Trygonometria

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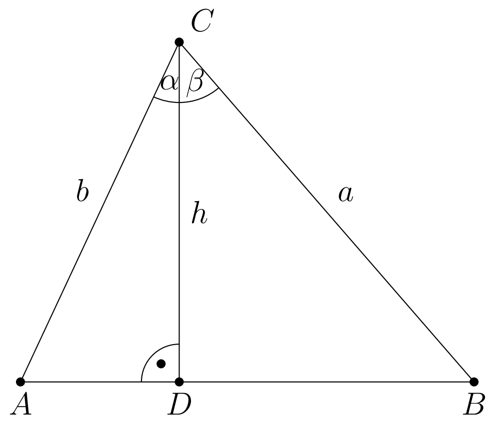
Spis treści

1 Tożsamości Trygonometryczne

1.1 Sinus sumy dwóch kątów

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

Dowód:



$$[ABC] = \frac{1}{2}ab\sin(\alpha+\beta)$$
$$[ADC] = \frac{1}{2}bh\sin\alpha$$
$$[DBC] = \frac{1}{2}ah\sin\beta$$

Z ΔADC i ΔBDC :

$\cos\alpha = \frac{h}{b}$ oraz $\cos\beta = \frac{h}{a}$

Stąd

$h = b\cos\alpha$ i $h = a\cos\beta$

Czyli

$$[ABC] = [ADC] + [BDC]$$
$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}bh\sin\alpha + \frac{1}{2}ah\sin\beta$$
$$\frac{1}{2}ab\sin(\alpha+\beta) = \frac{1}{2}ba\cos\beta\sin\alpha + \frac{1}{2}ab\cos\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

c.n.d.

1.2 Cosinus sumy dwóch kątów

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Dowód:

$$\begin{aligned}\cos(\alpha+\beta) &= \sin(90^\circ - (\alpha+\beta)) = \\ &= \sin(90^\circ - \alpha - \beta) = \\ &= \sin((90^\circ - \alpha) + (-\beta)) = \\ &= \sin(90^\circ - \alpha) cos(-\beta) + \cos(90^\circ - \alpha)\sin(-\beta) = \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

c.n.d.

1.3 Tangens sumy dwóch kątów

$$\operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \\ &= \frac{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \\ &= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 - \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}} = \\ &= \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}.\end{aligned}$$

c.n.d.

1.4 Cotangens sumy dwóch kątów

$$\operatorname{ctg}(\alpha+\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}(\alpha+\beta) &= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \\ &= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} = \\ &= \frac{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\sin\beta}} = \\ &= \frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} - 1}{\frac{\cos\beta}{\sin\beta} + \frac{\cos\alpha}{\sin\alpha}} = \\ &= \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}.\end{aligned}$$

c.n.d.

1.5 Sinus różnicy dwóch kątów

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

Dowód:

$$\begin{aligned}\sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) = \\ &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta) = \\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta.\end{aligned}$$

c.n.d.

1.6 Cosinus różnicy dwóch kątów

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Dowód:

$$\begin{aligned}\cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) = \\ &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) = \\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

c.n.d.

1.7 Tangens różnicy dwóch kątów

$$\operatorname{tg}(\alpha - \beta) = \frac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{1 + \operatorname{tg}\alpha\operatorname{tg}\beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}(\alpha - \beta) &= \operatorname{tg}(\alpha + (-\beta)) = \\ &= \frac{\operatorname{tg}\alpha + \operatorname{tg}(-\beta)}{1 - \operatorname{tg}\alpha\operatorname{tg}(-\beta)} = \\ &= \frac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{1 + \operatorname{tg}\alpha\operatorname{tg}\beta}.\end{aligned}$$

c.n.d.

