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Induced maps on n-fold symmetric product suspensions

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ABSTRACT

In a previous paper, we define the n-fold symmetric product suspensions of continua. Now, we investigate the induced maps between these spaces.

1. Introduction

In 1979 Sam B. Nadler Jr. introduced the hyperspace suspension of a continuum [1]. In 2004 S. Macías, defined the n-fold hyperspace suspension of a continuum [2]. For a continuum X and $n \geqslant 2$, in 2009, we define the n-fold symmetric product suspensions of X [3], denoted by $SF_n(X)$, as the quotient space $F_n(X)/F_1(X)$, where $F_n(X)$ is the hyperspace of nonempty subsets of X with at most n points. Given a map $f : X \to Y$ between continua and an integer $n \geqslant 2$, we let $F_n(f) : F_n(X) \to F_n(Y)$ and $SF_n(f) : SF_n(X) \to SF_n(Y)$ denote the corresponding induced maps. Let $\mathcal{M}$ be a class of maps between continua. As it was done with hyperspaces (see, for example [4–8]), in this paper we study the interrelations between the following three statements:

(1) $f \in \mathcal{M}$;
(2) $F_n(f) \in \mathcal{M}$;
(3) $SF_n(f) \in \mathcal{M}$.

The paper consists of ten sections. In Section 2, we give the basic definitions for understanding the paper. In Section 3, we study homeomorphisms. Section 4 is devoted to monotone maps. Section 5 is about open maps. In Section 6, we discuss confluent maps. The light maps are analyzed in Section 7. In Section 8, we consider the class of quasi-interior maps and the class of MO-maps. In Section 9, we prove results concerning to the class of quasi-monotone maps and the class of weakly monotone maps. Finally, in Section 10 we study the class of weakly confluent maps and the class of pseudo-confluent maps.

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2. Definitions

The symbols $\mathbb{N}$ and $\mathbb{R}$ will denote the set of positive integers and the set of real numbers, respectively. A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space X. A continuum X is said to be *irreducible* provided that no proper subcontinuum of X contains $\{p,q\}$ for some $p,q\in X$. If X is a continuum, then given $A\subset X$ and $\epsilon>0$, the open ball about A of radius ϵ is denoted by $\mathcal{V}_\epsilon(A)$, the closure of A in X by $\mathrm{Cl}_X(A)$, and the interior of A in X by $\mathrm{int}_X(A)$. A *map* means a continuous function. A surjective map $f:X\to Y$ between continua is said to be:

- *confluent* provided that for each subcontinuum B of Y and for each component C of $f^{-1}(B)$, we have $f(C)=B$;
- *light* provided that $f^{-1}(y)$ is totally disconnected for each $y\in Y$;
- *monotone* provided that $f^{-1}(y)$ is connected for each $y\in Y$;
- *open* provided $f(U)$ is open in Y for each open subset U of X;
- *MO* provided that there exist a continuum Z, an open map $h:X\to Z$, and a monotone map $g:Z\to Y$ such that $f=g\circ h$;
- *pseudo-confluent* provided that for each irreducible subcontinuum B of Y, there exists a component C of $f^{-1}(B)$ such that $f(C)=B$;
- *quasi-interior* provided that for each $y\in Y$ if U is an open subset of X containing a component of $f^{-1}(y)$, then $y\in \mathrm{int}_Y(f(U))$;
- *quasi-monotone* provided that for each subcontinuum B of Y having nonempty interior in Y, $f^{-1}(B)$ has only finitely many components and each of these components maps onto B under f;
- *weakly confluent* provided that for each subcontinuum B of Y, there exists a component C of $f^{-1}(B)$ such that $f(C)=B$;
- *weakly monotone* provided that for each subcontinuum B of Y having nonempty interior in Y, each component of $f^{-1}(B)$ is mapped by f onto B.

Given a continuum X and $n\in\mathbb{N}$, the product of X with itself n times will be denoted by X^n, the symbol $F_n(X)$ denotes the *n-fold symmetric product of X*; that is:

$$F_n(X)=\{A\subset X \mid A \text{ has at most } n \text{ points}\},$$

topologized with the Hausdorff metric, which is defined as follows:

$$\mathcal{H}(A,B)=\inf\bigl\{\epsilon>0 \bigm| A\subset\mathcal{V}_\epsilon(B) \text{ and } B\subset\mathcal{V}_\epsilon(A)\bigr\},$$

$\mathcal{H}$ always denotes the Hausdorff metric. Given a finite collection, $U_1,\dots,U_m$, of subsets of X, $\langle U_1,\dots,U_m\rangle_n$, denotes the following subset of $F_n(X)$:

$$\left\{A\in F_n(X) \Bigm| A\subset\bigcup_{k=1}^m U_k \text{ and } A\cap U_k\neq\emptyset \text{ for each } k\in\{1,\dots,m\}\right\}.$$

It is known that the family of all subsets of $F_n(X)$ of the form $\langle U_1,\dots,U_m\rangle_n$, where each U_i is an open subset of X, forms a basis for a topology for $F_n(X)$ (see [4, 0.11]) called the *Vietoris topology*. The Vietoris topology and the topology induced by the Hausdorff metric coincide [4, 0.13].

Given a continuum X and $n\in\mathbb{N}$, with $n\geqslant 2$, we define the *n-fold symmetric product suspension of the continuum X* [3], denoted by $SF_n(X)$, as the quotient space:

$$SF_n(X)=F_n(X)/F_1(X)$$

with the quotient topology. The fact that $SF_n(X)$ is a continuum follows from 3.10 of [9].

Notation 2.1. Given a continuum X, $q_X^n:F_n(X)\to SF_n(X)$ denotes the quotient map. Also, let F_X^n denote the point $q_X^n(F_1(X))$.

Remark 2.2. Note that $SF_n(X)\setminus\{F_X^n\}$ is homeomorphic to $F_n(X)\setminus F_1(X)$, using the appropriate restriction of q_X^n.

Given a map $f:X\to Y$ between continua and an integer $n\geqslant 2$, the function $F_n(f):F_n(X)\to F_n(Y)$ given by $F_n(f)(A)=f(A)$ is the *induced map by f between the n-fold symmetric products of X and Y*. By [10, Corollary 1.8.23] $F_n(f)$ is continuous. Also, we have an induced map $SF_n(f):SF_n(X)\to SF_n(Y)$ called the *induced map by f between the n-fold symmetric product suspensions of X and Y*, which can be defined by

$$SF_n(f)(\chi)=\begin{cases} q_Y^n(F_n(f)((q_X^n)^{-1}(\chi))), & \text{if } \chi\neq F_X^n;\\ F_Y^n, & \text{if } \chi=F_X^n.\end{cases}$$

We note that, by [11, Theorem 4.3, p. 126], $SF_n(f)$ is continuous. In addition, the following diagram

$$\begin{tikzcd} F_n(X) \arrow[r, "F_n(f)"] \arrow[d, "q_X^n"'] & F_n(Y) \arrow[d, "q_Y^n"] \\ SF_n(X) \arrow[r, "SF_n(f)"'] & SF_n(Y) \end{tikzcd} \tag{$*$}$$

is commutative.

Let X be a continuum and let $n \in \mathbb{N}$. We denote by $f_n^X : X^n \to F_n(X)$ the map given by $f_n^X((x_1, \dots, x_n)) = \{x_1, \dots, x_n\}$ (see [12, Lemma 1]), if there is no confusion, we write $f_n : X^n \to F_n(X)$. Given a map $f : X \to Y$ between continua, we denote by $f_{X,Y}^n : X^n \to Y^n$ the map given by $f_{X,Y}^n((x_1, \dots, x_n)) = (f(x_1), \dots, f(x_n))$, if there is no confusion, we write $f^n : X^n \to Y^n$. In addition, the following diagram

$$\begin{tikzcd} X^n \arrow[r, "f_{X,Y}^n"] \arrow[d, "f_n^X"'] & Y^n \arrow[d, "f_n^Y"] \\ F_n(X) \arrow[r, "F_n(f)"'] & F_n(Y) \end{tikzcd} \tag{$**$}$$

is commutative.

3. Homeomorphisms

We begin with some simple results.

Theorem 3.1. *Let $f : X \to Y$ be a map between continua, and let $n \geqslant 2$ be an integer. Then the following are equivalent*:

(1) $f : X \to Y$ *is injective*;
(2) $F_n(f) : F_n(X) \to F_n(Y)$ *is injective*;
(3) $SF_n(f) : SF_n(X) \to SF_n(Y)$ *is injective.*

Theorem 3.2. *Let $f : X \to Y$ be a map between continua, and let $n \geqslant 2$ be an integer. Then the following are equivalent*:

(1) $f : X \to Y$ *is surjective*;
(2) $F_n(f) : F_n(X) \to F_n(Y)$ *is surjective*;
(3) $SF_n(f) : SF_n(X) \to SF_n(Y)$ *is surjective.*

As an easy consequence of Theorems 3.1 and 3.2, we have the following:

Theorem 3.3. *Let $f : X \to Y$ be a map between continua, and let $n \geqslant 2$ be an integer. Then the following are equivalent*:

(1) $f : X \to Y$ *is a homeomorphism*;
(2) $F_n(f) : F_n(X) \to F_n(Y)$ *is a homeomorphism*;
(3) $SF_n(f) : SF_n(X) \to SF_n(Y)$ *is a homeomorphism.*

4. Monotone maps

We prove the equivalence of the monotonicity of all the maps we are considering.

