G is k-regular, we have

$$|E_1| = k|S| \quad ...(1)$$

Let E_2 denote the set of edges incident with vertices in $N(S)$. Then, since $N(S)$ is the set of vertices which are joined by edges to S, we have $E_1 \subset E_2$. Thus

$$|E_1| \leq |E_2|. \quad ...(2)$$

Again by the k-regularity of G we have

$$|E_2| = k|N(S)|. \quad ...(3)$$

From (1), (2) and (3), we get $k|N(S)| = |E_2| \geq |E_1| = k|S|$ and so

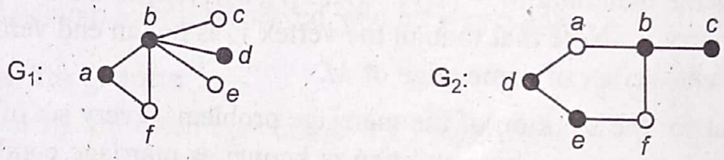
$$k|N(S)| \geq k|S|$$

Since $k > 0$ $|N(S)| \geq |S|$. Since S was an arbitrary subset of X, it follows from Hall's marriage theorem that G contains a matching M which saturates every vertex in X. Since $|X| = |Y|$ the edges in the matching M also saturate every vertex in Y. Thus M is a perfec[t] matching in G.

15.7. Graph Covering

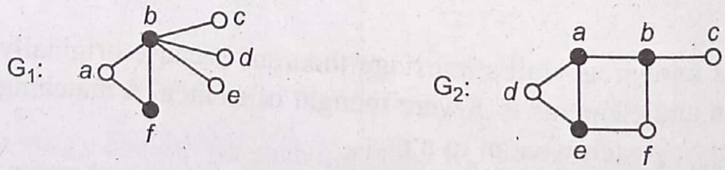
An edge (u, v) of a graph G covers the vertices u and v, and a vertex of G is said to cove[r] the edges with which it is incident. Similarly, a vertex G is said to cover the edges with which it i[s] incident. Covering with vertices and edges are closely related with independent set and matching

Vertex Covering: A vertex covering of a graph $G(V, E)$ is set of vertices $C \subseteq V$ such tha[t] each edge of G is incident to at least one vertex in C. The set C is said to cover the edges of G. The following figure shows examples of vertex covering in two graphs (the set C is marked dark



Here $C_1 = \{a; b, d\}$ in G_1 and $C_2 = \{b, c, d, e\}$ A vertex covering $c \subseteq v$ of a graph G is sai[d] to be minimal covering if no vertex can be removed without destroying its ability to cover G.

A graph may have many minimal coverings of different sizes. A **minimum vertex coverin[g]** is a minimal vertex covering of smallest possible size. The **vertex covering number** T is the siz[e] of a minimum vertex covering. The following figure shows examples of minimum vertex coverin[g] in the previous graphs



Note: (*i*) The set of all vertices for any graph G is trivially a vertex covering of G.

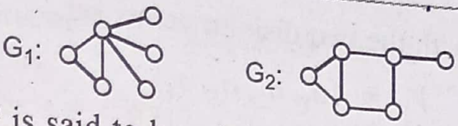
(*ii*) The end points of any maximal matching form a vertex covering.

(*iii*) The complete bipartite graph $K_{m,n}$ has a minimum covering of size $T(K_{m,n}) = \min(m,$

(*vi*) For any graph $G(V, E)$, $T(G)$ + maximum independent set = number of vertices in V.

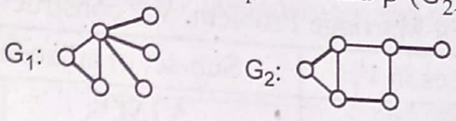
Edge Covering: An edge covering of a graph $G(V, E)$ is a set of edges $L \subseteq E$ such that ea[ch] vertex in G is incident with at least one edge in L. The set L is said to *cover* the vertices of G. T[he] following figure shows examples of edge coverings in two graphs.

Theorem 15.23. If G is a bipartite graph, then the maximum size of a matching in G is equ[al] to the minimum size of a vertex covering of G.



An edge covering L in G is said to be minimal covering if no edge of L can be removed without destroying its ability to cover the graph.

An **minimum edge covering** is a minimal edge covering of smallest possible size. The **edge covering number** $\rho(G)$ is the size of a minimum edge covering. The following figure shows examples of minimum edge coverings. Here $\rho(G_1) = 4$ and $\rho(G_2) = 3$.



