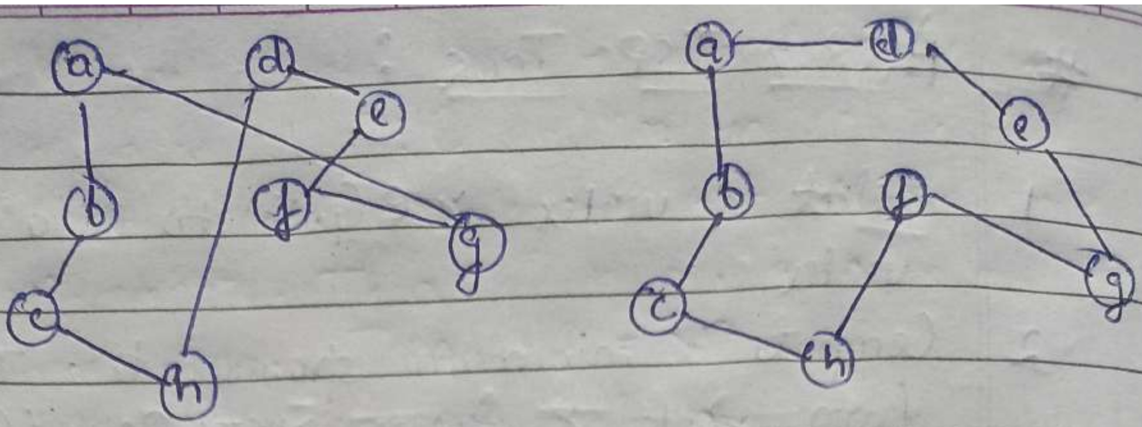


Approx TSP tour $\longrightarrow$ optimal tour

about

23% shorter

Thm- When the triangle inequality holds, APPROX TSP-Tour is a polynomial time 2-approximation algorithm (i.e, cost of Approx-TSP tour is at most twice the cost of optimal tour) for the TSP

Proof- Let H^* denote optimal tour for given set of vertices

& T be the minimum spanning tree

we know as

$$n(E_{H^*}) > n(E_T)$$

so

$$c(T) \leq c(H^*)$$

Let W be the full pre-order walk of T

W = a b c b h b a d e f e g e d a

(In prev. example (above))

$$c(W) = c(T) \times 2$$

$$c(W) \leq 2c(H^*)$$

let H be the cycle corresponding to this preorder walk

$$c(H) \leq c(W)$$

$$\therefore \quad c(H) \leq 2c(H^*) \quad \text{HP}$$

The General traveling-Salesman problem-
(when cost f^n c does not satisfy Δ inequality)

We will prove,

If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP.

Proof. Proving by contradiction -

Suppose for some $\rho \geq 1$, $\exists$ a polynomial-time approx. algo. A with approximation ratio ρ

Assuming ρ to be an integer (by rounding it up if necessary)

We will show how to use A to solve instances of HC problem in polynomial time.. Since we know that HC problem is NP-complete. Then if it has a polynomial time algo.., Then $P = NP$.
which is a contradiction.

<u>Now</u>,
Let $G = (V, E)$ be an instance for HCP

We will now show whether G contains a HC by making use of hypothesized approximation algo.

Let $G' = (V, E')$ be the complete graph on V
$E' = \{(u,v) : u,v \in V \;\&\; u \neq v\}$
Assigning cost f^n

$$c(u,v) = \begin{cases} 1 & , \text{if } (u,v) \in E \\ \rho(|V|)+1 & , \text{otherwise} \end{cases}$$

it will take polynomial time in $|V|$ & $|E|$ to create G' & c

Now consider $TSP(G', c)$.

If the graph G has a hamiltonian cycle H then cost f^n assigns to each edge of H a cost of 1 & so, cost of tour $= |V|$.

But if not, then any tour of G' must use some edge not in E & any such tour has a cost of atleast

$$(\rho|V|+1) + (|V|-1) = \rho|V| + |V| > \rho(|V|)$$

As edges not in G are so costly, there is a gap of atleast $\rho(|V|)$ b/w the cost of tour that is a HC in G (cost $= |V|$) & the cost of any other tour (cost $\geq \rho(|V|) + |V|$)

Therefore,

$$\frac{\text{cost of tour that is not HC}}{\text{cost of tour that is HC}} \geq \rho + 1$$