Theorem 4.1. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. Then the following are equivalent*:

(1) $f : X \to Y$ *is monotone*;
(2) $F_n(f) : F_n(X) \to F_n(Y)$ *is monotone*;
(3) $SF_n(f) : SF_n(X) \to SF_n(Y)$ *is monotone.*

Proof. Suppose that f is monotone. Let $B \in F_n(Y)$. We assume that $B = \{y_1, \dots, y_r\}$, where $r \leqslant n$. Note that

$$\big(F_n(f)\big)^{-1}(B) = \big\langle f^{-1}(y_1), \dots, f^{-1}(y_r)\big\rangle_n.$$

To prove this, take $A \in (F_n(f))^{-1}(B)$. Then $f(A) = B$. Thus, $A \subset \bigcup_{i=1}^r f^{-1}(y_i)$. For each $i \in \{1, \dots, r\}$, let $x_i \in A$ be such that $f(x_i) = y_i$. It follows that for each $i \in \{1, \dots, r\}$, $A \cap f^{-1}(y_i) \neq \emptyset$. Hence, $A \in \langle f^{-1}(y_1), \dots, f^{-1}(y_r)\rangle_n$. Thus, $(F_n(f))^{-1}(B) \subset$

$\langle f^{-1}(y_1), \ldots, f^{-1}(y_r)\rangle_n$. Now, let $D \in \langle f^{-1}(y_1), \ldots, f^{-1}(y_r)\rangle_n$. Then $D \subset \bigcup_{i=1}^{r} f^{-1}(y_i) = f^{-1}(B)$. It follows that $f(D) \subset B$. Let $y_i \in B$. Since $D \cap f^{-1}(y_i) \neq \emptyset$, we can find $d_i \in D \cap f^{-1}(y_i)$. Note that $f(d_i) = y_i \in f(D)$. Consequently, $B \subset f(D)$. This implies that $F_n(f)(D) = f(D) = B$. We have that $D \in (F_n(f))^{-1}(B)$. Thus, $\langle f^{-1}(y_1), \ldots, f^{-1}(y_r)\rangle_n \subset (F_n(f))^{-1}(B)$.

Since f is monotone, for each $i \in \{1, \ldots, r\}$, $f^{-1}(y_i)$ is connected. By [13, Lemma 1], $(F_n(f))^{-1}(B)$ is connected. Therefore, (1) implies (2).

Note that q_Y^n is a monotone map. If $F_n(f)$ is monotone, then, by [14, (5.1)], $q_Y^n \circ F_n(f)$ is monotone. By $(*)$, we obtain that $SF_n(f) \circ q_X^n$ is monotone. Hence, by [14, (5.15)], we have that $SF_n(f)$ is monotone. Thus, (2) implies (3).

Finally, we prove that (3) implies (1). Let $y \in Y$. Take a point $y' \in Y \setminus \{y\}$, and let $B = \{y, y'\}$. Let $x, x' \in X$ be such that $f(x) = y$ and $f(x') = y'$. Let $A = \{x, x'\}$. It follows that $F_n(f)(A) = f(A) = B$. Hence, $A \in (F_n(f))^{-1}(B)$.

Since q_X^n and $SF_n(f)$ are monotone maps, by [14, (5.1)], we have that $SF_n(f) \circ q_X^n$ is monotone. Thus, by $(*)$, $q_Y^n \circ F_n(f)$ is monotone. Then $(q_Y^n \circ F_n(f))^{-1}(q_Y^n(B))$ is connected. Since $B \in F_n(X) \setminus F_1(X)$, we obtain that $(q_Y^n \circ F_n(f))^{-1}(q_Y^n(B)) = (F_n(f))^{-1}(B)$. This implies that $(F_n(f))^{-1}(B)$ is connected.

Since $f^{-1}(y)$ and $f^{-1}(y')$ are closed subsets of X and $f^{-1}(y) \cap f^{-1}(y') = \emptyset$, there exists $\epsilon > 0$ such that $\mathcal{V}_\epsilon(f^{-1}(y)) \cap \mathcal{V}_\epsilon(f^{-1}(y')) = \emptyset$. We note that $A \in (F_n(f))^{-1}(B) \subset \langle \mathcal{V}_\epsilon(f^{-1}(y)), \mathcal{V}_\epsilon(f^{-1}(y'))\rangle_n$. Then, by [15, Lemma 6.1], it follows that $(\bigcup (F_n(f))^{-1}(B)) \cap \mathcal{V}_\epsilon(f^{-1}(y))$ is connected. Since $\bigcup (F_n(f))^{-1}(B) = f^{-1}(B) = f^{-1}(y) \cup f^{-1}(y')$, we have that $(\bigcup (F_n(f))^{-1}(B)) \cap \mathcal{V}_\epsilon(f^{-1}(y)) = f^{-1}(y)$. Hence, $f^{-1}(y)$ is connected. Therefore, f is monotone. □

5. Open maps

We study the relationship of openness between the considered maps.

Lemma 5.1. *Let X be a continuum, let $n, r \in \mathbb{N}$ be such that $r \leqslant n$, and let $U_1, \ldots, U_r$ be open subsets of X. Then $\bigcup \langle U_1, \ldots, U_r\rangle_n$ is an open subset of X.*

Proof. Let $x \in \bigcup \langle U_1, \ldots, U_r\rangle_n$. Then there exists $A_x \in \langle U_1, \ldots, U_r\rangle_n$ such that $x \in A_x$. Let $J = \{j \in \{1, \ldots, r\}\colon\ x \in U_j\}$. Since $x \in A_x \subset \bigcup_{i=1}^{r} U_i$, $J \neq \emptyset$, also $x \in \bigcap_{j \in J} U_j$. We see that $\bigcap_{j \in J} U_j \subset \bigcup \langle U_1, \ldots, U_r\rangle_n$. Let $y \in \bigcap_{j \in J} U_j$, and let $A = \{y\} \cup (A_x \setminus \{x\})$. Hence, $A \in \langle U_1, \ldots, U_r\rangle_n$. It follows that $y \in \bigcup \langle U_1, \ldots, U_r\rangle_n$. Thus, $x \in \bigcap_{j \in J} U_j \subset \bigcup \langle U_1, \ldots, U_r\rangle_n$. Therefore, $\bigcup \langle U_1, \ldots, U_r\rangle_n$ is an open subset of X. □

As a consequence of Lemma 5.1, we have the following:

Corollary 5.2. *Let X be a continuum and $n \in \mathbb{N}$. If $\mathcal{C}$ is an open subset of $F_n(X)$, then $\bigcup \mathcal{C}$ is an open subset of X.*

Theorem 5.3. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is an open map, then $f : X \to Y$ is an open map.*

Proof. Let U be an open subset of X. Then $\langle U\rangle_n$ is an open subset $F_n(X)$. It follows that $F_n(f)(\langle U\rangle_n)$ is an open subset of $F_n(Y)$. By Corollary 5.2, $\bigcup F_n(f)(\langle U\rangle_n)$ is an open subset of Y.

We see that $\bigcup F_n(f)(\langle U\rangle_n) = f(U)$. Let $y \in \cup F_n(f)(\langle U\rangle_n)$. Then there exists $B \in F_n(f)(\langle U\rangle_n)$ such that $y \in B$. Hence, there exists $A \in \langle U\rangle_n$ such that $F_n(f)(A) = B$. Thus, $B \subset f(U)$. It follows that $y \in f(U)$. This implies that $\bigcup F_n(f)(\langle U\rangle_n) \subset f(U)$. Now, let $z \in f(U)$. Take $a \in U$ such that $f(a) = z$. Let $C = \{a\}$ and $D = \{z\}$. It follows that $C \in \langle U\rangle_n$ and $F_n(f)(C) = D$. Hence, $D \in F_n(f)(\langle U\rangle_n)$ and $z \in D$. We have that $z \in \bigcup F_n(f)(\langle U\rangle_n)$. This implies that $f(U) \subset \bigcup F_n(f)(\langle U\rangle_n)$. Therefore, $\bigcup F_n(f)(\langle U\rangle_n) = f(U)$. Hence, we obtain that $f(U)$ is an open subset of Y. We conclude that f is an open map. □

Theorem 5.4. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $SF_n(f) : SF_n(X) \to SF_n(Y)$ is an open map, then $f : X \to Y$ is a homeomorphism.*

Proof. It is sufficient to prove that f is injective. Suppose that there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Let $A = \{x_1, x_2\}$ and $y = f(x_1) = f(x_2)$. Take a point z in Y different of y. Thus, we can take open subsets V and W of Y such that $y \in V$, $z \in W$ and $V \cap W = \emptyset$. It follows that, $f^{-1}(V)$ is an open subset of X. Moreover, $x_1, x_2 \in f^{-1}(V)$. Let U_1 and U_2 be open subsets of X such that $x_1 \in U_1 \subset f^{-1}(V)$, $x_2 \in U_2 \subset f^{-1}(V)$ and $U_1 \cap U_2 = \emptyset$. Let $\mathcal{U} = \langle U_1, U_2\rangle_n$. Hence, $\mathcal{U}$ is an open subset of $F_n(X)$ such that $A \in \mathcal{U}$ and $\mathcal{U} \cap F_1(X) = \emptyset$. This implies that $q_X^n(\mathcal{U})$ is open in $SF_n(X)$.

Now, for each $k \in \mathbb{N}$, let $z_k \in Y$ be such that $z_k \in W$, $z_k \neq z$ and such that the sequence $\{z_k\}_{k=1}^{\infty}$ converges to z in Y. For each $k \in \mathbb{N}$, let $A_k = \{z_k, z\}$. Hence, the sequence $\{A_k\}_{k=1}^{\infty}$ converges to $\{z\}$ in $F_n(Y)$. Note that, for each $k \in \mathbb{N}$, $A_k \notin \langle V\rangle_n$. Since $F_n(f)(\mathcal{U}) \subset \langle V\rangle_n$, we have that, for each $k \in \mathbb{N}$, $A_k \notin F_n(f)(\mathcal{U})$. This implies that, for each $k \in \mathbb{N}$, $A_k \notin [F_1(Y) \cup F_n(f)(\mathcal{U})]$. Since $[F_1(Y) \cup F_n(f)(\mathcal{U})] = (q_Y^n)^{-1}(q_Y^n(F_n(f)(\mathcal{U})))$, we obtain that, for each $k \in \mathbb{N}$, $A_k \notin (q_Y^n)^{-1}(q_Y^n(F_n(f)(\mathcal{U})))$.

Notice that $\{z\} \in (q_Y^n)^{-1}(q_Y^n(F_n(f)(\mathcal{U})))$. If $(q_Y^n)^{-1}(q_Y^n(F_n(f)(\mathcal{U})))$ is an open subset of $F_n(Y)$, there exists $N \in \mathbb{N}$ such that if $k \geqslant N$, $A_k \in (q_Y^n)^{-1}(q_Y^n(F_n(f)(\mathcal{U})))$, which is a contradiction. Then $(q_Y^n)^{-1}(q_Y^n(F_n(f)(\mathcal{U})))$ is not an open subset of $F_n(Y)$.

