

# Trygonometria

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# Spis treści

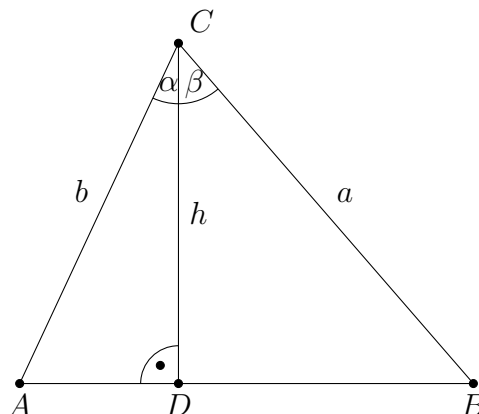
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# 1 Tożsamości Trygonometryczne

## 1.1 Sinus sumy dwóch kątów

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Dowód:



$$[ABC] = \frac{1}{2}ab \sin(\alpha + \beta)$$

$$[ADC] = \frac{1}{2}bh \sin \alpha$$

$$[DBC] = \frac{1}{2}ah \sin \beta$$

Z  $\triangle ADC$  i  $\triangle BDC$  :

$$\cos \alpha = \frac{h}{b} \text{ oraz } \cos \beta = \frac{h}{a}$$

Stąd

$$h = b \cos \alpha \text{ i } h = a \cos \beta$$

Czyli

$$[ABC] = [ADC] + [BDC]$$

$$\frac{1}{2}ab \sin(\alpha + \beta) = \frac{1}{2}bh \sin \alpha + \frac{1}{2}ah \sin \beta$$

$$\frac{1}{2}ab \sin(\alpha + \beta) = \frac{1}{2}ba \cos \beta \sin \alpha + \frac{1}{2}ab \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta.$$

c.n.d.

## 1.2 Cosinus sumy dwóch kątów

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Dowód:

$$\begin{aligned}\cos(\alpha + \beta) &= \sin(90^\circ - (\alpha + \beta)) = \\ &= \sin(90^\circ - \alpha - \beta) = \\ &= \sin((90^\circ - \alpha) + (-\beta)) = \\ &= \sin(90^\circ - \alpha)\cos(-\beta) + \cos(90^\circ - \alpha)\sin(-\beta) = \\ &= \cos \alpha \cos \beta - \sin \alpha \sin \beta.\end{aligned}$$

c.n.d.

## 1.3 Tangens sumy dwóch kątów

$$\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} = \\ &= \frac{\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \alpha \cos \beta}} = \\ &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 + \frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta}} = \\ &= \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}.\end{aligned}$$

c.n.d.

## 1.4 Cotangens sumy dwóch kątów

$$\operatorname{ctg}(\alpha + \beta) = \frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta - 1}{\operatorname{ctg} \alpha + \operatorname{ctg} \beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}(\alpha + \beta) &= \frac{\cos(\alpha + \beta)}{\sin(\alpha + \beta)} = \\ &= \frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\sin \alpha \cos \beta + \cos \alpha \sin \beta} = \\ &= \frac{\frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\sin \alpha \sin \beta}}{\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \sin \beta}} = \\ &= \frac{\frac{\cos \alpha \cos \beta}{\sin \alpha \sin \beta} - 1}{\frac{\cos \beta}{\sin \beta} + \frac{\cos \alpha}{\sin \alpha}} = \\ &= \frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta - 1}{\operatorname{ctg} \alpha + \operatorname{ctg} \beta}.\end{aligned}$$

c.n.d.

## 1.5 Sinus różnicy dwóch kątów

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

Dowód:

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) = \\ &= \sin \alpha \cos(-\beta) + \cos \alpha \sin(-\beta) = \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta.\end{aligned}$$

c.n.d.

## 1.6 Cosinus różnicy dwóch kątów

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

Dowód:

$$\begin{aligned}\cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) = \\ &= \cos \alpha \cos(-\beta) - \sin \alpha \sin(-\beta) = \\ &= \cos \alpha \cos \beta + \sin \alpha \sin \beta.\end{aligned}$$

c.n.d.

## 1.7 Tangens różnicy dwóch kątów

$$\operatorname{tg}(\alpha - \beta) = \frac{\operatorname{tg} \alpha - \operatorname{tg} \beta}{1 + \operatorname{tg} \alpha \operatorname{tg} \beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}(\alpha - \beta) &= \operatorname{tg}(\alpha + (-\beta)) = \\ &= \frac{\operatorname{tg} \alpha + \operatorname{tg}(-\beta)}{1 - \operatorname{tg} \alpha \operatorname{tg}(-\beta)} = \\ &= \frac{\operatorname{tg} \alpha - \operatorname{tg} \beta}{1 + \operatorname{tg} \alpha \operatorname{tg} \beta}.\end{aligned}$$

c.n.d.

