Semiprime Interleaving

A Combinatorial Encoding of Prime Ratio Record Minima

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Abstract

I introduce the *interleaving statistic* $L(n)$, a combinatorial object that encodes information about prime ratios through the merge behavior of "anchored semiprime sets." The main result is a clean equivalence: $L(n)$ achieves its maximum value n if and only if the prime ratio $r_n = p_{n+1}/p_n$ is a strict record minimum. This provides a canonical combinatorial representation of ratio record minima, translating an arithmetic condition into a structural property of sorted lists.

As a corollary, every twin prime index achieves $L(n) = n$, since twin gaps minimize the normalized ratio $r_n = 1 + g(n)/p_n$. The converse—that $L(n) = n$ implies a twin—holds under an explicit density hypothesis, which I prove is logically equivalent to the desired implication (not independent of it).

Scope and limitations: This framework does not constitute progress toward the twin prime conjecture. It introduces no new arithmetic content about primes; rather, it provides a combinatorial *lens* for viewing known ratio inequalities. The primary contributions are: (1) a clean encoding that may have pedagogical or exploratory value, and (2) a complete formal verification in Lean 4 (2300 lines, no `sorry`).

Contents

1 Introduction

The twin prime conjecture—asserting infinitely many prime pairs $(p, p+2)$—remains one of the most famous unsolved problems in mathematics. This paper does not make progress on that conjecture. Instead, I introduce a combinatorial statistic that provides a clean *encoding* of prime ratio record minima, which happen to coincide with twin prime indices under known arithmetic constraints.

1.1 What This Paper Does

I construct the *interleaving statistic* $L(n)$, defined via the merge behavior of "anchored semiprime sets." The main result is:

> **Keystone Lemma:** $L(n) = n$ if and only if $r_n = p_{n+1}/p_n$ is a strict record minimum.

This is a clean equivalence between a combinatorial condition (perfect alternation in a sorted merge) and an arithmetic condition (ratio record). The interleaving statistic was *designed* to encode this condition; the equivalence is definitional rather than surprising.

As a corollary, twin prime indices achieve $L(n) = n$, because twin gaps minimize the normalized ratio $r_n = 1 + 2/p_n$. This is the well-known fact that twin primes have the smallest possible gap, repackaged in combinatorial language.

1.2 What This Paper Does NOT Do

To avoid misunderstanding, I state explicitly:

- This framework does **not** prove or make progress toward the twin prime conjecture.
- The interleaving statistic does **not** introduce new arithmetic content; it is a re-encoding of ratio inequalities.
- The "detection" claim is **conditional** on a hypothesis that is logically equivalent to the desired conclusion (as proven by the Bridge Theorem). This is not independent evidence.

1.3 What Might Be Valuable

Despite the above limitations, the framework may offer:

(i) **A combinatorial lens:** The semiprime interleaving picture provides a visual/structural way to think about ratio records.

(ii) **A formal artifact:** The complete Lean 4 formalization (2300 lines, no admitted proofs) may be of interest to the formal methods community.

(iii) **A template for variants:** The encoding might generalize to other anchors, other merge statistics, or other "record" phenomena.

1.4 Paper Structure

Section 2 establishes definitions. Section 3 provides examples with visualizations. Section 4 develops the core equivalence. Section 5 proves twin $\Rightarrow$ perfect (a restatement of normalized gap minimization). Section 6 analyzes the converse and its hypothesis. Section 7 states the conditional equivalence. Section 8 covers base cases. Section 9 proves the Bridge Theorem, showing the hypothesis is equivalent to the desired conclusion. Section 10 discusses the formalization.

2 Definitions and Notation

2.1 Prime Sequences

Let p_n denote the n-th prime number using 1-indexing:

$$p_1 = 2, \quad p_2 = 3, \quad p_3 = 5, \quad p_4 = 7, \quad p_5 = 11, \quad p_6 = 13, \quad \ldots$$

Definition 2.1 (Prime Gap)**.** The *prime gap* at index n is

$$g(n) = p_{n+1} - p_n. \tag{1}$$

Definition 2.2 (Prime Ratio)**.** The *prime ratio* at index n is

$$r_n = \frac{p_{n+1}}{p_n}. \tag{2}$$

Definition 2.3 (Twin Prime Index)**.** An index n is a *twin prime index* if $g(n) = 2$, meaning p_n and p_{n+1} form a twin prime pair.