It follows that $q_Y^n(F_n(f)(\mathcal{U}))$ is not an open subset of $SF_n(Y)$. By $(*)$, we obtain that $SF_n(f)(q_X^n(\mathcal{U}))$ is not an open subset of $SF_n(Y)$. Therefore, $SF_n(f)$ is not an open map. □

Applying Theorems 5.4 and 3.3, we obtain the following:

Theorem 5.5. *Let $f: X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $SF_n(f): SF_n(X) \to SF_n(Y)$ is an open map, then $F_n(f): F_n(X) \to F_n(Y)$ is a homeomorphism.*

The following example proves that the converse of Theorem 5.4 is not true.

Example 5.6. Let $X = [-1, 1]$, $Y = [0, 1]$, and let $f: X \to Y$ be defined by $f(x) = |x|$. It is clear that f is an open map. If $SF_n(f): SF_n(X) \to SF_n(Y)$ were an open map, where $n \geqslant 2$, then, by Theorem 5.4, f would be a homeomorphism, which is not true. Hence, $SF_n(f)$ is not an open map.

Theorem 5.7. *Let $f: X \to Y$ be a surjective map between continua. Consider the following conditions*:

(1) $f: X \to Y$ *is open*;
(2) $F_2(f): F_2(X) \to F_2(Y)$ *is open*;
(3) $SF_2(f): SF_2(X) \to SF_2(Y)$ *is open.*

Then (1) *and* (2) *are equivalent,* (3) *implies* (1), (3) *implies* (2), (1) *does not imply* (3) *and* (2) *does not imply* (3).

Proof. By Theorem 5.3, we have that (2) implies (1). To prove that (1) implies (2). Consider the map $f^2_{X,Y}: X^2 \to Y^2$. Since f is open, we obtain that $f^2_{X,Y}$ is open. Also, by [12, Lemma 9], we have that f_2^Y is open. Hence, $f_2^Y \circ f^2_{X,Y}$ is open [14, (5.1)]. By $(**)$, it follows that $F_2(f) \circ f_2^X$ is open. Then, by [14, (5.15)], we conclude that $F_2(f)$ is open.

Applying Theorem 5.4, we have that (3) implies (1). Since (1) and (2) are equivalent, we obtain that (3) implies (2). By Example 5.6, it follows that (1) does not imply (3). Since (1) and (2) are equivalent, (2) does not imply (3). □

It is known that only f_2^X is an open map [12, Lemma 9]. So, we prove the following result:

Lemma 5.8. *Let $f: X \to Y$ be a surjective map between continua and let $n \geqslant 2$ be an integer. Let $U_1, \dots, U_n$ be open subsets of X such that for each $j, l \in \{1, \dots, n\}$, $U_j \cap U_l = \emptyset$ if $j \neq l$, and let $\mathcal{U} = U_1 \times \dots \times U_n$. Then $f_n^X(\mathcal{U})$ is an open subset of $F_n(X)$.*

Proof. Let $\{b_1, \dots, b_k\} \in f_n^X(\mathcal{U})$, where $k \leqslant n$. Then exists $(a_1, \dots, a_n) \in \mathcal{U}$ such that $f_n^X((a_1, \dots, a_n)) = \{b_1, \dots, b_k\}$. Thus, $\{a_1, \dots, a_n\} = \{b_1, \dots, b_k\}$. Notice that $k = n$. Since, for each $j \in \{1, \dots, n\}$, $a_j \in U_j$, we have that $\{b_1, \dots, b_n\} \in \langle U_1, \dots, U_n\rangle_n$. Now, we see that $\langle U_1, \dots, U_n\rangle_n \subset f_n^X(\mathcal{U})$. Let $\{c_1, \dots, c_n\} \in \langle U_1, \dots, U_n\rangle_n$. Since, for each $j, l \in \{1, \dots, n\}$, $U_j \cap U_l = \emptyset$ if $j \neq l$, without loss of generality, we assume that, for each $j \in \{1, \dots, n\}$, $c_j \in U_j$. Hence, $(c_1, \dots, c_n) \in \mathcal{U}$ and $f_n^X((c_1, \dots, c_n)) = \{c_1, \dots, c_n\}$. Thus, $\{c_1, \dots, c_n\} \in f_n^X(\mathcal{U})$. It follows that, $\langle U_1, \dots, U_n\rangle_n \subset f_n^X(\mathcal{U})$. Hence, we have that $\{b_1, \dots, b_n\} \in \langle U_1, \dots, U_n\rangle_n \subset f_n^X(\mathcal{U})$. Therefore, $f_n^X(\mathcal{U})$ is an open subset of $F_n(X)$. □

Theorem 5.9. *Let $f: X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be an integer. Then $F_n(f): F_n(X) \to F_n(Y)$ is an open map if and only if $f: X \to Y$ is a homeomorphism.*

Proof. Suppose that $F_n(f)$ is an open map. It is enough to prove that f is injective. Suppose, on the contrary, that there exist $x_1, x_2 \in X$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. Let $p_1 = f(x_1) = f(x_2)$. We can take $p_2, \dots, p_{n-1} \in Y \setminus \{p_1\}$ such that $p_j \neq p_l$ if $j \neq l$ and $j, l \in \{2, \dots, n-1\}$. Hence, for each $i \in \{1, \dots, n-1\}$, we take an open subset V_i of Y such that $p_i \in V_i$, and $V_j \cap V_l = \emptyset$ if $j \neq l$.

Let U_1 and U_2 be open subsets of X such that $x_1 \in U_1 \subset f^{-1}(V_1)$, $x_2 \in U_2 \subset f^{-1}(V_1)$ and $U_1 \cap U_2 = \emptyset$. For each $i \in \{2, \dots, n-1\}$, let $U_{i+1} = f^{-1}(V_i)$. Notice that, for each $j, l \in \{1, \dots, n\}$, $U_j \cap U_l = \emptyset$ if $j \neq l$. Let $\mathcal{U} = U_1 \times \dots \times U_n$. It follows that $\mathcal{U}$ is an open subset of X^n. By Lemma 5.8, $f_n^X(\mathcal{U})$ is an open subset of $F_n(X)$.

Since $F_n(f)$ is an open map, by Theorem 5.3, f is an open map. Hence, $f^n_{X,Y}$ is an open map. Then $f^n_{X,Y}(\mathcal{U}) = f(U_1) \times \dots \times f(U_n)$ is an open subset of Y^n such that $(p_1, p_1, p_2, \dots, p_{n-1}) \in f^n_{X,Y}(\mathcal{U})$.

Now let $\{y_k\}_{k=1}^{\infty}$ and $\{z_k\}_{k=1}^{\infty}$ be sequences converging to p_2 in Y such that $y_j \neq z_l$ for each $j, l \in \mathbb{N}$, for each $i \in \{1, \dots, n-1\}$ and for each $k \in \mathbb{N}$, $y_k \neq p_i$ and $z_k \neq p_i$. Note that $\{\{p_1, y_k, z_k, p_3, \dots, p_{n-1}\}\}_{k=1}^{\infty}$ is a sequence converging to the point $\{p_1, p_2, \dots, p_{n-1}\}$ in $F_n(Y)$. Since $(p_1, p_1, p_2, \dots, p_{n-1}) \in f^n_{X,Y}(\mathcal{U})$, it follows that $\{p_1, p_2, \dots, p_{n-1}\} \in f_n^Y(f^n_{X,Y}(\mathcal{U}))$.

Suppose that $f_n^Y(f^n_{X,Y}(\mathcal{U}))$ is an open subset of $F_n(Y)$. Then there exists $N \in \mathbb{N}$ such that $\{p_1, y_N, z_N, p_3, \dots, p_{n-1}\} \in f_n^Y(f^n_{X,Y}(\mathcal{U}))$ and $y_N, z_N \in f(U_3) = V_2$. Notice that $(f_n^Y)^{-1}(\{p_1, y_N, z_N, p_3, \dots, p_{n-1}\}) = \{(y_1, \dots, y_n) \in Y^n\colon \{y_1, \dots, y_n\} =$

$\{p_1, y_N, z_N, p_3, \ldots, p_{n-1}\}\}$. Since the points of $\{p_1, y_N, z_N, p_3, \ldots, p_{n-1}\}$ are pair wise distinct, any point of the set $(f_n^Y)^{-1}(\{p_1, y_N, z_N, p_3, \ldots, p_{n-1}\})$ has two different coordinates in $f(U_3)$ (y_N and z_N). Since $f(U_3) \cap f(U_i) = \emptyset$, for each $i \in \{1, \ldots, n\} \setminus \{3\}$, we obtain that $(f_n^Y)^{-1}(\{p_1, y_N, z_N, p_3, \ldots, p_{n-1}\}) \cap f_{X,Y}^n(\mathcal{U}) = \emptyset$. This implies that $\{p_1, y_N, z_N, p_3, \ldots, p_{n-1}\} \notin f_n^Y(f_{X,Y}^n(\mathcal{U}))$, which is a contradiction. Thus, $f_n^Y(f_{X,Y}^n(\mathcal{U}))$ is not an open subset of $F_n(Y)$. By $(**)$, it follows that $F_n(f)(f_n^X(\mathcal{U}))$ is not an open subset of $F_n(Y)$. Hence, $F_n(f)$ is not open map, which is a contradiction. Therefore, f is injective.

By Theorem 3.3, the reverse implication follows. $\square$

To see that the converse of Theorem 5.3 is not true, we have the following:

Example 5.10. Let $X = [-1, 1]$, $Y = [0, 1]$, and $f : X \to Y$ defined by $f(x) = |x|$. It is clear that f is an open map. If $F_n(f) : F_n(X) \to F_n(Y)$ were an open map, where $n \geqslant 3$, by Theorem 5.9, f would be a homeomorphism, which is not true. Hence, $F_n(f)$ is not an open map.

