Lectures on Symmetric Attractors

Article · July 1997

Source: CiteSeer

Michael Field
Rice University/Imperial College

LECTURES ON SYMMETRIC ATTRACTORS

MICHAEL FIELD

Date: June, 1997.

Research supported in part by NSF Grant DMS-9403624 and Texas Advanced Research Program Award 003652-757.

Contents

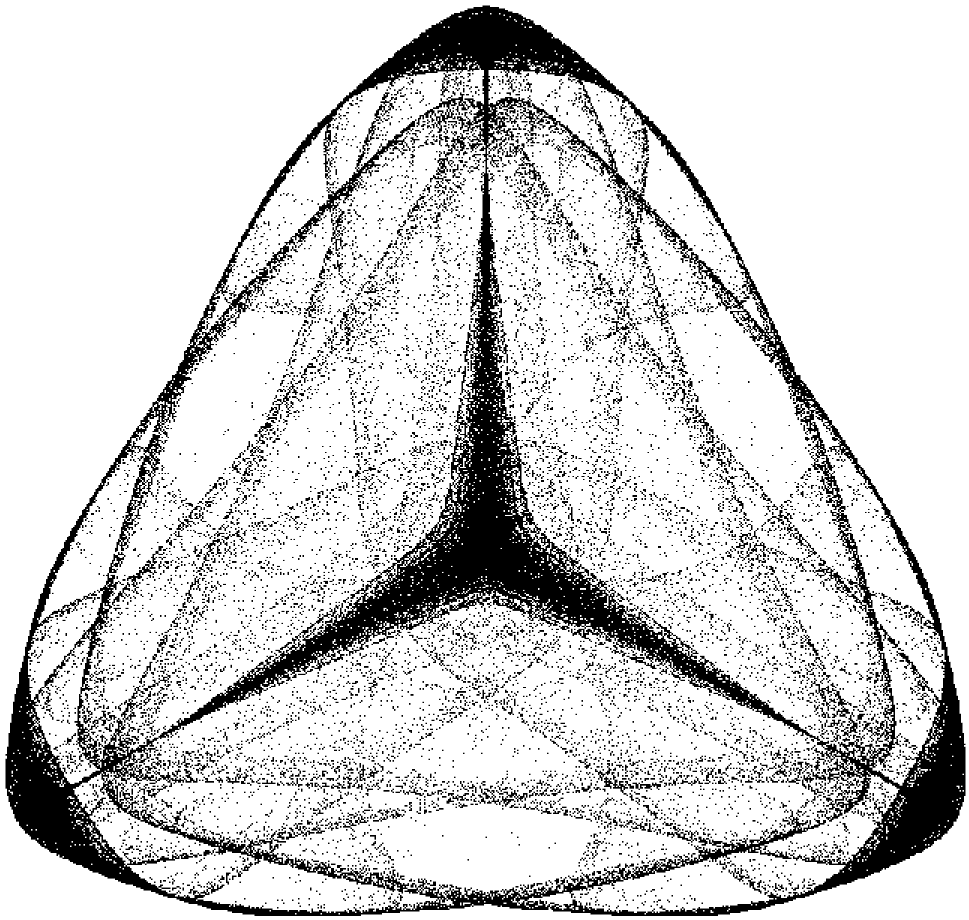

FIGURE 1. Attractor of a planar map with 3-fold symmetry

1. Introduction

These notes are an expanded version of three lectures given at the *DANSE* Workshop on Equivariant Dynamics held in Berlin, May, 1997.

The notes are intended to fill in some details not given in the lectures as well as to provide a source of references for further reading.

Sections 2–4 cover the first lecture. Section 2 contains a basic introduction to equivariant dynamics with a focus on attractors. In section 3, we describe one exotic example that displays, in a persistent fashion, a variety of features not seen in non-equivariant dynamics. In section 4 we review Williams' construction of a solenoidal attractor. In Lecture II (section 6), we show how to construct symmetric hyperbolic attractors with specified finite symmetry group. Finally, in lecture III (section 8), we describe some recent work, some joint with Parry (Warwick), on ergodic properties of attractors for equivariant dynamical systems equivariant by a general compact connected Lie group. We conclude with an example of a stably ergodic skew extension over a non-uniformly hyperbolic base.

At the end of each lecture, we provide additional comments and suggestions for further reading (sections 5, 7 & 9).

The notes are not intended to form a comprehensive survey of symmetric attractors or equivariant dynamics. Thus, we do not discuss any of the recent work on transverse instability of attractors contained in invariant subspaces (see [6] and also [3] for riddled basins). Nor do we discuss symmetry detectives (see, for example, [9, 31, 5]).

It is a pleasure to thank Reiner Lauterbach, and the other organizers of the DANSE Workshop, for their hospitality and for their organization of such a splendid and interesting workshop. Thanks also to Peter Ashwin, the EPSRC and the University of Surrey, where the work on these notes was completed.

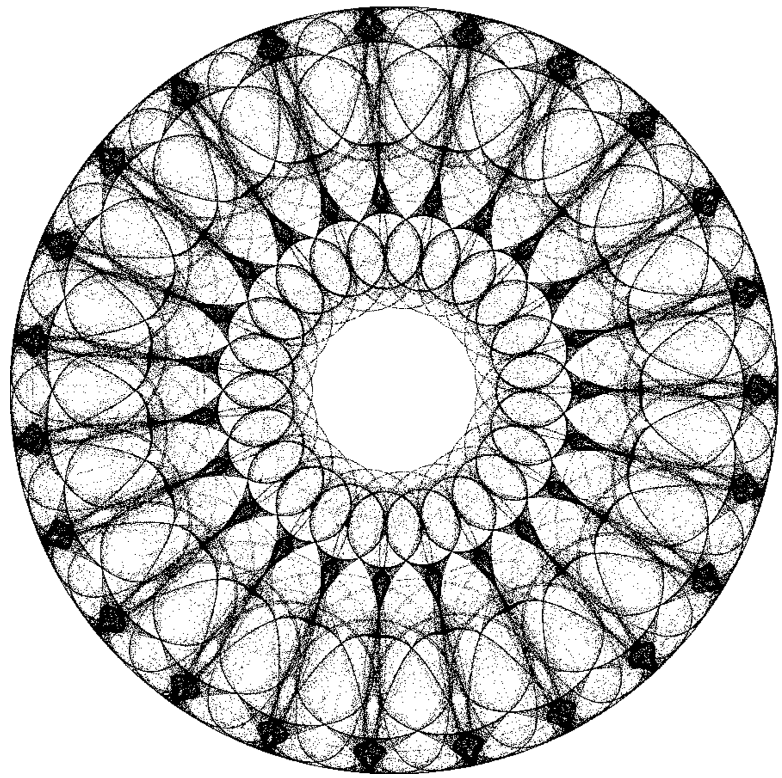

FIGURE 2. Attractor of a planar map with 23-fold symmetry

2. PRELIMINARIES

2.1. **Lie groups.** Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth (that is C^∞) mapping. If $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation of $\mathbb{R}^n$, we say that T is a (linear) *symmetry* of f if

$$f(Tx) = T(f(x)), \quad (x \in \mathbb{R}^n).$$

Obviously, if T, S are symmetries of f so are $T \circ S$ and T^{-1}. It follows that the set of linear symmetries of f forms a group, say $\mathcal{S}(f)$. Clearly $\mathcal{S}(f)$ is a subgroup of the general linear group $\mathrm{GL}(n, \mathbb{R})$. Since every element in the closure of $\mathcal{S}(f)$ is obviously also a symmetry of f, it follows that $\mathcal{S}(f)$ is a *closed* subgroup of $\mathrm{GL}(n, \mathbb{R})$.

The group $\mathcal{S}(f)$ is an example of a *Lie group*. That is, a group which has the structure of a differential manifold and for which composition and inversion are smooth maps. More 'nonlinear' examples can be constructed by looking at the group of all smooth diffeomorphisms which commute with a given smooth mapping f of a (compact) manifold M. In this case, $\mathcal{S}(f)$ will be a closed subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ of M.

For our purposes, we shall restrict attention to *compact* Lie groups. This class includes all *finite* groups. It is well-known that every compact Lie group can be represented as a (closed) subgroup of an orthogonal group $\mathrm{O}(n)$.

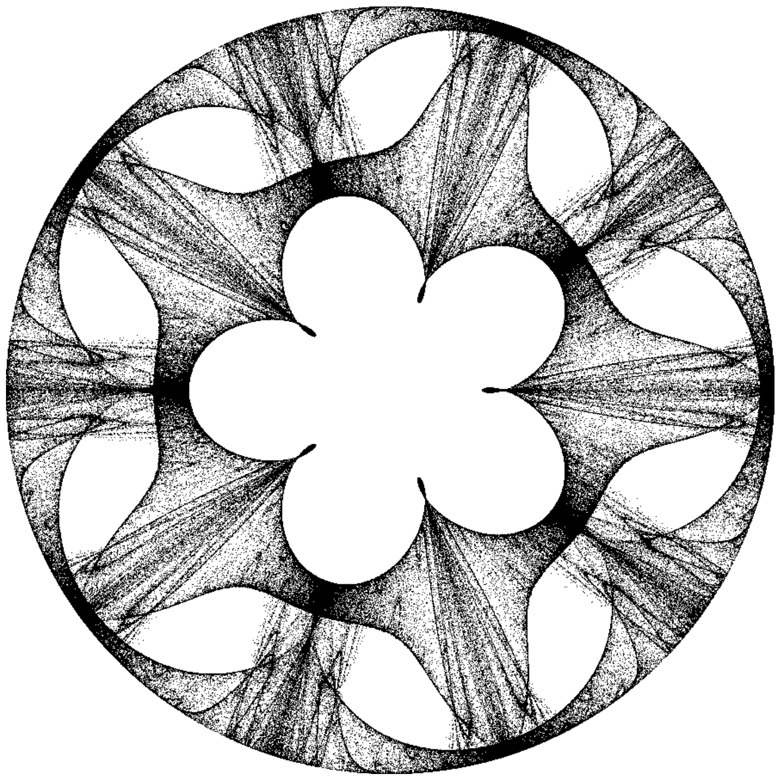

FIGURE 3. Attractor of a planar map with 5-fold symmetry

Rather than looking at the symmetries of a specific map f, our approach will be to specify a compact Lie group Γ, an *action* of Γ on a vector space or differential manifold, and then investigate a class of dynamical systems which has (at least) Γ symmetry.

2.2. **Linear group actions – representations.** Suppose that Γ is a subgroup of $\mathrm{O}(n)$. We have an associated (orthogonal) action of Γ on $\mathbb{R}^n$ defined by evaluation:

$$\mathbb{R}^n \times \Gamma \to \mathbb{R}^n, \ \ (x, \gamma) \mapsto \gamma x.$$

We refer to $(\mathbb{R}^n, \Gamma)$ as a *representation* of Γ (on $\mathbb{R}^n$).

Remark 2.1. More generally, we can look at homomorphisms $\rho : \Gamma \to \mathrm{O}(n)$ and the associated action of Γ on $\mathbb{R}^n$. Provided that that we study dynamics on $\mathbb{R}^n$ (as opposed to manifolds), all that will matter is the image $\rho(\Gamma) \subset \mathrm{O}(n)$ and so it is no loss of generality to regard Γ as a matrix group. $\diamondsuit$

Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$. We say f is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x), \ \ (x \in \mathbb{R}^n, \gamma \in \Gamma).$$

If f is Γ-equivariant then the group of linear symmetries of f is at least as big as Γ.

Example 2.2. Let $\mathbf{D}_n$ denote dihedral group of order $2n$. That is, $\mathbf{D}_n$ is the full symmetry group of the regular n-gon. Regard $\mathbf{D}_n$ as a subgroup of $\mathrm{O}(2)$ and identify $\mathbb{R}^2$ with $\mathbb{C}$ in the

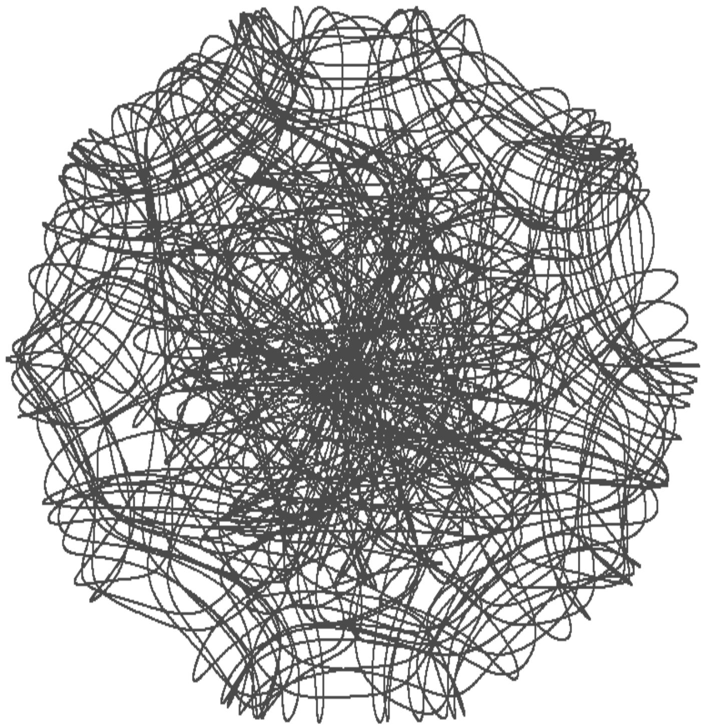

FIGURE 4. Attractor of a symmetric flow on $\mathbb{R}^4$

usual way. Then $\mathbf{D}_n = \langle \rho, \kappa \rangle$, where

$$\rho(z) = \exp(2\pi \imath/n)z, \quad \kappa(z) = \bar{z}.$$

Suppose that $p, q : \mathbb{C} \to \mathbb{R}$ are smooth maps. Then it is easy to verify that the map $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = p(|z|^2, \mathrm{Re}(z^n))z + q(|z|^2, \mathrm{Re}(z^n))\bar{z}^{n-1},$$

is $\mathbf{D}_n$-equivariant. The converse is also true. If f is a polynomial (or analytic) this follows straightforwardly from classical invariant theory (the maps p, q will then be polynomials or analytic). If f is smooth, the result is a consequence of Schwarz' theorem on smooth invariants. For more details on these matters see [32, Chapter XII, §4]. (Figures 1 – 3 were produced by taking $p(|z|^2, \mathrm{Re}(z^n)) = \lambda + \alpha|z|^2 + \gamma \mathrm{Re}(z^n)$ and $q(|z|^2, \mathrm{Re}(z^n)) = \delta$, where $\lambda, \ldots, \delta \in \mathbb{R}$. For explicit parameters, see [28].) ♡

2.3. **Dynamics.** We continue to suppose that we are given a linear action of a compact Lie group Γ on $\mathbb{R}^n$ (everything we say holds just as well for smooth actions on manifolds).