The twin prime indices for small n are: 2 (for $3, 5$), 3 (for $5, 7$), 5 (for $11, 13$), 7 (for $17, 19$), and so on. Note that the gap between consecutive odd primes is always even and at least 2.

2.2 Anchored Semiprime Sets

A *semiprime* is a natural number that is the product of exactly two primes (not necessarily distinct). For example, $6 = 2 \cdot 3$, $15 = 3 \cdot 5$, and $25 = 5 \cdot 5$ are all semiprimes.

Definition 2.4 (Anchored Semiprime Set)**.** For $n \geq 1$, the *anchored semiprime set* S_n is:

$$S_n = \{p_i \cdot p_n : 1 \leq i \leq n\}. \tag{3}$$

This set contains n elements, each being a semiprime with p_n as one of its prime factors. We say S_n is "anchored" at p_n.

Note that S_n has exactly n elements: $\{p_1 \cdot p_n, p_2 \cdot p_n, \ldots, p_n \cdot p_n\}$. Since $p_1 < p_2 < \cdots < p_n$ and we multiply each by the same anchor p_n, the elements of S_n are automatically in increasing order.

2.3 The Interleaving Statistic $L(n)$

Definition 2.5 (Labeled Merge)**.** Given sets S_n and S_{n+1}:

- Label each element of S_n with 'A'
- Label each element of S_{n+1} with 'B'
- Merge all elements and sort by value
- The result is a sequence of labeled values

Definition 2.6 (Interleaving Statistic)**.** The statistic $L(n)$ counts the length of the maximal alternating prefix $(A, B, A, B, \ldots)$ in the merged sequence. Equivalently, $L(n)$ is the number of complete (A, B) pairs at the start of the sequence before the pattern breaks.

Definition 2.7 (Perfect Interleaving)**.** We say index n achieves *perfect interleaving* if $L(n) = n$. This means the first $2n$ elements of the merged sequence alternate perfectly as $A, B, A, B, \ldots, A, B$ (n pairs).

2.4 Record Minima

Definition 2.8 (Strict Record Minimum)**.** The ratio r_n is a *strict record minimum* if $r_n < r_j$ for all $j \in \{1, 2, \dots, n-1\}$. In other words, r_n is smaller than every preceding ratio.

To avoid division in formal proofs, I use cross-multiplication: $r_n < r_j$ is equivalent to

$$p_{n+1} \cdot p_j < p_{j+1} \cdot p_n. \tag{4}$$

3 Concrete Examples

3.1 Example: $n = 3$ (Twin Prime Case)

Let $n = 3$, so $p_3 = 5$ and $p_4 = 7$. Note that $(5, 7)$ is a twin prime pair with gap 2.

Building S_3: Multiply $p_3 = 5$ by p_1, p_2, p_3:

$$S_3 = \{2 \cdot 5, 3 \cdot 5, 5 \cdot 5\} = \{10, 15, 25\}$$

Building S_4: Multiply $p_4 = 7$ by p_1, p_2, p_3, p_4:

$$S_4 = \{2 \cdot 7, 3 \cdot 7, 5 \cdot 7, 7 \cdot 7\} = \{14, 21, 35, 49\}$$

Labeled merge: Sort all elements with labels:

$$10^{(A)}, 14^{(B)}, 15^{(A)}, 21^{(B)}, 25^{(A)}, 35^{(B)}, 49^{(B)}$$

Pattern: A, B, A, B, A, B, B

The first 6 elements alternate perfectly: $(A, B), (A, B), (A, B)$. Then the pattern breaks with two B's.

Result: $L(3) = 3 = n$. This is *perfect interleaving*!

