Unit-I (Mathematical Logic)

(Continued here...)

To prove a statement using

★ Mathematical Induction:

Let $S(n)$ be a statement for n^{th} value.

Then, step :- 1) verify statement $S(1)$ is true.

2) suppose $S(K)$ is true.

3) verify $S(k+1)$ is true using $S(K)$.

→ Prove that:

$$1^2+2^2+3^2+ \text{-----------} +n^2 = \frac{n(n+1)(2n+1)}{6}$$

Let $S(n)$: $1^2+2^2+ \text{--------} +n^2 = \frac{n(n+1)(2n+1)}{6}$ —— (1)

$$S(1): 1^2 = \frac{1(1+1)\times(2\times1+1)}{6}$$

Hence, L.H.S. $= 1^2$ = R.H.S. $= 1$

$\Rightarrow S(1)$ is true.

Let $S(K)$ is true.

$S(K)$: $1^2+2^2+ \text{------} +k^2 = \frac{k(k+1)(2k+1)}{6}$ —— (2)

Now, verifying $S(k+1)$.

$S(k+1)$: $1^2+2^2+ \text{------} +(k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$ —— (3)

L.H.S. $= \underbrace{1^2+2^2+3^2+ \text{-----} +k^2}+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$ [using eq^n (2)]

PAGE NO.:
DATE: / /

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6}$$

$$= \frac{(k+1)[2k^2+7k+6]}{6}$$

$$= \frac{(k+1)[2k^2+4k+3k+6]}{6}$$

$$= \frac{(k+1)[2k(k+2)+3(k+2)]}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} = R.H.S.$$

$\therefore S(n)$ is true for all value of n.

→ Show that :- using mathematical induction :-

a) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

b) $3 + 33 + 333 + \cdots + 333\ldots3 = (10^{n+1} - 9n - 10)/27$

c) Show that : $n^2 > 2n+1$ for $n \geq 3$ is true.

d) $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each +ve integer.

e) $n! \geq 2^{n-1}$, $n \geq 1$

d) $S_n : 43 \mid 6^{n+2} + 7^{2n+1}$ —— ①

Verifying $S(1)$ is true :-

$S(1) : 43 \mid 6^{1+2} + 7^{2\times1+1}$ is true.

as $6^3 + 7^3 = 216 + 343 = 559$ and $559 = 43\times 13\cancel{4}0$

$\Rightarrow 43 \mid 559$

~~verifying~~ Let $S(k)$ is true :-

$S(k) : 43 \mid 6^{k+2} + 7^{2k+1}$ —— ②

Verifying $S(k+1)$ is true :-

$S(k+1) : 43 \mid 6^{k+1+2} + 7^{2(k+1)+1}$

$43 \times \alpha = 6^{k+3} + 7^{2k+3}$ —— ③ $\alpha \in z^+$

From eq^n ② :-

$43\beta = 6^{k+2} + 7^{2k+1}$ —— ④ $\beta \in z^+$

Let us consider :- $6^{k+3} + 7^{2k+3} = 6\times 6^{k+2} + 7^2 \times 7^{2k+1}$

$6^{k+3} + 7^{2k+3} = 6^{k+3} + 7^{2k+1}\times 7^2 = 6(6^{k+2} + 7^{2k+1}) + 43(7^{2k+1})$

$= 6\times 6^{k+2} + 49(7^{2k+1})$

$= 6\times 6^{k+2} + (6+43)(7^{2k+1})$

$= 6\times 6^{k+2} + 6\times 7^{2k+1} + 43\times 7^{2k+1}$

$= 6[6^{k+2} + 7^{2k+1}] + 43\times(7^{2k+1})$

$= 6\times 43\beta + 43\times(7^{2k+1})$ [using eq^n ④]

$= 43[6\beta + 7^{2k+1}] = 43\times\alpha$, $\alpha \in z^+$

$\Rightarrow 43 \mid 6^{k+3} + 7^{2k+3}$

$\Rightarrow S(n)$ is true for all value of n.

PAGE NO.:
DATE: / /

→ Prove that :- $7^{n+2} + 8^{2n+1}$ is divisible by 57 for $n \in z^{+}$.

Aim - $57 \mid 7^{k+3} + 8^{2k+3}$ Given - $57 \mid 7^{k+2} + 8^{2k+1}$

$S_n : 57 \mid 7^{n+2} + 8^{2n+1}$ ——— ①

Verifying S(1) is true :-

$57 \mid 7^{1+2} + 8^{2\times1+1}$ is true.

As $7^3 + 8^3 = 343 + 512 = 855$ and $855 = 57 \times 15 + 0$

$\Rightarrow 57 \mid 855$

Let S(k) is true.

$S(k) : 57 \mid 7^{k+2} + 8^{2k+1}$ ——— ②

Verifying S(k+1) is true :-

$S(k+1) : 57 \mid 7^{k+1+2} + 8^{2(k+1)+1}$

$57 \times \alpha = 7^{k+3} + 8^{2k+3}$ ——— ③ $\alpha \in z^{+}$

From eq^n ② :- $57\beta = 7^{k+2} + 8^{2k+1}$ ——— ④

Let's consider :- $7^{k+3} + 8^{2k+3} = 7 \times 7^{k+2} + 8^2 \times 8^{2k+1}$

$= 7 \times 7^{k+2} + 64(8^{2k+1}) = 7 \times 7^{k+2} + (7+57)\, 8^{2k+1}$

$= 7 \times 7^{k+2} + 7 \times 8^{2k+1} + 57 \times 8^{2k+1}$

$= 7(7^{k+2} + 8^{2k+1}) + 57(8^{2k+1})$

$= 7 \times 57\beta + 57(8^{2k+1})$ [using eq^n ④]

$= 57(7\beta + 8^{2k+1}) = 57 \times \alpha$, $\alpha \in z^{+}$

$57 \mid 7^{k+3} + 8^{2k+3}$

$\therefore$ S(n) is true for all value of n.