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth Γ-equivariant map. Suppose that $A \subset \mathbb{R}^n$ is a compact f-invariant set.

We shall assume that $f : A \to A$ is topologically transitive. That is, there exists $x_0 \in A$ such that the ω-limit point set $\omega(x_0) = A$.

Since f is Γ-invariant, it is natural to ask about the symmetry of the set A and to this end, we define the *symmetry group of* A by

$$\Sigma_A = \{\gamma \in \Gamma \mid \gamma(A) = A\}.$$

Obviously, Σ_A is a *closed* subgroup of Γ.

Example 2.3. Let $\mathbb{K}$ denote the group of complex numbers of unit modulus (also commonly denoted by SO(2) and S^1). Take the standard action of $\mathbb{K}$ on $\mathbb{C}$ defined by complex multiplication. Define $f(z) = c(1 + 0.5(1 - |z|^2))z$, where $|c| = 1$. Clearly $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{K}$-equivariant. Note that the unit circle $C \subset \mathbb{C}$ is f-invariant. For all non-zero $z \in \mathbb{C}$, $\omega(z) \subset C$. If $c = 1$, then every point $u \in C$ is fixed by f and so if we take $A = \{u\}$, $f : A \to A$ is trivially topologically transitive and $\Sigma_A = \{e\}$. Suppose $c \neq 1$. Fix $u \in C$ and let A denote the closure of the f-orbit of u. There are two possibilities. First suppose $c = \exp(2\pi\imath p/q)$, where $(p, q) = 1$, $p \neq 0$. Then A is a periodic orbit of f, prime period q and $\Sigma_A \cong \mathbb{Z}_q$ (cyclic group of order q). If instead $c = \exp(2\pi\imath\alpha)$, where α is irrational, then $A = C$ and $\Sigma_A = \mathbb{K}$. $\heartsuit$

Lemma 2.4. *Suppose that* $A = \omega(x_0)$ *as above. Then*

(1) $\omega(\gamma x_0) = \gamma\omega(x_0)$, $\gamma \in \Gamma$.
(2) $\gamma \in \Sigma_A$ *if and only if* $\gamma x_0 \in A$.
(3) $\Sigma_{\gamma A} = \gamma\Sigma_A\gamma^{-1}$.

Proof. Statement (1) follows by Γ-equivariance of f. The remaining statements follow immediately from (1). $\square$

Remark 2.5. In general the set $\{\omega(\gamma x_0) \mid \gamma \in \Gamma\}$ does not define a *partition* of $\Gamma(A)$ into closed sets. One interesting case where we obtain a partition is described in Chossat and Golubitsky [15]. We also obtain a partition if A has a hyperbolic or transversally hyperbolic structure (see §§6, 8.4). On the other hand, if A has a non-uniformly hyperbolic structure we generally do not obtain a partition (see §8.7). For another interesting class of counter-examples, we refer to the work of Ashwin on 'stuck on' attractors [4]. $\diamondsuit$

We say a compact f-invariant set $A \subset \mathbb{R}^n$ is an *attractor* if there exists an open neighborhood U of A such that

- $f(U) \subset U$.
- $\cap_{n \geq 0} f^n(U) = A$.

2.4. **Some basic questions.** (1) Suppose that (V, Γ) is a representation of a finite group Γ. What subgroups of Γ can occur as symmetry groups of attractors for Γ-equivariant differentiable dynamical systems on $V\Gamma$ (Similar question for smooth Γ-manifolds.)

Example 2.6. Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbf{D}_6$-equivariant. Can f have an attractor with $\mathbf{D}_3$ symmetryΓ Answer: NO - even for continuous maps - by results of Dellnitz, Golubitsky & Melbourne [18]. $\heartsuit$

Unlike Dellnitz et al., our emphasis will be on smooth invertible dynamical systems (diffeomorphisms and flows). For continuous maps, see [18, 7, 42].

(2) If Γ is finite, what can be said about the *structure* of attractors associated to Γ-equivariant dynamical systemsΓ

In general, this is a difficult, and not at all well understood, classification problem. We confine ourselves to describing a broad class of examples where there is either a uniform hyperbolic structure or controlled non-uniformly hyperbolic structure. These examples very much develop from the standard non-equivariant theory of differentiable dynamical systems. However, we shall shortly give one exotic example that depends crucially on the underlying symmetry ...

(3) If Γ is compact, but not finite, what can be said about the statistical properties of attractors (invariant sets) with full Γ-symmetryΓ Do there even exist interesting examples of attractors will full Γ-symmetryΓ

3. An exotic equivariant flow on $\mathbb{R}^4$

We describe some recent work with X Peng. More details and computations can be found in [25, Appendix].

Let the symmetric group S_5 act on $\mathbb{R}^5$ by permutation of coordinates. Let H be the zero locus of $x_1 + \ldots + x_5$. Obviously, H is S_5-invariant. Furthermore S_5 acts (absolutely) irreducibly on H (that is, there are no proper S_5-invariant subspaces of H). Identify H with $\mathbb{R}^4$.

Let $\hat{\Gamma} \subset S_5$ be the group generated by

$$s = (12345), \quad t = (2453).$$

It is straightforward to verify that $|\hat{\Gamma}| = 20$ and that

$$st = ts^2.$$

It is easy to show that $\hat{\Gamma}$ is isomorphic to $\mathrm{Aff}_1(\mathbb{F}_5)$ – the invertible affine linear transformations of the field with five elements. The representation of S_5 on $\mathbb{R}^4$ yields a representation of $\hat{\Gamma}$ on $\mathbb{R}^4$ and it follows from the double transitivity of $\hat{\Gamma}$ that the representation is absolutely irreducible. Let $\Gamma = \langle \hat{\Gamma}, -I \rangle$. It follows from the above that $|\Gamma| = 40$ and Γ acts absolutely irreducibly on $\mathbb{R}^4$.

3.1. A basis for the action of Γ on $\mathbb{R}^4 = \mathbb{C}^2$.

Set $V\mathbb{C}^2$. Let $\omega = \exp(\frac{2\pi\imath}{5})$. It may be shown[1] that the action of Γ on $\mathbb{R}^4$ is equivalent to the action of Γ on V defined by

$$\begin{aligned} s(z_1, z_2) &= (\omega z_1, \omega^2 z_2), \\ t(z_1, z_2) &= (\bar{z}_2, z_1), \\ -I(z_1, z_2) &= (-z_1, -z_2). \end{aligned}$$

[1] See [25] or observe that since $|\hat{\Gamma}| = 20 < 2 \times 4^2$, $\hat{\Gamma}$ only has one irreducible representation of degree four.

3.2. **Geometry of the representation** (V,Γ). We take (complex) coordinates (z_1,z_2) on V and corresponding real coordinates (x_1,y_1,x_2,y_2), where $z_i = x_i + \imath y_i$, $i=1,2$.

Lemma 3.1. *Representative proper fixed point spaces of* (V,Γ) *are given by*
(A) $\mathbf{A} = \mathbb{R}(1,0,1,0) = V^{(t)}$ *(axis of symmetry).*
(B) $\mathbf{B} = \mathbb{R}(1,0,-1,0) = V^{(-t)}$ *(axis of symmetry).*
(S) $\mathbf{S} = \{(x_1,0,x_2,0) \mid x_1,x_2 \in \mathbb{R}\} = V^{(t^2)}$ *(submaximal stratum).*
(P) $\mathbf{P} = \{(0,y_1,0,y_2) \mid y_1,y_2 \in \mathbb{R}\} = V^{(-t^2)}$ *(maximal orbit stratum).*
There are exactly five axes of type (A), five axes of type (B), five planes of type (S) and five planes of type (P). There are no three dimensional fixed point subspaces.

3.3. **Cubic equivariant vector fields on** V.

Lemma 3.2. *A basis for the homogeneous cubic equivariant polynomial maps from* V *to* V *is given by*

$$R(z_1,z_2) = (|z_1|^2+|z_2|^2)(z_1,z_2),\quad p(z_1,z_2) = (\bar{z}_1^2\bar{z}_2, z_1\bar{z}_2^2),\quad q(z_1,z_2) = (z_2^3,\bar{z}_1^3)$$

We consider the following Γ-equivariant family of vector fields on V.

$$z_1' = \lambda z_1 - (|z_1|^2+|z_2|^2)z_1 + \beta\bar{z}_1^2\bar{z}_2 + \gamma z_2^3, \tag{3.1}$$
$$z_2' = \lambda z_2 - (|z_1|^2+|z_2|^2)z_2 + \beta z_1\bar{z}_2^2 + \gamma\bar{z}_1^3. \tag{3.2}$$

It turns out that if $|\beta+\gamma| < 2$ then there is a supercritical bifurcation at $\lambda = 0$ and that for $\lambda > 0$, all non-zero trajectories are forward asymptotic to an attracting Γ-invariant 3-sphere. Henceforth, we assume $|\beta+\gamma| < 2$.

3.4. **Equilibria along axes of symmetry.** Rewriting the system (3.1,3.2) in real coordinates, we find that

$$\begin{aligned}
x_1' &= \lambda x_1 - \|x\|^2 x_1 + \beta(x_1^2x_2 - y_1^2x_2 - 2x_1y_1y_2) + \gamma(x_2^3 - 3x_2y_2^2),\\
y_1' &= \lambda y_1 - \|x\|^2 y_1 + \beta(-x_1^2y_2 + y_1^2y_2 - 2x_1y_1x_2) + \gamma(-y_2^3 + 3x_2^2y_2),\\
x_2' &= \lambda x_2 - \|x\|^2 x_2 + \beta(x_1x_2^2 - x_1y_2^2 + 2y_1x_2y_2) + \gamma(x_1^3 - 3x_1y_1^2),\\
y_2' &= \lambda y_2 - \|x\|^2 y_2 + \beta(y_1x_2^2 - y_1y_2^2 - 2x_1x_2y_2) + \gamma(y_1^3 - 3x_1^2y_1).
\end{aligned}$$

Equilibria on the axis $\mathbf{A}$. If $\lambda \ge 0$, then there is a pair of equilibria $\pm a(\lambda)$ on the axis $\mathbf{A}$. Computing, we find that

$$a(\lambda) = (\sqrt{\frac{-\lambda}{\beta+\gamma-2}}, 0, \sqrt{\frac{-\lambda}{\beta+\gamma-2}}, 0).$$

The eigenvalues of the linearization of the cubic system at $\pm a(\lambda)$ are

$$[-2\lambda : \frac{4\lambda\gamma}{\beta+\gamma-2} : \frac{\lambda}{\beta+\gamma-2}[(\gamma+3\beta) \pm \imath(3\gamma-\beta)].$$

Note that -2λ is the eigenvalue associated to the radial direction and that the eigenspace of the eigenvalue $\frac{4\lambda\gamma}{\beta+\gamma-2}$ lies in the x_1,x_2-plane $V^{(t^2)}$. The invariant space associated to the remaining pair of eigenvalues is the y_1,y_2-plane $V^{(-t^2)}$.

Equilibria on the axis $\mathbf{B}$. For $\lambda \geq 0$, there is a pair of equilibria $\pm b(\lambda)$ on the axis $\mathbf{B}$ given by

$$b(\lambda) = (\sqrt{\frac{\lambda}{\beta+\gamma+2}}, 0, -\sqrt{\frac{\lambda}{\beta+\gamma+2}}, 0).$$

The eigenvalues of the linearization of the cubic system at $\pm b(\lambda)$ are

$$[-2\lambda : \frac{4\lambda\gamma}{\beta+\gamma+2} : \frac{\lambda}{\beta+\gamma+2}[(\gamma+3\beta) \pm \imath(3\gamma-\beta)].$$

The eigenspace of the eigenvalue $\frac{4\lambda\gamma}{\beta+\gamma+2}$ lies in the x_1, x_2-plane $V^{(t^2)}$. The invariant space associated to the remaining pair of eigenvalues is the y_1, y_2-plane $V^{(-t^2)}$.

Dynamics on and near the x_1, x_2*-plane* $\mathbf{S}$. Dynamics on $\mathbf{S}$ are governed by the system

$$x_1' = \lambda x_1 - (x_1^2 + x_2^2)x_1 + \beta x_1^2 x_2 + \gamma x_2^3, \tag{3.3}$$

$$x_2' = \lambda x_2 - (x_1^2 + x_2^2)x_2 + \beta x_1 x_2^2 + \gamma x_1^3. \tag{3.4}$$

Since we are assuming that $|\beta+\gamma| < 2$, (3.3,3.4) has a globally attracting invariant circle $C(\lambda)$, for $\lambda > 0$. Necessarily, $\pm a(\lambda), \pm b(\lambda) \in C(\lambda)$.

Lemma 3.3. *Let $\lambda > 0$. If $\gamma \neq 0$ and $|\beta+\gamma| < 2$, then $\pm a(\lambda)$, $\pm b(\lambda)$ are the only nonzero equilibrium points of (3.3,3.4).*

Lemma 3.4. *Let $\lambda > 0$, $\gamma \neq 0$, and $|\beta+\gamma| < 2$. Stabilities of $a(\lambda)$, $b(\lambda)$ in the x_1, x_2-plane are as follows.*

(a) *If $\gamma > 0$, $\pm a(\lambda)$ are sinks, $\pm b(\lambda)$ are saddles and*

$$C(\lambda) = W^u(b(\lambda)) \cup W^u(-b(\lambda)) \cup \{a(\lambda), -a(\lambda)\}.$$

(b) *If $\gamma < 0$, $\pm b(\lambda)$ are sinks, $\pm a(\lambda)$ are saddles and*

$$C(\lambda) = W^u(a(\lambda)) \cup W^u(-a(\lambda)) \cup \{b(\lambda), -b(\lambda)\}.$$

Lemma 3.5. *Suppose that $\lambda > 0$ and and $|\beta+\gamma| < 2$.*

(1) *If $\gamma > 0$ and $\gamma + 3\beta < 0$, then $dim(W^u(a(\lambda))) = 2$, $dim(W^s(b(\lambda))) = 3$, $W^u(a(\lambda))$ and $W^s(b(\lambda))$ are transverse to $\mathbf{S}$ and there is a one-dimensional connection from $b(\lambda)$ to $a(\lambda)$ in $\mathbf{S}$.*

(2) *If $\gamma < 0$ and $\gamma + 3\beta > 0$, then $dim(W^u(b(\lambda))) = 2$, $dim(W^s(a(\lambda))) = 3$, $W^u(b(\lambda))$ and $W^s(a(\lambda))$ are transverse to $\mathbf{S}$ and there is a one-dimensional connection from $a(\lambda)$ to $b(\lambda)$ in $\mathbf{S}$.*

(3) *If the conditions of (1) and (2) are not satisfied and $\gamma(\gamma+3\beta) \neq 0$, then exactly one of $a(\lambda)$, $b(\lambda)$ is a sink.*

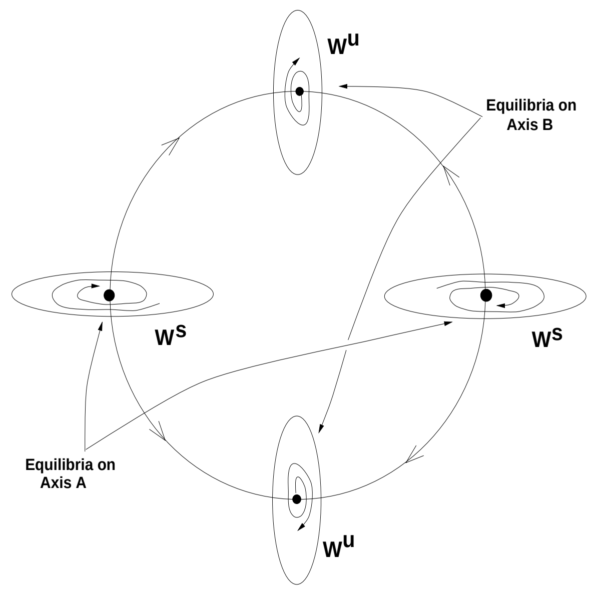

FIGURE 5. Dynamics near x_1, x_2 plane, case III, III*

Dynamics on the y_1, y_2-plane $\mathbf{P}$. We refer the reader to [25, Appendix] for detailed results. Suffice it to say that either there are 8 equilibria on an invariant circle or there is a limit cycle. In all cases, equilibria and cycles are saddles.

