2 CHAPTER Mathematical Logic

2.1. Introduction

Logic is concerned with the study of the principles and techniques of reasoning. The Greek philosopher and scientist Aristotle (381–322 BC) is said to be the first person to have studied logical reasoning. Logical reasoning is the essence of mathematics and is therefore an important starting point for study of discrete mathematics. Logic, among other things, have provided the theoretical basis for many areas of computer science such as digital logic design, automata theory and computability, artificial intelligence etc. One component of logic is **proposition calculus** which deals with statements with values true and false and is concerned with analysis of propositions. And the other part is **predicate calculus** which deals with the predicates which are propositions containing variables. In this chapter, we discuss a few of the basic ideas and define some of the logical concepts that are useful in computer science.

2.2. Proposition (Statement)

A **Proposition** or **Statement** is a declarative sentence that is either true or false, but not both. For example, "Three plus three equals six." and "Three plus three equals seven" are both statements, the first because it is true and the second because it is false. Similarly "$x + y > 1$" is not a statement because for some values of x and y the sentence is true, whereas for others it is false. For instance, if $x = 1$ and $y = 2$, the sentence is true, if $x = -3$ and $y = 1$, this is false. The truth or falsity of a statement is called its **truth value.** Since only two possible truth values are admitted this logic is sometimes called **two – valued logic.** Questions, exclamations and commands are not propositions. For example, consider the following sentences:

(*a*) The sun rises in the west.

(*b*) $2 + 4 = 6$

(*c*) $(5, 6) \subset (7, 6, 5)$

(*d*) Do you speak Hindi?

(*e*) $4 - x = 8$

(*f*) Close the door.

(*g*) What a hot day!

(*h*) We shall have chicken for dinner.

The sentences (*a*), (*b*) and (*c*) are statements, the first is false and second and third are true.

(*d*) is a question, not a declarative sentence, hence it is not a statement.

(*e*) is a declarative sentence, but not a statement, since it is true or false depends on the value of x.

(*f*) is not a statement, it is a command.

(*g*) is not a statement, it is exclamation.

(*h*) is a statement since it is either true or false but not both, although one has to wait until dinner to find out if it is true or false.

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It is customary to represent simple propositions by letters p, q, r..... known as **proposition variables** (Note that usually a real variable is represented by the symbol x. This means that x is not a real number but can take a real value. Similarly, a propositional variable is not a proposition but can be replaced by a proposition) Propositional variables can only assume two values: 'true' denoted by T or 1 and 'false' denoted by F or 0. If p denotes the proposition "The sun sets in the east", then instead of saying the proposition is false, one can simply say the value of p is F.

2.3. Compound Proposition

A proposition consisting of only a single propositional variable or a single propositional constant is called an **atomic (primary, primitive)** proposition or simply proposition; that is they can not be further subdivided. A proposition obtained from the combinations of two or more propositions by means of logical operators or connectives of two or more propositions or by negating a single proposition is referred to **molecular or composite or compound proposition.**

Connectives

The words and phrases (or symbols) used to form compound propositions are called **connectives.** There are five basic connectives called Negation, Conjunction, Disjunction, Conditional and Biconditional. The following symbols are used to represent connectives.

Symbol used	Connective word	Nature of the compound statement formed by the connective	Symbolic from	Negation
$\sim, \neg, \rceil. N$	not	Negation	$\sim p$	$\sim(\sim p) = p$
$\wedge$	and	Conjuction	$p \wedge q$	$(\sim p) \vee (\sim q)$
$\vee$	or	Disjunction	$p \vee q$	$(\sim p) \wedge (\sim q)$
$\rightarrow$	if,.....then	Conditional	$p \rightarrow q$	$p \wedge (\sim q)$
$<->$	if and only if	Bi-conditional	$p \leftrightarrow q$	$[p \wedge (\sim q)] \vee [q \wedge (\sim p)]$

Negation

If p is any proposition, the negation of p, denoted by $\sim p$ and read as not p, is a proposition which is false when p is true and true when p is false. Consider the statement

p : Paris is in France.

Then the negation of p is the statement

$\sim p$: It is not the case that Paris is in France.

Normally it is written as

$\sim p$: Paris is not in France.

Strictly speaking, not is not a connective, since it does not join two statements and $\sim p$ is not really a compound statement. However, not is a unary operation for the collection of statements, and $\sim p$ is a statement if p is considered a statement.

Note: 1. The following propositions all have the same meaning:

p : All people are intelligent.
q : Every person is intelligent.
r : Each person is intelligent.
s : Any person is intelligent.

2. The negation of the proposition

p : All students are intelligent.

is

$\sim p$: Some students are not intelligent.
$\sim p$: There exists a student who is not intelligent.
$\sim p$: At least one student is not intelligent.

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The negation of

q : No student is intelligent.

is

$\sim q$: Some students are intelligent.

Note that "No student is intelligent" is *not* the negation of p; "All students are intelligent" is *not* the negation of q.

Conjunction

If p and q are two statements, then conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. The truth values of $p \wedge q$ are given in the truth table shown in Table 2.1 (*a*).

Example 1. Form the conjunction of p and q for each of the following.

(*a*) p : Ram is healthy $\quad q$: He has blue eyes

(*b*) p : It is cold $\quad q$: It is raining

(*c*) $p : 5x + 6 = 26 \quad q : x > 3$.

Solution: (*a*) $p \wedge q$: Ram is healthy and he has blue eyes.

(*b*) $p \wedge q$: It is cold and raining.

(*c*) $p \wedge q : 5x + 6 = 26$ and $x > 3$.

Remark

In logic we may combine any two sentences to form a conjunction, there is no requirement that the two sentences be related in content or subject matter. Any combinations, however absurd, are permitted, of course, we are usually not interested in sentences like 'Tapas loves Rini, and 4 is divisible by 2'.

Disjunction

If p and q are two statements, the disjunction of p and q is the compound statement denoted by $p \vee q$ and read as "p or q". The statement $p \vee q$ is true if at least one of p or q is true. It is false when both p and q are false. The truth of $p \vee q$ are given in the truth table shown in Table 2.1 (*b*).

Example 2. Assign a truth value to each of the following statements.

(*i*) $5 < 5 \vee 5 < 6$

(*ii*) $5 \times 4 = 21 \vee 9 + 7 = 17$

(*iii*) $6 + 4 = 10 \vee 0 > 2$.

Solution: (*i*) True, since one of its components *viz.* $5 < 6$ is true.

(*ii*) False, since both of its components are false.

(*iii*) True, since one of its components *viz.* $6 + 4 = 10$ is true.

Example 3. If p : It is cold and q : It is raining.

Write simple verbal sentence which describes each of the following statements

(*a*) $\sim p$ (*b*) $p \wedge q$ (*c*) $p \vee q$ (*d*) $p \vee \sim q$.

Solution. (*a*) $\sim p$: It is not cold.

(*b*) $p \wedge q$: It is cold and raining

(*c*) $p \vee q$: It is cold or raining.

(*d*) $p \vee \sim q$: It is cold or it is not raining.

2.4. Propositions and Truth Tables

A truth table is a table that shows the truth value of a compound proposition for all possible cases.

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For example, consider the conjunction of any two propositions p and q. The con statement $p \wedge q$ is true when both p and q are true, otherwise it is false. There are four possible c

1. p is true and q is true.
2. p is true and q is false.
3. p is false and q is true.
4. p is false and q is false.

These four cases are listed in the first two coloumns and the truth values of $p \wedge q$ are shown in the third column of Table 2.1 (*a*). The truth tables for the other two connectives disjunction and negation discussed above are shown in Table 2.1 (*b*) and 2.1 (*c*).

p	q	$(p \wedge q)$
T	T	T
T	F	F
F	T	F
F	F	F

(*a*)

p	q	$(q \vee q)$
T	T	T
T	F	T
F	T	T
F	F	F

(*b*)

p	$\sim p$
T	F
F	T

(*c*)

Table 2.1 Truth tables for the three propositional connectives

The first columns of the table are for the variables $p, q, \ldots\ldots$ and the number of rows depends on the number of variables. For 2 variables, 4 rows are necessary ; for 3 variables, 8 rows are necessary ; and in general, for n variables, 2^n rows are required. There is then a column for each elementary stage of the construction of the proposition. The truth value at each step is determined from the previous stages by the definition of connectives. The truth value of the proposition appears in the last column.

Example 4. Construct a truth table for each compound proposition.

(*i*) $p \wedge (\sim q \vee q)$ (*ii*) $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

Solution. (*i*) Make columns labeled $p, q, \sim q, (\sim q \vee q)$ and $p \wedge (\sim q \vee q)$. Fill in the p and q columns with all the logically possible combinations of Ts and Fs. Then fill in the $\sim q$ and $\sim q \vee q$ columns with the appropriate truth values. Complete the table by considering the truth values of $p \wedge (\sim q \vee q)$.

p	q	$\sim q$	$\sim q \vee q$	$p \wedge (\sim q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

Table 2.2 Truth table for $p \wedge (\sim q \vee q)$

(*ii*) Set up columns labeled $p, q, \sim q\ p \vee q, \sim (p \vee q), (\sim p \wedge - q)$. Fill in the p and q columns with all the logically possible combinations of Ts and Fs. Then fill in the $\sim p, \sim q, p \vee q, \sim (p \vee q)$, $(\sim p \wedge \sim q)$ columns with appropriate truth values. Complete the table by considering the truth values of $\sim (p \vee q) \vee (\sim p \wedge \sim q)$.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$(\sim p \wedge \sim q)$	$\sim (p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

Table 2.3 Truth table for $\sim (p \vee q) \vee (\sim p \wedge \sim q)$

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2.5. Logical Equivalence

If two propositions $P(p, q, \ldots\ldots)$ and $Q(p, q, \ldots\ldots)$ where $p, q, \ldots\ldots$ are propositional variables have the same truth values in every possible case or $P \leftrightarrow Q$ is a tautology, then the propositions are called logically equivalent or simply equivalent, and denoted by

$P(p, q, \ldots\ldots) \equiv Q(p, q, \ldots\ldots)$ or $P(p, q, \ldots\ldots) \Leftrightarrow Q(p, q, \ldots\ldots)$

It is always permissible, and sometimes desirable to replace a given proposition by an equivalent one.

To test whether two propositions P and Q are logically equivalent the following steps are followed.

1. Construct the truth table for P.
2. Construct the truth table for Q using the same propositional variables.
3. Check each combinations of truth values of the propositional variables to see whether the value of P is the same as the truth value of Q. If in each row the truth value of P is the same as the truth value of Q, then P and Q are logically equivalent.

2.6. Algebra of Propositions

Propositions satisfy various laws which are listed in Table 2.4. These laws are useful to simplify expression. Note that, with the exception of the Involution law, all the laws of Table come in pairs, called **dual pairs.** For each expression, one finds the dual by replacing all T by F and all F by T and replacing all $\wedge$ by $\vee$ and all $\vee$ by $\wedge$.

Idempotent laws	
(1a) $p \vee p \equiv p$	(1b) $p \wedge p \equiv p$
Associative laws	
(2a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$	(2b) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws	
(3a) $p \vee q \equiv q \vee p$	(3b) $p \wedge q \equiv q \wedge p$
Distributive laws	
(4a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	(4b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws	
(5a) $p \vee F \equiv p$	(5b) $p \wedge T \equiv p$
(6a) $p \vee T \equiv T$	(6b) $p \wedge F \equiv F$
Complement laws	
(7a) $p \vee \sim p \equiv T$	(7b) $p \wedge \sim p \equiv F$
(8a) $\sim T \equiv F$	(8b) $\sim F \equiv T$
Involution law	
(9) $\sim(\sim p) \equiv p$	
DeMorgan's laws	
(10a) $\sim(p \vee q) \equiv \sim p \wedge \sim q$	(10b) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Table 2.4 Laws of the algebra of propositions

From the laws given in Table 2.4 one can derive further laws. Of particular importance are the **absorption laws,** which are

$$p \vee (p \wedge q) \equiv p \qquad 11\ (a)$$
$$p \wedge (p \vee q) \equiv p \qquad 11\ (b)$$

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Equivalence in 11 (*a*) can be proved as follows:

$$\begin{aligned} p \vee (p \wedge q) &\equiv (p \wedge T) \vee (p \wedge q) \\ &\equiv (p \wedge (T \vee q) \\ &\equiv p \wedge T && \text{using Identity law} \\ &\equiv p && \text{using Identity law} \end{aligned}$$

The proof of 11 (*b*) is similar. The absorption laws are very useful when expressions need to be simplified.

All the laws given in Table 2.4 can be proved with the help of truth table.

Example 5. Use truth table to prove the distributive law

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Solution. The truth table of the compound propositions is shown in the table. Since the entries in the 5th and last column of the table are the same, the two propositions are logically equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 2.5

2.7. Conditional Proposition

If p and q are proposition, the compound proposition " if p then q" denoted by $p \rightarrow q$ is called a **conditional proposition** and the connective is the **conditional connective.** The proposition p is called **antecedent** or **hypothesis,** and the proposition q is called the **consequent** or **conclusion.**

The only circumstances under which $p \rightarrow q$ is false when p is true and q is false.

Examples include

1. If tomorrow is Sunday then today is Saturday.
2. If it rains then I will carry an umbrella.

Here p : Tomorrow is Sunday — is antecedent.
q : Today is Saturday — is consequent.
and p : It rains — is antecedent.
q : I will carry an umbrella — is consequent.

The connective if ..... then can also be read as follows:

1. p is sufficient for q.
2. p only if q.
3. q is necessary for p.
4. q if p.
5. q follows from p.
7. q is consequence of p.

The truth table of $p \rightarrow q$ is given in Table 2.6

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 2.6 Truth table of $p \rightarrow q$.

Note: Some authors call $p \rightarrow q$ as an implication.

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Example 6. Which of the following propositions are true and which are false?

(*a*) If the earth is round then the earth travels round the sun.

(*b*) If Alexander Graham Bell invented telephone, then tigers have wings.

(*c*) If tigers have wings, then RDX is dangerous.

Solution. (*i*) True. Let

p : The earth is round

q : The earth travels round the sun

Here p is true and q is true and hence the conditional proposition is true.

(*ii*) False. Let

p : Alexander Graham Bell invented telephone.

q : Tigers have wings.

Here p is true and q is false and hence the conditional proposition is false.

(*iii*) True. Let

p : Tigers have wings.

q : RDX is dangerous.

Here p is false and q is true and hence the conditional proposition is true.

Example 7. Construct truth table for

(*i*) $p \vee \sim q \rightarrow p$ (*ii*) $(\sim (p \wedge q) \vee r) \rightarrow \sim p$

Solution. (*i*) The truth table of the given compound statement is shown below.

p	q	$\sim q$	$p \vee \sim q$	$p \vee \sim q \rightarrow p$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	F

Table 2.7

(*ii*) The truth table of the given compound statement is shown below.

p	q	r	$(p \wedge q)$	$\sim (p \wedge q)$	$(\sim (p \wedge q) \vee r)$	$(\sim (p \wedge q) \vee r) \rightarrow \sim p$
T	T	T	T	F	T	F
T	T	F	T	F	F	T
T	F	T	F	T	T	F
T	F	F	F	T	T	F
F	T	T	F	T	T	T
F	T	F	F	T	T	T
F	F	T	F	T	T	T
F	F	F	F	T	T	T

Table 2.8

Example 8. Use truth table to show that

$$p \rightarrow q \equiv \sim p \vee q$$

Solution. The truth table of $p \rightarrow q$ and $\sim p \vee q$ is shown below

p	q	$p \rightarrow q$	$\sim p$	$\sim p \vee q$
F	F	T	T	T
F	T	T	T	T
T	F	F	F	F
T	T	T	F	T

Table 2.9

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The logical equivalence is established by the 3rd and 5th column of the table which are identical.

2.8. Converse, Contrapositive and Inverse

If p and q are any two propositions, then some other conditional propositions related to $p \to q$ are

(*i*) Converse : The converse of $p \to q$ is $q \to p$.

(*ii*) Contrapositive : The contrapositive of $p \to q$ is $\sim q \to \sim p$.

(*iii*) Inverse : The inverse of $p \to q$ is $\sim p \to \sim q$.

The truth table of the four propositions follow:

p	q	Conditional $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$	Contrapositive $\sim q \to \sim p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Table 2.10 Truth Table for the four propositions

Consider the statement

p : It rains.

q : The crops will grow

The conditional proposition $p \to q$ states that,

$p \to q$: If it rains then the crops will grow.

The converse of $p \to q$ namely $q \to p$ states that,

$q \to p$: If the crops grow, then there has been rain.

The contrapositive of $p \to q$, namely $\sim q \to \sim p$ states that,

$\sim q \to \sim p$: If the crops do not grow then there has been no rain.

The inverse of $p \to q$, namely $\sim p \to \sim q$ states that,

$\sim p \to \sim q$: If it does not rain then the crops will not grow.

Notice that a conditional proposition and its converse or inverse are not logically equivalent. On the other hand, a conditional proposition and its contrapositive are logically equivalent.

Example 9. Show that contrapositive and conditional propositions are logically equivalent; that is

$$\sim q \to \sim p \equiv p \to q$$

Solution. The truth table of $\sim q \to \sim p$ and $p \to q$ are shown below and the logical equivalence is established by the last two columns of the table, which are identical.

p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$	$p \to q$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Table 2.11

Example 10. Prove that if x^2 is divisible by 4, then x is even.

Solution. Let p and q be the propositions such that

$p : x^2$ is divisible by 4.

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and q : x is even.

The conditional is of the form $p \to q$. The contrapositive is $\sim q \to \sim p$, which states in words

If x is odd, then x^2 is not divisible by 4.

The proof of contrapositive is easy.

Since x is odd, one can write $x = 2k + 1$, for some integer k. Hence

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4\,(k^2 + k + 1/4)$$

Since $k^2 + k$ is an integer, $k^2 + k + 1/4$ is not an integer; therefore x^2 is not divisible by 4.

2.9. Biconditional Statement

If p and q are statement, then the compound statement p if and only if q, denoted by $p \leftrightarrow q$ is called a biconditional statement and the connective if and only if is the **biconditional connective.** The biconditional statement $p \leftrightarrow q$ can also be stated as "p is a necessary and sufficient condition for q" Examples include

1. He swims if and only if the water is warm.
2. Sales of houses fall if and only if interest rate rises.

The truth table of $p \leftrightarrow q$ is given in Table 2.12. It may be noted $p \leftrightarrow q$ is true when both p and q are true or both p and q are false

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 2.12. Truth Table for the biconditional statement $p \leftrightarrow q$

Example 11. Show that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Solution. Table 2.13. shows that these two expressions are logically equivalent ; the two columns corresponding to the given two expressions have identical truth values.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Table 2.13

Example 12. Show that $p \leftrightarrow q \equiv (p \vee q) \to (p \wedge q)$, using

(*a*) truth table (*b*) algebra of propositions.

Solution. (*a*) Truth table shows that these two expressions are logically equivalent ; the two columns corresponding to the given two expressions have identical truth values.

p	q	$p \leftrightarrow q$	$p \vee q$	$p \wedge q$	$(p \vee q) \to (p \wedge q)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	F	F
F	F	T	F	F	T

Table 2.14

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(b) $p \leftrightarrow q$ $\equiv (p \to q) \wedge (q \to p)$

$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\equiv [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$ by Distributive law

$\equiv [\sim q \wedge (\sim p \vee q)] \vee [p \wedge (\sim p \vee q)]$ by Commutative law

$\equiv [(\sim q \wedge q) \vee (\sim q \wedge \sim p)] \vee [(p \wedge q) \vee (p \wedge \sim p]$ by Distributive law

$\equiv [F \vee (\sim q \wedge \sim p)] \vee [(p \wedge q) \vee F]$ by Complement law

$\equiv (\sim q \wedge \sim p) \vee (p \wedge q)$ by Identity law

$\equiv [\sim (p \vee q)] \vee (p \wedge q)$ by De Morgan's law

$\equiv (p \vee q) \to (p \wedge q)$

2.10. Negation of Compound Statements

Negation of Conjunction:

The negation of a conjunction $p \wedge q$ is the disjunction of the negation of p and the negation of q. Equivalently, we write

$$\sim (p \wedge q) \equiv \sim p \vee \sim q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Table 2.15

Example 13. Write the negation of each of the following conjunctions :

(a) Paris is in France and London is in England.

(b) $2 + 4 = 6$ and $7 < 12$.

Solution. (a) Let p : Paris is in France and q : London is in England.

Then, the conjunction in (a) is given by $(p \wedge q)$.

Now

$\sim p$: Paris is not in France and

$\sim q$: London is not in England.

Therefore, negation of $p \wedge q$ is given by

$\sim (p \wedge q)$: Paris is not in France or London is not in England.

(b) Let $p : 2 + 4 = 6$ and $q : 7 < 12$.

Then the conjunction in (b) is given by $p \wedge q$.

Now $\sim p : 2 + 4 \neq 6$ and $\sim q : 7 \nless 12$.

The negation of $p \wedge q$ is given by

$\sim (p \wedge q) : 2 + 4 \neq 6$ or $7 \nless 12$

Negation of Disjunction:

The negation of a disjunction $p \vee q$ is the conjunction of the negation of p and the negation of q. Equivalently, we write

$$\sim (p \vee q) \equiv \sim p \wedge \sim q$$

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In order to prove the above equivalence, we prepare the following table.

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$	$\sim (p \vee q)$
T	T	F	F	F	T	F
T	F	F	T	F	T	F
F	T	T	F	F	T	F
F	F	T	T	T	F	T

Table 2.16

Example 14. Write the negation of each of the following disjunction:

(*a*) Ram is in class XI or Arun is in Class XII.

(*b*) 9 is greater than 4 or 6 is less than 8.

Solution. (*a*) Let p: Ram is in class XI and q: Arun is in class XII.

Then, the disjunction in (*a*) is given by $p \vee q$.

Now $\sim p$: Ram is not in Class XI.

$\sim q$: Arun is not in Class XII.

Then, negation of $p \vee q$ is given by

$\sim (p \vee q)$: Ram is not in Class X and Arun is not in Class XII.

(*b*) Let p : 9 is greater than 4 and q : 6 is less than 8.

Then, negation of $p \vee q$ is given by

$\sim (p \vee q)$: 9 is not greater than 4 and 6 is not less than 8.

Negation of a Negation : A negation of negation of a statement is the statement itself. Equivalently, we write

$$\sim(\sim p) \equiv p$$

Example 15. Verify for the statement

p : Roses are red.

Solution. The negation of p is given by

$\sim p$: Roses are not red.

Therefore, the negation of negation of p is $\sim (\sim p)$:

It is not the case that Roses are not red.

or

It is false that Roses are not red.

or

Roses are red.

Negation of Conditional : If p and q are two statements, then

$$\sim (p \rightarrow q) \equiv p \wedge \sim q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	T	F

Table 2.17

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Example 16. Write the negation of each of the following statements :

(*a*) If it is raining, then the game is cancelled.

(*b*) If he studies, he will pass the examination.

Solution. (*a*) Let p : It is raining. q: The game is cancelled.

The given statement can be written as $p \to q$. The negation of $p \to q$ is written as

$$\sim (p \to q) \equiv p \wedge \sim q$$

Hence the negation of the given statement is it is raining and the game is not cancelled.

(*b*) Let p : He studies and q: He will pass the examination.

The given statement can be written as $p \to q$. The negation of $p \to q$ is written as

$$\sim (p \to q) \equiv p \wedge \sim q$$

Hence the negation of the given statement is he studies and he will not pass the examination.

Negation of Biconditional: If p and q are two statements, then

$$\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$$

In order to prove the above equivalence, we prepare the following table.

p	q	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$	$\sim p$	$\sim p \leftrightarrow q$	$\sim q$	$p \leftrightarrow \sim q$
T	T	T	F	F	F	F	F
T	F	F	T	F	T	T	T
F	T	F	T	T	T	F	T
F	F	T	F	T	F	T	F

Table 2.18

Note that $\sim (p \leftrightarrow q) \equiv \sim [(p \to q) \wedge (q \to p)] \equiv \sim [(\sim p \vee q) \wedge (\sim q \vee p)]$

$$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

Example 17. Write the negation of each of the following statements :

(*a*) He swims if and only if the water is warm.

(*b*) This computer program is correct if and only if, it produces the correct answer for all possible sets of input data.

Solution. (*a*) Let p : He swims and q : The water is warm.

The given statement can be written as $p \leftrightarrow q$. The negation of $p \leftrightarrow q$ is written as

$$\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$$

Hence the negation of the given statement is either of the following:

He swims if and only if the water is not warm.

He does not swim if and only if the water is warm.

(*b*) Let p : This computer program is correct and

q : It produces the correct answer for all possible sets of input data.

The given statement can be written as $p \leftrightarrow q$. The negation of $p \leftrightarrow q$ is written as

$$\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$$

Hence the negation of the given statement is either of the following

This program is correct if and only if it does not produce the correct answer for all possible sets of input data.

This program is not correct if and only if it produces the correct answer for all possible sets of input data.

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Derived Connectives

1. NAND : It means negation of conjuction of two statements. Assume p and q be two propositions. Nand of p and q is a proposition which is false when both p and q are true otherwise true. It is denoted by $p \uparrow q$ and $p \uparrow q \equiv \sim (p \wedge q)$

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Table 2.19. Truth table for NAND

2. NOR: It means negation of disjunction of two statements. Assume p and q be two propositons. Nor of p and q is a proposition which is true when both p and q are false, otherwise false. It is denoted by $p \downarrow q$ and $p \downarrow q \equiv \sim (p \vee q)$

p	q	$p \downarrow q$
T	T	F
T	F	F
F	T	F
F	F	T

Table 2.20 Truth table for NOR

Note that

(*i*) $\sim p \equiv p \downarrow p$

(*ii*) $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$

(*iii*) $p \vee q \equiv (p \downarrow q) \downarrow (p \downarrow q)$

3. XOR (Exclusive OR): Assume p and q be two proposition. The exclusive or (XOR) of p and q, denoted by $p \oplus q$ is the proposition that is true when exactly one of p and q is true but not both and is false otherwise

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 2.21 Truth table for XOR

Properties of Exclusive OR

(*i*) $p \oplus q \equiv q \oplus p$ (Commutative)

(*ii*) $(p \oplus q) \oplus r \equiv p \oplus (p \oplus r)$ (Associative)

(*iii*) $p \wedge (q \oplus r) \equiv (p \wedge q) \oplus (p \wedge r)$ (Distributive)

2.11. Tautologies and Contradictions

A compound proposition that is always true for all possible truth values of its variables or, in other words contain only T in the last column of its truth table is called a **tautology.** A compound proposition that is always false for all possible values of its variables or, in other words contain only F in the last column of its truth table is called a **contradiction.** Finally a proposition that is neither a tautology nor a contradiction is called a **contingency.**

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Example 18. Prove that the following propositions are tautology

(*a*) $p \vee \sim p$ (*b*) $\sim (p \wedge q) \vee q$ (*c*) $p \rightarrow (p \vee q)$

Solution. (*a*) The truth table of the given proposition is shown below. Since the truth value is TRUE for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Table 2.22

(*b*) We construct the truth table for the expression in question. It can be seen that for any possible assignment of p and q, the expression $\sim (p \wedge q) \vee q$ is true, which establishes that it is a tautology.

p	q	$p \wedge q$	$\sim (p \wedge q)$	$\sim (p \wedge q) \vee q$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Table 2.23

(*c*) We construct the truth table of the given expression. It can be seen from the last column of the truth table that the expression is true for all possible assignments of p and q. Hence the proposition is a tautology.

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Table 2.24

Example 19. Verify that the proposition $p \wedge (q \wedge \sim p)$ is a contradiction.

Solution. We construct the truth table of the given proposition. It can be seen from the truth Table 2.25 that for any possible assignment of p and q, the given proposition is false, which establishes that it is a contradiction.

p	q	$\sim p$	$q \wedge \sim p$	$p \wedge (q \wedge \sim p)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Table 2.25

Tautologically Implication

A compound proposition A (p, q, ...) is said to be tautologically imply or simply imply the compound proposition B (p, q, ...) if and only if A $\rightarrow$ B is a tautology. This is denoted by A $\Rightarrow$ B and read as 'A implies B'.

For example, $p \wedge q \Rightarrow q$ since $p \wedge q \rightarrow q$ is a tautology which can be seen from the following truth table.

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p	q	$p \wedge q$	$p \wedge q \rightarrow q$
T	T	T	T
T	F	F	T
F	T	F	T
F	T	F	T

Some important implications which can be proved by truth tables are given below:

1. $p \Rightarrow p \vee q$
2. $\sim p \Rightarrow p \rightarrow q$
3. $q \Rightarrow p \rightarrow q$
4. $\sim (p \rightarrow q) \Rightarrow p$
5. $\sim (p \rightarrow q) \Rightarrow \sim q$
6. $p \wedge (p \rightarrow q) \Rightarrow q$
7. $\sim q \wedge (p \rightarrow q) \Rightarrow \sim p$
8. $\sim p \wedge (p \vee q) \Rightarrow q$
9. $(p \rightarrow q) \wedge (q \rightarrow r) \Rightarrow p \rightarrow r$
10. $(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r) \Rightarrow r$

Well formed Formula

A statement formula is an expression which is a string consisting of variables, parentheses and connective symbols. A gramatically correct expression is called a well formed formula, which is abbreviated wff and can be pronounced as, 'woof'. A well formed formula can be generated by the following rules:

1. All variables and constants are well-formed formula.
2. If P is a well-formed formula, then $\rceil P$ is also a well-formed formula.
3. If P and Q are well-formed formulae, then $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$ and $(P \leftrightarrow Q)$ are all well-formed formulae.
4. A statement formula consists of variables, parentheses and connectives is recursively well formed formula iff it can be obtained by finitely applying the above rules.
5. Nothing else is a well-formed formula.

For example, $\rceil (P \vee Q)$ and $(P \rightarrow (P \wedge Q))$ are well-formed formulae where as $\wedge Q$ and $(P \wedge Q) \rightarrow (\wedge P)$ are not.

Functionally Complete Set of Connectives

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from the set is called a functionally complete set of connectives. A functionally complete set of connectives which does not contain a connective that can be expressed in term of the other connectives of the set is called a minimal functionally complete set of connectives.

There are five logical connectives $\wedge$, $\vee$, $\sim$, $\rightarrow$ and $\leftrightarrow$. We know the following logical equivalence:

$$p \rightarrow q \equiv \sim p \vee q \tag{1}$$

There are two ways to express the biconditional:

$$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q) \tag{2}$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p) \tag{3}$$

Using (1), (3) can be written as

$$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

Thus all the conditional and biconditional can be replaced by the three connectives $\wedge$, $\vee$, $\sim$. Again from De Morgan's law we have

$$p \wedge q \equiv \sim (\sim p \vee \sim q) \qquad p \vee q \equiv \sim (\sim p \wedge \sim q)$$

The first equivalence means that it is also possible to obtain a formula which is equivalent to a given formula in which conjunction is eliminated. A similar procedure is possible for the elimination of disjunction.

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Thus, in any formula, we can replace first all the biconditionals, then the conditionals, and finally all the conjunctions or all the disjunctions to obtain an equivalent formula. This formula will contain either the negation and disjunction or the negation and conjunction. Thus the set of connectives $\{\sim, \wedge\}$, $\{\sim, \vee\}$ are minimal functionally complete set.

Example 20. Write an equivalent expression for $(p \to q \wedge r) \vee (r \leftrightarrow s)$ which contains neither the biconditional nor the conditional.

Solution. First we replace the biconditional connective by its equivalent in the given expression. Then

$$\begin{aligned}(p \to q \wedge r) \vee (r \leftrightarrow s) &\equiv (p \to q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s)) \\ &\equiv (\sim p \vee q \wedge r) \vee ((\sim r \vee s) \wedge (r \vee \sim s)) \\ &\qquad [p \to q \wedge r \equiv \sim p \vee (q \wedge r)]\end{aligned}$$

This expression does not contain biconditional and conditional connectives.

2.12. Normal Forms

By comparing truth tables, one determine whether two logical expressions P and Q are equivalent. But the process is very tedious when the number of variables increases. A better method is to transform the expressions P and Q to some standard forms of expressions P' and Q' such that a simple comparison of P' and Q' shows whether $P \equiv Q$, The standard forms are called **normal forms** or **canonical forms.** There are two types of normal form: disjunctive normal form and conjunctive normal form.

It will be convenient to use the words product and sum in place of the logical connectives conjunction and disjunction.

Disjunctive Normal Form (dnf)

In an logical expression, a product of the variables and their negations is called an elementary product. For example, $p \wedge \sim q$, $\sim p \wedge \sim q$, $\sim p \wedge q$ are some elementary products in two variables. A sum of the variables and their negations is called an elementary sum. For example, $p \vee q$, $p \vee \sim q$, $\sim p \vee \sim q$ are elementary sums in two variables. A part of the elementary sum of product which is itself an one elementary sum of product is called a factor of the original sum or product. The elementary sums or products satisfy the following properties. We only state them without proof.

1. An elementary product is identically false if and only if it contains at least one pair of factors in which one is negation of the other.
2. An elementary sum is identically true if and only if it contains at least one pair of factors in which one is the negation of the other.

A logical expression is said to be in disjunctive normal form if it is the sum of elementary products. For example, $p \vee (q \wedge r)$ and $p \vee (\sim q \wedge r)$ are in disjunctive normal form, $p \wedge (q \vee r)$ is not in disjunctive normal form.

Procedure to obtain a disjunctive normal form of a given logical expression

Three steps are required to obtain a disjunctive normal form through algebraic manipulations.

1. Remove all $\to$ and $\leftrightarrow$ by an equivalent expression containing the connectives $\wedge, \vee, \sim$ only.
2. Eliminate $\sim$ before sums and products by using the double negation or by using De-Morgan's laws.
3. Apply the distributive law until a sum of elementary product is obtained.

Example 21. Obtain the disjunctive normal forms of the followings:

(a) $p \wedge (p \to q)$ (b) $p \vee (\sim p \to (q \vee (q \to \sim r)))$

(c) $p \to ((p \to q)\ [(\wedge \sim (\sim q \vee \sim p)]$

Solution. (a) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$

$$\equiv (p \wedge \sim p) \vee (p \wedge q)$$

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which is the required disjunctive normal form.

$$(b)\ p \vee (\sim p \to (q \vee (q \to \sim r))) \equiv p \vee (\sim p \to (q \vee (\sim q \vee \sim r)))$$
$$\equiv p \vee (p \vee (q \vee (\sim q \vee \sim r)))$$
$$\equiv p \vee p \vee q \vee \sim q \vee \sim r$$
$$\equiv p \vee q \vee \sim q \vee \sim r$$

which is the required disjunctive normal form.

$$(c)\ p \to ((p \to q) \wedge \sim (\sim q \vee \sim p)) \equiv \sim p \vee ((p \to q) \wedge \sim (\sim q \vee \sim p))$$
$$\equiv \sim p \vee ((\sim p \vee q) \wedge \sim (\sim q \vee \sim p))$$
$$\equiv \sim p \vee ((\sim p \vee q) \wedge (q \wedge p))$$
$$\equiv \sim p \vee [(\sim p \wedge (q \wedge p)) \vee (q \wedge (q \wedge p))]$$
$$\equiv \sim p \vee [(\sim p \wedge q) \wedge p] \vee [(q \wedge q) \wedge p]$$
$$\equiv \sim p \vee F \vee (p \wedge q)$$
$$\equiv \sim p \vee (p \wedge q)$$

which is the required disjunctive normal form.

Note : Disjunctive normal form of a given logical expression is not unique. For example consider $p \vee (q \wedge r)$. This is already in disjunctive normal form.

We can write $\quad p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

$$\equiv (p \wedge p) \vee (p \wedge q) \vee (p \wedge r) \vee (q \wedge r)$$

and this is another equivalent normal form.

Conjunctive Normal Form (cnf)

A logical expression is said to be in conjunctive normal form if it consists of a product of elementary sums.

Example 22. Obtain a conjunctive normal form of the followings:

(*a*) $p \wedge (p \to q)$ $\qquad$ (*b*) $[q \vee (p \wedge r)] \wedge \sim [(p \vee r) \wedge q]$

Solution. (*a*) $p \wedge (p \to q) \equiv p \wedge (\sim p \vee q)$ which is the required conjunctive normal form

$$(b)\ [q \vee (p \wedge r)] \wedge \sim [(p \vee r) \wedge q] \equiv [q \vee (p \wedge r)] \wedge [\sim (p \vee r) \vee \sim q]$$
$$\equiv [q \vee (p \wedge r)] \wedge [(\sim p \wedge \sim r) \vee \sim q]$$
$$\equiv (q \vee p) \wedge (q \vee r) \wedge (\sim p \vee \sim q) \wedge (\sim r \vee \sim q)$$

which is in conjunctive normal form.

Principal Disjunctive Normal Form (Pdnf)

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$ and $\sim p \wedge \sim q$ are called minterms of p and q. It may be noted that none of the minterms should contain both a variable and its negation. For given two variables, there are 2^2 minterms. The number of minterms in n variable is 2^n. For example, minterms for the three variable p, q and r are

$$p \wedge q \wedge r, p \wedge q \wedge \sim r, p \wedge \sim q \wedge r, p \wedge \sim q \wedge \sim r,$$
$$\sim p \wedge q \wedge r, \sim p \wedge q \wedge \sim r, \sim p \wedge \sim q \wedge r, \sim p \wedge \sim q \wedge \sim r$$

The truth table of the minterms of p and q are given below.

p	q	$p \wedge q$	$p \wedge \sim q$	$\sim p \wedge q$	$\sim p \wedge \sim q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Table 2.26

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Note that each minterm has the truth value T for exactly one combination of the truth values of the variables p and q. Also no two minterms are equivalent.

Principal disjunctive normal form of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms only. This is also called the **sum of products canonical form.** The process for obtaining principal disjunctive norm is discussed below.

(I) By truth table:

1. Construct a truth table of the given compound propositions.
2. For every truth value T of the given proposition, select the minterm, which also has the value T for the same combination of the truth value of the statement variables.
3. The disjunctive of the minterms selected in 2 is the required principal disjunctive normal form.

(II) Without constructing truth table.

1. Obtain a disjunctive normal form.
2. Drop elementary products which are contradictions (such as $p \wedge \sim p$).
3. If p_i and $\sim p_i$ are missing in an elementary product α, replace α by $(\alpha \wedge p_i) \vee (\alpha \wedge \sim p_i)$
4. Repeat step 3 until all elementary products are reduced to sum of minterms. Identical minterms appearing in the disjunction are deleted.

The advantages of obtaining principal disjunctive normal form are :

(*i*) The principal disjunctive normal of a given formula is unique.

(*ii*) Two formulas are equivalent if and only if their principal disjunctive normal forms coincide.

(*iii*) If the given compound proposition is a tautology, then its principal disjunctive normal form will contain all possible minterms of its components.

Example 23. Obtain the principal disjunctive normal form of

(*a*) $p \to q$ (*b*) $q \vee (p \vee \sim q)$

(*c*) $\sim p \vee q$ (*d*) $(p \wedge \sim q \wedge \sim r) \vee (q \wedge r)$

(*a*) and (*b*) using the truth table, (*c*) and (*d*) without using the truth table :

Solution. (*a*) The truth table of $p \to q$ is given below.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The column containing $p \to q$ has truth value T for three combinations of the truth values of p and q. Now T in the first row of $p \to q$ corresponds to minterm $p \wedge q$, T in the third row of $p \to q$ corresponds to minterm $\sim p \wedge q$ and T in the fourth row of $p \to q$ corresponds to minterms $\sim p \wedge \sim q$.

Thus, principal disjunctive normal form of $p \to q$ is $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$.

(*b*) The truth table of $q \vee (p \vee \sim q)$ is given below:

p	q	$\sim q$	$p \vee \sim q$	$q \vee (p \vee \sim q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	T

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The column containing $q \vee (p \vee \sim q)$ has truth value T for four combinations of the truth values of p and q i.e it is tautology. The minterm terms corresponding to first, second, third and fourth row are $p \wedge q, p \wedge \sim q, \sim p \wedge q, \sim p \wedge \sim q$.

Thus, the required principal disjunctive normal form is

$$(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$$

(c) $\sim p \vee q \equiv (\sim p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$ [$T \equiv q \vee \sim q$ is a tautology and $p \wedge T \equiv p$]

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$$

$$\equiv (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$$

which is the required principal disjunctive normal form.

(d) Let $A = (p \wedge \sim q \wedge - r) \vee (q \wedge r)$.

The given formula $(p \wedge \sim q \wedge - r)$ is already a minterm.

Now, $(q \wedge r) \equiv (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p)$

$$\equiv (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$$

So, the required principal disjunctive normal form is

$$(p \wedge \sim q \wedge \sim r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$$

Principal Conjunctive Normal Forms (pcnf)

The dual of a minterm is called a maxterm. For a given number of variables the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.

For example, for two variables p and q, there are maxterms given by

$$p \vee q, p \vee \sim q, \sim p \vee q \text{ and } \sim p \vee \sim q.$$

Maxterm for three variables p, q and r are :

$p \vee q \vee r \rightarrow$	$p \vee q \vee \sim r \rightarrow$	$p \vee \sim q \vee r \rightarrow$	$p \vee \sim q \vee \sim r \rightarrow$
$\sim p \vee q \vee r$	$\sim p \vee q \vee \sim r$	$\sim p \vee \sim q \vee r$	$\sim p \vee \sim q \vee \sim r$

each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Different maxterms have the truth value F for different combinations of the truth values of the variables.

