抽代基础：组合恒等式归纳练习

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1

$$\sum_{k=1}^{n} a_k \sum_{k=1}^{n} b_k = n\sum_{k=1}^{n} a_k b_k - \sum_{1\le j<k\le n} (a_k - a_j)(b_k - b_j)$$

证明：

① $n = 1$ 显然成立

② $n = m$ 假设：$\sum\limits_{k=1}^{m} a_k \sum\limits_{k=1}^{m} b_k = m\sum\limits_{k=1}^{m} a_k b_k - \sum\limits_{1\le j<k\le m} (a_k - a_j)(b_k - b_j)$

$n = m + 1$

$$\begin{aligned}\sum_{k=1}^{m+1} a_k \sum_{k=1}^{m+1} b_k &= \left(\sum_{k=1}^{m} a_k + a_{m+1}\right)\left(\sum_{k=1}^{m} b_k + b_{m+1}\right)\\ &=\sum_{k=1}^{m} a_k \sum_{k=1}^{m} b_k + b_{m+1}\sum_{k=1}^{m} a_k + a_{m+1}\sum_{k=1}^{m} b_k + a_{m+1}b_{m+1}\\ &=m\sum_{k=1}^{m} a_k b_k - \sum_{1\le j<k\le m+1} (a_k - a_j)(b_k - b_j)\\ &\quad + b_{m+1}\sum_{k=1}^{m} a_k + a_{m+1}\sum_{k=1}^{m} b_k + a_{m+1}b_{m+1}\end{aligned}$$

而我们的目标可化为：

$$\begin{aligned}&(m+1)\sum_{k=1}^{m+1} a_k b_k - \sum_{1\le j<k\le m+1} (a_k - j_j)(b_k - b_j)\\ =&(m+1)\left(\sum_{k=1}^{m} a_k b_k + a_{m+1}b_{m+1}\right) + \Bigg[\sum_{1\le j<k\le m} (a_k - a_j)(b_k - b_j)\\ &+ \sum_{\substack{1\le j\le m-1\\ k=m+1}} (a_k - a_j)(b_k - b_j) + \sum_{\substack{j=m\\ k=m+1}} (a_k - a_j)(b_k - b_j)\Bigg]\end{aligned}$$

$$
\begin{aligned}
=&m\sum_{k=1}^{m}a_kb_k+ma_{m+1}b_{m+1}+\sum_{k=1}^{m}a_kb_k+a_{m+1}b_{m+1}\\
&-\sum_{1\le j<k\le m}(a_k-a_j)(b_k-b_j)-\sum_{1\le j\le m-1}(a_{m+1}-a_j)(b_{m+1}-b_j)-(a_{m+1}-a_m)(b_{m+1}-b_m)
\end{aligned}
$$

对比上面两个结果，已经有3个相等项：$m\sum\limits_{k=1}^{m}a_kb_k,-\sum\limits_{1\le j<k\le m}(a_k-a_j)(b_k-b_j),a_{m+1}b_{m+1}$，而第二个结果中的其余项可化为：

$$
\begin{aligned}
&ma_{m+1}b_{m+1}+\sum_{k=1}^{m}a_kb_k-(a_{m+1}b_{m+1})(m-1)+a_{m+1}\sum_{1\le j\le m-1}b_j+b_{m+1}\sum_{1\le j\le m-1}a_j\\
&-\sum_{1\le j\le m-1}a_jb_j-a_{m+1}b_{m+1}+a_{m+1}b_m+a_mb_{m+1}-a_mb_m\\
=&a_{m+1}\sum_{k=1}^{m-1}b_k+b_{m+1}\sum_{k=1}^{m-1}a_k+a_{m+1}b_m+a_mb_{m+1}\\
=&a_{m+1}\sum_{k=1}^{m}b_k+b_{m+1}\sum_{k=1}^{m}a_k
\end{aligned}
$$

这与第一个结果中的剩余项相同。

2

切比雪夫单调不等式：

$$
\begin{array}{lll}
\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k\le n\sum_{k=1}^{n}a_kb_k & a_1\le\ldots\le a_n & b_1\le\ldots\le b_n\\
\sum_{k=1}^{n}a_k\sum_{k=1}^{n}b_k\ge n\sum_{k=1}^{n}a_kb_k & a_1\le\ldots\le a_n & b_1\ge\ldots\ge b_n
\end{array}
$$