3.5. **Equilibria.**

Lemma 3.6. *Let $\lambda > 0$. Provided that $\gamma(3\beta+\gamma) \neq 0$, the equilibria of (3.1,3.2) are hyperbolic and lie on the Γ-orbits of the axes* $\mathbf{A}$*,* $\mathbf{B}$ *and the plane* $\mathbf{P}$*.*

In Figure 6, we have divided the β,γ-plane into six stability regions. In regions I and II, (3.1,3.2) has ten hyperbolic sinks, all lying on the Γ-orbit of the axis $\mathbf{A}$. In regions III there are no sinks (or sources). In regions II and III there are five limit cycles lying in the Γ orbit of $\mathbf{P}$. We have an analogous situation in the regions I*, II* and III*, with A and B interchanged.

3.6. **Dynamics.** Using the dynamical systems package *dstool* [8], we have investigated the dynamics of (3.1,3.2). Granted our assumption $|\beta+\gamma| < 2$ and the homogeneity of higher order terms, we may fix $\lambda = 1$.

3.7. **The regions I,I*, and II, II*.** In these regions, there are always ten hyperbolic sinks lying on the Γ-orbits of one or other of the axes $\mathbf{A}$, $\mathbf{B}$. Typically, dynamics are asymptotic to one of these sinks.

3.8. **The regions III, III*.** For $\beta,\gamma \in \mathrm{III} \cup \mathrm{III}^\star$, there are no equilibria which are sinks (or sources). Moreover, the limit cycle in the plane $\mathbf{P}$ is neither attracting nor repelling.

Restricting to the attracting flow-invariant three sphere, we see that there is a possibility that there are heteroclinic cycles connecting equilibria on axes of type $\mathbf{A}$ and $\mathbf{B}$. Indeed, there is always a one-dimensional connection between $a(1)$ and $b(1)$ in the plane $\mathbf{S}$. If, for example,

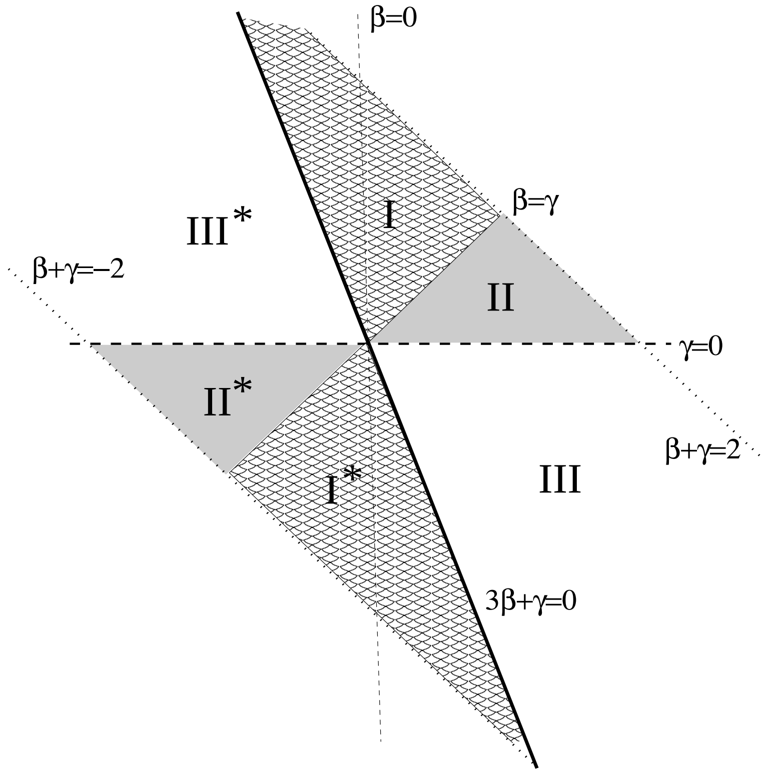

FIGURE 6. Stability of equilibria

$a(1)$ is a sink for the dynamics restricted to $\mathbf{S}$, then $\dim(W^u(a(1))) = 2$, $\dim(W^s(b(1))) = 2$ (restricted to the invariant sphere) and so there is the possibility of a persistent (transverse) intersection between $W^u(a(1))$ and (the Γ-orbit) of $W^s(b(1))$. This situation, similar to what occurs in the Shilnikov bifurcation [34, 6.5], can be expected to lead to very complex dynamics. In our case, the symmetry is likely to result in the formation of a 'Shilnikov network' of heteroclinic cycles. Since the limit cycle in the plane $\mathbf{P}$ is a hyperbolic saddle, there are the additional possibilities of horseshoe dynamics, due to transverse intersections between the stable and unstable manifolds of the limit cycles, as well as intersections between the invariant manifolds of equilibria and limit cycles.

Numerical simulations confirm the complexity of the dynamics that arise in this system. In particular, it is likely that there are heteroclinic cycles connecting equilibria on axes of type **A** and **B**. In Figures 7, 8, we show time series obtained using *dstool*. In Figure 7, we show the plot of the variable x_1 against time when β, γ is close to the boundary of the region III*. In the second plot, we show the time series when β, γ is in region III and well away from the boundary. In both cases, we have used a time step of $\Delta t = 0.002$ and plotted the result of 200,000 steps of a 4th order Runge-Kutta algorithm. Finally, we note that for least some values of $\beta, \gamma \in \mathrm{III} \cup \mathrm{III}^\star$, (3.1,3.2) has an attracting asymmetric limit cycle.

Finally, in Figure 4, we show the projection into the x_1, y_1-plane of a characteristic flow (parameters $\beta = 1$, $\gamma = -0.6$)

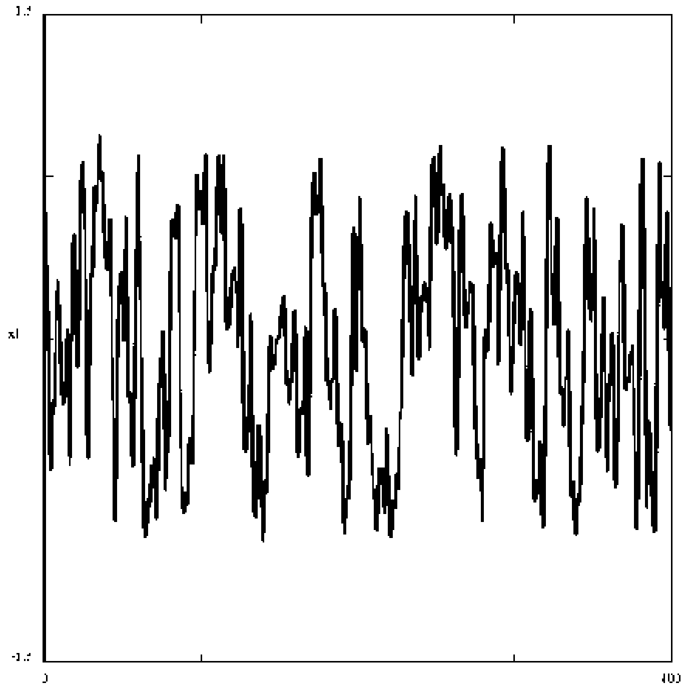

FIGURE 7. Time series for x_1 when $\beta = -0.35$, $\gamma = 1$

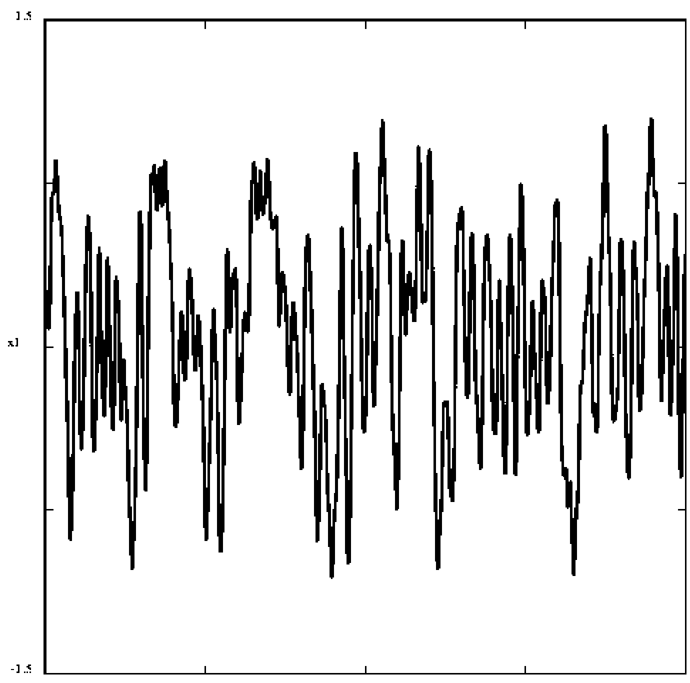

FIGURE 8. Time series for x_1 when $\beta = 1$, $\gamma = -0.6$

4. An attractor in $\mathbb{R}^3$

For the remainder of this lecture, we describe Williams' construction of expanding hyperbolic attractors [50] (see also the lecture notes by Newhouse [43]). Rather than giving the general theory, we look at one specific example in $\mathbb{R}^3$ (no symmetry) that contains most of the features of the general case.

4.1. **Abstract set up.** Let $f : S^1 \to S^1$ be the map $f(z) = z^2$. Obviously f is an *expanding map*: $\|Tf(X)\| = 2\|X\|$, all $X \in TS^1$. We define the classical *solenoid* $\mathcal{S} \subset \Pi^\infty S^1$ as the inverse limit

$$\mathcal{S} = \{(z_i)_{i \geq 0} \mid f(z_i) = z_{i-1}, i \geq 1\}.$$

The map f induces a *shift homeomorphism* $\sigma : \mathcal{S} \to \mathcal{S}$ by

$$\sigma(z)_j = f(z_j), \ \ (z \in \mathcal{S}, j \geq 0).$$

It is straightforward to verify that periodic points are dense in $\mathcal{S}$ and that $\sigma : \mathcal{S} \to \mathcal{S}$ is topologically transitive (this is already true for $f : S^1 \to S^1$).

4.2. **Geometric realization.** We want to realize $\sigma : \mathcal{S} \to \mathcal{S}$ as a hyperbolic attractor of a smooth diffeomorphism of $\mathbb{R}^3$.

Let $\mathbf{D} \subset \mathbb{C}$ denote the closed disk of radius one half. Consider the solid torus $\mathbf{T} = \mathbf{D} \times S^1$ Regard $\mathbf{T}$ as embedded in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ so that the core $\{0\} \times S^1$ is the unit circle in the (x, y)-plane.

Denoting coordinates on $\mathbf{T}$ by (z, θ), let $f^\star : \mathbf{T} \to \mathbf{T}$ be defined by $f^\star(z, \theta) = (0, 2\theta)$. Clearly, $f^\star$ is a smooth extension of $f : S^1 \to S^1$ to $\mathbf{T}$. Obviously, $f^\star$ is not injective and our next step will be to deform $f^\star$ to an embedding $F : \mathbf{T} \to \mathbf{T}$. Specifically, we define

$$F(z, \theta) = (\frac{z}{8} + \frac{1}{4}\exp(\imath\theta), 2\theta).$$

It is easy to verify that F is a smooth embedding. Indeed, we can make an arbitrarily C^∞-small perturbation of $f^\star$ to an embedding of $\mathbf{T}$ in $\mathbf{T}$.

For $\theta \in S^1$, let $\mathbf{D}_\theta \subset \mathbf{T}$ denote the transverse disk $\mathbf{D} \times \{\theta\}$. It follows from our construction of F that F preserves the transverse disks. That is, for each $\theta \in S^1$, we have

$$F(\mathbf{D}_\theta) \subset \mathbf{D}_{2\theta}.$$

Indeed, $F(\mathbf{T}) \cap \mathbf{D}_{2\theta} = F(\mathbf{D}_\theta) \cup F(\mathbf{D}_{\theta+\pi})$.

Define $\Lambda = \cap_{n \geq 0} F^n(\mathbf{T})$. Certainly Λ is an F-invariant compact subset of $\mathbf{T}$. It follows easily from the construction that Λ is homeomorphic to the solenoid $\mathcal{S}$ we defined above and that $F|\Lambda$ is conjugate to the shift map (points in $\mathcal{S}$ or Λ have unique history). Locally, Λ is homeomorphic to a Cantor set cross an open interval (see Figure 10).

