# 2021-2022 第二学期高等数学 A(二)试卷 A 参考答案

**一、选择题（每题 3 分，共 15 分）**

**1. C． 2. C． 3. D． 4. A． 5. B.**

**二、填空题（每题 3 分，共 15 分）**

**6.** 5. **7.** $\dfrac{xdy-ydx}{x^2+y^2}$. **8.** $\dfrac{1}{2}$. **9.** $4\pi$ **10.** $\dfrac{\pi^2}{2}$.

**三、计算题（每题 9 分，共 63 分）**

**11. 解：** 令$F(x,y,z)=x^2+y^2-z$，于是该曲面在点$(1,1,2)$处切平面的法向量为

$n=(F_x,F_y,F_z)\big|_{(1,1,2)}=(2x,2y,-1)\big|_{(1,1,2)}=(2,2,-1)$,

故所求切平面方程为

$$2(x-1)+2(y-1)-(z-2)=0,$$

即

$$2x+2y-z=2.$$

故所求法线方程为

$$\frac{x-1}{2}=\frac{y-1}{2}=\frac{z-2}{-1}.$$

**12. 解：**在方程两边关于$x$求偏导数得$1-\dfrac{\partial z}{\partial x}=e^z\dfrac{\partial z}{\partial x}$，

当$(x,y)=(1,0)$时，$z=0$，代入上式，得$\left.\dfrac{\partial z}{\partial x}\right|_{(1,0)}=\dfrac{1}{2}$．类似可得$\left.\dfrac{\partial z}{\partial y}\right|_{(1,0)}=\dfrac{1}{2}$．

两边关于$y$求偏导数得$-\dfrac{\partial^2 z}{\partial x\partial y}=e^z\dfrac{\partial z}{\partial x}\cdot\dfrac{\partial z}{\partial y}+e^z\dfrac{\partial^2 z}{\partial x\partial y}$，代入$x=1,y=0,z=0$，$\left.\dfrac{\partial z}{\partial x}\right|_{(1,0)}=\dfrac{1}{2}$及$\left.\dfrac{\partial z}{\partial y}\right|_{(1,0)}=\dfrac{1}{2}$，

解得$\left.\dfrac{\partial^2 z}{\partial x\partial y}\right|_{(1,0)}=-\dfrac{1}{8}$.

或者：计算得$\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial y}=\dfrac{1}{1+e^z}$，$\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{-e^z}{(1+e^z)^3}$，同理可得$\left.\dfrac{\partial^2 z}{\partial x\partial y}\right|_{(1,0)}=-\dfrac{1}{8}$.

**13. 解：** 令$\begin{cases} f_x'(x,y) = -2x+6=0, \\ f_y'(x,y) = 3y^2-12=0, \end{cases}$ 得驻点$(3,2)$, $(3,—2)$．又

$$f''_{xx}(x,y) = -2,\quad f_{xy}''(x,y) = 0,\quad f''_{yy}(x,y) = 6y.$$

在驻点$(3,2)$ 处，$A = f''_{xx}(3,2) = —2, B = f_{xy}''(3,2) = 0, C = f''_{yy}(3,2) = 12$，

$AC - B^2 = -24 < 0$，故$(3,2)$ 不是极值点；

在驻点$(3,-2)$ 处，$A = f''_{xx}(3,-2) = -2, B = f''_{xy}(3,-2) = 0, C = f''_{yy}(3,-2) = -12$，

$AC - B^2 = 24 > 0$，且 $A < 0$，故$(3,-2)$ 是极大值点，且极大值为$f(3,-2) = -18$.

**14.** 解：$D = D_1 \cup D_2$，其中$D_1: 0 \le x \le 1,\ x^2 \le y \le 1$，$D_2: 0 \le x \le 1,\ 0 \le y \le x^2$.

$$I = \iint_{D_1} + \iint_{D_2} = \int_0^1 dx \int_{x^2}^1 (y - x^2) dy + \int_0^1 dx \int_0^{x^2} (x^2 - y) dy$$

$$= \int_0^1 (\frac{1}{2} - x^2 + \frac{1}{2}x^4) dx + \int_0^1 \frac{1}{2} x^4 dx = \frac{4}{15} + \frac{1}{10} = \frac{11}{30}.$$

**15. 解：** 令$P = 2 + xe^{2y}, Q = x^2 e^{2y} —1$，

$$\frac{\partial P}{\partial y} = 2xe^{2y} = \frac{\partial Q}{\partial x}$$ ，故积分与路径无关，

取路径$OA: y = 0, x: 0 \to 4$

$$I = \int_{OA} (2 + xe^{2y}) dx + (x^2 e^{2y} - 1) dy = \int_0^4 (2 + x) dx = 16.$$

**16.** 解法一：补充曲面 $\Sigma_1: z = 1(x^2 + y^2 \le 1)$，取上侧；$\Sigma_2: z = 0(x^2 + y^2 \le 1)$，取下侧，则 $\Sigma, \Sigma_1, \Sigma_2$ 构成封闭曲面，取外侧，它们所围区域记为 $\Omega$.