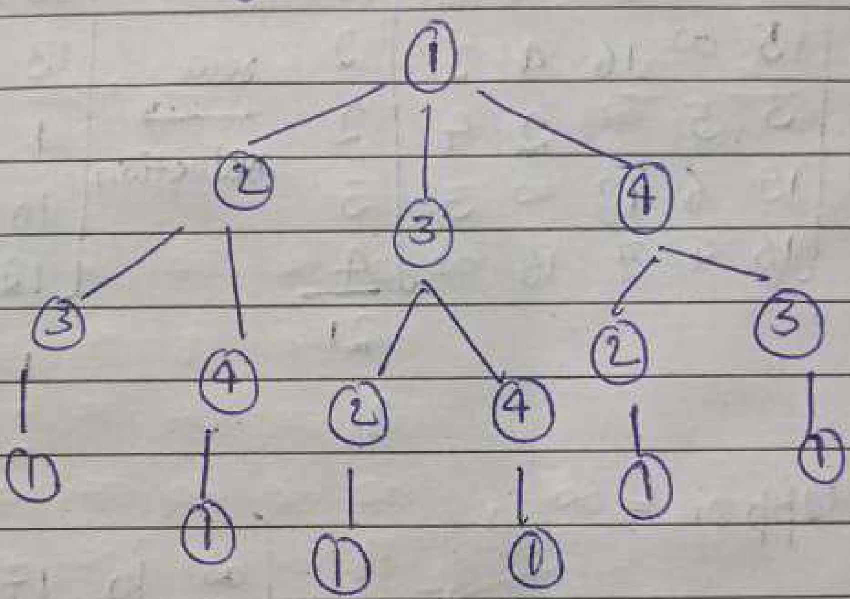
Travelling Salesman Problem :

① **Dynamic Programming Approach - (Back-Tracking)**

$$g(v_1, \{v_2, v_3, v_4 \cdots v_n\}) = \min_{v_k \in V} \left\{ C_{1k} + g\left(v_k, \{v_2, v_3 \cdots v_n\} - \{v_k\}\right) \right\}$$

for, $G = (V, E)$

General way : $g(i, S) = \min_{k \in S} \left\{ C_{ik} + g(k, S - \{k\}) \right\}$

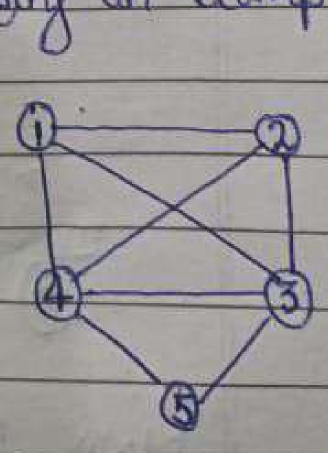


② **Branch & Bound Approach -**

Let us ~~see~~ see this approach using an example -

The following adjacency matrix is :

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 14 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix}
\end{array} \longrightarrow (i)$$



Step 1 : Reduction → select the smallest element & then subtract it from each element of that particular row / column

Perform row reduction & column reduction on (i) we get

$$\begin{array}{c|c}
\begin{bmatrix} \infty & 20 & 30 & 10 & 11 \\ 15 & \infty & 16 & 4 & 2 \\ 3 & 5 & \infty & 2 & 4 \\ 19 & 6 & 18 & \infty & 3 \\ 16 & 4 & 7 & 16 & \infty \end{bmatrix} & \begin{matrix} 10 \\ 2 \\ 2 \\ 3 \\ \underline{4} \\ 21 \end{matrix}
\end{array}
\xrightarrow[\text{reduction}]{\text{row}}
\begin{array}{c}
\begin{bmatrix} \infty & 10 & 20 & 0 & 1 \\ 13 & \infty & 14 & 2 & 0 \\ 1 & 3 & \infty & 0 & 2 \\ 16 & 3 & 15 & \infty & 0 \\ 12 & 0 & 3 & 12 & \infty \end{bmatrix} \\ \hline
\begin{matrix} 1 & 0 & 3 & 0 & 0 \end{matrix}
\end{array}$$

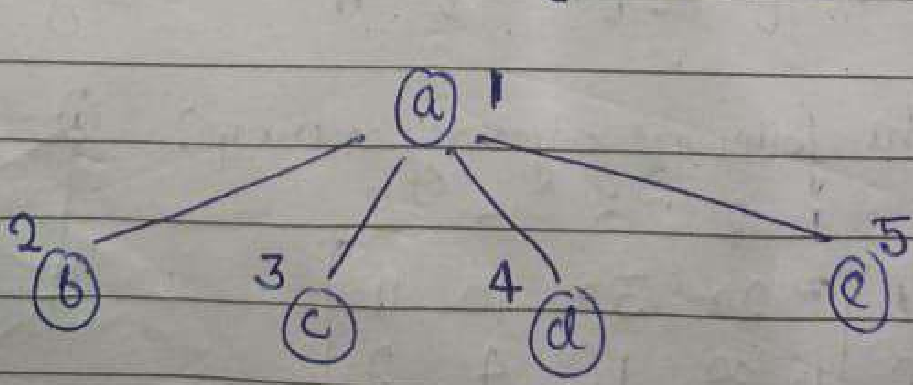
$\downarrow$ column reduction

Bound : Upper $= \infty$

$$\begin{bmatrix} \infty & 10 & 17 & 0 & 1 \\ 12 & \infty & 11 & 2 & 0 \\ 0 & 3 & \infty & 0 & 2 \\ 15 & 3 & 12 & \infty & 0 \\ 11 & 0 & 0 & 12 & \infty \end{bmatrix}$$

Cost at (a) → (a) —

$C(a) = 21 + 4 = 25$



Now for 1 → 2 reduce (a) ← matrix w.r.t 1
respectively

Steps for the reduction $p \to q$

S-1 make p^{th} row & q^{th} column ∞

S-2 make (q,p) as ∞.