我们只证明第一个，第二个同理：

① $n=1$ 左边 $=a_1b_1=$ 右边

② $n=m$ 假设：$\sum_{k=1}^{m}a_k\sum_{k=1}^{m}b_k\le m\sum_{k=1}^{m}a_kb_k$
$n=m+1$

$$
\begin{aligned}
&\sum_{k=1}^{m+1}a_k+\sum_{k=1}^{m+1}b_k\\
=&\left(\sum_{k=1}^{m}a_k+a_{m+1}\right)\left(\sum_{k=1}^{m}b_k+b_{m+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{k=1}^{m} a_k \sum_{k=1}^{m} b_k + b_{m+1} \sum_{k=1}^{m} a_k + a_{m+1} \sum_{k=1}^{m} b_k + a_{m+1}b_{m+1} \\
&\le m \sum_{k=1}^{m} a_k b_k + b_{m+1} \sum_{k=1}^{m} a_k + a_{m+1} \sum_{k=1}^{m} b_k + a_{m+1}b_{m+1}
\end{aligned}
$$

而我们要证明原式：

$$
\begin{aligned}
&\le (m+1) \sum_{k=1}^{m+1} a_k b_k \\
&= m \sum_{k=1}^{m} a_k b_k + m a_{m+1} b_{m+1} + \sum_{k=1}^{m} a_k b_k + a_{m+1} b_{m+1}
\end{aligned}
$$

对比上面两个结果，已经有2个相等项$m\sum_{k=1}^{m} a_k b_k$和$a_{m+1}b_{m+1}$，将两个结果中的剩余项做差：

$$
\begin{aligned}
&\left(b_{m+1} \sum_{k=1}^{m} a_k + a_{m+1} \sum_{k=1}^{m} b_k\right) - \left(m a_{m+1} b_{m+1} + \sum_{k=1}^{m} a_k b_k\right) \\
=&\sum_{k=1}^{m} \left[(b_{m+1}a_k + a_{m+1}b_k) - (a_{m+1}b_{m+1} + a_k b_k)\right] \\
=&\sum_{k=1}^{m} \left[a_k(b_{m+1} - b_k) + a_{m+1}(b_k - b_{m+1})\right] \\
=&\sum_{k=1}^{m} (a_k - a_{m+1})(b_{m+1} - b_k) \le 0
\end{aligned}
$$

3

对于下降阶乘幂，我们也有类似于二项展开的结论：

$$
(x+y)^{\underline{m}} = \sum_{i=0}^{m} \binom{m}{i} x^{\underline{i}} y^{\underline{m-i}}
$$

证明：

① $m=1$ $(x+y)^{\underline{1}} = x + y = \binom{1}{0} x^{\underline{0}} y^{\underline{1}} + \binom{1}{1} x^{\underline{1}} y^{\underline{0}} = y + x$

② $m=k$ 假设：$(x+y)^{\underline{k}}=\sum_{i=0}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i}}$

$m=k+1$

$$
\begin{aligned}
(x+y)^{\underline{k+1}} &= (x+y)^{\underline{k}}(x+y-k) \\
&= \sum_{i=0}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i}}(x+y-k)
\end{aligned}
$$

下面从目标结果回推：

$$
\begin{aligned}
&\sum_{i=0}^{k+1}\binom{k+1}{i}x^{\underline{i}}y^{\underline{k-i+1}} \\
=&\sum_{i=1}^{k}\binom{k+1}{i}x^{\underline{i}}y^{\underline{k-i+1}}+y^{\underline{k+1}}+x^{\underline{k+1}} \\
=&\sum_{i=1}^{k}\left(\binom{k}{i-1}+\binom{k}{i}\right)x^{\underline{i}}y^{\underline{k-i+1}}+x^{\underline{k+1}}+y^{\underline{k+1}} \\
=&\sum_{i=1}^{k}\binom{k}{i-1}x^{\underline{i}}y^{\underline{k-i+1}}+\sum_{i=1}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i+1}}+x^{\underline{k+1}}+y^{\underline{k+1}} \\
=&\sum_{i=0}^{k-1}\binom{k}{i}x^{\underline{i+1}}y^{\underline{k-i}}+\sum_{i=1}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i+1}}+x^{\underline{k+1}}+y^{\underline{k+1}} \\
=&\sum_{i=0}^{k}\binom{k}{i}x^{\underline{i+1}}y^{\underline{k-i}}+\sum_{i=0}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i+1}} \\
=&\sum_{i=0}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i}}\left[(x-i)+(y-k+i)\right] \\
=&\sum_{i=0}^{k}\binom{k}{i}x^{\underline{i}}y^{\underline{k-i}}(x+y-k)=\text{原式}
\end{aligned}
$$