Principal conjunctive normal form of a given formula can be defined as an equivalent formula consists of conjunctive of maxterm only. This is also called the **product of sums canonical form.** The process for obtaining principal conjunctive norm form is similar to the one followed for principal disjunctive normal form. For obtaining principal conjunctive norm of α, one can also construct the principal disjunctive normal of $\sim \alpha$ and apply negation.

The advantages of obtaining principal conjunctive normal form are :

(*i*) The principal conjunctive normal form is unique.

(*ii*) Every compound proposition, which is not a tautology, has an equivalent principal conjunctive normal form.

(*iii*) If the given compound proposition is a contradiction, then its principal conjunctive normal form will contain all possible maxterms of its components.

Example 24. Obtain the principal conjunctive normal form

(*a*) $p \wedge q$ using truth table.

(*b*) $(\sim p \rightarrow r) \wedge (q \leftrightarrow p)$ without using truth table.

(*c*) $A = (p \wedge q) \vee (\sim p \wedge q) \vee (q \wedge r)$ by constructing principal disjunctive normal form

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Solution (*a*): Truth table of $p \wedge q$ is given below:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

From the column $p \wedge q$ has truth value F for three combinations of the truth value of p and q. Now F in the second row of $p \wedge q$ correspond to maxterm $\sim p \vee q$. F in the third row $p \wedge q$ correspond to maxterm $p \vee \sim q$ and F in the fourth row of $p \wedge q$ corresponds to maxterm $\sim p \wedge \sim q$. Hence the required principal conjunctive normal form of $p \wedge q$ is

$$(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$$

(*b*) $(\sim p \to r) \wedge (q \leftrightarrow p) \equiv (p \vee r) \wedge [(q \to p) \wedge (p \to q)]$

$\equiv (p \vee r) \wedge [(\sim q \vee p) \wedge (\sim p \vee q)]$

$\equiv [(p \vee r) \vee (\sim q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)]$

$\equiv (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

$\equiv (p \vee r \vee q) \wedge (p \vee r \vee \sim q) \wedge (\sim q \vee p \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

[dropping the identical term]

which is the required principal conjunctive normal form

(*c*) Let $\sim A = \sim ((p \wedge q) \vee (\sim p \wedge q) \wedge (q \wedge r))$

$\equiv \sim ((p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (\sim (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge \sim p))$

$\equiv (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (\sim q \wedge \sim r \wedge \sim p)$

$\vee (\sim q \wedge \sim r \wedge p) \equiv (\sim p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r) \vee (p \wedge \sim q \wedge r)$

Now $\sim (\sim A) \equiv (p \vee q \vee r) \wedge (p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

which is the required principal conjunctive normal form.

2.13. Logic in Proof

A theorem is a proposition that can be proved to be true. An argument that establishes the truth of a theorem is called a proof. There are many different types of proof. In this section we shall look at some of the more common type.

Arguments

An argument is a process by which a conclusion is drawn from a set of propositions. The given set of propositions are called **premises or hypotheses.** The final proposition derived from the given proposition is called **conclusion.**

Sometimes an argument is written in the follwing form

$$\left.\begin{array}{l} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_n \end{array}\right\} \text{premises}$$

$$\therefore \quad c \longleftarrow \text{conclusion}$$

An argument is said to be logically **valid argument** if and only if the conjunction of the premises implies the conclusion i.e., if the premises are all true, the conclusion must also be true. The argument which yield a conclusion c from the premises $p_1, p_2, p_3, \ldots\ldots, p_n$ is valid if and only if the statement

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$p_1 \wedge p_2 \wedge p_3 \wedge \ldots\ldots\ldots, \wedge p_n \rightarrow c$ is a tautology.

Procedure for testing the validity of an Argument using Truth Table.

1. Identify the premises and conclusion of the argument.
2. Construct a truth table showing the truth values of all premises and the conclusion.
3. Find the rows (called **critical rows**) in which all the premises are true.
4. In each critical row, determine whether the conclusion of the argument is also true.
   (*a*) If in each critical row the conclusion is also true, then the argument form is valid.
   (*b*) If there is at least one critical row in which the conclusion is false, the argument form is invalid.

The method to determine whether the conclusion logically follows from the given premises by constructing the relevent truth table is called truth table technique.

Rules of Inference.

The rules of inference are criteria for determining the validity of an argument. Any conclusion which is arrived by following these rules is called a valid conclusion, and the argument is called valid argument. The most familiar type of proof uses two fundamental rules of inference.

Fundamental Rule 1. If the statement in p is assumed as true and also the statement $p \rightarrow q$ is accepted as true, then, q must be true.

Symbolically it is written in the following pattern, where we use the familiar symbol

$$\begin{array}{l} p \rightarrow q \\ \underline{\quad p \quad} \\ \therefore\ q \end{array}$$

In this presentation of an argument, the assertions $p \rightarrow q$ and p above the horizontal line are the **hypotheses** or **premises** and the assertion q below the line is the **conclusion.** The rule depicted is known as **modus ponens** or the **rule of detachment.** The term modus ponens is Latin word meaning "method of affirming" (since the conclusion is an affirmation). The validity of the argument can also be seen from the following truth table.

Premises			Conclusion	
p	q	$p \rightarrow q$	p	q
T	T	T	T	T
T	F	F	T	F
F	T	T	F	T
F	F	T	F	F

Table 2.27

We see in Table 2.27 that there is only one case in which both premises are true, namely, the first case, and that in this case the conclusion is also true, hence the argument is valid.

Another way of stating that the above argument is valid is that $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology.

Example 25. Represent the argument

If I study hard, then I get A' s
I study hard.
.....
I get A's

symbolically and determine whether the argument is valid.

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Solution. If we let

p : I study hard, q : I get A's.

The argument may be written symbolically as

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore\ q \end{array}$$

Hence, by modus ponens the argument is valid.

Example 26. Suppose that the implication "If the last digit of this number is a 5, then this number is divisible by 5" and its hypothesis, "The last digit of this number is a 5" are true. Then, by modus ponens, it follows that the conclusion of the implication, "This number is divisible by 5" is true.

Fundamental Rule 2. Whenever the two statements $p \to q$ and $q \to r$ are accepted as true, then the statement $p \to r$ is accepted as true. Symbolically it can be represented as

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

This argument is known as a **hypothetical syllogism.**

The truth table of the argument appears in Table 2.28

p	q	r	$p \to q$	$q \to r$	$p \to r$
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	T
F	F	T	T	T	T
F	F	F	T	T	T

Table 2.28

Both premises are true as seen in the first, fifth, seventh, and eighth rows of the truth table. Since in each case the conclusion is also true, the argument is valid.

This rule is a valid rule of inference because

$$(p \to q) \wedge (q \to r) \to (p \to r) \text{ is a tautology.}$$

Example 27. Represent the argument

If it rains today, then we will not have a party today.

If we do not have party today, then we will have a party tomorrow.

.....

Therefore, if it rains today, then we will have a party tomorrow.

Symbolically and determine whether the argument is valid.

Solution. If we let

p : It is raining today, q : we will not have a party today,

r : We will have a party tomorrow.

The argument is of the form

$$\begin{array}{c} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

Hence the argument is a hypothetical sylloglism and thus the argument is valid.

Additional Valid Argument Forms

There are other valid inferences which state that certain form of arguments are valid. Some of these rules of inference are nothing more than reinterpretation of the two fundamental rules in the light of the law of contraposition.

Modus Tollens

The argument of the form

$$\begin{array}{c} p \to q \\ \sim q \\ \hline \therefore\ \sim p \end{array}$$

This argument is valid and is called modus tollens. Modus tollens is a Latin word meaning "method of denying" (since the conclusion is a denial). It can be established by using a truth table.

Example 28. Represent the argument

If this number is divisible 6, then it is divisible by 3.
This number is not divisible by 3.

.....

This number is not divisible by 6.

symbolically and determine whether the argument is valid.

Solution. If we let

p : The number is divisible by 6.

q : It is divisible by 3.

The argument may be written as

$$\begin{array}{c} p \to q \\ \sim q \\ \hline \therefore\ \sim p \end{array}$$

Thus by modus tollens the argument is valid.

Addition

The following argument form is valid.

$$\begin{array}{c} p \\ \hline \therefore\ p \vee q \end{array}$$

This argument form is used for making generalizations. If p is true, then, more generally, p or q is true for any other statement q.

Disjunctive Syllogism

The following statement form is valid :

$$\begin{array}{c} p \vee q \\ \sim q \\ \hline \therefore\ p \end{array}$$

This argument states that when there are two possibilities and one can rule one out, the other must be the case.

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Example 29. Represent the argument

Either Ram is not guilty or Shyam is telling the truth.
Shaym is not telling the truth.

.....

$\therefore$ Ram is not guilty

symbolically and determine whether the argument is valid.

Solution. If we let

p : Ram is not guilty, q : Shyam is telling the truth.

The argument can be written as

$$\begin{array}{l} p \vee q \\ \sim q \\ \hline \therefore \quad p \end{array}$$

Thus by disjunctive syllogism, the argument is valid.

We list below some of the more important valid inferences along with the inferences discussed above in the following Table 2.29. Most of the rules follow from the two fundamental rules, De- Morgan's laws and the law of contraposition.

Rule of Inference	Tautological Form	Name
a. $\begin{array}{l} p \\ \hline \therefore\ p \vee q \end{array}$ b. $\begin{array}{l} q \\ \hline \therefore\ p \vee q \end{array}$	a. $p \to (p \vee q)$ b. $q \to (p \vee q)$	Addition
a. $\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$ b. $\begin{array}{l} p \wedge q \\ \hline q \end{array}$	a. $(p \wedge q) \to p$ b. $(p \wedge q) \to q$	Simplification
$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$\begin{array}{l} p \to q \\ p \\ \hline \therefore\ q \end{array}$	$[(p \to q) \wedge p] \to q$	Modus ponens
$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore\ \sim p \end{array}$	$[(p \to q) \wedge \sim q] \to \sim p$	Modus tollens
$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical syllogism
$\begin{array}{l} p \vee q \\ \sim p \\ \hline \therefore\ q \end{array}$	$[(p \vee q) \wedge \sim p] \to q$	Disjunction syllogism
$\begin{array}{l} p \to q \wedge (r \to s) \\ p \vee r \\ \hline \therefore\ q \vee s \end{array}$	$(p \to q) \wedge (r \to s) \wedge (p \vee r) \to (q \vee s)$	Constructive dilemma
$\begin{array}{l} p \to q \wedge (r \to s) \\ \sim q \vee \sim s \\ \hline \therefore\ \sim p \vee \sim r \end{array}$	$(p \to q) \wedge (r \to s) \wedge (\sim q \vee \sim s)$ $\to (\sim p \vee \sim r)$	Destructive dilemma

Table 2.29 Rules of Inference

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Example 30. Show that t is a valid conclusion from the premises $p \to q$, $q \to r$, $r \to s$, $\sim s$ and $p \vee t$.

Solution. The valid argument for deducting t from the given five premises is given as a sequence.

1.	$p \to q$	Premise (Given)
2.	$q \to r$	Premise (Given)
3.	$r \to s$	Premise (Given)
4.	$p \to r$	Hypothetical syllogism using 1 and 2
5.	$p \to s$	Hypothetical syllogism using 3 and 4
6.	$\sim s$	Premise (Given)
7.	$\sim p$	Modus tollens using 5 and 6
8.	$p \vee t$	Premise (Given)
9.	t	disjunctive syllogism using 7 and 8.

Thus we can conclude t from the given premises.

Example 31. s is a valid conclusion from the premises $p \to q$, $p \to r$, $\sim (q \wedge r)$ and $s \vee p$.

Solution. The valid agreement for deducting s from the given premises is given as a sequence.

1.	$p \to q$	Premise (Given)
2.	$p \to r$	Premise (Given)
3.	$(p \to q) \wedge (p \to r)$	Using 1 and 2
4.	$\sim (q \wedge r)$	Premise (Given)
5.	$\sim q \vee \sim r$	Demorgan's law using 4.
6.	$\sim p \vee \sim p$	Destructive dilema using 3 and 5
7.	$\sim p$	Idempotent law using 6
8.	$s \vee p$	Premise (Given)
9.	s	Disjunctive syllogism using 7 and 8

Thus we can conclude s from the given premises.

Example 32. Prove the validity of the following argument "If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard"

Solution. Let

p : I get the job
q : I work hard
r : I get promoted
s : I will be happy

Then the given arguments can be written in symbolic form as

$$(p \wedge q) \to r$$
$$r \to s$$
$$\sim s$$

So 1.	$(p \wedge q) \to r$	Premise (Given)
2.	$r \to s$	Premise (Given)
3.	$(p \wedge q) \to s$	Hypothetical syllogism using 1 and 2
4.	$\sim s$	Premise (Given)
5.	$\sim (p \wedge q)$	Modus tollens using 3 and 4
6.	$\sim p \vee \sim q$	Conclusion

Hence the argument is valid.

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In addition to the two basic rules of inference. There is one more basic rule for proving a statement called Rule CP.

Conditional Proof (CP).

If a formula s can be derived from another formula r and a set of premises, then the statement $(r \to s)$ can be derived from the set premises alone.

Note: If the conclusion is of the form $r \to s$, we will take r as and additional and derive s using the given premises and r.

Example 33. Establish the validity of the following by conditional proof.

$$p \wedge ((p \to \sim q) \vee (r \wedge s)) \to (q \to r)$$

Solution: We assume q as an additional premise

1.	p	Premise (Given)
2.	$(p \to \sim q) \vee (r \wedge s)$	Premise (Given)
3.	q	Additional premise
4.	$p \wedge q$	Using 1, 3 and conjunction
5.	$\sim(\sim p \vee \sim q)$	Using 4 and De Morgen's law
6.	$\sim(p \to \sim q)$	Using 5 and $p \to q \equiv \sim p \vee q$
7.	$r \wedge s$	Using 2, 6 and disjunctive syllogism
8.	r	Using 7 and simplification
9.	$q \to r$	Using 3, 8 and CP

2.14. Methods of Proof

In previous sections we discussed proofs in the setting and symbolism of the propositional calculus. The proofs used in everyday working mathematics are based on the same logical framework as the propositional calculus but their structure is not usually displayed in the format.

Direct Proof

We are typically faced with a set of hypothesis $H_1, H_2, \ldots H_n$ from which we want to infer a conclusion C. One of the most natural sorts of proof is the direct proof in which we show

$$H_1 \wedge H_2 \wedge \ldots\ldots\ldots\ldots H_n \to C.$$

We give a direct proof of the following example.

Example 34 (*a*). Prove that if x and y are rational numbers then $x + y$ is rational.

Solution. Since x and y are rational numbers, we can find integers p, q, m, n such that $x = p/q$ and $y = m/n$ Then

$$x + y = p/q + m/n = (pn + mq)/qn$$

Since $pn + mq$ and qn are both integers, we conclude that $x + y$ is a rational number.

Example 34 (*b*). Prove that product of two odd integers is an odd integer.

Solution. Assume m and n are two odd integers. Then there exists two integers r and t, so that $m = 2r + 1$ and $n = 2t + 1$. Then

$$\begin{aligned} mn &= (2r + 1)(2t + 1) = 4rt + 2r + 2t + 1 \\ &= 2(2rt + r + t) + 1 \text{ which is odd.} \end{aligned}$$

Hence the proof.

Indirect Proof

Proofs that are not direct are called indirect. The two main types of indirect proof both use the negation of the conclusion, so they are often suitable when that negation is easy to state. The first type of proof is **contrapositive proof.** We give a proof by contrapositive of the statement in the following example.

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Example 35. Let n be an integer. Prove that if n^2 is odd then n is also odd.

Solution. Let p: n^2 is odd and q : n is odd. We have to prove $p \to q$ is true whenever both p and q are true. We prove the contrapositive i.e., $\sim q \to \sim p$. Suppose n is not odd i.e., n is even. Let $n = 2k$, k is any integer. Then $n^2 = (2k)^2 = 2(2k^2)$, so n^2 is even. This shows that if n is even then n^2 is also even, which is the contrapositive of the given statement. Hence, the given statement is true.

The second type of indirect proof is known as **proof by contradiction (reductio ad absurdum)**. In this type of proof, we assume the opposite of what we are trying to prove and get a logical contradiction. Hence our assumption must have been false, and therefore what we originally required to prove must be true. Hence in order to infer a conclusion c from the premises $H_1, H_2, \ldots, H_n$ by this method, we include $\sim c$ as an additional premise and derive a logical contradiction (F) using $\sim c$ and given premises. We give a proof of the statement of an example by contradiction.

Example 36. Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.

Solution. Suppose that $\sqrt{2}$ is rational. We will show that this leads to a contradiction.

Under assumption that $\sqrt{2}$ is rational, there exist integers a and b such that $\sqrt{2} = a/b$, where a and b have no common factors. Squaring both sides, we get

$$2 = a^2/b^2 \Rightarrow 2b^2 = a^2$$

Hence a^2 is a multiple of 2, and therefore even. This implies a is even. Hence $a = 2k$ for some integer k. Then $2\ b^2 = (2k)^2$ and therefore $b^2 = 2k^2$. Thus b^2 is even, b is even. But now a and b have a common factor of 2, which is a contradiction to the statement that a and b have no common factors.

Hence, our initial assumption that $\sqrt{2}$ is rational is false. Thus, $\sqrt{2}$ is irrational.

Proof by Cases

An implication of the form

$$H_1 \vee H_2 \vee \ldots\ldots\ldots\ldots \vee H_n \to C$$

is equivalent to

$$(H_1 \to C) \text{ and } (H_2 \to C) \text{ and } \ldots\ldots \text{ and } (H_n \to C)$$

and hence can be proved by cases *i.e.*, by proving each of $H_1 \to C, \ldots\ldots, H_n \to C$ separately. The following example illustrates the process.

Example 37. Prove that for every positive integer n, $n^3 + n$ is even.

Solution. Case (*i*) Suppose, n is even, Then $n = 2k$ for some positive integer k.

Now $n^3 + n = 8\ k^3 + 2k = 2\ (4k^3 + k)$ which is even.

Case. (*ii*) Suppose n is odd. Then $n = 2k + 1$ for some positive integer k.

Now $n^3 + n = (8k^3 + 12k^2 + 6k + 1) + (2k + 1) = 2\ (4k^3 + 6k^2 + 4k + 1)$ which is even.

Hence the sum $n^3 + n$ is even.

2.15. Fallacies

A fallacy is an error in reasoning that results in an invalid argument. Several common fallacies arise in incorrect arguments. They superficially resemble those that are valid by rules of inference, but are not in fact valid. In this section we discuss two types of fallacies.

1. The fallacy of affirming the consequent (or affirming the converse).
2. The fallacy of denying the hypothesis (or assuming the inverse).

The fallacy of affirming the consequent has the following form

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$$\begin{array}{l} p \to q \\ q \\ \hline \therefore p \end{array}$$

Example 38. Show that the following argument is invalid:

If Siddhartha solved this problem, then he obtained the answer 5

Siddhartha obtained the answer 5.

Therefore, Siddhartha solved this problem correctly.

Solution. Let p and q be the propositions as

p : Siddhartha solved this problem.

q : Siddhartha obtained the answer 5.

Then this argument is of the form: if $p \to q$ and q, then p. This argument is faulty because the conclusion can be false even though $p \to q$ and q are true. That is, $[(p \to q) \wedge q] \to p$ is not a tautology. It is possible, Siddhartha obtained the correct answer 5 by luck, guessing or prior knowledge but the arguments and intermediate steps are wrong. Hence the argument is invalid.

The fallacy under lying this invalid argument form is called the **fallacy of affirming the consequent** because the conclusion of the argument would follow from the premises $p \to q$ when replaced by its converse. Such a replacement is not allowed, because a conditional statement is not logically equivalent to its converse.

Example 39. Test the validity of the following argument

If two sides of a triangle are equal, then the opposite angles are equal.

Two sides of a triangle are not equal.

Therefore, the opposite angles are not equal.

Solution. Let p and q be the propositions as

p : Two sides of a triangle are equal.

q : The opposite angles are equal.

Hence $\sim p$: Two side of a triangle are not equal. Then this argument is of the form.

$$\begin{array}{l} p \to q \\ \sim q \\ \hline \therefore \sim p \end{array}$$

The proposition $[(p \to q) \wedge \sim p] \to \sim q$ is not a tautology, the argument is invalid.

The fallacy under lying this invalid argument form is called the **fallacy of denying the hypothesis** assuming the inverse because the incorrect argument is of the form $p \to q$ and $\sim p$ imply $\sim q$ which is not allowed.

2.16. Mathematical Induction

The word induction means the method of inferring a general statement from the validity of particular cases. Mathematical induction is a technique by which one can prove mathematical statements involving positive integers.

Before describing the method of mathematical induction, let us try to understand its power. To do this, let us consider the statement :

$$1 + 2 + 3 + \ldots\ldots\ldots\ldots + n = n(n+1)/2$$

It is easy to check that this statement is true for $n = 1$, for $n = 2$ and $n = 3$ etc.

From the above, one can not conclude that the statement is true for all positive n as one can never be sure that the statement does not fail for some untried value of n. But it is also impossible

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to substitute infinite number of possible values of n. Mathematical induction reduce the proof to finite number of steps and guarantee that there is no positive n for which the statement fails to determined.

A formal statement of **Principle of Mathematical Induction** can be stated as follows.

Let S (n) be a statement that involves positive integer $n = 1, 2, 3, \ldots\ldots$ Then S (n) is true for all positive integer n provided that

1. S (1) is true
2. S ($k + 1$) is true whenever S (k) is true.

So, there are 3 steps of proof using the principle of mathematical induction.

Step 1. (Inductive base) Verify that S (1) is true.

Step 2. (Inductive hypothesis) Assume that S (k) is true for an arbitrary value of k.

Step 3. (Inductive step) Verify that S ($k + 1$) is true on basis of the inductive hypothesis.

Note. (Change of inductive base): The principle of mathematical induction defined above begins at $n = 1$ and proves that S (n) is true for $n \geq 1$. One can also begin with an integer different from 1, say, at $n = n_0$ and prove that for $n = k + 1$ assuming that the statement is true for $n = k$ ($k \geq n_0$).

Example 40. Show that

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + n^2 = n(n+1)(2n+1)/6, n \geq 1$$

by mathematical induction.

Solution. Let S (n) be the given statement

1. **Inductive base:** For $n = 1$ we have

$$1^2 = 1(1+1)(2+1)/6 = 1$$

So, S (1) is true.

2. **Inductive hypothesis :** Assume that S (k) is true *i.e.*,

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + k^2 = k(k+1)(2k+1)/6$$

3. **Inductive Step:** We wish to show the truth of S ($k + 1$) *i.e.*,

$$1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + (k+1)^2 = (k+1)(k+2)(2k+3)/6$$

which has been obtained by substituting $k + 1$ for n is S (n).

Now, $1^2 + 2^2 + 3^2 + \ldots\ldots\ldots\ldots + (k+1)^2 = (1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + k^2) + (k+1)^2$

$$= k(k+1)(2k+1)/6 + (k+1)]$$

$$= (k+1)[(2k^2 + 7k + 6)/6]$$

$$= (k+1)(k+2)(2k+3)/6$$

which is S ($k + 1$). That is S ($k + 1$) is true whenever S(k) is true.

By the principal of mathematical induction S (n) is true for all positive integer n.

Example 41. Show that

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Solution. Let the given statement be denoted by S (n).

1. **Inductive base:** For $n = 1$

$$\frac{1}{1.2} = \frac{1}{1+1} = \frac{1}{2}$$ Hence S (1) is true.

2. **Inductive hypothesis:** Assume that S (k) is true *i.e.*,

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots\ldots\ldots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

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3. **Inductive Step:** We wish to show that the statement is true for $n = k + 1$ *i.e.*,

$$\frac{1}{1.2}+\frac{1}{2.3}+\ldots\ldots\ldots+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$$

Now, $$\frac{1}{1.2}+\frac{1}{2.3}+\ldots\ldots\ldots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{k+1}{k+2}$$

Thus S $(k + 1)$ is true whenever S (k) is True. By principle of mathematical induction S (n) is true for all positive integer n.

Example 42. Show that

$$3 + 33 + 333 + \ldots\ldots\ldots\ldots\ldots\ldots + 33 \ldots\ldots\ldots\ldots\ldots 3 = (10^{n+1} - 9n - 10)/27$$

Solution. Let the given statement be denoted by S (n).

1. **Inductive base:** For $n = 1$

$3 = (10^2 - 9.1 - 10)/27 = 3$. Hence S (1) is true.

2. **Inductive hypothesis:** Assume that S (k) is true *i.e.*,

$$3 + 33 + \ldots\ldots\ldots\ldots\ldots\ldots + 33 \ldots\ldots\ldots\ldots\ldots\ldots 3 = (10^{k+1} - 9k - 10)/27$$

3. **Inductive Step:** We wish to show the truth of S $(k + 1)$ *i.e.*,

$$3 + 33 + \ldots\ldots\ldots\ldots\ldots\ldots + 33 \ldots\ldots\ldots\ldots\ldots\ldots 3 = (10^{k+2} - 9(k+1) - 10)/27$$

Now, $3 + 33 + \ldots\ldots\ldots\ldots\ldots\ldots + 33 \ldots\ldots\ldots\ldots\ldots\ldots 3$

$$= 3 + 33 + \ldots\ldots\ldots\ldots\ldots + 33 \ldots\ldots\ldots 3 + 33 \ldots\ldots\ldots\ldots 3$$

$$= (10^{k+1} - 9k - 10)/27 + (99 \ldots\ldots\ldots\ldots 9)/3$$

$$= (10^{k+1} - 9k - 10)/27 + (10^{k+1} - 1)/3$$

$$= (10^{k+1} - 9k - 10 + 9.10^{k+1} - 9)/27$$

$$= (10^{k+2} - 9(k+1) - 10)/27$$

Thus S $(k + 1)$ is true whenever S (k) is true. By the principle of mathematical induction S (n) is true for all positive integer n.

Example 43. Show that $n^2 > 2n + 1$ for $n \geq 3$ by mathematical induction.

Solution. Let the given statement be denoted by S (n). Here S (n) is false for $n = 1$ and $n = 2$.

1. **Inductive base:** For $n = 3$

$$3^2 = 9 > 2.3 + 1 = 7 \text{ Hence S (3) is true.}$$

2. **Inductive hypothesis:** Assume that S (k) is true *i.e.*,

$$k^2 > 2k + 1$$

3. **Inductive step:** We wish to show the truth of S $(k + 1)$ *i.e.*,

$$(k+1)^2 > 2(k+1) + 1 = 2k + 3$$

Now, $$(k+1)^2 = k^2 + 2k + 1$$

$$> 2k + 1 + 2k + 1 = 2k + 2 + 2k$$

$$> 2k + 2 + 1$$

$$> 2k + 3$$

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s the hypothesis

Thus S $(k + 1)$ is true whenever S (k) is true. By the principal of mathematical induction S (n) is true for all positive integer if $n \geq 3$.

Note: The above example shows how to use induction to prove statements S (n) which a false for the first few values of n. The starting value *i.e.*, the basis for induction $n = 1$ is not real a requirement.

Principle of strong mathematical induction

It is sometimes convenient to replace the induction hypothesis S (k) by the stronger assumptio S (1), S (2), S (3), ... S (k) are true. The resulting principle known as the principle of stron mathematical induction which can be stated as follows.

For a given statement S (n) involving a natural number n is true for all positive integer $n \geq n_0$

1. S (1) is true
2. S $(k + 1)$ is true when S (k) is true for $1 \leq n \leq k$.

Thus, just as before there are 3 steps of proof by strong mathematical induction.

1. **Inductive base:** Show S (1) is true
2. **Strong inductive hypothesis:** Assume S (n) is true for all integer $1 \leq n \leq k$.
3. **Inductive step :** Show that S $(k + 1)$ is true on basis of the strong inductive hypothesis.

One can also begin at $n = n_0$ and prove the statement is true for $n = k + 1$ assuming that the statement is true for $n_0 \leq n \leq k$. In a proof using the principle of mathematical induction, we are allowed to assume S (k) in order to establish S $(k + 1)$. But in using strong mathematical induction, we assume not only S (k) but also S $(k - 1)$, S $(k - 2)$, ...S (1) as well, to establish S $(k + 1)$. The reason why strong induction is sometimes used is that by strengthening the inductive step, in other words, assuming S (1), S (2), ... S (k) rather than just S (k), proof can be simplified.

Solved Examples

Example 44. Prove by mathematical induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.

Solution. Let S (n) be the given statement.

1. **Inductive base:** For $n = 1$ we have

$$6^{1+2} + 7^{2+1} = 6^3 + 7^3 = 216 + 343 = 559 = 43 \,.\, 13$$

which is divisible by 43. So, S (1) is true

2. **Inductive hypothesis:** Assume that S (k) is true *i.e.*, $6^{k+2} + 7^{2k+1} = 43.m$ for some integer.

3. **Inductive step:** We wish to show the truth of S $(k + 1)$ *i.e.*, $6^{(k+1)+2} + 7^{2(k+1)+1} = 6^{k+3} + 7^{2k+3}$ is divisible by 43

Now, $6^{k+3}\, 7^{2k+3} = 6^{k+3} + 7^{2k+1+2} = 6^{k+3} + 7^{2k+1}\, 7^2 = 6(6^{k+2} + 7^{2k+1}) + 43\,(7^{2k+1})$

$$= 6\ 43\ m + 43\,(7^{2k+1}) = 43\,(6m + 7^{2k+1})$$

which is divisible by 43.

Hence S $(k + 1)$ is true whenever S (k) is true. By the principle of mathematical induction S (n) is true for all positive integer of n.

Example 45. If A is a set with n elements, show that A has 2^n subsets.

Solution. Let S (n) be the statement that a set with n elements has exactly 2^n subsets.

1. **Inductive base:** For $n = 1$ the set S contains only one element, say $A = \{a_1\}$ and it has only two (2^1) subsets, *viz.*, ϕ and $\{a_1\}$. Hence S (1) is true.

2. **Inductive hypothesis:** Assume that S (k) is true *i.e.*, the set $A = \{a_1, a_2, a_3, \ldots\ldots a_k\}$ has 2^k subsets.

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3. Inductive step: We wish to show the truth of S $(k + 1)$ *i.e.*, the set $A = \{a_1, a_2, a_3, \ldots\ldots\ldots, a_k, a_{k+1}\}$ has 2^{k+1} subsets.

Now the subset of A containing $(k + 1)$ elements can be divided into two categories.

Category 1. Subsets not containing a_{k+1} and this number is 2^k by inductive hypothesis

Category 2. Subsets containing a_{k+1} which can be represented as $\{a_{k+1}\}$, $\{a_1, a_{k+1}\}$, $\{a_2\ a_{k+1}\}$ ..... $\{a_1, a_2, a_{k+1}\}$, $\{a_1, a_3, a_{k+1}\}$, ... $\{a_1, a_2 \ldots\ldots\ldots a_{k+1}\}$ and this number is 2^k.

Thus the number of subsets of A = The number of subsets not containing a_{k+1} + The number of subsets containing $a_{k+1} = 2^k + 2^k = 2^{k+1}$ which is S $(k + 1)$

That is S $(k + 1)$ is true whenever S (k) is true. By the principle of mathematical induction S (n) is true for all positive of n.

Example 46. If $a_n = a_{n-1} + a_{n-2}$ for all $n > 3$ and $a_1 = a_2 = 1$

Prove by strong mathematical induction $2^{n-1}\, a_n = n \pmod 5$ for all $n > 1$

Solution. Let S (n) be the statement $2^{n-1}\, a_n = n \pmod 5$

1. Inductive base: For $n = 1$, $2^0 a_1 = 1 \pmod 5 \Rightarrow a_1 = 1 \pmod 5$ which is true since $a_1 = 1$

2. Inductive hypothesis: Let S (n) is true for all $n = 1, 2, \ldots\ldots\ldots k$

3. Induction step: Now we wish to prove S (n) for $n = k + 1$

i.e., $\quad 2^{k+1-1}\, a_{k+1} = k + 1 \pmod 5$

But $\quad a_{k+1} = a_k + a_{k-1}$

or $\quad 2^k\, a_{k+1} = 2^k\, a_k + 2^k\, a_{k-1}$

$$= 2\,(2^{k-1}\, a_k) + 2^2\,(2^{k-2}\, a_{k-1})$$

By hypothesis, we have $2^{k-1}\, a_k = k \pmod 5$ and $2^{k-2}\, a_{k-1} = k - 1 \pmod 5$

Therefore,
$$\begin{aligned} 2^{k+1-1}\, a_{k+1} &= 2^k\, a_{k+1} \\ &= 2^k\, a_k + 2^k a_{k-1} \\ &= 2 \,.\, k + 2^2\,(k - 1) \pmod 5 \\ &= k + 1 \pmod 5 \qquad [6 = 1 \pmod 5 \text{ and } -1 = 4 \pmod 5] \end{aligned}$$

Thus S $(k + 1)$ is true. Hence by mathematical induction S (n) is true for all $n > 1$.

Example 47. The nth Fibonacci number denoted by F_n is related by the reccurence relation

$F_{n+1} = F_n + F_{n-1} \qquad n \geq 2 \qquad$ and $\qquad F_1 = F_2 = 1.$

Use strong mathematical induction *prove* the inequality

$$F_n < [(1 + \sqrt{5})/2]^{n-1}, \qquad n = 1, 2, 3, \ldots\ldots\ldots$$

Solution. Let S (n) be the statement that the inequality is true for all positive integer n.

1. Inductive base: For $n = 1$, S (1) is true, since $1 \leq 1$.

In this case it is also useful to verify S (2) directly *i.e.*,

$1 \leq [(1 + \sqrt{5})/2]$ which holds true because $\sqrt{5}$ is greater than 1.

2. Strong Inductive hypothesis: We assume that S (n) is true for all $n = 1, 2, 3, \ldots. k$.

i.e., $F_k \leq t^{k-1}$, $F_{k-1} \leq t^{k-2}$ and let $t = (1 + \sqrt{5})/2$ and we observe that t is one of the solution of the equation $x^2 - x - 1 = 0$, we have $t^2 - t - 1 = 0$ *i.e.*, $t^2 = t + 1$

3. Induction step: We wish to *prove* that S $(k + 1)$ is true. Now

$$\begin{aligned} F_{k+1} = F_k + F_{k-1} &\leq t^{k-1} + t^{k-2} \\ &= t^{k-2}\,(t + 1) \\ &= t^{k-2} \,.\, t^2 = t^k \text{ which shows that S } (k + 1) \text{ is true.} \end{aligned}$$

Hence by mathematical induction S (n) is true for $n \geq 2$

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Example 48. If x and y denote any pair of real numbers for which $0 < x < y$, prove by mathematical induction $0 < x^n < y^n$ for all natural numbers n.

Solution. Let S (n) be the given statement.

1. **Inductive base:** For $n = 1$; $0 < x < y$ which is true, hence S (1) is true.
2. **Inductive hypothesis:** we assume that S (k) is true *i.e.*,

$$0 < x^k < y^k$$

3. **Induction step:** We wish to prove that S $(k + 1)$ is true.

Now we have $$0 < x^k < y^k \quad (1)$$

Multiplying both sides by y, we get

$$0 < yx^k < y^{k+1} \quad (2)$$

Again multiplying $0 < x < y$ by x^k, we get

$$0 < x^{k+1} < yx^k$$

Combining (1) and (2), we get

$$0 < x^{k+1} < yx^k < y^{k+1}$$

Thus S $(k + 1)$ is true, and by the principal of mathematical induction S (n) is true for all n.

Example 49. Show that $(1 + 2/3)^n \geq 1 + 2n/3$ for al $n \geq 1$

Solution. Let S (n) be the given statement.

1. **Inductive base:** For $n = 1$ $(1 + 2/3)^1 = 1 + 2/3$ So, S (1) is true.
2. **Inductive hypothesis:** Let the statement is true for S (k) *i.e.*,

$$(1 + 2/3)^k \geq 1 + 2k/3$$

3. **Induction step:** We wish to show that the statement is true for $n = k + 1$

Now $(1 + 2/3)^{k+1} = (1 + 2/3)^k (1 + 2/3) \geq (1 + 2k/3)(1 + 2/3)$ (by hypothesis)

$$\geq 1 + (2k/3) + 2/3 + (4k/9)$$

$$\geq 1 + (2(k + 1(/3) + (4k/9) > 1 + (2(k + 1)/3) \text{ since } (4k/9) > 0$$

Hence S $(k + 1)$ is true. Thus by the principal of mathematical induction S (n) is true for all $n \geq 1$.

Example 50. Use induction to show that

$$n! \geq 2^{n-1} \text{ for } n \geq 1$$

Solution. Let S (n) be the given statement

1. **Inductive base:** For $n = 1$, $1! = 1 \geq 1 = 2^{1-1}$, hence S (1) is true.
2. **Inductive hypothesis:** We assume that the statement is true for $n = k$

i.e., $k! \geq 2^{k-1}$

3. **Inductive step:** We wish to prove that the statement is true for $n = k + 1$

Now
$$\begin{aligned}(k + 1)! &= (k + 1)(k!)\\ &\geq (k + 1)\, 2^{k-1}\\ &\geq 2.2^{k-1} \text{ since } k + 1 \geq 2\\ &= 2^k\end{aligned}$$

Hence S $(k + 1)$ is true. Thus, by mathematical induction S (n) is true for all $n \geq 1$

2.17. Predicate Calculus

The propositional calculus does not allow us to represent many of the statements that we use in mathematics, computes science and in every day life. Predicate calculus is a generalization

of propositional calculus. It contains all the components of propositional calculus, including propositional variables and constants. Predicate calculus is important for several reasons, this has application in expert system, in database and also basis for the Prolog language.

A part of a declarative sentence describing the properties of an object or relation among objects is called a predicate. For example, consider two propositions:

Ram is a bachelor.

Shyam is a bachelor.

Both Ram and Shyam has the same property of being bachelor. In the propositional calculus there is no symbolic representation of "is a bachelor" since this phrase, or predicate, is not a sentence. The two propositions can be replaced by a single proposition " x is a bachelor'. By replacing x by Ram, Shaym (or by any other name), we get many propositions. In logic, predicates can be obtained by removing any nouns from a statement. A predicate is symbolised by a capital letter and the names of individuals or objects in general by small letters. The sentence " x is a bachelor" is symbolised as $P(x)$, where x is a **predicate variable.** When concrete values are substituted in place of x (predicate variable), a statement results. $P(x)$ is also called a **propositional function,** because each choice of x produces a proposition $P(x)$ that is either true or false. Thus a **predicate** is a sentence that contains a finite number of variables and becomes a proposition when specific values are substituted for the variables. The **domain (universe of discourse or simply universe)** of a predicate variable is the set of all possible values that may be substituted in place of variables. For example, The domain for $P(x)$: "x is a bachelor", can be taken as the set of all human names.

Universal and Existential Quantifier

One obvious way to change predicates into statements is to assign specific values to their variables. Another way to obtain statements from predicates is to add quantifiers. Quantifiers are words that refer to quantities such as some, few, many, all, none and indicate how frequently a certain statement is true.

The phrase "for all" (denoted by $\forall$) is called the universal quantifier. For example, consider the sentence " All human beings are mortal".

Let $P(x)$ denote " x is mortal".

Then the above sentence can be written as

$$(\forall x \in U)\, P(x) \qquad \text{or} \qquad \forall x\, P(x) \tag{1}$$

where U is the Universe of discourse denoting the set of all human beings. $\forall x$ represent each of the following phrases, since they have essentially the same

for all x

for every x

for each x.

The statement (1) is called a **universal statement.** The expression $P(x)$ by itself is an open sentence and therefore has no truth value. However $\forall x\, P(x)$ does have a truth value and is assigned truth values as follows:

$\forall x\, P(x)$ is true if $P(x)$ is true for every x in U;

$\forall x\, P(x)$ is false, if and only if, $P(x)$ is false for at least one x in U.

A value for which $P(x)$ is false is called a **counter example** to the universal statement.

The phrase "there exists" (denoted by $\exists$) is called the existential quantifier. For Example, consider the sentence

"there exists x such that $x^2 = 5$".

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This sentence can be written as

$$(\exists\, x \in \mathrm{R})\, P(x) \text{ or } \exists\, x\, P(x) \qquad (2)$$

where $P(x)$ denotes "$x^2 = 5$".

$\exists\, x$ represents each of the following phrases

There exists an x

There is an x

For some x

There is at least one x.

The statement (2) is called an existential statement and $\exists\, x\, P(x)$ has truth values,

$\exists\, x\, P(x)$ is true if $P(x)$ is true for at least one x in U, where U is the domain or universe of

$\exists\, x\, P(x)$ is false if $P(x)$ is false for every x in U.

When the quantifiers are used, one should specify the universe of discourse. If the universe of discourse is changed, the truth value may change. For example

$\mathrm{R}(x) : x^2 = 3$

If the universe of discourse is the set of all integers, then $\exists\, x\, \mathrm{R}(x)$ is false. If the universe of discourse is the set of all real numbers, then $\exists\, x\, \mathrm{R}(x)$ is true.