S-3 In the trailing path before make any connection b/w 'q' & above node ∞.

S-4 Reduce the matrix like before

we get,

$$(B) \leftarrow \begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & 2 & 0 \\ 0 & \infty & \infty & 0 & 2 \\ 15 & \infty & 12 & \infty & 0 \\ 11 & \infty & 0 & 12 & \infty \end{bmatrix}$$

lly, we get reduced matrices for c, d, e

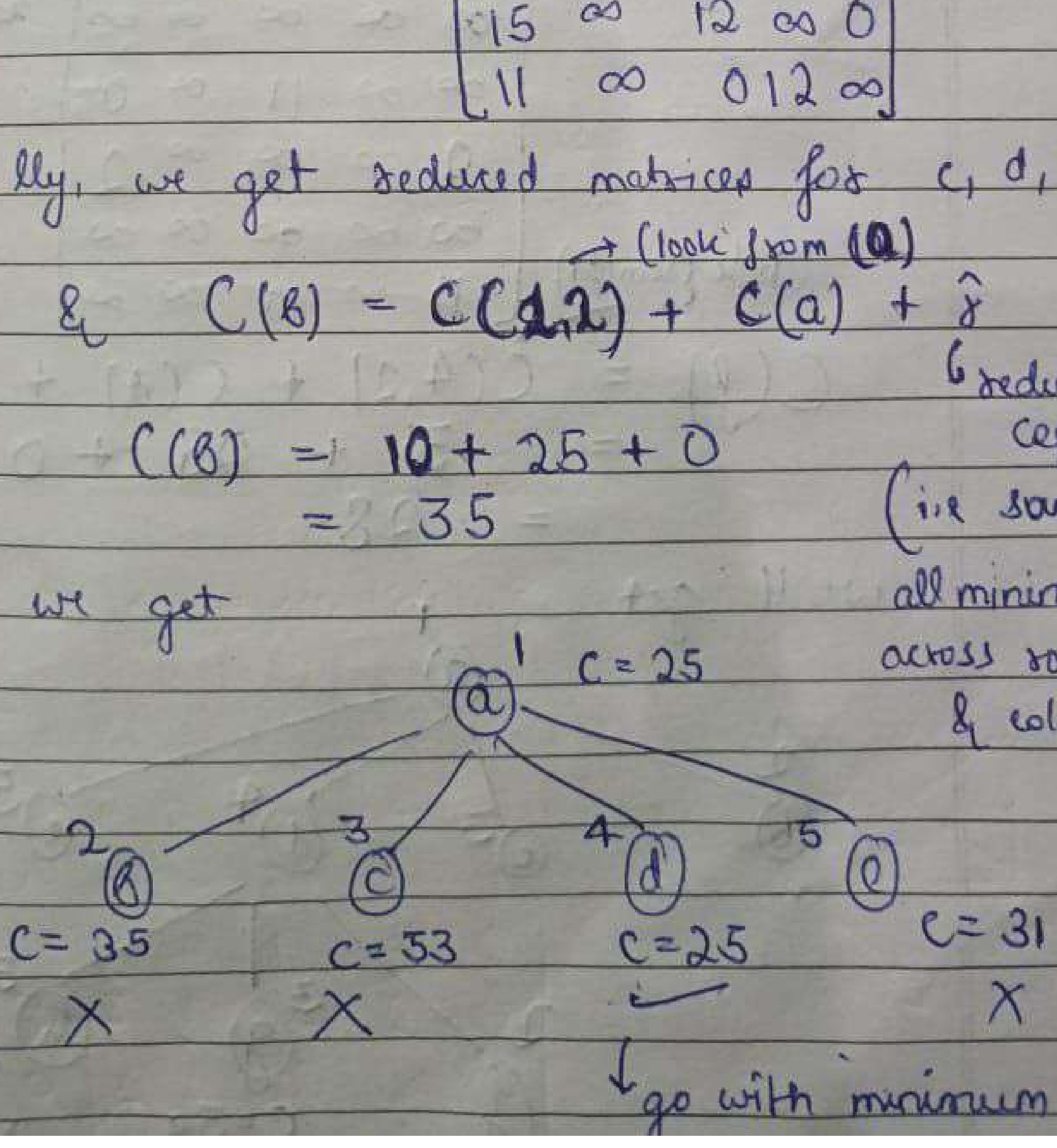
$\rightarrow$ (look from (a))

