Trygonometria

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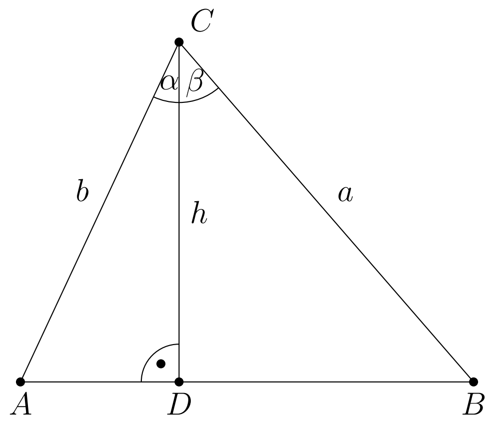
Spis treści

1 Tożsamości Trygonometryczne

1.1 Sinus sumy dwóch kątów

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

<u>Dowód:</u>



$$[ABC] = \frac{1}{2}ab\sin(\alpha + \beta)$$
$$[ADC] = \frac{1}{2}bh\sin\alpha$$
$$[DBC] = \frac{1}{2}ah\sin\beta$$

Z ΔADC i ΔBDC :

$\cos\alpha = \frac{h}{b}$ oraz $\cos\beta = \frac{h}{a}$

Stąd

$h = b\cos\alpha$ i $h = a\cos\beta$

Czyli

$$[ABC] = [ADC] + [BDC]$$
$$\frac{1}{2}ab\sin(\alpha + \beta) = \frac{1}{2}bh\sin\alpha + \frac{1}{2}ah\sin\beta$$
$$\frac{1}{2}ab\sin(\alpha + \beta) = \frac{1}{2}ba\cos\beta\sin\alpha + \frac{1}{2}ab\cos\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

c.n.d.

1.2 Cosinus sumy dwóch kątów

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Dowód:

$$\begin{aligned}\cos(\alpha+\beta) &= \sin(90^\circ - (\alpha+\beta)) = \\ &= \sin(90^\circ - \alpha - \beta) = \\ &= \sin((90^\circ - \alpha) + (-\beta)) = \\ &= \sin(90^\circ - \alpha) cos(-\beta) + \cos(90^\circ - \alpha)\sin(-\beta) = \\ &= \cos\alpha\cos\beta - sin\alpha\sin\beta.\end{aligned}$$

c.n.d.

1.3 Tangens sumy dwóch kątów

$$\operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}$$

Dowód:

$$\begin{aligned}\operatorname{tg}(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta} = \\ &= \frac{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \\ &= \frac{\frac{\sin\alpha}{\cos\alpha} + \frac{\sin\beta}{\cos\beta}}{1 + \frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}} = \\ &= \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}.\end{aligned}$$

c.n.d.

1.4 Cotangens sumy dwóch kątów

$$\operatorname{ctg}(\alpha+\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}(\alpha+\beta) &= \frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)} = \\ &= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\sin\alpha\cos\beta + \cos\alpha\sin\beta} = \\ &= \frac{\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\sin\alpha\sin\beta}}{\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\sin\alpha\sin\beta}} = \\ &= \frac{\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} - 1}{\frac{\cos\beta}{\sin\beta} + \frac{\cos\alpha}{\sin\alpha}} = \\ &= \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}.\end{aligned}$$

c.n.d.

1.5 Sinus różnicy dwóch kątów

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

Dowód:

$$\begin{aligned}\sin(\alpha-\beta)&=\sin(\alpha+(-\beta))=\\&=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)=\\&=\sin\alpha\cos\beta-\cos\alpha\sin\beta.\end{aligned}$$

c.n.d.

1.6 Cosinus różnicy dwóch kątów

$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$

Dowód:

$$\begin{aligned}\cos(\alpha-\beta)&=\cos(\alpha+(-\beta))=\\&=\cos\alpha\cos(-\beta)-\sin\alpha\sin(-\beta)=\\&=\cos\alpha\cos\beta+\sin\alpha\sin\beta.\end{aligned}$$

c.n.d.

1.7 Tangens różnicy dwóch kątów

$\operatorname{tg}(\alpha-\beta)=\dfrac{\operatorname{tg}\alpha-\operatorname{tg}\beta}{1+\operatorname{tg}\alpha\operatorname{tg}\beta}$

Dowód:

$$\begin{aligned}\operatorname{tg}(\alpha-\beta)&=\operatorname{tg}(\alpha+(-\beta))=\\&=\frac{\operatorname{tg}\alpha+\operatorname{tg}(-\beta)}{1-\operatorname{tg}\alpha\operatorname{tg}(-\beta)}=\\&=\frac{\operatorname{tg}\alpha-\operatorname{tg}\beta}{1+\operatorname{tg}\alpha\operatorname{tg}\beta}.\end{aligned}$$

c.n.d.

