Mathematics of Origami

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Introduction

- Origami
  - *ori* + *kami*, "folding paper"
  - Tools: one uncut square of paper, mountain and valley folds
  - Goal: create art with elegance, balance, detail

- Outline
  - History
  - Applications
  - Foldability
  - Design

History of Origami

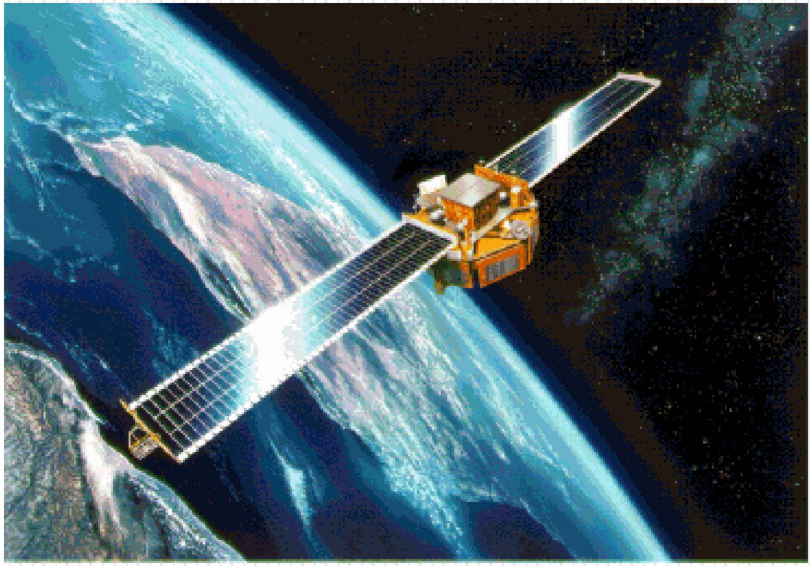
- 105 A.D.: Invention of paper in China
  - Paper-folding begins shortly after in China, Korea, Japan
- 800s: Japanese develop basic models for ceremonial folding
- 1200s: Origami globalized throughout Japan
- 1682: Earliest book to describe origami
- 1797: *How to fold 1,000 cranes* published
- 1954: Yoshizawa's book formalizes a notational system
- 1940s-1960s: Origami popularized in the U.S. and throughout the world