## 1.8 Cotangens różnicy dwóch kątów

$$\operatorname{ctg}(\alpha - \beta) = \frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta + 1}{-\operatorname{ctg} \alpha + \operatorname{ctg} \beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}(\alpha - \beta) &= \operatorname{ctg}(\alpha + (-\beta)) = \\ &= \frac{\operatorname{ctg} \alpha \operatorname{ctg}(-\beta) - 1}{\operatorname{ctg} \alpha + \operatorname{ctg}(-\beta)} = \\ &= \frac{-\operatorname{ctg} \alpha \operatorname{ctg} \beta - 1}{\operatorname{ctg} \alpha - \operatorname{ctg} \beta} = \\ &= \frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta + 1}{-\operatorname{ctg} \alpha + \operatorname{ctg} \beta}.\end{aligned}$$

c.n.d.

## 1.9 Sinus dwukrotności kąta

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

Dowód:

$$\begin{aligned}\sin 2\alpha &= \sin(\alpha + \alpha) = \\ &= \sin \alpha \cos \alpha + \cos \alpha \sin \alpha = \\ &= 2 \sin \alpha \cos \alpha.\end{aligned}$$

c.n.d.

## 1.10 Cosinus dwukrotności kąta

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

Dowód:

$$\begin{aligned}\cos 2\alpha &= \cos(\alpha + \alpha) = \\ &= \cos \alpha \cos \alpha - \sin \alpha \sin \alpha = \\ &= \cos^2 \alpha - \sin^2 \alpha.\end{aligned}$$

c.n.d.

## 1.11 Tangens dwukrotności kąta

$$\operatorname{tg} 2\alpha = \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}$$

Dowód:

$$\begin{aligned}\operatorname{tg} 2\alpha &= \operatorname{tg}(\alpha + \alpha) = \\ &= \frac{\operatorname{tg} \alpha + \operatorname{tg} \alpha}{1 - \operatorname{tg} \alpha \operatorname{tg} \alpha} = \\ &= \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}.\end{aligned}$$

c.n.d.

## 1.12 Cotangens dwukrotności kąta

$$\operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha}$$

Dowód:

$$\begin{aligned}\operatorname{ctg} 2\alpha &= \operatorname{ctg}(\alpha + \alpha) = \\ &= \frac{\operatorname{ctg} \alpha \operatorname{ctg} \alpha - 1}{\operatorname{ctg} \alpha + \operatorname{ctg} \alpha} = \\ &= \frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha}.\end{aligned}$$

c.n.d.

### 1.13 Sinus trzykrotności kąta

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$

Dowód:

$$\begin{aligned}\sin 3\alpha &= \sin(2\alpha + \alpha) = \\&= \sin 2\alpha \cos \alpha + \cos 2\alpha \sin \alpha = \\&= (2 \sin \alpha \cos \alpha) \cos \alpha + (\cos^2 \alpha - \sin^2 \alpha) \sin \alpha = \\&= 2 \sin \alpha \cos^2 \alpha + \cos^2 \alpha \sin \alpha - \sin^3 \alpha = \\&= \cos^2 \alpha (3 \sin \alpha) - \sin^3 \alpha = \\&= (1 - \sin^2 \alpha)(3 \sin \alpha) - \sin^3 \alpha = \\&= 3 \sin \alpha - 4 \sin^3 \alpha.\end{aligned}$$

c.n.d.

### 1.14 Cosinus trzykrotności kąta

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

Dowód:

$$\begin{aligned}\cos 3\alpha &= \cos(2\alpha + \alpha) = \\&= \cos 2\alpha \cos \alpha - \sin 2\alpha \sin \alpha = \\&= (2 \cos^2 \alpha - 1) \cos \alpha - (2 \sin \alpha \cos \alpha) \sin \alpha = \\&= (2 \cos^2 \alpha - 1) \cos \alpha - 2 \sin^2 \alpha \cos \alpha = \\&= (2 \cos^2 \alpha - 1) \cos \alpha - 2(1 - \cos^2 \alpha) \cos \alpha = \\&= 2 \cos^3 \alpha - \cos \alpha - 2 \cos \alpha + 2 \cos^3 \alpha = \\&= 4 \cos^3 \alpha - 3 \cos \alpha.\end{aligned}$$

c.n.d.