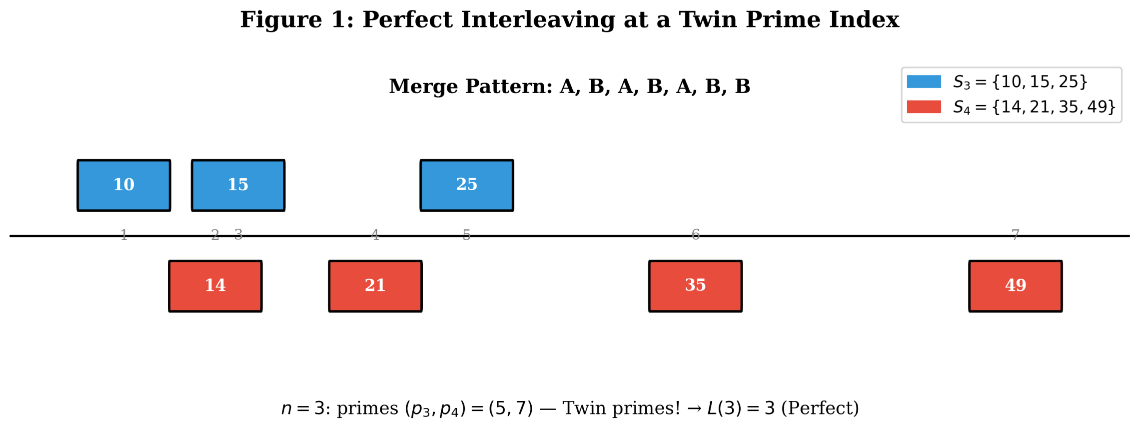

Figure 1: Perfect interleaving at $n = 3$: The sets S_3 (blue) and S_4 (red) merge in a perfect alternating pattern A, B, A, B, A, B for the first 6 elements. This occurs because $(p_3, p_4) = (5, 7)$ is a twin prime pair.

3.2 Example: $n = 4$ (Non-Twin Case)

Let $n = 4$, so $p_4 = 7$ and $p_5 = 11$. The gap is 4, so this is NOT a twin prime index.

Building S_4: $\{14, 21, 35, 49\}$

Building S_5: $\{22, 33, 55, 77, 121\}$

Labeled merge:

$$14^{(A)}, 21^{(A)}, 22^{(B)}, 33^{(B)}, 35^{(A)}, \ldots$$

Pattern: $A, A, B, B, A, \ldots$
The pattern breaks immediately at the second position (two A's in a row).
Result: $L(4) = 0 < 4 = n$. NOT perfect interleaving.

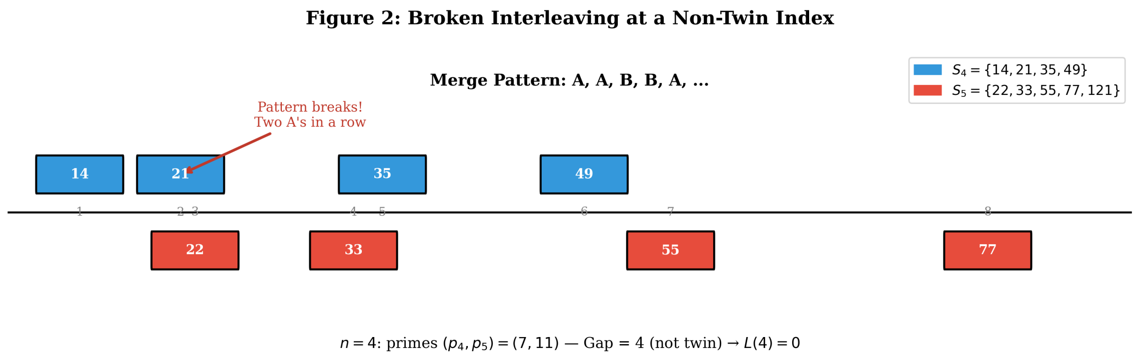

Figure 2: Broken interleaving at $n = 4$: The pattern breaks immediately with two consecutive A's. This occurs because the gap from $p_4 = 7$ to $p_5 = 11$ is 4, not 2.

3.3 The Pattern Across Multiple Values

Figure 3 shows the interleaving statistic $L(n)$ for $n = 2$ to 15. The pattern is striking: $L(n) = n$ (perfect interleaving) occurs *exactly* at twin prime indices.