1.8 Cotangens różnicy dwóch kątów

$$\operatorname{ctg}(\alpha - \beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta + 1}{-\operatorname{ctg}\alpha + \operatorname{ctg}\beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}(\alpha - \beta) &= \operatorname{ctg}(\alpha + (-\beta)) = \\ &= \frac{\operatorname{ctg}\alpha\operatorname{ctg}(-\beta) - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}(-\beta)} = \\ &= \frac{-\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha - \operatorname{ctg}\beta} = \\ &= \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta + 1}{-\operatorname{ctg}\alpha + \operatorname{ctg}\beta}.\end{aligned}$$

c.n.d.

1.9 Sinus dwukrotności kąta

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

Dowód:

$$\begin{aligned}\sin 2\alpha &= \sin(\alpha+\alpha) = \\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = \\ &= 2\sin\alpha\cos\alpha.\end{aligned}$$

c.n.d.

1.10 Cosinus dwukrotności kąta

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

Dowód:

$$\begin{aligned}\cos 2\alpha &= \cos(\alpha+\alpha) = \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \\ &= \cos^2\alpha - \sin^2\alpha.\end{aligned}$$

c.n.d.

1.11 Tangens dwukrotności kąta

$\operatorname{tg} 2\alpha = \dfrac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$

Dowód:

$$\begin{aligned}\operatorname{tg} 2\alpha &= \operatorname{tg}(\alpha+\alpha) = \\ &= \frac{\operatorname{tg}\alpha + \operatorname{tg}\alpha}{1-\operatorname{tg}\alpha\operatorname{tg}\alpha} = \\ &= \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}.\end{aligned}$$

c.n.d.

1.12 Cotangens dwukrotności kąta

$\operatorname{ctg} 2\alpha = \dfrac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}$

Dowód:

$$\begin{aligned}\operatorname{ctg} 2\alpha &= \operatorname{ctg}(\alpha+\alpha) = \\ &= \frac{\operatorname{ctg}\alpha\operatorname{ctg}\alpha - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\alpha} = \\ &= \frac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}.\end{aligned}$$

c.n.d.

1.13 Sinus trzykrotności kąta

$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$

<u>Dowód:</u>

$$\begin{aligned}
\sin 3\alpha &= \sin(2\alpha + \alpha) = \\
&= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha = \\
&= (2\sin\alpha\cos\alpha)\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha = \\
&= 2\sin\alpha\cos^2\alpha + \cos^2\alpha\sin\alpha - \sin^3\alpha = \\
&= \cos^2\alpha(3\sin\alpha) - \sin^3\alpha = \\
&= (1 - \sin^2\alpha)(3\sin\alpha) - \sin^3\alpha = \\
&= 3\sin\alpha - 4\sin^3\alpha.
\end{aligned}$$

c.n.d.

1.14 Cosinus trzykrotności kąta

$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

<u>Dowód:</u>

$$\begin{aligned}
\cos 3\alpha &= \cos(2\alpha + \alpha) = \\
&= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha = \\
&= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha = \\
&= (2\cos^2\alpha - 1)\cos\alpha - 2\sin^2\alpha\cos\alpha = \\
&= (2\cos^2\alpha - 1)\cos\alpha - 2(1 - \cos^2\alpha)\cos\alpha = \\
&= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha = \\
&= 4\cos^3\alpha - 3\cos\alpha.
\end{aligned}$$

c.n.d.

1.15 Tangens trzykrotności kąta

$\operatorname{tg} 3\alpha = \dfrac{3\operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1 - 3\operatorname{tg}^2\alpha}$

<u>Dowód:</u>

$$\begin{aligned}
\operatorname{tg} 3\alpha &= \operatorname{tg}(2\alpha + \alpha) = \\
&= \frac{\operatorname{tg} 2\alpha + \operatorname{tg}\alpha}{1 - \operatorname{tg} 2\alpha\operatorname{tg}\alpha} = \\
&= \frac{\frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha} + \operatorname{tg}\alpha}{1 - \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}\operatorname{tg}\alpha} = \\
&= \frac{\frac{2\operatorname{tg}\alpha + \operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1-\operatorname{tg}^2\alpha}}{\frac{1-\operatorname{tg}^2\alpha - 2\operatorname{tg}\alpha\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}} = \\
&= \frac{3\operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1 - 3\operatorname{tg}^2\alpha}.
\end{aligned}$$

c.n.d.