Furthermore, Λ has a *hyperbolic* structure. The contracting subspaces are given by the transverse disks $\mathbf{D}_\theta$. More precisely, given $\lambda = (z, \theta) \in \Lambda$, let $\mathbb{E}^s_\lambda$ be the 2-dimensional linear subspace of $\mathbb{R}^3$ tangent to $\mathbf{D}_\theta$ and $\mathbb{E}^u_\lambda$ be the 1-dimensional subspace tangent to Λ at λ. In

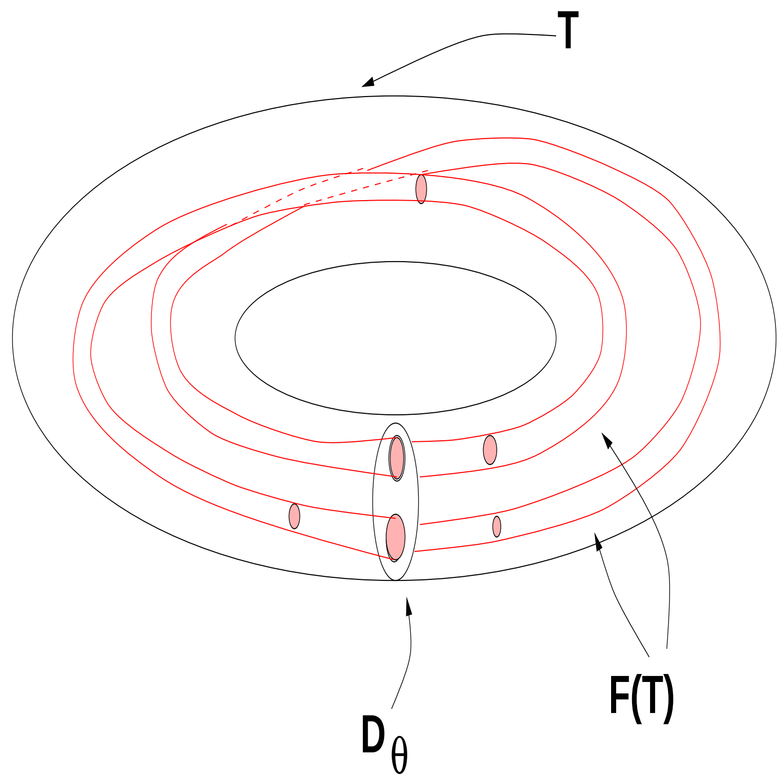

FIGURE 9. The image $F(\mathbf{T})$

this way, we define a 2-dimensional TF-invariant contracting bundle $\mathbb{E}^u$ and 1-dimensional TF-invariant expanding bundle $\mathbb{E}^u$ over Λ such that

$$T_\Lambda \mathbb{R}^3 = \mathbb{E}^u \oplus \mathbb{E}^s.$$

We also have the hyperbolic estimates

$$\begin{aligned} \|TF(X)\| &= \frac{1}{8}\|X\|, \ (X \in \mathbb{E}^s), \\ \|TF(X)\| &= 2\|X\|, \ (X \in \mathbb{E}^u). \end{aligned}$$

The problem remains of extending F to a diffeomorphism of $\mathbb{R}^3$. For this we need the isotopy theorem of differential topology. In our case, we are given a smooth embedding $F : \mathbf{T} \rightarrow \mathbb{R}^3$. It follows from the isotopy theorem that F extends to a diffeomorphism of $\mathbb{R}^3$ if F is smoothly isotopic[2] to the identity map of $\mathbf{T}$. This is easy to verify for our example ($F(\mathbf{T})$ is unknotted) and so F extends to a smooth diffeomorphism of $\mathbb{R}^3$.

To summarize the method: Start with a smooth expanding map of a 1-dimensional space. Embed in Euclidean space. Thicken to a manifold with boundary, foliated by transverse disks (tubular neighborhood). Perturb to an embedding. Make sure that the perturbed map is isotopic to the identity. Extend to ambient space. The contracting directions correspond to

[2]That is, there exists a smooth map $H : \mathbf{T} \times [0,1] \rightarrow \mathbb{R}^3$ such that $H_t : \mathbf{T} \rightarrow \mathbb{R}^3$ is an embedding, $t \in [0,1]$, and $H_0 = F$, $H_1 = I_{\mathbf{T}}$.

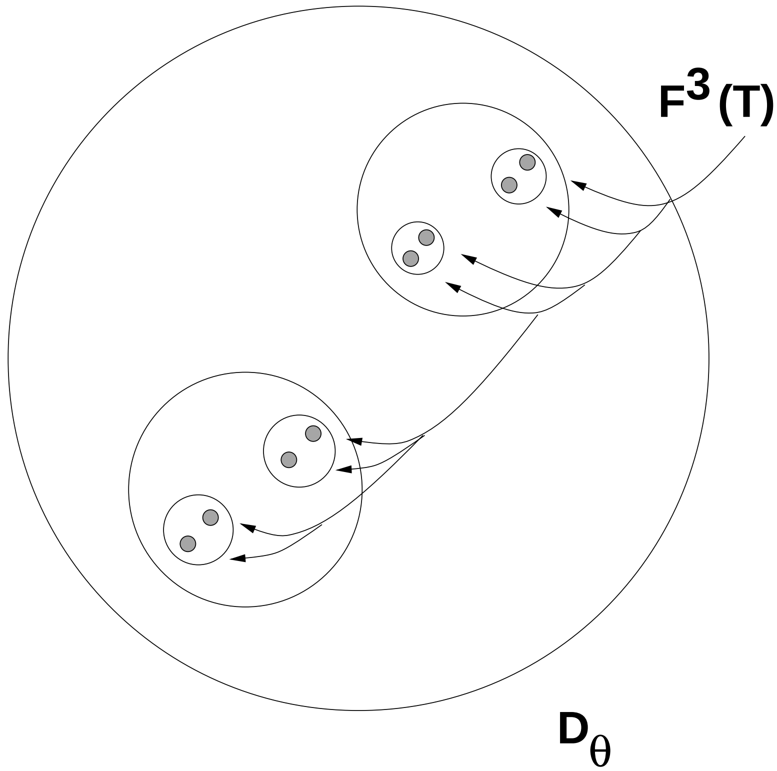

FIGURE 10. $F^3(\mathbf{T}) \cap \mathbf{D}_\theta$

transverse disks, the expanding directions are approximately those of the core 1-dimensional manifold.

5. Notes on Lecture 1

Useful references on compact (non-finite) Lie groups and their representations include the books by Adams [1] and Bröcker & tom Dieck [13].

5.1. **Haar measure.** Every compact Lie group Γ carries a unique left and right translation invariant probability measure $d\mu$ – Haar measure. Haar measure provides a vital tool in constructing symmetric objects. For example, suppose X is a smooth vector field on a Γ-manifold M. If we define

$$\tilde{X}(x) = \int_\Gamma \gamma^{-1} X(\gamma x)\, d\mu, \quad (x \in M),$$

then it is easy to see that $\tilde{X}$ is smooth and Γ-equivariant. Using Haar measure, we can average any Riemannian metric on M to obtain a Γ-invariant Riemannian metric on M. In case $M = \mathbb{R}^n$ and $\Gamma \subset \mathrm{GL}(n, \mathbb{R})$, it follows that we can always choose a Γ-invariant inner product on $\mathbb{R}^n$ relative to which $\Gamma \subset \mathrm{O}(n)$.

5.2. **Lie groups.** Let $\mathrm{O}(n)$ denote the group of $n \times n$ orthogonal matrices and $\mathrm{SO}(n)$ denote the determinant one subgroup of $\mathrm{O}(n)$ – the special orthogonal group. Both $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ are compact *submanifolds* of $\mathrm{GL}(n, \mathbb{R})$. In particular, group composition and inversion define smooth (in fact algebraic) operations on $\mathrm{O}(n)$ and $\mathrm{SO}(n)$.

It may be shown that every closed subgroup of a Lie group is a Lie group. In particular, every closed subgroup of $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. Furthermore, if Γ is a *compact* Lie group then Γ may be represented as a subgroup of an orthogonal group. This follows from the fact that every compact Lie group has a *faithful* representation.

Example 5.1. The group $\mathbb{K}$ is Abelian. For $m \geq 1$, let $\mathbb{K}^m$ denote the m-fold product of $\mathbb{K}$. We refer to $\mathbb{K}^m$ as the m-torus. Every compact connected m-dimensional Abelian Lie group is (smoothly) isomorphic to $\mathbb{K}^m$. ♡

Let Γ be a connected Lie group with center $Z(\Gamma)$. We recall that

- Γ is *simple* if Γ has no nontrivial normal subgroups.
- Γ is *almost simple* if every normal subgroup of Γ is finite.
- Γ is semisimple if $Z(\Gamma)$ is finite.

Remark 5.2. Let $\mathfrak{g}$ denote the Lie algebra of Γ. Since Γ is connected, every *finite* normal subgroup of Γ is a subgroup of $Z(\Gamma)$. Consequently, Γ is simple as a group if and only if $\mathfrak{g}$ is a simple Lie algebra and $Z(\Gamma)$ is trivial. If $\mathfrak{g}$ is simple, then Γ is almost simple. ♢

Examples 5.3. (1) For $n > 2$, the groups $\mathrm{SO}(n)$ are not Abelian. In fact, $\mathrm{SO}(2n)$ is almost simple, $n > 1$, and $\mathrm{SO}(2n+1)$ is simple $n > 0$.
(2) For $n \geq 2$, the special unitary groups $\mathrm{SU}(n)$ are almost simple.
(3) For $n \geq 2$, the symplectic groups $\mathrm{Sp}(n)$ are compact, connected and almost simple ($\mathrm{Sp}(1) = \mathrm{SU}(2)$).
(4) There are five 'exceptional' almost simple Lie groups: $\mathbf{G}_2$, $\mathbf{F}_4$, $\mathbf{E}_6$, $\mathbf{E}_7$ and $\mathbf{E}_8$.
(5) If we let $G^\star$ denote the universal cover of G, then it is known that every compact connected Lie group Γ is a quotient by a finite group of a product $\mathbb{K}^m \times G_1^\star \times \dots G_N^\star$, where $G_1, \dots, G_N$ are groups listed in (1–4). ♡

5.3. **Representations and actions.** Suppose that Γ is a compact Lie group. A representation of Γ on $\mathbb{R}^n$ is a homomorphism $\rho : \Gamma \rightarrow \mathrm{O}(n)$. Associated to ρ, we have an action of Γ on $\mathbb{R}^n$ defined by

$$\gamma(X) = \rho(\gamma)(X), \ (\gamma \in \Gamma, X \in \mathbb{R}^n).$$

In the sequel, we usually write γX rather than $\gamma(X)$.

More generally, if M is a smooth manifold with diffeomorphism group $\mathrm{Diff}(M)$, and $\rho : \Gamma \rightarrow \mathrm{Diff}(M)$ is a homomorphism then we may define an action of Γ on M by

$$\gamma(m) = \rho(\gamma)(m), \ (\gamma \in \Gamma, m \in M).$$

Example 5.4. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$. If we are given a representation of Γ on $\mathbb{R}^{n+1}$ then this action restricts to give a smooth action of Γ on S^n. ♡

We say that a map $f : M \rightarrow M$ is Γ-*equivariant* (or just equivariant) if

$$f(\gamma m) = \gamma f(m), \quad (m \in M, \gamma \in \Gamma).$$

If TM denotes the tangent bundle of M, we have a natural action of Γ on TM defined by $\gamma(X) = T\gamma(X)$, $X \in TM$, $\gamma \in \Gamma$. We say that a vector field X on M is equivariant if $X : M \rightarrow TM$ is Γ-equivariant with respect to the actions of Γ on M and TM.

5.4. **Orbits and isotropy groups.** Suppose that the compact Lie Γ acts smoothly on M. Given $m \in M$, define

$$\Gamma m = \{\gamma m \mid \gamma \in \Gamma\}.$$

We refer to Γm as the Γ-*orbit* through m or just the Γ-orbit of m. Γm is invariant by Γ and so we have an induced action of Γ on Γm.

Given $m \in M$, we define the *isotropy subgroup* of Γ at m by

$$\Gamma_m = \{\gamma \in \Gamma \mid \gamma m = m\}.$$

Obviously, Γ_m is a closed (therefore Lie) subgroup of Γ.

Using the easily proven fact that the homogeneous space $\Gamma \rightarrow \Gamma/\Gamma_m$ admits local smooth sections, it is not hard to show that Γm is a smooth submanifold of M, all $m \in M$. For this, and other details about compact Lie group actions, see [10], especially Chapter 6.

We have the following elementary properties relating group orbits and isotropy groups.

- For all $\gamma \in \Gamma$, $m \in M$, $\Gamma_{\gamma m} = \gamma \Gamma_m \gamma^{-1}$.
- Γm is Γ-equivariantly diffeomorphic to Γ/Γ_m, where we let Γ act on Γ/Γ_m by left translation. Conversely, Γ_m is Γ-equivariantly diffeomorphic to Γ_n if and only if Γ_m and Γ_n are *conjugate* subgroups of Γ: $\exists \gamma \in \Gamma$ such that $\Gamma_n = \gamma \Gamma_m \gamma^{-1}$.

If H is an isotropy group for the action of Γ on M, let (H) denote the conjugacy class of H in Γ. It follows easily from the differentiable slice theorem (see below) that if M is compact or a Γ-representation, then there are only finitely many isotropy types.

5.5. **Mappings and isotropy groups.** Suppose M and N are Γ-manifolds and that $f : M \rightarrow N$ is Γ-equivariant. It follows from the equivariance of f that

$$\Gamma_x \subset \Gamma_{f(x)}, \quad (x \in M). \tag{5.5}$$

Furthermore, if f is 1:1 then $\Gamma_x = \Gamma_{f(x)}$, all $x \in M$.

If H is a subgroup of Γ, let M^H denote the fixed point space of the action of H on M. That is,

$$M^H = \{x \in M \mid hx = x, \text{ for all } h \in H\}.$$

It follows from (5.5) that if $f : M \rightarrow N$ is Γ-equivariant, then $f(M^H) \subset N^H$ for all subgroups H of Γ. This observation has been made the basis of a small industry in case f is a vector field on M and M^H is 1-dimensional [32].

5.6. **Slice theorem.** ¿From the technical point of view, the most important property of the smooth action of a compact Lie group is the existence of *slices*. Roughly speaking, the differentiable slice theorem says that compact Lie group actions are locally linearizable transverse to the group action.

Theorem 5.5. *Let M be a Γ-manifold. Given $m \in M$, there exists a Γ_m-representation (V, Γ_m) and a smooth Γ_m-equivariant embedding $j : V \to M$ satisfying the following properties:*

(1) $j(0) = m$.
(2) $T_m M = T_m \Gamma m \oplus T_m j(V)$ *(j is transverse to Γm at m).*
(3) $\gamma j(V) \cap j(V) \neq \emptyset$ *if and only if* $\gamma \in \Gamma_m$.
(4) $\cup_{\gamma \in \Gamma} \gamma j(V)$ *is an open neighborhood of* Γ_m.

The set $j(V)$ is called a differentiable slice for the action at m.

It is useful to reformulate the slice theorem. To this end, observe that if (V, Γ_m) is a Γ_m-representation, then we can define a free action of Γ_m on the product $\Gamma \times V$ by

$$\gamma(g, v) = (g\gamma^{-1}, \gamma v), \quad ((g, v) \in \Gamma \times V, \gamma \in \Gamma_m).$$

Let $\Gamma \times_{\Gamma_m} V$ denote the orbit space $(\Gamma \times V)/\Gamma_m$. Since the action of Γ_m on $\Gamma \times V$ is free, $\Gamma \times_{\Gamma_m} V$ is smooth[3]. Since the action of Γ on $\Gamma \times V$ defined by left translation on the first factor Γ commutes with the action of Γ_m, it follows that $\Gamma \times_{\Gamma_m} V$ has the natural structure of a smooth Γ-manifold. The space $\Gamma \times_{\Gamma_m} V$ is usually called the *twisted product* of Γ and V.