4

$$
\sum_{k\le n}\binom{r+k}{k}=\binom{r}{0}+\binom{r+1}{1}+\ldots+\binom{r+n}{n}=\binom{r+n+1}{n}
$$

证明：

1. $n=0$ 左 $=\binom{r}{0}=$ 右

2. $n=m$ 假设: $\sum\limits_{k\le m}\binom{r+k}{k}=\binom{r+m+1}{m}$

$n=m+1$

$$\sum_{k\le m+1}\binom{r+k}{k}=\sum_{k\le m}\binom{r+k}{k}+\binom{r+m+1}{m+1}$$
$$=\binom{r+m+1}{m}+\binom{r+m+1}{m+1}=\binom{r+m+2}{m+1}$$

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$$\sum_{0\le k\le n}\binom{k}{m}=\binom{0}{m}+\binom{1}{m}+\ldots+\binom{n}{m}=\binom{n+1}{m+1}$$

证明:

① $n=0$

左 $=\binom{0}{m}$ 右 $=\binom{1}{m+1}$

$m>0$ 左 $=$ 右 $=0$

$m=0$ 左 $=$ 右 $=1$

② $n=p$ 假设: $\sum\limits_{0\le k\le p}\binom{k}{m}=\binom{p+1}{m+1}$

$n=p+1$

$$\sum_{0\le k\le p+1}\binom{k}{m}=\sum_{0\le k\le p}\binom{k}{m}+\binom{p+1}{m}$$
$$=\binom{p+1}{m+1}+\binom{p+1}{m}=\binom{p+2}{m+1}$$

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$$\binom{r}{k}=(-1)^k\binom{k-r-1}{k}$$

证明:

① $r=0$ 显然成立

② $r=m$ 假设: $\binom{m}{k}=(-1)^k\binom{k-m-1}{k}$

$r=m+1$

$$\binom{m+1}{k}=\binom{m}{k}+\binom{m}{k-1}=(-1)^k\binom{k-m-1}{k}+\binom{m}{k-1}$$

$$
\begin{aligned}
&=(-1)^k\binom{k-m-1}{k}+(-1)^{k-1}\binom{k-m-2}{k-1}\\
&=(-1)^k\binom{k-m-2}{k}+(-1)^k\binom{k-m-2}{k-1}-(-1)^k\binom{k-m-2}{k-1}\\
&=(-1)^k\binom{k-m-2}{k}
\end{aligned}
$$

7

$$\sum_{k\le m}\binom{r}{k}(-1)^k=(-1)^m\binom{r-1}{m}$$

证明:

7.1 方法1：对m归纳

① $m=0$ $\binom{r}{0}=\binom{r-1}{0}$

② $m=p$ 假设: $\sum_{k\le p}\binom{r}{k}(-1)^k=(-1)^p\binom{r-1}{p}$

$m=p+1$

$$
\begin{aligned}
\sum_{k\le p+1}\binom{r}{k}(-1)^k=&\sum_{k\le p}\binom{r}{p}(-1)^k+\binom{r}{p+1}(-1)^{p+1}=(-1)^p\binom{r-1}{p}-(-1)^p\binom{r}{p+1}\\
=&(-1)^p\binom{r-1}{p}-(-1)^p\binom{r-1}{p+1}-(-1)^p\binom{r-1}{p}=(-1)^{p+1}\binom{r-1}{p+1}
\end{aligned}
$$

7.2 方法2：对r归纳

① $r=1$ 成立

② $r=p$ 假设 $\sum_{k\le m}\binom{p}{k}(-1)^k=(-1)^m\binom{p-1}{m}$

$r=p+1$

$$
\begin{aligned}
\sum_{k\le m}\binom{p+1}{k}(-1)^k&=\sum_{1\le k\le m}\binom{p+1}{k}(-1)^k+\binom{p+1}{0}(-1)^0\\
&=\sum_{1\le k\le m}\binom{p}{k-1}(-1)^k+\sum_{1\le k\le m}\binom{p}{k}(-1)^k+1
\end{aligned}
$$

$$
\begin{aligned}
&=\sum_{1\le k\le m}\binom{p}{k-1}(-1)^k+\sum_{0\le k\le m}\binom{p}{k}(-1)^k\\
&=\sum_{1\le k\le m}\binom{p}{k-1}(-1)^k+(-1)^m\binom{p-1}{m}\\
&=\sum_{0\le k\le m-1}\binom{p}{k}(-1)^{k+1}+(-1)^m\binom{p-1}{m}\\
&=-\sum_{0\le k\le m-1}\binom{p}{k}(-1)^k+(-1)^m\binom{p-1}{m}\\
&=-(-1)^{m-1}\binom{p-1}{m-1}+(-1)^m\binom{p-1}{m}\\
&=(-1)^m\left[\binom{p-1}{m-1}+\binom{p-1}{m}\right]=(-1)^m\binom{p}{m}
\end{aligned}
$$

