REPRESENTING CONVEX STRUCTURES WITH CHESS

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Abstract.

1. Introduction

2. Notation

3. Moore Families

Definition 3.1. Let X be a set. Let $\mathscr{C} \subseteq \mathscr{P}(X)$. We call $\mathscr{C}$ a **Moore family** if and only if the following conditions are satisfied:

(1) $\varnothing, X \in \mathscr{C}$.
(2) $\forall \mathscr{D} \subseteq \mathscr{C} \left((\mathscr{D} \neq \varnothing) \implies \bigcap \mathscr{D} \in \mathscr{C} \right)$

Definition 3.2. Let X be a set. Let $\alpha : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$. We call α a **closure operator** if and only if the following conditions are satisfied:

(1) $\alpha(\varnothing) = \varnothing$.
(2) $\forall A \in \mathscr{P}(X)(A \subseteq \alpha(A))$.
(3) $\forall A, B \in \mathscr{P}(X)(A \subseteq B \implies \alpha(A) \subseteq \alpha(B))$.
(4) $\forall A \in \mathscr{P}(X)(\alpha(\alpha(A)) = A)$.

Definition 3.3. We denote

$$\begin{array}{rcl} MF_X & = & \{\mathscr{D} \subseteq \mathscr{P}(X) : \mathscr{D} \text{ is a Moore family}\}, \\ CL_X & = & \{\alpha \in \mathscr{P}(X)^{\mathscr{P}(X)} : \alpha \text{ is a closure operator}\}. \end{array}$$

Lemma 3.4. *Let $\alpha \in CL_X$. Then*

(a) $Im(\alpha) \in MF_X$.
(b) $\forall \mathscr{D} \subseteq Im(\alpha)$, $A = \bigcap \mathscr{D}$ *is such that* $\alpha(A) = A$.

Proof. We verify that $Im(\alpha) \in MF_X$.

(1) $\varnothing = \alpha(\varnothing) \in Im(\alpha)$, and $X \subseteq \alpha(X) \subseteq X \implies X = \alpha(X) \in Im(\alpha)$
(2) Let $\varnothing \neq \mathscr{D} \subseteq Im(\alpha)$. Define $A = \bigcap \mathscr{D}$. We must have $A \in Im(\alpha)$. Notice that, $\forall Z \in \mathscr{D} \subseteq Im(\alpha)$, $\exists B \in \mathscr{P}(X)$ such that $Z = \alpha(B)$. Then
$$\alpha(Z) = \alpha(\alpha(B)) = \alpha(B) = Z.$$
Let $Z \in \mathscr{D}$. Then $A \subseteq Z$, and $\alpha(A) \subseteq \alpha(Z)$. Therefore
$$A \subseteq \alpha(A) \subseteq \bigcap_{Z \in \mathscr{D}} \alpha(Z) = \bigcap_{Z \in \mathscr{D}} Z = A.$$
Finally $A = \alpha(A) \in Im(\alpha)$, which proves (a) and (b).

$\square$

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Lemma 3.5. *Let $\mathscr{D} \in MF_X$. Then $\Phi_{\mathscr{D}} : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ defined by*

$$\Phi_{\mathscr{D}}(A) = \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z,$$

is such that $\Phi_{\mathscr{D}} \in CL_X$.

Proof. (1) $\Phi_{\mathscr{D}}(\varnothing) = \bigcap_{\substack{A \in \mathscr{D} \\ \varnothing \subseteq A}} A = \varnothing$.