Fix a Γ-invariant Riemannian metric on M. Let $V = T_m \Gamma m^{-}$. Since $T_m \Gamma m$ is obviously Γ_m-invariant, it follows that V is a Γ_m-representation. The normal bundle of Γm is easily seen to be isomorphic to the twisted product $\Gamma \times_{\Gamma_m} V$. Hence the slice theorem follows from the equivariant version of the tubular neighborhood theorem. Of course, the existence of slices follows rather easily if M is a Γ-representation.

Remark 5.6. If X is a Γ-equivariant vector field defined on a neighborhood of Γm, then X may be regarded as defined on the twisted product $\Gamma \times_{\Gamma_m} V$, where V is the normal fiber at m. Moreover, X lifts equivariantly to a skew product vector field on $\Gamma \times V$. For a nice presentation of this lifting, in the context of proper actions of non-compact Lie groups, see [19, §2]. Similar results hold for equivariant diffeomorphisms which are equivariantly isotopic to the identity [24, §6], [22]. $\diamondsuit$

Let M be a smooth Γ-manifold. We say that a point $m \in M$ has *principal isotropy type* if given any $x \in M$ there exists $\gamma \in \Gamma$ such that $\Gamma_m \subset \gamma \Gamma_x \gamma^{-1}$. That is, the isotropy of m is as small as possible.

An important, and easy, consequence of the differentiable slice theorem is the following result.

Theorem 5.7. *Let M be a connected smooth Γ-manifold. The set of points in M with principal isotropy type form an open and dense subset of M.*

[3]In fact, $\Gamma \times_{\Gamma_m} V$ has the structure of a real non-singular algebraic variety.

Remark 5.8. If Γ is finite and M is connected, it is no loss of generality to assume that points with principal isotropy type have trivial isotropy. Indeed, suppose $m \in M$ has principal isotropy. Let $U \subset M$ be the subset consisting of points with isotropy group Γ_m. It follows from the slice theorem that U is an open subset of M. If $x \in \partial U$ and S is a slice at x, then Γ_m fixes all points in the open subset $U \cap S$ and so must fix all points in S. It follows that Γ_m fixes all points in M. Since $\Gamma_{\gamma m} = \gamma \Gamma_m \gamma^{-1}$, it follows that Γ_m is a normal subgroup of Γ. Now replace Γ by Γ / Γ_m.

Although this result does not generalize to general compact Lie groups, it does hold for 'most' representations. $\diamondsuit$

A second important application of the differentiable slice theorem is in the construction of equivariant vector fields and maps. For example, suppose that $j : V \to M$ is a slice at m. Let X be a Γ_m-equivariant vector field defined on $j(V)$ (that is, a smooth section of $T_{j(V)}M$. The vector field X extends by Γ-equivariance to all of $\Gamma(j(V))$. Indeed, for $v \in j(V)$, $\gamma \in \Gamma$, define $\tilde{X}(\gamma v) = \gamma X(v)$. If X has compact support, the same is true for $\tilde{X}$ and so $\tilde{X}$ extends by zero to all of M.

5.7. **Isotopy theorem.** Let M be a smooth Γ-manifold and suppose that N is a closed Γ-invariant subset of M. We shall shortly encounter situations where we have defined a smooth equivariant embedding $i : N \to M$ and we want to extend i to a smooth Γ-equivariant diffeomorphism of M. The equivariant version of the isotopy theorem tells us when we can do this (for the detailed proof, see Bredon [10, Chapter VI]).

Theorem 5.9. *Let N be a compact Γ-invariant subset of the Γ-manifold M. Suppose that $i : N \to M$ is a smooth equivariant embedding and that i is smoothly equivariantly isotopic to the identity map of N. Then i extends to a smooth equivariant diffeomorphism of M.*

Proof. We follow the proof of the standard isotopy theorem in differential topology. The main part of that proof is to construct a smooth vector field on $M \times [0, 1]$ which extends the vector field defined by the isotopy. Average this vector field over Γ and use the fact that the flow of an equivariant vector field is equivariant. $\square$

6. Lecture II: Constructing hyperbolic symmetric attractors

In this lecture we want to describe some recent work by Field, Melbourne & Nicol [29] on the construction of symmetric hyperbolic expanding attractors.

Suppose that $\Gamma \subset \mathrm{O}(n)$ is finite. Let H be a subgroup of Γ. We ask for conditions on $(\mathbb{R}^n, \Gamma)$ that yield the existence of a Γ-equivariant diffeomorphism or flow on $\mathbb{R}^n$ which has a hyperbolic attractor A with $\Sigma_A = H$. (Recall that we always assume that A is topologically transitive.)

As we indicated in Lecture I, if Γ is finite it is no loss of generality to assume that principal isotropy groups are trivial. Granted this, we will attempt to construct A so that it consists of points of trivial isotropy.

An involution $\gamma \in \Gamma$ is called a *reflection* if the fixed point space of γ is $(n-1)$-dimensional. We call an $(n-1)$-dimensional linear subspace V of $\mathbb{R}^n$ a *reflection hyperplane* of Γ contains a reflection with fixed point space V. Reflections provide the only obstruction to constructing attractors with specified symmetry group.

Let $\mathcal{L}$ denote the union of all the reflection hyperplanes of Γ. The connected components of $\mathbb{R}^n \setminus \mathcal{L}$ are permuted by Γ. Obviously, an equivariant flow on $\mathbb{R}^n$ fixes the components of $\mathbb{R}^n \setminus \mathcal{L}$ (as the reflection hyperplanes are invariant by the flow). Consequently, if A is the attractor of a flow on $\mathbb{R}^n$ then A must lie in (the closure of) a single connected component C of $\mathbb{R}^n \setminus \mathcal{L}$. In particular, C is fixed by the subgroup Σ_A. Consequently, if H is a subgroup of Γ it follows that a necessary condition for there to exist a Γ-equivariant flow with an H-symmetric attractor is that H fixes a connected component of $\mathbb{R}^n \setminus \mathcal{L}$. It turns out that this condition is also sufficient. Furthermore, if $n \geq 5$, then given any subgroup $H \subset \Gamma$, there exists a Γ-equivariant flow on $\mathbb{R}^n$ which has a hyperbolic attractor A with $\Sigma_A = H$.

The situation for equivariant diffeomorphisms is a little more complicated. To simplify matters, we shall assume that $(\mathbb{R}^n, \Gamma)$ has no reflections (for the general case, see [29]). Under these conditions we have

Theorem 6.1. *Let H be a subgroup of Γ and suppose $n \geq 4$. There exists a smooth Γ-equivariant diffeomorphism of $\mathbb{R}^n$ which has a hyperbolic attractor A with $\Sigma_A = H$.*

Remark 6.2. Suppose the attractor A is a periodic orbit of prime period p. Let H denote the isotropy group of any point of A. It is easy to verify that $\Sigma_A/H \cong \mathbb{Z}_p$. We shall only construct attractors all of whose points have trivial isotropy. Consequently, if Σ_A is not cyclic then A cannot be a periodic orbit. ♢

Rather than give the full proof of Theorem 6.1, we shall describe one example that indicates all the techniques needed to prove the general case[4].

6.1. **An example in $\mathbb{R}^3$.** We suppose that $\Gamma = \mathbb{Z}_2 = \langle \kappa \rangle$ acts on $\mathbb{R}^3$ by

$$\kappa(x, y, z) = (-x, -y, z), \quad ((x, y, z) \in \mathbb{R}^3).$$

Note that the fixed point set of the action is the z-axis.

[4] In fact, this example is, in some ways, harder than the general case in dimensions greater than 3.

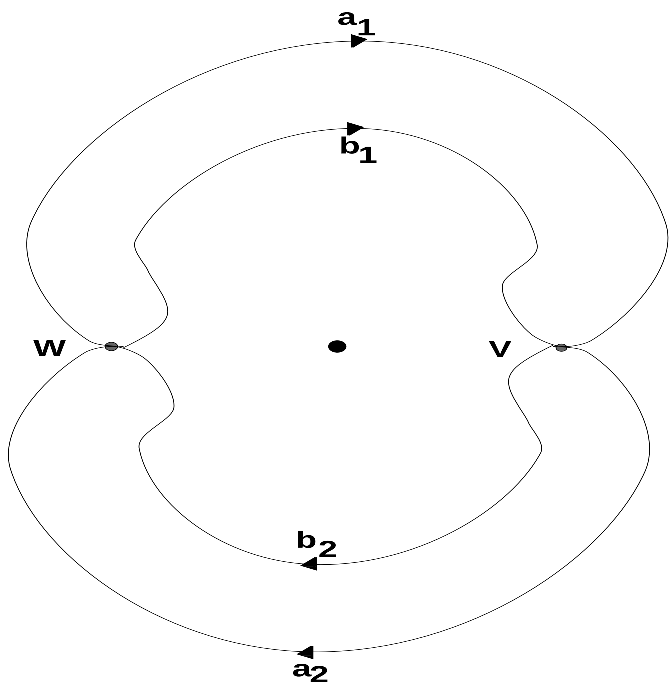

FIGURE 11. Smooth $\mathbb{Z}_2$-invariant graph $\mathcal{G} \subset \mathbb{R}^2$

We construct a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$ which has an infinite $\mathbb{Z}_2$-symmetric hyperbolic attractor.

Our strategy will be to construct a smooth $\mathbb{Z}_2$-symmetric graph $\mathcal{G}$ and $\mathbb{Z}_2$-equivariant expanding map $f : \mathcal{G} \to \mathcal{G}$ and then attempt to mimic the proof we gave for the solenoid. The main problems we encounter will be to define the $\mathbb{Z}_2$-symmetric graph $\mathcal{G}$ and expanding map f in such a way that we can use the $\mathbb{Z}_2$-equivariant version of the isotopy theorem to obtain a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$.

Example 6.3. The obvious way to attempt to construct a $\mathbb{Z}_2$-equivariant hyperbolic attractor in $\mathbb{R}^3$ is to mimic the construction of the classical solenoid given in the last lecture. Regard $\mathbb{C} \subset \mathbb{R}^3$. Although the map $u \mapsto u^2$ is not $\mathbb{Z}_2$-equivariant, the map $f(u) = u^3$ is odd and so $\mathbb{Z}_2$-equivariant. We can thicken the unit circle in $\mathbb{C}$ to a solid torus $\mathbf{T}$, extend f $\mathbb{Z}_2$-equivariantly to $f : \mathbf{T} \to \mathbf{T}$ and then perturb f to a $\mathbb{Z}_2$-equivariant embedding $\tilde{f} : \mathbf{T} \to \mathbf{T}$. It is easy to see that we can do this construction so that $\tilde{f}$ has a $\mathbb{Z}_2$-invariant hyperbolic attractor $\Lambda \subset \mathbf{T}$. The problem lies in finding an extension of $\tilde{f}$ to a $\mathbb{Z}_2$ equivariant diffeomorphism of $\mathbb{R}^3$. In fact, no such extension exists. Rather than proving this, we just observe that the isotopy theorem cannot be applied to the map $\tilde{f}$ as $\tilde{f}$ is not *equivariantly* isotopic to the identity map of $\mathbf{T}$: The image of any isotopy joining $\tilde{f}$ to $I_{\mathbf{T}}$ would have to cross the z-axis and this would violate the preservation of isotopy type by embeddings. ♡

Construction of the graph. Regard $\mathbb{Z}_2$ as acting on the (x, y)-plane $\mathbb{R}^2 \subset \mathbb{R}^3$ as multiplication by $\pm I$.

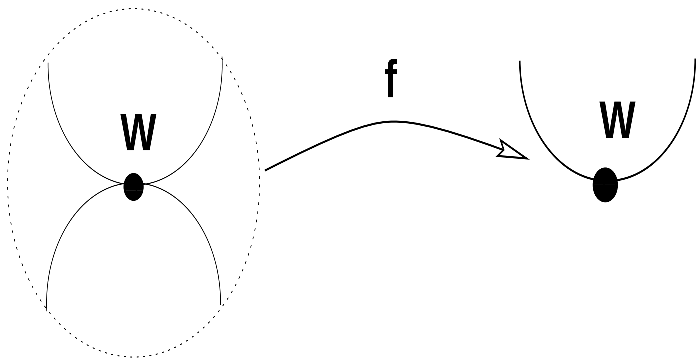

FIGURE 12. Image of a neighborhood of W

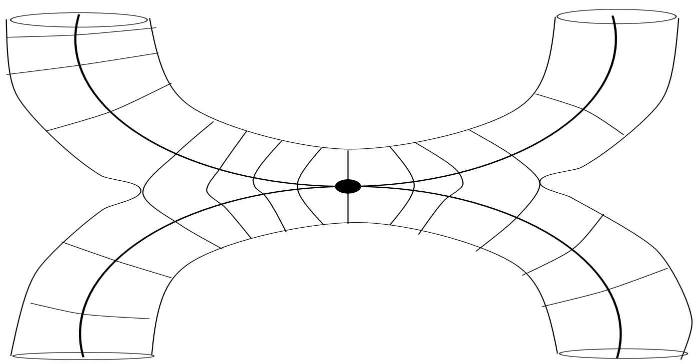

FIGURE 13. Fibers near a branch point

In Figure 11 we show a $\mathbb{Z}_2$-invariant graph $\mathcal{G} \subset \mathbb{R}^2$. Following the terminology of Williams [50], $\mathcal{G}$ is a *branched manifold*. The two branches (both diffeomorphic to circles) have infinite order contact at the vertices V, W. Note that $\mathbb{Z}_2$ maps vertices to vertices, edges to edges. A map $f : \mathcal{G} \rightarrow \mathcal{G}$ will be smooth if (and only if), f extends to a smooth map of $\mathbb{R}^2$.

We may define a smooth $\mathbb{Z}_2$-equivariant expanding map $f : \mathcal{G} \rightarrow \mathcal{G}$ by the rules:

$$
\begin{aligned}
a_1 &\mapsto a_2^{-1} b_2 a_1 \\
b_1 &\mapsto b_2^{-1} a_2 b_1 \\
a_2 &\mapsto a_1^{-1} b_1 a_2 \\
b_2 &\mapsto b_1^{-1} a_1 b_2
\end{aligned}
$$

Observe that the f-image of a small connected neighborhood of V (respectively, W) is an arc through V (respectively W). That is, the image of a neighborhood of a branch point is non-singular. We refer to Figure 12.

We now construct a $\mathbb{Z}_2$-invariant (smooth) tubular neighborhood $\mathbf{G}$ of $\mathcal{G} \subset \mathbb{R}^3$. The only potential difficulty here lies with the fibers (2-disks) of the tubular neighborhood near the branch points W, V. However, our flatness condition on the branches at W, V enables us to construct $\mathbf{G}$ so that all fibers are transverse to the core $\mathcal{G}$. In Figure 13, we show the intersection of the transverse discs with the x, y-plane near a branch point.