### 1.15 Tangens trzykrotności kąta

$$\operatorname{tg} 3\alpha = \frac{3 \operatorname{tg} \alpha - \operatorname{tg}^3 \alpha}{1 - 3 \operatorname{tg}^2 \alpha}$$

Dowód:

$$\begin{aligned}\operatorname{tg} 3\alpha &= \operatorname{tg}(2\alpha + \alpha) = \\&= \frac{\operatorname{tg} 2\alpha + \operatorname{tg} \alpha}{1 - \operatorname{tg} 2\alpha \operatorname{tg} \alpha} = \\&= \frac{\frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha} + \operatorname{tg} \alpha}{1 - \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha} \operatorname{tg} \alpha} = \\&= \frac{\frac{2 \operatorname{tg} \alpha + \operatorname{tg} \alpha - \operatorname{tg}^3 \alpha}{1 - \operatorname{tg}^2 \alpha}}{\frac{1 - \operatorname{tg}^2 \alpha - 2 \operatorname{tg} \alpha \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}} = \\&= \frac{3 \operatorname{tg} \alpha - \operatorname{tg}^3 \alpha}{1 - 3 \operatorname{tg}^2 \alpha}.\end{aligned}$$

c.n.d.

### 1.16 Cotangens trzykrotności kąta

$$\operatorname{ctg} 3\alpha = \frac{\operatorname{ctg}^3 \alpha - 3 \operatorname{ctg} \alpha}{3 \operatorname{ctg}^2 \alpha - 1}$$

Dowód:

$$\begin{aligned}\operatorname{ctg} 3\alpha &= \operatorname{ctg}(2\alpha + \alpha) = \\ &= \frac{\operatorname{ctg} 2\alpha \operatorname{ctg} \alpha - 1}{\operatorname{ctg} 2\alpha + \operatorname{ctg} \alpha} = \\ &= \frac{\frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha} \operatorname{ctg} \alpha - 1}{\frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha} + \operatorname{ctg} \alpha} = \\ &= \frac{\frac{(\operatorname{ctg}^2 \alpha - 1) \operatorname{ctg} \alpha - 2 \operatorname{ctg} \alpha}{2 \operatorname{ctg} \alpha}}{\frac{\operatorname{ctg}^2 \alpha - 1 + 2 \operatorname{ctg}^2 \alpha}{2 \operatorname{ctg} \alpha}} = \\ &= \frac{(\operatorname{ctg}^2 \alpha - 1) \operatorname{ctg} \alpha - 2 \operatorname{ctg} \alpha}{\operatorname{ctg}^2 \alpha - 1 + 2 \operatorname{ctg}^2 \alpha} = \\ &= \frac{\operatorname{ctg}^3 \alpha - \operatorname{ctg} \alpha - 2 \operatorname{ctg} \alpha}{3 \operatorname{ctg}^2 \alpha - 1} = \\ &= \frac{\operatorname{ctg}^3 \alpha - 3 \operatorname{ctg} \alpha}{3 \operatorname{ctg}^2 \alpha - 1}.\end{aligned}$$

c.n.d.

### 1.17 Sinus kąta połówkowego

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

Dowód:

$$\begin{aligned}\cos 2\alpha &= 1 - 2 \sin^2 \alpha \\ \sin \alpha &= \pm \sqrt{\frac{1 - \cos 2\alpha}{2}}\end{aligned}$$

Zatem

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

c.n.d.

### 1.18 Cosinus kąta połówkowego

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

Dowód:

$$\begin{aligned}\cos 2\alpha &= 2 \cos^2 \alpha - 1 \\ \cos \alpha &= \pm \sqrt{\frac{1 + \cos 2\alpha}{2}}\end{aligned}$$



Zatem

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

c.n.d.

### 1.19 Tangens kąta połówkowego

$$\operatorname{tg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

Dowód:

$$\begin{aligned} \operatorname{tg} \frac{\alpha}{2} &= \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \\ &= \frac{\pm \sqrt{\frac{1 - \cos \alpha}{2}}}{\pm \sqrt{\frac{1 + \cos \alpha}{2}}} = \\ &= \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}. \end{aligned}$$

c.n.d.