(2) Let $A \in \mathscr{P}(X)$. We will show that $A \subseteq \Phi_{\mathscr{D}}(A)$.
Let $a \in A$, and $Z \in \mathscr{D}$, such that $A \subseteq Z$. Then $a \in Z$, and

$$\therefore \quad a \in \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z \implies A \subseteq \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z = \Phi_{\mathscr{D}}(A).$$

(3) Let $A, B \in \mathscr{P}(X)$, such that $A \subseteq B$. We require $\Phi_{\mathscr{D}}(A) \subseteq \Phi_{\mathscr{D}}(B)$.
Notice that $\{Z \in \mathscr{D} \,:\, B \subseteq Z\} \subseteq \{Z \in \mathscr{D} \,:\, A \subseteq Z\}$. Therefore,

$$\Phi_{\mathscr{D}}(A) = \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z \subseteq \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z = \Phi_{\mathscr{D}}(B).$$

(4) We need to show $\Phi_{\mathscr{D}}(A) = \Phi_{\mathscr{D}}(\Phi_{\mathscr{D}}(A))$. By (2), $\Phi_{\mathscr{D}}(A) \subseteq \Phi_{\mathscr{D}}(\Phi_{\mathscr{D}}(A))$.
Let $Z_0 \in \mathscr{D}$ such that $A \subseteq Z_0$. Therefore $\Phi_{\mathscr{D}}(A) = \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z \subseteq Z_0$.

$$\begin{array}{lrcl}
\therefore & \forall Z \in \mathscr{D}(A \subseteq Z & \implies & \Phi_{\mathscr{D}}(A) \subseteq Z) \\
\therefore & \{Z \in \mathscr{D} \,:\, A \subseteq Z\} & \subseteq & \{Z \in \mathscr{D} \,:\, \Phi_{\mathscr{D}}(A) \subseteq Z\} \\
\therefore & \Phi_{\mathscr{D}}(\Phi_{\mathscr{D}}(A)) = \displaystyle\bigcap_{\substack{Z \in \mathscr{D} \\ \Phi_{\mathscr{D}}(A) \subseteq Z}} Z & \subseteq & \displaystyle\bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z = \Phi_{\mathscr{D}}(A).
\end{array}$$

□

Lemma 3.6. *Let $\mathscr{D} \in MF_X$. Then $Im(\Phi_{\mathscr{D}}) = \mathscr{D}$.*

Proof. Since $\mathscr{D} \in MF_X$, $\forall A \subseteq X$, $\Phi_{\mathscr{D}}(A) \in \mathscr{D}$. Then $Im(\Phi_{\mathscr{D}}) \subseteq \mathscr{D}$. Let $A \in \mathscr{D}$. So $A \in \mathscr{D}$ is such that $A \subseteq A$, $A \subseteq \Phi_{\mathscr{D}}(A) = \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z \subseteq A$. Then $A = \Phi_{\mathscr{D}}(A)$, and $\mathscr{D} \subseteq Im(\Phi_{\mathscr{D}})$. □

Proposition 3.7. *Let $\Phi : MF_X \longrightarrow CL_X$ be defined by:*

$$\Phi_{\mathscr{D}} : \mathscr{P}(X) \longrightarrow \mathscr{D}\ ,\ \Phi_{\mathscr{D}}(A) = \bigcap_{\substack{Z \in \mathscr{D} \\ A \subseteq Z}} Z.$$

Then Φ is a bijection.

Proof. We prove injection. Let $\mathscr{C}, \mathscr{D} \in MF_X$, such that $\Phi_{\mathscr{C}} = \Phi_{\mathscr{D}}$. By Lemma 3.6,

$$\mathscr{C} = Im(\Phi_{\mathscr{C}}) = Im(\Phi_{\mathscr{D}}) = \mathscr{D}.$$

It remains verify surjection. Let $\alpha \in CL_X$. Consider $\Phi_{Im(\alpha)} : \mathscr{P}(X) \longrightarrow Im(\alpha)$. By Lemma 3.4(a), $Im(\alpha) \in MF_X$. Let $A \subseteq X$. Notice that $A \subseteq \alpha(A) \in Im(\alpha)$.

$$\therefore \quad A \subseteq \Phi_{Im(\alpha)}(A) = \bigcap_{\substack{Z \in Im(\alpha) \\ A \subseteq Z}} Z \subseteq \alpha(A)$$

$$\therefore \quad \alpha(A) \subseteq \alpha\big(\Phi_{Im(\alpha)}(A)\big) \subseteq \alpha(\alpha(A)) = \alpha(A) \implies \alpha(\Phi_{Im(\alpha)}(A)) = \alpha(A).$$