Example 51. Let Z, the set of integer, be the universe of discourse and consider the statements

$$(\forall\, x \in \mathrm{Z})\, x^2 = x$$

$$(\exists\, x \in \mathrm{Z})\, x^2 = x.$$

Find the truth values of each of the statements.

Solution. Let $P(x)$ be the proposition $x^2 = x$.

Then $\forall\, x\, P(x)$ is false because, $P(3)$ *i.e.*, $3^2 = 3$ is false.

And $\exists\, P(x)$ is true, because at least one proposition $P(x)$ is true: infact, exactly two of them are true, namely $P(0)$ and $P(1)$.

Example 52. Let $\mathrm{D} = \{1, 2, 3, \ldots 9\}$. Determine the truth value of each of the following statements.

(*a*) $(\forall\, x \in \mathrm{D})\, x + 4 < 15$

(*b*) $(\exists\, x \in \mathrm{D})\, x + 4 = 10$,

(*c*) $(\forall\, x \in \mathrm{D})\, x + 4 \leq 10$,

(*d*) $(\exists\, x \in \mathrm{D})\, x + 4 > 15$.

Solution. (*a*) True, for every number in D satisfies $x + 4 < 15$.

(*b*) True, for if $x = 6$, then $6 + 4 = 10$

(*c*) False, for $x = 7$, $7 + 4 \nleq 10$

(*d*) False, for every number in D, $x + 4 > 15$ is false.

Translating Sentences into Logical Expressions

The logical operators and quantifiers can be used to express English sentences into logical expressions. Consider for instance, the sentence.

Every person is precious (1)

We rephrase this,

For every x, if x is person then x is precious (2)

Thus if $\mathrm{M}(x)$: x is a person and $\mathrm{A}(x)$: x is precious. We may then write

For every x, $M(x) \rightarrow A(x)$

or $$(\forall x)(M(x) \rightarrow A(x))$$

Statements containing words like every, each and every one usually indicate universal quantifier. Such statements must typically be reworded such that they start with for every x, which is then denoted by $\forall x$.

Consider the sentence:

Some student of this college passed MCA entrance examination.

We rephrase this,

There exist student of this college who passed MCA entrance examination :

Let P be the property *passed* MCA entrance examination. Then the sentence can be written as $\exists x\, P(x)$.

Statements containing phrase as some and at least one suggest existential quantification. They should be rephrased as there is an x such that which is translated by $\exists x$.

Example 53. Let $K(x)$: x is man, $L(x)$: x is mortal

$M(x)$: x is an integer, $N(x)$: either positive or negative

Express the following using quantifiers

(*a*) All men are mortal

(*b*) Any integer is either positive or negative

Solution. (*a*) The given sentence can be written as

For all x, if x is a man, then x is mortal and this can be expressed as

$$(\forall x)(K(x) \rightarrow L(x))$$

(*b*) The given sentence can be written as

For all x, if x is an integer, then x is either positive or negative

$$(\forall x)(M(x) \rightarrow N(x)).$$

Example 54. Let $K(x)$: x is student, $M(x)$: x is clever, $N(x)$: x is successful

Express the followings using quantifiers

(*a*) There exists a student

(*b*) Some students are clever

(*c*) Some students are not successful

Solution. (*a*) $(\exists x)(K(x))$

(*b*) There exists an x such that x is student and x is clever.

$$(\exists x)(K(x) \wedge M(x))$$

(*c*) There exists an x such that x is student and x is not successful.

$$(\exists x)(K(x) \wedge \sim N(x)).$$

Negations of Quantified Statements

Consider the statement

"All students in the class have taken a course in discrete mathematics."

This statement can be written as

$$\forall x\, P(x),$$

where $P(x)$ is the statement "x has taken a course in discrete mathematics".

Its negation reads:

"It is not the case that all students in the class have taken a course in discrete mathematics".

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This is equivalent to

"There is a student in the class who has not taken a course in discrete mathematics".

This is simply the existential quantification of the negation of the original propositional function, namely

$$\exists x \sim P(x)$$

This example illustrates the following equivalence:

$$\sim \forall x P(x) \equiv \exists x \sim P(x)$$

Thus, the negation of a universal statement (all are) is logically equivalent to an existential (some are not) statement.

Now consider the statement

"There is a student in this class who has taken a course in discrete mathematics". The statement can be written as

$$\exists x \; Q(x)$$

where $Q(x)$ is the statement " x has taken a course in discrete mathematics".

Its negation reads as

"It is not the case that there is a student in this class who has taken a course in discrete mathematics". This is equivalent to

"All students in this class has not taken discrete mathematics".

which is just the universal quantification of the negation of the original propositional function *i.e.*,

$$\forall x \sim Q(x).$$

This example illustrates the equivalence

$$\sim \exists x \, Q(x) \equiv \forall x \sim Q(x)$$

Thus, the negation of an existential statement (some are) is logically equivalent to a universal statement (all are not).

We list these facts as follows :

Statement		Negation	
all true	$\forall x \, P(x)$	$\exists x [\sim P(x)]$	at least one false
at least one false	$\exists x [\sim P(x)]$	$\forall x \, P(x)$	all true
all false	$\forall x [\sim P(x)]$	$\exists x \; P(x)$	at least one true
at least one true	$\exists x \, P(x)$	$\forall x [\sim P(x)]$	all false

In general the negation of a quantified predicate is logically equivalent to the proposition obtained by replacing each $\forall$ by $\exists$, replacing each $\exists$ by $\forall$, and by replacing the predicate itself by its negation.

Example 55. Negate the statements.

(*i*) All integers are greater than 8.

(*ii*) For all real numbers x, if $x > 3$ then $x^2 > 9$

Solution. (*i*) Let $P(x) : x > 8$. The given statement can be written as $\forall x \, (P(x))$. The negation of the statement $\sim \forall x \, (P(x)) \equiv \exists x (\sim P(x))$.

There exists an integer x such that x less than or equals to 8.

(*ii*) Let $P(x)$ and $Q(x)$ denote " $x > 3$" and "$x^2 > 9$" respectively.

Then the given statement can be written as

$$\forall x\,(P(x) \rightarrow Q(x))$$

which is equivalent $\forall x\,(\sim P(x) \vee Q(x))$.

The negation of the statement is

$$\exists x\,(P(x \wedge \sim Q(x))$$

There exists a real number x such that $x > 3$ and $x^2 \leq 9$.

Bound and free Variables

If a quantifier is applied on the variable x, then this occurrence of the variable is called bounded and the variable is called bound variable. The quantified variables, either by $\forall$ or $\exists$ are bound by values of universe of discourse on which the function is defined.

An occurrence of a variable that is not bound by a quantifier is said to be free.

The smallest formula immediately following $\forall x$, or $\exists x$ is called the **scope** of the quantifier. Consider the following examples:

(*i*) $\forall x\, P(x, y)$ (*ii*) $\exists x\,(P(x)) \rightarrow Q(x)$

(*iii*) $\forall x[P(x) \rightarrow (\exists y)\, Q(x, y)]$

Solution. (*i*) x is a bound variable, where as y is a free variable since quantifier is applied only on x not on y. The scope of $\forall x$ is $P(x, y)$.

(*ii*) Here $\exists x$ quantifies only $P(x)$, and therefore x is bound variable for $P(x)$, in $Q(x)$ it is free variable. The scope of $\exists x$ is $P(x)$.

(*iii*) Both x and y are bound variable since $\forall x$ quanitify $P(x) \rightarrow (\exists y)\, Q(x, y)$ and $\exists y$ quantifies $Q(x, y)$. The scope of $\forall x$ is $P(x) \rightarrow \exists y\, Q(x, y)$ and the **scope** of $\exists y$ is $Q(x, y)$.

Rules of Inference of Predicate Calculus

Before discussing the rules of inference, we note that: (*a*) proposition formulas are also predicate formulas; (*b*) predicate formulas where all the variables are quantified are proposition formulas. Therefore, all the rules of inference for proposition formulas are also applicable for predicate calculus where ever necessary (also known as universal specification). (Four more additional rules of inference for predicate calculus are) is the rule of inference that indicates the truth of a property in a particular case follows as a special instance of its more general or universal truth *i.e.*, $P(c)$ is true, when c is a particular number of the universe of discourse, given the premise $\forall x\, P(x)$. Universal instantiation is used when we conclude from the statement

Universal instantiation

Every man is mortal

Socrates is a man

$\therefore$ Socrates is mortal

when Socrates is a member of the universe of discourse of all men. This form of an argument is called instantiation because it consists of taking a particular instance of a statement.

Universal generalization is the rule of inference which states that $\forall x\, P(x)$ is true, given the premise that $P(c)$ is true for all elements c in the universe of discourse. The element c that we select must be an arbitrary and not a specific element of the universe of discourse.

Existential instantiation (also known as existential specification) is the rule which allows us to conclude that there is an element c in the universe of discourse for which $P(c)$ is true if we know that $\exists x\, P(x)$ is true.

Existential generalization is the rule of inference which is used to conclude that $\exists x\, P(x)$ is true when a particular element c with $P(c)$ true is known. That is, if we know one element c in the universe of discourse for which $P(c)$ is true, then we know that $\exists x\, P(x)$ is true.

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We summarise these rules of inference in Table 2.30.

1. Universal instantiation:

$$\frac{\forall x\, P(x)}{\therefore\ P(c)}$$

c is some element of the universe.

2. Existential instantiation

$$\frac{\exists x\, P(x)}{\therefore\ P(c)}$$

c is some element for which $P(c)$ is true.

3. Universal generalisation

$$\frac{P(x)}{\forall x\, P(x)}$$

x should not be free in any of the given premises.

4. Existential generalisation

$$\frac{P(c)}{\therefore\ \exists x\, P(x)}$$

c is some element of the universe.

Table 2.30

Universal Modus Ponens

The rule of universal instantiation can be combined with modus ponens to obtain the rule called universal modus ponens.

$$\frac{\begin{array}{l}\forall x, (P(x) \to Q(x)\\ P(a)\end{array}}{\therefore Q(a)}$$

a is some element of universe

The argument form consists of two premises and a conclusion and at least one premise is quantified. The first and second premises are called it **major** and **minor** premises, respectively.

Universal Modus Tollens

The rule of universal instantiation can be combined with modus tollens to obtain the rule called universal modus tollens.

$$\frac{\begin{array}{l}\forall x\,(P(x) \to Q(x))\\ \sim Q(a)\end{array}}{\therefore \sim P(a)}$$

a is some element of universe

Universe Hypothetical Syllogism

$$\frac{\begin{array}{l}\forall x\,(P(x) \to Q(x))\\ Q(a) \to R(a)\end{array}}{\therefore\ P(a) \to R(a)}$$

a is some element of the universe.

Example 56. Rewrite the following argument using quantifiers, variables and predicate symbols. Prove the validity of the argument.

If a number is odd, then its square is odd.
K is a particular number that is odd.
$\therefore$ K^2 is odd.

Solution. $\forall x$, if x is odd, then x^2 is odd.

Let $P(x)$: x is odd.

$Q(x)$: x^2 is odd.

And let k stands for a particular member that is odd. Then the argument has the following form

$$\begin{array}{l} \forall x\,(P(x) \to Q(x)) \\ P(k) \\ \hline \therefore\ Q(k) \end{array}$$

By universal modus ponens this argument is valid.

Example 57. Rewrite the following argument using quantifiers, variables and predicate symbols. Prove the validity of the argument.

All healthy people eat an apple a day.
Ram does not eat apple a day
Ram is not a healthy person.

Solution. $\forall x$, if x is healthy person, then x eats an apple a day.

Let $P(x)$: x is a healthy person.

$Q(x)$: x eats an apple a day.

Let c stands for Ram

$$\begin{array}{l} \forall x\,(P(x) \to Q(x)) \\ \sim Q(c) \\ \hline \therefore\ \sim P(c) \end{array}$$

By universal modus tollens this argument is valid.

Note that the following arguments are invalid.

Converse Error	**Inverse Error**
$\forall x$, if $P(x)$ then $Q(x)$	$\forall x$, if $P(x)$ then $Q(x)$
$Q(a)$ for a particular a	$\sim P(a)$, for a particular a.
$\therefore$ $P(a)$	$\therefore$ $\sim Q(a)$

Multiple Quantifiers

We can also consider predicates which are functions of more than one variable, perhaps from more than one universe of discourse and in such cases multiple use of quantifiers is natural. It is also possible to continue universal and existential quantifiers within a single statement involving more than one variable. There are eight ways to apply the two quantifiers to the two variables, $\forall x\ \forall y$, $\forall y\ \forall x$, $\forall x\ \exists y$, $\exists y\ \forall x$, $\exists x\ \forall y$, $\forall y\ \exists x$, $\exists y\ \exists x$, $\exists x\ \exists y$.

Let us consider the statement $P(x, y)$: x admires y. Then

$(\forall x)\,(\forall y)\,P(x, y)$	:	Every one admires everybody.
$(\forall x)\,(\exists y)\,P(x, y)$	:	Every one admires some one.
$(\exists x)\,(\forall y)\,P(x, y)$	:	Some one admires every one.
$(\exists x)\,(\exists y)\,P(x, y)$	:	Some one admires somebody.
$(\forall y)\,(\forall x)\,P(x, y)$	:	Everybody is admired by every one.
$(\forall y)\,(\exists x)\,P(x, y)$	:	Every body is admired by some one.
$(\exists y)\,(\forall x)\,P(x, y)$	:	There is someone whom every one admires.
$(\exists y)\,(\exists x)\,P(x, y)$	:	There is someone whom some one admires.

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The first and fifth, and fourth and eighth are logically equivalent, one is allowed to interchange universal quantifiers, and the same is true for existential quantifiers. This is indicated by the following laws:

$$\forall x \forall y P(x, y) \equiv \forall y \forall x P(x, y) \quad (1)$$

$$\exists x \exists y P(x, y) \equiv \exists y \exists x P(x, y) \quad (2)$$

Both the sides of (1) are true if and only if $P(x, y)$ is true for all possible pairs x and y and false if there is a pair x, y for $P(x, y)$ is false. Both side of (2) are true if there is a single pair of x, y that makes $P(x, y)$ true and false if $P(x, y)$ is false for every pair x, y.

To verify rule (2), that is, to verify that it is tautology (the proposition has the value true for all possible universe of discourse). We need to check that $\exists x \exists y P(x, y)$ has the value true for a given universe if and only if $\exists y \exists x P(x, y)$ has the value true for that universe. Suppose $\exists x \exists y P(x, y)$ is true. The $\exists y P(x_0, y)$ is true for some x_0 in the universe, so $P(x_0, y_0)$ is true for some y_0 in the universe. Hence $\exists P(x, y_0)$ is true and thus $\exists y \exists x P(x, y)$ is true. The implication in other direction follows similarly.

When both the universal and existential quantifiers appear the order of quantifiers is very important. The statement $\exists x \forall y P(x, y)$ and $\forall y \exists x P(x, y)$ are not logically equivalent. A different ordering of the quantifiers may yield a different statement. The statement $\exists x \forall y P(x, y)$ is true if there is an x, for which $P(x, y)$ is true for every y and false, if for every x there is a y for which $P(x, y)$ is false. So, for this statement to be true, no matter which x one choose, there must be a value of y (possibly depending on x) for which $P(x, y)$ is true. The statement $\exists y \forall x P(x, y)$ is true if and only if there is a y that makes $P(x, y)$ true for every x. So, for this statement to be true, there must be a particular value of y for which $P(x, y)$ is true regardless of the choice of x.

Example 58. (*a*) Consider the statement for every y, there exist a x such that $2 + x = y$ that is

$$\forall y \exists x, 2 + x = y$$

This is true, for example, if y is 7, then x is 5, if y is 0, then x is -2 and so on.

Thus if we consider a fixed y, then, $2 + x = y$ is true for all y.

However, if the order of the quantifies is reversed it becomes.

$$\exists x \forall y, 2 + x = y$$

That is there exists a single value of x such that for every y, $2 + x = y$ which is not true; hence the statement is false.

The difference in the two statement is that the choice of x is made after the choice of y (and dependent on it) in the first case, but before it (and thus independent of it) in the second case. Thus the order of quantifiers is very significant. A different ordering of the quantifiers may yield a different statement.

(*b*) Let $Q(x, y, z)$ be the statement $x + y = z$. What are the truth values of the statement $\forall x \forall y \exists z Q(x, y, z)$ and $\exists z \forall x \forall y Q(x, y, z)$ where $x, y, z \in R$.

Solution. $(\forall x \forall y \exists z) Q(x, y, z)$ says for all real number x and y, there is a real number z such that $x + y = z$ which is true since the sum of two real numbers is a real number.

$\exists z \forall x \forall y Q(x, y, z)$ says these is a real number z, such that for all real numbers x and y $x + y = z$ which is false since there is no value of z that satisfies the equation $x + y = z$ for all values of x and y.

Example 59. Show that

$$\exists y \forall x P(x, y) \Rightarrow \forall x \exists y P(x, y)$$

Solution.

$\exists y \forall x P(x, y)$	Premise (Given)
$\forall x P(x, b)$	by Existential instantiation

$P(x, b)$ by Universal instantiation

$\exists y\, P(x, y)$ by Existential generalization

$\forall x \exists y\, P(x, y)$ by Universal generalization

Note: The converse is not true.

One can prove the following implications using the method of derivation.

$$\forall y \forall x\, P(x, y) \Rightarrow \exists x \forall y\, p(x, y)$$
$$\exists x \forall y\, P(x, y) \Rightarrow \forall y \exists x\, P(x, y)$$
$$\forall x \exists y\, P(x, y) \Rightarrow \exists y \exists x\, P(x, y)$$
$$\forall y \exists x\, P(x, y) \Rightarrow \exists x \exists y\, P(x, y)$$

Negations of Quantified Statements with More Than One Variable

Quantified statements with more than one variable may be obtained by successively applying the rule for negating a statement with only one quantifier. Thus each $\forall$ is changed to $\exists$ and each $\exists$ is changed to $\forall$ as the negation symbol passes through the statement from left to right. For example,

$$\sim [\forall x \exists y\, P(x, y)] \equiv \exists x [\sim \exists y\, P(x, y)]$$
$$\equiv \exists x \forall y [\sim P(x, y)]$$

and $\sim (\exists x \exists y \sim P(x, y) \wedge \forall x \forall y\, Q(x, y)) \equiv \forall x \forall y\, P(x, y) \vee (\exists x \exists y \sim Q(x, y))$

SOLVED EXAMPLES

Example 60. Show tha $R \wedge (P \vee Q)$ is a valid conclusion from the premise $P \vee Q$, $Q \to R$, $P \to M$ and $\sim M$.

Solution. The valid argument for deductin $R \wedge (P \vee Q)$ from the given premises is given as a sequence.

1.	$P \to M$	Premise (Given)
2.	$\sim M$	Premise (Given)
3.	$\sim P$	Modus tollens using 1 and 2
4.	$P \vee Q$	Premise (Given)
5.	Q	Disjunctive syllogism using 3 and 4
6.	$Q \to R$	Premise (Given)
7.	R	Modus ponens using 5 and 6
8.	$R \wedge (P \vee Q)$	Conjunction using 7 and 4

Hence proved.

Example 61. *Show that $\sim r$ is a valid conclusion from the premises $p \to \sim q$, $r \to p$, q*

(a) with truth table *(b) without truth table.*

Solution. (*a*) We construct the following truth table :

Table 2.31

p	q	r	$p \to \sim q$	$r \to p$	$\sim r$
T	T	T	F	T	F
T	T	F	F	T	T
T	F	T	T	T	F
T	F	F	T	T	T
F	T	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	F
F	F	F	T	T	T

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Now premises $p \to \sim q$, $r \to p$, q are true simultaneously in row 6, where $\sim r$ is also true. This proves the validity of the conclusion.

(*b*) The valid argument for deducting $\sim r$ from the given premises is given as a sequence.

1.	$r \to p$	Premise (Given)
2.	$p \to \sim q$	Premise (Given)
3.	$r \to \sim q$	Hypothetical syllogism using 1 and 2
4.	q	Premise (Given)
5.	$\sim r$	Modus tollens using 3 and 4

Example 62. *Show that* $(p \vee q) \wedge (\sim p \wedge \sim q)$ *is a contradiction.*

Solution. The truth table for the given statement is as given below:

Table 2.32

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

It is evident from the truth table that the given statement is a contradiction.

Example 63. *Write down the negation of the following proposition for every number x there is a number y such that* $y < x$.

Solution. The given proposition can be written in symbolic form as

$$\forall x\ \exists y\ p(x, y) \text{ where } p(x, y) : y < x$$

Now
$$\sim [\forall x\ \exists y\ p(x, y)]$$
$$\equiv \exists x[\sim \exists y\ p(x, y)]$$
$$\equiv \exists x\ \forall y\ [\sim p(x, y)]$$

Hence the negation of the given proposition is

There is a number x such that for every number y, $y \geq x$.

Example 64. *Using rule of inference, determine whether the following inference patterns are valid or not.*

$$\begin{array}{l} \sim t \to \sim r \\ \sim s \\ t \to w \\ r \vee s \\ \hline \therefore\ w \end{array}$$

Solution.

1.	$r \vee s$	Premise (Given)
2.	$s \vee r$	Commutative law
3.	$\sim s$	Premise (Given)
4.	r	Disjunction syllogism using 2 and 3.
5.	$\sim t \to \sim r$	Premise (Given)
6.	$\sim (\sim t)$	Modus tollens using 5 and 4.
7.	t	

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8.	$t \to w$	Premise (Given)
9.	w	Modus ponens using 8 and 7

Hence the given inference pattern is valid.

Example 65. *Determine whether the conclusion C is valid in the following premises :*

H_1 : $P \to (Q \to R)$

H_2 $P \wedge Q$

C : R

Solution.

1.	$P \wedge Q$	Premise (Given)
2.	P	Simplification using 1
3.	$P \to (Q \to R)$	Premise (Given)
4.	$Q \to R$	Modus Ponens using 3 and 2
5.	$Q \wedge P$	Commutative law of $\wedge$ using 1.
6.	Q	Simplification using 1
7.	R	Modus ponens using 4 and 6

Hence the conclusion C is valid.

Example 66. *Show that* $[(p \vee q) \wedge \sim (\sim p \wedge (\sim q \vee \sim r))] \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$ *is tautology by using laws of logic.*

Solution. We have

$[(p \vee q) \wedge \sim (\sim p \wedge (\sim q \vee \sim r))] \vee (\sim p \wedge \sim q) \vee (\sim p \wedge \sim r)$

$\equiv [(p \vee q) \wedge \sim (\sim p \wedge \sim (q \wedge r))] \vee (\sim (p \vee q) \vee \sim (p \vee r)$ (Using De Morgan's law)

$\equiv [(p \vee q) \wedge (p \vee (q \wedge r))] \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv [(p \vee q) \wedge (p \vee q) \wedge (p \vee r)] \vee \sim ((p \vee q) \wedge (p \vee r))$ (Using Distributive law)

$\equiv [((p \vee q) \wedge (p \vee q)) \wedge (p \vee r)] \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv ((p \vee q) \wedge (p \vee r)) \vee \sim ((p \vee q) \wedge (p \vee r))$

$\equiv x \vee \sim x$ where $x = (p \vee q) \wedge (p \vee r)$

$\equiv$ T (Complement law)

Hence Proved.

Example 67. *By the principal of mathematical induction, prove that*

$3^{2n+1} + (-1)^n 2 \equiv 0 \pmod 5$

Solution. Let the given statement be denoted by $S(n)$.

1. **Inductive base:** For $n = 1$

   $3^{2.1+1} + (-1)^1 . 2 = 27 - 2 = 25 \equiv 0 \pmod 5$. Hence $S(1)$ is true.

2. **Inductive hypothesis** : Assume that $S(k)$ is true, i.e.

   $3^{2k+1} + (-1)^k 2 \equiv 0 \pmod 5$

3. **Inductive step** : We wish to show the truth of $S(k+1)$, *i.e.*

$$3^{2(k+1)1} + (-1)^{k+1} 2 = 3^{2k+1+2} - (-1)^k 2$$

$$= 3^{2k+1}.3^2 - (1)^k 2$$

$$= 9(3^{2k+1} + (-1)^k 2) - (-1)^k 18 - (-1)^k .2$$

$$= 9(3^{2k+1} + (-1)^k 2) - (-1)^k 20$$

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Since $9(3^{2k+1} + (-1)^k.2) = 0 \pmod 5$ and $2 = 0 \pmod 5$, it follows that $S(k+1)$ is true whenever $S(k)$ is true. By the principal of mathematical induction $S(n)$ is true for all positive integer.

Example 68. *Give the symbolic form of the following statements.*

(i) Some men are genius.

(ii) For every x, there exists a y such that $x^2 + y^2 \geq 100$.

(iii) Given any positive integer, there is a greater positive integer.

(iv) Every one who likes fun will enjoy each of these plays.

Solution.

(*i*) Let $M(x)$: x is a man

$G(x)$: x is genious

Hence, the given statement can be written symbolically,

$(\exists x)\ (M(x) \wedge G(x))$

(*ii*) $(\forall x)\ (\exists y)\ (x^2 + y^2 \geq 100)$

(*iii*) Let x and y be two variables belong to Z^+. The given statement can be written as: For all x, there exists *a* y such that y is greater than x.

Let $G(x, y)$: x is greater than y

The given statement is

$(\forall x)\ (\exists y)\ G(y, x)$

(*iv*) Let $L(x)$: x likes fun.

$P(x)$: x is a play

$E(x, y)$: x will enjoy y.

The given statement in symbolice form is

$(\forall x)\ (\forall y)\ [(L(x) \wedge P(y) \Rightarrow E(x, y))]$

Example 69. *Verify the validity of the inference. If one person is more successful than another, then he has worked harder to deserve success. John has not worked harder than Peter. Therefore, John is not successful than Peter.*

Solution. Universe : All persons.

Let $S(x, y)$: x is more successful than y.

$W(x, y)$: x has worked harder than y to deserve success.

The given premises are

$(\forall x)\ (\forall y)\ [S(x, y) \rightarrow W(x, y)]$, $\rceil W(a, b)$,

Conclusion : $\rceil S(a, b)$

1. $\rceil W(a, b)$ Premise (Given)

2. $(\forall x)\ (\forall y)\ [S(x, y) \rightarrow W(x, y)]$, Premise (Given)

3. $(\forall y)\ [S(a, y) \rightarrow W(a, y)]$ (Universal instantiation and 2)

4. $S(a, b) \rightarrow W(a, b)]$ (Universal instantiation and 3)

5. $\rceil S(a, b)$ (From 1 and 4.)

Example 70. *Obtain pcnf and pdnf of the formula $(\rceil p \vee \rceil q) \rightarrow (p \leftrightarrow \rceil q)$.*

Solution: Let $S \equiv (\rceil p \vee \rceil q) \rightarrow (p \leftrightarrow \rceil q)$. The truth table of S is given below:

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p	q	$\rceil p$	$\rceil q$	$\rceil p \vee \rceil q$	$p \leftrightarrow \rceil q$	S	$\rceil S$
T	T	F	F	F	F	T	F
T	F	F	T	T	T	T	F
F	T	T	F	T	T	T	F
F	F	T	T	T	F	F	T

$\therefore$ The pdnf of S is $(p \wedge q)$ $(p \wedge \rceil q)$ $(\rceil p \wedge q)$ since the minterm corresponding to the 3T values of the column S are $p \wedge q$, $p \wedge \rceil q$ and $\rceil p \wedge q$.

The pcnf $= \rceil(\rceil pdnf) = \rceil$(pdnf of $\rceil S$)

$= \rceil(\rceil p \wedge \rceil q) = p \vee q$.

Example 71. Obtain the pdnf and pcnf of the formula

$$S \equiv (\rceil Q \vee \rceil R) \rightarrow \rceil P) \wedge ((Q \vee R) \rightarrow P)$$

Solution.

$S \equiv (\rceil(\rceil Q \vee \rceil R) \vee \rceil P) \wedge (\rceil(Q \vee R) \vee P)$

$\equiv (Q \wedge R) \vee \rceil P) \wedge ((\rceil Q \wedge \rceil R) \vee P)$

$\equiv [(Q \wedge R) \wedge (\rceil Q \wedge \rceil R)] \vee (Q \wedge R \wedge P) \vee (\rceil P \wedge \rceil Q \wedge \rceil R) \vee (\rceil P \wedge P)$

by extended distributive law

$\equiv [(Q \wedge \rceil Q) \wedge (R \wedge \rceil R)] \vee (P \wedge Q \wedge R) \vee (\rceil P \wedge \rceil Q \wedge \rceil R) \vee F$

$\equiv (F \wedge F) \vee (P \wedge Q \wedge R) \vee (\rceil P \wedge \rceil Q \wedge \rceil R)$

$\equiv (P \wedge Q \wedge R) \vee (\rceil P \wedge \rceil Q \wedge \rceil R)$ which is the pdnf of S.

$\therefore$ pcnf of $S = \rceil$(pdnf of $\rceil S$)

$= \rceil(P \wedge Q \wedge \rceil R) \vee (P \wedge \rceil Q \wedge R) \vee (P \wedge \rceil Q \wedge \rceil R) \vee (\rceil P \wedge Q \wedge R) \vee (\rceil P \wedge Q \wedge \rceil R)$
$\vee (\rceil P \wedge \rceil Q \wedge R)]$

$= (\rceil P \vee \rceil Q \vee R) \wedge (\rceil P \vee Q \vee \rceil R) \wedge (\rceil P \vee Q \vee R) \wedge (P \vee \rceil Q \vee R) \wedge (P \vee \rceil Q \vee R)$
$\wedge (P \vee Q \vee \rceil R)$

Example 72. Prove that for any integer $n > 1$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{n}} \geq \sqrt{n}.$$

Solution. Let the given statement be denoted by $S(n)$.

Inductive base: For $n = 1$

$\frac{1}{\sqrt{1}} = 1 = \sqrt{1}$. Hence $S(1)$ is true.

Inductive hypothesis: Assume the result is true for $n = k$

i.e., $S(k)$ is true, Hence

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots\ldots + \frac{1}{\sqrt{k}} \geq \sqrt{k} \qquad \ldots(1)$$

Inductive step: We wish to show that the result is true for $n = k + 1$.

Adding $\frac{1}{\sqrt{k+1}}$ on both sides of (1), we get

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$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots\ldots\frac{1}{\sqrt{k}}+\frac{1}{k+1} \geq \sqrt{k}+\frac{1}{\sqrt{k+1}}$$

$$\geq \sqrt{k}+\frac{1}{\sqrt{k+1}+\sqrt{k}}$$

$$\geq \sqrt{k}+\frac{\sqrt{k+1}-\sqrt{k}}{k+1-k}=\sqrt{k+1}$$

i.e the result is true for $n = k + 1$.

Hence by induction $S(n)$ is true $\forall n \in N$.

Example 73. Using mathematical induction find a general formula on the basis of observations of the following statements.

$$1^3 = 1$$
$$2^3 = 3 + 5$$
$$3^3 = 7 + 9 + 11$$
$$4^3 = 13 + 15 + 17 + 19$$

Solution. Let $S(1) = 1^3 = 1$

$$S(2) = 2^3 = 3 + 5 = (2^2 - 2 + 1) + \{(2^2 - 2 + 1) + 2\}$$

$$S(3) = 3^3 = 7 + 9 + 11 = (3^2 - 3 + 1) + \{(3^2 - 3 + 1) + 2\} + \{(3^2 - 3 + 1) + 4\}$$

Hence by induction

$$S(n) = n^3 = (n^2 - n + 1) + \{(n^2 - n + 1) + 2\} + \{n^2 - n + 1) + 4\} + \ldots\ldots\ldots\ldots\ldots\ldots + \{(n^2 - n + 1) + (2n - 2)\}$$

(Total number of terms are n)

$$S(n) = (n^2 - n + 1) + (n^2 - n + 3) + (n^2 - n + 5) + \ldots\ldots\ldots\ldots + (n^2 + n - 1)$$

which is the general formula.

Example 74. *Let p: He is intelligent and q:* He is *tall* be two propositions. Write each of the following statement in symbolic form using p and q:

(*i*) He is tall but not intelligent.

(*ii*) He is neither tall nor intelligent.

(*iii*) He is intelligent or he is tall.

(*iv*) It is not true that he is intelligent or tall.

(*v*) It is not true that he is not tall or not intelligent.

Solution.

(*i*) $q \wedge \sim p$

(*ii*) $\sim q \wedge \sim p$

(*iii*) $p \vee q$

(*iv*) $\sim (p \vee q)$

(*v*) $\sim (\sim q \vee \sim p)$

Example 75. There are two restaurants next to each other one has signboard that says "Good food is not cheap" and other has signboard that says 'Cheap food is not good'. Are the signboards saying the same thing'.

Solution. Let p : Food good and q : Food is cheap. Then the statement 'Good food is not cheap' is written as

$$p \to \sim q$$

and the statement 'Cheap food is not good' is written as

$$q \to \sim p.$$

The truth table for the statement are given below

p	q	$\sim p$	$\sim q$	$p \to \sim q$	$q \to \sim p$
T	T	F	F	F	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

Since the two columns of the statements are same, they are saying the same thing.

Example 76. Given the value of $p \to q$ is false, determine the value of $(\sim p \vee \sim q) \to q$.

Solution. The truth table for the given two statement is given below:

p	q	$\sim p$	$\sim q$	$p \to q$	$(\sim p) \vee (\sim q)$	$(\sim p \vee \sim q) \to q$
T	T	F	F	T	F	T
T	F	F	T	F	T	F
F	T	T	F	T	T	T
F	F	T	T	T	T	F

From the table it follows that $p \to q$ is false, then the value of $(\sim p \vee \sim q) \to q$ is false.

Example 77. Prove the following equivalences by using laws of propositional algebra.

(*a*) $(p \to q) \to q \equiv p \vee q$

(*b*) $P \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$

(*c*) $(\sim p \vee q) \wedge (p \wedge (p \wedge q)) \equiv p \wedge q$

Solution. (*a*)

$$\begin{aligned}
(p \to q) \to q &\equiv (\sim p \vee q) \to q \\
&\equiv \sim(\sim p \vee q) \vee q && [\because r \to s \equiv \sim r \vee s] \\
&\equiv (p \wedge \sim q) \vee q && \text{by De Morgain's law} \\
&\equiv (p \vee q) \wedge (\sim q \vee q) && \text{by Distributive law} \\
&\equiv (p \vee q) \wedge T && [\because \sim q \vee q \equiv T] \\
&\equiv p \vee q && [\because p \wedge T \equiv p]
\end{aligned}$$

(*b*)

$$\begin{aligned}
\text{RHS} = (P \to Q) \vee (P \to R) &\equiv (\sim P \vee Q) \vee (\sim P \vee R) && \text{by Commutative law} \\
&\equiv Q \vee \{\sim P \vee (\sim P \vee R)\} && \text{by Associative law} \\
&\equiv Q \vee \{(\sim P \vee \sim P) \vee R\} && \text{" " "} \\
&\equiv Q \vee \{\sim P \vee R\} && \text{by Idempotent law} \\
&\equiv Q \vee \{R \vee \sim P\} && \text{by Commutative law} \\
&\equiv (Q \vee R) \vee \sim P && \text{by Associative law} \\
&\equiv P \vee (Q \vee R) && \text{by Commutative law} \\
&\equiv P \to (Q \vee R) && [\because P \to Q \equiv \sim P \vee Q]
\end{aligned}$$

(*c*)

$$\begin{aligned}
(\sim p \vee q) \wedge (p \wedge (p \wedge q)) &\equiv (\sim p \vee q) \wedge ((p \wedge p) \wedge q) && \text{by Associative law} \\
&\equiv (\sim p \vee q) \wedge (p \wedge q) && \text{by Idempotent law} \\
&\equiv ((p \wedge q) \wedge (\sim p \vee q) && \text{by Commutative law} \\
&\equiv ((p \wedge q) \wedge \sim p) \vee ((p \vee q) \wedge q) && \text{by Distributive law} \\
&\equiv (\sim p \wedge (p \wedge q)) \vee ((p \wedge q) \wedge q) && \text{by Commutative law}
\end{aligned}$$

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$\equiv ((\sim p \wedge p) \wedge q) \vee (p \wedge (q \wedge q)$ by Associative law

$\equiv (F \wedge q) \vee (p \wedge q)$ by Commutative and Idompotent law

$\equiv F \vee (p \wedge q)$ $\because F \wedge p \equiv F$ and $F \vee p \equiv p$

$\equiv p \wedge q$

Example 78. Obtain the product of sums canonical form for $(P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge\neg R)$

Solution. The minterms of P, Q, R are $P\wedge Q\wedge R$, $P\wedge Q\wedge\neg R$, $P\wedge\neg Q\wedge R$, $\neg P\wedge Q\wedge R$, $P\wedge\neg Q\wedge\neg R$, $\neg P\wedge Q\wedge\neg R$, $\neg R\wedge\neg Q\wedge R$, $\neg P\wedge\neg Q\wedge\neg R$.

Let $A \Leftrightarrow (P\wedge Q\wedge R)\vee(\neg\wedge Q\wedge R)\vee(\neg P\wedge\neg Q\wedge\neg R)$

$\neg A \Leftrightarrow (P\wedge Q\wedge\neg R)\vee(P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)\vee(\neg P\wedge Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)$

(remaining minterms)

$A \Leftrightarrow \neg(\neg A)$ By applying duality principle to $\neg A$, we get pcnf

$\Leftrightarrow (\neg P\wedge\neg Q\vee R)\wedge(\neg P\wedge Q\wedge\neg R)\wedge(\neg P\vee Q\vee R)\wedge(P\vee\neg Q\vee R)\wedge(P\vee Q\vee\neg R)$

Example 79. Obtain pdnf and pcnf of the formula

$(\neg P\vee\neg Q)\rightarrow(P \leftrightarrow \neg Q)$

Solution. The truth of the given formula is

p	Q	$\neg P$	$\neg Q$	$(\neg p \vee \neg Q) \equiv A$	$P \leftrightarrow \neg Q \equiv \neg B$	$A \rightarrow B$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	T

Since the minterms corresponding of the 3 T values of the last columns are $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, the pdnf of the given formula is $(P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Now pdnf of $\neg(A \rightarrow B) \equiv \neg P \wedge \neg Q$

$\therefore$ pcnf of $(A \rightarrow B) \equiv \neg(\neg P \wedge \neg Q) \equiv P \vee Q$.

Example 80. Write dnf of the following statement.

$$p \rightarrow (p \wedge (q \rightarrow p))$$

Solution. $p \rightarrow (p \wedge (q \rightarrow p)) \equiv \sim p \vee \{p \wedge (q \vee p)\}$ $\because p \rightarrow q = \sim p \vee q$

$\equiv (\sim p \vee p) \wedge \{\sim p \vee (\sim q \vee p)\}$

$\equiv \wedge \{\sim p \vee (\sim q \vee p)\}$ by complement law

$\equiv \sim p \vee (\sim q \vee p)$ by Identity law

$\equiv \sim p \vee (p \vee \sim q)$ by Commutative law

$\equiv (\sim p \vee p) \vee \sim q$ by Associative law

$\equiv T \vee \sim q \equiv T$

$\equiv p \vee \sim p$ which is a dnf

Example 81. Find cnf of the expression.

$$\sim (p \vee q) \leftrightarrow (p \wedge q)$$

Solution. $(p \vee q) \leftrightarrow (p \wedge q)$

$\equiv (\sim (p \vee q) \rightarrow (p \wedge q)) \wedge ((p \wedge q) \rightarrow \sim (p \vee q))$

$\equiv (\sim (\sim (p \vee q)) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee (\sim (p \vee q)))$

$\equiv ((p \vee q) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee (\sim p \wedge \sim q))$ $[\because r \rightarrow s \equiv \sim r \vee s]$

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$\equiv ((p \vee q \vee p) \wedge (p \vee q \vee q)) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q))$ [by Distributive Law]

$\equiv ((p \vee q \vee p) \wedge (p \vee q \vee q)) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q))$

[by Distributive Law]

$\equiv (p \vee q \vee p) \wedge (p \vee q \vee q) \wedge (\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q)$

which is the required cnf.

Example 82. Using truth table find the principal disjunctive normal form (pdnf) and the principal conjunctive normal form (pcnf) of $(\sim p \vee \sim q) \rightarrow (p \leftrightarrow \sim q)$

Solution. The truth table of the given proposition is

p	q	$\sim p$	$\sim q$	$(\sim p \vee \sim q) \equiv a$	$(p \leftrightarrow \sim p) \equiv b$	$a \rightarrow b$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	F

Since the minterms corresponding to the 3 T values of the last column are $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$ the pdnf of the given proposition is $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$

Now pdnf of $\sim (a \rightarrow b) \equiv \sim p \wedge \sim q$

$\therefore$ pcnf of $(a \rightarrow b) \equiv \sim (\sim p \wedge \sim q) \equiv p \vee q$

Example 83. Prove that $(p \rightarrow \neg q) \leftrightarrow (q \rightarrow p)$ is a contingency.