### 1.20 Cotangens kąta połówkowego

$$\operatorname{ctg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$$

Dowód:

$$\begin{aligned} \operatorname{ctg} \frac{\alpha}{2} &= \frac{\cos \frac{\alpha}{2}}{\sin \frac{\alpha}{2}} = \\ &= \frac{\pm \sqrt{\frac{1 + \cos \alpha}{2}}}{\pm \sqrt{\frac{1 - \cos \alpha}{2}}} = \\ &= \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}. \end{aligned}$$

c.n.d.

### 1.21 Sinus wyrażony tangensem kąta połówkowego

$$\sin \alpha = \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}$$

Dowód(1):

$$\begin{aligned}
 P &= \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}} = \\
 &= \frac{2 \cdot \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}}{1 + \frac{1 - \cos \alpha}{1 + \cos \alpha}} = \\
 &= \frac{\pm 2 \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}}{\frac{1 + \cos \alpha + 1 - \cos \alpha}{1 + \cos \alpha}} = \\
 &= \frac{\pm 2 \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}}{\frac{2}{1 + \cos \alpha}} = \\
 &= \pm (1 + \cos \alpha) \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \\
 &= \pm \sqrt{\frac{(1 + \cos \alpha)^2 (1 - \cos \alpha)}{1 + \cos \alpha}} = \\
 &= \pm \sqrt{(1 + \cos \alpha)(1 - \cos \alpha)} = \\
 &= \pm \sqrt{1 - \cos^2 \alpha} = \\
 &= \pm \sqrt{\sin^2 \alpha} = \\
 &= \sin \alpha = L.
 \end{aligned}$$

Dowód(2):

$$\begin{aligned}
 P &= \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}} = \\
 &= \frac{2 \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}}{1 + \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}} = \\
 &= \frac{\frac{2 \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}}{\frac{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}} = \\
 &= \frac{\frac{2 \sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}}}{\frac{1}{\cos^2 \frac{\alpha}{2}}} = \\
 &= 2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \\
 &= \sin \alpha = L.
 \end{aligned}$$

c.n.d.

## 1.22 Cosinus wyrażony tangensem kąta połówkowego

$$\cos \alpha = \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}$$

Dowód(1):

$$\begin{aligned}
 P &= \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}} = \\
 &= \frac{1 - (\pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}})^2}{1 + (\pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}})^2} = \\
 &= \frac{1 - \frac{1 - \cos \alpha}{1 + \cos \alpha}}{1 + \frac{1 - \cos \alpha}{1 + \cos \alpha}} = \\
 &= \frac{\frac{1 + \cos \alpha - 1 + \cos \alpha}{1 + \cos \alpha}}{\frac{1 + \cos \alpha + 1 - \cos \alpha}{1 + \cos \alpha}} = \\
 &= \frac{1 + \cos \alpha - 1 + \cos \alpha}{1 + \cos \alpha + 1 - \cos \alpha} = \\
 &= \cos \alpha = L.
 \end{aligned}$$

Dowód(2):

$$\begin{aligned}
 P &= \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}} = \\
 &= \frac{1 - \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}}{1 + \frac{\sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}} = \\
 &= \frac{\frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}}{\frac{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2}}} = \\
 &= \frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{\cos^2 \frac{\alpha}{2} + \sin^2 \frac{\alpha}{2}} = \\
 &= \frac{\cos^2 \frac{\alpha}{2} - \sin^2 \frac{\alpha}{2}}{1} = \\
 &= \cos \alpha = L.
 \end{aligned}$$

c.n.d.

## 1.23 Tangens wyrażony tangensem kąta połówkowego

$$\operatorname{tg} \alpha = \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 - \operatorname{tg}^2 \frac{\alpha}{2}}$$

Dowód:

$$\begin{aligned}
 \operatorname{tg} \alpha &= \frac{\sin \alpha}{\cos \alpha} = \\
 &= \frac{\frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}}{\frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}} = \\
 &= \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 - \operatorname{tg}^2 \frac{\alpha}{2}}.
 \end{aligned}$$

c.n.d.

## 1.24 Cotangens wyrażony tangensem kąta połówkowego

$$\operatorname{ctg} \alpha = \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{2 \operatorname{tg} \frac{\alpha}{2}}$$

Dowód:

$$\begin{aligned} \operatorname{ctg} \alpha &= \frac{\cos \alpha}{\sin \alpha} = \\ &= \frac{\frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}}{\frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}} = \\ &= \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{2 \operatorname{tg} \frac{\alpha}{2}}. \end{aligned}$$

c.n.d.