Note that the figure on the right is not only an edge cover but also a matching. In particular, it is a perfect matching: a matching M in which each vertex is incident with exactly one edge in M. A perfect matching is always a minimum edge covering.

Note: (*i*) The set of all edges is an edge cover, assuming that there are no degree-0 vertices.

(*ii*) The complete bipartite graph $K_{m,n}$ has edge covering number $L = \max(m, n)$.

(*iii*) Every edge covering of G includes all the pendant edges of the graph.

Theorem 15.23 Let a graph have a matching M and covering C. Then $|M| \leq |C|$. Moreover, if $|M| = |C|$, then M is a maximum matching and C is a minimum covering.

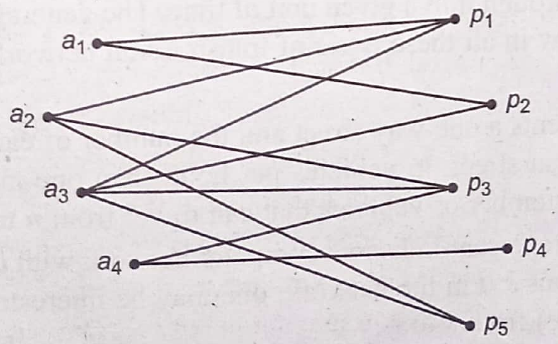
Proof. From the definition of a covering, every edge of the graph, and in particular every edge in M, is incident with some vertex in C. If the edge e is in M, let $v(e)$ be a vertex in C incident with e. Now if e_1 and e_2 are distinct edges in M, then $v(e_1)$ and $v(e_2)$ are also distinct, since, by definition, two edges in a matching cannot share a vertex. Thus there are at least as many vertices in C as edges in M, and so $|M| \leq |C|$.

Now suppose $|M| = |C|$. If M were not a maximum matching, there would be a matching M' with $|M'| > |M| = |C|$, contradicting the first part of the theorem. Similarly, if C were not a minimum covering, these would be a covering with fewer than $|M|$ vertices which leads to the same contradiction.

Example 27. Applicants a_1, a_2, a_3 and a_4 apply for five posts p_1, p_2, p_3, p_4 and p_5. The application are done as follows: $a_1 \to \{p_1, p_2\}$, $a_2 \to \{p_1, p_3, p_5\}$, $a_3 \to \{p_1, p_2, p_3, p_5\}$ and $a_4 \to \{p_3, p_4\}$. Using graph theory find (*i*) whether there is any perfect matching of the set of applicants into the set of posts.

Find matching (*ii*) whether every applicant can be offered a single post.

Solution. The given correspondence can be represented in the following graphical form:



This is a bipartite graph with the two disjoint vertex set

$$V_1 = \{a_1, a_2, a_3, a_4\}$$

and $$V_2 = \{p_1, p_2, p_3, p_4, p_5\}.$$

(*i*) Here we see the edge set $\{a_1p_1, a_2p_3, a_3p_5, a_4p_4\}$ is maximum matching which is not perfect since every vertex of V_1 is not incident in some edge in V_2.

This graph has no perfect matching

(*ii*) This can be treated as a Marriage Problem. We construct the following table:

Subset of vertices in V_1		Sub-set of adjacent vertices in V_2	
S	$\|S\|$	$N(S)$	$\|N(S)\|$
$\{a_1\}$	1	$\{p_1, p_2\}$	2
$\{a_2\}$	1	$\{p_1, p_3, p_5\}$	3
$\{a_3\}$	1	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_4\}$	1	$\{p_3, p_4\}$	2
$\{a_1a_2\}$	2	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_1a_3\}$	2	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_1a_4\}$	2	$\{p_1, p_2, p_3, p_4\}$	4
$\{a_2a_3\}$	2	$\{p_1, p_3, p_5, p_2\}$	4
$\{a_2a_4\}$	2	$\{p_1, p_3, p_5, p_4\}$	4
$\{a_3a_4\}$	2	$\{p_1, p_2, p_3, p_5, p_4\}$	5
$\{a_1a_2a_3\}$	3	$\{p_1, p_2, p_3, p_5\}$	4
$\{a_1a_2a_4\}$	3	$\{p_1, p_2, p_3, p_5, p_4\}$	5
$\{a_2a_3a_4\}$	3	$\{p_1, p_3, p_5, p_2, p_4\}$	5
$\{a_1a_3a_4\}$	3	$\{p_1, p_2, p_3, p_5, p_4\}$	5
$\{a_1a_2a_3a_4\}$	4	$\{p_1, p_2, p_3, p_5, p_4\}$	5

In the above table we see, in every row cardinal number of every set in 4th column is ≥ cardinal number of the set in 2nd column.