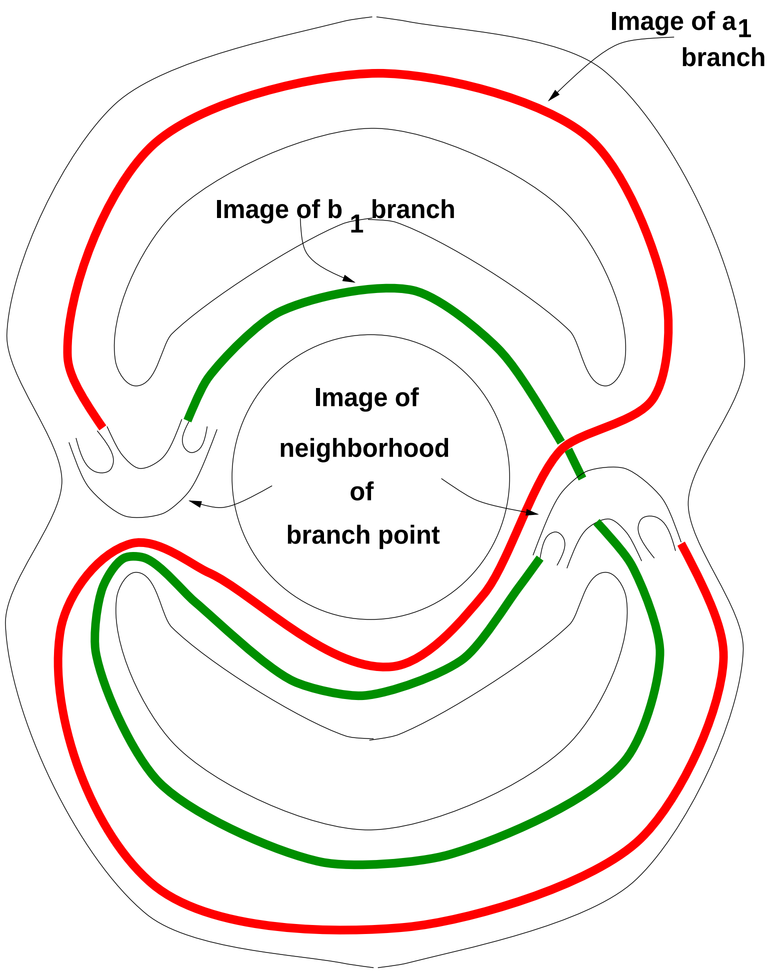

FIGURE 14. The image of F in $\mathbf{G}$

Given $u \in \mathcal{G}$, let D_u denote the transverse disk through u. Since the f-image of a small connected neighborhood of a branch point is an arc, it follows that we can choose $\mathbf{G}$ so that

$$\text{If } v \in D_u \cap \mathcal{G}, \text{ then } f(u) = f(v). \tag{6.6}$$

(Alternatively, we can fix $\mathbf{G}$ and then choose f.) Extend f smoothly and $\mathbb{Z}_2$-equivariantly to a map $\tilde{f} : \mathbf{G} \to \mathbf{G}$ such that

(a) $\tilde{f}(\mathbf{G}) = \mathcal{G}$.
(b) $\tilde{f}(D_u) = \{\tilde{f}(u)\}$, all $u \in \mathcal{G}$.

Note that we can satisfy (b) because of (6.6).

We deform $\tilde{f} : \mathbf{G} \to \mathbf{G}$ to a $\mathbb{Z}_2$-equivariant embedding $F : \mathbf{G} \to \mathbf{G}$ which preserves transverse disks (that is, for all $u \in \mathcal{G}$, $F(D_u) \subset D_{\tilde{f}(u)}$). Furthermore, we can suppose that F contract disks and expands in a complementary direction. More precisely, we may require that there exists a TF-invariant continuous splitting $T_{\mathbf{G}}\mathbb{R}^3 = \mathbb{E}^s \oplus \mathbb{E}^u$ and constants $C > 0$, $\lambda \in (0, 1)$ such that if $X \in \mathbf{G}$, $n \in \mathbb{N}$ and $F^n(X) \in \mathbf{G}$ then

(u) $\|TF^{-n}(v)\| \leq C\lambda^n\|v\|$, $v \in \mathbf{E}^u_X$,
(s) $\|TF^n(v)\| \leq C\lambda^n\|v\|$, $v \in \mathbf{E}^s_X$.

The contracting spaces $\mathbf{E}^s$ will be tangent to the disk foliation of $\mathbf{G}$.

In Figure 14, we show the image of the branches associated to the edges a_1 and b_1 in $\mathbf{G}$ (the remainder of the image is obtained by reflection in the z-axis). We may choose F so that the image of the a_1-branch lies below the (x, y)-plane in the upper half of the picture and above the (x, y)-plane in the lower half. Similarly, we may suppose that the image of the b_1-branch lies above the (x, y)-plane in the upper half of the picture and below (as shown) in the lower half. With this configuration, we can ensure that no knots or links are created between the images of the branches associated to the four edges.

We define $\Lambda = \cap_{n\geq 0} F^n(\mathbf{G})$. Just as we showed for the classical solenoid, $F : \Lambda \to \Lambda$ is ($\mathbb{Z}_2$-equivariantly) conjugate to the solenoid defined by the inverse limit of $f : \mathcal{G} \to \mathcal{G}$. Since Λ is a compact F-invariant subset of $\mathbf{G}$, it follows from estimates (u,s) that Λ is a hyperbolic attractor.

Finally, it remains to extend F to a $\mathbb{Z}_2$-equivariant diffeomorphism of $\mathbb{R}^3$. For this it suffices to note that that the F-images of the branches of $\mathbf{G}$ corresponding to the edges $a_1, \ldots, b_2$ do not wind round the z-axis. Since we constructed F so that there were no knots or links between the images of the edges, it follows easily that F is $\mathbb{Z}_2$-equivariant to the identity map of $\mathbf{G}$.

Remark 6.4. In dimensions greater than three, we no longer have to worry about the possibility of the embedded image of a tubular neighborhood having knots or links. If the codimension of the singular set[5] is greater than two, then the image of the tubular neighborhood cannot form a non-trivial link with the singular set. In particular, there will be no obstructions to applying the equivariant isotopy theorem. $\diamondsuit$

7. Notes on Lecture II

For foundational results on smooth equivariant dynamical systems see [20, 21]. In Lecture II, we ignored the simplest examples of attractors: Invariant Γ-orbits (or relative equilibria) and their periodic generalizations.

Suppose first that Γ is finite. Let x be a point of prime period p for $f : M \to M$. For simplicity, assume x has trivial isotropy. Let Σ denote the isotropy group of $o(x)$ (the f-orbit of x):

$$\Sigma = \{\gamma \in \Gamma \mid \gamma o(x) = o(x)\}.$$

It is easy to verify that $\Sigma \cong \mathbb{Z}_p$ for some $p \geq 1$. If $p = 1$, we say the orbit is *asymmetric*. Otherwise, we say it is *symmetric*. Whether symmetric or asymmetric, there is no obstruction to perturbing f so that $o(x)$ is hyperbolic [21].

If A is an attractor or ω-limit point set of an equivariant diffeomorphism, and periodic points are dense in A, one can often show that asymmetric and symmetric period points are dense. Results along these lines are given in [17]. If A is an equivariant subshift of

[5]The set of points which are not of principal isotropy type.

finite type [22], or admits an equivariant Markov partition [27], it is easy to prove density of symmetric and asymmetric periodic points directly.

The case when Γ is compact, non-finite, is more interesting. The generalization of a fixed point is a 'relatively fixed point' ('relative equilibrium' for a vector field). That is, a Γ-orbit which is left invariant by f. The natural hyperbolicity concept is then that of *normal hyperbolicity* – hyperbolicity transverse to the group orbit. Suppose then that $\alpha = \Gamma x$ is an invariant f-orbit, $f : M \rightarrow M$. Using the differentiable slice theorem, it is easy to show that we can always perturb f to require that α is normally hyperbolic [21]. It remains to understand the dynamics of $f|\alpha$. If α consists of points of trivial isotropy and Γ is connected, one may show that α is foliated by f-invariant tori [21]. For generic f, these tori will be of dimension equal to the rank of Γ. Similar results hold for flows and periodic points. If Γ is not connected or α does not consist of points of trivial isotropy, one has to examine maximal Abelian subgroups which are not tori. For more details on all of this, we refer to the articles by Krupa [37] and Field [21, 24].

In [22] constructions are given for equivariant diffeomorphisms which have subshifts of finite type with specified symmetry groups. For example, it is shown that every equivariant diffeomorphism can be C^0-approximated by an equivariantly structurally stable diffeomorphism with Ω-set consisting of subshifts of finite type. We give one simple example in the next lecture.

8. Lecture III: Stable ergodicity of skew extensions

Let f be a diffeomorphism of the compact Riemannian manifold M. We say f is *partially hyperbolic* [12] if there is a continuous Tf-invariant splitting

$$TM = \mathbb{E}^u \oplus \mathbb{E}^c \oplus \mathbb{E}^s$$

such that Tf expands $\mathbb{E}^u$, contracts $\mathbb{E}^s$ and

$$\sup \|T^s_p f\| < \inf m(T^c_p f), \quad \sup \|T^c_p f\| < \inf m(T^u_p f).$$

(Here $m(A) = \|A^{-1}\|^{-1}$.)

If the *center bundle* $\mathbb{E}^c$ is tangent to a C^1-foliation $\mathcal{F}$ of M, then partial hyperbolicity can be thought of as hyperbolicity *transverse* to the foliation $\mathcal{F}$.

Grayson, Pugh & Shub [33] have suggested that partial hyperbolicity provides a natural setting for *stable ergodicity*. For example, they proved that the time one-map of the geodesic flow on the unit tangent bundle of surface of constant negative curvature is stably ergodic within the class of volume preserving diffeomorphisms. Thus ergodicity holds on an open subset of volume preserving diffeomorphisms even though these diffeomorphisms will typically not be structurally stable. More recently, Pugh & Shub [47] extended this result to general manifolds of constant negative curvature and Wilkinson [49] has proved stable ergodicity of the time one map for all negatively curved surfaces.

All of these results are difficult to prove on account of the fact that leaves of the center foliation are typically non compact and so verification of ergodicity under perturbation requires delicate estimates.

8.1. Skew extensions.

Partially hyperbolic sets arise naturally in the study of diffeomorphisms equivariant with respect to a compact non-finite Lie group Γ.

The set of all Γ-orbits determines a (singular) foliation $\mathcal{G}$ of M. If $f : M \to M$ is Γ-equivariant, then $\mathcal{G}$ is f-invariant.

Example 8.1. Suppose that $f(\Gamma x) = \Gamma x$. It is easy to show that one can choose a Γ-invariant Riemannian metric on M with respect to which $Tf : T\Gamma x \to T\Gamma x$ is an isometry. In this situation, it natural to require hyperbolicity transverse to the Γ-orbit (*normal hyperbolicity*) – see Figure 15. ♡

More generally, if all Γ-orbits have the same dimension – and so determine a non-singular foliation – it is natural to ask about hyperbolicity transverse to Γ-orbits on all of M.

We start by reviewing some recent results that apply when the action of Γ is *free* and Γ is *Abelian*.

8.2. Result of Adler-Kitchens-Shub.

Let $\mathbf{T}^n$ denote the n-dimensional torus (no group structure). Suppose that $\phi : \mathbf{T}^n \to \mathbf{T}^n$ is *Anosov*. Let λ denote the associated (unique) C^1-invariant ergodic measure [35, §19.2].

Let $C^\infty(\mathbf{T}^n, \mathbb{K})$ denote the space of smooth $\mathbb{K}$-valued maps on $\mathbf{T}^n$. Let h denote Haar measure on $\mathbb{K}$. Given $f \in C^\infty(\mathbf{T}^n, \mathbb{K})$, define

$$\phi_f : \mathbb{K} \times \mathbf{T}^n \to \mathbb{K} \times \mathbf{T}^n$$

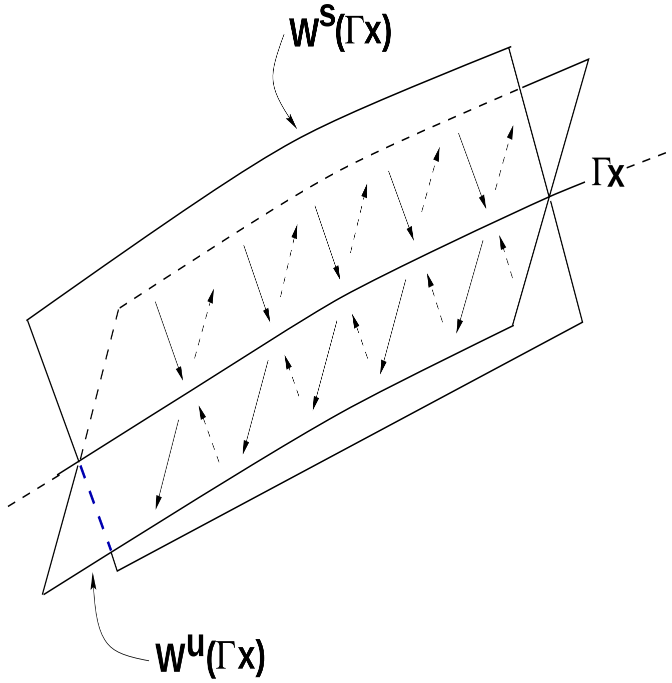

FIGURE 15. A normally hyperbolic invariant group orbit

by

$$\phi_f(k,t) = (kf(t), \phi(t)).$$

(We call ϕ_f a $\mathbb{K}$-extension or a *skew extension* of ϕ by f.) Observe that ϕ_f is measure preserving, relative to the product measure $m = h \times \lambda$ on $\mathbb{K} \times \mathbf{T}^n$. However, ϕ_f may not be ergodic.

Theorem 8.2 ([2]). *There is an open (C^0-topology) and dense (C^∞-topology) subset $\mathcal{U}$ of $C^\infty(\mathbf{T}^n, \mathbb{K})$ such that for all $f \in \mathcal{U}$, ϕ_f is (stably) ergodic.*

Proof. We sketch the main ideas used in the proof. Let $f \in C^\infty(\mathbf{T}^n, \mathbb{K})$. It follows from an old result of Brin [11], that there exists a closed subgroup H of $\mathbb{K}$ such that the ergodic components of ϕ_f define a *partition* of $\mathbb{K} \times \mathbf{T}^n$ into closed C^1 H-principal subbundles. The ergodic components are permuted by the group action. In particular, if $E \subset \mathbb{K} \times \mathbf{T}^n$ is an ergodic component then

(a) $\Sigma_E = H$.
(b) $\phi_f|E : E \to E$ is ergodic.