Solution. The truth table of the given statement is

p	q	$\neg q$	$p \rightarrow \neg q \equiv a$	$q \rightarrow p \equiv b$	$a \leftrightarrow b$
T	T	F	F	T	F
T	F	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T

The last column shows that the statement is neither tautology nor a contradiction, so this is a contingency.

Example 84. Show that $\neg P$ follows from the set of premises $(R \rightarrow \neg Q)$, $R \vee S$, $S \rightarrow \neg Q$, $P \rightarrow Q$ using indirect method of proof.

Solution. To use indirect method, we will include $\neg (\neg P) = P$ as an additional premise and prove a contradiction.

1.	P	Additional premise
2	$P \rightarrow Q$	Premise given
3.	Q	1, 2 and modus poneus
4.	$R \rightarrow \neg Q$	Premise given
5.	$S \rightarrow \neg Q$	Premise given
6.	$(R \vee S) \rightarrow \neg Q$	4, 5 and equivalence
7.	$R \vee S$	Premise given
8.	$\rightarrow Q$	6, 7 and modus poneus
9.	$Q \wedge \neg Q$	3, 8 and conjunction
10.	F	9 and Negation law

Hence proved.

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Example 85. Shows that the following statements are logically equivalent without using truth table.

(*i*) $(P \to Q) \wedge (P \to R) \Leftrightarrow P \to (Q \wedge R)$

(*ii*) $\neg (P \vee (-P \wedge Q) \Leftrightarrow \neg P \wedge \neg Q)$

Solution. (*i*)

$$(P \to Q) \wedge (P \to R)$$
$$\Leftrightarrow (\neg P \vee Q) \wedge (\neg P \vee R)$$
$$\Leftrightarrow \neg P \vee (Q \vee R) \quad \text{by Distributive law}$$
$$\Leftrightarrow P \to (Q \wedge R)$$

Hence proved

(*ii*)

$$\neg (P \vee (\neg P \wedge Q)) \Leftrightarrow \neg P \wedge \neg (\neg P \wedge Q) \quad \text{by De-Morgain's law}$$
$$\Leftrightarrow \neg P \wedge (\neg (\neg P) \vee \neg Q) \quad \text{by De-Morgain's law}$$
$$\Leftrightarrow \neg P \wedge (P \vee \neg Q) \quad \text{by Involution law}$$
$$\Leftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q) \quad \text{by Distributive law}$$
$$\Leftrightarrow F \vee (\neg P \wedge \neg Q) \quad \text{by Negation law}$$
$$\Leftrightarrow (\neg P \wedge \neg Q) \vee F \quad \text{by Commutative law}$$
$$\Leftrightarrow \neg P \wedge \neg Q \quad \text{by Identity law}$$

Hence proved.

Example 86. Show that the following statement is a tautology.

$((P \vee Q) \wedge (P \to R) \wedge (Q \to R)) \to R$

Solution. The truth table of the given statement is

P	Q	R	$P \vee Q \equiv X$	$P \to R \equiv Y$	$X \wedge Y$	$Q \to R \equiv Z$	$X \wedge Y \wedge Z \equiv A$	$(A) \to R$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	T	T	T	T	T	T
T	F	F	T	F	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	T	F	F	T
F	F	T	F	T	F	T	F	T
F	F	F	F	T	F	T	F	T

Since all the entries in the last column are T, the given statement is a tautology.

Example 87. Show that $(\neg p \wedge (\neg q \wedge r)) \vee q \wedge r) \vee (p \wedge r) \Leftrightarrow r$.

Solution.

$$(\neg p \wedge (\neg q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$$
$$\Leftrightarrow (\neg p \wedge (\neg q \wedge r)) \vee ((q \vee p) \wedge r) \quad \text{by Distributive law}$$
$$\Leftrightarrow ((\neg p \wedge \neg q) \wedge r)) \vee ((q \vee p) \wedge r) \quad \text{by Associative law}$$
$$\Leftrightarrow ((\neg p \wedge \neg q) \wedge r) \vee ((p \vee q) \wedge r) \quad \text{by Commutative law}$$
$$\Leftrightarrow (\neg (p \vee q) \vee (p \vee q)) \wedge r \quad \text{by Distribution law}$$
$$\Leftrightarrow T \wedge r \quad \text{by Negation law}$$
$$\Leftrightarrow r \quad \text{by Identity law}$$

Hence proved.

Example 88. Prove that the following set of premises is inconsistent:

$$p \to q, p \to r, q \to \sim r, p$$

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Solution. If we derive a contradiction by using the given premises, then they are inconsistent.

1.	p	Premise (Given)
2.	$p \to q$	Premise (Given)
3.	q	Modus ponens using 1 and 2
4.	$q \to \sim r$	Premise (Given)
5.	$\sim r$	Modus ponens using 3 and 4
6.	$p \to r$	Premise (Given)
7.	r	Modus ponens using 1 and 6
8.	$r \wedge \sim r$	Conjunction using 7 and 5
9.	F	

Hence $p \to q$, $p \to r$, $q \to \neg r$ and p are inconsistent.

Example 89. Prove that the premises $a \to (b \to c)$, $d \to (b \wedge \neg c)$ and $a \wedge d$ are inconsistent.

Solution.

1.	$a \wedge d$	premise given
2.	a	1 and simplification
3.	d	1 and simplification
4.	$a \to (b \to c)$	premise given
5.	$b \to c$	2, 4 and modus penens
6.	$\neg b \vee c$	5 equivalence
7.	$d \to (b \wedge \neg c)$	premise given
8.	$\neg (b \wedge \neg c) \to \neg d$	7 and contrapositive
9.	$\neg b \vee c \to \neg d$	8 and equivalence
10.	$\neg d$	6, 9 and modus ponens
11.	$d \wedge \neg d$	3, 10 and conjunction
12.	F	11 and Negation law

Hence the given premises are inconsistent.

Example 90. Is the following conclusion validly derivable from the given primes? Verify.

If $\forall (x) (P(x) \to Q(x)))$, (exists y) $P(y)$ then (exists z) $Q(z)$.

Solution. We use the indirect method by assuming that the conclusion $(\exists z)\, Q(z)$ is same and prove contradiction.

1.	$\neg \exists (z)\, Q(z)$	Premise (assumed)
2.	$(\forall z) \neg Q(z)$	1 and equivalence
3.	$(\exists y)\, P(y)$	Premise given
4.	$P(a)$	3 and ES
5.	$\neg Q(a)$	2 and US
6.	$P(a) \wedge \neg Q(a)$	4, 5 and conjunctive
7.	$\neg (P(a) \to Q(a))$	6 and equivalence
8.	$(\forall x)\, (P(x) \to Q(x))$	Premise given
9.	$P(a) \to Q(a)$	8 and US
10.	$(P(a) \to Q(a)) . \wedge \neg (P(a) \to Q(a))$	7, 9 and conjunction
11.	F	A conjunctive

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Example 91. Show that $\exists x\, Q(x)$ is a valid conclusion from the premises.

$$\forall (x)\,(P(x) \to Q(x)) \text{ and } \exists x\, P(x)$$

Solution.

1.	$\exists x\, P(x)$	Premise (Given)
2.	$P(x)$	1 and EI
3.	$\forall x\,(P(x) \to Q(x))$	Premise (Given)
4.	$P(y) \to Q(y)$	3 and UI
5.	$Q(y)$	4 and Modus Ponens
6.	$\exists x\, Q(x)$	5 and EG.

Hence $\exists x\, Q(x)$ is a valid conclusion.

Example 92. Show that $\forall x\,(P(x) \vee Q(x)) \to \forall x\, P(x) \vee \exists\, Q(x)$ using indirect method.

Solution. In indirect method, we assume $\neg(\forall x\, P(x) \vee \exists x\, Q(x))$ as the additional premise and prove contradiction (F).

1.	$\neg(\forall x\,(P(x) \vee \exists x\, Q(x))$	Additional Premise
2.	$\neg(\forall x\, P(x)) \wedge \neg(\exists x\, Q(x))$	1 and De Morgan's rule
3.	$\exists x\,(\neg P(x)) \wedge \forall x\,(\neg Q(x))$	2 and Negation equivalence
4.	$\exists x\,(\neg\, P(x))$	3 and simplification
5.	$\forall x\,(\neg Q(x))$	3 and simplification
6.	$\neg P(a)$	4 and EI
7.	$\neg Q(a)$	5 and UI
8.	$\neg P(a) \wedge \neg Q(a)$	6, 7 and Conjunction
9.	$\neg(P(a) \vee Q(a))$	Q De-Morgan's law
10.	$\forall x\,(P(x) \vee Q(x))$	Premise (Given)
11.	$P(a) \vee Q(a)$	10 and UI
12.	$(P(a) \vee Q(a)) \neg (P(a) \vee Q(a))$	9, 11 and conjunction
13.	F	12 Negation law

Hence proved.

Example 93. Prove that $(\exists x)\,(P(x) \wedge Q(x)) \to (\exists x)\, P(x) \wedge (\exists (x)\, Q(x)$

Solution.

1.	$(\exists x)\,(P(x) \wedge Q(x))$	Premise
2.	$P(y) \wedge Q(y)$	1, ES, y fixed
3.	$P(y)$	2, Simplification
4.	$Q(y)$	3, Simplification
5.	$(\exists x)\, P(x)$	3, EG
6.	$(\exists x)\, Q(x)$	4, EG
7.	$(\exists x)\, P(x) \wedge (\exists x)\, Q(x)$	5, 6, $(P, Q \to P \wedge Q)$

Hence proved.

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MATHEMATICAL LOGIC PROBLEM SET

Problem Set 2.1

1. Which of the following sentences are propositions. What are the truth values of those that are proposition?
   (*a*) Is this true? — NS
   (*b*) Ram is a name. S
   (*c*) Please submit your proposal as soon as possible. NS
   (*d*) Four is even. S
   (*e*) $5 \in \{1, 6, 7\}$. S
   (*f*) What a hit ! S
   (*g*) Answer this question. NS
   (*h*) $5 + 6 = 12$. S
   (*i*) Buy two cinema tickets for Friday. N.S
   (*j*) May God bless you. N.S
2. What is the negation of each of the following propositions?
   (*a*) Today is Tuesday.
   (*b*) A cow is an animal.
   (*c*) No one wants to buy my house.
   (*d*) $5 + 1 = 6$.
   (*e*) Some people have no scooter.
   (*f*) Every even integer greater than 4 is the sum of two *primes*
3. Determine the truth value of each of the following statements
   (*a*) $6 + 2 = 7$ and $4 + 4 = 8$ F
   (*b*) $3 + 1 = 4$ and $4 + 5 = 7$
   (*c*) $4 + 3 = 7$ and $6 + 2 = 8$ T
   (*d*) $2 + 3 = 5$ and $3 + 1 = 2$
4. Consider the following
   p : Anil is rich
   q : Kanchan is poor
   Write each of the following statements in symbolic forms
   (*a*) Anil and Kanchan are both rich. → $p \wedge \sim q$
   (*b*) Anil is poor and Kanchan is rich. → $\sim p \wedge \sim q$
   (*c*) Anil is not rich and Kanchan is poor. — $(\sim p) \wedge q$
   (*d*) Neither Anil nor Kanchan is poor. $\sim p \wedge \sim q$
   (*e*) It is not true that Anil and Kanchan are both rich. $[\sim p \sim (q)]$
   (*f*) Either Anil is poor or Kanchan is poor. $\sim p \vee q$
   (*g*) Either Anil or Kanchan is rich. $p \vee (q)$
5. Consider the following
   p : This computer is good
   q : This computer is cheap
   Write each of the following statements in symbolic forms
   (*a*) This computer is good and cheap. $p \wedge q$
   (*b*) This computer is not good but cheap. $\sim p \wedge q$
   (*c*) This computer is costly but good. $p \wedge \sim q$
   (*d*) This computer is neither good nor cheap. $(\sim p \vee \sim q)$
   (*e*) This computer is good or cheap. $p \vee q$
6. Consider the following
   p : It is cold day
   q : The temperature is 5°C
   Write in simple sentences the meaning of the following
   (*a*) $\sim p$
   (*b*) $p \vee q$
   (*c*) $p \wedge q$
   (*d*) $\sim (p \vee q)$
   (*e*) $\sim (p \wedge q)$
   (*f*) $\sim p \wedge \sim q$
   (*g*) $\sim (\sim p \vee \sim q)$

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7. Given p and q are true and r and s are false, find the truth value of the following expressions
   (a) $p \vee (q \vee r)$ (b) $(\wedge (\wedge r)) \vee \sim ((p \vee q \wedge (r \vee s))$
   (c) $(\sim (p \wedge q) \vee \sim r) \vee (((\sim p \wedge q) \vee \sim r) \wedge s)$
8. Find the truth table of the following propositions.
   (a) $\sim p \wedge q$. (b) $p \vee \sim q$ (c) $\sim p \wedge \sim q$.
   (d) $(p \vee \sim q) \wedge p$. (e) $\sim (p \vee q) \vee (\sim p \wedge - q)$.
   (f) $p \wedge (q \vee r)$. (g) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (q \wedge r)$
   (h) $(p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$
9. Restate each proposition in if-then form.
   (a) Manju will pass in History exam, if she studies hard.
   (b) The program is readable only if it is well structured.
   (c) A sufficient condition for Sita's team to win the championship is that it wins the rest of its games.
   (d) Payment will be made on the fifth unless a new hearing is granted.
10. Determine the truth value of each of the following statements
   (a) If $3 \times 5 = 24$, then $3 + 5 = 8$ (b) If $3 \times 5 = 15$ then $3 + 5 = 12$
   (c) If $3 \times 5 = 17$, then $3 + 5 = 35$ (d) If $3 \times 5 = 15$ then $3 + 5 = 8$
11. Consider the following

   p : Today is Tuesday
   q : It is raining
   r : It is cold

   Write in simple sentences the meaning of the following
   (a) $p \to q$ (b) $\sim q \to (r \wedge p)$ (c) $\sim p \to (q \vee r)$ (d) $(p \vee q) \leftrightarrow r$
12. Consider the following

   p : you take a course in Discrete Mathematics
   q : You understand logic
   r : You get an A on the final exam.

   Write in simple sentences the meaning of the following
   (a) $q \to r$ (b) $\sim p \to \sim q$ (c) $(p \wedge q) \to r$
   (d) $(p \wedge \sim q) \to \sim r$ (e) $\sim (\sim r)$
13. Consider the following

   p : x is even
   q : x is divisible by 2

   Write in simple sentences the meaning of the following.
   (a) $\sim p$ (b) $\sim p \wedge \sim q$ (c) $p \vee q$ (d) $p \to q$
   (e) $\sim p \to \sim q$ (f) $p \to q$
14. If p is true and q is false, find the truth values of the following:
   (a) $\sim (p \to \sim q)$ (b) $(p \wedge q) \to (p \vee q)$ (c) $\sim (p \wedge q) \vee \sim (q \leftrightarrow p)$
   (d) $(p \to q) \vee \sim (p \leftrightarrow \sim q)$
15. If p and q are true and s is false, find the truth values of the followings:
   (a) $p \to q$ (b) $p \to r$ (c) $(p \wedge \sim q) \to r$
   (d) $(p \vee q) \leftrightarrow (p \to \sim r)$ (e) $(p \leftrightarrow r) \to r$.
16. Write the negation of the following
   (a) If she studies, she will pass in exam.
   (b) If it rains, then they will not go for picnic.
   (c) If the determinant of a system of linear equations is zero then either the system has no solution or it has an infinite number of solutions.

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17. Construct the truth tables for the following:
(a) $\sim p \vee q \rightarrow \sim q$
(b) $\wedge \sim q \rightarrow r$
(c) $(\sim q \rightarrow \sim p) \rightarrow (p \rightarrow q)$
(d) $(\sim p \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

18. Construct the truth tables for the following:
(a) $\sim (p \wedge q) \vee \sim (q \leftrightarrow p)$
(b) $(p \rightarrow q) \vee \sim (p \leftrightarrow \sim q)$
(c) $p \wedge \sim r \leftrightarrow q \vee r$
(d) $[(p \wedge q) \vee (\sim r)] \leftrightarrow p$

19. Prove that the following propositions are tautologies.
(a) $p \vee \sim p$
(b) $\sim (p \vee q) \vee [(\sim p) \wedge q] \vee p$
(c) $(p \wedge q) \rightarrow p$
(d) $\sim p \rightarrow (p \rightarrow q)$
(e) $[p \wedge (p \rightarrow q)] \rightarrow q$
(f) $(p \wedge q) \rightarrow (p \rightarrow q)$
(g) $[(p \rightarrow \wedge)\ (q \rightarrow r)] \Rightarrow (p \rightarrow r)$

20. Which of the following logical forms represent tautologies
(a) $p \rightarrow p$
(b) $p \rightarrow (q \rightarrow p)$
(c) $(q \rightarrow p) \rightarrow p$
(d) $(\sim p \rightarrow \sim q) \rightarrow (q \rightarrow p)$
(e) $(p \rightarrow (q \rightarrow r) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

21. Prove that the following propositions are tautologies
(a) $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$
(b) $((p \rightarrow q) \rightarrow r) \leftrightarrow ((p \leftrightarrow r) \vee (q \rightarrow r))$
(c) $\wedge (p \vee q) \leftrightarrow p$
(d) $p \wedge (p \wedge r) \leftrightarrow (p \wedge q) \wedge r$
(e) $(p \wedge (p \leftrightarrow q)) \rightarrow q$
(f) $[(p \vee q) \wedge (p \rightarrow r) \wedge (q \rightarrow r)] \rightarrow r$

22. Show that the following pairs of propositions are logically equivalent.
(a) $\sim (p \wedge q)$ and $\sim p \vee \sim q$.
(b) $p \wedge (\sim q \vee q)$ and p
(c) $p \vee (p \wedge \sim q)$ and p.
(d) $p \wedge q$ and $q \wedge p$.
(e) $p \vee (p \wedge q)$ and q.

23. Establish the equivalences
(a) $\sim (p \leftrightarrow) \equiv (p \wedge \sim q) \vee (\sim q \wedge q)$
(b) $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$
(c) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$
(d) $p \rightarrow \equiv \sim (p \vee q) \vee (p \wedge q)$

24. Using laws of algebra of propositions, show that
(a) $(p \wedge q) \vee p \equiv p$
(b) $(p \wedge q) \vee (p \wedge \sim q) \equiv p$
(c) $\sim (p \vee (\sim p \wedge q) \equiv \sim p \wedge q$
(d) $(p \rightarrow q) \wedge (r \rightarrow q) \equiv (p \vee r) \rightarrow q$
(e) $(\sim p \rightarrow (\sim p \rightarrow (\sim p \wedge q))) \equiv p \vee q$
(f) $\sim ((p \sim \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$

25. Write an equivalent formula for $p \wedge (q \leftrightarrow r) \vee (r \leftrightarrow p)$ which does not involve biconditional.

26. Write an equivalent formula for $p \wedge (q \leftrightarrow r)$ which is free from biconditional as well as conditional.

27. State the converse, inverse and contrapositive of the following.
(a) If today is Easter, then tomorrow is Monday.
(b) If triangle ABC is a triangle them $|AB|^2 + |BC|^2 = |AC|^2$.
(c) If P is a square, then P is a rectangle.
(d) If n is prime, then n is odd or n is 2.
(e) If a triangle is not isosceles, then it is not equilateral.

28. Restate the following implication $p \Rightarrow q$ in the equivalent form $(\sim p) \vee q$
(a) If the work is not finished on time, then I am in trouble.
(b) If it is cold, then he wears a hat.
(c) If an integer is a multiple of 2, then it is even.

29. Establish the equivalence
$$p \rightarrow q \vee r \equiv p \wedge \sim p \rightarrow r \equiv \wedge \sim r \rightarrow q$$
hence rewrite the following sentence in two different ways.
If n is prime, then n is odd or n is 2.

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30. Show that the proposition $\wedge \sim q$ and $(p \vee q) \wedge (\sim p) \wedge (\sim q)$ are contradiction.

31. Obtain the disjunctive normal form of the followings.

(a) $\sim (P \to (Q \wedge R))$ (b) $\sim (P \vee Q) \leftrightarrow P \wedge Q$

(c) $\sim (\sim (P \leftrightarrow Q \wedge R)$ (d) $(\sim P \to R \wedge (P \leftrightarrow Q)$

(e) $(P \wedge \sim (Q \wedge R)) \vee (P \to Q)$

32. Obtain the conjunctive normal form of the followings.

(a) $\sim ((P \vee \sim Q) \wedge \sim R)$ (b) $\sim (P \vee Q) \leftrightarrow P \wedge Q$

(c) $[Q \vee (P \wedge R) \wedge \sim (P \vee R \wedge Q)]$

33. Using truth table, find the principal disjunctive normal form of

(a) $p \vee q$ (b) $p \leftrightarrow q$ (c) $\sim (p \wedge q)$ (d) $(p \wedge \sim r \wedge \sim s) \vee (r \wedge s)$

34. Find the principal disjunctive normal form without using truth table

(a) $(\sim p \vee \sim p) \to (\sim p \wedge r)$ (b) $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$

35. Using truth table, find the principal conjunctive normal form of

$$q \wedge (q \vee \sim q)$$

36. Find the principal conjunctive normal form without using truth table.

$$(p \wedge q) \vee (\sim q \wedge r)$$

37. The principal disjunctive normal form of A is $(p \wedge q) \vee (\sim p \wedge r) \vee (q \wedge r)$, find principal conjunctive normal form.

Problem Set 2.2

1. Show that $p \to (s \vee t)$ is a valid conclusion from the given premises

$$p \to (q \vee r),\ q \to s \text{ and } r \to t$$

2. Show that $t \to s$ is a valid conclusion from the given premises

$$(p \wedge q) \vee (r \to s),\ t \to r,\ \sim (p \wedge q)$$

3. Show that s is a valid conclusion from the given premises

$$p \to \sim q,\ q \vee r,\ \sim s \to p,\ \sim r$$

4. Show that $\sim r$ is valid conclusion from the given premises

$$p \vee \sim q,\ \sim q \to r,\ q$$

5. Show that s is valid conclusion from the given premises

$$p \to q,\ q \to r,\ r \to s \wedge t,\ p$$

6. (a) Show that $p \to \sim r$ is valid conclusion from the given premises

$$p \to q,\ r \to \sim q$$

(b) Show that t is valid conclusion from the given premises

$$\sim p \wedge q,\ r \to p,\ \sim r \to s \text{ and } s \to t$$

7. Test the validity of the arguments using CP

(a) $[(P \to Q) \wedge (P \to (Q \to R))] \to (P \to R)$

(b) $[(P \to (Q \to R) \wedge (Q \to (R \to S))] \to [P \to (Q \to S)]$

(c) $(P \to Q) \to [P \to P \wedge Q]$

(d) $[(P \to Q) \to P] \to [(P \to Q) \to Q]$

8. Check the validity of the following arguments

(a) "If Roli has completed MCA or MBA, then she is assured of a good job. If Roli is assured of a good job, she is happy. Roli is not happy. So Roli has not completed MBA".

(b) "If I study, then I will pass in examination. If I do not go to cinema, then I will study. But I failed in examination. Therefore, I went to cinema."

(c) "If today is Sunday, then yesterday was Saturday. Yesterday was Saturday. Today is Sunday."

(d) "If I drive to work then I will arrive in time. I do not drive to work. Therefore, I will not arrive in time".

(e) "If I try hard and I have a talent, then I will become a scientist. If I become scientist, then I will be happy. Therefore, if I will not be happy, then I did not try hard or I do not have talent."

(f) If my program runs successfully then I will submit my project. I can appear in the examination only if I submit my project. Either my program runs successfully or the computer crashes. Therefore, if the computer does not crash then I can appear at the examination.

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9. Prove or disprove
   (*a*) The sum of two even integers is an even integer.
   (*b*) The sum of three odd integers is an odd integer.
   (*c*) The sum of two primes is never a prime integer.
10. Prove that $\sqrt{3}$ is irrational.
11. Prove that $|xy| = |x||y|$.
12. Show that
   (*a*) $\left\lfloor \frac{n}{3} \right\rfloor + \left\lceil \frac{2n}{3} \right\rceil = n$ for all positive integer $n \geq 1$.
   (*b*) $\left\lfloor \frac{m}{2} \right\rfloor + \left\lceil \frac{m}{2} \right\rceil$ for all integers m
13. Use recursive formula of summation together with mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots\ldots, a_n$ and c are real numbers, then

$$\sum_{i=1}^{n} ca_i = c\left(\sum_{i=1}^{n} a_i\right)$$

14. Prove by contradiction that the difference of any rational number and any irrational number is irrational.
15. Prove by contradiction that $6 - 7\sqrt{2}$ is irrational.
16. Prove that there is at most one real number a with the property $a + r = r$ for all real number a. Give direct and also proof by contradiction.
17. Prove that the structure [even integer, +, x] is closed with respect to multiplication.
18. Show that each of the following inferences is fallacy
   (*a*) If today is Megha's birthday, then today is Febraury 16. Today is Febraury 16. Hence today is Megha's birthday.
   (*b*) If the client is guilty, then he was at the scene of the crime. The client was at the scene of the crime. Hence the client is not guilty.

Problem Set 2.3

1. Using principle of mathematical establish the following formulae for all integers $n \geq 1$
   (*i*) $1 + 2 + 3 + \ldots\ldots + n = n(n+1)/2$
   (*ii*) $1 + 3 + 5 + \ldots\ldots + (2n-1) = n^2$
   (*iii*) $2 + 4 + 6 + \ldots\ldots + 2n = n(n+1)$
   (*iv*) $1.2 + 2.3 + 3.4 + \ldots\ldots + n(n+1) = n(n+1)(n+2)/3$
   (*v*) $1.2 + 3.4 + 5.6 + \ldots\ldots + (2n-1)2n = n(n+1)(4n-1)/3$
   (*vi*) $3.6 + 6.9 + 9.12 + \ldots\ldots + 3n(3n+3) = 3n(n+1)(n+2)$
   (*vii*) $1.2.3 + 2.3.4 + \ldots\ldots + n(n+1)(n+2) = n(n+1)(n+2)$
   (*viii*) $1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = [n(n+1)]2$
   (*ix*) $1(1!) + 2(2!) + \ldots\ldots + n(n!) = (n+1)! - 1$
   (*x*) $1^2 + 3^2 + 5^2 + \ldots\ldots + (2n-1)^2 = (4n^3 - n)/3$
2. Using mathematical induction show that for all positive integers n
   (*i*) $n^3 - 4n + 6$ is divisible by 3
   (*ii*) $11^n - 4^n$ is divisible by 7
   (*iii*) $10^{2n-1} + 1$ is divisible by 11
   (*iv*) $3.5^{2n+1} + 2^{3n+1}$ is divisible by 17
   (*v*) $7^{2n} + 16n - 1$ is divisible by 64
   (*vi*) $n(n+1)(n+2)$ is divisible by 6
   (*vii*) $n^3 + 2n$ is divisible by 3
   (*viii*) $n(n^2+5)$ is an integral multiple of 6.
   (*ix*) $4^{2n+1} + 3^{n+2}$ is an integer multiple of 13
   (*x*) $7^n - 2^n$ is divisible by 5

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3. (*i*) Show that $(1 + 1/2)^n \geq 1 + n/2$
   (*ii*) Show that $3^n > n^3$ for each integer $n \geq 4$
   (*iii*) Show that $(1 + 5/2)^n \geq 1 + 5n/2$
   (*iv*) Show that $n^3 + 1 > n^2$ for each integer $n \geq 2$
   (*v*) Show that $4n > 3n$ for all $n \geq 0$.
   (*vi*) Show that $2n > n^2$, for $n \geq 5$.
   (*vii*) Show that $n^2 > 2n + 1$ for all $n \geq 3$
4. (*i*) $1/(1)(2) + 1/(2)(3) + 1/(3)(4) + \ldots\ldots + 1/n(n+1) = n/(n+1)$
   (*ii*) $1/(1)(3) + 1/(3)(5) + 1/(5)(7) + \ldots\ldots + 1/(2n-1)(2n+1) = 1/(2n+1)$
   (*iii*) $1^2/(1)(3) + 2^2/(3)(5) + 3^2/(5)(7) + \ldots\ldots + n^2/(2n-1)(2n+1) = n(n+1)/2(2n+1)$
5. Use induction to show that if $r \neq 1$,
   $a + ar + ar^2 + ar^3 + \ldots\ldots ar^n = a(r^{n+1} - 1)/(r - 1)$ for $n = 0, 1, 2, \ldots\ldots$
6. Use induction to show that for the arithmetic progression with initial value a and common difference d, $f(n) = a + d(n - 1)$
7. Let $A_1, A_2, \ldots\ldots, A_n$ be any n sets. Show by mathematical induction

$$\left(\bigcup_{i=1}^{n} A_i\right)' = \bigcap_{i=1}^{n} A_i'$$

Problem Set 2.4

1. Consider the following statement

$$\forall x \in R \text{ such that } x^2 = 3$$

   Which of the following are equivalent ways of expressing this statement?
   (*a*) The square of each real number is 3.
   (*b*) Some real numbers have square 3.
   (*c*) The number x has square 3, for some real number x.
   (*d*) If x is a real number, then $x^2 = 3$.
   (*e*) There is at least one real number whose square is 3.
2. Consider the following statement:
   $\forall$ integers n, n^2 is odd then n is odd.
   Which of the following are equivalent ways of expressing this statement.
   (*a*) All integers have odd squares and are odd.
   (*b*) Given any integer whose square is odd, that integer is itself odd.
   (*c*) For all integers, there are some whose square is odd.
   (*d*) Any integer with an odd square is odd.
   (*e*) If the square of an odd integer is odd, then that integer is odd.
   (*f*) All odd integers have odd squares.
3. Write negations for each of the following:
   (*a*) Real number x, if $x > 3$ then $x^2 > 9$.
   (*b*) $x \in R$ if $x(x + 1) > 0$ then $x > 0$ or $x < -1$.
   (*c*) integers a, b and c if $a - b$ even and $b - c$ is even, then $a - c$ is even.
4. Rewrite the following argument using quantifiers, variables, and predicate symbols
   (*a*) All birds can fly.
   (*b*) Not all birds can fly.
   (*c*) Some men are genius.
   (*d*) Some numbers are not rational
   (*e*) There is a student who likes mathematics but not Geography.

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5. Remove all negations of the quantifiers of the following expression.

$$\sim \forall y (\exists x P(x, y \wedge \sim \forall x Q(x, y))$$

6. Translate each of the statement into symbols, using quantifiers, variables and predicate symbols.
Let P (x): x can speak Tamil and Q (x) : x knows the language C + +
(*a*) There is a student who can speak Tamil and who knows C + +
(*b*) There is a student who can speak Tamil but does not know C + +
(*c*) Every student either can speak Tamil or knows C + +.
(*d*) No student can speak Tamil or knows C + +

7. Consider the following premises:

$$\exists x M(x), \forall x (M(x)) \rightarrow \exists y C(x, y)$$

and

$$\forall x (\exists y C(x, y) \rightarrow F(x)).$$

Give a formal derivation to prove that these premises lead one to the conclusion and there must be *a y* such that F (y) is true. State the rules of inference in the derivation.

8. (*a*) Consider the following premises:

$$\forall x \forall y [P(x, y) \Rightarrow Q(x, y)], \sim Q(a, b)$$

Prove that these premises lead to a conclusion $\sim P(a, b)$.
(*b*) Prove that $(\exists x)(R(x) \wedge S(x))$ is a valid conclusion from the premises, $(\exists x)(P(x) \wedge S(x))$, $(\forall x)(P(x) \Rightarrow R(x))$.

9. Use predicate calculus to prove the following agreement.
(*a*) All mammals are animals. Some mammals are two-legged. Therefore, some animals are two–legged.
(*b*) All animals are mortal. All human being are animals.
Therefore, all human beings are mortal.
(*c*) No human being are quadrupeds. All women are human beings. Therefore, no women are quadrupeds.
(*d*) Lions are dangerous animals. There are lions. Therefore, they are dangerous animals.
(*e*) Everyone is this class owns a personal computer. Someone in this class has never used the internet. Therefore, Someone who owns a personal computer has never used the internet.

ANSWERS 2.1

1. (*a*) Not a proposition (*b*) Proposition (*c*) Not a proposition (*d*) Proposition
(*e*) Proposition (*f*) Not a proposition (*g*) Not a proposition (*h*) Proposition
(*i*) Not a proposition (*j*) Not a proposition
2. (*a*) Today is not Tuesday (*b*) A cow is not an animal
(*c*) Some one wants to buy my house (*d*) $5 + 1 \neq 6$
(*e*) Every person has a scooter.
(*f*) Some even integer greater than 4 is not the sum of two primes.
3. (*a*) F (*b*) F (*c*) T (*d*) F
4. (*a*) $p \not\sim \wedge q$ (*b*) $(\sim p) \wedge (\sim q)$ (*c*) $(\sim p) \wedge q$ (*d*) $(\sim p) \wedge q$
(*e*) $\sim [p \wedge (-q)]$ (*f*) $(\sim p) \vee q$ (*g*) $p \vee (\sim q)$
5. (*a*) $p \wedge q$ (*b*) $(\sim) p \wedge q$ (*c*) $(\sim q) \wedge p$ (*d*) $(\sim p) \wedge (\sim q)$
(*e*) $p \vee q$
6. (*a*) It is not a cold day
(*b*) It is a cold day or temperature is 5°C
(*c*) It is a cold day and temperature is 5°C
(*d*) It is false that it is cold day or temperature is 5°C

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(*e*) It is not true that it is cold day and temperature is 5°C.

(*f*) It is neither a cold day nor temperature is 5°C

(*g*) It is false that it is not a cold day or temperature is not 5°C.

7. (*a*) T (*b*) T (*c*) T

8. (*a*)

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(*b*)

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(*c*)

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(*d*)

p	q	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge p$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	F

(*e*)

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$(\sim p \wedge \sim q)$	$\sim (p \vee q) \vee (\sim p \wedge \sim q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

(*f*)

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

(*g*) Let $A = (\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

p	q	r	$(\sim q \wedge r)$	$(\sim p \wedge (\sim q \wedge r))$	$(q \wedge r)$	$(p \wedge r)$	A
T	T	T	F	F	T	T	T
T	T	F	F	F	F	F	F
T	F	T	T	F	F	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	T
F	T	F	F	F	F	F	F
F	F	T	T	T	F	F	T
F	F	F	F	F	F	F	F

(*h*) Let $A = (p \wedge q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge p \sim q)$

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p	q	$(p \wedge q)$	$(\sim p \wedge q)$	$(p \wedge \sim q)$	$(\sim p \wedge \sim q)$	A
T	T	T	F	F	F	T
T	F	F	F	T	F	T
F	T	F	T	F	F	T
F	F	F	F	F	T	T

9. (*a*) If Manju studies hard, then she will pass in History exam.
   (*b*) If the program is readable, then it is well structured.
   (*c*) If sita's team wins the rest of its games, then it will win the championship.
   (*d*) If a new hearing is not granted, then payment will be made on fifth.
10. (*a*) T (*b*) F (*c*) T (*d*) T
11. (*a*) If today is Tuesday, then it is raining
   (*b*) If it is not raining, then it is cold and today is Tuesday.
   (*c*) If today is not Tuesday, then it is raining or it is cold.
   (*d*) It is not the case that today is Tuesday or it is raining if and only if it is cold.
12. (*a*) If you understand logic. then you get an A on the final exam.
   (*b*) If you do not understand logic, then you do not get an A on the final exam.
   (*c*) If you take a course in Discrete Mathematics and understand logic, then you get an A on the final exam.
   (*d*) If you take a course in Discrete Mathematics and do not understand logic, then you do not get an A on the final exam.
   (*e*) It is not true that you do not get an A on the final exam.
13. (*a*) x is not an even
   (*b*) x is neither even nor divisible by 2
   (*c*) x is even or divisible by 2.
   (*d*) If x is even, then it is divisible by 2
   (*e*) If x is not even, then it is not divisible by 2.
   (*f*) x is even if and only if it is divisible by 2
14. (*a*) F (*b*) T (*c*) T (*d*) F
15. (*a*) T (*b*) F (*c*) T (*d*) T
   (*e*) T
16. (*a*) She studies and she will not pass in exam.
   (*b*) It rains and they will go for picnic.
   (*c*) The determinant of a system of linear equations is zero and the system has a solution and does not have infinite number of solutions.
17. (*a*)

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$\sim p \vee q \rightarrow \sim q$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	F	T	F
F	F	T	T	T	T

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(*b*)

p	q	r	$\sim p$	$p \wedge \sim q$	$p \wedge \sim q \to r$
T	T	T	F	F	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	T	F	F	F	T
F	F	T	T	F	T
F	F	F	T	F	T

(*c*)

p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$	$p \to q$	$(\sim q \to \sim p) \to (p \to q)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

18. (*a*)

p	q	$\sim (p \wedge q) \vee \sim (q \leftrightarrow p)$
T	T	F
T	F	T
F	T	T
F	F	T

(*b*)

p	q	$(p \to q) \vee \sim (p \leftrightarrow \sim q)$
T	T	T
T	F	F
F	T	T
F	F	T

(*c*)

p	q	r	$p \wedge \sim r$	$q \vee r$	$p \wedge \sim r \leftrightarrow q \vee r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	F	F
F	T	T	F	T	F
F	T	F	F	T	F
F	F	T	F	T	F
F	F	F	F	F	T

(*d*)

p	q	r	$[p \wedge q) \vee (\sim r) \leftrightarrow p$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	T
F	F	F	F

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20. (*a*) Tautology (*b*) Tautology (*c*) Not a tautology (*d*) Tautology (*e*) Tautology.

25. $[p \wedge ((q \to r) \wedge (r \to q)) \vee ((r \to p) \wedge (p \to r))]$

26. $p \wedge ((\sim q \vee r) \wedge (\sim r \vee q))$.

27. (*a*) (*i*) If tomorrow is Monday, then today is Easter

(*ii*) If today is not Easter, then tomorrow is not Monday

(*iii*) If tomorrow is not Monday, then today is not Easter.

(*b*) (*i*) If $|AB|^2 + |BC|^2 = |AC|^2$, then triangle ABC is a right triangle.

(*ii*) If triangle ABC is not a right triangle, then $|AB|^2 + |BC|^2 \neq |AC|^2$

(*iii*) If $|AB|^2 + |BC|^2 \neq |AC|^2$, then triangle ABC is not a right angle.

(*c*) (*i*) If P is a rectangle, then *P* is a square.

(*ii*) If P is not a square, then *P* is not a rectangle.

(*iii*) If P is not a rectangle, then *P* is not a square.

(*d*) (*i*) If *n* is odd or *n* is 2, then n is prime.

(*ii*) If *n* is not prime, then *n* is not odd, and *n* is not 2.

(*iii*) If *n* is not odd and *n* is not 2, then *n* is not prime.

(*e*) (*i*) If a triangle is not equilateral, then it is not isosceles.

(*ii*) If a triangle is isosceles, then it is equilateral.

(*iii*) If a triangle is equilateral, then it is isosceles.

28. (*a*) The work is finished on time or I am in trouble.

(*b*) It is not cold or he wears a hat.

(*c*) An integer is not a multiple of 2 or it is even.

31. (*a*) $(P \wedge \sim Q) \vee (P \wedge \sim R)$

(*b*) $(\sim P \wedge \sim Q \wedge P \wedge Q) \vee (P \wedge \sim P) \vee (Q \wedge \sim P) \vee (P \wedge \sim Q) \vee (Q \wedge \sim Q)$

(*c*) $(P \wedge \sim Q) \vee (P \wedge \sim R) \vee (\sim P \wedge \sim Q) \vee (\sim Q \wedge \sim R) \vee (\sim P \wedge \sim R) \vee (Q \wedge \sim R) \vee \sim R$

(*d*) $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \sim R)$

32. (*a*) $(\sim P \vee R) \wedge (Q \vee R)$

(*b*) $(P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\sim P \vee \sim Q \vee \sim P) \wedge (\sim P \vee \sim Q \vee \sim Q)$

(*c*) $(Q \vee P) \wedge (Q \vee R) \wedge (\sim P \vee \sim Q) \wedge (\sim R \wedge \sim Q)$.

33. (*a*) $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$ (*b*) $(b \wedge q) \vee (\sim p \wedge \sim q)$

(*c*) $(p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$ (*d*) $(q \wedge \sim r \wedge r \wedge \sim s) \vee (q \wedge r \wedge s) \vee (-q \wedge r \wedge s)$

34. (*a*) $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r) \vee (\sim p \sim \wedge \sim q \wedge r)$

(*b*) $(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

35. $(\sim p \vee q) \wedge (p \vee \sim q) \wedge (p \vee q)$

36. $(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r) \wedge (p \vee q \vee r) \wedge (p \vee \sim q \vee r)$

37. $(\sim p \vee q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee q \vee r)$

ANSWERS 2.2

8. (*a*) Valid (*b*) Valid (*c*) Not valid (*d*) Not valid (*e*) Valid (*f*) Valid

9. (*a*) Direct Proof (*b*) False, give an example

(*c*) False, the sum of four consecutive integers in never divisible by 4.

ANSWERS 2.4

1. (*b*), (*c*), (*e*), (*f*) **2.** (*b*), (*d*), (*e*)

3. (*a*) $\exists$ a real number x such that $x > 3$ and $x^2 \leq 9$.