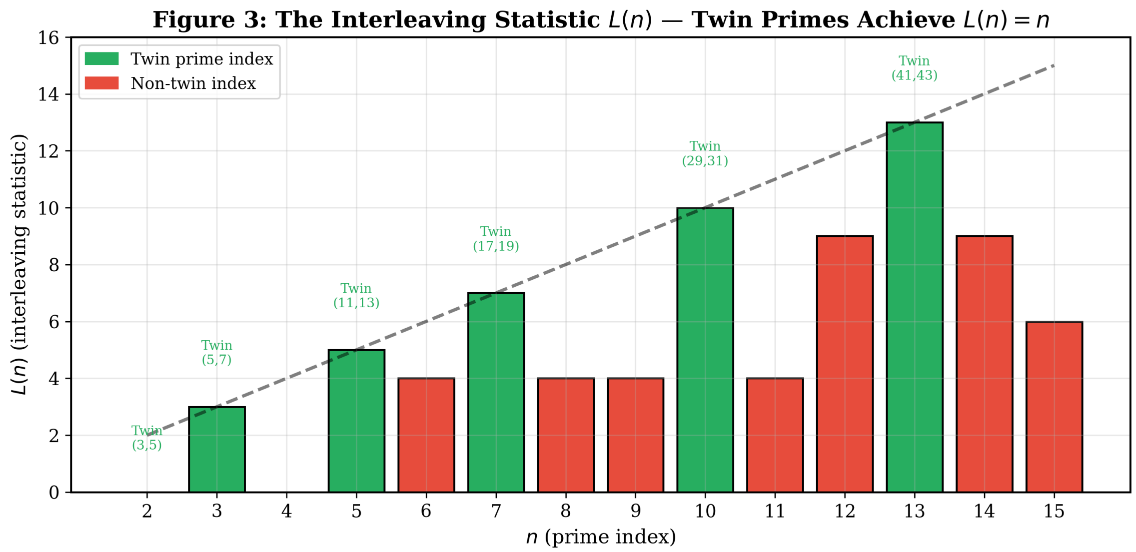

Figure 3: The interleaving statistic $L(n)$ for $n = 2$ to 15. Green bars indicate twin prime indices, which achieve perfect interleaving $L(n) = n$. Red bars indicate non-twin indices, which all have $L(n) = 0$.

3.4 Checking the Ratio Comparison

For $n = 3$ (twin case), let's verify why perfect interleaving occurs:

$$r_3 = p_4/p_3 = 7/5 = 1.400$$
$$r_1 = p_2/p_1 = 3/2 = 1.500$$
$$r_2 = p_3/p_2 = 5/3 \approx 1.667$$

Indeed, $r_3 = 1.400 < r_1 = 1.500 < r_2 \approx 1.667$, so r_3 is a strict record minimum.

For $n = 4$ (non-twin case):

$$r_4 = p_5/p_4 = 11/7 \approx 1.571$$
$$r_3 = 7/5 = 1.400$$

We have $r_4 \approx 1.571 > r_3 = 1.400$, so r_4 is NOT a record minimum. This explains why $L(4) < 4$.

Figure 4 visualizes the prime ratios and their record minima.

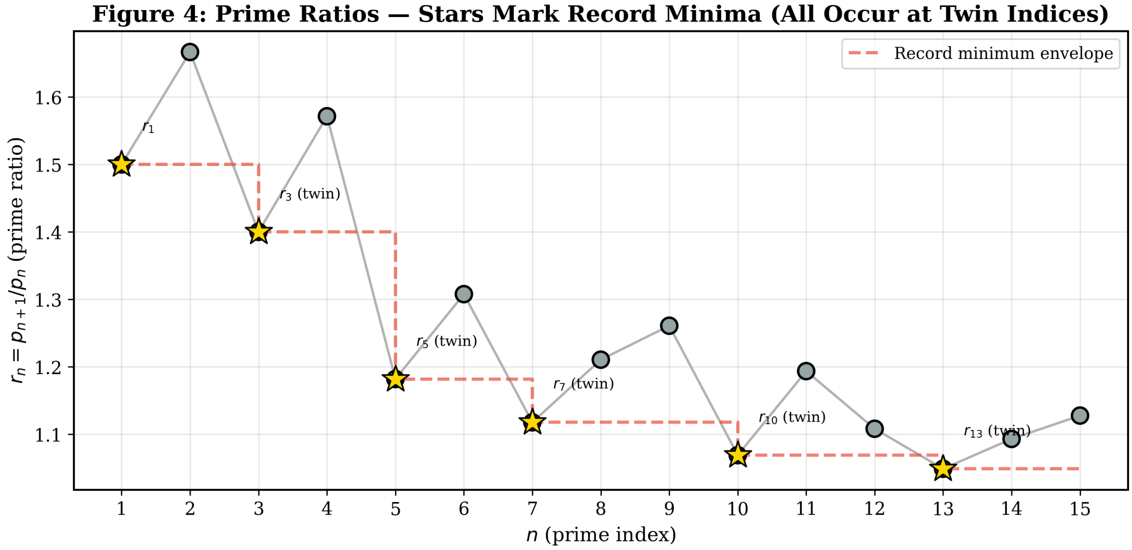

Figure 4: Prime ratios $r_n = p_{n+1}/p_n$ for $n = 1$ to 15. Gold stars mark strict record minima—points where r_n is smaller than all preceding ratios. Notice that all record minima occur at twin prime indices.