由高斯公式可得，$\displaystyle\oiint_{\Sigma+\Sigma_1+\Sigma_2} = \iiint_\Omega (2xy + 2y\sin x + 2z) \mathrm{d}V$，

根据奇偶对称性可知 $\displaystyle\iiint_\Omega 2xy\mathrm{d}V = \iiint_\Omega 2y\sin x\mathrm{d}V = 0$，所以

$$\oiint_{\Sigma+\Sigma_1+\Sigma_2} = 2\iiint_\Omega z\mathrm{d}V = 2\iint_{x^2+y^2\le1} \mathrm{d}x\mathrm{d}y \int_0^1 z\mathrm{d}z = \pi.$$

而 $\displaystyle\iint_{\Sigma_1} = \iint_{\Sigma_1} z^2 \mathrm{d}x\mathrm{d}y = \iint_{x^2+y^2\le1} 1^2 \mathrm{d}x\mathrm{d}y = \pi$，$\displaystyle\iint_{\Sigma_2} = \iint_{\Sigma_2} z^2 \mathrm{d}x\mathrm{d}y = -\iint_{x^2+y^2\le1} 0^2 \mathrm{d}x\mathrm{d}y = 0,$

所以 $\displaystyle I = \oiint_{\Sigma+\Sigma_1+\Sigma_2} - \iint_{\Sigma_1} - \iint_{\Sigma_2} = \pi - \pi - 0 = 0.$

解法二：由于 $\iint\limits_{\Sigma} z^2 \mathrm{d}x\mathrm{d}y = 0$，所以 $I = \iint\limits_{\Sigma} x^2 y\mathrm{d}y\mathrm{d}z + y^2 \sin x\mathrm{d}z\mathrm{d}x$ .

补充曲面 $\Sigma_1 : z = 1(x^2 + y^2 \le 1)$ ，取上侧； $\Sigma_2 : z = 0(x^2 + y^2 \le 1)$ ，取下侧，则 $\Sigma, \Sigma_1, \Sigma_2$ 构成封闭曲面，所围区域为 $\Omega$ , 取外侧.

由高斯公式可得，$\oiint\limits_{\Sigma+\Sigma_1+\Sigma_2} = \iiint\limits_{\Omega} (2xy + 2y\sin x)\mathrm{d}V$ ,

根据奇偶对称性可知 $\iiint\limits_{\Omega} 2xy\mathrm{d}V = \iiint\limits_{\Omega} 2y\sin x\mathrm{d}V = 0$，所以 $\oiint\limits_{\Sigma+\Sigma_1+\Sigma_2} = 0$ .

而 $\iint\limits_{\Sigma_1} x^2 y\mathrm{d}y\mathrm{d}z + y^2 \sin x\mathrm{d}z\mathrm{d}x = \iint\limits_{\Sigma_2} x^2 y\mathrm{d}y\mathrm{d}z + y^2 \sin x\mathrm{d}z\mathrm{d}x = 0$，所以 $I = 0$ .

**17. 解.** 收敛半径 $R = \lim\limits_{n\to\infty} \frac{2n+1}{2n+3} = 1$， 故收敛区间为$(-1,1)$ 。

显然$x = \pm 1$的时候，原级数发散，从而收敛域为$(-1,1)$

设 $S(x) = \sum\limits_{n=0}^{\infty} (2n+1)x^n, x \in (-1,1)$

又 $\sum\limits_{n=0}^{\infty} (2n+1)x^n = 2\sum\limits_{n=0}^{\infty} nx^n + \sum\limits_{n=0}^{\infty} x^n$

对 $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$ 逐项求导得 $\sum\limits_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$ ,

故 $\sum\limits_{n=0}^{\infty} nx^n = x\sum\limits_{n=0}^{\infty} nx^{n-1} = \frac{x}{(1-x)^2}$ ,

于是 $S(x) = \sum\limits_{n=0}^{\infty} (2n+1)x^n = \frac{2x}{(1-x)^2} + \frac{1}{1-x} = \frac{1+x}{(1-x)^2}, x \in (-1,1).$

**四、证明题（本题 7 分）**

18. **证明**：设 $\sum\limits_{n=1}^{\infty} (b_{n+1} - b_n) = s$ .

由于其前$n$ 项部分和$s_n = b_2 - b_1 + b_3 - b_2 + \ldots + b_{n+1} - b_n = b_{n+1} - b_1$ ,

所以 $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} (b_{n+1} - b_1) = s$， 得 $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} b_{n+1} = s + b_1$， 从而数列$\{b_n\}$ 有界.

不妨令 $|b_n| \le M$ ，则 $0 \le |a_n b_n| \le Ma_n$ . 因为 $\sum\limits_{n=1}^{\infty} Ma_n$ 收敛， 所以由正项级数的比较判别法可知

$\sum\limits_{n=1}^{\infty} |a_n b_n|$ 收敛，即 $\sum\limits_{n=1}^{\infty} a_n b_n$ 绝对收敛.