(*b*) $\exists$ a real number x such that $x(x+1) > 0$ then $x \leq 0$ and $x \geq -1$.

(*c*) $\exists$ integers a, b and c such that $a - b$ is even and $b - c$ is even and $a - c$ is even.

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4. (*a*) $\forall x\,[B(x) \to F(x)]$ (*b*) $\sim[\forall x\,(B(x) \to F(x))]$
   (*c*) $\exists x\,[M(x) \wedge G(x)]$ (*d*) $\sim[\exists x\,(N(x) \wedge R(x))]$
   (*e*) $\exists x\,[S(x) \wedge M(x) \wedge \sim G(x)]$
5. $\exists y\,(\forall x \sim P(x, y) \vee \forall x\, Q(x, y)$
6. (*a*) $\exists x\,(P(x) \wedge Q(x))$ (*b*) $\exists x\,(P(x) \wedge \sim Q(x))$
   (*c*) $\forall x\,(P(x) \vee Q(x))$ (*d*) $\forall x \sim (P(x) \vee Q(x))$

SHORT/MULTIPLE CHOICE QUESTIONS

1. Find out which statements are propositions:
   (*a*) It is raining today.
   (*b*) Close the door.
   (*c*) This statement is true.
   (*d*) What are you doing ?
   (*e*) It may rain to day.
   (*f*) Sheela is inside her house.
   (*g*) $3 + x = 10$
   (*h*) Do Your home work.
   (*i*) Listen to me, Mukesh.
   (*j*) Beware of dogs.
2. Negate the following propositions:
   (*a*) All the triangles have the sum of their anglesequal to 180°.
   (*b*) All integers are greater than 6.
   (*c*) There is a dog without a tail.
   (*d*) There is an integer between 4 and 8 inclusive.
   (*e*) Every student has a book which that student reads.
   (*f*) All students live in the hostels.
   (*g*) Some students are 28 or older.
   (*h*) No one wants to buy my house.
   (*i*) For all x in S, $0 + x = 0$.
   (*j*) $3 < 8$.
   (*k*) $\forall\, n \in N,\ n + 1 > 6$.
   (*l*) All differentiable functions are continuous.
   (*m*) None of the students of this Class has passed.
   (*n*) There is a real number which is not a Complex number.
3. Determine the truth value of each of the following statements:
   (*a*) If $6 < 2$, then $-2 > -7$
   (*b*) If $3 + 3 = 6$,then $6 + 5 = 14$
   (*c*) $6 + 6 = 12$, then $6 + (-6) = -17$
   (*d*) Real part of $(1 + 2i)^3$ is 4
   (*e*) Every Complex number is a rational number.
   (*f*) 2 divides 9 and $6 + 10 = 16$
   (*g*) 2 divides 8 and $4 + 7 = 11$
4. Let p : Manju is intelligent.
   q : Manju is hardworking.
   Write each statement in symbolic form using p and q
   (*a*) Manju is intelligent and Manju is hard working.

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(*b*) Manju is intelligent and Manju is not hard working.

(*c*) Manju is neither intelligent nor hard working.

5. Let p: He is rich q : He is happy.

Write each statement in symbolic form using p and q:

(*a*) If he is reach, then he is unhappy.

(*b*) He is neither rich nor happy.

(*c*) It is necessary to be poor in order to be happy.

(*d*) To be poor is to be unhappy.

6. Let p be 'He is old' and q be 'He is clever'. Write in symbolic form:

(*a*) He is old and clever.

(*b*) He is old but not clever.

(*c*) He is neither old nor clever.

(*d*) It is not true that he is young or not clever.

7. Let p be 'the examination is hard' and q be 'the grades are low'. Write in symbolic form:

(*a*) If the examination is hard, the grades are low.

(*b*) The grades are not low, and the examination is not hard.

(*c*) The grades are low only if the examination is hard.

8. Rewrite the statements in if – then form:

(*a*) This loop will repeat exactly n times if it does not contain a stop or a go to.

(*b*) Stand up or I'll shoot.

(*c*) Catching the 9.10 local train is a sufficient condition for my being on time for work.

9. The proposition $p \wedge p$ is equivalent to

(*a*) 1

(*b*) p

(*c*) $-p$

(*d*) None of these

10. The proposition $\neg (p \vee q)$ is equivalent to

(*a*) $p \vee q$

(*b*) $\neg p \wedge \neg q$

(*c*) $\neg p \vee \neg q$

(*d*) None of the above.

11. The proposition $\neg (p \wedge q)$ is equivalent to

(*a*) $\neg p \wedge \neg q$

(*b*) $\neg p \vee \neg q$

(*c*) $p \vee q$

(*d*) None of the above.

12. Write the Converse, Inverse and Contrapositive to each of the following Statements :

(*a*) If $x + 4 = 10$ then $x = 6$

(*b*) If you are guilty then you are punished.

(*c*) If I run fast then I can win the race.

(*d*) If ABC is righy angled at B then $AB^2 + BC^2 = AC^2$

13. Which of the following proposition is a tautology?

(*a*) $(p \vee q) \rightarrow q$

(*b*) $p \vee (q \rightarrow p)$

(*c*) $p \vee (p \rightarrow q)$

(*d*) $p \rightarrow (p \rightarrow q)$

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14. Using the propositions p : I am bored , q : I am waiting for one hour and r : There is no bus. Translate the following into English.
   (a) $(q \vee r) \rightarrow p$
   (b) $\sim q \rightarrow p$
   (c) $(q \rightarrow p) \vee (r \rightarrow p)$
15. If p is necessary for q then which of the following is true (i) $p \rightarrow q$ (ii) $q \rightarrow p$
16. If p is necessary for q then which of the following is true : (i) $p \rightarrow q$ (ii) $q \rightarrow p$
17. Which of the following expressions are predicates (and what are the free variables), and which are propositions (and what are the truth values ?).
   (a) $4 + 2 = 7$
   (b) $x + 2 = 7$
   (c) $x + y = z$
   (d) $2 = 2$
   (e) $x = x$
18. Let the universe of discourse be the set of integers, determine the truth values of the following sentences:
   (a) $\forall x\ (x^2 \geq 0)$
   (b) $\forall x\ (x^2 - 8x + 15 = 0)$
   (c) $\exists x\ (x^2 - 5 = 0)$
   (d) $\forall y\ \exists x\ (x^2 = y)$
19. Let $p(x)$ be the open statement "$x^2 = 3x$", where the universe comprises all integers. Determine whether each of the following statements is true or false.
   (a) $p(0)$ (b) $p(2)$ (c) $p(-3)$ (d) $p(3)$
   (e) $\exists x\ p(x)$ (f) $\forall xp\ (x)$
20. Determine the truth value of each Statement when the universe comprises all nonzero integer.
   (a) $\exists x\ \exists y\ [xy = 2]$ (b) $\exists x\ \forall y\ [xy = 2]$ (c) $\forall x\ \exists y\ [xy = 2]$
   (d) $\exists x\ \exists y\ [9 + y = 3 \wedge x - y = 5]$
21. What are the two types of quantifiers?
22. $p \vee (\sim p \wedge q)$ is equivalent to
   (a) $p \vee q$ (b) $p \wedge q$ (c) p (d) q
23. $p \vee (p \wedge (p \vee (q \wedge p)))$ is equivalent to
   (a) p (b) q (c) T (d) F
24. $p \uparrow q$ is equivalent to
   (a) p (b) $\sim p$ (c) T (d) F
25. The proposition $p \wedge (q \wedge \sim q)$ is a
   (a) contradiction (b) tautology (c) both (a) and (b) (d) None of these
26. Negation of $\exists x\ \forall y\ P(x, y)$ is
   (a) $\forall x\ \forall y \sim P(x, y)$ (b) $\forall x\ \exists y \sim P(x, y)$ (c) $\exists x\ \exists y \sim P(x, y)$ (d) None of these
27. The negation of "some students like football" is
   (a) Some students dislike football (b) Every student dislikes football
   (c) Every student likes football (d) None of these
28. If T(x) : x is teach , M(x) : is male, then the symbolic representation of the statement "All teachers are male" is
   (a) $\forall x\ (T(x) \rightarrow M(x))$ (b) $\forall x\ (T(x) \vee M(x))$ (c) $\forall x\ (T(x) \wedge M(x))$ (d) $\exists x\ (T(x) \rightarrow M(x))$

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ANSWERS

1. (*a*) Yes (*b*) No (*c*) Yes (*f*) No
   (*e*) No (*f*) Yes (*g*) No (*h*) No
   (*i*) No (*j*) No

2. (*a*) There is a triangles whose sum of angles $\neq 180°$.
   (*b*) There is an integer less than or equal to 6.
   (*c*) All dogs have tails.
   (*d*) All integers are not between 4 and 8 inclusive.
   (*e*) There is a student who does not read any books .
   (*f*) Some students do not live in hostels.
   (*g*) None of the students are 28 or older.
   (*h*) Someone wants to buy my house.
   (*g*) There exists an x in S such that $0 + x \neq 0$
   (*i*) $3 \geq 8$
   (*j*) $\exists\, n \in N$, such that $n + 7 < 6$
   (*l*) Not all differentiable Functions are Continuous or There exists a differentiable function which is not Continuous.
   (*m*) At least one student of this class has passed.
   (*n*) Every real number is a Complex number.

3. (*a*) True (*b*) True (*c*) False (*d*) True (*e*) False (*f*) False (*g*) True

4. (*a*) $p \wedge q$ (*b*) $p \wedge -q$ (*c*) $-p \wedge -q$

5. (*a*) $p \rightarrow q$ (*b*) $-p \wedge -q$ (*c*) $p \rightarrow \sim p$ (*d*) $-p \leftrightarrow -q$

6. (*a*) $p \wedge q$ (*b*) $p \wedge -q$ (*c*) $-p \wedge -q$ (*d*) $-(-p \vee -q)$

7. (*a*) $p = <q$ (*b*) $-p \wedge -q$ (*c*) $q \rightarrow p$

8. (*a*) If this loop does not contain stop or a go to , then it will repeat exactly n times.
   (*b*) If you do not stand up, then I w'll shoot.
   (*c*) If I catch the 9.10 local train, then I am on time for work.

9. (*b*) **10.** (*b*) **11.** (*b*)

12. (*a*) If $x = 6$ then $x + 4 = 10$
    If $x + 4 \neq 10$ then $x \neq 6$
    If $x \neq 6$ then $x + 4 \neq 10$
    (*b*) If you are punished then you are guilty.
    If you are not guilty then you are not punished.
    If you are not punished then you are not guilty.
    (*c*) If I can win the race then I run fast.
    If I do not run fast then I can not win the race.
    If I can not win the race then do not run fast.
    (*d*) If $AB^2 + BC^2 = AC^2$ then ΔABC is right angled at B.
    If ΔABC is not right angled at B then $AB^2 + BC^2 \neq AC^2$.
    If $AB^2 + BC^2 \neq AC^2$ then ΔABC is not right angled at B.

13. (*c*)

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14. (*a*) If I am waiting for one hour or there is no bus, then I get bored.
(*b*) If I am not waiting for one hour, then I am not bored.
(*c*) If I am waiting for one hour then I am bored or If there is no bus, then I am bored.

15. (*ii*) 16. (*i*)

17. (*a*) Proposition, False (*b*) Predicate, Free variable x
(*c*) Predicate, Free variable x, y, z (*d*) Proposition, True
(*e*) Predicate, Free variable x

18. (*a*) True (*b*) False (*c*) False (*d*) False

19. (*a*), (*d*), (*e*) True. (*b*), (*c*), (*f*) False

20. (*a*) True (*b*) False (*c*) False (*d*) True

21. (1) The existential quantifier and (2) the universal quantifier.

22. (*a*) 23. (*a*) 24. (*b*) 25. (*a*)

26. (*b*) 27. (*b*) 28. (*a*)

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3 CHAPTER

Boolean Algebra and Logic Circuits

3.1. Introduction

Boolean algebra is an algebra of logic. One of the earliest investigators of symbolic logic was George Boole (1815 – 1864) who invented a systematic way of manipulating logic symbols which became known as Boolean algebra. Till 1939, there was no practical application of Boolean algebra. In 1938 Shanon's work gave an idea that Boolean algebra could be applied to digital electronics. Indeed, it has become now an indispensable tool to computer scientists because of its direct applicability to switching theory and the logical design of digital computers.

Although the symbols 0 and 1 used in this chapter look like the binary numbers, they are not. They do not represent any numerical values. They only have logical significance.

3.2. Boolean Algebra

A non-empty set B with two binary operations + and . , a unary operation ′, and two distinct elements 0 and 1 is called a Boolean algebra, denoted by (B, + , . , ′, 0, 1) if and only if the following properties are satisfied.

Axioms of Boolean Algebra

If $a, b, c \in$ B, then

1. Commutative Laws:
   (*a*) $a + b = b + a$ (*b*) $a.b = b.a$
2. Distributive Law:
   (*a*) $a + (b.c) = (a + b).(a + c)$ (*b*) $a.(b + c) = (a.b) + (a.c)$
3. Identity Laws:
   (*a*) $a + 0 = a$ (*b*) $a.1 = a$
4. Complement Laws:
   (*a*) $a + a' = 1$ (*b*) $a.a' = 0$

Basic Theorems

Let $a, b, c \in$ B, then

1. Idempotent Laws:
   (*a*) $a + a = a$ (*b*) $a.a. = a$
2. Boundedness Laws:
   (*a*) $a + 1 = 1$ (*b*) $a.0 = 0$
3. Absorption Laws:
   (*a*) $a + (a.b) = a$ (*b*) $a.(a + b) = a$
4. Associative Laws:
   (*a*) $(a + b) + c = a + (b + c)$ (*b*) $(a.b).c = a.(b.c)$
5. Uniqueness of Complement:
   $a + x = 1$ and $a.x = 0$, then $x = a'$

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6. Involution Law: $(a')' = a$
7. (a) $0' = 1$ (b) $1' = 0$
8. De-Morgan's Laws:
   (a) $(a + b)' = a'.b'$ (b) $(a.b)' = a' + b'$

Note: Where there is no danger of confusion, the symbol . can be deleted, just as in writing algebraic products *i.e.* $a.b$ as ab.

Algebraic proofs of some laws are given below and the rest are left as exrcises for the readers. Each of the laws can also be proved using a truth table.

Theorem 3.1 $a + a = a$

Proof.

$$\begin{aligned} a &= a + 0 && \text{by Identity law} \\ &= a + aa' && \text{using Complement law} \\ &= (a + a).(a + a') && \text{using Distributive law} \\ &= (a + a).1 && \text{using Complement law} \\ &= a + a && \text{using Identity law} \end{aligned}$$

Theorem 3.2 $a.a = a$

Proof.

$$\begin{aligned} a.a &= a.a + 0 && \text{by Identity law} \\ &= a.a + a.a' && \text{using Complement law} \\ &= a.(a + a') && \text{using Distributive law} \\ &= a.1 = a && \text{using Complement and Identity law} \end{aligned}$$

Theorem 3.3 $a + 1 = 1$

Proof.

$$\begin{aligned} a + 1 &= a + a + a' && \text{by Complement law} \\ &= (a + a) + a' && \text{using Associative law} \\ &= a + a' && \text{using Idempotent law} \\ &= 1 && \text{using complement law} \end{aligned}$$

Theorem 3.4 $a + (a.b) = a$

Proof.

$$\begin{aligned} a + (a.b) &= a.1 + a.b && \text{by Identity law} \\ &= a.(1 + b) && \text{using Distributive law} \\ &= a.1 && \text{using Boundness law} \\ &= a && \text{using Identity law} \end{aligned}$$

Theorem 3.5 For each $a \in B$, a' is unique.

Proof. If possible, Let a' and a'' be two complements of a.

Then $a + a' = a' + a = 1$ and $a.a' = a'.a = 0$

$a + a'' = a'' + a = 1$ and $a.a'' = a''.a = 0$

Now

$$\begin{aligned} a' = a'.1 &= a'.(a + a'') \\ &= a'.a + a'.a'' \\ &= 0 + a'.a'' \\ &= a.a'' + a'.a'' \\ &= (a + a').a'' \\ &= 1.a'' \\ &= a'' \end{aligned}$$

So $a' = a''$

Hence the complement a' for $a \in B$ is unique.

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Theorem 3.6. The elements 0 and 1 are unique.

Proof. Suppose that there are two zero elements, 0_1 and 0_2. For each element $a_1 \in B$ and $a_2 \in B$ we have

$$a_1 + 0_1 = a_1 \quad \text{and} \quad a_2 + 0_2 = a_2$$

Let

$$a_1 = 0_2 \quad \text{and} \quad a_2 = 0_1. \text{ Thus,}$$

$$0_2 + 0_1 = 0_2 \quad \text{and} \quad 0_1 + 0_2 = 0_1$$

But

$$0_2 + 0_1 = 0_1 + 0_2 \quad \text{and} \quad \text{thus } 0_1 = 0_2$$

Similarly, the reader can easily show that the element 1 is unique.

Theorem 3.7 (*i*) $(a + b)' = a'.b'$ (*ii*) $(a.b)' = a' + b'$.

Proof. (*i*) The method of proof here is to show that

$$(a + b) + a'b' = 1 \text{ and } (a + b).a'b' = 0.$$

This shows that $(a + b)$ and $a'b'$ are complements and by the theorem 3.5 we establish the theorem.

Now

$$\begin{aligned}
(a + b) + a'b' &= [(a + b) + a'].[(a + b) + b'] && \text{by Distributive law}\\
&= [a' + (a + b)].[a + (b + b')] && \text{using Associative law}\\
&= [(a' + a) + b].[a + (b + b')]\\
&= (1 + b).(a + 1) && \text{using Complement law}\\
&= 1.1 && \text{using Boundness law}\\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
(a + b).a'b' &= a.(a'b') + b.(a'b') && \text{by Distributive law}\\
&= (aa').b' + b.(b'a') && \text{using Associative law}\\
&= 0.b' + (bb').a && \text{using Complement and Associative law}\\
&= 0 + 0.a' && \text{using Boundness and complementary law}\\
&= 0 + 0\\
&= 0
\end{aligned}$$

So

$$(a + b)' = a'.b'$$

(*ii*) To show that the complement of $(a.b)$ is $a' + b'$ we have to show that

$$(a.b) + (a' + b') = 1 \quad \text{and} \quad (a.b).(a' + b') = 0$$

Now

$$\begin{aligned}
(a.b) + (a' + b') &= [a + (a' + b')].[b + (a' + b')] && \text{by Distributive law}\\
&= [(a + a') + b'].[b + (b' + a')]\\
&= [1 + b'].[(b + b') + a']\\
&= (1 + b').(1 + a')\\
&= 1.1 && \text{using Boundness law}\\
&= 1
\end{aligned}$$

Again

$$\begin{aligned}
(a.b).(a' + b') &= (a.b).a' + (a.b).b' && \text{by Distributive law}\\
&= (b.a).a' + (a.b).b' && \text{using Commutative law}\\
&= b.(a.a') + a.(b.b') && \text{using Associative law}\\
&= b.0 + a.0 && \text{using Complement law}\\
&= 0 + 0 && \text{using Boundness law}\\
&= 0
\end{aligned}$$

So

$$(a.b)' = a' + b'$$

3.3. Unique Features

The striking unique features of Boolean algebra, which distinguishes it from the other branches of algebra are

(*a*) Boolean algebra has only a finite set of elements, namely 1 and 0, which are usually called true and false, where as ordinary algebra deals with a set of an infinite number of elements.

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Thus all the variables and constants are allowed to have only two possible values 0 and 1. Hence it provides the operations and rules for working with the set {0, 1};

(*b*) Boolean algebra does not have operations equivalent to subtraction and division;

(*c*) In ordinary algebra, there is no equivalent of the unary operation of bar or/over a variable known as complementing;

(*d*) In Boolean algebra the cancellation law does not hold, that is, $a + b = a + c \nRightarrow b = c$ and $a \,.\, b = a \,.\, c \nRightarrow b = c$.

Boolean Subalgebra

A non-empty subset A of a Boolean algebra $(B, +, ., ', 0, 1)$ is called a Boollean subalgebra if A is a Boolean algebra under the same set of binary and unary operations *i.e.* $a, b \in A \Rightarrow a + b$, $a.b$, $a' \in A$.

For example, consider a set $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$, the divisor of 30 and let +, . and $'$ be defined on D_{30} as follows :

$$a + b = \text{lcm}\,(a, b)$$

$$a.b = gcd\,(a, b) \text{ and } a' = 30/a$$

Then D_{30} is a Boolean algebra (see example 22) and $A_1 = \{1, 5, 6, 30\}$, $A_2 = \{1, 3, 10, 30\}$. and $A_3 = \{1, 5, 6, 30\}$ are Boolean sub algebra of D_{30} since A_1, A_2 and A_3 are closed under +, . and $'$. But $A_4 = \{1, 2, 3, 6\}$ is not a Boolean subalgebra because $6' = 30/6 = 5$ does not belong to A_4.

To show that A is a Boolean subalgebra it is not necessary to verify the closureness with respect to all three operations. It is enough to show that A is closed under the set of operations $\{+, '\}$ or $\{., '\}$.

3.4. Basic operations

The basic operations used in Boolean algebra are Logical addition, logical multiplication and complementation. These basic operations are called **logic operations.** Boolean 1 and 0 do not represent actual numbers. In the digital logic field several terms are used synonymously with 0 and 1. Some of the common ones are shown in Table 3.1.

Logic 0	False	Off	Low	Open	No
Logic 1	True	On	High	Closed	Yes

Table 3.1

Logical Addition (OR operation)

Each variable in Boolean algebra has either two values: true or false (1 or 0), The operation can be defined as follows

$0 + 0 = 0$ $\quad 0 + 1 = 1$ $\quad 1 + 0 = 1$ $\quad 1 + 1 = 1$

where logical addition is denoted by + or by OR. If we write C = A + B, one can determine C by listing all possible combinations for A and B in a tabular form which is known as truth table of logical addition as in Table 3.2.

Input		*Output*
A	*B*	*C = A + B*
0	0	0
1	0	1
0	1	1
1	1	1

Table 3.2. Truth table of logical addition (OR operations)

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The symbol + does not have the normal meaning, but is a logical addition and is referred to as OR operation. The equation A + B = C can be read as A OR B equals C. This concept can be extended to any number of variables. To avoid ambiguity, a number of other symbols have been recommended as replacement for + sign, for example, ∪ and ∨.

Logical Multiplication (AND operation)

The logical multiplication can be defined as follows

$$0.0 = 0 \qquad 0.1 = 0 \qquad 1.0 = 0 \qquad 1.1 = 1$$

where logical multiplication is denoted by. or by AND. If we write C = A.B, we can determine C by listing all possible combination of A and B. The truth table for logical multiplication of two variables is shown in Table 3.3.

Input		*Output*
A	*B*	*C = A . B*
0	0	0
1	0	0
0	1	0
1	1	1

Table 3.3 Truth table of logical multiplication (AND operations)

It is apparent from the table that the AND operation is exactly the same as ordinary multiplication. The equation A.B = C can be read as A AND B equals C.

Complementation

In Boolean algebra, an operation called complementation is used for which symbol is – (bar over) or ′ and this can be defined as 0′ = 1 and 1′ = 0.

Here A′ means the complement of A and read as NOT A. It is opposite to the logic value of A. The truth Table 3.4 clarifies this for the two cases A = 0 and A = 1.

Input	*Output*
A	*B = A′*
0	1
1	0

Table 3.4 Truth table for logical complementation

Switching Circuit

There are systems whose behaviour can be described in terms of Boolean algebra. An important example is provided by electrical circuits consisting of networks of wires and switches.

A switch is a device for opening and closing an electric circuit and it can be idealised as an element that has only two stable states, either open or closed. Here we shall consider electric circuit that has only switches and they are referred to as a switching circuits or network.

The simplest network consists of a wire containing single switch X indicated in Fig 3.1. If switch X is open, then we will define the value of X to be 0; if switch X is closed, then we will define the value of X to be 1. When the switch is closed the current will flow and the current will not flow if the switch is open.

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Thus all the variables and constants are allowed to have only two possible values 0 and 1. Hence it provides the operations and rules for working with the set {0, 1};

(*b*) Boolean algebra does not have operations equivalent to subtraction and division;

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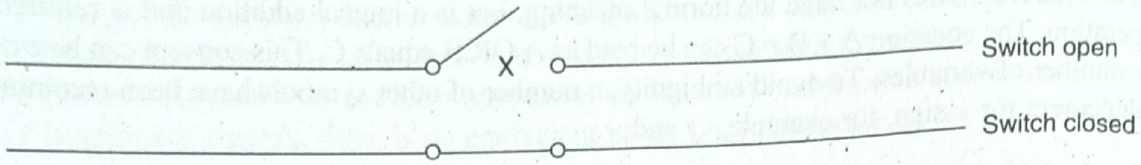

Fig. 3.1

As there is only one switch, the circuit has the same value as the switch has. Thus, the circuit has value 1 if the current flows and 0 if the current does not flow.

Now consider two switches connected in series as shown in Fig. 3.2. It is clear that the current will flow, that is, the network has value 1 if and only if both switch are closed (*i.e.*, both switches have value 1) and the current will not flow if either switch is open (*i.e.*, either switch has value 0)

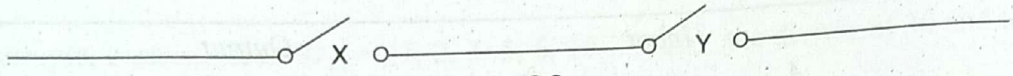

Fig. 3.2

Thus the state of the series network is completely determined by the state of two switches X and Y. If f_1 represents the state of the network, then $f_1 = f_1$ (X, Y). The values of f_1 for different values of X and Y are shown in the table 3.5.

Switch X	*Switch Y*	f_1 (X, Y)
Open (0)	Open (0)	OFF (0)
Closed (1)	Open (0)	OFF (0)
Open (0)	Closed (1)	OFF (0)
Closed (1)	Closed (1)	ON (1)

Table 3.5. Behaviour of two switches in series circuits.

This is precisely the property described by the truth table of logical multiplication (AND operation). Thus the switches in series perform the AND operation.

Next consider a network composed of two switches in parallel.

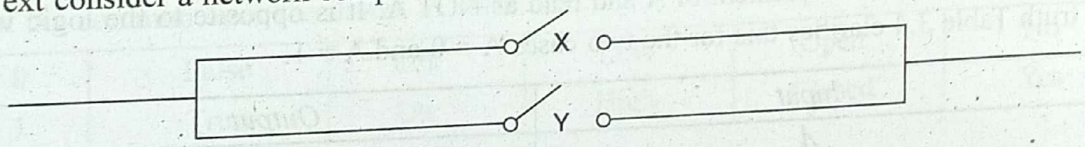

Fig. 3.3

It is clear that the current will flow between two terminals iff switch X is closed or switch Y is closed. If f_2 represent the state of the networks the value of f_2 for different values of X and Y are shown in the Table 3.6.

Switch X	*Switch Y*	f_2 (X, Y)
Open (0)	Open (0)	OFF (0)
Closed (1)	Open (0)	ON (1)
Open (0)	Closed (1)	ON (1)
Closed (1)	Closed (1)	ON (1)

Table 3.6. Behaviour of two switches in parallel circuits.

This is precisely the property described by the truth table of logical addition (OR operation). Thus the switches in parallel perform the OR Operation.

Equivalent Switching Circuit of Axioms and Theorems

It is easy to construct switching circuits corresponding to Boolean expression. If two Boolean expressions are equivalent to one another, then it follows that the corresponding switching circuits are also equivalent.

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Here we will illustrate some of the axioms and basic theorems of Boolean algebra by network switches. Two switching networks S_1 and S_2 are equivalent, denoted by $S_1 \sim S_2$ if both are open or both are closed for the same state of switches in S_1 and S_2.

Boolean Algebra **Switching network**

$a + 0 = a$

A switch in parallel with an open circuit is equivalent to the switch alone.

$a.1 = a$

$a + 1 = 1$

A switch in parallel with a closed circuit is equivalent to the closed circuit.

$a + a = a$

$a.a = a$

$a + a' = 1$

$a + b = b + a$

$a.b = b.a$

$a + (b + c) = (a + b) + c$

$a\,(b.c) = (a.b).c$

$a.(b + c) = a.b + a.c$

$a + (b.c) = (a + b).(a + c)$

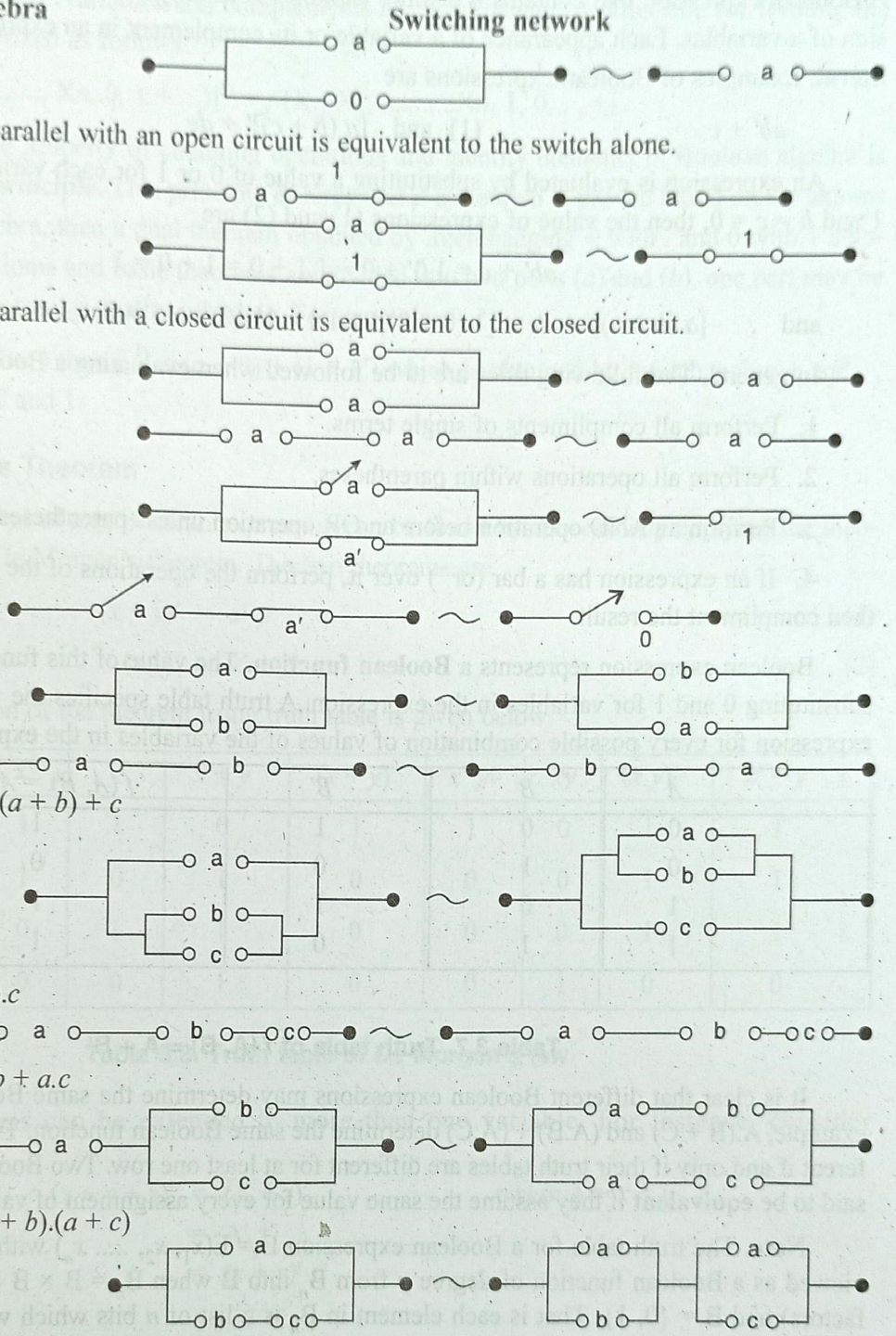

Fig. 3.4

3.5. Boolean Function

A variable x is called a **Boolean variable** if it assumes values either 0 or 1

Boolean expressions are formed by application of the basic operations to one or more variables or constants. The simplest expression consists of a single constant or variable, such as 0, x or x'. The Boolean expression in the variables $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ are defined recursively as follows:

1. $0, 1, x_1, x_2 \ldots\ldots\ldots\ldots x_n$ are Boolean expressions.

If E_1 and E_2 are Boolean expressions, then E_1', $(E_1.E_2)$, and $(E_1 + E_2)$ are Boolean expressions. A Boolean expression that contains n distinct variables is usually referred to as a Boolean expression of n variables. Each appearance of a variable or its complement in an expression is referred as **literal.** Examples of Boolean expressions are

$$ab' + c \qquad (1) \quad \text{and} \quad [a\,(b + c)]' + de \qquad (2)$$

An expression is evaluated by substituting a value of 0 or 1 for each variable. If $a = d = e = 1$ and $b = c = 0$, then the value of expressions (1) and (2) are

$$ab' + c = 1.0' + 0 = 1.1 + 0 = 1 + 0 = 1$$

and $$[a\,(b + c)]' + de = [1\,(0 + 0)]' + 1.1 = [1\,(0)]' + 1 = 1 + 1 = 1$$

In general, The following rules are to be followed when evaluating a Boolean expression:

1. Perform all compliments of single terms.
2. Perform all operations within parentheses.
3. Perform an AND operation before an OR operation unless parentheses indicate otherwise.
4. If an expression has a bar (or ′) over it, perform the operations of the expression first and then compliment the result.

Boolean expression represents a **Boolean function.** The value of this function is obtained by substituting 0 and 1 for variables in the expression. A truth table specifies the values of a Boolean expression for every possible combination of values of the variables in the expression.

A	B	B'	$f(A, B) = A + B'$
0	0	1	1
0	1	0	0
1	0	1	1
1	1	0	1

Table 3.7. Truth table of f (A, B) = A + B′

It is clear that different Boolean expressions may determine the same Boolean function. For example, A.(B + C) and (A.B) + (A.C) determine the same Boolean function. Two functions are different if and only if their truth tables are different for at least one row. Two Boolean expressions are said to be **equivalent** if they assume the same value for every assignment of values to the variables.

Note. The truth table for a Boolean expression $E = E(x_1, x_2, \ldots\ldots x_n)$ with n variables may be viewed as a Boolean function of degree n from B_n into B when $B_n = B \times B \times B \times \ldots\ldots\ldots\ldots \times B$ (n factors) and $B = \{0, 1\}$. That is each element in B_n is a list of n bits which when assigned to the list of variables in E produces an element in B.

A Boolean function of n variable be represented by a table having 2^n rows.

Example 1. Find the number of different Boolean functions of degree n.

Solution. By applying the product rule for counting, we get 2^n different n-tuples of 0s and 1s. Since a Bolean function is an assignment of 0 and 1 to each of these 2^n different n-tuples, hence there are 2^{2^n} different Boolean functions.

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Duality

Given a Boolean expression, the dual is formed by replacing AND with OR, OR with AND, 0 with 1, and 1 with 0. Variables and complements are left unchanged. This rule for forming the dual can be summerized as follows:

$$[f(X_1, X_2, \ldots\ldots, Xn, 0, 1 +, .)]^D = f(X_1, X_2, \ldots\ldots Xn, 1, 0, ., +)$$

This important property of changing operations and identity elements in Boolean algebra is known as **duality principle.** This principle ensures that if a theorem is proved based on the axioms of the Boolean algebra, then a dual theorem obtained by interchanging + with . and 0 with 1 automatically holds. Axioms and basic theorems are divided into two parts (*a*) and (*b*), one part may be obtained by other by using duality principle. For example,

If $f = (a.1) . (0.a')$ then $f^D = (a + 0) + (1 + a')$ which is obtained by interchanging + and . , and interchanging 0 and 1.

3.6. De-Morgan's Theorem

The complement of any Boolean expression can be found by successively applying the theorem referred to as De-Morgan's theorem. The two theorems are

$$(x + y)' = x' y' \quad \ldots (1)$$

$$(x y)' = x' + y' \quad \ldots (2)$$

The verification of the theorem using truth table is given below.

x	y	x'	y'	$x + y$	$(x + y)'$	$x' y'$	xy	$(x y)'$	$x' + y'$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

Table 3.8. Truth table of De-Morgan's law

These theorems can be extended to more than one variable. For instance, consider $(x + y + z)'$

$$\begin{aligned}(x + y + z)' &= (A + z)' \qquad \text{where } A = x + y \\ &= A' z' \\ &= (x + y)' z' \\ &= x' y' z'.\end{aligned}$$

Similarly, it can be proved $(xyz)' = x' + y' + z'$.

Proceeding similarly, we can extend the results to any number of variables.

The theorems are equally valid where x and / or y are expressions that contain more than one variable. For example,

$$\begin{aligned}(AB' + C)' &= (AB')' C' \\ &= (A' + (B')') C'\end{aligned}$$

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$$= (A' + B)\, C'$$
$$= A'\, C' + BC'$$

We can combine the two forms of De-Morgan's theorem into a single rule which will enable us to complement an expression. To obtain the complement of a Boolean expression replace

(*i*) 0 with 1

(*ii*) 1 with 0

(*iii*) + with •

(*iv*) • with +

(*v*) each variable with its complement.

which can be summerized as

$$[f(x_1, x_2, \ldots\ldots\ldots, x_n, 0, 1, +, .)]' = f(x_1', x_2', \ldots\ldots\ldots x_n', 1, 0, \ .\ , +).$$

Example 2. Obtain an equivalent expression for [(x.y) (z′ + xy′)]

Solution. Applying general form of De-Morgan's theorem, we get

$$\begin{aligned}
[(x.y)\,(z' + xy')]' &= (x.y)' + (z' + xy')' \\
&= x' + y' + [z.(x' + y)] \\
&= x' + y' + zx' + zy \\
&= x' + x'.z + y' + yz \\
&= x' + y' + z' \qquad [\text{applying } x + xy = x \text{ and } x + x'y = x + y]
\end{aligned}$$

Note. The method of forming the dual is similar to the method of forming the complement except that the variables are not complemented when forming the dual.

3.7. Logic Gate

A logic gate is an electronic circuit that operates on one or more input signals to produce an output signal. Gates are digital circuits and often called logic circuits. Gates are represented by symbols, the lines entering the symbol from left are inputs, and the line on the right is the output. Placing a small circle on an input or output complements the signal on that line.

OR gate

The OR gate is an electronic device which receives two or more inputs and produces an output equal to the OR sum of the inputs. Fig. 3.5. is the symbol for two input OR gate and its truth table is shown in Table 3.10. An input signal applied to gate has only two stable states either 1 (high) or 0 (low) and output is 1 (high) if at least one of the inputs is high. The output is 0 if all inputs are 0.

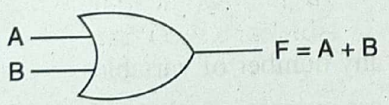

Fig. 3.5. Symbol for an OR gate

Input		*Output*
A	*B*	*F = A + B*
1	1	1
1	0	1
0	1	1
0	0	0

Table 3.9. The truth table for an OR gate

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The same idea can be extended to more than two inputs. A three input OR gate and its truth table is shown in Fig. 3.6 and Table 3.10.

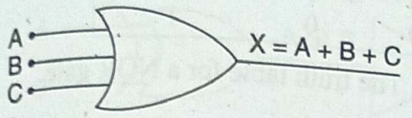

Fig. 3.6. Symbol for three input OR gate

Input A	B	C	Output X = A + B + C
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Table 3.10. The truth table for an three input OR gate

AND gate

AND gate is an electronic device that accepts the values of two or more Boolean variables as input and produces their Boolean sum as output. The symbol of the gate is shown in Fig. 3.7 and the truth table is shown in Table 3.11. Same operation holds good for an AND gate with more than the two inputs.

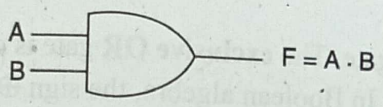

Fig. 3.7. Symbol for an AND gate

Input A	B	Output F = A . B
1	1	1
1	0	0
0	1	0
0	0	0

Table 3.11. The truth table for an AND gate

NOT gate

A NOT gate has only one input and one outzput. It is also called an inverter. Its output is the complement of the input. The symbol of the gate is shown in Fig. 3.8. and the truth table is shown in Table 3.12.

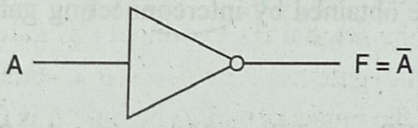

Fig. 3.8. Symbol for an inverter

A	F = A′
1	0
0	1

Table 3.12 The truth table for a NOT gate

NOR gate

This gate is logically equivalent to a NOT gate with an OR gate as shown in Fig. 3.9. The NOR function is the complement of OR function. It is represented by the symbol shown in Fig. 3.10. and has truth table defined in Table 3.13.

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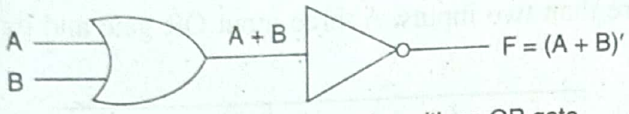

Fig. 3.9. A NOT gate in series with an OR gate.