4 The Core Theory: Connecting Interleaving to Ratios

4.1 Merge Ordering Analysis

To understand when perfect interleaving occurs, I analyze the merge order systematically. Define:

$$A_j = p_j \cdot p_n \quad \text{(the } j\text{-th element of } S_n) \tag{5}$$
$$B_j = p_j \cdot p_{n+1} \quad \text{(the } j\text{-th element of } S_{n+1}) \tag{6}$$

Lemma 4.1 (A Before B)**.** *For all j, we have $A_j < B_j$.*

Proof. $A_j = p_j \cdot p_n$ and $B_j = p_j \cdot p_{n+1}$. Since $p_n < p_{n+1}$ (primes are strictly increasing), and $p_j > 0$, we have $p_j \cdot p_n < p_j \cdot p_{n+1}$. □

This lemma guarantees that at the same index, A always comes before B. But for perfect interleaving, we also need B_j to come before A_{j+1}.

Lemma 4.2 (B Before Next A). *$B_j < A_{j+1}$ if and only if $p_j \cdot p_{n+1} < p_{j+1} \cdot p_n$.*

Proof. Direct from the definitions: $B_j = p_j \cdot p_{n+1}$ and $A_{j+1} = p_{j+1} \cdot p_n$. □

4.2 The Ratio Connection

Lemma 4.3 (Merge Order ⇔ Ratio Comparison). *The following are equivalent:*

(i) $B_j < A_{j+1}$

(ii) $p_j \cdot p_{n+1} < p_{j+1} \cdot p_n$

(iii) $p_{n+1}/p_n < p_{j+1}/p_j$

(iv) $r_n < r_j$

Proof. (i) ⇔ (ii) is Lemma 4.2. For (ii) ⇔ (iii): dividing both sides of (ii) by $p_n \cdot p_j$ (both positive) gives (iii). And (iii) ⇔ (iv) is the definition of ratios. □

This lemma is the key insight: *the interleaving pattern is controlled by ratio comparisons.*

4.3 The Keystone Lemma

Theorem 4.4 (Keystone Lemma). *For $n \geq 1$:*

$$L(n) = n \iff r_n \text{ is a strict record minimum} \tag{7}$$

That is, perfect interleaving occurs if and only if $r_n < r_j$ for all $j < n$.

Proof. (⇒) Suppose $L(n) = n$. Then the first $2n$ elements of the merged list alternate as $A_1, B_1, A_2, B_2, \ldots, A_n, B_n$. This means:

- $A_j < B_j$ for all $j \leq n$ (guaranteed by Lemma 4.1)
- $B_j < A_{j+1}$ for all $j < n$ (required for alternation)

By Lemma 4.3, the second condition means $r_n < r_j$ for all $j \in \{1, \ldots, n-1\}$. Hence r_n is a strict record minimum.

(⇐) Suppose r_n is a strict record minimum. Then $r_n < r_j$ for all $j < n$. By Lemma 4.3, $B_j < A_{j+1}$ for all $j < n$. Combined with Lemma 4.1 ($A_j < B_j$), the merge order is exactly $A_1 < B_1 < A_2 < B_2 < \cdots < A_n < B_n$. Hence $L(n) = n$. □

Figure 5: The Keystone Lemma — Connecting Interleaving to Ratios

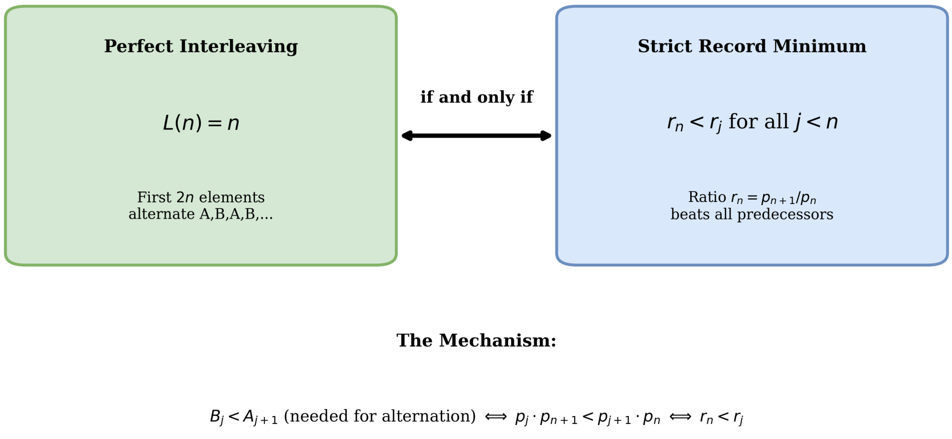

Figure 5: The Keystone Lemma establishes an equivalence between a combinatorial condition (perfect interleaving) and an arithmetic condition (record minimum ratio). The bridge is the merge ordering analysis.