8

$$
\sum_{k\le m}\binom{r}{k}(\frac{r}{2}-k)=\frac{m+1}{2}\binom{r}{m+1}
$$

证明:

8.1 方法1：对m归纳

① $m=0$ $\binom{r}{0}\frac{r}{2}=\frac{r}{2}=\frac{1}{2}\binom{r}{1}$

② $m=p$ 假设: $\sum_{k\le p}\binom{r}{k}\left(\frac{r}{2}-k\right)=\frac{p+1}{2}\binom{r}{p+1}$

$m=p+1$

$$
\begin{aligned}
\sum_{k\le p+1}\binom{r}{k}\left(\frac{r}{2}-k\right)=&\sum_{k\le p}\binom{r}{k}\left(\frac{r}{2}-k\right)+\binom{r}{p+1}\left(\frac{r}{2}-p-1\right)\\
=&\frac{p+1}{2}\binom{r}{p+1}+\binom{r}{p+1}\left(\frac{r}{2}-p-1\right)\\
=&\frac{r-p-1}{2}\binom{r}{p+1}\\
=&\frac{r-p-1}{2}\binom{r-1}{p+1}+\frac{r-p-1}{2}\binom{r-1}{p-1}\\
=&\frac{r-p-1}{2}\binom{r-1}{p}-\frac{p+1}{2}\binom{r-1}{p+1}+\frac{r}{2}\binom{r-1}{p+1}
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{r}{2}\binom{r-1}{p}-\frac{p+1}{2}\left[\binom{r-1}{p}+\binom{r-1}{p+1}\right]+\frac{r}{2}\binom{r-1}{p+1}\\
&=\frac{r}{2}\binom{r-1}{p}-\frac{p+1}{2}\binom{r}{p+1}+\frac{r}{2}\binom{r-1}{p+1}\\
&=\frac{r}{2}\binom{r-1}{p}-\frac{p+1}{2}\frac{r}{p+1}\binom{r-1}{p}+\frac{r}{2}\binom{r-1}{p+1}\\
&=\frac{r}{2}\binom{r-1}{p+1}
\end{aligned}
$$

8.2 方法2：对r归纳

① $r=0$ $\sum_{k\le m}\binom{0}{k}(-k)=\frac{m+1}{2}\binom{0}{m+1}=0$

② $r=p$ 假设：$\sum_{k\le m}\binom{p}{k}\left(\frac{p}{2}-k\right)=\frac{m+1}{2}\binom{p}{m+1}$