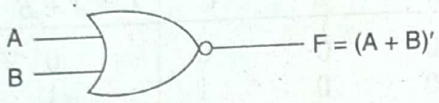

Fig. 3.10. Symbol for a NOR gate.

A	B	$F = (A + B)'$
1	1	0
1	0	0
0	1	0
0	0	1

Table 3.13. The truth table for a NOR gate

NAND gate

This gate is logically equivalent to a NOT gate in series with an AND gate as shown in Fig. 3.11. It is represented by the symbol shown in Fig. 3.12. And has the truth table defined by Table 3.14.

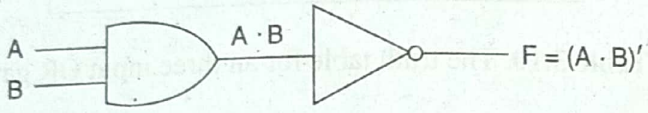

Fig. 3.11. A NOT gate series with an AND gate

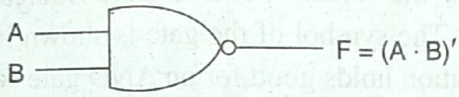

Fig. 3.12. Symbol for a NAND gate

A	B	$F = (A \cdot B)'$
1	1	0
1	0	1
0	1	1
0	0	1

Table 3.14. The truth table for a NAND gate.

Exclusive OR gate

The full name of OR gate is the **inclusive OR** gate. The **exclusive OR** gate is different. It gives an output of 1 when either but not both inputs are 1. In Boolean algebra, the sign ⊕ stands for XOR operation. The truth table for this gate is given in Table 3.15. and its symbol is shown in Fig. 3.13.

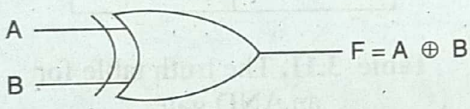

Fig. 3.13. Symbol for a XOR gate

Input A	B	*Output* $F = A'B + AB'$
0	0	0
0	1	1
1	0	1
1	1	0

Table 3.15. The Truth Table for XOR gate

Using the truth table, the disjunctive normal form for the gate is:

$$F = A.B' + A'.B = A \oplus B$$

A logic circuit for XOR operation can be obtained by interconnecting gates which has been shown in next section.

Exclusive–NOR(XNOR) Gate

XNOR gate is equivalent to an Exclusive OR gate followed by an inverter. The output is 1 only when both inputs are either 0 or 1. The truth table for this gate is shown in Table 3.16.

Input A	B	*Output* $F = AB + A'B'$
0	0	1
0	1	0
1	0	0
1	1	1

Table 3.16. Truth Table for XNOR Gate

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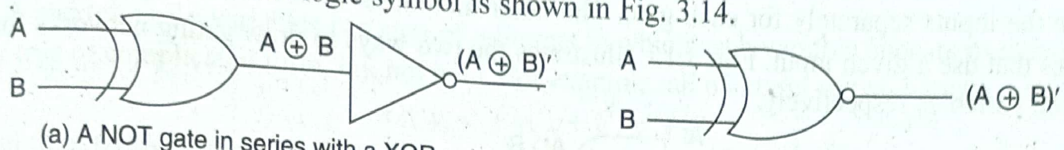
XNOR operation can be written as follows:

$$F = AB + A'B'$$
$$= A \text{ XNOR } B$$
$$= (A \oplus B)'$$

The logic diagram and logic symbol is shown in Fig. 3.14.



(a) A NOT gate in series with a XOR gate (b) Symbol for XNOR gate

Fig. 3.14.

Seven important logic gates with their truth tables are summerized in Table 3.17.

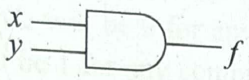
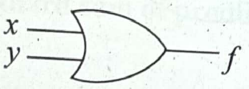
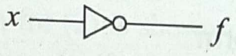
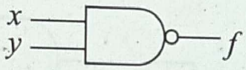
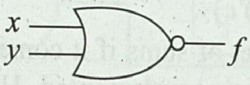
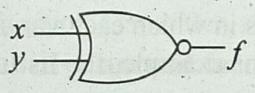
Name	*Symbol*	*Truth Table*	*Boolean Formula*
AND		x y f 0 0 0 0 1 0 1 0 0 1 1 1	$f = xy$
OR		x y f 0 0 0 0 1 1 1 0 1 1 1 1	$f = x + y$
NOT (inverter)		x f 0 1 1 0	$f = x'$
NAND		x y f 0 0 1 0 1 1 1 0 1 1 1 0	$f = (xy)'$
NOR		x y f 0 0 1 0 1 0 1 0 0 1 1 0	$f = (x + y)'$
XOR (Exclusive-OR)		x y f 0 0 0 0 1 1 1 0 1 1 1 0	$f = x \oplus y$ $= x'y + xy'$
XNOR (Exclusive-NOR)		x y f 0 0 1 0 1 0 1 0 0 1 1 1	$f = (x \oplus y)'$ $= xy + x'y'$

Table 3.17

Interconnecting Gates

The gates can be interconnected to form **gating or logic networks.** The Boolean expression corresponding to given network, can be derived by systematically progressing from input to output on the gates. When gating network is formed, some gates may share inputs. One method is to indicate the inputs separately for each gate. The other method is to use branching that indicate all the gates that use a given input. Fig. 3.15. illustrates the two ways to draw gating networks for two and three variables respectively.

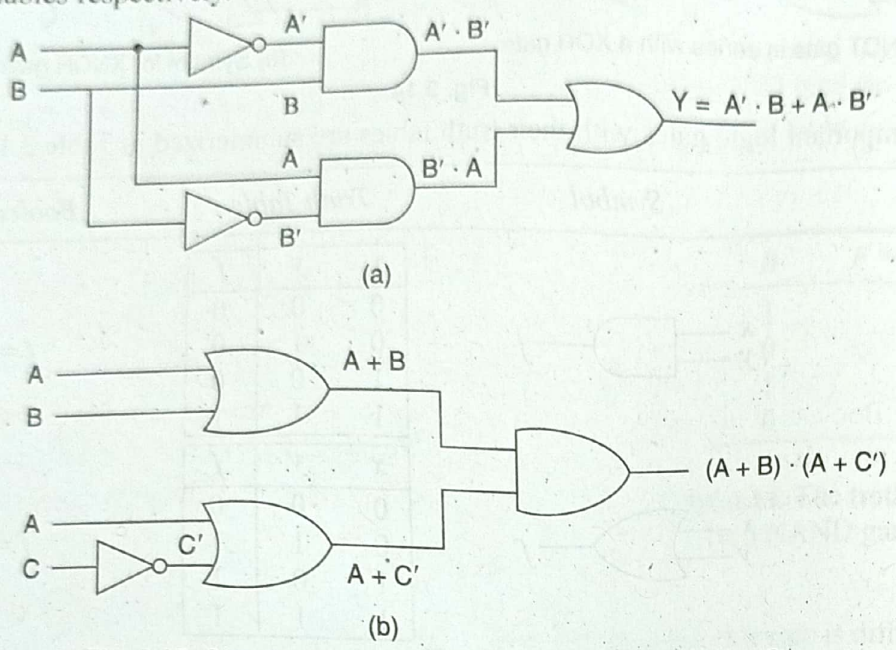


Fig. 3.15. Circuits to implement A′B + AB′ and (A + B) . (A + C′)

3.8. Sum of Products and Product of Sums Form

A Boolean expression E is said to be in a sum of products form if E is a sum of two or more products of variables (complemented or uncomplemented), none of which is included in another. Some examples of this form are

1. $xz' + x'yz' + xy'z$
2. $abc + ac + a'bc'$
3. $xy + x'yz' + y'z' + w$

Note. In a sum of product expression, one complementation sign can not cover more than one variable in a term (*e.g.*, we can not have $(xyz)'$ or $(ab)'c$)

A Boolean expression E is said to be in a product of sums if it consists of several sum terms logically multiplied. The variables may or may not be complemented. Here are some product of sum expression

1. $(x + y)(x + y)$
2. $(x + y' + z)(x + z)$
3. $(a + b')(c' + d)f$

3.9. Normal Form

Minterm

A minterm of n variables is a product of n literals in which each variable appears exactly once in either true or complimented form, but not both. For example, the list of all of the minterms of the two variables x and y are

$$xy, x'y, xy', x'y'$$

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The list of all of the minterms of three variables x, y and z are

$$xyz,\ xyz',\ xy'z,\ x'yz,\ xy'z',\ x'y'z,\ x'yz',\ x'y'z'$$

In a similar way, n variables can be combined to form 2^n minterms.

Maxterm

A maxterm of n variables is a sum of n literals in which each variable appears exactly once in either true or complicated form, but not both. For example, all maxterm of two variables x and y are

$$x + y,\ x' + y,\ x + y',\ x' + y'$$

and all max terms of three variables x, y and z are

$$x' + y' + z',\ x' + y' + z,\ x' + y + z',\ x + y' + z',\ x + y' + z,\ x + y + z',\ x' + y + z,\ x + y + z$$

In a similar manner, n variables forming maxterms, with each variable being complemented or uncomplemented, provide 2^n possible combination.

Note. Any single minterm will be zero except for a single combination of the values of the literals while any single maxterm will be 1 except for a particular combination of the values of the literals. For instances, $xy'z$ will be 0 for any combinations of x, y and z except $x = 1$, $y = 0$ and $z = 1$. Also, $x + y' + z$ will be 1 for any combination for x, y and z except for $x = 0$, $y = 1$ and $z = 0$.

When a Boolean function f is written as a sum of minterms, it is referred to as a **minterm expansion** or the **disjunctive normal form** of the Boolean function. It is also called **canonical sum of products** or **standard sum of products.** The following are minterm expansions of Boolean functions

(1) $f(x, y) = xy + x'y$

(2) $f(x, y, z) = x'y'z + x'yz + xy'z' + xyz$

The canonical sum of products form can be implemented simply by an interconnection of gates. Such a realisation for (2) is shown in Fig. 3.16. Note that each mean term is realised using single AND gate. The outputs of all the minterms are connected to the inputs of a single OR gate whose output is the desired function. Therefore, if there are n_1 minterms in the canonical sum of products, there will be $n_1 + 1$ gates used in its realisation.

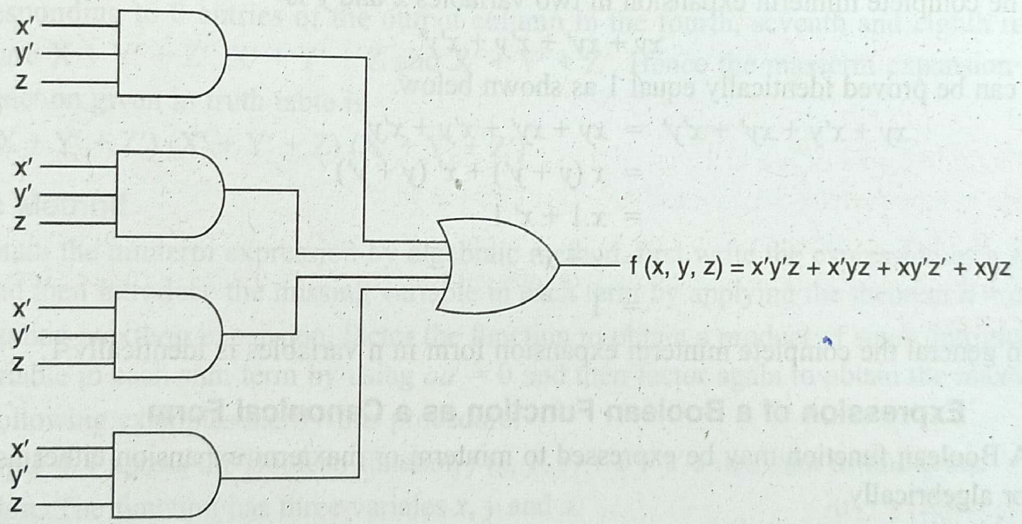

Fig. 3.16. An AND-OR gate Realisation

When a Boolean function f is written as a product of maxterm, it is referred to as a **maxterm expansion** or **the conjunctive normal form of** the Boolean function. It is also called **canonical product of sums** and **standard product of sums.** The following are maxterm expansions of Boolean functions

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(1) $f(x, y) = (x + y)(x' + y)$

(2) $f(x, y, z) = (x + y\ z)(x + y' + z)(x' + y + z')(x' + y' + z)$

The gate circuit realisation of (2) is shown in Fig. 3.17.

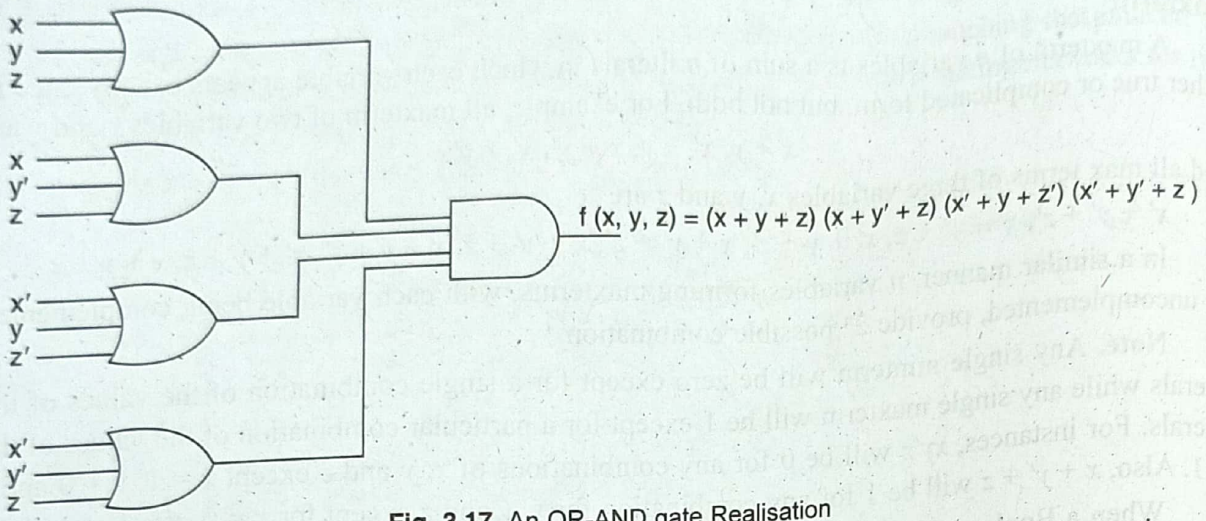

Fig. 3.17. An OR-AND gate Realisation

Actually either canonical form may use many more gates than is necessary. In later section we shall discuss a procedure for minimizing the number of gates required.

Boolean functions expressed as a sum of minterms or product of maxterms are said to be in **canonical form.**

A Boolean function when expressed as a sum of all 2^n minterms of n variables is called the **complete minterm expansion** or the **complete disjunctive normal form.** Similarly, a Boolean function when expressed as a product of all 2^n maxterms of n variables is called the **complete maxterm expansion** or **complete conjunctive normal form.**

Boolean function when expressed as a complete minterm expansion or complete maxterm expansion form is called **complete canonical form**

The complete minterm expansion in two variables x and y is

$$xy + xy' + x'y + x'y'.$$

which can be proved identically equal 1 as shown below.

$$\begin{aligned} xy + x'y + xy' + x'y' &= xy + xy' + x'y + x'y' \\ &= x(y + y') + x'(y + y') \\ &= x.1 + x'.1 \\ &= x + x' \\ &= 1 \end{aligned}$$

In general the complete minterm expansion form in n variables is identically 1.

3.10. Expression of a Boolean Function as a Canonical Form

A Boolean function may be expressed to minterm or maxterm expansion either using a truth table or algebrically.

From a Given Truth Table.

If a truth table to describe the behaviour of logic network is given, the corresponding logic expression in the minterm and maxterm expansions can be obtained from the truth table by the method described below.

(*a*) Inspect the column corresponding to dependent variable from the top row and pick the row with 1 entry.

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(*b*) A term in the Boolean expression corresponding to this row is obtained ANDing all the independent variables in the truth table.

(*c*) Repeat steps (*a*) and (*b*) till all the entries in the dependent variable column is exhausted. Obtain a Boolean expression by ORing the terms corresponding to 1 entries of dependent variable.

The method is applied for obtaining minterm expansion of a Boolean function. The method for obtaining the maxterm expansion of a Boolean is similar, instead of picking the rows of 1 entry in dependent variable, the rows of 0 entry are to be picked up. The individual term is formed by taking the sum of independent variables. The expression is obtained by ANDing the individual terms. The following example clarifies this procedure.

Example 3. Consider the truth table given in Table 3.18. and find out the corresponding Boolean expression in sum of minterms and product of maxterms.

Input			*Output*
X	*Y*	*Z*	*F*
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	0
1	1	1	0

Table 3.18.

Solution. Corresponding to entry 1 of the output column in the first, second, third, fifth and sixth row the minterms are X′Y′Z′, X′Y′Z, X′YZ′, XY′Z′, XY′Z. Hence the minterm expansion of the Boolean function given in truth table is

$$F = X'Y'Z' + X'Y'Z + X'YZ' + XY'Z' + XY'Z$$

Corresponding to 0 entries of the output column in the fourth, seventh and eighth row the maxterms are X + Y′ + Z′, X′ + Y′ + Z and X′ + Y′ + Z′. Hence the maxterm expansion of the Boolean function given in truth table is

$$F = (X + Y' + Z')\,(X' + Y' + Z)\,(X' + Y' + Z')$$

Algebraic Method

To obtain the minterm expression by algebraic method, first write the expression as a sum of products and then introduce the missing variable in each term by applying the theorem $a + a' = 1$.

For finding maxterm expansion, factor the function to obtain a product of sums, introduce the missing variable in each sum term by using $aa' = 0$ and then factor again to obtain the maxterms.

The following examples clarify this procedure.

Example 4. Express the Boolean function $f(x, y, z) = x + y'z$ in a sum of minterms.

Solution. The function has three variales x, y and z.

The first term x is missing two variables; therefore applying $a + a' = 1$

$$x = x(y + y') = xy + xy'$$

This is still missing the variable z:

$$x = xy(z + z') + xy'(z + z')$$

$$= xyz + xyz' + xy'z + xy'z' \qquad (1)$$

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The second term $y'z$ is missing one variable x:

$$y'z = y'z(x + x') = xy'z + x'y'z \quad (2)$$

Hence

$$\begin{aligned} f(x, y, z) &= x + y'z \\ &= xyz + xyz' + xy'z + xy'z' + xy'z + x'y'z \text{ using (1) and (2)} \\ &= xyz + xyz' + xy'z + xy'z' + x'y'z \text{ applying the theorem} \\ &\qquad a + a = a \text{ on } xy'z \end{aligned}$$

Example 5. Express the Boolean function $f(a, b, c) = ab + a'c$ as a product of maxterms.

Solution. First convert the function into factor to obtain the product of sum forms by using distributive law $x + y.z = (x + y)(x + z)$

$$\begin{aligned} f(a, b, c) = ab + a'c &= (ab + a')(ab + c) \\ &= (a + a')(b + a')(a + c)(b + c) \\ &= (b + a')(a + c)(b + c) \qquad \because a + a' = 1 \end{aligned}$$

Since each sum term is missing one variable, applying $aa' = 0$ we get

$$\begin{aligned} a' + b = a' + b + cc' &= (a' + b + c)(a' + b + c') \\ a + c = a + c + bb' &= (a + b + c)(a + b' + c) \\ b + c = b + c + aa' &= (a + b + c)(a' + b + c) \end{aligned}$$

Now removing those terms that appear more than once $(a.a = a)$,

$$f(a, b, c) = (a + b + c)(a + b' + c)(a' + b + c)(a' + b + c')$$

which is a product of maxterms.

Conversion of Disjunctive Normal Form (DNF) of a Boolean function to its Conjunctive Normal Form (CNF) and vice-versa.

The following steps can be followed to convert a Boolean function given in disjunctive normal form to conjunctive normal form.

Conversion of DNF to CNF

1. Denote the given function which is in DNF by f.
2. Find f', which is the sum of those terms of the complete DNF in the variables of f, that are not present in the given function.
3. Take complement f'' of f'.
4. Simplify *R.H.S.* of f'' by using De Morgan's laws. This will transform the *R.H.S.* into its *CNF*.
5. Since $f'' = f$, the *R.H.S.* is the required *CNF* of the given function in *DNF*.

The following steps can be followed to convert a Boolean function given in conjunctive normal form to disjunctive normal form.

Conversion of CNF to DNF

1. Denote the given function which is in *CNF* by f.
2. Find f', which is the product of those terms of the complete *CNF* in the variables of f, that are not present in the given function.
3. Take complement f'' of f'.
4. Simplify the *R.H.S.* of f' by using De Morgan's laws. This will transform the *R.H.S.* in-to its *DNF*.
5. Since $f' = f$, the *R.H.S.* is the required *DNF* of the given function in *CNF*.

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Example 6. *Convert the Boolean function $f(x, y) = x.y' + x'.y + x'.y'$ to its conjunctive normal form.*

Solution. Here $f(x, y) = x.y' + x'.y + x'.y'$

The complete *DNF* in two variables $x, y = x.y + x.y' + x'.y + x'.y'$

Hence $f'(x, y) = x.y$

So, $f''(x, y) = (x.y)' = x' + y'$

which is the required *CNF* of given function.

Example 7. Convert the Boolean function

$f(x, y, z) = (x' + y + z') . (x' + y + z) (x + y' + z)$ in disjunctive normal form.

Solution. Here $f(x, y, z) = (x' + y + z') . (x' + y + z) . (x + y' + z)$

The complete *CNF* in three variables $(x, y, z) = (x + y + z) (x + y + z') (x + y' + z)$

$.(x' + y + z) . (x + y' + z') . (x' + y + z') . (x' + y' + z) . (x' + y' + z')$

Hence $f'(x, y, z) = (x + y + z) (x + y + z') . (x + y' + z') . (x' + y' + z) . (x' + y' + z')$

$\therefore\ (f'(x, y, z)' = [(x + y + z) (x + y + z') . (x + y' + z') . (x' + y' + z) . (x' + y' + z')]'$

$= [(x + y + z) (x + y + z') . (x + y' + z') . (x' + y' + z)]' + x.y.z$

$= [(x + y + z) (x + y + z') . (x + y' + z')]' + x.y.z' + x.y.z$

$= [(x + y + z) (x + y + z')]' + x'.y.z + x.y.z' + x.y.z$

$= (x + y + z)' + (x + y + z')' + x'.y.z + x.y.z' + x.y.z$

$= x' . y' . z' + x' . y' . z + x'.y.z + x.y.z' + x.y.z$

which is the required *DNF* of the given function.

3.11. Simplification of Boolean Expression by Algebraic Method

A Boolean expression can be reduced into the simplest form by using Boolean algebraic laws and the consensus theorem. Since each expression corresponds to a network of logic gates, simplifying an expression leads to simplifying the corresponding logic network which is less complex then the one which the original expression would have produced. It is not always obvious which theorem should be applied to produce the simplest result. Further, there is no other way to test whether the simplified expression is in its simplest form or whether it could have been simplified further.

Consensus theorem: The theorem is stated as follows

$$xy + x'z + yz = xy + x'z$$

Proof.

$$\begin{aligned} xy + x'z + yz &= xy + x'z + (x + x')\, yz \\ &= xy + x'z + xyz + x'yz \\ &= (xy + xyz) + (x'z + x'yz) \\ &= xy\,(1 + z) + x'z\,(1 + y) \\ &= xy + x'z \end{aligned}$$

Example 8. Simplify $f(x, y) = xy' + xy + x'y$

Solution.

$$\begin{aligned} f(x, y) &= xy' + xy + x'y \\ &= x\,(y' + y) + x'y \\ &= x.1 + x'y \\ &= x + x'y \\ &= x + y \end{aligned}$$

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Note that the original expression would require four gates (*e.g.*, there are three AND gates and an OR gate) while the simplified expression require a OR gate which is a considerable reduction in the number of gates.

3.12. Boolean Expression from Logic and Switching Network

The Boolean expression corresponding to any logic and switching network, may be completely obtained using the Boolean operations OR, AND and NOT. For example, consider the network in Fig. 3.18.

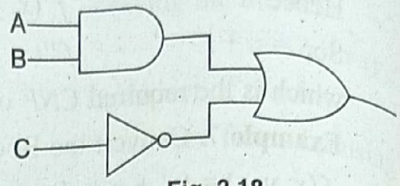

Fig. 3.18

This network has three inputs, A, B, C and a single output. Utilizing the Boolean expression for each gate, one can describe logic network algebraically. The expression for the AND gate output with inputs A and B is written A.B. The output of NOT gate with input C is written is C′. These two outputs are inputs of the OR gate so that output is the OR sum of the inputs. Thus we can write the output of the network as

$$f = A.B + C'$$

The algebraic description at each step of the logic network is shown in Fig. 3.19.

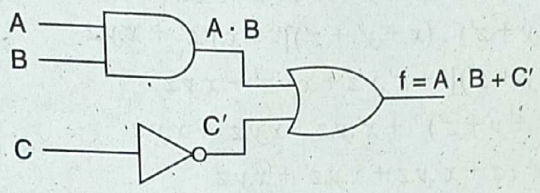

Fig. 3.19

To illustrate further, consider the network given in Fig. 3.20.

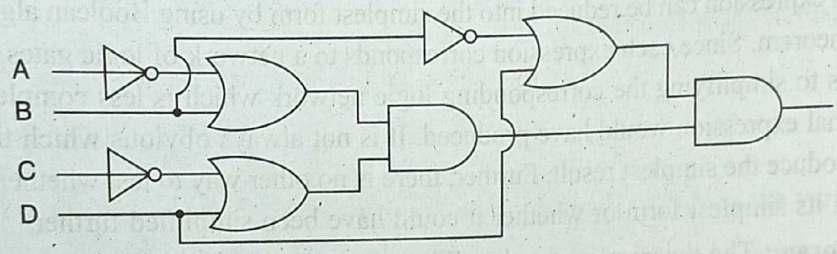

Fig. 3.20

This network has four inputs A, B, C, D and single output. The inputs A and C after passing through NOT gate produce outputs A′ and C′. The outputs of two OR gates with inputs A′, B and C′, D are A′ + B and C′ + D respectively. These outputs serve as input of AND gate and produces output as (A′ + B).(C′ + D). Again B after passing NOT gate is input with D to OR gate and its output is B′ + D. Finally B′ + D and (A′ + B). (C′ + D) are inputs to AND gate and final output is (B′ + D) . (A′ + B). (C′ + D). The algebraic description of the network at each stage is shown in Fig. 3.21

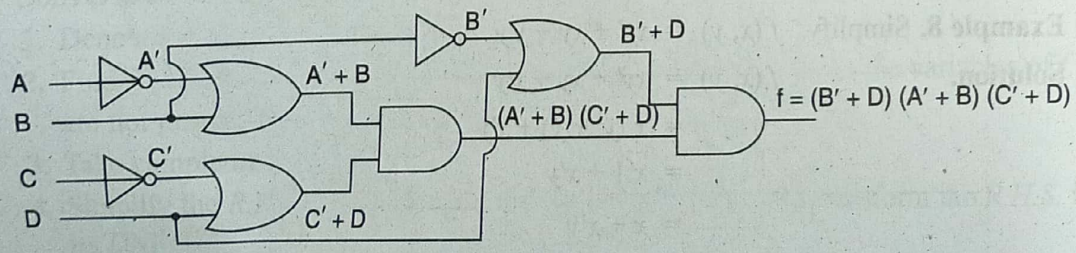

Fig. 3.21

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In swithcing network, the expression of output when x and y is in series is written $x.y$ and when x, y is in parallel is written as $x + y$. For example, consider the switching network in Fig. 3.22.

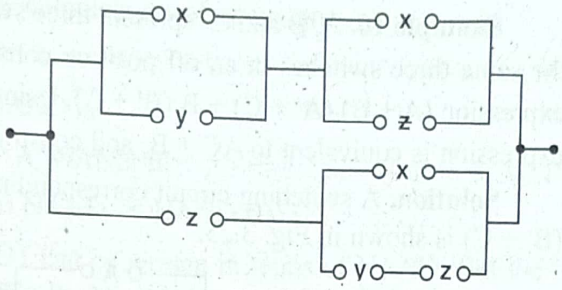

Fig. 3.22

The upper branch consists of two parallel networks. They are represented by the Boolean expression $(x + y) . (x + z)$. The lower branch consists of one parallel network and one in series. They are equivalent to $z. (x + y.z)$. These two in turn form a parallel network. Therefore, the wole circuit is represented by

$$(x + y).(x + z) + z.(x + y.z)$$

which is the Boolean expression of the switching network.

3.13. Implementation of Logic Expressions With Logic Gates and Switching Circuits

A Boolean expression may be transformed from an algebraic expression into a logic diagram composed of AND, OR and NOT gates. The logic diagram include an inverter circuit for every variable present in its complement form. For example, if a logic expression is defined by $f = A + B.C$ we would immediately know that the corresponding logic network will require one AND and one OR gate. If we need a circuit that is defined by $f = A + B'$, we would use a two input OR gate with a NOT gate on one of the inputs. To find simpler circuits. One must know how to obtain equal and simple expression.

Example 9. Find the logic networks corresponding to Boolean expressions (*i*) AB + CD (*ii*) X′ Y′Z + X′YZ + XY′.

Solution. Expression (*i*) will require two AND gates and one OR gate and (*ii*) will require three AND gates, one OR gate and two NOT gates.

Figs. 3.23. and 3.24 show the implementation of the logic expressions with logic gates.

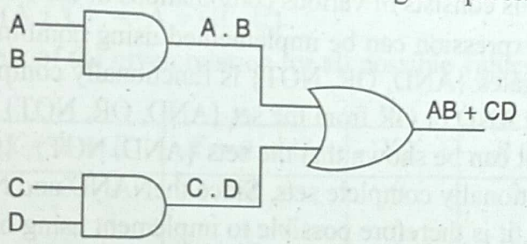

Fig. 3.23. Implementation of F_1 = AB + CD with AND - OR network

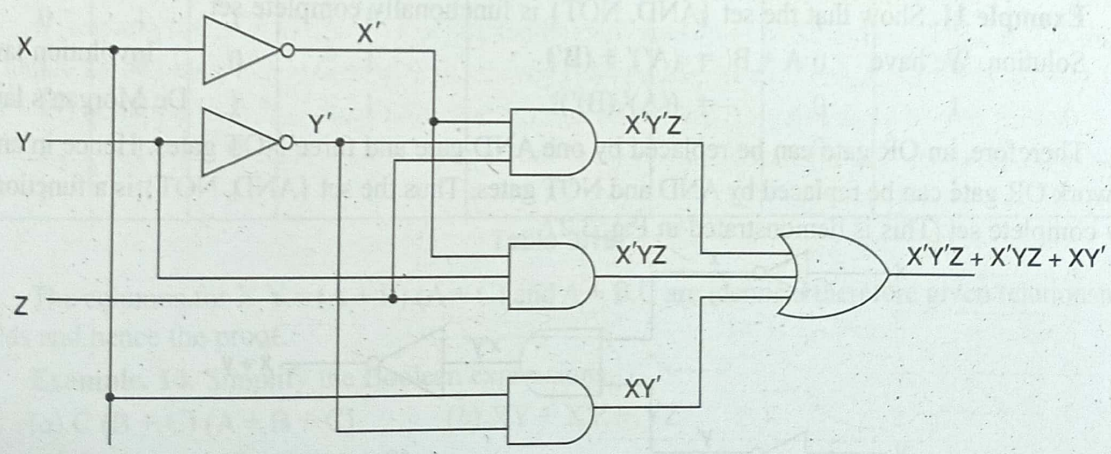

Fig. 3.24. Implementation of F_2 = X′Y′Z + X′YZ + XY′ with AND, OR and NOT network.

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Example 10. A, B and C represent three switches in an on position and A′, B′ and C′ represent the same three switches in an off position construct a switching circuit representing the Boolean expression (A + B) (A′ + C) + B (B′ + C). Using the laws of Boolean algebra, show that the above expression is equivalent to AC + B, and construct an equivalent switching circuit.

Solution. A switching circuit corresponding to the Boolean expression (A + B) (A′ + C) + B (B′ + C) is shown in Fig. 3.25.

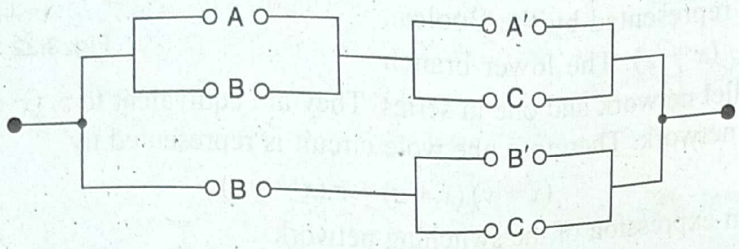

Fig. 3.25

Now (A + B) (A′ + C) + B (B′ + C)

$$= AA' + AC + BA' + BC + BB' + BC'$$
$$= AC + BA' + BC + BC'$$
$$= AC + BA' + B(C + C') \qquad C + C' = 1$$
$$= AC + BA' + B \qquad A + 1 = 1$$
$$= AC + B(A' + 1) = AC + B.$$

The equivalent switching circuit is shown in fig. 3.26.

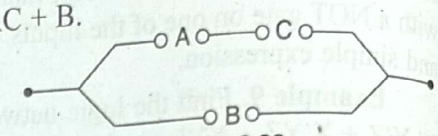

Fig. 3.26

3.14. Functionally Complete Sets

A set of operations is called a functionally complete set if and only if any Boolean function can be expressed by operations belonging to the set only.

All Boolean expressions consists of various combinations of the basic operations of AND, OR, and NOT. Therefore, any expression can be implemented using combinations of these operations. This shows that the set of gates {AND, OR, NOT} is functionally complete. It is interesting to see that we can eliminate either AND or OR from the set {AND, OR, NOT} and still obtain a functionally complete set of gates. It can be shown that the sets {AND, NOT}, {OR, NOT}, {XOR, AND}, {NAND}, {NOR} are functionally complete sets. Since theNAND and NOR operations alone form a functionally complete set, it is therefore possible to implement using only one type of gate, either NAND or NOR. Hence these two operations gates are also known as **universal operations**/gates.

Example 11. Show that the set {AND, NOT} is functionally complete set.

Solution. We have $A + B = (A')' + (B')$ Involution law

$= ((A)'.(B)')'$ De Morgan's law

Therefore, an OR gate can be replaced by one AND gate and three NOT gates. Hence in any network OR gate can be replaced by AND and NOT gates. Thus the set {AND, NOT} is a functionally complete set. This is demonstrated in Fig. 3.27.

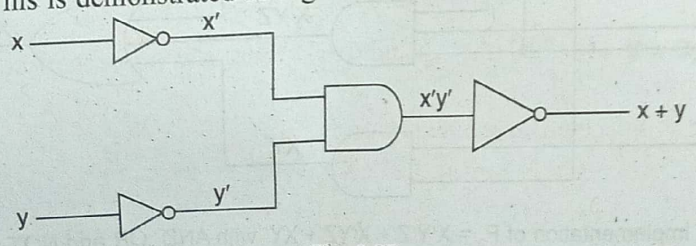

Fig. 3.27

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Example 12. The set {NAND} is a functionally complete set of operations

Solution. We observe that A NAND B = (AB)′

Therefore,

$$A' = (AA)' = A \text{ NAND } A$$

$$A + B = ((A)'(B)')' = A' \text{ NAND } B'$$

$$= (A \text{ NAND } A) \text{ NAND } (B \text{ NAND } B)$$

The above relations show that both OR and NOT can be written in terms of NAND. But the set {OR, NOT} is functionally complete. It follows that the set {NAND} is functionally complete. This is demonstrated in Fig. 3.28. In Fig. (*a*), the NAND simply acts as NOT operation. In Fig. (*b*), two NAND gates are connected so that the AND operation is performed. In Fig. (*c*), the OR operation is performed by using NAND gates.

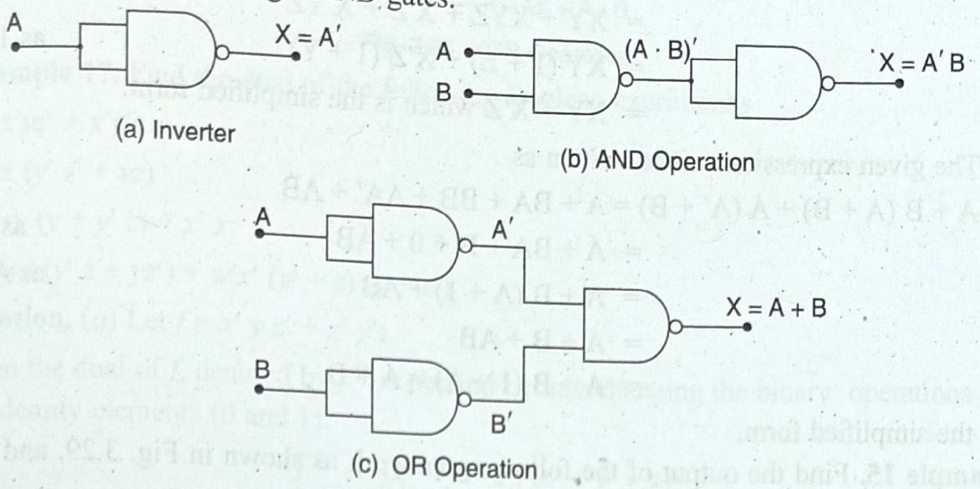

Fig. 3.28

SOLVED EXAMPLES

Example 13. Use truth table to prove that

$$(A + B) . (A + C) = A + (B . C)$$

Solution. The truth table of the given relation for all possible values of A, B and C are shown in the table given below.

A	*B*	*C*	*X = A + B*	*Y = A + C*	*X.Y*	*B.C*	*A + (B.C)*
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

Table 3.19.

The columns for X.Y = (A + B).(A + C) and A + B.C are identical therefore given relationship holds and hence the proof.

Example. 14. Simplify the Boolean expressions

(*a*) C (B + C) (A + B + C) (*b*) XY + X′Z + YZ

(*c*) A + B (A + B) + A (A′ + B)

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Solution. The given expression (*a*) can be written as

$$
\begin{aligned}
(CB + CC)(A + B + C) &= (CB + C)(A + B + C) && \text{as } CC = C \\
&= C(A + B + C) && \text{as } C + CB = C \\
&= CA + CB + CC \\
&= CA + CB + C \\
&= CA + C \\
&= C && \text{which is the simplified form.}
\end{aligned}
$$

(*b*)

$$
\begin{aligned}
XY + X'Z + YZ &= XY + X'Z + YZ(X + X') && \text{as } X + X' = 1 \\
&= XY + X'Z + XYZ + X'YZ \\
&= XY + XYZ + X'Z + X'YZ \\
&= XY(1 + Z) + X'Z(1 + Y) && \text{as } 1 + Z = 1 \\
&= XY + X'Z \text{ which is the simplified form.}
\end{aligned}
$$

(*c*) The given expression can be written as

$$
\begin{aligned}
A + B(A + B) + A(A' + B) &= A + BA + BB + AA' + AB \\
&= A + BA + B + 0 + AB && \text{as } AA' = 0 \\
&= A + B(A + 1) + AB && \text{as } A + 1 = 1 \\
&= A + B + AB \\
&= A + B(1 + A) = A + B
\end{aligned}
$$

which is the simplified form.

Example 15. Find the output of the following network as shown in Fig. 3.29. and design a simple network having the same output.

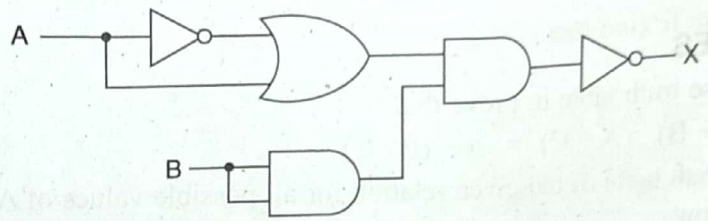

Fig. 3.29

The network output can be found out by finding the output of each gate, going from left to right. The NOT gate with input A produces an output A′. The OR gate with inputs A and A′ results output A + A′ and the inputs B and B through AND gate has output B.B. These two outputs are inputs of the next AND gate and the output is (A + A′).(B.B) which after passing through NOT gate becomes ((A + A′).(B.B))′. Hence the final output is the Boolean expression ((A + A′).(B.B))′.

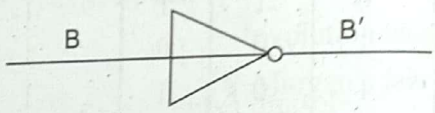

Fig. 3.30

Now A + A′ = 1 and B.B = B, the output expression can be simplified as ((A + A′).(B.B))′ = (1.B)′ = (B)′ = B′ which can be represented by logic gate as

Example 16. Show that A ⊕ B = ((A.B)′)′.(A′.B)′)′ and hence design a logic circuit of XOR gate using NAND only.

Solution.

$$
\begin{aligned}
((A.B')'.(A'.B)')' &= ((A' + (B')').((A')' + (B)'))' \\
&= ((A' + B).(A + B'))' \\
&= (A' + B)' + (A + B')'
\end{aligned}
$$

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$$= (A')'.B' + A'.(B')' = A.B' + A'.B$$
$$= A \oplus B$$

The NAND gate implementations of the XOR function is shown in Fig. 3.31.