Theorem 5.11. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be a positive integer. Consider the following conditions*:

(1) *$f : X \to Y$ is open*;
(2) *$F_n(f) : F_n(X) \to F_n(Y)$ is open*;
(3) *$SF_n(f) : SF_n(X) \to SF_n(Y)$ is open.*

Then (2) *and* (3) *are equivalent,* (2) *implies* (1), (3) *implies* (1), (1) *does not implies* (3) *and* (1) *does not implies* (2).

Proof. Applying Theorems 5.3, 5.4, and 5.5, we have that (2) implies (1), (3) implies (1) and (3) implies (2), respectively. Now, if $F_n(f)$ is open, then, by Theorem 5.9, f is a homeomorphism. Hence, by Theorem 3.3, $SF_n(f)$ is a homeomorphism, in particular $SF_n(f)$ is open. Thus, (2) implies (3). By Example 5.6, (1) does not imply (3). Moreover, applying Example 5.10, we obtain that (1) does not imply (2). $\square$

6. Confluent maps

Lemma 6.1. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer, and let B be a subcontinuum of Y. If C is a component of $f^{-1}(B)$, then $F_n(C)$ is a component of $(F_n(f))^{-1}(F_n(B))$.*

Proof. We note that, by [16, p. 877], $F_n(C)$ is connected. Furthermore, $F_n(C) \subset (F_n(f))^{-1}(F_n(B))$. Let $\mathcal{C}$ the component of $(F_n(f))^{-1}(F_n(B))$ such that $F_n(C) \subset \mathcal{C}$. By [15, Lemma 2.2], $\bigcup \mathcal{C}$ is a subcontinuum of X such that $C \subset \bigcup \mathcal{C} \subset f^{-1}(B)$. Hence, $C = \bigcup \mathcal{C}$. We see that $\mathcal{C} = F_n(C)$. Let $A \in \mathcal{C}$. Thus, $A \subset C$. This implies that $A \in F_n(C)$. Then, $\mathcal{C} \subset F_n(C)$. Therefore, $\mathcal{C} = F_n(C)$. $\square$

Theorem 6.2. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a confluent map, then $f : X \to Y$ is a confluent map.*

Proof. Let B be a subcontinuum of Y, and let C be a component of $f^{-1}(B)$. By Lemma 6.1, we have that $F_n(C)$ is a component of $(F_n(f))^{-1}(F_n(B))$. It follows that $F_n(f)(F_n(C)) = F_n(B)$. Now, we prove that $B \subset f(C)$. Let $b \in B$. Then there exists $H \in F_n(C)$ such that $f(H) = \{b\}$. This implies that $b \in f(C)$. Thus, we have that $B \subset f(C)$. Hence, we obtain that $f(C) = B$. Therefore, f is a confluent map. $\square$

Theorem 6.3. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a confluent map, then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a confluent map.*

Proof. Since q_n^Y is monotone, by [9, Theorem 13.15], q_n^Y is confluent. Then, by [17, 2.6], we have that $q_n^Y \circ F_n(f)$ is confluent. Hence, by $(*)$, $SF_n(f) \circ q_n^X$ is confluent. By [14, 5.16], we obtain that $SF_n(f)$ is confluent. $\square$

The proof of the following theorem is similar to the proof [6, Theorem 18]. We include the details for the convenience of the reader.

Theorem 6.4. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be an integer. Then $F_n(f) : F_n(X) \to F_n(Y)$ is a confluent map if and only if $f : X \to Y$ is a monotone map.*

Proof. If f is a monotone map, then, by Theorem 4.1, $F_n(f)$ is a monotone map. Hence, by [9, Theorem 13.15], $F_n(f)$ is a confluent map.

Now, we assume that f is not monotone. Then there exists $y \in Y$ such that $f^{-1}(y)$ is not a connected subset of X. Let P and Q be different components of $f^{-1}(y)$, and let $p \in P$ and $q \in Q$. We take a point $z \in Y \setminus \{y\}$. Let R be a subcontinuum of Y such that $\{z\} \subsetneq R \subset Y \setminus \{y\}$ [9, Corollary 5.5]. Let

$$\mathcal{K} = \big\{\{y\} \cup K\colon\ K \in F_{n-1}(R)\big\}.$$

It is clear that $\mathcal{K}$ is a subcontinuum of $F_n(Y)$. We take a point $s \in X$ such that $f(s) = z$. Let S be the component of $f^{-1}(R)$ such that $s \in S$. Let $\mathcal{C}$ be the component of $(F_n(f))^{-1}(\mathcal{K})$ such that $\{p, q, s\} \in \mathcal{C}$. Let $Z = \bigcup \mathcal{C}$. Then, by [15, Lemma 2.2], $Z \in C_n(X)$. We note that $P \cup Q \cup S \subset Z$.

We see that P is a component of Z. Let P' be the component of Z such that $P \subset P'$. We assume that $P \subsetneq P'$. Let $w \in P' \setminus f^{-1}(y)$. Then $f(w) \in f(P') \cap R$. We note that $y \in \{y\} \cap f(P')$ and $f(P') \subset \{y\} \cup R$. Thus, we have that $f(P')$ is not connected, a contradiction. Hence, P is component of Z. Similarly, Q is component of Z.

Now, we prove that $F_n(f)(\mathcal{C}) \subset \{\{y\} \cup K\colon\ K \in F_{n-2}(R)\}$. Let $A \in \mathcal{C}$. Since $\mathcal{C}$ is a connected subset of 2^Z, by [18, Lemma 3.1, p. 241], $A \cap P \neq \emptyset$ and $A \cap Q \neq \emptyset$. Thus, $f(A) \in \{\{y\} \cup K\colon\ K \in F_{n-2}(R)\}$.

Since $\{\{y\} \cup K\colon\ K \in F_{n-2}(R)\} \subsetneq \mathcal{K}$, we have that $F_n(f)(\mathcal{C}) \subsetneq \mathcal{K}$. Therefore, $F_n(f)$ is not confluent. □

As a consequence of the proof of Theorem 6.4, we have the following:

Theorem 6.5. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be an integer. Then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a confluent map if and only if $f : X \to Y$ is a monotone map.*

Applying Theorems 4.1, 6.4 and 6.5, we obtain the following:

Theorem 6.6. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be an integer. Then the following are equivalent*:

(1) *$f : X \to Y$ is monotone*;
(2) *$F_n(f) : F_n(X) \to F_n(Y)$ is monotone*;
(3) *$SF_n(f) : SF_n(X) \to SF_n(Y)$ is monotone*;
(4) *$F_n(f) : F_n(X) \to F_n(Y)$ is confluent*;
(5) *$SF_n(f) : SF_n(X) \to SF_n(Y)$ is confluent.*

The following example proves that the converse of Theorem 6.2 is not true.

Example 6.7. Let $X = [-1, 1]$, $Y = [0, 1]$, and $f : X \to Y$ defined by $f(x) = |x|$. It is readily seen that f is confluent, but f is not monotone. By Theorem 6.6, $F_n(f) : F_n(X) \to F_n(Y)$, where $n \geqslant 3$, is not confluent.

Theorem 6.8. *Let $f : X \to Y$ be a surjective map between continua. If $SF_2(f) : SF_2(X) \to SF_2(Y)$ is a confluent map, then $f : X \to Y$ is a confluent map.*

Proof. Let B be a subcontinuum of Y, and let C be a component of $f^{-1}(B)$. If $B = Y$, then $C = X$ and $f(C) = B$. We assume that there exists a point $y \in Y \setminus B$. Let

$$\mathcal{K} = \big\{\{y\} \cup K\colon\ K \in F_1(B)\big\}.$$

We note that $\mathcal{K}$ is a subcontinuum of $F_2(Y)$ such that $\mathcal{K} \cap F_1(Y) = \emptyset$. Hence, $q_Y^2(\mathcal{K})$ is a subcontinuum of $SF_2(Y)$.

Now let $z \in f^{-1}(y)$, and we define $\mathcal{A} = \{\{z\} \cup A\colon\ A \in F_1(C)\}$. Then $\mathcal{A}$ is a connected subset of $(F_2(f))^{-1}(\mathcal{K})$. Let $\mathcal{C}$ be the component of $(F_2(f))^{-1}(\mathcal{K})$ such that $\mathcal{A} \subset \mathcal{C}$. Thus, $q_X^2(\mathcal{C})$ is a component of $(SF_2(f))^{-1}(q_Y^2(\mathcal{K}))$. Since $SF_2(f)$ is a confluent map, we have that $SF_2(f)(q_X^2(\mathcal{C})) = q_Y^2(\mathcal{K})$. By $(*)$, $q_Y^2(F_2(f)(\mathcal{C})) = q_Y^2(\mathcal{K})$. Since $\mathcal{K} \cap F_1(Y) = \emptyset$, $F_2(f)(\mathcal{C}) = \mathcal{K}$.

On the other hand, since $\bigcup \mathcal{C} \subset f^{-1}(y) \cup f^{-1}(B)$, $\bigcup \mathcal{C}$ is not connected. Moreover, by [15, Lemma 2.2], $\bigcup \mathcal{C} = C_1 \cup C_2$, where C_1 and C_2 are the components of $\bigcup \mathcal{C}$. Without loss of generality, we assume that $C_1 \subset f^{-1}(y)$ and $C_2 \subset f^{-1}(B)$. It follows that $C_2 = C$.