1.8 Cotangens różnicy dwóch kątów

$\operatorname{ctg}(\alpha-\beta)=\dfrac{\operatorname{ctg}\alpha\operatorname{ctg}\beta+1}{-\operatorname{ctg}\alpha+\operatorname{ctg}\beta}$

Dowód:

$$\begin{aligned}\operatorname{ctg}(\alpha-\beta)&=\operatorname{ctg}(\alpha+(-\beta))=\\&=\frac{\operatorname{ctg}\alpha\operatorname{ctg}(-\beta)-1}{\operatorname{ctg}\alpha+\operatorname{ctg}(-\beta)}=\\&=\frac{-\operatorname{ctg}\alpha\operatorname{ctg}\beta-1}{\operatorname{ctg}\alpha-\operatorname{ctg}\beta}=\\&=\frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta+1}{-\operatorname{ctg}\alpha+\operatorname{ctg}\beta}.\end{aligned}$$

c.n.d.

1.9 Sinus dwukrotności kąta

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

Dowód:

$$\begin{aligned}\sin 2\alpha &= \sin(\alpha+\alpha) = \\ &= \sin\alpha\cos\alpha + \cos\alpha\sin\alpha = \\ &= 2\sin\alpha\cos\alpha.\end{aligned}$$

c.n.d.

1.10 Cosinus dwukrotności kąta

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

Dowód:

$$\begin{aligned}\cos 2\alpha &= \cos(\alpha+\alpha) = \\ &= \cos\alpha\cos\alpha - \sin\alpha\sin\alpha = \\ &= \cos^2\alpha - \sin^2\alpha.\end{aligned}$$

c.n.d.

1.11 Tangens dwukrotności kąta

$$\operatorname{tg} 2\alpha = \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$$

Dowód:

$$\begin{aligned}\operatorname{tg} 2\alpha &= \operatorname{tg}(\alpha+\alpha) = \\ &= \frac{\operatorname{tg}\alpha + \operatorname{tg}\alpha}{1-\operatorname{tg}\alpha\operatorname{tg}\alpha} = \\ &= \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}.\end{aligned}$$

c.n.d.

1.12 Cotangens dwukrotności kąta

$$\operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}$$

Dowód:

$$\begin{aligned}\operatorname{ctg} 2\alpha &= \operatorname{ctg}(\alpha+\alpha) = \\ &= \frac{\operatorname{ctg}\alpha\operatorname{ctg}\alpha - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\alpha} = \\ &= \frac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}.\end{aligned}$$

c.n.d.

1.13 Sinus trzykrotności kąta

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$

Dowód:

$$\begin{aligned}
\sin 3\alpha &= \sin(2\alpha + \alpha) = \\
&= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha = \\
&= (2\sin\alpha\cos\alpha)\cos\alpha + (\cos^2\alpha - \sin^2\alpha)\sin\alpha = \\
&= 2\sin\alpha\cos^2\alpha + \cos^2\alpha\sin\alpha - \sin^3\alpha = \\
&= \cos^2\alpha(3\sin\alpha) - \sin^3\alpha = \\
&= (1-\sin^2\alpha)(3\sin\alpha) - \sin^3\alpha = \\
&= 3\sin\alpha - 4\sin^3\alpha.
\end{aligned}$$

c.n.d.

1.14 Cosinus trzykrotności kąta

$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$

Dowód:

$$\begin{aligned}
\cos 3\alpha &= \cos(2\alpha + \alpha) = \\
&= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha = \\
&= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha = \\
&= (2\cos^2\alpha - 1)\cos\alpha - 2\sin^2\alpha\cos\alpha = \\
&= (2\cos^2\alpha - 1)\cos\alpha - 2(1-\cos^2\alpha)\cos\alpha = \\
&= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha = \\
&= 4\cos^3\alpha - 3\cos\alpha.
\end{aligned}$$

c.n.d.

1.15 Tangens trzykrotności kąta

$$\operatorname{tg} 3\alpha = \frac{3\operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1 - 3\operatorname{tg}^2\alpha}$$

Dowód:

$$\begin{aligned}
\operatorname{tg} 3\alpha &= \operatorname{tg}(2\alpha + \alpha) = \\
&= \frac{\operatorname{tg} 2\alpha + \operatorname{tg}\alpha}{1 - \operatorname{tg} 2\alpha \operatorname{tg}\alpha} = \\
&= \frac{\frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha} + \operatorname{tg}\alpha}{1 - \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}\operatorname{tg}\alpha} = \\
&= \frac{\frac{2\operatorname{tg}\alpha + \operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1-\operatorname{tg}^2\alpha}}{\frac{1-\operatorname{tg}^2\alpha - 2\operatorname{tg}\alpha\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}} = \\
&= \frac{3\operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1 - 3\operatorname{tg}^2\alpha}.
\end{aligned}$$

c.n.d.