If $H = \mathbb{K}$, then there is just one ergodic component and so ϕ_f is ergodic. If $H \neq \mathbb{K}$, then $H \cong \mathbb{Z}_r$, some $r \geq 1$. Using the known result that if ϕ is a toral Anosov map then $1 - \phi^\star : H^1(\mathbf{T}, \mathbb{Z}) \to H^1(\mathbf{T}, \mathbb{Z})$ is an isomorphism, Adler et al. show that if $H \cong \mathbb{Z}_r$, then f is cohomologous to a constant function $\exp(2\pi\imath\frac{s}{r})$, $(s,r) = 1$. That is, there exists a continuous function $h : \mathbf{T}^n \to \mathbb{K}$ such that

$$f(t) = \exp(2\pi\imath\frac{s}{r})h(\phi(t))/h(t), \quad (t \in \mathbf{T}^n).$$

(It follows that ϕ_f is conjugate to the transformation $\Phi(k,t) = (k\exp(2\pi\imath\frac{s}{r}), \phi(t))$ and that ergodic components are always trivial H-principal subbundles. But not necessarily connected!)

Suppose that $x \in \mathbf{T}^n$ is a point of prime period p for T. Since f is cohomologous to the constant function $\exp(2\pi\imath\frac{s}{r})$ it follows easily that

$$\Pi_{j=0}^{p-1} f(\phi^j(x)) = \exp(2\pi\imath\frac{ps}{r}). \tag{8.7}$$

Observe that this product does not depend on x - only on the period of x.

Now choose a pair of periodic points of same prime period but lying on different ϕ-orbits. It follows from (8.7) that a necessary condition for $H \cong \mathbb{Z}_r$ is

$$\Pi_{j=0}^{p-1} f(\phi^j(x)) = \Pi_{j=0}^{p-1} f(\phi^j(y)), \tag{8.8}$$

This condition does not depend on r. Hence if (8.8) fails then ϕ_f is ergodic. But now the set of cocycles f satisfying

$$\Pi_{j=0}^{p-1} f(\phi^j(x)) \neq \Pi_{j=0}^{p-1} f(\phi^j(y)),$$

clearly defines a C^0-open and C^∞-dense subset of $C^\infty(\mathbf{T}^n, \mathbb{K})$. □

Remark 8.3. It is possible for (8.8) to hold for all pairs of periodic orbits with the same period and for ϕ_f still to be ergodic. For example, this will happen if f is cohomologous to the constant function $\exp(2\pi\imath\alpha)$, where α is irrational. In this case, ϕ_f will be unstably ergodic. That is, ergodic but not stably ergodic. ◊

8.3. **Results of Parry, Parry-Pollicott.** Using results of Parry [44], Parry and Pollicott [45] have recently extended the result of Adler et al. [2] to a larger class of $\mathbb{K}^m$-extensions.

Definition 8.4. Let ϕ be a diffeomorphism of M. An infinite ϕ-invariant closed subset Λ of M is *hyperbolic* if $T_\Lambda M = \mathbb{E}^u \oplus \mathbb{E}^s$ (with usual $T\phi$-invariance and estimates) and Λ has a local product structure (equivalently, Λ is maximal and isolated).

In what follows, we assume that Λ is hyperbolic and $\phi : \Lambda \to \Lambda$ is topologically mixing. We fix a Hölder equilibrium measure λ on Λ (see [35] for details and background). It follows that ϕ is measure theoretically mixing. As usual, we let h denote Haar measure on $\mathbb{K}^m$.

Let $m \geq 1$. Given the cocycle $f : \Lambda \to \mathbb{K}^m$, let $\phi_f : \mathbb{K}^m \times \Lambda \to \mathbb{K}^m \times \Lambda$ denote the skew extension defined by $\phi_f(\gamma, x) = (\gamma f(x), \phi(x))$. Note that ϕ_f is $\mathbb{K}^m$-equivariant and preserves the product measure $\lambda \times h$.

Theorem 8.5 ([45]). *Suppose that Λ is a hyperbolic set. Assume that* either *Λ is a subshift of finite type* or *Λ is connected and $\phi^\star : H^1(\Lambda, \mathbb{Z}) \to H^1(\Lambda, \mathbb{Z})$ does not have one as an eigenvalue. Then there is an open (C^0-topology) and dense (C^∞-topology) subset $\mathcal{U}$ of $C^\infty(\Lambda, \mathbb{K}^m)$ such that for all $f \in \mathcal{U}$, ϕ_f is ergodic and mixing.*

The proof of this result depends on Livšic regularity results proved by Parry [44].

8.4. **Skew Extensions by general compact connected Lie groups.** Suppose that Γ is a compact connected Lie group. Let $\phi : \Lambda \rightarrow \Lambda$ and suppose that Λ is a hyperbolic set with equilibrium measure λ.

Let $f : \Lambda \rightarrow \Gamma$ and let $\phi_f : \Gamma \times \Lambda \rightarrow \Gamma \times \Lambda$ be the associated skew extension.

More generally, we may consider a principal Γ-bundle $\pi : E \rightarrow \Lambda$ and Γ-equivariant map $\Phi : E \rightarrow E$ covering ϕ.

Given $\phi : \Lambda \rightarrow \Lambda$, let $\phi^\star : H^1(\Lambda, \mathbb{Z}) \rightarrow H^1(\Lambda, \mathbb{Z})$ denote the induced map on first cohomology.

Consider the following conditions on Λ and Γ.

(H) Λ is connected, $\dim(\Lambda) \neq 0$ and $\mathrm{rank}(\ker(I - \phi^\star)) < \infty$.
(Z) $\dim(\Lambda) = 0$ ($\phi : \Lambda \rightarrow \Lambda$ is a subshift of finite type).
(S) Γ is semisimple.

Remark 8.6. If Λ is an *attractor* then it can be shown [30] that $\mathrm{rank}(\ker(I - \phi^\star)) < \infty$. $\diamondsuit$

Theorem 8.7 (Field & Parry [30]). *Let Γ be a compact connected Lie group. Suppose that $\phi : \Lambda \rightarrow \Lambda$ is hyperbolic. Assume that one of the conditions* **(H,Z,S)** *holds. Then ϕ_f will be ergodic and mixing for f in an open and dense subset of $C^\infty(\Lambda, \Gamma)$. The same result holds for principal bundle extensions of ϕ.*

Remark 8.8. If $\Gamma = \mathbb{K}^m$ it is possible for ϕ_f to be ergodic but not stably ergodic – see Remark 8.3. A characterization of ergodic but unstably ergodic extensions is given in [30]. On the other hand if Γ is semisimple, Wilkinson has very recently shown that ergodicity and stable ergodicity are equivalent. $\diamondsuit$

The proof of Theorem 8.7 breaks into four steps: Proof of stable ergodicity when (1) $\Gamma = \mathbb{K}^m$ and either condition **(H)** or **(Z)** holds; (2) Γ is semisimple; and (3) Case of general Γ. (4) Stable ergodicity $\Longrightarrow$ Stable mixing.

Step (3) follows easily from (1,2). Step (4) is straightforward (trivial if Γ is semisimple!).

Surprisingly, perhaps, step (2) is much easier than step (1).

The proof of (2) depends on the following results

Theorem 8.9 ([26]). *Let Γ be a compact connected semisimple Lie group. Then the set of pairs $(g, h) \in \Gamma^2$ which topologically generate Γ form a non-empty (*Zariski*) open subset of Γ^2.*

Remark 8.10. We note that the *density* of generating pairs is a relatively old result proved by Kuranishi [38]. The openness does not seem to have been noticed before. $\diamondsuit$

Theorem 8.11 ([45, 30]). *The ergodic components of ϕ_f are* closed *principal subbundles of $\Gamma \times \Lambda$.*

Remark 8.12. Results along the lines of Theorem 8.11 were first obtained by Brin [10] in the context of principal bundle extensions of Anosov diffeomorphisms. $\diamondsuit$

8.5. **Sketch proof of Theorem 8.7 - Γ semisimple.** Fix $\alpha \in (0, 1)$. Generally, we work with the C^α-topology on cocycles.

Replacing ϕ by a power of ϕ, it is no loss of generality to suppose that ϕ has a fixed point, say x_0. Set $z_0 = (e, x_0) \in \Gamma \times \Lambda$. It follows from Theorem 8.11 that for each $u \in \Gamma z_0$ there is

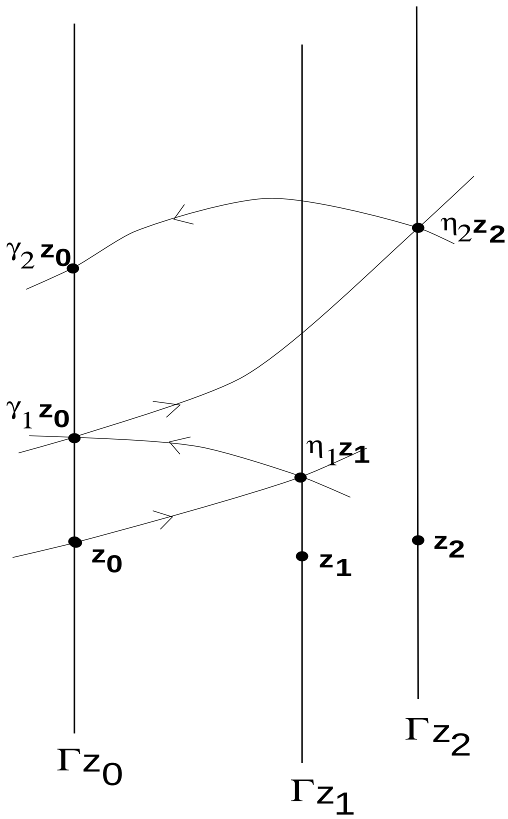

FIGURE 16. Intersections of strong stable and unstable manifolds

a unique closed ergodic component $\mathcal{E}_u$ of ϕ_f containing the point u and that $\gamma\mathcal{E}_u = \mathcal{E}_{\gamma u}$ for all $\gamma \in \Gamma$. Let $\Sigma_0 \subset \Gamma$ denote the isotropy group of the ergodic component $\mathcal{E}_{z_0}$. Then ϕ_f is ergodic if and only if $\Sigma_0 = \Gamma$.

Since the ergodic components are closed ϕ_f-invariant sets,

$$W^{ss}(u), W^{uu}(u) \subset \mathcal{E}_u, \ \ (u \in \Gamma z_0).$$

Choose $x_1, x_2 \in W^u(x_0) \cap W^s(x_0) \setminus \{x_0\}$ with distinct ϕ-orbits. Set $z_i = (e, x_i)$ and $\gamma_0 = e$. For $i = 1, 2$, there exist unique $\eta_i, \gamma_i \in \Gamma$ such that

$$\eta_i z_i \in W^{uu}(\gamma_{i-1} z_0) \cap W^{ss}(\gamma_i z_0). \tag{8.9}$$

We refer to Figure 16.

Since the ergodic components $\mathcal{E}_{\gamma z_0}$ define a partition of $\Gamma \times \Lambda$, it follows from (8.9) that $\gamma_i z_0 \in \mathcal{E}_{z_0}$, $i = 0, 1, 2$. Hence $\Sigma_0 \supset \langle \gamma_1, \gamma_2 \rangle$. In particular, if $\langle \gamma_1, \gamma_2 \rangle = \Gamma$, then $\Sigma_0 = \Gamma$ and ϕ_f is ergodic. It may be shown that η_i, γ_i depend continuously on f, C^α-topology. Since the set of pairs of topological generators is open in Γ^2, it follows that if $\langle \gamma_1, \gamma_2 \rangle = \Gamma$ the same will be true for $f' \in C^\infty(\Lambda, \Gamma)$ sufficiently C^α-close to f. Conversely, we can always make an arbitrarily C^∞-small perturbation f' of f, supported on small neighborhoods of x_1, x_2, so that the corresponding pair (γ_1', γ_2') topologically generates Γ. Hence the set of $f \in C^\infty(\Lambda, \Gamma)$ for which ϕ_f is ergodic contains an open and dense set. Finally, since Γ is

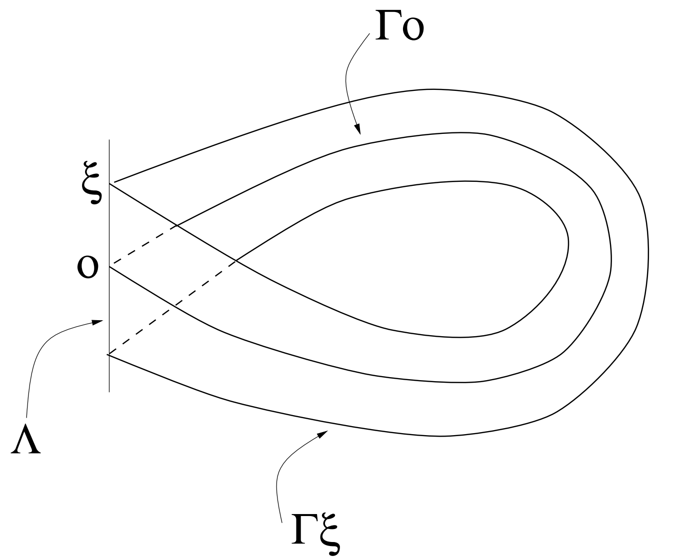

FIGURE 17. The twisted product $\mathbb{K} \times_{\mathbb{Z}_2} 3^{\mathbb{Z}}$

semisimple, it follows that if ϕ is mixing and ϕ_f is ergodic, then ϕ_f is automatically weak mixing and therefore mixing by Rudolf's theorem [48]. □

8.6. **Hyperbolicity for equivariant diffeomorphisms.** Assume that M is a Γ-manifold. Let $\mathbb{E}^c \subset TM$ be the Tf-invariant (singular) subbundle of TM defined by

$$\mathbb{E}^c = \cup_{x \in M} T_x \Gamma x.$$

Suppose that Λ be a compact f- and Γ-invariant subset of M. We say that Λ is *transversally hyperbolic for f* if

(a) All Γ-orbits in Λ have the same dimension.
(b) There exists a continuous Tf-invariant splitting

$$T_\Lambda M = \mathbb{E}^u \oplus \mathbb{E}^c_\Lambda \oplus \mathbb{E}^s,$$

and constants $C > 0$, $\lambda > 1$, such that for all $n \in \mathbb{N}$,

$$\begin{aligned} \|T_x \phi^n(v)\| &\leq C\lambda^{-n}\|v\|, \ (v \in \mathbb{E}^s_x, \ x \in \Lambda), \\ \|T_x \phi^n(v)\| &\leq C\lambda^{n}\|v\|, \ (v \in \mathbb{E}^u_x, \ x \in \Lambda). \end{aligned}$$

Just as for hyperbolic sets, we can require a local product structure on a transversally hyperbolic set.

A transversally hyperbolic set Λ is *Γ-hyperbolic* if Λ has a local product structure and the induced map $\tilde{f}$ on the orbit space Λ/Γ is topologically transitive.

If Λ is Γ-hyperbolic, then $\tilde{f} : \Lambda/\Gamma \to \Lambda/\Gamma$ admits a Markov partition [27].