We prove that $B \subset f(C)$. Let $b \in B$. Then $\{y, b\} \in \mathcal{K}$. Thus, there exists $D \in \mathcal{C}$ such that $F_2(f)(D) = \{y, b\}$. Hence, $b \in f(D) \subset f(C_1) \cup f(C)$. If $b \in f(C_1)$, then $b = y$, a contradiction. This implies that $b \in f(C)$. Then $B \subset f(C)$. Therefore, f is a confluent map. □

Lemma 6.9. *If $f : X \to Y$ is a confluent map between continua and Y is locally connected, then f is a quasi-interior map.*

Proof. Let $y \in Y$, let C be a component of $f^{-1}(y)$ and let U be an open subset of X such that $C \subset U$. Since Y is locally connected, there exists a sequence $\{B_i\}_{i=1}^{\infty}$ of subcontinua of Y such that $\bigcap_{i=1}^{\infty} B_i = \{y\}$ and, for each i, $B_{i+1} \subset B_i$ and

$y \in \operatorname{int}(B_i)$. For each i, let C_i be the component of $f^{-1}(B_i)$ such that $C \subset C_i$. Since, for each i, $f^{-1}(B_{i+1}) \subset f^{-1}(B_i)$, we have that $C \subset C_{i+1} \subset f^{-1}(B_i)$. Hence, for each i, $C_{i+1} \subset C_i$. Let $K = \bigcap_{i=1}^{\infty} C_i$. By [9, Theorem 1.8], K is a continuum. We note that $C \subset K$. Moreover, $K \subset f^{-1}(y)$, because if $x \in K$, then $f(x) \in \bigcap_{i=1}^{\infty} B_i = \{y\}$, thus $x \in f^{-1}(y)$. It follows that $C = K$. By [9, Proposition 1.7], there exists a positive integer N such that for each $i \geqslant N$, $C_i \subset U$. Let $j \geqslant N$. Hence, $f(C_j) \subset f(U)$. Since f is a confluent map, we have that $B_j \subset f(U)$. Since $y \in \operatorname{int}(B_j)$, we conclude that $y \in \operatorname{int}(f(U))$. Therefore, f is a quasi-interior. □

Theorem 6.10. *Let $f : X \to Y$ be a surjective map between continua, and let Y be locally connected. Then the following are equivalent*:

(1) *$f : X \to Y$ is confluent*;
(2) *$F_2(f) : F_2(X) \to F_2(Y)$ is confluent*;
(3) *$SF_2(f) : SF_2(X) \to SF_2(Y)$ is confluent.*

Proof. If f is a confluent map then, by Lemma 6.9, f is a quasi-interior map. By [17, 3.1], there exist a continuum Z, a monotone map $h : X \to Z$, and an open map $g : Z \to Y$ such that $f = g \circ h$. Hence, by Theorems 4.1 and 5.7, it follows that $F_2(h)$ is a monotone map and $F_2(g)$ is an open map, respectively. Furthermore, $F_2(f) = F_2(g) \circ F_2(h)$. Then, by [17, 3.1], $F_2(f)$ is a quasi-interior map. By [17, 2.7], $SF_2(f)$ is a confluent map. Thus (1) implies (2).

By Theorems 6.3 and 6.8, it follows that (2) implies (3) and (3) implies (1), respectively. □

Question 6.11. Let $f : X \to Y$ be a surjective map between continua such that Y is not locally connected. If $SF_2(f) : SF_2(X) \to SF_2(Y)$ is a confluent map, then is $F_2(f) : F_2(X) \to F_2(Y)$ a confluent map?

7. Light maps

It is easy to verify that if $f : X \to Y$ is a light map between continua and A is a subcontinuum of X, then $f|_A : A \to f(A)$ is a light map. Therefore, we have the following:

Lemma 7.1. *Let $f : X \to Y$ be a surjective map between continua, and let $n \in \mathbb{N}$. If $F_n(f) : F_n(X) \to F_n(Y)$ is a light map, then $F_m(f) : F_m(X) \to F_m(Y)$ is a light map, for each $m \in \mathbb{N}$ with $m \leqslant n$.*

Lemma 7.2. *Let X be a continuum, and let $n \geqslant 2$ be an integer. If $C_1, \ldots, C_k$, where $k \leqslant n$, are totally disconnected closed subsets of X and $C_i \cap C_j = \emptyset$, for each $i \neq j$, then $\langle C_1, \ldots, C_k \rangle_n$ is a totally disconnected subset of $F_n(X)$.*

Proof. Suppose that there exists a nondegenerate component $\mathcal{C}$ of $\langle C_1, \ldots, C_k \rangle_n$. By [15, Lemma 2.2], $\bigcup \mathcal{C}$ has at most n components, we say that $\bigcup \mathcal{C} = \bigcup_{i=1}^{s} K_i$, where K_i is a component of $\bigcup \mathcal{C}$ and $s \leqslant n$. If K_i is degenerate, for each $i \in \{1, \ldots, s\}$, then $\mathcal{C}$ is degenerate, which is a contradiction. Hence, there exists $j \in \{1, \ldots, s\}$ such that K_j is not degenerate. Since $\bigcup \mathcal{C} \subset \bigcup_{i=1}^{k} C_i$, there exists $l \in \{1, \ldots, k\}$ such that $K_j \subset C_l$, which is a contradiction. Therefore, $\langle C_1, \ldots, C_k \rangle_n$ is a totally disconnected subset of $F_n(X)$. □

Theorem 7.3. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. Then $F_n(f) : F_n(X) \to F_n(Y)$ is a light map if and only if $f : X \to Y$ is a light map.*

Proof. Suppose that $F_n(f) : F_n(X) \to F_n(Y)$ is a light map. Applying Lemma 7.1, we have that $F_1(f) : F_1(X) \to F_1(Y)$ is a light map. On the other hand, since X and Y are homeomorphic to $F_1(X)$ and $F_1(Y)$, respectively, we can consider two homeomorphisms $h : X \to F_1(X)$ and $k : Y \to F_1(Y)$ such that $k \circ f = F_1(f) \circ h$. It follows that f is a light map.

Now assume that $f : X \to Y$ is a light map. Let $B \in F_n(Y)$. Let $B = \{b_1, \ldots, b_k\}$, where $k \leqslant n$ and $b_i \neq b_j$, for each $i \neq j$. It is easy to see that $(F_n(f))^{-1}(B) = \langle f^{-1}(b_1), \ldots, f^{-1}(b_k) \rangle$. Hence, by Lemma 7.2, we conclude that $(F_n(f))^{-1}(B)$ is a totally disconnected subset of $F_n(X)$. Thus, $F_n(f)$ is a light map. □

Theorem 7.4. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a light map, then $F_n(f) : F_n(X) \to F_n(Y)$ is a light map.*

Proof. Let $B \in F_n(Y)$. Then we have the following cases:

Case (1). $B \in F_1(Y)$.

It follows that $q_Y(B) = F_n^Y$. Since $SF_n(f)$ is light, we obtain that $(SF_n(f))^{-1}(F_Y^n)$ is a totally disconnected subset of $SF_n(X)$. Suppose that there exists a nondegenerate component $\mathcal{C}$ of $(F_n(f))^{-1}(B)$. Then, by [15, Lemma 2.2], $\bigcup \mathcal{C}$ has at most n components, say $C_1, \ldots, C_r$, where $r \leqslant n$. We note that there exists $j \in \{1, \ldots, r\}$ such that C_j is nondegenerate. Hence, $F_n(C_j)$ is a nondegenerate subcontinuum of $F_n(X)$ such that $F_n(C_j) \cap (F_n(X) \setminus F_1(X)) \neq \emptyset$. Since $F_n(C_j) \subset (F_n(f))^{-1}(B) \subset$

$(q_X^n)^{-1}((SF_n(f))^{-1}(F_Y^n))$, it follows that $q_X^n(F_n(C_j))$ is a nondegenerate subcontinuum of $(SF_n(f))^{-1}(F_Y^n)$. Which is a contradiction. Thus, $\mathcal{C}$ is degenerate. This implies that $(F_n(f))^{-1}(B)$ is a totally disconnected subset of $F_n(X)$.

Case (2). $B \in F_n(Y) \setminus F_1(Y)$.

Then $q_Y(B) \in SF_n(X) \setminus \{F_Y^n\}$. Hence $(SF_n(f))^{-1}(q_Y(B))$ is totally disconnected and $(SF_n(f))^{-1}(q_Y(B)) \subset SF_n(X) \setminus \{F_X^n\}$. It follows that $(q_X)^{-1}((SF_n(f))^{-1}(q_Y(B)))$ is totally disconnected. Since $(q_X)^{-1}((SF_n(f))^{-1}(q_Y(B))) = (F_n(f))^{-1}(B)$, we conclude that $(F_n(f))^{-1}(B)$ is totally disconnected. Therefore, $F_n(f)$ is a light map. □

Applying Theorems 7.3 and 7.4, we obtain the following:

Theorem 7.5. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a light map, then $f : X \to Y$ is a light map.*

The following example proves that the converse of Theorems 7.4 and 7.5 are not true.

Example 7.6. Let $S^1 = \{e^{it}\colon t \in \mathbb{R}\}$, $f : S^1 \to S^1$ such that $f(z) = z^2$, and let $n \geqslant 2$ be an integer. We note that f is a light map. By Theorem 7.3, $F_n(f)$ is a light map. Let $\mathcal{A} = \{\{e^{it}, e^{i(\pi+t)}\}\colon t \in [0, \frac{\pi}{2}]\}$. Then $\mathcal{A}$ is a nondegenerate connected subset of $F_n(S^1)$ such that $\mathcal{A} \cap F_1(S^1) = \emptyset$ and $F_n(f)(\mathcal{A}) \subset F_1(S^1)$. It follows that $q_{S^1}^n(\mathcal{A})$ is a nondegenerate connected subset of $SF_n(S^1)$ such that $q_{S^1}^n(\mathcal{A}) \subset (SF_n(f))^{-1}(F_{S^1}^n)$. This implies that $SF_n(f)$ is not a light map.

8. Quasi-interior maps and MO-maps

Theorem 8.1. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a quasi-interior map, then f is a quasi-interior map.*

Proof. Let $y \in Y$, let U be an open subset of X, and let C be a component of $f^{-1}(y)$ such that $C \subset U$. Then, by Lemma 6.1, $F_n(C)$ is a component of $(F_n(f))^{-1}(\{y\})$. Moreover, $\langle U\rangle_n$ is open in $F_n(X)$ and $F_n(C) \subset \langle U\rangle_n$. Hence, $\{y\} \in \mathrm{int}_{F_n(Y)}(F_n(f)(\langle U\rangle_n))$. Let $U_1, \ldots, U_m$ be open subsets of Y such that $\{y\} \in \langle U_1, \ldots, U_m\rangle_n \subset F_n(f)(\langle U\rangle_n)$. We define $V = \bigcap_{i=1}^m U_i$. It follows that $y \in V \subset f(U)$. Thus, $y \in \mathrm{int}_Y(f(U))$. Then f is a quasi-interior map. □

Theorem 8.2. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a quasi-interior map, then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a quasi-interior map.*

Proof. Since q_Y^n is a monotone map, by [17, 2.1], it follows that q_Y^n is a quasi-interior map. Then, by [17, 2.8], $q_Y^n \circ F_n(f)$ is a quasi-interior map. By $(*)$, $SF_n(f) \circ q_X^n$ is a quasi-interior map. Then, by [14, 5.20] and [17, 3.1], $SF_n(f)$ is a quasi-interior map. □

Theorem 8.3. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be an integer. Then the following are equivalent:*

(1) *$f : X \to Y$ is monotone;*
(2) *$F_n(f) : F_n(X) \to F_n(Y)$ is quasi-interior;*
(3) *$SF_n(f) : SF_n(X) \to SF_n(Y)$ is quasi-interior.*

Proof. If f is a monotone map then, by Theorem 4.1, $F_n(f)$ is a monotone map. Hence, by [17, 2.1], $F_n(f)$ is a quasi-interior map. Thus, (1) implies (2).