1.16 Cotangens trzykrotności kąta

$$\operatorname{ctg} 3\alpha = \frac{\operatorname{ctg}^3\alpha - 3\operatorname{ctg}\alpha}{3\operatorname{ctg}^2\alpha - 1}$$

Dowód:

$$\begin{aligned}
\operatorname{ctg} 3\alpha &= \operatorname{ctg}(2\alpha+\alpha) = \\
&= \frac{\operatorname{ctg} 2\alpha \operatorname{ctg}\alpha - 1}{\operatorname{ctg} 2\alpha + \operatorname{ctg}\alpha} = \\
&= \frac{\frac{\operatorname{ctg}^2\alpha-1}{2\operatorname{ctg}\alpha}\operatorname{ctg}\alpha - 1}{\frac{\operatorname{ctg}^2\alpha-1}{2\operatorname{ctg}\alpha} + \operatorname{ctg}\alpha} = \\
&= \frac{\frac{(\operatorname{ctg}^2\alpha-1)\operatorname{ctg}\alpha-2\operatorname{ctg}\alpha}{2\operatorname{ctg}\alpha}}{\frac{\operatorname{ctg}^2\alpha-1+2\operatorname{ctg}^2\alpha}{2\operatorname{ctg}\alpha}} = \\
&= \frac{(\operatorname{ctg}^2\alpha-1)\operatorname{ctg}\alpha-2\operatorname{ctg}\alpha}{\operatorname{ctg}^2\alpha-1+2\operatorname{ctg}^2\alpha} = \\
&= \frac{\operatorname{ctg}^3\alpha-\operatorname{ctg}\alpha-2\operatorname{ctg}\alpha}{3\operatorname{ctg}^2\alpha-1} = \\
&= \frac{\operatorname{ctg}^3\alpha-3\operatorname{ctg}\alpha}{3\operatorname{ctg}^2\alpha-1}.
\end{aligned}$$

c.n.d.

1.17 Sinus kąta połówkowego

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

Dowód:

$$\cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\sin\alpha = \pm\sqrt{\frac{1-\cos 2\alpha}{2}}$$

Zatem

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}.$$

c.n.d.

1.18 Cosinus kąta połówkowego

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

Dowód:

$$\cos 2\alpha = 2\cos^2\alpha - 1$$

$$\cos\alpha = \pm\sqrt{\frac{1+\cos 2\alpha}{2}}$$

Zatem

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

c.n.d.

1.19 Tangens kąta połówkowego

$$\operatorname{tg}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Dowód:

$$\begin{aligned}\operatorname{tg}\frac{\alpha}{2} &= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \\ &= \frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}} = \\ &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.\end{aligned}$$

c.n.d.

1.20 Cotangens kąta połówkowego

$$\operatorname{ctg}\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}\frac{\alpha}{2} &= \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = \\ &= \frac{\pm\sqrt{\frac{1+\cos\alpha}{2}}}{\pm\sqrt{\frac{1-\cos\alpha}{2}}} = \\ &= \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}.\end{aligned}$$

c.n.d.

1.21 Sinus wyrażony tangensem kąta połówkowego

$$\sin\alpha = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}$$

Dowód(1):

$$
\begin{aligned}
P &= \frac{2\,\mathrm{tg}\,\frac{\alpha}{2}}{1+\mathrm{tg}^2\,\frac{\alpha}{2}} = \\
&= \frac{2\cdot\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}{1+\frac{1-\cos\alpha}{1+\cos\alpha}} = \\
&= \frac{\pm 2\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}{\frac{1+\cos\alpha+1-\cos\alpha}{1+\cos\alpha}} = \\
&= \frac{\pm 2\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}{\frac{2}{1+\cos\alpha}} = \\
&= \pm(1+\cos\alpha)\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \\
&= \pm\sqrt{\frac{(1+\cos\alpha)^2(1-\cos\alpha)}{1+\cos\alpha}} = \\
&= \pm\sqrt{(1+\cos\alpha)(1-\cos\alpha)} = \\
&= \pm\sqrt{(1-\cos^2\alpha)} = \\
&= \pm\sqrt{\sin^2\alpha} = \\
&= \sin\alpha = L.
\end{aligned}
$$