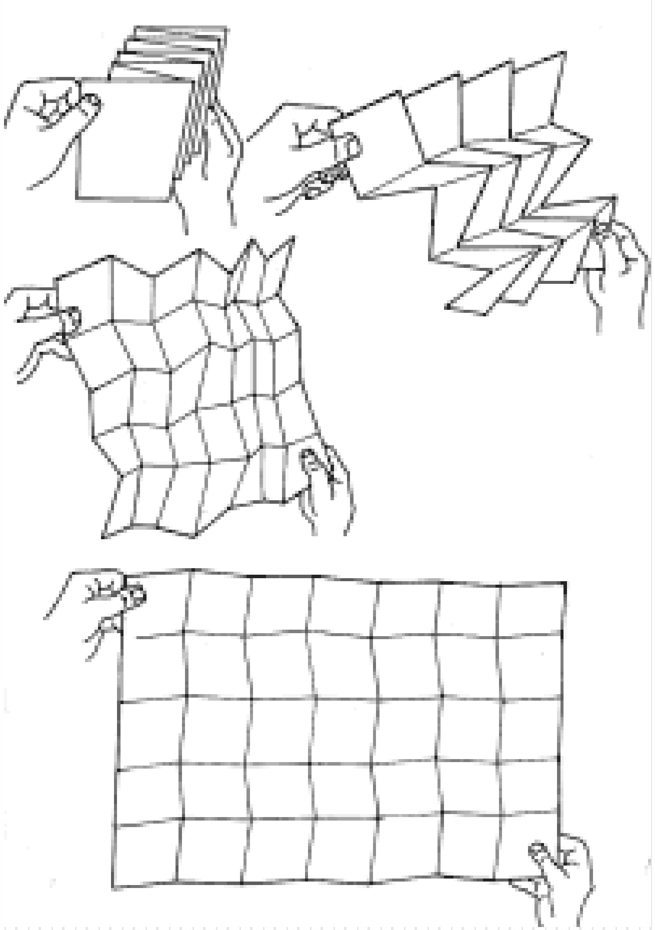
History of Origami Mathematics

- 1893: *Geometric exercises in paper folding* by Row
- 1936: Origami first analyzed according to axioms by Beloch
- 1989-present:
  - Huzita-Hatori axioms
  - Flat-folding theorems: Maekawa, Kawasaki, Justin, Hull
  - TreeMaker designed by Lang
  - *Origami sekkei* – "technical origami"
  - Rigid origami
  - Applications from the large to very small