Since $\mathscr{D}' = \{Z \in Im(\alpha) : A \subseteq Z\} \subseteq Im(\alpha)$, then $\Phi_{Im(\alpha)}(A) = \bigcap \mathscr{D}'$.
Applying the Lemma 3.4(b) to $\alpha \in CL_X$, and $\Phi_{Im(\alpha)}(A) = \bigcap \mathscr{D}'$, we obtain

$$\alpha(A) = \alpha(\Phi_{Im(\alpha)}(A)) = \Phi_{Im(\alpha)}(A) \implies \Phi_{Im(\alpha)} = \alpha.$$

$\square$

Lemma 3.8. *Let $X \neq \varnothing$. For each $\alpha \in I$, let $M_\alpha \in MF_X$. Then* $\bigcap_{\alpha \in I} M_\alpha \in MF_X$.

Definition 3.9. Let $X \neq \varnothing$ be a set. Let $\mathscr{A} \subseteq \mathscr{P}(X)$. Define the Moore family generated by $\mathscr{A}$:

$$M(\mathscr{A}) = \bigcap_{\substack{M \in MF_X \\ \mathscr{A} \subseteq M}} M = \Big\{\bigcap \mathscr{D} : \mathscr{D} \subseteq \mathscr{A}\Big\} \cup \{\varnothing, X\}.$$

Remark 3.10. We have defined the following system of operators.

$$\begin{tikzcd}
MF_X \arrow[rr, leftrightarrow, "\Phi"] \arrow[d, hook, shift left] & & CL_X \arrow[d, hook] \\
\mathscr{P}(\mathscr{P}(X)) \arrow[u, shift left, "M"] & & \mathscr{P}(X)^{\mathscr{P}(X)} \arrow[ll, "Im(\cdot)"']
\end{tikzcd}$$

Proposition 3.11. *Let $X \neq \varnothing$ be a set. Then*

(1) $M(\varnothing) = \{\varnothing, X\}$.
(2) $\forall \mathscr{A} \subseteq \mathscr{P}(X)(\mathscr{A} \subseteq M(\mathscr{A}))$.
(3) $\forall \mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(X)(\mathscr{A} \subseteq \mathscr{B} \implies M(\mathscr{A}) \subseteq M(\mathscr{B}))$.
(4) $\forall \mathscr{A} \subseteq \mathscr{P}(X)(M(M(\mathscr{A})) = M(\mathscr{A}))$.

Proof. (1) $M(\varnothing) = \Big\{\bigcap \mathscr{D} : \mathscr{D} \subseteq \varnothing\Big\} \cup \{\varnothing, X\} = \Big\{\bigcap \varnothing\Big\} \cup \{\varnothing, X\} = \{\varnothing, X\}$.

(2) $\mathscr{A} = \Big\{\bigcap \{A\} : A \in \mathscr{A}\Big\} \subseteq \Big\{\bigcap \mathscr{D} : \mathscr{D} \subseteq \mathscr{A}\Big\} \cup \{\varnothing, X\} = M(\mathscr{A})$.

(3) Let $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(X)$, such that $\mathscr{A} \subseteq \mathscr{B}$.
Then $\{M \in MF_X : \mathscr{B} \subseteq M\} \subseteq \{M \in MF_X : \mathscr{A} \subseteq M\}$, implies

$$M(\mathscr{A}) = \bigcap_{\substack{M \in MF_X \\ \mathscr{A} \subseteq M}} M \subseteq \bigcap_{\substack{M \in MF_X \\ \mathscr{B} \subseteq M}} M = M(\mathscr{B}).$$

(4) Since $\forall \mathscr{M} \in MF_X$, $M(\mathscr{M}) = \mathscr{M}$, it follows that, $M(M(\mathscr{A})) = M(\mathscr{A})$.