1.16 Cotangens trzykrotności kąta

$$\operatorname{ctg} 3\alpha = \frac{\operatorname{ctg}^3 \alpha - 3 \operatorname{ctg} \alpha}{3 \operatorname{ctg}^2 \alpha - 1}$$

Dowód:

$$\begin{aligned}
\operatorname{ctg} 3\alpha &= \operatorname{ctg}(2\alpha + \alpha) = \\
&= \frac{\operatorname{ctg} 2\alpha \operatorname{ctg} \alpha - 1}{\operatorname{ctg} 2\alpha + \operatorname{ctg} \alpha} = \\
&= \frac{\frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha} \operatorname{ctg} \alpha - 1}{\frac{\operatorname{ctg}^2 \alpha - 1}{2 \operatorname{ctg} \alpha} + \operatorname{ctg} \alpha} = \\
&= \frac{\frac{(\operatorname{ctg}^2 \alpha - 1) \operatorname{ctg} \alpha - 2 \operatorname{ctg} \alpha}{2 \operatorname{ctg} \alpha}}{\frac{\operatorname{ctg}^2 \alpha - 1 + 2 \operatorname{ctg}^2 \alpha}{2 \operatorname{ctg} \alpha}} = \\
&= \frac{(\operatorname{ctg}^2 \alpha - 1) \operatorname{ctg} \alpha - 2 \operatorname{ctg} \alpha}{\operatorname{ctg}^2 \alpha - 1 + 2 \operatorname{ctg}^2 \alpha} = \\
&= \frac{\operatorname{ctg}^3 \alpha - \operatorname{ctg} \alpha - 2 \operatorname{ctg} \alpha}{3 \operatorname{ctg}^2 \alpha - 1} = \\
&= \frac{3 \operatorname{tg} \alpha - \operatorname{tg}^3 \alpha}{1 - 3 \operatorname{tg}^2 \alpha}.
\end{aligned}$$

c.n.d.

1.17 Sinus kąta połówkowego

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}$$

Dowód:

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha$$

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos 2\alpha}{2}}$$

Zatem

$$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}}.$$

c.n.d.

1.18 Cosinus kąta połówkowego

$$\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}}$$

Dowód:

$$\cos 2\alpha = 2 \cos^2 \alpha - 1$$

$$\cos \alpha = \pm \sqrt{\frac{1 + \cos 2\alpha}{2}}$$

Zatem

$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

c.n.d.

1.19 Tangens kąta połówkowego

$$\operatorname{tg}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

Dowód:

$$\begin{aligned}\operatorname{tg}\frac{\alpha}{2} &= \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \\ &= \frac{\pm\sqrt{\frac{1-\cos\alpha}{2}}}{\pm\sqrt{\frac{1+\cos\alpha}{2}}} = \\ &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.\end{aligned}$$

c.n.d.

1.20 Cotangens kąta połówkowego

$$\operatorname{ctg}\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

Dowód:

$$\begin{aligned}\operatorname{ctg}\frac{\alpha}{2} &= \frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}} = \\ &= \frac{\pm\sqrt{\frac{1+\cos\alpha}{2}}}{\pm\sqrt{\frac{1-\cos\alpha}{2}}} = \\ &= \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}.\end{aligned}$$

c.n.d.

2 Podsumowanie

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\operatorname{tg}(\alpha+\beta) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}$$
$$\operatorname{ctg}(\alpha+\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta - 1}{\operatorname{ctg}\alpha + \operatorname{ctg}\beta}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\operatorname{tg}(\alpha-\beta) = \frac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{1 + \operatorname{tg}\alpha\operatorname{tg}\beta}$$
$$\operatorname{ctg}(\alpha-\beta) = \frac{\operatorname{ctg}\alpha\operatorname{ctg}\beta + 1}{-\operatorname{ctg}\alpha + \operatorname{ctg}\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$
$$\operatorname{tg} 2\alpha = \frac{2\operatorname{tg}\alpha}{1 - \operatorname{tg}^2\alpha}$$
$$\operatorname{ctg} 2\alpha = \frac{\operatorname{ctg}^2\alpha - 1}{2\operatorname{ctg}\alpha}$$

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$$
$$\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$$
$$\operatorname{tg} 3\alpha = \frac{3\operatorname{tg}\alpha - \operatorname{tg}^3\alpha}{1 - 3\operatorname{tg}^2\alpha}$$
$$\operatorname{ctg} 3\alpha = \frac{\operatorname{ctg}^3\alpha - 3\operatorname{ctg}\alpha}{3\operatorname{ctg}^2\alpha - 1}$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$
$$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$
$$\operatorname{tg}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$
$$\operatorname{ctg}\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$