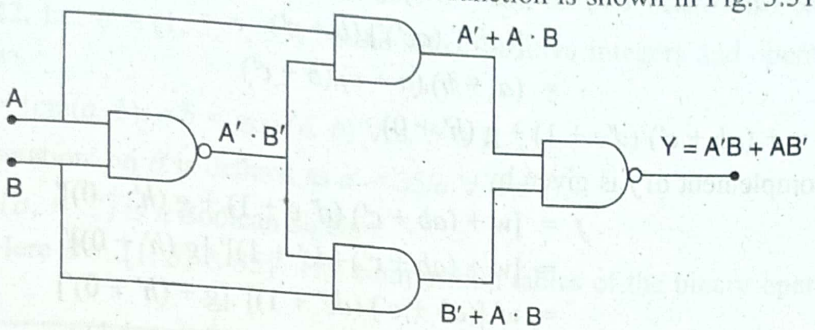

Fig. 3.31. XOR Operation

Example 17. Find the dual of the following Boolean expressions

(*a*) $x'yz' + x'y'z$

(*b*) $x\,(y'\,z' + yz)$

(*c*) $xy\,(y + y'\,z) + x'\,z$

(*d*) $wx\,(y'\,z + yz') + w'x'\,(y' + z)\,(y + z')$

Solution. (*a*) Let $f = x'\,y\,z' + x'\,y'z$

Then the dual of f, denoted by f^D is obtaind by interchanging the binary operations (+ and .) and the identity elements (0 and 1).

$$\therefore \quad f^D = (x' + y + z')\,(x' + y' + z)$$

(*b*) Let $f = x\,(y'z' + yz)$

Then the dual of f, denoted by f^D is

$$f^D = x + (y' + z')\,(y + z)$$

(*c*) Let $f = xy\,(y + y'\,z) + x'\,z$

Then the dual of f, denoted by f^D is

$$f^D = [x + y + y\,(y' + z)]\,(x' + z)$$

(*d*) Let $f = wx\,(y'z + yz') + w'x'\,(y' + z)\,(y + z')$

Then the dual of f denoted by f^D is

$$f^D = [(w + x) + (y' + z)\,(y + z')]\,[(w' + x') + y'z + yz']$$

Example 18. Find the complement of the following Boolean expressions

(*a*) $xy' + x'z$ (*b*) $x\,(y'z' + yz)$

(*c*) $ab' + ac + b'c$ (*d*) $w + (ab + c')\,(d'\,e + 1) + g\,(h' + 0)$

Solution. (*a*) Let $f = xy' + x'\,z$

Then the complement of f is given by

$$f' = (xy' + x'z)' = (xy')'.(x'z)'$$
$$= (x' + (y')').((x')' + z) = (x' + y).(x + z')$$

(*b*) Let $f = x\,(y'z' + yz)$

Then the complement of f is given by

$$f' = [x\,(y'z' + yz)]' = x' + (y'z' + yz)'$$
$$= x' + (y'z').(yz)'$$
$$= x' + (y + z).(y' + z')$$

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(*c*) Let $f = ab' + ac + b'c$

Then the complement of f is given by

$$\begin{aligned} f' = (ab' + ac + b'c)' &= (ab' + ac)'.(b'c)' \\ &= [(ab')'.(ac')'].(b + c') \\ &= (a' + b).(a + c).(b + c') \end{aligned}$$

(*d*) Let $f = w + (ab + c')(d'e + 1) + g(h' + 0)$

Then the complement of f is given by

$$\begin{aligned} f &= [w + (ab + c')(d'e + 1) + g(h' + 0)]' \\ &= [w + (ab + c')(de' + 1)]'.[g(h' + 0)]' \\ &= w'.[(ab + c')(de' + 1)]'.[g + (h' + 0)'] \\ &= w'.[(ab + c')' + (de' + 1)'].(g + h.1) \\ &= w'.[(ab)'.c + (de')'.0].(g + h.1) \\ &= w'.[(a' + b').c + (d' + c).0].(g + h.1) \end{aligned}$$

Example 19. Prove the Boolean identity $A \oplus B \oplus A.B = A + B$

Solution. L.H.S. $= A \oplus B \oplus A.B$

$$\begin{aligned} &= [A \oplus B] \oplus A.B \\ &= C \oplus A.B \qquad \text{(where } C = A \oplus B) \\ &= C(AB)' + AB(C') = C(A' + B') + AB(C') \\ &= (AB' + A'B)(A' + B') + AB[A'B + AB']' \\ &= AB' + A'B + AB[(A'B)'(AB')'] \text{ as } AA' = 0, (AB)' = A' + B' \\ &= AB' + A'B + AB[(A + B')(A' + B)] \\ &= AB' + A'B + AB(AB + B'A') \\ &= AB' + A'B + AB \\ &= A'B + A(B' + B) = A'B + A \qquad B + B' = 1 \\ &= (A + B)(A' + A) = A + B = \text{R.H.S.} \end{aligned}$$

Example 20. Simplify the following expression using Boolean algebra:

(*i*) $(AB'C' + AB'C + ABC + AB'C)(A + B)$

(ii) $P + P'QR' + (Q + R)'$

Solution. (*i*) $(AB'C' + AB'C + ABC + ABC')(A + B)$

$$\begin{aligned} &= [AB'(C' + C) + AB(C + C')](A + B) \qquad [\because C + C' = 1] \\ &= (AB' + AB)(A + B) = A(B' + B)(A + B) \\ &= A(A + B) = AA + AB = A + AB = A(1 + B) = A \end{aligned}$$

(*ii*)

$$\begin{aligned} P + P'QR' + (Q + R)' &= P + P'QR' + Q'R' \\ &= P + R'(P'Q + Q') \\ &= P + R'(Q' + P') = P + R'Q' + R'P' \\ &= (P + R'P') + R'Q' = P + R' + R'Q' \\ &= P + R'(1 + Q') = P + R'. \end{aligned}$$

Example 21. Show that the sheffer stroke function

$$f(x_1, x_2) = x_1 \mid x_2 = (x_1 x_2)'$$

is functionally complete.

Solution. The set of operation is functionally complete set if one can generate the sets (+, ′) or (., ′) from a set since it is well known that these two sets are functionally complete.

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Now $x_1 \mid x_1 = (x_1\, x_1)' = x_1'$

and $(x_1 \mid x_1) \mid (x_2 \mid x_2) = x' \mid x_2 = (x'_i\, x'_2)' = x_1 + x_2$

So, the given operation can generate the set (+, ′) and hence functionally complete.

Example 22. Let $B = (1, 5, 7, 35)$ be the set of positive integers and operations + and. are defined as follows :

$$a + b = \text{lcm}(a, b),\ a.b = gcd\ (a, b)\ \forall\ a, b \in B.$$

A unary operation′ on B is defined as $a' = 35/a\ \forall\ a \in B$.

Show that $(B, +, ., ')$ is a Boolean algebra.

Solution. Here $B = \{1, 5, 7, 35\}$. The composition tables of the binary operations are given below.

+	1	5	7	35
1	1	5	7	35
5	5	5	35	35
7	7	35	7	35
35	35	35	35	35

.	1	5	7	35
1	1	1	1	1
5	1	5	1	5
7	1	1	7	7
35	1	5	7	35

It is clear form the composition tables that + and. are binary operations on B.

From the composition tables, we have

$a + b = b + a$ and $a\,.\,b = b\,.\,a\ \forall\ a, b \in B$

Hence the commutative law holds.

Again, for all $a, b, c, \in B$, we have

$a + (b.c) = (a + b)\,.\,(a + c)$ and $a.\ (b + c) = (a.\ b) + (a.\ c)$

Hence the distributive law holds.

For arbitrary element $a = 7, b = 5, c = 1$

$a + (b.c) = 7 + (5.1) = 7 + 1 = 7$ and $(a + b).(a + c) = (7 + 5).(7 + 1) = 35.7 = 7$

$a.\ (b + c) = 7.\ (5 + 1) = 7.5 = 1$ and $(a.\ b) + (a.\ c) = (7\,.\,5) + (7\,.\,5) + (7\,.\,1)$
$= 1 + 1 = 1$

a	1	$a + 1$
1	1	1
5	1	5
7	1	7
35	1	35

a	35	$a.\ 35$
1	35	1
5	35	5
7	35	7
35	35	35

$\therefore\ a + 1 = a\ \forall\ a \in B$ $\qquad$ $a\,.\,35 = a\ \forall\ a \in B.$

Hence identity law holds. Here 1 is the zero element and 35 is the unit element

a	a'	$a + a'$
1	35	35
5	7	35
7	5	35
35	1	35

a	a'	aa'
1	35	1
5	7	1
7	5	1
35	1	1

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$\therefore \quad a + a' = 35 \ \forall \ a \in B$

$aa' = 1 \ \forall \ a \in B$

Hence complement law holds.

$\therefore (B, +, ., ')$ is a Boolean algebra.

Note that $B = \{1, 5, 7, 35\}$ can be denoted by D_{35} which contains all the positive divisiors of 35.

Example 23. Show that the power set $P(X)$ of a non-empty set X, equipped with the binary operations $\cup$ and $\cap$ and the unary operation $'$ of complementation is a Boolean algebra.

Solution. It is known from the set theory that

$A \cup B, A \cap B$ and $A' \in P(X)$ for $A, B \in P(X)$

So the closure property holds.

Further for any two sets $A, B \in P(X)$,

$A \cup B = B \cup A, \qquad A \cap B = B \cap A$

Hence the commutative law holds.

The distributive law also holds as

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

For $A \in P(X)$, we have

$A \cup \phi = A$ and $A \cap X = A$

So, identity law holds. Here ϕ is the zero element and X is the unit element of $P(X)$.

Finally, for $A \in P(X)$, we have

$A \cup A' = X$ and $A \cap A' = \phi$

Hence $(P(X), \cap, \cup, ')$ is a Boolean algebra.

Example 24. In the Boolean algebra $(B, +, ., ')$ express the Boolean function $f(x, y, z) = x.(y'.z)'$ in its disjunctive normal form.

Solution.

$$\begin{aligned} f(x, y, z) &= x.(y'.z)' = x.(y')' + z') && \text{(By De Morgan's law)} \\ &= x.(y + z') = x.y + x.z' \\ &= x.y.(z + z') + x.z'.(y + y') && (\therefore z + z' = 1, y + y' = 1) \\ &= x.y.z + x.y.z' + x.z'.y + x.z'.y' \\ &= x.y.z + (x.y.z' + x.y.z') + x.y'.z' \\ &= x.y.z + x.y.z' + x.y'.z' && \text{(By idempotent law)} \end{aligned}$$

which is the required disjunctive normal form.

Example 25. In the Boolean algebra $(B, +, ., ')$, express the Boolean function $f(x, y, z) = (x + y).(x + z) + y + z'$ in its disjunctive normal form.

Solution.

$$\begin{aligned} f(x, y, z) &= (x + y).(x + z') + y + z' \\ &= x + y.z' + y + z' && [\therefore a + (b.c) = (a + b).(a + c)] \\ &= x. + y + z'.(y + 1) \\ &= x + y + z' && [\therefore a + 1 = 1] \\ &= x.(y + y').(z + z') + y'.(x + x').(z + z') + z'.(x + x').(y + y') \\ &= (x.y + x.y').(z + z') + (y.x + y.x').(z + z') + (z'.x + z'x').(y + y') \\ &= (x.y + x.y').z\ (x.y + x.y')z' + (y.x + y.x')z'\ (z'.x + z'.x').y + (z'.x + z'.x').y' \\ &= x.y.z + x.y'.z + x.y.z' + x.y'.z' + y.x.z + y.x'.z + y.x.z' + y.x'.z \\ &\qquad + z'.x.y + z'.x'.y + z'.x.y' + z'.x'.y' \\ &= x.y.z + x.y'.z + x.y.z' + x.y'.z' + x.y.z + x'.y.z + x.y.z' + x'.y.z' \\ &\qquad + x.y.z' + x'.y.z' + x.y'.z' + x'.y.z' \\ &= (x.y.z + x.y.z) + x.y'.z + (x.y.z' + x.y.z' + x.y.z') + x'.y.z + (x'.y.z' \\ &\qquad + x'.y.z') + (x.y'.z' + x.y'.z') + x'.y'.z' \\ &= x.y.z + x.y'.z + x.y.z' + x'.y.z + x'.y.z' + x.y'.z' + x.y'.z' \end{aligned}$$

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which is the required disjunctive normal form.

Example 26. Prove that the complete disjunctive normal form and complete conjunctive normal form in three variables in a Boolean algebra is equal to the unit and zero element of the Boolean algebra.

Solution. Let the Boolean algebra be $(B, +, ., ', 0, 1)$ and the variables be x, y and z. Complete disjunctive normal form in x, y and z

$$= x.y.z + x.y.z' + x.y'.z + x'.y.z + x.y'.z' + x'.y.z' + x'.y'.z + x'.y'.z'$$
$$= x.(y.z + y.z' + y'.z + y'.z') + x'.(y.z + y.z' + y'.z + y'.z')$$
$$= (x + x').(y.z + y.z' + y'.z + y'.z')$$
$$= 1.(y.z + y.z' + y'.z + y'.z') = (y.z + y.z' + y'.z + y'.z')$$
$$= y.(z + z') + y'(z + z')$$
$$= (y + y').(z + z') \qquad \therefore z + z' = 1$$
$$= 1.1 = 1, \text{ unit element of the Boolean algebra.}$$

Again, complete conjunctive normal form in x, y and z

$$= (x + y + z)(x + y + z')(x + y' + z)(x' + y + z)(x + y' + z')(x' + y + z')$$
$$(x' + y' + z)(x' + y' + z')$$
$$= [(x + y + z)(x + y + z')][(x + y' + z)(x + y' + z')][(x' + y + z)(x' + y + z')]$$
$$[(x' + y' + z)(x' + y' + z')]$$
$$= (x + y + z.z')(x + y' + z.z')(x' + y + zz')(x' + y' + zz')$$
$$= (x + y)(x + y')(x' + y)(x' + y') = (x + yy').(x + yy')$$
$$xx' + yy' = 0, \text{ zero element of the Boolean algebra.}$$

Hence the result holds.

Example 27. In the Boolean algebra $(B, +, ., ')$, express the Boolean function $f(x, y, z) = (x.y' - x.z)' + z'$ in its conjunctive normal form.

Solution. $f(x, y, z) = (x.y' - x.z)' + z'$

$$= (x.y')'.(x.z)' + z'$$
$$= (x' + y).(x' + z') + z' \qquad (\therefore a + b.c = (a + b).(a + c))$$
$$= (x' + y + z').(x' + z' + z')$$
$$= (x' + y + z').(x' + z' + yy') \qquad (\therefore a.a' = 0)$$
$$= (x' + y + z')(x' + z' + y).(x' + z' + y')$$

which is the required conjunctive normal form.

3.15. Karnaugh Map Method for Simplification of Boolean Polynomial Expression

The Karnaugh map method is a graphical technique which provides a simple straightforward procedure for simplification of Boolean expressions of two, three or four variables, it can also be extended to functions of five, six, or more variables. But as the number of variables increases, the excessive number of squares prevents a reasonable selection of adjacent squares and it is difficult to be sure that the best selection has been made.

A Karnaugh map (K-map) is a diagram made up of a number of squares. If the expression contains n variables, the map will have 2^n squares. Each square represents a minterm and 1s are written in the corresponding squares for the minterms present in the expression and 0s are written in those squares which correspond to the minterms not present in the expression. Once the map is filled with 0s and 1s, the canonical sum of products expression for the output can be obtained by ORing together those squares that contain 1. It is a two dimensional representation of a truth table, as it can display in a visual form all the information contained in a truth table.

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2–Variable Karnaugh Maps

Since the number of variables are 2, the map will have $2^2 = 4$ squares The values of one variable, say A, are listed above the top horizontal line and the values of other variable, say B, are listed on the left side. Four possible minterms with two variables A and B

$$AB, AB', A'B, A'B'$$

are represented by the four squares in the map as labelled in Fig. 3.32 (*a*) and alternative representation is shown in Fig. 3.32 (*b*) Squares are said to be **adjacent** if the minterms that they represent differ in exactly one literal. For instance the square representing A′B is adjacent to the squares representing AB and A′B′.

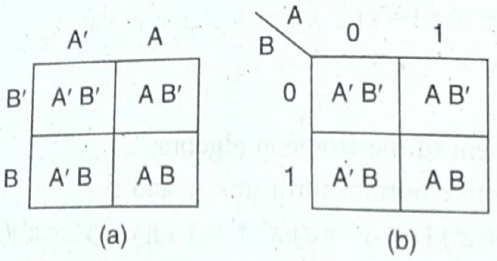

Fig. 3.32. Two variables Karnaugh map

The expression can be simplified by properly combining those squares in the K-map which contains 1s. The process for combining 1s is called **looping.** Whenever there are 1s in two adjacent squares in the K-map, the minterms represented by these squares can be looped and it eliminates the variable that appear in complemented and uncomplemented form.

Example 28. Find Karnaugh maps and simplify the expressions

(*i*) $AB' + A'B'$ (*ii*) $AB' + A'B$ (*iii*) $AB' + A'B + A'B'$

Solution. Since each expression contains two variables, three K-maps of four squares are shown below.

(*i*) In Fig. 3.33 two adjacent squares A′B′ and AB′ containing 1 have been grouped together. They have been encircled. These two terms can be looped that eliminates the A variable since it appears both in complemented and uncomplemented forms. This can be verified algebrically as follows:

$$\begin{aligned} AB' + A'B' &= (A + A')B' \\ &= 1.B' = B' \end{aligned}$$

(*ii*) In Fig. 3.34, K-map does not contain any adjacent square of minterms containing 1, hence the expression can not be simplified further.

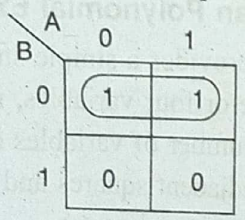

Fig. 3.33

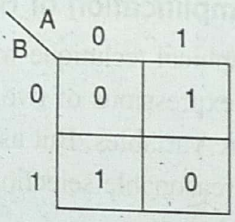

Fig. 3.34

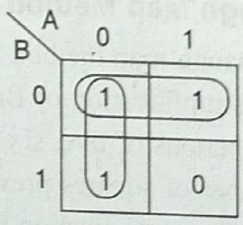

Fig. 3.35

(*iii*) There are two pairs of 1s and they can be combined as shown in Fig. 3.35. Notice that 1 in first column and first row has been enclosed twice, as it is permissible to use the same 1 more than once. Looping of horizontal 1-squares gives the result B′ and vertical 1-squares gives A′. Hence the given expression reduces to the simplified form as A′ + B′. This can also be verified algebrically as follows:

$$\begin{aligned} AB' + A'B + A'B' &= AB' + A'(B + B') \\ &= AB' + A' \\ &= A' + B' \end{aligned}$$

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3-Variable Karnaugh Map

A Karnaugh map in three variables is a rectangle divided into eight squares. One of the ways that eight possible minterms are labled in squares is shown in Fig. 3.36 (*a*) and Fig. 3.36 (*b*) shows the alternate way of representing three variables. In addition to square which are physically adjacent, leftmost and rightmost columns of K-map differ in only one variable. Thus A′B′C′ and AB′C′ are adjacent, and so are A′B′C and AB′C.

	A′B′	A′B	AB	AB′
C′	A′B′C′	A′BC′	ABC′	AB′C′
C	A′B′C	A′BC	ABC	AB′C

(a)

C \ AB	00	01	11	10
0	A′B′C′	A′BC′	ABC′	AB′C′
1	A′B′C	A′BC	ABC	AB′C

(b)

Fig. 3.36. Karnaugh map for three variables

Given a minterm expansion of a function, it can be plotted on a map by placing 1s in the squares which corresponds to minterms present in the expressions and 0s in the remaining squares. Karnaugh map of the Boolean expression A′B′C′ + A′B′C + A′BC′ + ABC′ is shown in Fig. 3.37.

C \ AB	00	01	11	10
0	1	1	1	0
1	1	0	0	0

Fig. 3.37

To simplify a sum-of-products expansion in three variables, one has to identify groups of minterms that can be combined. While forming groups of squares containing 1s the following considerations must be kept in mind.

(*i*) The number of squares in a group must be equal to 2^n such that 2, 4, 8, 16.

(*ii*) A square containing 1 can be included in as many groups as desired.

(*iii*) Group must be largest possible groups, a group of two squares containing 1 should not be made if these squares can be included in a group of four squares. A K-map that contains a group of four 1s that are adjacent to each other is called a **quad.** Looping a quad of 1s eliminates the two variables that appear in both complemented and uncomplemented form.

Example 29. Use Karnaugh map to simplify the followings

(*a*) X = ABC′ + ABC

(*b*) X = A′B′C′ + AB′C′

(*c*) X = A′B′C′ + A′BC′ + ABC′ + AB′C.

(*d*) X = A′B′C′ + A′B′C + A′BC + A′BC′ + AB′C + ABC

Solution. (*a*) Fig. 3.38. represents three variables Karnaugh map of the given Boolean function.

C \ AB	00	01	11	10
0	0	0	1	0
1	1	0	1	0

Fig. 3.38. Karnaugh map of X = ABC′ + ABC

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The adjacent squares representing ABC′ and ABC are grouped together. This eliminates the C variable since it appears in both uncomplemented and complemented forms. The simplification function will be X = AB.

(*b*) The Boolean function X = A′B′C′ + AB′C′ is shown in fig. 3.39. In a K-map, the left and right column of squares are considered to be adjacent. Thus, the two 1s in this map can be looped to provide a simplified result X = B′C′ (variable A is eliminated).

C \ AB	00	01	11	10
0	1	0	0	1
1	0	0	0	0

Fig. 3.39. Karnaugh map of X = A′ B′ C′ + A B′ C′

(*c*) Figure 3.40. represents three variables Karnaugh map of the given Boolean function There are two pairs of 1s and they can be combined as shown in Fig. 3.40. Only 1 representing the term AB′C is isolated. The group of first two horizontal 1 square gives A′C′ and the group second and third horizontal 1 square gives BC′. Hence, the simplified result is

$$X = A'C' + BC' + AB'C$$

C \ AB	00	01	11	10
0	1	1	1	0
1	0	0	0	1

Fig. 3.40. Karnaugh map of X = A′ B′ C′ + A′ B C′ + A B C′ + A B′ C

(*d*) The Karnaugh map of the given function is shown in Fig. 3.41.

C \ AB	00	01	11	10
0	1	1	0	0
1	1	1	1	1

Fig. 3.41. Karnaugh map of X = A′ B′ C′ + A′ B′ C + A′ B C + A′ B C′ + A B′ C + A B C

Here two quads have been formed. Notice that two 1s are a part of both the quads. The quad formed by A′B′C′, A′B′C′, A′BC′ and A′BC produces the resultant as A′ and the quad formed by A′B′C, A′BC, ABC and AB′C produces the resultant as C. Hence the final resultant expression is X = A′ + C.

4-Variable Karnaugh Map

A Karnaugh map in four variables is a square divided into 16 (= 2^4) squares. The squares represent the 16 possible minterms in four variables. The definition of adjacent squares must be extended so that not only are leftmost and rightmost column adjacent as in the 3-variables map, but also the first and last rows are adjacent. Fig. 3.42 (*a*) shows Karnaugh map for four variables A, B, C and D and Fig. 3.42 (*b*) shows the alternate way of representing four variables. Note that A′B′C′D′ square in the first row is adjacent to the square A′B′C′D′ in the last row, since they differ only in the variable C. One can think of the top of the map as being wrapped around to touch the bottom of the map.

	A′ B′	A′ B	A B	A B′
C′D′	A′B′C′D′	A′B C′D′	ABC′D′	AB′C′D′
C′D	A′B′C′D	A′BC′D	ABC′D	AB′C′D
CD	A′B′CD	A′BCD	ABCD	AB′CD
CD′	A′B′CD′	A′BCD′	ABCD′	AB′CD′

(a)

CD \ AB	00	01	11	10
00	A′B′C′D′	A′BC′D′	ABC′D′	AB′C′D′
01	A′B′C′D	A′BC′D	ABC′D	AB′C′D
11	A′B′CD	A′BCD	ABCD	AB′CD
10	A′B′CD′	A′BCD′	ABCD′	AB′CD′

(b)

Fig. 3.42. Karnaugh map for four variables

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To simplify a sum-of-product expansion in four variables, one has to identify groups of minterms of squares 2, 4, 8 or 16 containing 1*s* that can be combined. A group of eight 1*s* that are adjacent to one another is called an **octet.** Looping an octet of 1*s* eliminates three variables that appear in both complemented and uncomplemented form. Several examples of octet are shown in Fig. 3.43.

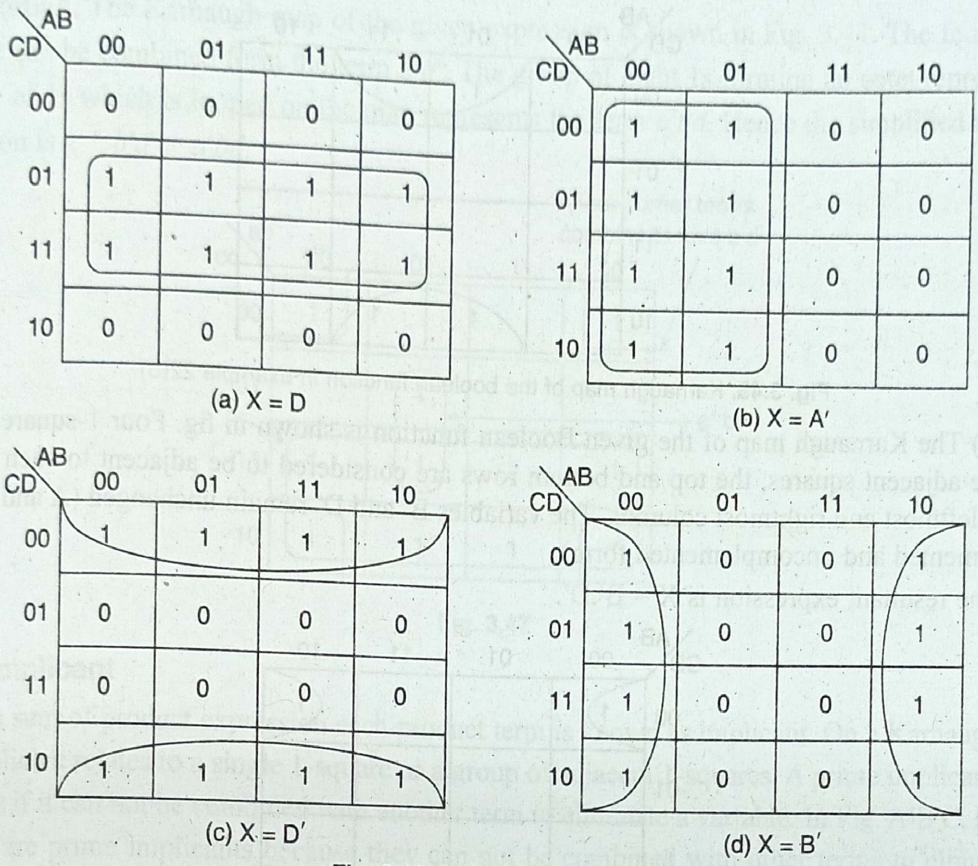

Fig. 3.43. Examples of octets

Example 30. Use Karnaugh map to simplify the following expression

(*a*) X = A′B′CD + A′B′CD′ + AB′C′D′ + AB′CD′

(*b*) X = A′BC′D′ + ABC′D′ + A′BCD′ + ABCD′

(*c*) X = A′B′C′D′ + AB′C′D′ + A′B′CD′ + AB′CD′

Solution. (*a*) The Karnaugh map of the given function is shown in Fig. 3.44. Two pairs of adjacent 1s are grouped together. The pair in the first column gives term A′B′C and the pair of top row and bottom row produces the term AB′D′. Hence, the simplified result is

$$X = A'B'C + AB'D'$$

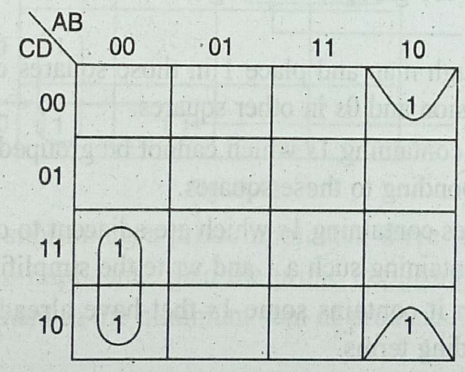

Fig. 3.44. Karnaugh map of X = A′ B′ C D + A′ B′ C D′ + A B′ C′ D′ + A B′ C D′

(*b*) The Karnaugh map of the given Boolean function is shown in Fig. 3.45. Two 1-squares in the first row and two 1-square in the last row f are adjacent squares and hence they form a quad. The variables B and D′ remain unchanged (A and C being complemented and uncomplemented form). The resultant expression is X = B.D′.

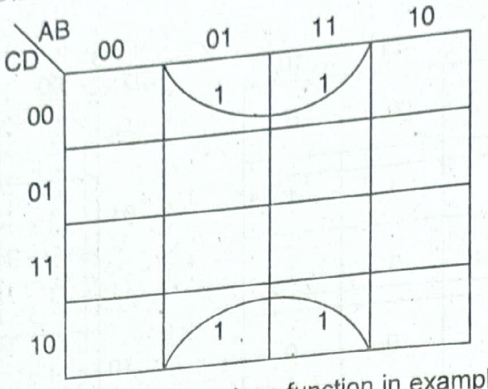

Fig. 3.45. Karnaugh map of the boolean function in example 22(b)

(*c*) The Karnaugh map of the given Boolean function is shown in fig. Four 1-squares of Fig. 3.46 are adjacent squares, the top and bottom rows are considered to be adjacent to each other as are the leftmost and rightmost columns. The variables B′ and D′ remain unchanged (A and C are in complemented and uncomplemented form).

The resultant expression is X = B′.D′.

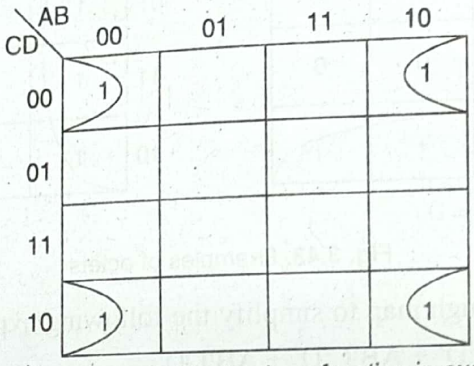

Fig. 3.46. Karnaugh map of the boolean function in example 22(c)

Complete Simplification Process

We have seen that on a K-map, a loop of two eliminates one variable, a loop of four eliminates two variables, are a loop of eight eliminates four variables. Therefore, to eliminate maximum number of variables, one must form the grouping with maximum number of consecutive terms having marked 1. However, number of such consecutive terms, forming a group, must be expressed in powers of 2. The steps below are grouped in using the K-map method for simplifying a Boolean expression.

1. Construct the Karnaugh map and place 1 in those squares corresponding to the minterm present in the expression and 0*s* in other squares.
2. Find out the squares containing 1*s* which cannot be grouped with any other square. Write the minterms corresponding to those squares.
3. Look for those squares containing 1*s* which are adjacent to only one square containing 1*s*. Loop for any pair containing such a 1 and write the simplified corresponding terms.
4. Loop any octet even it contains some 1*s* that have already been looped and write the simplified corresponding terms.
5. Loop any quad. that contains one or more 1*s* which have not been looped, making sure to use the minimum number of loops.

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6. The simplified expression of the given Boolean expression is the sum of all the terms derived during the process from (2) to (4).

Example 31. Use Karnaugh map to simplify the Boolean expression $a'b'c'd' + ab'c'd' + a'bc'd' + a'b'cd + a'bcd + abcd + ab'cd + a'b'cd' + a'bcd' + abcd' + ab'cd'$

Solution. The Karnaugh map of the given expression is shown in Fig. 3.47. The four corner 1-square can be combined form the term $b'd'$. The group of eight 1s forming an octet represents c. The pair of 1s which is looped on the map represents the term $a'bd$. Hence the simplified Boolean expression is $c + b'd' + a'bd$.

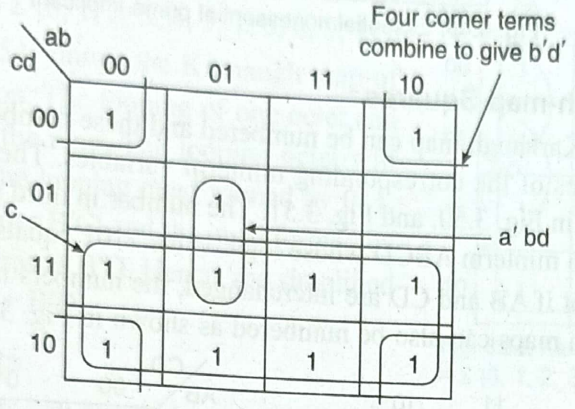

Fig. 3.47

Prime Implicant

In a sum of product expression each product term is known as implicant. On a Karnaugh map each implicant relates to a single 1-square or a group of adjacent 1-squares. A prime implicant is an implicant if it can not be combined with another term to eliminate a variable. In Fig. A'B'C, A'CD', and AC' are prime implicants because they can not be combined with other terms to eliminate a variable. All of the prime implications of a function can be obtained from a Karnaugh map. A single 1 on a map represents a prime implicant if it is not adjacent to any other 1s. Two adjacent 1s on a map form a prime implicant if they are not contained in a group of four 1s., four adjacent 1s form a prime implication if they are not contained in a group of eight 1s etc.

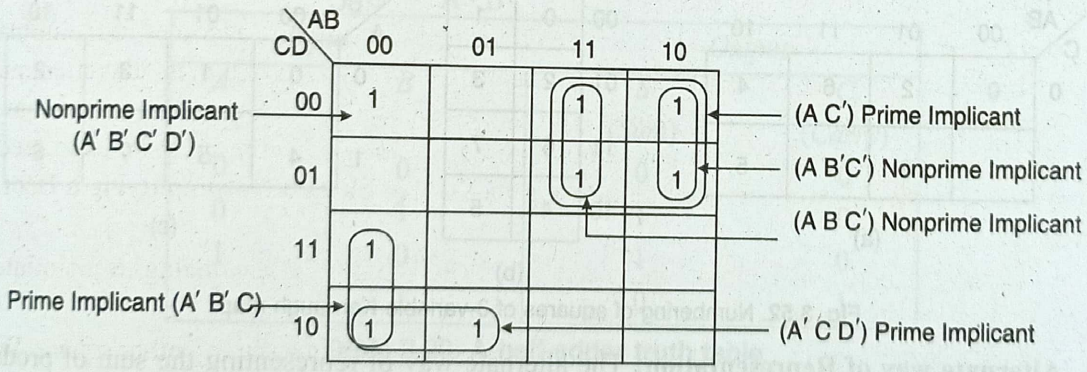

Fig. 3.48

If among the minterms subsuming a prime implicant, there is at least one that is covered by this and only by this prime implicant, then the prime implicant is called an **essential prime implicant** and it must be included in the minimum sum of products. Fig. 3.49 shows essential and nonessential prime implicants.

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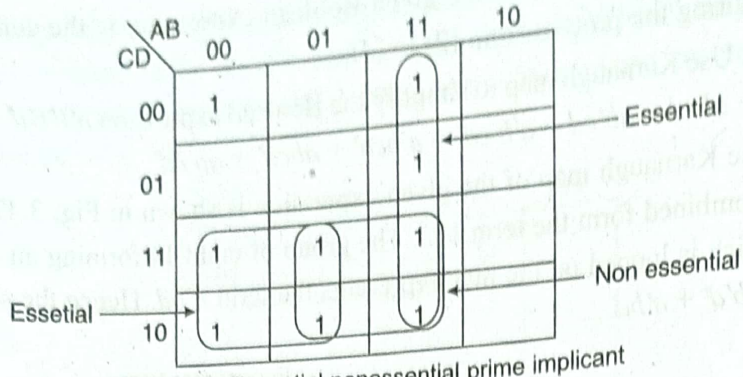

Fig. 3.49. Essential nonessential prime implicant

Labelling of Karnaugh-map Squares

The sequence of a Karnaugh map can be numbered and these numbers are equal to the binary equivalent of logic values of the corresponding minterm variables. The numbering of 4-variable Karnaugh map is shown in Fig. 3.50. and Fig. 3.51. The number in third column and second row is 13, which corresponds to minterm ABCD whose logic value 1101 equals to 13 in decimal system.

It is to be noted that if AB and CD are interchanged, the numbers in square will also change. Three-variable Karnaugh maps can also be numbered as shown in Fig. 3.52.

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

Fig. 3.50. Labelling of four-variable Karnaugh map squares

AB \ CD	00	01	11	10
00	0	1	3	2
01	4	5	7	6
11	12	13	15	14
10	8	6	11	10

Fig. 3.51. Labelling of 4-variable Karnaugh map squares

C \ AB	00	01	11	10
0	0	2	6	4
1	1	3	7	5

(a)

AB \ C	0	1
00	0	1
01	2	3
11	6	7
10	4	5

(b)

A \ BC	00	01	11	10
0	0	1	3	2
1	4	5	7	6

(c)

Fig. 3.52. Numbering of squares of 3-variable Karnaugh map

Alternate way of Representation: The alternate way of representing the sum of products expression is explained with the help of an example.

Consider the Boolean function

$$X = A'B'C' + ABC' + A'BC$$

This function can be represented as

$$X(A, B, C) = \Sigma(0, 6, 3) \text{ or } \Sigma m(0, 6, 3)$$

where m stands for minterms.

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This means the minterms corresponding to squares 0, 6 and 3 of the Karnaugh map are present in the Boolean function and marked by 1. Other squares are marked by 0. This is shown in Fig. 3.53.

C \ AB	00	01	11	10
0	1	0	1	0
1	0	1	0	0

Fig. 3.53. Karnaugh map of X (A, B, C) = Σ (0, 6, 3)

Example 32. Simplify the Boolean function

F (A, B, C, D) = Σ (0, 1, 2, 3, 4, 5, 7, 6, 8, 9, 11).

Solution. Figure 3.54. shows the Karnaugh map of the given Boolean function. The looping of one octet and two quads are shown in the map. The looping octet will give the minterm as A′ The looping quad formed by 0, 1, 8, 9 produces the minterm as B′C′ and the quad formed by 1, 3, 9, 11 gives the minterm as B′D. Hence, the simplified function is F = A′ + B′C′ + B′D.

CD \ AB	00	01	11	10
00	1	1		1
01	1	1		1
11	1	1		1
10	1	1		

Fig. 3.54. Karnaugh map of F(A, B, C, D) = Σ (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11)

3.16. Arithmetic Circuits

A computer's circuits can respond only to binary numbers. Arithmetic operations, such as addition, subtraction, multiplication, division, etc., are performed in the binary form in a digital computer. Logic circuits of some basic arithmetic operations are discussed in the following sections.

Half Adder

A half adder is a logic circuit that adds 2 bits. Table 3.21 shows the addition of two bits. Columns 1 and 2 of Table 3.12 give the values of two input bits, column 3 gives the sum of these two bits, and column 4 the carry bit. This table is called a *half adder truth table*. Even though the inputs and outputs are binary numbers, they may be taken as depicting truth values of 0 and 1 for developing the boolean expressions for C and S. By inspection of Table 3.20.

$$S = A'.B + A.B' = A \oplus B$$

$$C = A \cdot B$$

Input		*Output*	
A	*B*	*S* (*Sum*)	*C* (*Carry*)
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Table 3.20. A half adder truth table

A logic circuit which uses logic gates to implement the half adder is shown in Fig. 3.55.

It can also be seen from Table 3.21 that the sum of two binary digits can be represented by the output of an XOR gate, and the carry output can be represented by the output of an AND gate, *i.e.*, if same two inputs are applied to XOR and AND gates, the output of XOR gate will represent the sum and the AND gate will represent the carry.

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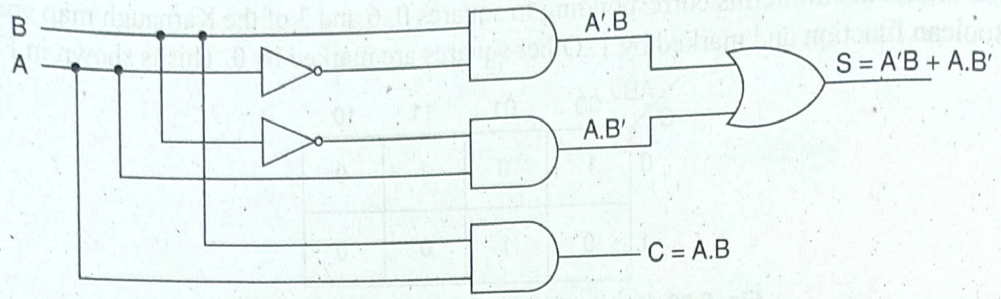

Fig. 3.55. A logic circuit realising half adder

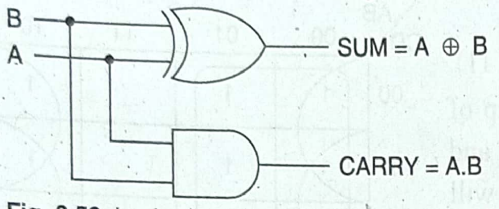

Fig. 3.56. Logic circuit for half adder using XOR gate

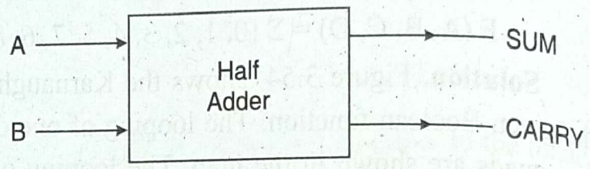

Fig. 3.57. Block diagram of half adder

Full Adder

When adding two binary numbers, we may have a carry from one column to the next. The carry coming out from one column is to be added to the next column. A half adder cannot add 3 bits as it has only 2 input terminals.