If $F_n(f)$ is a quasi-interior map, by Theorem 8.2, $SF_n(f)$ is a quasi-interior map. We obtain that (2) implies (3).

Finally, if $SF_n(f)$ is a quasi-interior map then, by [17, 2.7], $SF_n(f)$ is a confluent map. By Theorem 6.6, we have that f is a monotone map. □

Theorem 8.4. *Let $f : X \to Y$ be a surjective map between continua. Then the following are equivalent:*

(1) *$f : X \to Y$ is quasi-interior;*
(2) *$F_2(f) : F_2(X) \to F_2(Y)$ is quasi-interior;*
(3) *$SF_2(f) : SF_2(X) \to SF_2(Y)$ is quasi-interior.*

Proof. We assume that f is a quasi-interior map. Then, by [17, 3.1], there exist a continuum Z, a monotone map $h : X \to Z$, and an open map $g : Z \to Y$ such that $f = g \circ h$. Using Theorems 4.1 and 5.7, we have that $F_2(h)$ is a monotone map and $F_2(g)$ is an open map. Furthermore, $F_2(f) = F_2(g) \circ F_2(h)$. Hence, by [17, 3.1], $F_2(f)$ is a quasi-interior map. Thus, (1) implies (2).

By Theorem 8.2, it follows that (2) implies (3).

Now we prove that (3) implies (1). We assume that $SF_2(f)$ is quasi-interior. Let $y \in Y$, let U be an open subset of X, and let C be a component of $f^{-1}(y)$ such that $C \subset U$. If $f(U) = Y$, then $y \in \mathrm{int}_Y(f(U)) = Y$. We suppose that there exists a point $z \in Y \setminus f(U)$. Let V and W be open subsets of Y such that $y \in V$, $z \in W$ and $V \cap W = \emptyset$. We define $U_1 = U \cap f^{-1}(V)$ and $U_2 = f^{-1}(W)$. Then U_1 and U_2 are open subsets of X such that $U_1 \cap U_2 = \emptyset$. Let $\mathcal{U} = \langle U_1, U_2\rangle_2$. It follows that $\mathcal{U}$ is an open subset of $F_2(X)$ such that $\mathcal{U} \cap F_1(X) = \emptyset$.

We take a component K of $f^{-1}(z)$. Let $\mathcal{A} = \langle C, K\rangle_2$ and $B = \{y, z\}$. By [13, Lemma 1], $\mathcal{A}$ is connected. We note that $\mathcal{A} \subset \mathcal{U}$ and $\mathcal{A} \subset (F_2(f))^{-1}(B)$.

Next we prove that $\mathcal{A}$ is component of $(F_2(f))^{-1}(B)$. Let $\mathcal{C}$ be the component of $(F_2(f))^{-1}(B)$ such that $\mathcal{A} \subset \mathcal{C}$. Then $C \cup K \subset \bigcup \mathcal{A} \subset \bigcup \mathcal{C}$. Since $\bigcup \mathcal{C} \subset f^{-1}(y) \cup f^{-1}(z)$, $(\bigcup \mathcal{C}) \cap f^{-1}(y) \neq \emptyset$ and $(\bigcup \mathcal{C}) \cap f^{-1}(z) \neq \emptyset$, we have that $\bigcup \mathcal{C}$ is not connected. By [15, Lemma 2.2], $\bigcup \mathcal{C}$ has exactly two components. Let $\bigcup \mathcal{C} = C_1 \cup C_2$, where C_1 and C_2 are the components of $\bigcup \mathcal{C}$. Without loss of generality, we assume that $C_1 \subset f^{-1}(y)$ and $C_2 \subset f^{-1}(z)$. It follows that $C_1 = C$ and $C_2 = K$. Hence, $\bigcup \mathcal{C} = C \cup K$. To see that $\mathcal{C} \subset \mathcal{A}$, let $D \in \mathcal{C}$. Thus, $D \subset C \cup K$. We suppose that $D \cap K = \emptyset$. This implies that $D \subset C$. Then $B = f(D) \subset f(C) = \{y\}$, a contradiction. Hence, $D \cap K \neq \emptyset$. Similarly, $D \cap C \neq \emptyset$. We obtain that $D \in \mathcal{A}$. It follows that $\mathcal{C} \subset \mathcal{A}$. Therefore, $\mathcal{A}$ is component of $(F_2(f))^{-1}(B)$.

We note that $q^2_X(\mathcal{U})$ is an open subset of $SF_2(X)$, $q^2_X(\mathcal{A}) \subset q^2_X(\mathcal{U})$ and $q^2_X(\mathcal{A})$ is a component of $(SF_2(f))^{-1}(q^2_Y(B))$. Since $SF_2(f)$ is a quasi-interior map, we have that $q^2_Y(B) \in \mathrm{int}_{SF_2(Y)}(SF_2(f)(q^2_X(\mathcal{U})))$. By (∗), $q^2_Y(B) \in \mathrm{int}_{SF_2(Y)} q^2_Y(F_2(f)(\mathcal{U}))$. Since $F_2(f)(\mathcal{U}) \subset F_2(Y) \setminus F_1(Y)$ and $q^2_Y|_{F_2(Y)\setminus F_1(Y)} : F_2(Y) \setminus F_1(Y) \to SF_2(Y) \setminus \{F^2_Y\}$ is a homeomorphism, we obtain that $B \in \mathrm{int}_{F_2(Y)}(F_2(f)(\mathcal{U}))$.

Now we prove that $y \in \mathrm{int}_Y(f(U))$. Let $V_1, \ldots, V_m$ be open subsets of Y such that $B \in \langle V_1, \ldots, V_m\rangle_2 \subset F_2(f)(\mathcal{U})$. We define $I = \{i \in \{1, \ldots, m\}\colon\ y \in V_i\}$ and $G = (\bigcap_{i\in I} V_i) \cap V$. To see that $G \subset f(U)$, let $x \in G$. We put $D = \{z, x\}$. Then $D \in \langle V_1, \ldots, V_m\rangle_2$. Let $E \in \mathcal{U}$ be such that $f(E) = D$. We have that $x \in f(E) \subset f(U_1) \cup f(U_2)$. If $x \in f(U_2)$, then $x \in V \cap W$, a contradiction. Hence, $x \in f(U_1) \subset f(U)$. Thus, $y \in G \subset f(U)$. This implies that $y \in \mathrm{int}_Y(f(U))$. Therefore, f is a quasi-interior map. □

Theorem 8.5. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 3$ be an integer. Then the following are equivalent*:

(1) *$f : X \to Y$ is monotone*;
(2) *$F_n(f) : F_n(X) \to F_n(Y)$ is an MO-map*;
(3) *$SF_n(f) : SF_n(X) \to SF_n(Y)$ is an MO-map.*

Proof. We assume that f is a monotone map. Then, by Theorem 4.1, $F_n(f)$ is a monotone map. Hence, $F_n(f)$ is an MO-map [14, p. 16].

If $F_n(f)$ is an MO-map then, by [17, 3.2] and [17, 3.1], $F_n(f)$ is a quasi-interior map. By Theorem 8.3, f is a monotone map. This implies, by Theorem 4.1, that $SF_n(f)$ is a monotone map. Thus, $SF_n(f)$ is an MO-map [14, p. 16].

If $SF_n(f)$ is an MO-map then, by [17, 3.2] and [17, 3.1], $SF_n(f)$ is a quasi-interior map. By Theorem 8.3, we have that f is a monotone map. □

Theorem 8.6. *Let $f : X \to Y$ be a surjective map between continua. If f is an MO-map, then $F_2(f) : F_2(X) \to F_2(Y)$ is an MO-map.*

Proof. We assume that f is an MO-map. Let Z be a continuum, let $h : X \to Z$ be an open map, and let $g : Z \to Y$ be a monotone map such that $f = g \circ h$. By Theorems 5.7 and 4.1, we have that $F_2(h)$ is an open map and $F_2(g)$ is a monotone map. Furthermore, $F_2(f) = F_2(g) \circ F_2(h)$. Hence, $F_2(f)$ is an MO-map. □

9. Quasi-monotone maps and weakly monotone maps

Theorem 9.1. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a quasi-monotone map, then f is a quasi-monotone map.*

Proof. Let B be a subcontinuum of Y such that $\mathrm{int}_Y(B) \neq \emptyset$. Let $y \in \mathrm{int}_Y(B)$ and let U be an open subset of Y such that $y \in U \subset B$. This implies that $\{y\} \in \langle U\rangle_n \subset F_n(B)$. Hence, $F_n(B)$ is a subcontinuum of $F_n(Y)$ such that $\mathrm{int}_{F_n(Y)}(F_n(B)) \neq \emptyset$. Then $(F_n(f))^{-1}(F_n(B))$ has only finitely many components, $\mathcal{K}_1, \ldots, \mathcal{K}_m$, and $F_n(f)(\mathcal{K}_i) = F_n(B)$ for each $i \in \{1, \ldots, m\}$.