Dowód(2):

$$
\begin{aligned}
P &= \frac{2\,\mathrm{tg}\,\frac{\alpha}{2}}{1+\mathrm{tg}^2\,\frac{\alpha}{2}} = \\
&= \frac{2\frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{1+\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}} = \\
&= \frac{\frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{\frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}} = \\
&= \frac{\frac{2\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}}}{\frac{1}{\cos^2\frac{\alpha}{2}}} = \\
&= 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = \\
&= \sin\alpha = L.
\end{aligned}
$$

c.n.d.

1.22 Cosinus wyrażony tangensem kąta połówkowego

$$
\cos\alpha = \frac{1-\mathrm{tg}^2\,\frac{\alpha}{2}}{1+\mathrm{tg}^2\,\frac{\alpha}{2}}
$$

Dowód(1):

$$
\begin{aligned}
P &= \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} = \\
&= \frac{1-(\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}})^2}{1+(\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}})^2} = \\
&= \frac{1-\frac{1-\cos\alpha}{1+\cos\alpha}}{1+\frac{1-\cos\alpha}{1+\cos\alpha}} = \\
&= \frac{\frac{1+\cos\alpha-1+\cos\alpha}{1+\cos\alpha}}{\frac{1+\cos\alpha+1-\cos\alpha}{1+\cos\alpha}} = \\
&= \frac{1+\cos\alpha-1+\cos\alpha}{1+\cos\alpha+1-\cos\alpha} = \\
&= \cos\alpha = L.
\end{aligned}
$$

Dowód(2):

$$
\begin{aligned}
P &= \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} = \\
&= \frac{1-\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}}{1+\frac{\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}} = \\
&= \frac{\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}}{\frac{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}} = \\
&= \frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}} = \\
&= \frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{1} = \\
&= \cos\alpha = L.
\end{aligned}
$$

c.n.d.

1.23 Tangens wyrażony tangensem kąta połówkowego

$$\operatorname{tg}\alpha = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}^2\frac{\alpha}{2}}$$

Dowód:

$$
\begin{aligned}
\operatorname{tg}\alpha &= \frac{\sin\alpha}{\cos\alpha} = \\
&= \frac{\frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}}{\frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}} = \\
&= \frac{2\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}^2\frac{\alpha}{2}}.
\end{aligned}
$$

c.n.d.

1.24 Cotangens wyrażony tangensem kąta połówkowego

$$\operatorname{ctg}\alpha = \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{2\operatorname{tg}\frac{\alpha}{2}}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}\alpha &= \frac{\cos\alpha}{\sin\alpha} = \\ &= \frac{\frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}}{\frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}} = \\ &= \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{2\operatorname{tg}\frac{\alpha}{2}}.\end{aligned}$$

c.n.d.

1.25 Suma dwóch sinusów

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Dowód:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y \\ \sin(x-y) &= \sin x\cos y - \cos x\sin y\end{aligned}$$

Dodając stronami otrzymujemy:

$$\sin(x+y) + \sin(x-y) = 2\sin x\cos y$$

Niech:

$$\begin{cases} x+y=\alpha \\ x-y=\beta \end{cases}$$

Czyli

$$\begin{cases} x=\frac{\alpha+\beta}{2} \\ y=\frac{\alpha-\beta}{2} \end{cases}$$

Zatem

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

c.n.d.