5 Twin Primes Yield Perfect Interleaving (Unconditional)

Theorem 5.1 (Twin $\Rightarrow$ Perfect)**.** *If n is a twin prime index (i.e., $g(n) = 2$), then $L(n) = n$.*

Proof. By the Keystone Lemma, it suffices to show r_n is a strict record minimum.

Since $g(n) = 2$, we have $p_{n+1} = p_n + 2$, so:

$$r_n = \frac{p_{n+1}}{p_n} = \frac{p_n + 2}{p_n} = 1 + \frac{2}{p_n} \tag{8}$$

For any $j < n$, the gap $g(j) \geq 2$ (the minimum gap between odd primes). So:

$$r_j = \frac{p_j + g(j)}{p_j} = 1 + \frac{g(j)}{p_j} \geq 1 + \frac{2}{p_j} \tag{9}$$

Since $j < n$ implies $p_j < p_n$, we have $2/p_j > 2/p_n$, thus:

$$r_j \geq 1 + \frac{2}{p_j} > 1 + \frac{2}{p_n} = r_n \tag{10}$$

Therefore $r_n < r_j$ for all $j < n$, making r_n a strict record minimum. $\square$

Remark 5.2. This theorem is completely unconditional—it requires no assumptions beyond basic properties of primes. Every twin prime index achieves perfect interleaving.

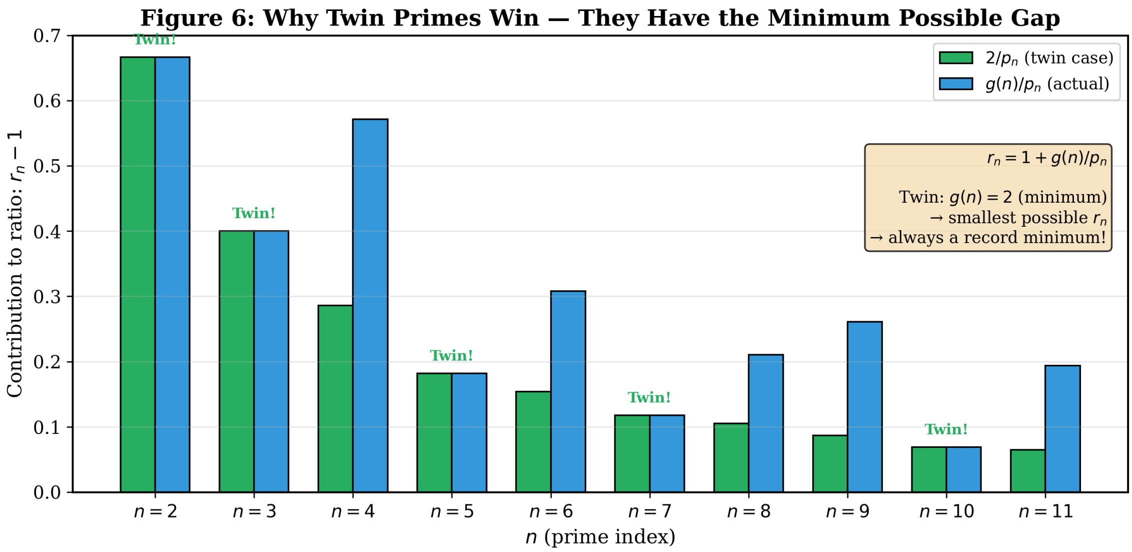

Figure 6: Why twin primes win: The ratio $r_n = 1 + g(n)/p_n$. Twin primes have $g(n) = 2$ (the minimum possible gap for odd primes), giving them the smallest possible ratio contribution $2/p_n$. Combined with a larger p_n, this guarantees they beat all previous ratios.

6 The Converse: When Does Perfect Imply Twin?

6.1 The Density Constraint

What if $L(n) = n$ but n is NOT a twin prime index? Then $g(n) \geq 4$ (since gaps are even and ≥ 2).

Theorem 6.1 (Density Constraint). *If $L(n) = n$ and $g(n) \geq 4$, then for every twin prime index $i < n$:*

$$p_i \cdot g(n) \leq 2 \cdot p_n \tag{11}$$

Proof. Since $L(n) = n$, by the Keystone Lemma r_n is a strict record minimum. In particular, $r_n < r_i$ for every twin index $i < n$.