Now let C be a component of $f^{-1}(B)$. Then, by Lemma 6.1, $F_n(C)$ is a component of $(F_n(f))^{-1}(F_n(B))$. Thus, there exists $j \in \{1, \ldots, m\}$ such that $F_n(C) = \mathcal{K}_j$. We note that, by [15, Lemma 2.2], $\bigcup \mathcal{K}_j$ is connected; also $C \subset \bigcup \mathcal{K}_J \subset f^{-1}(B)$. It follows that $C = \bigcup \mathcal{K}_j$. Hence, $f^{-1}(B)$ has at most m components. Furthermore, if $b \in B$, then there exists $D \in F_n(C)$ such that $f(D) = \{b\}$. Then $b \in f(C)$. Hence, $B \subset f(C)$. Thus, we obtain that $f(C) = B$. Therefore, f is a quasi-monotone map. □

With similar arguments to the ones given to prove Theorem 9.1, we can verify the following theorem:

Theorem 9.2. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a weakly monotone map, then f is a weakly monotone map.*

Theorem 9.3. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a quasi-monotone map, then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a quasi-monotone map.*

Proof. We note that q_Y^n is a monotone map. Then, by [9, Proposition 13.18], q_Y^n is a quasi-monotone map. Hence, by [14, Theorem 5.6], $q_Y^n \circ F_n(f)$ is a quasi-monotone map. By $(*)$, $SF_n(f) \circ q_X^n$ is a quasi-monotone map. By [14, 5.19], it follows that $SF_n(f)$ is a quasi-monotone map. □

It is known that the composition of weakly monotone maps may not be weakly monotone [14, 5.9]. However, the following is true.

Lemma 9.4. *Let $f : X \to Z$ be a weakly monotone map between continua and let $g : Z \to Y$ be a monotone map between continua. Then $g \circ f : X \to Y$ is a weakly monotone map.*

Proof. Let B be a subcontinuum of Y such that $\mathrm{int}_Y(B) \neq \emptyset$ and let C be a component of $(g \circ f)^{-1}(B)$. Since g is a monotone map, by [19, 2.2, p. 138], $g^{-1}(B)$ is a subcontinuum of Z. Furthermore, $\mathrm{int}_Z(f^{-1}(B)) \neq \emptyset$. Since f is a weakly monotone map, $f(C) = g^{-1}(B)$. Then $(g \circ f)(C) = B$. Hence, $g \circ f$ is a weakly monotone map. □

Theorem 9.5. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a weakly monotone map, then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a weakly monotone map.*

Proof. Since q_Y^n is a monotone map. Then, by Lemma 9.4, $q_Y^n \circ F_n(f)$ is a weakly monotone map. Hence, by $(*)$, $SF_n(f) \circ q_X^n$ is a weakly monotone map. By [14, 5.19], it follows that $SF_n(f)$ is a weakly monotone map. □

Example 9.6. Let $f : [-1, 1] \to [0, 1]$ be defined by $f(x) = |x|$ and let $n \geqslant 3$ be an integer. We note that f is a quasi-monotone map and, by [14, 4.40], f is a weakly monotone map, but f is not a monotone map. We assume that $SF_n(f)$ is a weakly monotone map. By [3, Theorem 5.2], $SF_n([0, 1])$ is locally connected. Then, by [18, Lemma 7.1], $SF_n(f)$ is a confluent map. Hence, by Theorem 6.5, f is a monotone map, a contradiction. Therefore, $SF_n(f)$ is not a weakly monotone map. Thus, by Theorem 9.5, $F_n(f)$ is not a weakly monotone map. By [14, 4.40], it follows that neither $SF_n(f)$ nor $F_n(f)$ is a quasi-monotone map.

We note that, by [9, Proposition 13.18] and [9, Theorem 13.20], the class of weakly monotone maps whose range is locally connected coincide with the class of confluent maps. Thus, we have the following:

Question 9.7. Let $f : X \to Y$ be a surjective map between continua such that Y is not locally connected.

(1) If $SF_2(f) : SF_2(X) \to SF_2(Y)$ is a weakly monotone map, then is $F_2(f) : F_2(X) \to F_2(Y)$ a weakly monotone map?
(2) If $SF_2(f) : SF_2(X) \to SF_2(Y)$ is a weakly monotone map, then is f a weakly monotone map?

Also, by [9, Theorem 13.23], the class of quasi-monotone maps whose domain is locally connected coincide with the class of confluent maps. Thus, we have the following:

Question 9.8. Let $f : X \to Y$ be a surjective map between continua such that X is not locally connected.

(1) If $SF_2(f) : SF_2(X) \to SF_2(Y)$ is a quasi-monotone map, then is $F_2(f) : F_2(X) \to F_2(Y)$ a quasi-monotone map?
(2) If $SF_2(f) : SF_2(X) \to SF_2(Y)$ is a quasi-monotone map, then is f a quasi-monotone map?

10. Weakly confluent and pseudo-confluent maps

Theorem 10.1. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a weakly confluent map, then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a weakly confluent map.*

Proof. Since q_Y^n is a monotone map, it is clear that q_Y^n is a weakly confluent map. Hence, by [14, 5.4], $q_Y^n \circ F_n(f)$ is a weakly confluent map. By $(*)$, $SF_n(f) \circ q_X^n$ is a weakly confluent map. By [14, 5.16], it follows that $SF_n(f)$ is a weakly confluent map. □

The proof of the following theorem is similar to the one given to prove Theorem 10.1, we need to use [14, 5.9] and [14, 5.4].

Theorem 10.2. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a pseudo-confluent map, then $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a pseudo-confluent map.*

Theorem 10.3. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a weakly confluent map, then $f : X \to Y$ is a weakly confluent map.*

Proof. Let B be a subcontinuum of Y. We assume that $Y \setminus B \neq \emptyset$. Let $y_1, \ldots, y_{n-1}$ be points in $Y \setminus B$ such that $y_i \neq y_j$ if $i \neq j$. Let $D = \{y_1, \ldots, y_{n-1}\}$. We define

$$\mathcal{K} = \{D \cup K\colon\ K \in F_1(B)\}.$$

Then $\mathcal{K}$ is a subcontinuum of $F_n(Y)$ such that $\mathcal{K} \cap F_1(Y) = \emptyset$. This implies that $q^n_Y(\mathcal{K})$ is a subcontinuum of $SF_n(Y)$ such that $F^n_Y \notin q^n_Y(\mathcal{K})$. Then there exists a component $\mathfrak{C}$ of $(SF_n(f))^{-1}(q^n_Y(\mathcal{K}))$ such that $SF_n(f)(\mathfrak{C}) = q^n_Y(\mathcal{K})$. Since $F^n_X \notin \mathfrak{C}$, $(q^n_X)^{-1}(\mathfrak{C})$ is a component of $(q^n_X)^{-1}((SF_n(f))^{-1}(q^n_Y(\mathcal{K}))) = (F_n(f))^{-1}(\mathcal{K})$. We note that $F_n(f)((q^n_X)^{-1}(\mathfrak{C})) = \mathcal{K}$.

On the other hand, we have that $\bigcup (q^n_X)^{-1}(\mathfrak{C}) \subset f^{-1}(y_1) \cup \cdots \cup f^{-1}(y_{n-1}) \cup f^{-1}(B)$, $(\bigcup (q^n_X)^{-1}(\mathfrak{C})) \cap f^{-1}(y_i) \neq \emptyset$ and $(\bigcup (q^n_X)^{-1}(\mathfrak{C})) \cap f^{-1}(B) \neq \emptyset$. Since, by [15, Lemma 2.2], $\bigcup (q^n_X)^{-1}(\mathfrak{C})$ has at most n components, it follows that $\bigcup (q^n_X)^{-1}(\mathfrak{C})$ has exactly n components, $C_1, \ldots, C_n$. Without loss of generality, we assume that $C_1 \subset f^{-1}(y_1), \ldots, C_{n-1} \subset f^{-1}(y_{n-1})$ and $C_n \subset f^{-1}(B)$. Let C the component of $f^{-1}(B)$ such that $C_n \subset C$.

We prove that $f(C) = B$. Let $b \in B$ and let $E = D \cup \{b\}$. Then $E \in \mathcal{K}$. Hence, there exists $A \in (q^n_X)^{-1}(\mathfrak{C})$ such that $f(A) = E$. This implies that $b \in f(A) \subset f(\bigcup (q^n_X)^{-1}(\mathfrak{C})) = f(C_1) \cup \cdots \cup f(C_n)$. If there exists $j \in \{1, \ldots, n-1\}$ such that $b \in f(C_j)$, then $b = y_j$, a contradiction. Hence, $b \in f(C_n)$. Thus, $b \in f(C)$. It follows that $B \subset f(C)$. Therefore, f is a weakly confluent map. $\square$

Given a continuum X, an integer $n \geqslant 2$, B a subcontinuum of X and $D = \{x_1, \ldots, x_{n-1}\} \in F_n(X)$, it is easy to see that $\mathcal{K} = \{D \cup K\colon\ K \in F_1(B)\}$ is homeomorphic to B. Hence, if B is an irreducible continuum, then $\mathcal{K}$ is an irreducible continuum. Therefore, with similar arguments to the ones given to prove Theorem 10.3, we can verify the following:

Theorem 10.4. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a pseudo-confluent map, then $f : X \to Y$ is a pseudo-confluent map.*

By Theorems 10.1 and 10.3, we obtain the following:

Theorem 10.5. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a weakly confluent map, then $f : X \to Y$ is a weakly confluent map.*

As a consequence of Theorems 10.2 and 10.4, we have the following theorem:

Theorem 10.6. *Let $f : X \to Y$ be a surjective map between continua, and let $n \geqslant 2$ be an integer. If $F_n(f) : F_n(X) \to F_n(Y)$ is a pseudo-confluent map, then $f : X \to Y$ is a pseudo-confluent map.*

The following example shows that the converse of Theorems 6.2 (for $n = 2$), 6.8, 10.3, 10.4, 10.5 and 10.6 are not true.