Thus by Hall's theorem a solution of this Marriage problem exist which shows that matching is perfect that is every applicant can be offered a single post.

15.8 Network Flows

In this section we consider the arcs of a digraph as pipes through which some commodity (such as number of cars, gallons of oils, bits of information, etc.) is transported from one place to another. The weight on an arc represents the capacity of the pipe, the maximum amount of some commodity that can flow through it in a given unit of time. The general problem in such a situation is to find the maximum flow in all these types of transmission network.

For example,

(*i*) If each arc represents a one-way street and the number of each arc is the maximum flow of traffic along that street, in vehicles per hour, then one may be interested to find the greatest possible number of vehicles that can travel from u to v in one hour.

(*ii*) The network can represent links in a computer network with data transmission capacities. Given two locations s, t in the network, one may be interested to find maximum flow of data (per unit time) from s to t.

Definitions

A **network or transport network** is a simple, connected, weighted directed graph $N(V, E)$ satisfying the following conditions :

(*i*) There is a unique vertex $s \in V$ if it has in-degree 0. This vertex is called the **source.**

(*ii*) There exists a unique vertex $t \in V$ if it has out-degree 0. This vertex is called the **sink.**

(*iii*) Every directed edge $e = (v, w) \in E$ has been assigned a non-negative number called the **capacity** of e, denoted by $c(e) = c(v, w)$. We can think of $c(e)$ as representing the maximum rate at which a commodity can be transported along the edge e.

We assume that the network has exactly one source s and one sink t. Any other vertex of N is called an **intermediate vertex.**

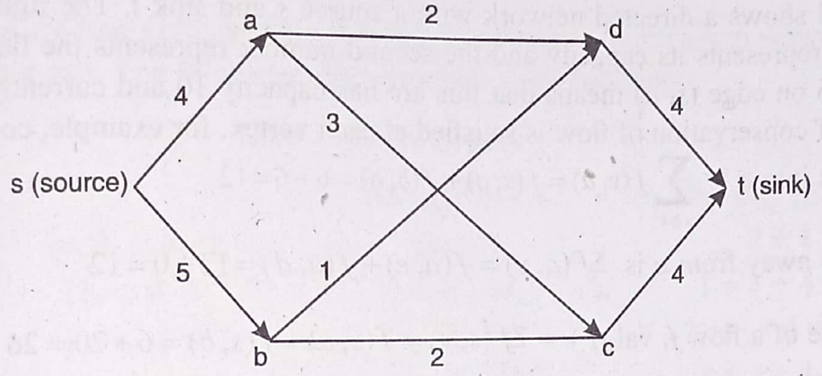


Fig. 15.7. A network showing the capacity on each arc.

A **flow** in a network is a function f that assigns to each arc $e = (v, w) \in E$ a non negative real number such that

(*i*) **(Capacity constraint)** $f(e) \le c(e)$ for each $e \in E$

i.e., $f(v, w) \le c(v, w)$ for each edge (v, w)

(*ii*) **(Flow conservation)** For any intermediate vertex x, total flow into x equals to the total flow out of x. Symbolically

$$\sum_{w \in V} f(w, x) = \sum_{v \in V} f(x, v)$$

(*iii*) $f(e) = 0$ for any edge e incident to the source s or incident from the sink t.

Conditions (*i*) ensures that the amount of material transported along a given edge cannot exceed the given capacity of that edge. (*ii*) enforces that the sum of all incoming flows at vertex x is equal to the sum of all outgoing flows at x and (*iii*) ensures that the flow moves from source to sink and not in the opposite direction.

The flow along an edge $e(v, w)$ is said to be **saturated** if $f(e) = c(e)$ *i.e.*, $f(v, w) = c(v, w)$ and if $f(e) < c(e)$, then the flow is said to be **unsaturated.** If an edge is unsaturated, then the **slack or** residual capacity of e in a flow f is defined to be $s(e) = c(e) - f(e)$.

The **value of a flow** is the net amount of flow per unit time leaving the source or equivalently, the net amount of flow per unit time entering the sink. Symbolically

$$val(f) = \sum_{v \in V} f(s, v) \text{ where } s \text{ is the source.}$$

$$= \sum_{v \in V} f(v, t) \text{ where } t \text{ is the sink}$$

i.e., the total outgoing flow at the source is equal to the total incoming flow at the sink.

Our problem is to calculate the maximum value of a flow for a given network without exceeding the capacity of each edge.