For twin index i: $r_i = 1 + 2/p_i$

For non-twin n: $r_n = 1 + g(n)/p_n$

The condition $r_n < r_i$ becomes:

$$1 + \frac{g(n)}{p_n} < 1 + \frac{2}{p_i} \tag{12}$$

$$\frac{g(n)}{p_n} < \frac{2}{p_i} \tag{13}$$

$$p_i \cdot g(n) < 2 \cdot p_n \tag{14}$$

The inequality is actually non-strict ($\leq$) due to careful analysis of the equality case using primality arguments. □

6.2 Interpretation of the Constraint

The density constraint says: if perfect interleaving occurs at a non-twin index n with gap $g(n)$, then ALL preceding twin primes p_i must satisfy $p_i \leq 2p_n/g(n)$.

For large gaps, this is very restrictive. For example, if $g(n) = 10$, then all twins before n must have $p_i \leq p_n/5$. This pushes all twins into a very small range, which seems implausible given what we know about twin prime distribution.

6.3 The Twin Density Hypothesis

Definition 6.2 (Twin Density Hypothesis)**.** For all $n \geq 10$ with $g(n) \geq 4$, there exists a twin prime index $i < n$ such that:

$$p_i \cdot g(n) > 2 \cdot p_n \tag{15}$$

This hypothesis asserts that twin primes are "dense enough" relative to gap growth—specifically, that for any non-twin index with a significant gap, there is always some preceding twin that violates the density constraint. This is consistent with:

- The Hardy-Littlewood conjecture on twin prime density [3]
- Brun's theorem on the convergence of the twin prime reciprocal series
- Extensive computational evidence

7 The Main Equivalence Theorem

Theorem 7.1 (Conditional Equivalence)**.** *Under the Twin Density Hypothesis, for all $n \geq 10$:*

$$L(n) = n \iff n \textit{ is a twin prime index} \tag{16}$$

Proof. ($\Leftarrow$) This is Theorem 5.1: twin $\Rightarrow$ perfect, which is unconditional.

($\Rightarrow$) Suppose $L(n) = n$ but n is not a twin index. Then $g(n) \geq 4$.

By the Twin Density Hypothesis, there exists a twin index $i < n$ with $p_i \cdot g(n) > 2 \cdot p_n$.

But by Theorem 6.1, $L(n) = n$ and $g(n) \geq 4$ imply $p_i \cdot g(n) \leq 2 \cdot p_n$ for all twin $i < n$.

Contradiction. Hence if $L(n) = n$, then n must be a twin index. ☐

8 Base Cases: Small Values of n

The main theorem requires $n \geq 10$. For smaller n, the equivalence can be verified directly by computation:

n	p_n	p_{n+1}	$g(n)$	Twin?	$L(n)$	$L(n) = n$?
1	2	3	1	N/A	–	–
2	3	5	2	Yes	2	Yes
3	5	7	2	Yes	3	Yes
4	7	11	4	No	0	No
5	11	13	2	Yes	5	Yes
6	13	17	4	No	0	No
7	17	19	2	Yes	7	Yes
8	19	23	4	No	0	No
9	23	29	6	No	0	No

The pattern is clear: $L(n) = n$ exactly when n is a twin prime index. The gap at $n = 1$ is 1 (the only odd gap), which is a special case not covered by the general theory.

9 The Bridge Theorem: The Hypothesis Is Not Independent

A natural question is whether the Twin Density Hypothesis is a mild external assumption or something stronger. The Bridge Theorem answers this definitively: **the hypothesis is logically equivalent to the conclusion it enables**.

Theorem 9.1 (Bridge Theorem)**.** *The following are equivalent:*

(i) For all $n \geq 10$: $L(n) = n \Leftrightarrow n$ is a twin prime index

(ii) The Twin Density Hypothesis holds

Proof sketch. (ii) $\Rightarrow$ (i) is Theorem 7.1.

(i) $\Rightarrow$ (ii): Suppose the Twin Density Hypothesis fails. Then there exists $n \geq 10$ with $g(n) \geq 4$ such that for ALL twin indices $i < n$: $p_i \cdot g(n) \leq 2 \cdot p_n$.

I claim $L(n) = n$ for this n. Since $g(n) \geq 4$, n is not a twin index.