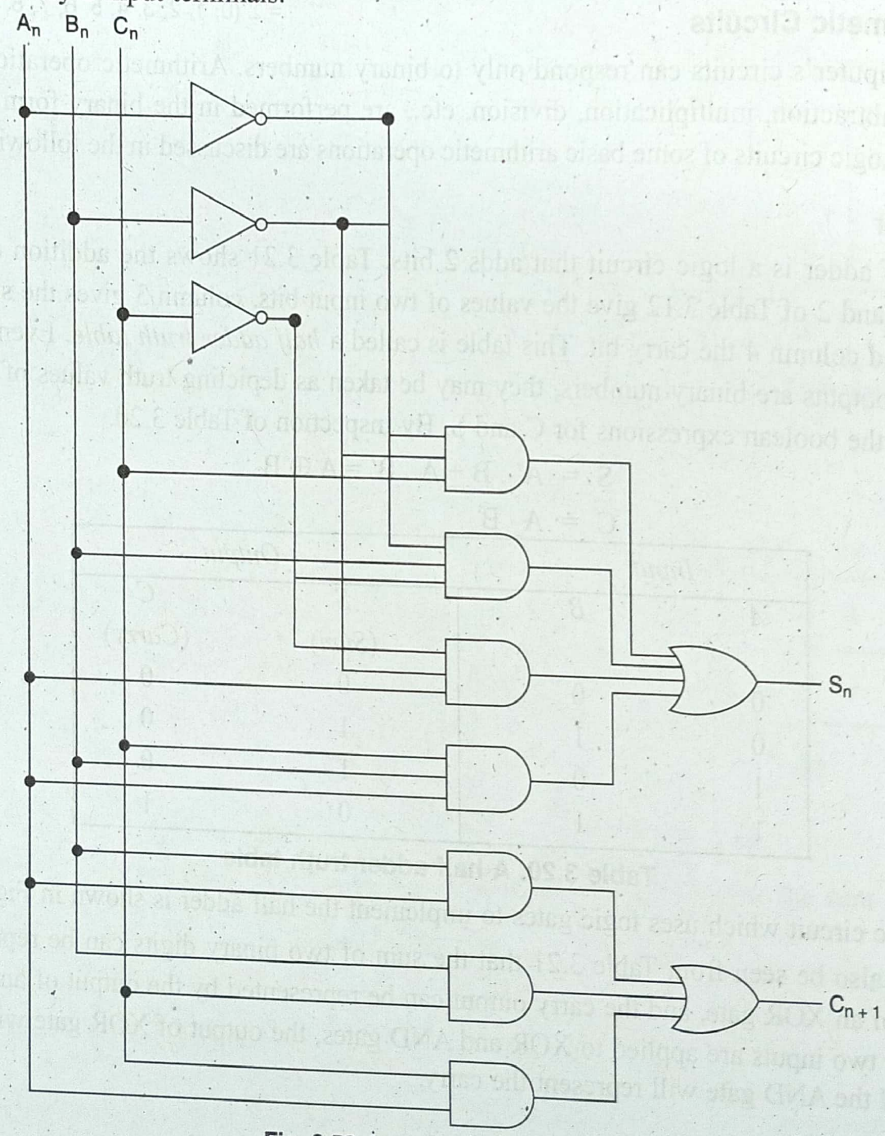

Fig. 3.58. Logic circuit for full adder

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A full adder is a logic circuit that can add 3 bits at a time. Again, here are two outputs: sum and carry. Table 3.21. gives the full adder truth table.

Input			Output	
A_n	B_n	C_n (Carry in)	S_n (Sum)	C_{n+1} (Carry Out)
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Table 3.21. Full Adder Truth Table

The Boolean expression for S_n and C_{n+1} are obtained by looking at the 1s in the S_n and C_{n+1} columns, respectively and writing the Boolean terms corresponding to their SUM is 1 when the number of input is odd, CARRY is 1 when two or more inputs are 1s.

$$S_n = A_n . B_n . C_n + A_n . B_n . C_n + A_n . B_n . C_n + A_n . B_n . C_n$$

$$= \bar{A}_n (\bar{B}_n C_n + B_n \bar{C}_n) + A_n (\bar{B}_n \bar{C}_n + B_n C_n)$$

$$= A_n (B_n \oplus C_n) + A_n (B_n \oplus C_n)$$

$$= A_n \oplus B_n \oplus C_n$$

$$C_{n+1} = \bar{A}_n . B_n . C_n + A_n . \bar{B}_n . C_n + A_n B_n . \bar{C}_n + A_n . B_n . C_n$$

$$= \bar{A}_n . B_n . C_n + A_n . B_n . C_n + A_n . \bar{B}_n . C_n + A_n . B_n . C_n + A_n . B_n . \bar{C}_n + A_n . B_n . C_n$$

$$= B_n . C_n (A_n + \bar{A}_n) + A_n . C_n (\bar{B}_n + B_n) + A_n . B_n (C_n + \bar{C}_n)$$

$$= B_n . C_n + A_n . C_n + A_n . B_n \qquad [\text{Using } a + a' = 1]$$

The logic circuit for the full adder is shown in Fig. 3.58. Also, a full adder using 3 input XOR gate and a block diagram of a full adder are shown in Figs. 3.59 and 3.60 respectively.

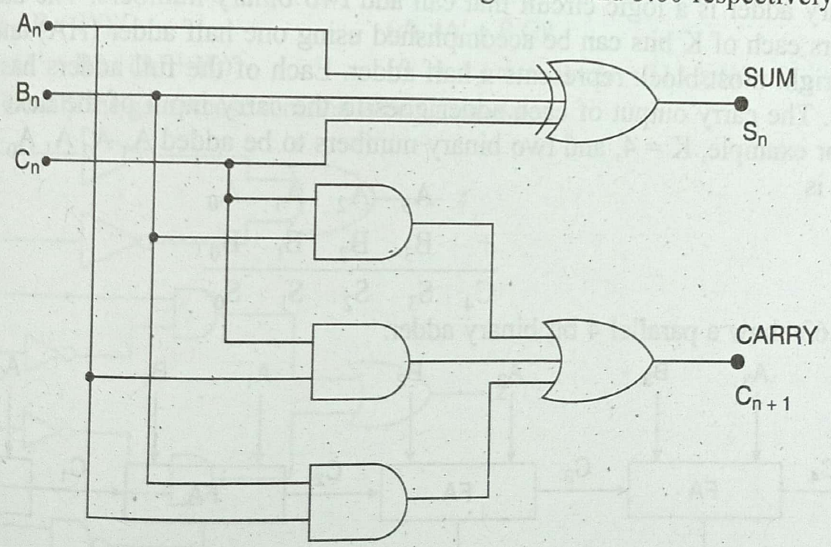

Fig. 3.59. A full adder using 3 input XOR gate

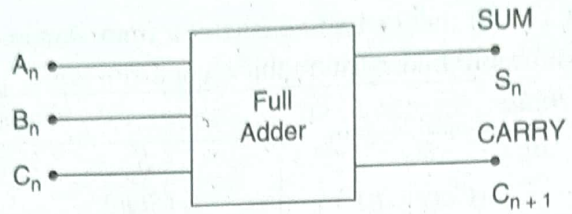

Fig. 3.60. Block diagram of a full adder

Circuit Diagram of a Full Adder using Two Half Adders and OR Gate.

By connecting two half adders and an OR gate, we get a full adder, Fig. 3.61 shows a full adder using two half adders.

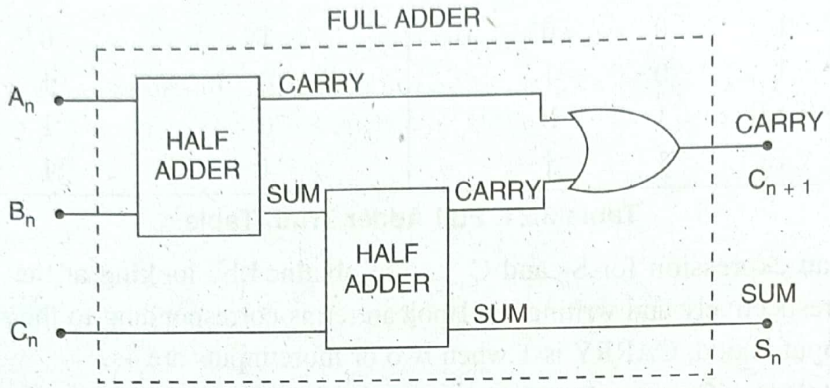

Fig. 3.61. A full adder realized from two half adders

For instance, suppose A = 1, B = 1, and C = 0. Fig. 3.62 (*a*) shows the full adder with these inputs. The first half adder (HA) has a sum of 0 with a carry of 1. The second half adder has a sum of 0 with a carry of 0. Therefore, the final output is a sum of 0 with a carry of 1.

If the inputs are A = 1, B = 1, and C = 1, Fig. 3.62 (*b*) shows the full adder with these inputs. The final output is a sum of 1 with a carry of 1.

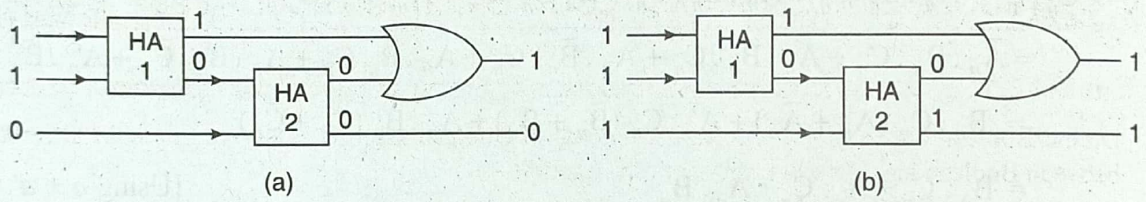

Fig. 3.62. Examples of working a full adder

Binary Adder

A binary adder is a logic circuit that can add two binary numbers. The addition of two binary numbers each of K bits can be accomplished using one half adder (HA) and K-1 full adders (FAs). The right most block represents a half adder. Each of the full adders has three inputs and two outputs. The carry output of each adder goes to the carry input of the next adder to the left. Consider, for example, K = 4, and two binary numbers to be added $A_3\ A_2\ A_1\ A_0$ and $B_3\ B_2\ B_1\ B_0$. The answer is

$$\begin{array}{rccccc} & & A_3 & A_2 & A_1 & A_0 \\ + & & B_3 & B_2 & B_1 & B_0 \\ \hline & C_4 & S_3 & S_2 & S_1 & S_0 \end{array}$$

Fig. 3.63 show a parallel 4 bit binary adder.

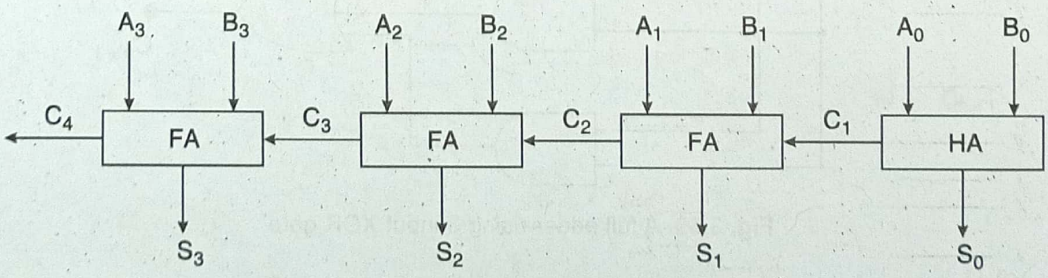

Fig. 3.63. Parallel 4 bit binary adder

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Suppose A = 1100 and B = 1011. The half adder produces a sum of 1 and carry of 0, then first full adder produces a sum of 1 and a carry of 0, the second full adder produces a sum of 1 and a carry of 0, and the third full adder produces a sum of 0 and a carry of 1. The overall output is 10111.

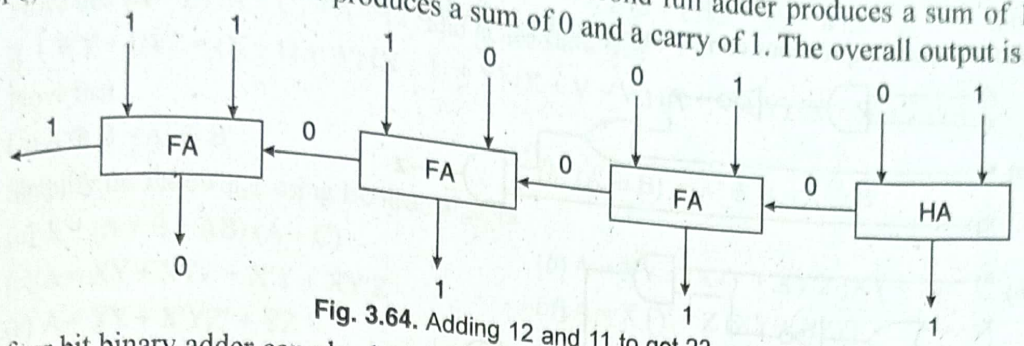


Fig. 3.64. Adding 12 and 11 to get 23

A four bit binary adder can also be built by combining four full adders (Fig. 3.65) the first full adder acts as a half adder as its carry input is held at logic 0 level.

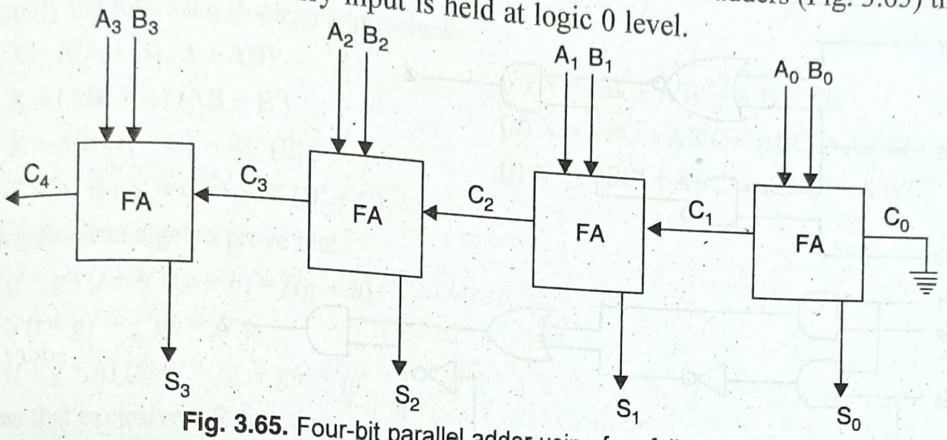


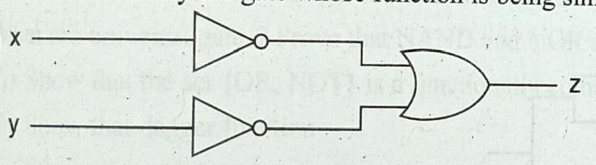
Fig. 3.65. Four-bit parallel adder using four full adders

BOOLEAN ALGEBRA AND LOGIC CIRCUITS PROBLEM SET

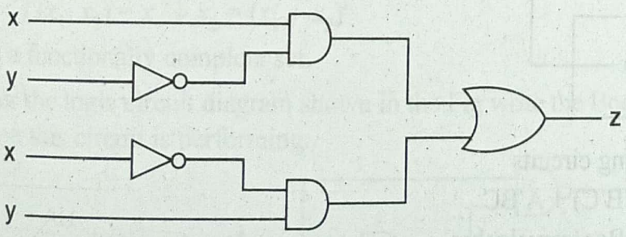
Problem Set 3.1

1. Define Boolean algebra. Write down the axioms of Boolean algebra. What are the main differences between Boolean algebra and algebra of real numbers.
2. State and prove De Morgan's theorem in both form.
3. Complement the following expressions by applying De-Morgan's theorem.
   (*a*) $(A + A' B)'$ (*b*) $((A + B') C)'$
   (*c*) $((A' + C) . (B + D'))'$ (*d*) $(A' + B'C)'$
   (*e*) $((A + B) . (A' + B') . (A' + B))'$ (*f*) $w + (ab + c') (d'e + 1) + g (h' + 0)$
4. In each case identify the gate whose function is being simulated

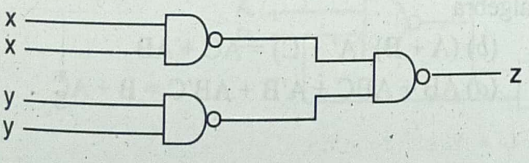
(*a*)



(*b*)



(*c*)



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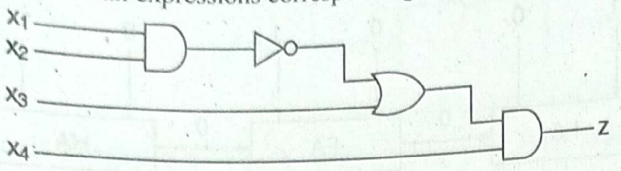
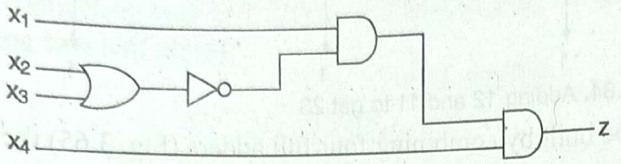
5. Use NAND gates to construct circuits with following outputs
   (*a*) x' (*ii*) $x + y$ (*iii*) xy (*iv*) $x \oplus y$
6. Write Boolean expressions corresponding to the following logic circuits

(*a*)


(*b*)


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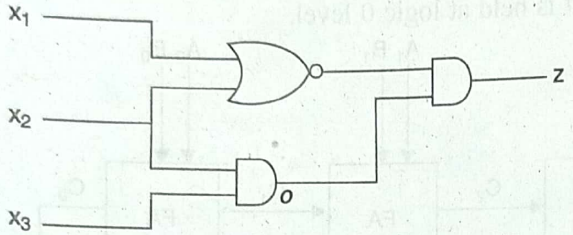
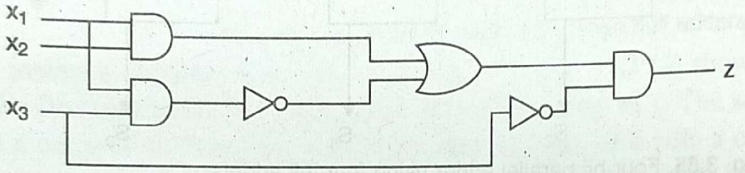
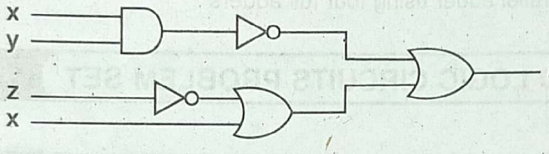
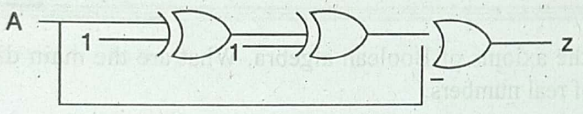
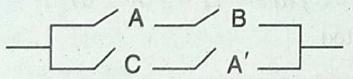


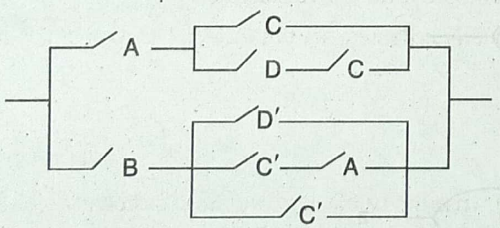
(*d*)


(*e*)


(*f*)


7. Draw logic diagram to represent the Boolean expressions
   (*a*) $(x_1' x_2)' + x_1 x_2$ (*b*) $(xy)' + xyz$
   (*c*) $(x_1 + x_2) x_3 + x_4$
8. Write each switching circuit symbolically

(*a*)


(*b*)


9. Represent the expressions as switching circuits
   (*a*) $(A + B')A$ (*b*) $A(BC' + B'C) + A'BC$
10. Prove the following identities using Boolean algebra
   (*a*) $C + BC' = B + C$ (*b*) $(A + B)(A' + C) = AC + AB$
   (*c*) $(A' + B)(A + B) = B$ (*d*) $AB + ABC + A'B + AB'C = B + AC$

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(*e*) $A \oplus B \oplus AB = A + B$

(*f*) $(c' + abd + b'd + a'b)(c + ab + bd)$
$= b(a + c)(a' + c') + d(b + c)$

11. Prove that $(A + B)(A + C) = A + BC$ and hence show that
$X + WY + UVZ = (X + U + W)(X + U + Y)(X + V + W)(X + V + Y)(X + W + Z)(X + Y + Z)$

12. Prove that
(*a*) $A \oplus B = A' \oplus B'$
(*b*) $(A \oplus B)' = A' \oplus B = A \oplus B'$

13. Simplify the followings using Boolean algebra
(*a*) $X = (A + B + AB)(A + C)$
(*b*) $A = XY + (XZ)' + XY'Z(XY + Z)$
(*c*) $A = XY + XYZ + X'Y + XY'Z$
(*d*) $A = X[Y + Z(XY + XZ)']$
(*e*) $A = XY + X'YZ' + YZ$
(*f*) $A = (XY' + Z)(X + Y')Z$
(*g*) $A = XY' + Z + (X' + Y)Z'$

14. Simplify the following Boolean expressions:
(*a*) $X = A(A + B)(A + AB)$
(*b*) $X = AB' + A'B' + A'B + AB$
(*c*) $X = (AB' + B)(AB + B')$
(*d*) $X = ABC + AB'C + ABC' + AB'C'$
(*e*) $X = A'B(B + C) + BC(B' + A')$
(*f*) $X = A'BC' + ABC' + AB'C' + A'B'C'$
(*g*) $X = A'B(A' + C) + AB'(B' + C)$

15. Using Boolean algebra prove that
(*a*) $(f + g')(f + h')(g + h) = f(g + h)$
(*b*) $h(f + g)' + f'gh = f'h$
(*c*) $(f + g + h)(fgh)' = fg' + gh' + hf'$.

16. Show that exclesive-OR function
$X = A \oplus B \oplus C \oplus D$
is an odd function.

17. Using truth table verify De-Morgan's laws.

18. Using truth table, prove the identity
(*a*) $x \oplus y = (x + y)(xy)'$
(*b*) $x \oplus y = xy' + y'x$.

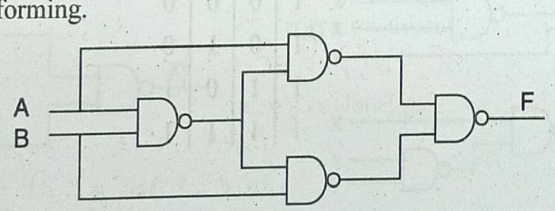
19. Find the dual form of each of the followings:
(*a*) $1 + xy$
(*b*) $0 + (x_1' + x_2 . x_3)$
(*c*) $x(y + 0)$
(*d*) $x(y + y'z) + y'z = xy + yz$
(*e*) $(x + y)(x + 1) = x + xy + y$
(*f*) $xy' = 0$ if and only if $xy = x$

20. How can you connect NAND get to get an OR gate?

21. What are universal gates? Prove that NAND and NOR are universal gates.

22. (*a*) Show that the set {OR, NOT} is a functionally complete set but the set {AND, OR} is not.
(*b*) Show that dagger function
$f(x_1, x_2) = x_1 \downarrow x_2 = (x_1 + x_2)'$
is a functionally complete set.

23. For the logic circuit diagram shown in the Fig write the Boolean expression. Indicate which logic operation this circuit is performing.



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24. Represent each of the following Boolean function in disjunctive normal form.

(i) $f(x_1, x_2) = (x_1 + x_2)(x_1' + x_2')$

(ii) $f(x_1, x_2) = x_1 \oplus x_2$

(iii) $f(x, y) = x + xy$

(iv) $f(x, y, z) = (yz + xz')(xy' + z)'$

(v) $f(x_1, x_2, x_3) = (x_1' x_2)' \cdot (x_1 + x_3)$

(vi) $f(x, y, z) = (x + x'y + x'yz')(xy + (xz)')(y + xyz')$

(vii) $f(x_1, x_2, x_3) = x_1 x_2' + x_3$

25. Find the disjunctive normal forms of the functions whose truth table are given

(a)

x	y	$f(x,y)$
1	1	1
1	0	0
0	1	0
0	0	1

(b)

x	y	z	$f(x,y,z)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	0

(c)

x	y	z	$f(x,y)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

26. Find the product of sum expansion of each of the following

(a) $f(x, y, z) = (x + z)\, y$

(b) $f(x, y, z) = x$

(c) $f(x, y, z) = xy'$

Problem Set 3.2

1. Plot the given truth table on the map. Then loop two pairs of 1's on the map and write the simplified form of the corresponding Boolean expression.

A	B	F
0	0	1
0	1	1
1	0	0
1	1	1

B \ A	0	1
0		
1		

2. From the truth table from three variables given, write the corresponding Boolean expression and draw the Karnaugh map

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	1

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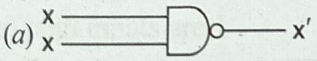
3. For each of the following Boolean expressions. find the minimum sum of products using Karnaugh map
   (*i*) $A'B'C' + AB'C' + A'BC' + ABC'$
   (*ii*) $AB + AB'C + ABC$
   (*iii*) $AB' + AB'C'D + CD + BC'D + ABCD$
4. Draw Karnaugh map and simplify the following Boolean expressions:
   (*a*) $AB + A\bar{B} + \bar{A}\bar{B}$
   (*b*) $ABC + \bar{A}B\bar{C} + AB\bar{C} + \bar{A}BC$
   (*c*) $\bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C} + AB\bar{C} + \bar{A}B\bar{C}$
   (*d*) $A\bar{B}\bar{C} + AB\bar{C} + ABC + A\bar{B}C + A\bar{B}\bar{C}$
   (*e*) $AB\bar{C}\bar{D} + A\bar{B}\bar{C}\bar{D} + AB\bar{C}\bar{D} + A\bar{B}\bar{C}D$
   (*f*) $\bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}B\bar{C}D + \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + \bar{A}BCD$
5. For a 4 variable Karnaugh map, which squares are adjacent to square 8 and square 14?
6. Draw Karnaugh map and simplify the following Boolean expressions:
   (*a*) $F(A, B, C) = \Sigma(2, 3, 6, 7)$
   (*b*) $F(A, B, C) = \Sigma(0, 1, 3, 4, 5)$
   (*c*) $F(A, B, C, D) = \Sigma(7, 13, 14, 15)$
   (*d*) $F(A, B, C, D) = \Sigma(0, 2, 6, 8, 10, 12, 14, 15)$
7. Obtain the simplified expressions in product of sums form:
   (*a*) $F(A, B, C) = \Pi(0, 1, 4, 5)$
   (*b*) $F(A, B, C, D) = \Pi(1, 3, 5, 7, 13, 15)$

Problem Set 3.3

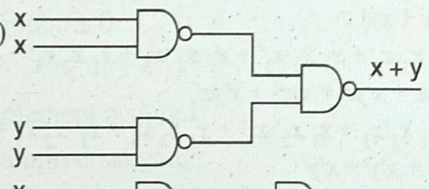
1. What is a half adder? Write truth table for a half adder and develop its logic circuit.
2. What is a full adder? How is a full adder built using half adders?
3. Determine the output of a full adder for inputs A = 1, B = 1 and (*i*) $C_{in} = 0$ (*ii*) $C_{in} = 1$.
4. Develop the expressions of the sum and carry for adding two bit numbers.
5. Give the schematic diagram of a binary 2's complement adder/subtractor. Breifly explain the operation of the circuit.
6. What do you mean by block diagram? Draw the block diagrams of serial adder and illustrate the function with two 15-bit operands.
7. How many half adders and full adders do you need to make parallel binary adder to add decimal numbers 11 and 7?
8. Determine the output of a half adder for inputs
   (*a*) A = 1, B = 0
   (*b*) A = 1, B = 1
9. Can you convert a full adder into a full subtractors? If yes, how?

ANSWERS 3.1

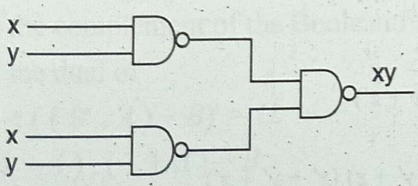
3. (*a*) $A'B'$
   (*b*) $A'B + C$
   (*c*) $AC' + B'D$
   (*d*) $A(B + C')$
   (*e*) $A' . B' + AB + AB'$
   (*f*) $[(a' + b')c + (d + e')0][g' + h.1][w']$.
4. (*a*) NAND
   (*b*) XOR
   (*c*) OR
5. (*a*)



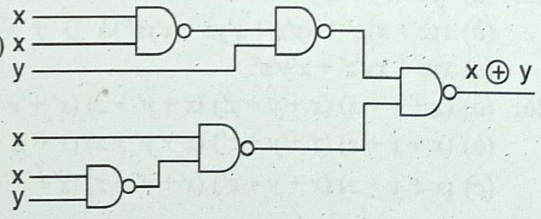
(*b*)



(*c*)



(*d*)



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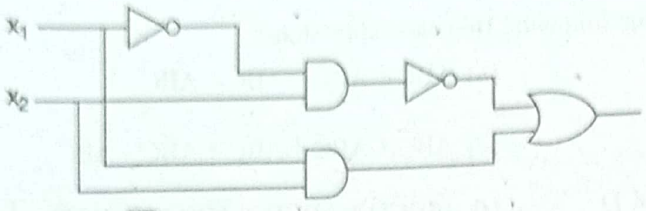
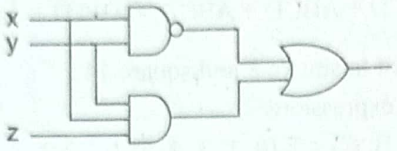
6. (*a*) $z = ((x_1 x_2)' + x_3)x_4$ (*b*) $z = ((x_2 + x_3)' x_1)x_4$
(*c*) $z = (x_1 + x_2)' (x_2 x_3)'$ (*d*) $z = ((x_1 x_2) + (x_2 x_3)') x_3'$
(*e*) $t = (xy)' + (z' + x)$ (*f*) $z = 0$

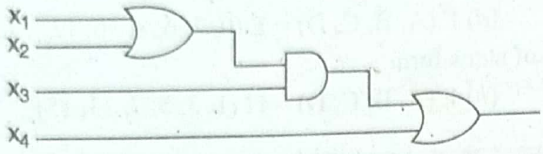
7. (*a*)



(*b*)

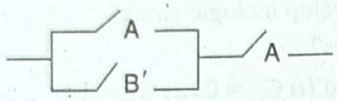


(*c*)

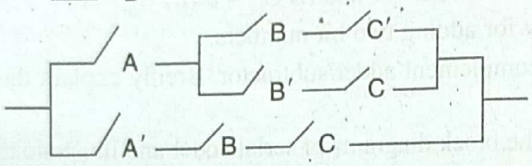


8. (*a*) AB + CA′ (*b*) (A (C + (DC))) + (B (D′ + (CA) + C′))

9. (*a*)



(*b*)

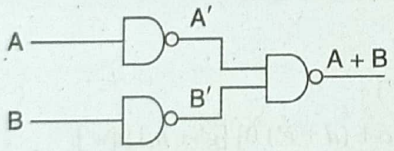


13. (*a*) X = A + BC (*b*) 1 (*c*) Y + XZ (*d*) XY
(*e*) Y (*f*) XZ + Y′Z (*g*) 1

14. (*a*) X = A (*b*) X = 1 (*c*) X = A (*d*) X = A
(*e*) X = A′B (*f*) X = C′. (*f*) X = A′ B + B′A

19. (*a*) $0\,(x + y)$ (*b*) $1(x_1' (x_2 + x_3))$ (*c*) $x + (y.1)$
(*d*) $(x + (y (y' + z))) (y' + z) = (x + y) (y' + z)$
(*e*) $xy + x.0 = x (x + y)y$ (*f*) $x + y' = 1$ if and only if $x + y = x$.

20.



23. [(A (AB′)′ . (B(A . B)′)′]′ = AB′ + A′B (XOR gate)

24. (*i*) $x_1' x_2 + x_2' x_1$ (*ii*) $x_1 x_2' + x_1' x_2$
(*iii*) $xy + xy'$ (*iv*) xyz'
(*v*) $x_1 x_2 x_3 + x_1 x_2 x_3' + x_1 x_2' x_3 + x_1 x_2' x_3' + x_1' x_2' x_3$.
(*vi*) $xyz + x'yz + xyz' + x'yz'$
(*vii*) $x_1 x_2' x_3 + x_1 x_2' x_3' + x_1 x_2 x_3 + x_1' x_2 x_3 + x_1' x_2 x_3'$

25. (*a*) $xy + x'y + x'y'$
(*b*) $xyz + xyz' + xy'z + x'yz + x'yz'$
(*c*) $xyz + xy'z' + x'y'z'$.

26. (*a*) $(x + y + z) (x + y + z') (x + y' + z) (x' + y + z) (x' + y + z')$
(*b*) $(x + y + z) (x + y + z') (x + y' + z) (x + y' + z')$
(*c*) $(x + y + z) (x + y + z') (x + y' + z) (x + y' + z') (x' + y' + z) (x' + y' + z')$

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ANSWERS 3.2

1.

B \ A	0	1
0	1	0
1	1	1

F = A + B

2.

C \ AB	A′B′	A′B	AB	AB′
C	0	0	1	0
C′	0	0	1	0

F = ABC′ + ABC

3. (*i*) F = C′ (*ii*) F = AB + AC (*iii*) F = AB′ + BD + CD.

4. (*a*) A + B′ (*b*) B (*c*) BC′ + A′B′ (*d*) A + B′C
(*e*) AC′ (*f*) A′B′ + A′CD

5. 0, 12, 9, 10 ; 6, 10, 12, 15

6. (*a*) B (*b*) B′ + A′C (*c*) ABD + ABC + BCD (*d*) D′ + ABC

7. (*a*) B (*b*) BC′ + B′D + AB′C.

ANSWERS 3.3

3. Sum = 0, carry = 1
Sum = 1 and carry = 1.

7. 1 half adder and 3 full adder

8. (*a*) Sum = 1, carry = 0 (*b*) Sum = 0, Carry = 1

9. The Bollean expression for the Sum in a full adder and the Difference in a full subtractor are the same. The difference is only between Carry and Borrow. Hence a full adder can be converted into a full subtractor by complementing the input A for Borrow in a substractor.

SHORT/MULTIPLE CHOICE QUESTIONS

1. The output of a 2 input OR gate is 0 only when its
(*a*) both inputs are 0 (*b*) either input is 1
(*c*) both inputs are 1 (*d*) either input is 0

2. The output of a 2 input AND gate is 1 only when its
(*a*) both inputs are 0 (*b*) either input is 0
(*c*) both inputs are 1 (*d*) none of the above

3. A logic gate is an electronic circuit which
(*a*) makes logic decisions (*b*) allows election flow only in one direction
(*c*) works on binary algebra (*d*) alternates between 0 and 1 values

4. When an input electrical signal A = 10100 is applied to a NOT gate, its output signal is
(*a*) both inputs are 0 (*b*) either input is 1
(*c*) both inputs are 1 (*d*) either input is 0

5. The only function of a NOT gate is to
(*a*) stop a signal (*b*) recomplement a signal
(*c*) invert an output signal (*d*) act as a universal gate

6. Find the complement of the Boolean function $AB' + C'D'$

7. Find the dual of
(*i*) $A + ((B' . A) + B)' = U$
(*ii*) $(1 + A) . (1 + B) = B$
(*iii*) If $A + B = 0$ then $A = 0 + B$.

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8. Find the value of the following Boolean expressions

$(A, B' + C) . A'$ if

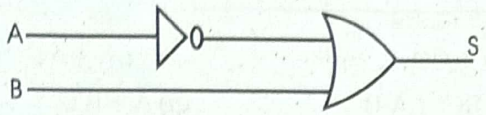
(*i*) $A = 1$, $B = 0$, $C = 0$

(*ii*) $A = 0$, $B = 1$, $C = 1$

(*iii*) $A = 0$, $B = 1$, $C = 1$

9. Find the output from the circuits

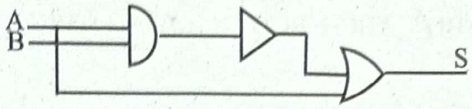
(*i*)



(*a*) $A = 0$, $B = 1$

(*b*) $A = 1$, $B = 1$

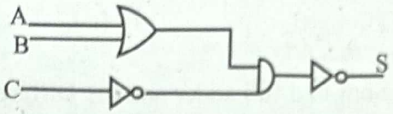
(*ii*)



(*a*) $A = 1$, $B = 0$

(*b*) $A = 1$, $B = 1$

(*iii*)



(*a*) $A = 1$, $B = 0$, $C = 1$

(*b*) $A = 0$, $B = 1$, $C = 0$

10. What logic function is produced by adding an inverter to the output of an AND gate?.

(*a*) NOR (*b*) NAND (*c*) XOR (*d*) OR

11. The operation which is commutative but not associative.

(*a*) AND (*b*) EX_NOR (*c*) OR (*d*) NAND

12. What is the number of binary digits which can be added by a full adder ?

(*a*) 2 (*b*) 3 (*c*) 4 (*d*) 8

13. What is the name of the logic circuit which can add two binary digits ?

(*a*) full adder (*b*) half adder (*c*) parallel adder (*d*) flat register

14. The number of two -input NAND gate required to produce the two-input OR function is

(*a*) 1 (*b*) 2 (*c*) 3 (*d*) 4

15. State True or False of the following statements

(*i*) A pair on a Karnaugh map can eliminte one variable.

(*ii*) A quad on a Karnaugh map can eliminate two variables.

(*iii*) A octet on a Karnaugh map can eliminate four variables.

(*iv*) Half adder can add binary numbers with more than one digit.

(*v*) The opration OR and AND are distribute.

(*vi*) Boolean algebra is a two state algebra.

16. $a. [a + b . (a + b)]$ is equal to

(*a*) a (*b*) c (*c*) $a + b$ (*d*) $c . b$

17. $a + b + \overline{(\bar{a} + b + c)}$ is equal to

(*a*) $\bar{a} + b$ (*b*) $a + b$ (*c*) $a + \bar{b}$ (*d*) $b + c$

18. Find the output if the bits 0,1 and 1 are inputs in each gate.

(*i*) AND gate (*ii*) OR gate

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19. Compute the NOR gate output by the input of the pairs of bits
(*a*) 0, 0 (*b*) 1, 0

20. Simplify the following Boolean expressions
(*i*) $y(x + y')$ (*ii*) $(x + y)(x + y')$

21. In D_{70}, lcm (2′, 14) and gcd (2, 14′) are
(*a*) 14, 2 (*b*) 70, 1 (*c*) 2, 14 (*d*) 7, 28

22. The minterms adjacent ot $xy'z'$ in the Karnaugh map are
(*a*) $xy'z, x'y'z'$ (*b*) $x'y'z', xyz'$ (*c*) $xy'z, xyz'$ (*d*) $xy'z, x'y'z', xyz'$

23. The Boolean expression corresponding is

$x \backslash y$	0	1
0	1	1
1	0	1

(*a*) $x + y$ (*b*) $x \cdot y$
(*c*) $x + y'$ (*d*) $x \cdot y'$

24. The sum of products form of $ab + bc'$ is
(*a*) $abc + abc' + a'bc'$ (*b*) $abc + abc' + a'b'c'$
(*c*) $abc' + a'bc' + a'b'c'$ (*d*) $abc' + a'bc' + a'b'c$

ANSWERS

1. (*a*) 2. (*c*) 3. (*a*) 4. (*a*) 5. (*c*)
6. $(A + B')$, $(C + D)$
7. (*i*) $A.((B'.A) + B)' = \phi$
(*ii*) $(0.A) + (1.B) = B$ (*iii*) If $A.B = 1$, then $A = 1 = B$
8. (*i*) 0 (*ii*) 1
9. (*i*) (*a*) 1 (*b*) 1 (*ii*) (*a*) 1 (*b*) 1 (*iii*) (*a*) 1 (*b*) 0
10. (*b*)
11. (*d*)
12. (*b*)
13. (*b*)
14. (*c*)
15. (*i*) *T* (*ii*) *T* (*iii*) *F* (*iv*) *F* (*v*) *T* (*vi*) *T*
16. (*a*) 17. (*b*) 18. (*i*) 0 (*ii*) 1 19. (*a*) 1 (*b*) 0 20. (*i*) xy (*ii*) x
21. (*b*) 22. (*d*) 23. (*c*) 24. (*a*)

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