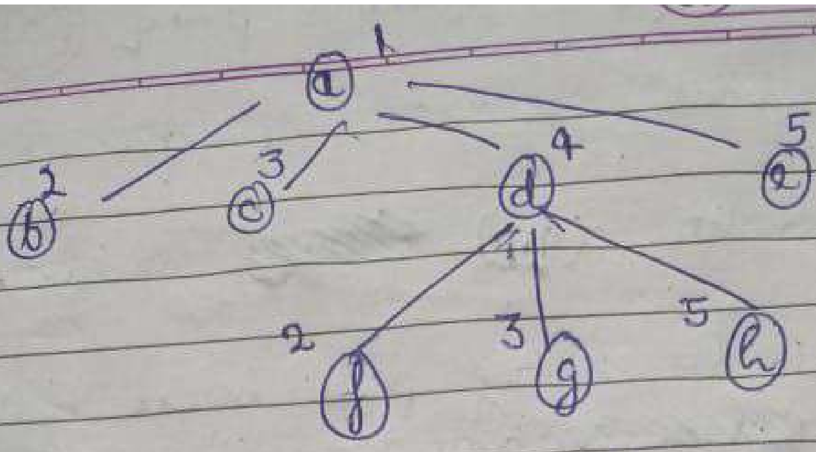
& $C(B) = C(1,2) + C(a) + \hat{r}$

$\hat{r}$ ↳ reduction cost (i.e sum of all minimums across rows & columns)

$$C(B) = 10 + 25 + 0$$
$$= 35$$

we get





Now calculate the costs for f, g & h node in similar manner

for ex. reduced matrix for $4 \to 2$ will be

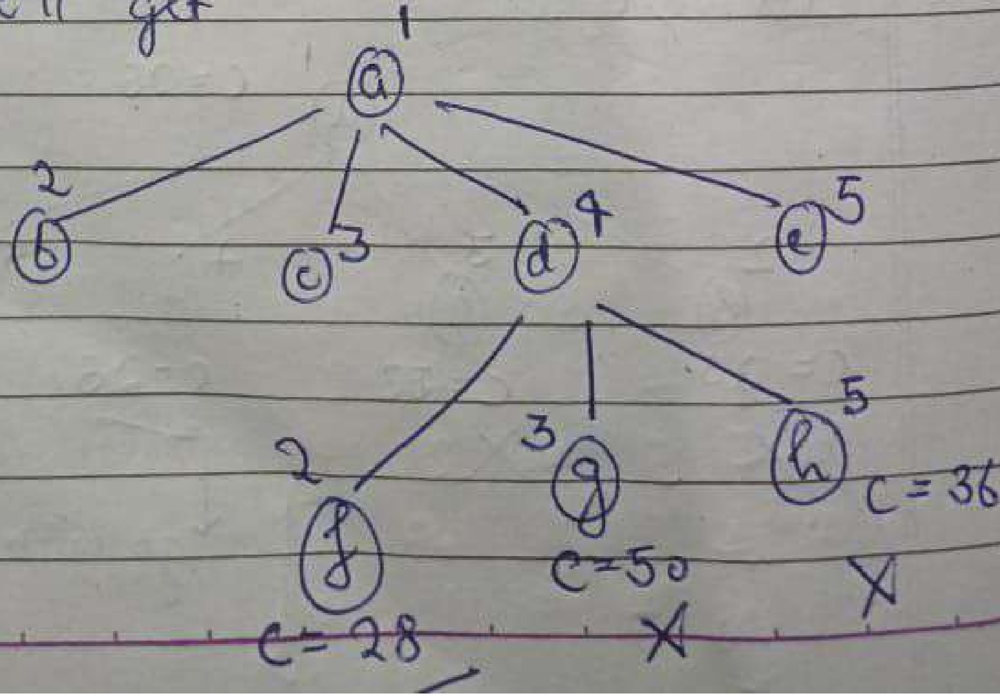
$$\begin{bmatrix} \infty & \infty & \infty & \infty & \infty \\ \infty & \infty & 11 & \infty & 0 \\ 0 & \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty & \infty \\ 11 & \infty & 0 & \infty & \infty \end{bmatrix}$$

← from (d)

(look from (d))

$$C(f) = C(4,2) + C(d) + \hat{r}$$
$$= 3 + 25 + 0$$
$$= 28$$

we'll get



And so on till path is completed

<u>Here</u>, the final & optimized path will be

$1 \longrightarrow 4 \rightarrow 2 \longrightarrow 5 \rightarrow 3$ <u>ans</u>