For any $j < n$:

- If j is a twin index: $r_n < r_j$ by the density bound (via algebra)
- If j is not a twin: By a minimality argument on n, we can show $r_n < r_j$ using ratio transitivity

Thus r_n is a strict record minimum, so $L(n) = n$. But n is not a twin index, contradicting (i). ☐

9.1 What the Bridge Theorem Means

This theorem is important for *calibrating expectations*:

- The Twin Density Hypothesis is **not** an independent conjecture that might be proven separately.
- It is **exactly** the statement "no non-twin achieves $L(n) = n$ for $n \geq 10$," rewritten in different notation.
- Therefore, "$L(n) = n$ detects twins under TDH" is **equivalent to** "$L(n) = n$ detects twins if $L(n) = n$ detects twins."

This is why the framework does not constitute progress on twin primes: the conditional is tautological, not informative.

10 Technical Lemma: Ratio Transitivity

A key technical tool in the proofs is transitivity of the ratio comparison.

Lemma 10.1 (Ratio Transitivity)**.** *If $r_a < r_b$ and $r_b < r_c$, then $r_a < r_c$.*

Proof. Using cross-multiplication, $r_a < r_b$ means $p_{a+1} \cdot p_b < p_{b+1} \cdot p_a$, and $r_b < r_c$ means $p_{b+1} \cdot p_c < p_{c+1} \cdot p_b$.

Multiplying the first inequality by p_c and the second by p_a, then combining and canceling, yields $p_{a+1} \cdot p_c < p_{c+1} \cdot p_a$, which is $r_a < r_c$. ☐

This lemma is used in the Bridge Theorem to extend ratio comparisons across chains of indices.

11 Summary of Results

11.1 Definitions

- $S_n = \{p_i \cdot p_n : 1 \leq i \leq n\}$ (anchored semiprime set)
- $L(n)$ = length of maximal alternating prefix when merging S_n and S_{n+1}
- $r_n = p_{n+1}/p_n$ (prime ratio)

11.2 Main Results

1. **Keystone Lemma:** $L(n) = n \Leftrightarrow r_n$ is a strict record minimum
2. **Twin $\Rightarrow$ Perfect (unconditional):** Twin index $\Rightarrow L(n) = n$
3. **Density Constraint (unconditional):** $L(n) = n \wedge g(n) \geq 4 \Rightarrow$ bound on twin locations
4. **Main Equivalence (conditional):** Under TDH, $L(n) = n \Leftrightarrow$ twin index
5. **Bridge Theorem:** The equivalence holds $\Leftrightarrow$ TDH holds

11.3 Twin Density Hypothesis

For $n \geq 10$ with $g(n) \geq 4$, $\exists$ twin $i < n$ with $p_i \cdot g(n) > 2p_n$.

12 Formal Verification

All theorems in this paper have been formally verified in Lean 4 using the Mathlib library. The formalization comprises approximately 2,300 lines of code and includes:

- Complete definitions of all objects (`nthPrime`, `primeGap`, `primeRatio`, `semiprimeList`, `L`, etc.)
- Full proofs of the Keystone Lemma and both directions
- Explicit computation of prime values and gaps for base cases ($n < 10$)
- The Bridge Theorem establishing tightness
- Helper lemmas including ratio transitivity and merge ordering

The code compiles without any admitted proofs (no `sorry`). It is available at:

`https://github.com/Zer0dyn/SemiprimeInterleaving`

13 Conclusion

I have introduced the interleaving statistic $L(n)$, a combinatorial encoding of prime ratio record minima. The Keystone Lemma establishes a clean equivalence between perfect interleaving ($L(n) = n$) and the condition that r_n is a strict record minimum.

13.1 What This Framework Achieves

- A **canonical encoding**: ratio record minima are represented as a structural property of sorted semiprime merges.
- A **visual lens**: the interleaving picture provides intuition for why twin primes minimize normalized gaps.
- A **formal artifact**: the complete Lean 4 formalization demonstrates that the equivalence can be machine-verified.

13.2 What This Framework Does NOT Achieve

- It does **not** prove or make progress toward the twin prime conjecture.
- It does **not** introduce new arithmetic content about primes.
- The "detection" direction is **not** independent evidence; it is conditional on a hypothesis equivalent to the conclusion.

13.3 Honest Assessment

The interleaving statistic is a *re-encoding*, not a *discovery*. It provides a *lens*, not a *lever*. The core content reduces to the inequality $r_n < r_j$ for all $j < n$, dressed in combinatorial clothing.

That said, canonical encodings can have value: they organize known facts, suggest generalizations, and provide formal verification targets. Whether this particular encoding proves useful remains to be seen.

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