Example 10.7. Let

$$\begin{aligned}
A_1 &= \{(x, y) \in \mathbb{R}^2\colon\ y = \sin(\tfrac{1}{x}) - 1,\ 0 < x \leqslant 1\},\\
B_1 &= \{(x, y) \in \mathbb{R}^2\colon\ x = 0,\ -3 \leqslant y \leqslant 0\},\\
A_2 &= \{(y, x) \in \mathbb{R}^2\colon\ (x, y) \in A_1\},\\
B_2 &= \{(y, x) \in \mathbb{R}^2\colon\ (x, y) \in B_1\},\\
A_3 &= \{(-x, y) \in \mathbb{R}^2\colon\ (x, y) \in A_1\}.
\end{aligned}$$

We define $X = A_1 \cup B_1 \cup A_2 \cup B_2$, $Y = A_1 \cup B_1 \cup A_3$ (see Fig. 1), and $f : X \to Y$ such that

$$f\big((x, y)\big) = \begin{cases} (x, y), & \text{if } (x, y) \in A_1 \cup B_1;\\ (-y, x), & \text{if } (x, y) \in A_2 \cup B_2. \end{cases}$$

Then f is a confluent map (weakly confluent map, pseudo-confluent map). However, $F_2(f)$ is not a pseudo-confluent map (either weakly confluent map or confluent map).

We prove that $F_2(f)$ is not a pseudo-confluent map. Define

$$\begin{aligned}
\mathcal{A} &= \{\{(0, y - 1), (0, y)\} \in F_2(X)\colon\ -2 \leqslant y \leqslant 0\},\\
\mathcal{B} &= \{\{(y - 1, 0), (y, 0)\} \in F_2(X)\colon\ -2 \leqslant y \leqslant 0\}.
\end{aligned}$$

It follows that $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $F_2(X)$ such that $\mathcal{A} \subset F_2(B_1)$, $\mathcal{B} \subset F_2(B_2)$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$.

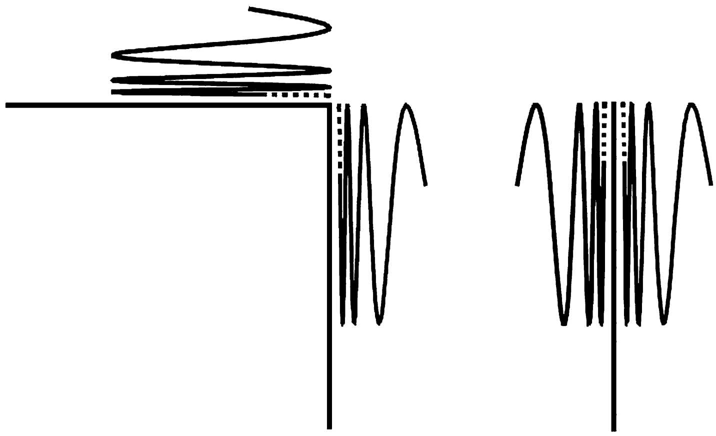

Fig. 1. Continua X and Y.

Let $\alpha_1 : [0,1) \to A_1$ be a homeomorphism. Let $\alpha_1(t) = (x(t), y(t))$. Hence, $\alpha_2 : [0,1) \to A_2$ given by $\alpha_2(t) = (y(t), x(t))$ is a homeomorphism. We take the maps $\delta_1 : [0,1) \to F_2(X)$ given by $\delta_1(t) = \{\alpha_1(t), (0, y(t) - 1)\}$ and $\delta_2 : [0,1) \to F_2(X)$ given by $\delta_2(t) = \{\alpha_2(t), (y(t) - 1, 0)\}$ Let $\mathcal{H} = \delta_1([0,1))$ and $\mathcal{K} = \delta_2([0,1))$. Then $\mathcal{H}$ and $\mathcal{K}$ are disjoint connected subsets of $F_2(X)$. Moreover, these subsets of $F_2(X)$ do not intersect $\mathcal{A}$ (or $\mathcal{B}$). Since $\mathcal{A} \subset \mathrm{Cl}_{F_2(X)}(\mathcal{H})$, we have that $\mathcal{A} \cup \mathcal{H} \subset \mathrm{Cl}_{F_2(X)}(\mathcal{H})$. On the other hand, we note that $\mathrm{Cl}_{F_2(X)}(\mathcal{H}) \cap F_2(B_1) = \mathcal{A}$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{H}) \setminus \mathcal{H} \subset F_2(B_1)$. This implies that $\mathrm{Cl}_{F_2(X)}(\mathcal{H}) \subset \mathcal{A} \cup \mathcal{H}$. Hence, $\mathrm{Cl}_{F_2(X)}(\mathcal{H}) = \mathcal{A} \cup \mathcal{H}$. Similarly, $\mathrm{Cl}_{F_2(X)}(\mathcal{K}) = \mathcal{B} \cup \mathcal{K}$. Therefore, $\mathrm{Cl}_{F_2(X)}(\mathcal{H})$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K})$ are disjoint subcontinua of $F_2(X)$. Note each continuum $\mathrm{Cl}_{F_2(X)}(\mathcal{H})$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K})$ is homeomorphic to the $\sin\frac{1}{x}$-continuum, with remainder $\mathcal{A}$ and $\mathcal{B}$, respectively. Hence, $\mathrm{Cl}_{F_2(X)}(\mathcal{H})$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K})$ are irreducible continua. Also, each continuum $F_2(f)(\mathrm{Cl}_{F_2(X)}(\mathcal{H}))$ and $F_2(f)(\mathrm{Cl}_{F_2(X)}(\mathcal{K}))$ is homeomorphic to the $\sin\frac{1}{x}$-continuum, with remainder $F_2(f)(\mathcal{A})$ and $F_2(f)(\mathcal{B})$, respectively. Let

$$\mathcal{D} = F_2(f)\big(\mathrm{Cl}_{F_2(X)}(\mathcal{H})\big) \cup F_2(f)\big(\mathrm{Cl}_{F_2(X)}(\mathcal{K})\big).$$

Since $F_2(f)$ identifies $\mathcal{A}$ and $\mathcal{B}$ only ($F_2(f)(\mathcal{A}) = F_2(f)(\mathcal{B}) = \mathcal{A}$), it follows that $\mathcal{D}$ is an irreducible subcontinuum of $F_2(Y)$. We note that $\mathcal{D} = \mathcal{H} \cup \mathcal{A} \cup F_2(f)(\mathcal{K})$. Let

$$\mathcal{A}_1 = \{\{(y-1, 0), (0, y)\} \in F_2(X)\colon\ -2 \leqslant y \leqslant 0\},$$
$$\mathcal{B}_1 = \{\{(0, y-1), (y, 0)\} \in F_2(X)\colon\ -2 \leqslant y \leqslant 0\}.$$

Then $\mathcal{A}_1$ and $\mathcal{B}_2$ are disjoint subcontinua of $F_2(X)$ each of which is disjoint to $\mathrm{Cl}_{F_2(X)}(\mathcal{H})$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K})$. We note that $(F_2(f))^{-1}(\mathcal{A}) = \mathcal{A} \cup \mathcal{B} \cup \mathcal{A}_1 \cup \mathcal{B}_1$. Now we put

$$\mathcal{H}_1 = \{\{\alpha_1(t), (y(t) - 1, 0)\} \in F_2(X)\colon\ 0 \leqslant t < 1\},$$
$$\mathcal{K}_1 = \{\{\alpha_2(t), (0, y(t) - 1)\} \in F_2(X)\colon\ 0 \leqslant t < 1\}.$$

It follows that $\mathcal{H}_1$ and $\mathcal{K}_1$ are connected sets such that $\mathcal{H}_1 \cap \mathcal{K}_1 = \emptyset$, each of which is disjoint to $\mathrm{Cl}_{F_2(X)}(\mathcal{H})$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K})$. Furthermore, $(F_2(f))^{-1}(\mathcal{H}) = \mathcal{H} \cup \mathcal{H}_1$ and $(F_2(f))^{-1}(F_2(f)(\mathcal{K})) = \mathcal{K} \cup \mathcal{K}_1$. We note that $\mathrm{Cl}_{F_2(X)}(\mathcal{H}_1) = \mathcal{H}_1 \cup \mathcal{A}_1$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K}_1) = \mathcal{K}_1 \cup \mathcal{B}_1$. Hence, $\mathrm{Cl}_{F_2(X)}(\mathcal{H})$, $\mathrm{Cl}_{F_2(X)}(\mathcal{K})$, $\mathrm{Cl}_{F_2(X)}(\mathcal{H}_1)$ and $\mathrm{Cl}_{F_2(X)}(\mathcal{K}_1)$ are the components of $(F_2(f))^{-1}(\mathcal{D})$. We observe that neither of these components is mapped onto $\mathcal{D}$ under $F_2(f)$. Therefore, $F_2(f)$ is not a pseudo-confluent map. Hence, $F_2(f)$ is not either a weakly confluent map or a confluent map.

To see that $SF_2(f)$ is not a pseudo-confluent map, we note that $\mathcal{D} \cap F_1(Y) = \emptyset$ and $(F_2(f))^{-1}(\mathcal{D}) \cap F_1(X) = \emptyset$. Since $F_2(Y) \setminus F_1(Y)$ is homeomorphic to $SF_2(Y) \setminus \{F_Y^2\}$, it follows that $q_Y^2(\mathcal{D})$ is an irreducible subcontinuum of $SF_2(Y)$. Similarly, we obtain that $q_X^2(\mathrm{Cl}_{F_2(X)}(\mathcal{H}))$, $q_X^2(\mathrm{Cl}_{F_2(X)}(\mathcal{H}_1))$, $q_X^2(\mathrm{Cl}_{F_2(X)}(\mathcal{K}))$ and $q_X^2(\mathrm{Cl}_{F_2(X)}(\mathcal{K}_1))$ are the components of $(SF_2(f))^{-1}(q_Y^2(\mathcal{D}))$ neither of which is mapped onto $q_Y^2(\mathcal{D})$ under $SF_2(f)$. Therefore, $SF_2(f)$ is not a pseudo-confluent map. Hence, $SF_2(f)$ is not either a weakly confluent map or a confluent map.

Example 10.7 shows a confluent map f whose range is not locally connected such that neither $F_2(f)$ or $SF_2(f)$ is a confluent map. However, Theorem 6.10 is true.

Question 10.8. Let $f : X \to Y$ be a surjective map between continua and let $n \geqslant 2$ be an integer. If $SF_n(f) : SF_n(X) \to SF_n(Y)$ is a weakly confluent map (pseudo-confluent map), then is $F_n(f) : F_n(X) \to F_n(Y)$ a weakly confluent map (pseudo-confluent map)?

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