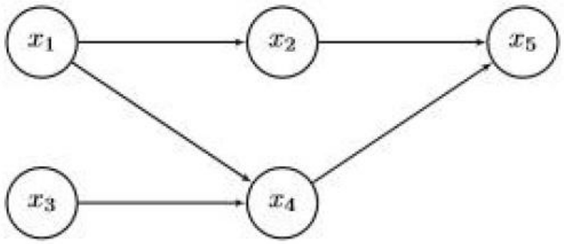
1 Probability (10 points)

Assume you are a contestant of a game show in which you are presented with three closed doors. Behind one of the doors is a car which will be yours if you choose the right door. After you have randomly (as you have no prior information) selected a door, the game host opens one of the other two doors which has nothing inside, while keeping the rest of doors closed. The host then asks whether you want to stay with your original selection or switch to another door. Should you change? Calculate the relevant posterior probabilities to explain your choice.

2 Bayesian Network Conditional Independence (10 points)

Consider the Bayesian network below where binary variables represent the following assertions: x_1 student is intelligent, x_2 student is good at taking tests, x_3 student is hard working, x_4 student understands the material, and x_5 student gets good grade.

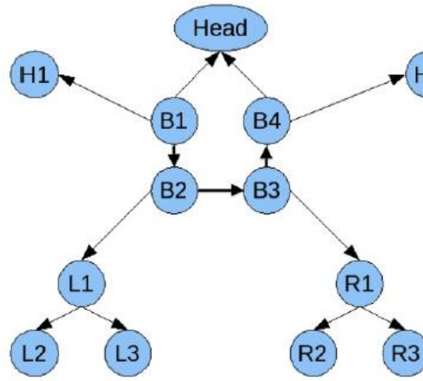


Write out the factorization of the probability distribution $p(x_1, \ldots, x_5)$ implied by this directed graph. Then, using the Bayes ball algorithm, answer the following questions as True or False.

1. x_2 and x_4 are independent.
2. x_2 and x_4 are conditionally independent given x_1, x_3, and x_5.
3. x_2 and x_4 are conditionally independent given x_1 and x_3.
4. x_5 and x_3 are conditionally independent given x_4.
5. x_5 and x_3 are conditionally independent given x_1, x_2, and x_4.
6. x_1 and x_3 are conditionally independent given x_5.
7. x_1 and x_3 are conditionally independent given x_2.
8. x_2 and x_3 are independent.
9. x_2 and x_3 are conditionally independent given x_5.
10. x_2 and x_3 are conditionally independent given x_5 and x_4.

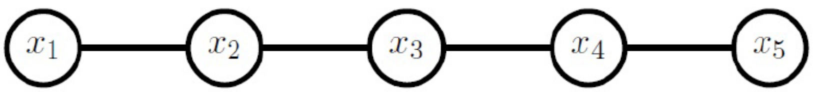
3 Junction Tree Construction (5 points)

Eve is looking for WallE using her cameras but can't find WallE. Eve has small circuits for performing the sum-product junction-tree algorithm. Help her out by helping in building a WallE classifier! Design a junction-tree from the graph above which Eve has in her mind for WallE.



4 Junction Tree Algorithm (15 points)

Consider the family of undirected graphical models known as Markov chains as shown below.



For simplicity, assume all variables in the model are binary. The probability distribution implied by the undirected graph is $p(x_1, ..., x_5) = \frac{1}{Z}\psi(x_1, x_2)\psi(x_2, x_3)\psi(x_3, x_4)\psi(x_4, x_5)$. Write an implementation of the junction tree algorithm that computes all the pairwise marginals $p(x_i, x_{i+1})$ for such a Markov chain for any number of variables n and any initialization of the clique potential functions. The initial clique potentials will serve as inputs to your junction tree algorithm and should be a cell array of $n-1$ elements, each of which is a 2 x 2 matrix of non-negative values as shown in the following Matlab code:

```
n = 5;
psis = cell(n-1, 1);
for i = 1:(n-1)
  psis{i} = rand(2,2);
end
```

The output of your junction tree algorithm (JTA) should be an identical data structure. It should contain consistent marginals that sum to unity appropriately and agree pairwise. Since the tree is only a chain, you don't have to implement a recursive algorithm (i.e. the Collect and Distribute steps in the Jordan book). Instead, you only need to perform left to right message passing and then right to left message passing by using a for loop or standard iteration. In other words, the JTA should process the cliques for $i = 1 : n-1$ and then for $i = n-1 : -1 : 1$ to do all the necessary messages in the JTA.

Test your algorithm by recovering the pairwise marginals when given the following values for the potential functions in the graphical model. Show code, results, and discussion.

$$\psi(x_1,x_2) = \left\{ \begin{array}{c|cc} & x_2=0 & x_2=1 \\ \hline x_1=0 & 0.1 & 0.7 \\ x_1=1 & 0.8 & 0.3 \end{array} \right\}$$

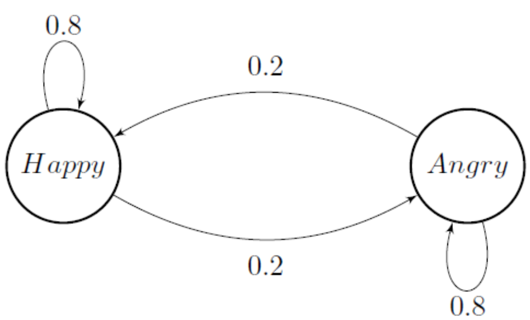
$$\psi(x_2,x_3) = \left\{ \begin{array}{c|cc} & x_3=0 & x_3=1 \\ \hline x_2=0 & 0.5 & 0.1 \\ x_2=1 & 0.1 & 0.5 \end{array} \right\}$$

$$\psi(x_3,x_4) = \left\{ \begin{array}{c|cc} & x_4=0 & x_4=1 \\ \hline x_3=0 & 0.1 & 0.5 \\ x_3=1 & 0.5 & 0.1 \end{array} \right\}$$

$$\psi(x_4,x_5) = \left\{ \begin{array}{c|cc} & x_5=0 & x_5=1 \\ \hline x_4=0 & 0.9 & 0.3 \\ x_4=1 & 0.1 & 0.3 \end{array} \right\}$$

5 Hidden Markov Model (10 points)

Your friend Super Mario has a simple life. In some days he is Happy and in some days he is Angry. But he won't tell you about his emotional state, and so all you can observe is whether he smiles, frowns, laughs, or yells. Let's start on day 1 with Happy state, and assume there is one transition per day. The transition and emission probabilities are as presented in the following figure and table. Define q_t to be emotional state on day t, and O_t is the observation on day t.



	smile	frown	laugh	yell
Happy	0.4	0.1	0.3	0.2
Angry	0.1	0.4	0.2	0.3

Assume that for the first five days, your observations of Mario are:

Day 1	Day 2	Day 3	Day 4	Day 5
smile	yell	frown	frown	laugh

What is the most likely sequence of his emotional states for the first five days? Show your work.