1.26 Różnica dwóch sinusów

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Dowód:

$$\begin{aligned}\sin(x+y) &= \sin x\cos y + \cos x\sin y \\ \sin(x-y) &= \sin x\cos y - \cos x\sin y\end{aligned}$$

Odejmując stronami otrzymujemy:

$$\sin(x+y) - \sin(x-y) = 2\cos x \sin y$$

Niech:

$$\begin{cases} x+y=\alpha \\ x-y=\beta \end{cases}$$

Czyli

$$\begin{cases} x=\frac{\alpha+\beta}{2} \\ y=\frac{\alpha-\beta}{2} \end{cases}$$

Zatem

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

c.n.d.

1.27 Suma dwóch cosinusów

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

Dowód:

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

Dodając stronami otrzymujemy:

$$\cos(x+y) + \cos(x-y) = 2\cos x\cos y$$

Niech:

$$\begin{cases} x+y=\alpha \\ x-y=\beta \end{cases}$$

Czyli

$$\begin{cases} x=\frac{\alpha+\beta}{2} \\ y=\frac{\alpha-\beta}{2} \end{cases}$$

Zatem

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

c.n.d.

1.28 Różnica dwóch cosinusów

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Dowód:

$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$

Odejmując stronami otrzymujemy:

$$\cos(x+y) - \cos(x-y) = -2\sin x\sin y$$

Niech:

$$\begin{cases} x+y=\alpha \\ x-y=\beta \end{cases}$$

Czyli

$$\begin{cases} x=\frac{\alpha+\beta}{2} \\ y=\frac{\alpha-\beta}{2} \end{cases}$$

Zatem

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$$

c.n.d.

1.29 Suma dwóch tangensów

$$\operatorname{tg}\alpha + \operatorname{tg}\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}\alpha + \operatorname{tg}\beta &= \frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta} = \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \\ &= \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}.\end{aligned}$$

c.n.d.

1.30 Różnica dwóch tangensów

$$\operatorname{tg}\alpha - \operatorname{tg}\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}\alpha - \operatorname{tg}\beta &= \frac{\sin\alpha}{\cos\alpha} - \frac{\sin\beta}{\cos\beta} = \\ &= \frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\cos\alpha\cos\beta} = \\ &= \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}.\end{aligned}$$

c.n.d.

1.31 Suma dwóch cotangensów

$$\operatorname{ctg}\alpha + \operatorname{ctg}\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}\alpha + \operatorname{ctg}\beta &= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos\beta}{\sin\beta} = \\ &= \frac{\cos\alpha\sin\beta + \sin\alpha\cos\beta}{\sin\alpha\sin\beta} = \\ &= \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}.\end{aligned}$$

c.n.d.

1.32 Różnica dwóch cotangensów

$$\operatorname{ctg}\alpha - \operatorname{ctg}\beta = \frac{-\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}\alpha - \operatorname{ctg}\beta &= \frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta} = \\ &= \frac{\cos\alpha\sin\beta - \sin\alpha\cos\beta}{\sin\alpha\sin\beta} = \\ &= \frac{-\sin(\alpha-\beta)}{\sin\alpha\sin\beta}.\end{aligned}$$

c.n.d.

1.33 Suma sinusa i cosinusa tego samego kąta

$$\sin\alpha + \cos\alpha = \sqrt{2}\sin(45^\circ + \alpha) = \sqrt{2}\cos(45^\circ - \alpha)$$

Dowód:

$$\begin{aligned}P_1 &= \sqrt{2}\sin(45^\circ + \alpha) = \\ &= \sqrt{2}(\sin 45^\circ \cos\alpha + \cos 45^\circ \sin\alpha) = \\ &= \sqrt{2}(\frac{\sqrt{2}}{2}\cos\alpha + \frac{\sqrt{2}}{2}\sin\alpha) = \\ &= \sqrt{2}\frac{\sqrt{2}}{2}(\sin\alpha + \cos\alpha) = \\ &= \sin\alpha + \cos\alpha = L.\end{aligned}$$

$$\begin{aligned}P_2 &= \sqrt{2}\cos(45^\circ - \alpha) = \\ &= \sqrt{2}(\cos 45^\circ \cos\alpha + \sin 45^\circ \sin\alpha) = \\ &= \sqrt{2}(\frac{\sqrt{2}}{2}\cos\alpha + \frac{\sqrt{2}}{2}\sin\alpha) = \\ &= \sin\alpha + \cos\alpha = L.\end{aligned}$$