$r=p+1$

$$
\begin{aligned}
&\sum_{k\le m}\binom{p+1}{m}\left(\frac{p+1}{2}-k\right)\\
&=\binom{p+1}{0}\frac{p+1}{2}+\sum_{1\le k\le m}\binom{p+1}{k}\left(\frac{p+1}{2}-k\right)\\
&=\frac{p+1}{2}+\sum_{1\le k\le m}\left[\binom{p}{k}+\binom{p}{k-1}\right]\left(\frac{p+1}{2}-k\right)\\
&=\frac{p+1}{2}+\sum_{1\le k\le m}\binom{p}{k}\left(\frac{p}{2}-k\right)+\sum_{1\le k\le m}\frac{1}{2}\binom{p}{m}+\sum_{1\le k\le m}\binom{p}{k-1}\left(\frac{p}{2}-k+\frac{1}{2}\right)\\
&=\frac{1}{2}+\sum_{0\le k\le m}\binom{p}{k}\left(\frac{p}{2}-k\right)+\sum_{1\le k\le m}\frac{1}{2}\binom{p+1}{k}+\sum_{1\le k\le m}\binom{p}{k-1}\left(\frac{p}{2}-k\right)\\
&=\frac{m+1}{2}\binom{p}{m+1}+\sum_{0\le k\le m}\frac{1}{2}\binom{p+1}{k}+\sum_{1\le k\le m}\binom{p}{k-1}\left(\frac{p}{2}-k\right)\\
&=\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\sum_{k\le m}\binom{p+1}{k}+\sum_{0\le k\le m-1}\binom{p}{k}\left(\frac{p}{2}-k-1\right)\\
&=\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\sum_{k\le m}\binom{p+1}{k}+\sum_{0\le k\le m-1}\binom{p}{k}\left(\frac{p}{2}-k\right)+\sum_{k\le m-1}\binom{p}{k}\\
&=\frac{m+1}{2}\binom{p}{m+1}+\frac{m+1}{2}\binom{p}{m}+\frac{1}{2}\sum_{k\le m}\binom{p+1}{k}-\sum_{k\le m}\binom{p}{k}+\frac{1}{2}\binom{p}{m}\\
&=\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\binom{p}{m}+\sum_{k\le m}\left[\binom{p+1}{k}\frac{1}{2}-\binom{p}{k}\right]
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\binom{p}{m}+\sum_{k\le m}\left[\frac{1}{2}\binom{p+1}{k}-\frac{1}{2}\binom{p}{k}\right]-\frac{1}{2}\sum_{k\le m}\binom{p}{k}\\
=&\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\binom{p}{m}+\sum_{1\le k\le m}\frac{1}{2}\left[\binom{p+1}{k}-\binom{p}{k}\right]-\frac{1}{2}\sum_{k\le m}\binom{p}{k}\\
=&\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\binom{p}{m}+\frac{1}{2}\sum_{1\le k\le m}\binom{p}{k-1}-\frac{1}{2}\sum_{k\le m}\binom{p}{k}\\
=&\frac{m+1}{2}\binom{p}{m+1}+\frac{1}{2}\binom{p}{m}+\frac{1}{2}\sum_{0\le k\le m-1}\binom{p}{k}-\frac{1}{2}\sum_{k\le m}\binom{p}{k}\\
=&\frac{m+1}{2}\binom{p}{m+1}
\end{aligned}
$$

9

$$
\sum_{k\le m}\binom{m+r}{k}x^k y^{m-k}=\sum_{k\le m}\binom{-r}{k}(-x)^k(x+y)^{m-k}
$$

证明:

① $m=0$ 左 $=1=$ 右

② $m=n$ 假设: $\sum\limits_{k\le n}\binom{n+r}{k}x^k y^{n-k}=\sum\limits_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}$

$m=n+1$

$$
\begin{aligned}
&\sum_{k\le n+1}\binom{n+1+r}{k}x^k y^{n+1-k}\\
=&\sum_{1\le k\le n}\binom{n+1+r}{k}x^k y^{n+1-k}+\binom{n+r+1}{n+1}x^{n+1}+y^{n+1}\\
=&y\left[\sum_{1\le k\le n}\binom{n+r}{k}x^k y^{n-k}+\sum_{1\le k\le n}\binom{n+r}{k-1}x^k y^{n-k}\right]+\binom{n+r+1}{n+1}x^{n+1}+y^{n+1}\\
=&y\left[\sum_{k\le n}\binom{n+r}{k}x^k y^{n-k}+\sum_{1\le k\le n}\binom{n+r}{k-1}x^k y^{n-k}\right]+\binom{n+r+1}{n+1}x^{n+1}\\
=&y\left[\sum_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}+\sum_{1\le k\le n}\binom{n+r}{k-1}x^k y^{n-k}\right]+\binom{n+r}{n+1}x^{n+1}+\binom{n+r}{n}x^{n+1}\\
=&y\left[\sum_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}+\sum_{1\le k\le n}\binom{n+r}{k-1}x^k y^{n-k}\right]+\binom{-r}{n+1}(-x)^{n+1}+\binom{n+r}{n}x^{n+1}
\end{aligned}
$$

而我们往证：

$$
\begin{aligned}
&\sum_{k\le n+1}\binom{-r}{k}(-x)^k(x+y)^{n+1-k}\\
=&(x+y)\sum_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}+\binom{-r}{n+1}(-x)^{n+1}\\
=&y\sum_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}+x\sum_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}+\binom{-r}{n+1}(-x)^{n+1}
\end{aligned}
$$

可以看到，两个结果中已经有2个相等项：$y\sum_{k\le n}\binom{-r}{k}(-x)^k(x+y)^{n-k}$，$\binom{-r}{n+1}(-x)^{n+1}$，因此我们只需证：

$$
\begin{aligned}
&y\sum_{1\le k\le n}\binom{n+r}{k-1}x^ky^{n-k}+\binom{n+r}{n}x^{n+1}\\
=&x\sum_{k\le n-1}\binom{n+r}{k}x^ky^{n-k}+\binom{n+r}{n}x^{n+1}\\
=&x\sum_{k\le n}\binom{n+r}{k}x^ky^{n-k}
\end{aligned}
$$