## 1.25 Suma dwóch sinusów

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

Dowód:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Dodając stronami otrzymujemy:

$$\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$$

Niech:

$$\begin{cases} x + y = \alpha \\ x - y = \beta \end{cases}$$

Czyli

$$\begin{cases} x = \frac{\alpha + \beta}{2} \\ y = \frac{\alpha - \beta}{2} \end{cases}$$

Zatem

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.$$

c.n.d.

## 1.26 Różnica dwóch sinusów

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

Dowód:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Odejmując stronami otrzymujemy:

$$\sin(x+y) - \sin(x-y) = 2 \cos x \sin y$$

Niech:

$$\begin{cases} x+y = \alpha \\ x-y = \beta \end{cases}$$

Czyli

$$\begin{cases} x = \frac{\alpha+\beta}{2} \\ y = \frac{\alpha-\beta}{2} \end{cases}$$

Zatem

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2}.$$

c.n.d.

### 1.27 Suma dwóch cosinusów

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

Dowód:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Dodając stronami otrzymujemy:

$$\cos(x + y) + \cos(x - y) = 2 \cos x \cos y$$

Niech:

$$\begin{cases} x + y = \alpha \\ x - y = \beta \end{cases}$$

Czyli

$$\begin{cases} x = \frac{\alpha + \beta}{2} \\ y = \frac{\alpha - \beta}{2} \end{cases}$$

Zatem

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}.$$

c.n.d.

### 1.28 Różnica dwóch cosinusów

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

Dowód:

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Odejmując stronami otrzymujemy:

$$\cos(x + y) - \cos(x - y) = -2 \sin x \sin y$$

Niech:

$$\begin{cases} x + y = \alpha \\ x - y = \beta \end{cases}$$

Czyli

$$\begin{cases} x = \frac{\alpha + \beta}{2} \\ y = \frac{\alpha - \beta}{2} \end{cases}$$

Zatem

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}.$$

c.n.d.

### 1.29 Suma dwóch tangensów

$$\operatorname{tg} \alpha + \operatorname{tg} \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg} \alpha + \operatorname{tg} \beta &= \frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta} = \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} = \\ &= \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}.\end{aligned}$$

c.n.d.

### 1.30 Różnica dwóch tangensów

$$\operatorname{tg} \alpha - \operatorname{tg} \beta = \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg} \alpha - \operatorname{tg} \beta &= \frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta} = \\ &= \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta} = \\ &= \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta}.\end{aligned}$$

c.n.d.

### 1.31 Suma dwóch cotangensów

$$\operatorname{ctg} \alpha + \operatorname{ctg} \beta = \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg} \alpha + \operatorname{ctg} \beta &= \frac{\cos \alpha}{\sin \alpha} + \frac{\cos \beta}{\sin \beta} = \\ &= \frac{\cos \alpha \sin \beta + \sin \alpha \cos \beta}{\sin \alpha \sin \beta} = \\ &= \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}.\end{aligned}$$

c.n.d.

### 1.32 Różnica dwóch cotangensów

$$\operatorname{ctg} \alpha - \operatorname{ctg} \beta = \frac{-\sin(\alpha - \beta)}{\sin \alpha \sin \beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg} \alpha - \operatorname{ctg} \beta &= \frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta} = \\ &= \frac{\cos \alpha \sin \beta - \sin \alpha \cos \beta}{\sin \alpha \sin \beta} = \\ &= \frac{-\sin(\alpha - \beta)}{\sin \alpha \sin \beta}.\end{aligned}$$

c.n.d.

### 1.33 Suma sinusa i cosinusa tego samego kąta

$$\sin \alpha + \cos \alpha = \sqrt{2} \sin(45^\circ + \alpha) = \sqrt{2} \cos(45^\circ - \alpha)$$

Dowód:

$$\begin{aligned}P_1 &= \sqrt{2} \sin(45^\circ + \alpha) = \\ &= \sqrt{2}(\sin 45^\circ \cos \alpha + \cos 45^\circ \sin \alpha) = \\ &= \sqrt{2}\left(\frac{\sqrt{2}}{2} \cos \alpha + \frac{\sqrt{2}}{2} \sin \alpha\right) = \\ &= \sqrt{2} \frac{\sqrt{2}}{2} (\sin \alpha + \cos \alpha) = \\ &= \sin \alpha + \cos \alpha = L.\end{aligned}$$

$$\begin{aligned}P_2 &= \sqrt{2} \cos(45^\circ - \alpha) = \\ &= \sqrt{2}(\cos 45^\circ \cos \alpha + \sin 45^\circ \sin \alpha) = \\ &= \sqrt{2}\left(\frac{\sqrt{2}}{2} \cos \alpha + \frac{\sqrt{2}}{2} \sin \alpha\right) = \\ &= \sin \alpha + \cos \alpha = L.\end{aligned}$$