PAGE NO.:
DATE: / /

a) Let $S(n): \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\text{-----------}+\frac{1}{n(n+1)}=\frac{n}{n+1}$ —— ①

Verifying S(1) is true :-

$S(1): \frac{1}{1\cdot2}=\frac{1}{1+1}=\frac{1}{2}$

Hence, $L.H.S.=\frac{1}{1\cdot2}=R.H.S.=\frac{1}{2}$

$\Rightarrow S(1)$ is true.

Let S(k) is true.

$S(k): \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\text{-----------}+\frac{1}{k(k+1)}=\frac{k}{k+1}$ —— ②

Verifying S(k+1) is true :-

$S(k+1): \frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\text{-----------}+\frac{1}{(k+1)(k+2)}=\frac{k+1}{k+2}$ —— ③

$$L.H.S.=\underbrace{\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\text{---------}+\frac{1}{k(k+1)}}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}$$

$$=\frac{k(k+2)+1}{(k+1)(k+2)}=\frac{k^2+2k+1}{k^2+3k+2}$$

$$=\frac{(k+1)^2}{k^2+2k+k+2}=\frac{(k+1)^2}{k(k+2)+1(k+2)}$$

$$=\frac{(k+1)^{\cancel{2}}}{\cancel{(k+1)}(k+2)}=\frac{k+1}{k+2}=R.H.S.$$

$\therefore$ S(n) is true for all value of n.

PAGE NO.:
DATE: / /

b) Let $S(n):\ 3+33+333+\cdots+333\ldots33 = (10^{n+1}-9n-10)/27$ —— ①

Verifying S(1) is true :-

$S(1):\ 3 = (10^{1+1}-9(1)-10)/27 = (100-9-10)/27$

$= \frac{100-19}{27} = \frac{81}{27} = 3$

Hence, L.H.S. = 3 = R.H.S. = 3

$\Rightarrow$ S(1) is true.

Let S(k) is true.

$S(k):\ 3+33+333+\cdots+\underbrace{333\ldots33}_{k\ \text{digits}} = (10^{k+1}-9k-10)/27$ —— ②

Verifying S(k+1) is true :-

$S(k+1):\ 3+33+333+\cdots+\underbrace{333\ldots33}_{k+1\ \text{digits}} = [10^{k+2}-9(k+1)-10]/27$ —— ③

$L.H.S. = \frac{10^{k+1}-9k-10}{27} + 3\left(\frac{10^{k+1}-1}{9}\right)$

$= \frac{10^{k+1}-9k-10+9(10^{k+1}-1)}{27}$

$= \frac{10^{k+1}-9k-10+9\times10^{k+1}-9}{27} = \frac{10^{k+1}+9\times10^{k+1}-9k-19}{27}$

$= \frac{10^{k+1}(1+9)-9(k+1)-10}{27} = \frac{[10^{k+2}-9(k+1)-10]}{27}$ = R.H.S.

$\therefore$ S(n) is true for all value of n.

PAGE NO.:
DATE: / /

c) Let $S(n): n^2 > 2n+1$ for $n \geq 3$. ——— ①

verifying $S(3)$ is true :-
$S(3): 3^2 > 2(3)+1 \Rightarrow 9 > 7$
$\Rightarrow S(3)$ is true.

Let $S(k)$ is true.
$S(k): k^2 > 2k+1$ ——— ②

verifying $S(k+1)$ is true :-
$S(k+1): (k+1)^2 > 2(k+1)+1$ ——— ③

$(k+1)^2 = k^2 + (2k+1) > (2k+1) + (2k+1)$ [using eq^n ②]
$> 2k+2+2k > 2k+2+1$
$(k+1)^2 > 2(k+1)+1$ is true.

As $2k > 1$
$(2k+2)+2k > (2k+2)+1$

$\therefore S(n)$ is true for $n \geq 3$.

e) let $S(n): n! \geq 2^{n-1}$ for $n \geq 1$ ——— ①

verifying $S(1)$ is true :-
$S(1): 1! \geq 2^{1-1} \Rightarrow 1 \geq 1$
$\Rightarrow S(1)$ is true.

Let $S(k)$ is true.
$S(k): k! \geq 2^{k-1}$ ——— ②

verifying S(k+1) is true :-

$S(k+1)$: $(k+1)! \geq 2^{k}$ ——— ③

$(k+1)! = (k+1)\, k! \geq (k+1)\, 2^{k-1}$ [using eq^n ②]

$\geq \dfrac{(k+1)}{2}\, 2^{k} \geq 1 \cdot 2^{k}$

As $k \geq 1$

$k+1 \geq 2$

$\boxed{\dfrac{k+1}{2} \geq 1}$

$(k+1)! \geq 2^{k}$ is true.

$\therefore S(n)$ is true for $n \geq 1$.