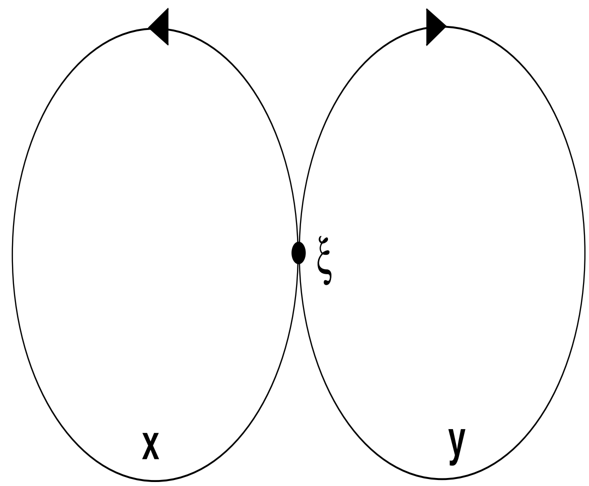

FIGURE 18. The space X

Example 8.13. Let $\Lambda = 3^{\mathbb{Z}}$ denote the full shift on three symbols $\{0,1,2\}$. We have a non-free action of $\mathbb{Z}_2 = \langle\kappa\rangle$ on Λ defined by $\kappa(1) = 2$, $\kappa(0) = 0$. Embed $\mathbb{Z}_2$ in $\mathbb{K}$ by $\kappa \mapsto \exp(\imath\pi)$.

Let $\mathbf{X} = \mathbb{K} \times_{\mathbb{Z}_2} \Lambda$ (that is, the twisted product – see Figure 17 and notes for Lecture I).

If $f : \Lambda \to \mathbb{K}$, then we get an induced map $\phi_f : \mathbb{K} \times_{\mathbb{Z}_2} \Lambda \to \mathbb{K} \times_{\mathbb{Z}_2} \Lambda$. In this case, it is easily seen that ϕ_f is generically stably ergodic. ♡

Generally, all Γ-hyperbolic sets are locally twisted products. This follows easily from the differentiable slice theorem.

Conjecture

If $\phi : \Lambda \to \Lambda$ is Γ-hyperbolic, then there exist C^∞-small Γ-equivariant perturbations of ϕ to a stably ergodic diffeomorphism.

8.7. **A non-uniformly hyperbolic base.** We conclude by describing a simple example based on a non-uniformly hyperbolic attractor constructed by Coelho et al. [16]. Let X denote the unit interval with $0, \frac{1}{2}, 1$ identified, say to $\xi \in X$. Let x and y respectively denote the oriented circles defined as the images of $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ in X. We may regard X as the 'figure of eight' consisting of the x, y with infinite order contact at ξ. Regard the circle group $\mathbb{K}$ as $[0,1]$, with $0,1$ identified. See Figure 18. Define the smooth map $\phi : X \to X$ by $\phi(x) = 3x$, mod 1. Let $f : X \to \mathbb{K}$ be the smooth cocycle defined by $f(x) = 2x$ and $g : X \to \mathbb{K}$ be the measurable cocycle defined by $g(x) = x$, $x \neq \xi$. Since

$$g(\phi x) - g(x) = 3x - x = 2x = f(x), \;\; x \neq \xi,$$

f is a coboundary (in the class of measurable cycles). On the other hand, there is obviously no continuous solution to the cohomology equation and so f is not a continuous (let alone smooth) coboundary.

Let $\phi_f : \mathbb{K} \times X \to \mathbb{K} \times X$ denote the $\mathbb{K}$-extension of ϕ. Given $\theta \in \mathbb{K}$, define $E_\theta = \{(x + \theta, x) \mid x \in X\}$. Clearly, $\mathbf{E} = \{E_\theta \mid \theta \in \mathbb{K}\}$ is a family of closed ϕ_f-invariant subsets of $\mathbb{K} \times X$ permuted by the action of $\mathbb{K}$. Although $\cup_{\theta \in \mathbb{K}} E_\theta = \mathbb{K} \times X$, $\mathbf{E}$ does not define a partition of $\mathbb{K} \times X$: $E_\theta \cap E_{\theta+\pi} = \{(\theta, \xi), (\theta + \pi, \xi)\}$. For all $\theta \in \mathbb{K}$, $E_\theta \cong \mathbb{K}$ and $\phi_f | E_\theta$ is smoothly conjugate to the expanding map $z \mapsto z^3$.

Let $\Phi : \Lambda \to \Lambda$ denote the non-uniformly hyperbolic attractor obtained as the inverse limit of $\phi : X \to X$ and $\tilde{f}$ denote the 'lift' of f to Λ. The skew extension $\Phi_{\tilde{f}} : \mathbb{K} \times \Lambda \to \mathbb{K} \times \Lambda$ is measure theoretically trivial but not continuously trivial. We describe closed ergodic components of Φ_f. For $\theta \in \mathbb{K}$, let $\tilde{E}_\theta \subset \mathbb{K} \times \Lambda$ denote the classical solenoid defined as the inverse limit of $\phi_f : E_\theta \to E_\theta$. Up to sets of measure zero, the set of ergodic components of Φ_f is given by $\tilde{\mathbf{E}} = \{\tilde{E}_\theta \mid \theta \in \mathbb{K}\}$. Although the components are closed and permuted by the action of $\mathbb{K}$, $\tilde{\mathbf{E}}$ does not define a *partition* of $\mathbb{K} \times \Lambda$ since $\tilde{E}_\theta \cap \tilde{E}_{\theta+\pi} \neq \emptyset$. For this example, it is not hard to show that there is no partition of $\mathbb{K} \times \Lambda$ into closed ergodic components.

There is an action of $\mathbb{Z}_2$ on X defined by reflection in a line tangent to X at ξ. Obviously, the action interchanges x and y, preserving orientations. The action of $\mathbb{Z}_2$ extends to Λ and the fixed point set of the action is given by $\Lambda^{\mathbb{Z}_2} = (\xi)$. If we form the twisted product $\mathbb{K} \times_{\mathbb{Z}_2} \Lambda$, we find that the quotients of the components $\tilde{E}_\theta$ give the closed ergodic components for the induced map on the twisted product. Of course, the ergodic components still do not give a *partition* of $\mathbb{K} \times_{\mathbb{Z}_2} \Lambda$ into closed sets.

Finally, we can ask about the stability of ergodicity of Γ-extensions over Λ. In spite of the fact that ergodic components will in general no longer yield a partition of $\Gamma \times \Lambda$ by closed sets, stable ergodicity of skew extensions is still generic. This follows easily using the fact that $\phi : \Lambda \to \Lambda$ is covered by the 'triadic' solenoid determined by $z \mapsto z^3$. Indeed, stable ergodicity will be generic within the class of $\mathbb{Z}_2$-invariant cocycles on Λ and so stable ergodicity is also generic for the twisted product.

9. Notes on Lecture III

There is an extensive and varied literature on skew extensions and the problem of lifting ergodicity and mixing. ¿From our perspective, the most interesting results develop from the work of Brin [11], on extensions over Anosov systems, and that of Livšic [39, 40]. The results of Livšic give precise criteria for extensions to be trivial. In addition, there are Livšic regularity theorems. These results allow one to make contact with work on measurable cocycles (see, for example, [36]). This type of analysis is basic to the proof of Theorem 8.7. Various constructions for realizing skew extensions as basic sets of equivariant dynamical systems may be found in [22, 23].

It is not yet clear what happens when the base is not hyperbolic. Melbourne has recently proved some results on lifting transitivity and weak mixing [41]. If Γ is semisimple, it is possible to prove genericity of stable transitivity without having to assume hyperbolicity of the base. For example, if the base is non-uniformly hyperbolic and Γ is semisimple, then extensions are generically stably transitive and weak mixing (up to a cycle). In a different direction, Bonatti and Diaz have constructed classes of stably transitive, non hyperbolic, dynamical systems [14].

References

[1] J F Adams. *Lectures on Lie Groups*, (Benjamin, New York, 1969).

[2] R Adler, B Kitchens and M Shub. 'Stably ergodic skew products', *Discrete and Continuous Dynamical Systems*, to appear.

[3] J C Alexander, I Kan, J A Yorke and Z You. Riddled basins, *Int. J. of Bif. and Chaos* **2** (1992), 795–813.

[4] P Ashwin. 'Attractors stuck onto invariant subspaces', *Phy. Lett. A* **209** (1995), 238–344.

[5] P Ashwin and M Nicol. 'Detection of symmetry of attractors from Observations I. Theory', *Physica D* **100** (1997), 58–70.

[6] P Ashwin, J Buescu and I N Stewart. 'From attractor to chaotic saddle: a tale of transverse instability', *Nonlinearity*, **9** (1996), 703–737.

[7] P Ashwin and I Melbourne. Symmetry groups of attractors. *Arch. Rat. Mech. Anal.* **126** (1994) 59–78.

[8] A Back, J Guckenheimer, M Myers, F Wicklin, and P Worfolk. '*dstool*: Computer Assisted Exploration of Dynamical Systems', *Notices AMS* **39**(4) (1992), 303–309.

[9] E Barany, M Dellnitz and M Golubitsky. 'Detecting the symmetry of attractors', *Physica D* **67** (1993), 66–87.

[10] G. E. Bredon. *Introduction to Compact Transformation Groups*. Pure & Appl. Math. **46**, Academic Press, New York, 1972.

[11] M I Brin. 'Topology of group extensions of Anosov systems', *Mathematical Notes of the Acad. of Sciences of the USSR*, **18**(3) (1975), 858–864.

[12] M Brin and Ya Pesin. 'Partially hyperbolic dynamical systems', *Math. USSR Izvestija* **8** (1974), 177-218.

[13] T Bröcker T tom Dieck. *Representations of Compact Lie Groups*, (Graduate Texts in Mathematics, Springer, New York, 1985).

[14] C Bonatti and L J Diaz. 'Persistent nonhyperbolic transitive diffeomorphisms', *Annals of Math.* **140** (1995), 357–396.

[15] P Chossat and M Golubitsky. Symmetry-increasing bifurcation of chaotic attractors, *Physica D***32** (1988), 423–436.

[16] Z Coelho, W Parry and R Williams. 'A note on Livšic's periodic point theorem', Warwick Univ. preprint, 1996.

[17] M Dellnitz and I Melbourne. 'A note on the shadowing lemma and symmetric periodic points', *Nonlinearity* **8** (1995), 1067–1075.

[18] M Dellnitz, M Golubitsky and I Melbourne. 'Mechanisms of symmetry creation', in *Bifurcation and Symmetry* (eds. E. Allgower et al) ISNM 104, Birkhäuser, Basel (1992), 99-109.

[19] B Fiedler, B Sandstede, A Scheel and C Wulff. 'Bifurcation from relative equilibria of noncompact Group actions: Skew products, Meanders and Drifts', preprint, 1996.

[20] M J Field. 'Equivariant Dynamical Systems', *Bull. Amer. Math. Soc.*, 76(1970), 1314-1318.

[21] M J Field. 'Equivariant Dynamical Systems', *Trans. Amer. Math. Soc.*, 259(1980),185-205.

[22] M J Field. 'Isotopy and stability of equivariant diffeomorphisms', *Proc. London Math. Soc.*, **46**(3), (1983), 487–516.

[23] M J Field. 'Equivariant diffeomorphisms hyperbolic transverse to a G-action', *J. London Math. Soc.*, **27**(2), (1983), 563–576.

[24] M J Field. 'Local structure of equivariant dynamics', in *Singularity Theory and its Applications, II*, eds. M. Roberts and I. Stewart, Springer Lecture Notes in Math., **1463** (1991), 168–195.

[25] M J Field. *Lectures on Dynamics, Bifurcations and Symmetry*, Pitman Research Notes in Mathematics, **356**, 1996, Pitman.

[26] M J Field. 'Generating sets for compact semisimple Lie groups', preprint, University of Houston, 1996.

[27] M J Field. 'The structure of transversally hyperbolic basic sets for equivariant diffeomorphisms', in preparation.

[28] M Field and M Golubitsky. *Symmetry in Chaos*, Oxford University Press, 1992.

[29] M J Field, I Melbourne and M Nicol. 'Symmetric Attractors for Diffeomorphisms and Flows', *Proc. London Math. Soc.*, (3) **72** (1996), 657–696.

[30] M J Field and W Parry. 'Stable ergodicity of skew extensions by compact Lie groups', preprint, 1997.

[31] M Golubitsky and N Nicol. 'Symmetry detectives for SBR attractors', *Nonlinearity* **8** (1995), 1027–1038.

[32] M Golubitsky, I N Stewart and D G Schaeffer. *Singularities and Groups in Bifurcation Theory*, Vol 2. Appl. Math. Sci. **69** Springer, New York, 1988.

[33] M Grayson, C Pugh and M Shub. 'Stably ergodic diffeomorphisms', *Annals of Math.* **140** (1994), 295–329.

[34] J Guckenheimer and P Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*. (Springer-Verlag, Applied Mathematical Sciences, **42**, 1983.)

[35] A Katok and B Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*, (Encyclopedia of Mathematics and its Applications **54**), Cambridge University Press, 1995.

[36] H B Keynes and D Newton. 'Ergodic measures for non-abelian compact group extensions', *Compositio Math.* **32** (1976), 53–70.

[37] M Krupa. 'Bifurcations of relative equilibria', *SIAM J. MATH. ANAL.*, **21**(6) (1990), 1453–1486.

[38] Kuranishi. 'Two element generations on semi-simple Lie groups', *Kodai math. Sem. Report*, (1949), 9–10.

[39] A N Livšic. 'Homology properties of Y-systems', *Mathematical Notes of the Acad. of Sc. of the USSR* **10** (1971), 758–763.

[40] A N Livšic. 'Cohomology of dynamical systems', *Math. USSR Izvestija* **6**(6) (1972), 1278–1301.

[41] I Melbourne. 'Compact group extensions of topologically transitive sets for smooth dynamical systems' preprint, University of Houston, 1997.

[42] I Melbourne, M Dellnitz and M Golubitsky. The structure of symmetric attractors. *Arch. Rat. Mech. Anal.* **123** (1993), 75–98.

[43] S E Newhouse. 'Lectures on Dynamical Systems', in *Dynamical Systems*, Progress in Mathematics **8**, Birkhäuser, 1980.

[44] W Parry. 'Skew-products of shifts with a compact Lie group', Warwick Univ. preprint, 1995.

[45] W Parry and M Pollicott. 'Stability of mixing for toral extensions of hyperbolic systems', Warwick Univ. preprint, 1996.

[46] W Parry and M Pollicott. 'The Livšic cocycle equation for compact Lie group extensions of hyperbolic systems', Warwick Univ. preprint, 1995.

[47] C Pugh and M Shub. 'Stably ergodic dynamical systems and partial hyperbolicity', preprint, 1996.

[48] D J Rudolph. 'Classifying the isometric extensions of a Bernoulli shift', *Journal d'Analyse Math.* **34** (1978), 36-60.

[49] A Wilkinson. 'Stable ergodicity of the time-one map of a geodesic flow', preprint, 1996.

[50] R F Williams. One-dimensional non-wandering sets, *Topology* **6** (1967), 473–487.