right]-\frac{1}{6}\right\}h^3y'''+O(h^4)\\&=-\frac{3}{8}h^3y'''+O(h^4)\end{aligned}$$