1.34 Różnica sinusa i cosinusa tego samego kąta

$$\sin\alpha - \cos\alpha = -\sqrt{2}\cos(45^\circ + \alpha) = -\sqrt{2}\sin(45^\circ - \alpha)$$

Dowód:

$$\begin{aligned}P_1 &= -\sqrt{2}\cos(45^\circ + \alpha) = \\ &= -\sqrt{2}(\cos 45^\circ \cos\alpha - \sin 45^\circ \sin\alpha) = \\ &= -\sqrt{2}(\frac{\sqrt{2}}{2}\cos\alpha - \frac{\sqrt{2}}{2}\sin\alpha) = \\ &= \sin\alpha - \cos\alpha = L.\end{aligned}$$

$$\begin{aligned}P_2 &= -\sqrt{2}\sin(45^\circ - \alpha) = \\ &= -\sqrt{2}(\sin 45^\circ \cos\alpha - \cos 45^\circ \sin\alpha) = \\ &= -\sqrt{2}(\frac{\sqrt{2}}{2}\cos\alpha - \frac{\sqrt{2}}{2}\sin\alpha) = \\ &= -\cos\alpha + \sin\alpha = L.\end{aligned}$$

2 Podsumowanie

$$
\begin{aligned}
&\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
&\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
&\operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta} \\
&\operatorname{ctg}(\alpha+\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}
\end{aligned}
$$

$$
\begin{aligned}
&\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
&\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
&\operatorname{tg}(\alpha-\beta) = \frac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{1 + \operatorname{tg}\alpha\operatorname{tg}\beta} \\
&\operatorname{ctg}(\alpha-\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta + 1}{-\operatorname{ctg}\alpha + \operatorname{ctg}\beta}
\end{aligned}
$$

$$
\begin{aligned}
&\sin 2\alpha = 2\sin\alpha\cos\alpha \\
&\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\
&\operatorname{tg} 2\alpha = \frac{2\operatorname{tg}\alpha}{1 - \operatorname{tg}^2\alpha} \\
&\operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \\
&\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha \\
&\operatorname{tg} 3\alpha = \frac{3\operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1 - 3\operatorname{tg}^2\alpha} \\
&\operatorname{ctg} 3\alpha = \frac{\operatorname{ctg}^3\alpha - 3\operatorname{ctg}\alpha}{3\operatorname{ctg}^2\alpha - 1}
\end{aligned}
$$

$$
\begin{aligned}
&\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \\
&\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \\
&\operatorname{tg}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} \\
&\operatorname{ctg}\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}
\end{aligned}
$$

$$\sin\alpha = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}$$
$$\cos\alpha = \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}$$
$$\operatorname{tg}\alpha = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1-\operatorname{tg}^2\frac{\alpha}{2}}$$
$$\operatorname{ctg}\alpha = \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{2\operatorname{tg}\frac{\alpha}{2}}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$
$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$
$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\operatorname{tg}\alpha + \operatorname{tg}\beta = \frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$$
$$\operatorname{tg}\alpha - \operatorname{tg}\beta = \frac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$$
$$\operatorname{ctg}\alpha + \operatorname{ctg}\beta = \frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$$
$$\operatorname{ctg}\alpha - \operatorname{ctg}\beta = \frac{-\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$$

$$\sin\alpha + \cos\alpha = \sqrt{2}\sin(45^\circ+\alpha) = \sqrt{2}\cos(45^\circ-\alpha)$$
$$\sin\alpha - \cos\alpha = -\sqrt{2}\cos(45^\circ+\alpha) = -\sqrt{2}\sin(45^\circ-\alpha)$$