$\square$

Proposition 3.12. *Let* $X \neq \varnothing$. *Let* $\mathscr{A}, \mathscr{B} \subseteq \mathscr{P}(X)$. *Define*

$$\mathscr{A} \sim \mathscr{B} \iff M(\mathscr{A}) = M(\mathscr{B}).$$

Then $\sim$ *is an equivalence relation on* $\mathscr{P}(X)$.

4. Convex Structures

Definition 4.1. Let X be set. Let $\mathscr{C} \subseteq \mathscr{P}(X)$. We call $\mathscr{C}$ a **convex structure** if and only if the following conditions are satisfied:

(1) $\varnothing, X \in \mathscr{C}$.

(2) $\forall \mathscr{D} \subseteq \mathscr{C} \left(\mathscr{D} \neq \varnothing \implies \bigcap \mathscr{D} \in \mathscr{C} \right)$.

(3) $\forall \mathscr{D} \subseteq \mathscr{C} \left(\forall D_1, D_2 \in \mathscr{D} (D_1 \subseteq D_2 \vee D_2 \subseteq D_1) \implies \bigcup \mathscr{D} \in \mathscr{C} \right)$

Definition 4.2. We call $\alpha : \mathscr{P}(X) \longrightarrow \mathscr{P}(X)$ a **domain finite** or **algebraic** if and only if

$$\forall A \in \mathscr{P}(X) \left(\alpha(A) = \bigcup_{F \in [A]^{<\aleph_0}} \alpha(F) \right).$$

Definition 4.3. Let

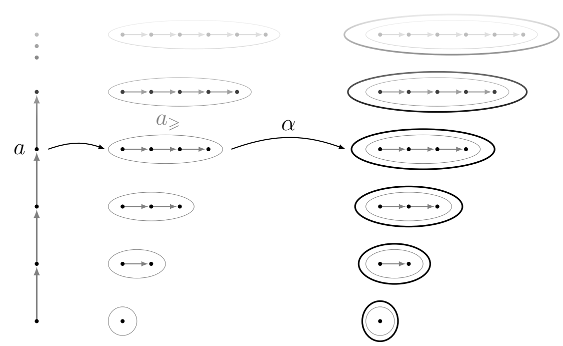
$$\begin{aligned} CS_X &= \{\mathscr{C} \subseteq \mathscr{P}(X) \ : \ \mathscr{C} \text{ is a convex structure}\}, \\ CLA_X &= \{\alpha \in \mathscr{P}(X)^{\mathscr{P}(X)} \ : \ \alpha \text{ is an algebraic closure operator}\}. \end{aligned}$$

Lemma 4.4. *Let* X *be a set. Let* $\mathscr{C} \subseteq \mathscr{P}(X)$ *be a convex structure. Then*

$$F \in \left[\bigcup \mathscr{C} \right]^{<\aleph_0} \iff \exists Z \in \mathscr{C} (F \in [Z]^{<\aleph_0}).$$

Theorem 4.5 ([1])**.** $\Phi_{\mathscr{D}} \in CLA_X \iff \mathscr{D} \in CS_X$.

Proof. Let $\mathscr{D} \in MF_X$. We note that $\mathscr{D} = Im(\Phi_{\mathscr{D}})$.
We must verify that $\mathscr{D}$ is closed under unions of chains.
Let $\mathscr{C} \subseteq \mathscr{D}$ a chain. Let $Z \in \mathscr{C}$, then $Z = \bigcup_{F \in [Z]^{<\aleph_0}} \Phi_{\mathscr{D}}(F)$. Therefore



$$\bigcup \mathscr{C} = \bigcup_{Z \in \mathscr{C}} \bigcup_{F \in [Z]^{<\aleph_0}} \Phi_{\mathscr{D}}(F).$$

(incomplete) $\square$

5. Finite Convex Structures

6. Chess

7. Further Work

References

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