### 1.34 Różnica sinusa i cosinusa tego samego kąta

$$\sin \alpha - \cos \alpha = -\sqrt{2} \cos(45^\circ + \alpha) = -\sqrt{2} \sin(45^\circ - \alpha)$$

Dowód:

$$\begin{aligned}P_1 &= -\sqrt{2} \cos(45^\circ + \alpha) = \\ &= -\sqrt{2}(\cos 45^\circ \cos \alpha - \sin 45^\circ \sin \alpha) = \\ &= -\sqrt{2}\left(\frac{\sqrt{2}}{2} \cos \alpha - \frac{\sqrt{2}}{2} \sin \alpha\right) = \\ &= \sin \alpha - \cos \alpha = L.\end{aligned}$$

$$\begin{aligned}P_2 &= -\sqrt{2} \sin(45^\circ - \alpha) = \\ &= -\sqrt{2}(\sin 45^\circ \cos \alpha - \cos 45^\circ \sin \alpha) = \\ &= -\sqrt{2}\left(\frac{\sqrt{2}}{2} \cos \alpha - \frac{\sqrt{2}}{2} \sin \alpha\right) = \\ &= -\cos \alpha + \sin \alpha = L.\end{aligned}$$



## 2 Podsumowanie

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\operatorname{tg}(\alpha + \beta) = \frac{\operatorname{tg} \alpha + \operatorname{tg} \beta}{1 - \operatorname{tg} \alpha \operatorname{tg} \beta}$$

$$\operatorname{ctg}(\alpha + \beta) = \frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta - 1}{\operatorname{ctg} \alpha + \operatorname{ctg} \beta}$$

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$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\operatorname{tg}(\alpha - \beta) = \frac{\operatorname{tg} \alpha - \operatorname{tg} \beta}{1 + \operatorname{tg} \alpha \operatorname{tg} \beta}$$

$$\operatorname{ctg}(\alpha - \beta) = \frac{\operatorname{ctg} \alpha \operatorname{ctg} \beta + 1}{-\operatorname{ctg} \alpha + \operatorname{ctg} \beta}$$

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$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cos^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

$$\operatorname{tg} 2\alpha = \frac{2 \operatorname{tg} \alpha}{1 - \operatorname{tg}^2 \alpha}$$

$$\operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha}$$

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$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha$$

$$\cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

$$\operatorname{tg} 3\alpha = \frac{3 \operatorname{tg} \alpha - \operatorname{tg}^3 \alpha}{1 - 3 \operatorname{tg}^2 \alpha}$$

$$\operatorname{ctg} 3\alpha = \frac{\operatorname{ctg}^3 \alpha - 3 \operatorname{ctg} \alpha}{3 \operatorname{ctg}^2 \alpha - 1}$$

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$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

$$\operatorname{tg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$\operatorname{ctg} \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}}$$

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$$\sin \alpha = \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}$$

$$\cos \alpha = \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{1 + \operatorname{tg}^2 \frac{\alpha}{2}}$$

$$\operatorname{tg} \alpha = \frac{2 \operatorname{tg} \frac{\alpha}{2}}{1 - \operatorname{tg}^2 \frac{\alpha}{2}}$$

$$\operatorname{ctg} \alpha = \frac{1 - \operatorname{tg}^2 \frac{\alpha}{2}}{2 \operatorname{tg} \frac{\alpha}{2}}$$


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$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$


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$$\operatorname{tg} \alpha + \operatorname{tg} \beta = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$$

$$\operatorname{tg} \alpha - \operatorname{tg} \beta = \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta}$$

$$\operatorname{ctg} \alpha + \operatorname{ctg} \beta = \frac{\sin(\alpha + \beta)}{\sin \alpha \sin \beta}$$

$$\operatorname{ctg} \alpha - \operatorname{ctg} \beta = \frac{-\sin(\alpha - \beta)}{\sin \alpha \sin \beta}$$


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$$\sin \alpha + \cos \alpha = \sqrt{2} \sin(45^\circ + \alpha) = \sqrt{2} \cos(45^\circ - \alpha)$$

$$\sin \alpha - \cos \alpha = -\sqrt{2} \cos(45^\circ + \alpha) = -\sqrt{2} \sin(45^\circ - \alpha)$$