Miura-Ori

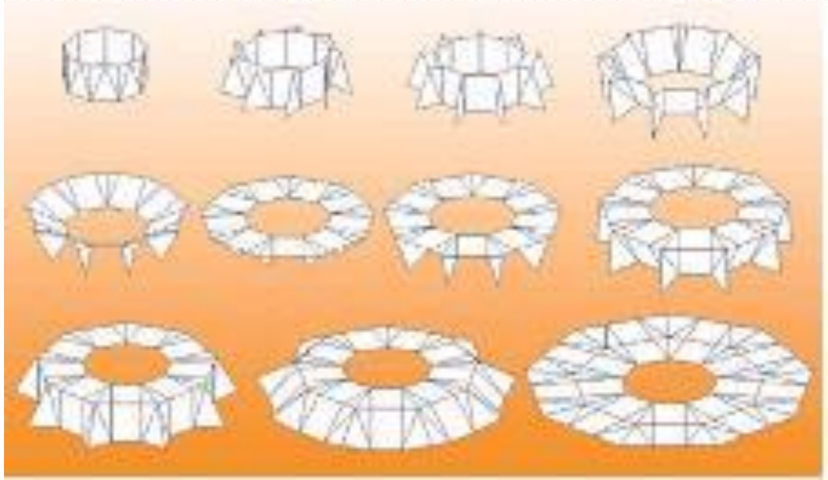
- Japanese solar sail

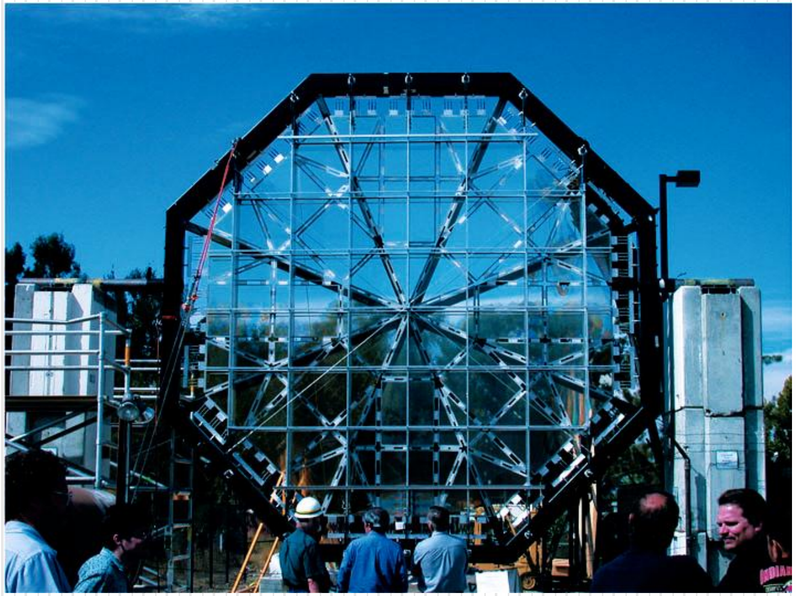




“Eyeglass” space telescope

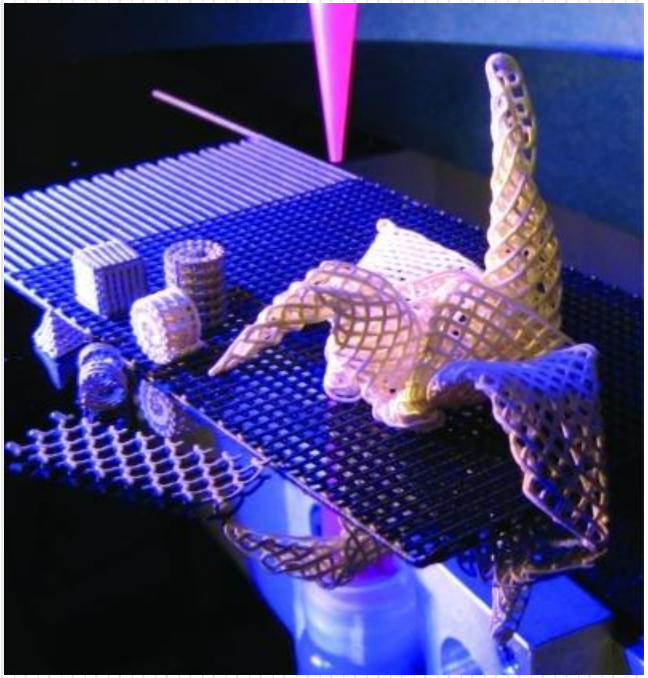
- Lawrence Livermore National Laboratory





Science of the small

- Heart stents
- Titanium hydride printing

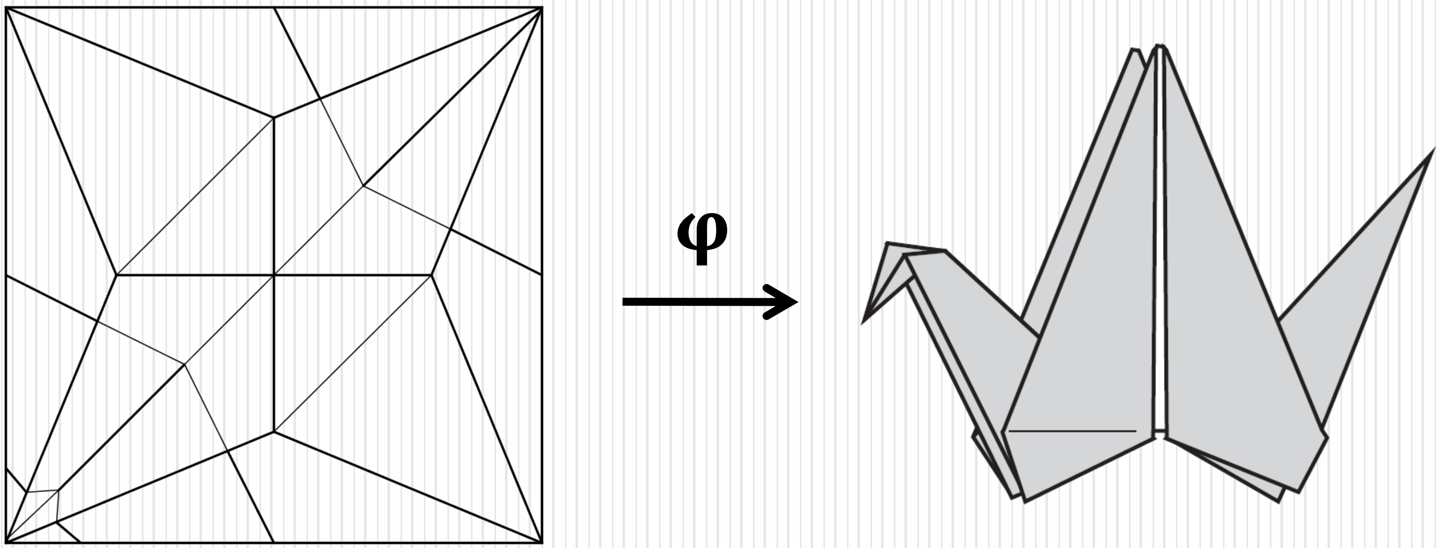


- DNA origami
- Protein-folding

Two broad categories

- Foldability (discrete, computational complexity)
  - Given a pattern of creases, when does the folded model lie flat?
- Design (geometry, optimization)
  - How much detail can added to an origami model, and how efficiently can this be done?

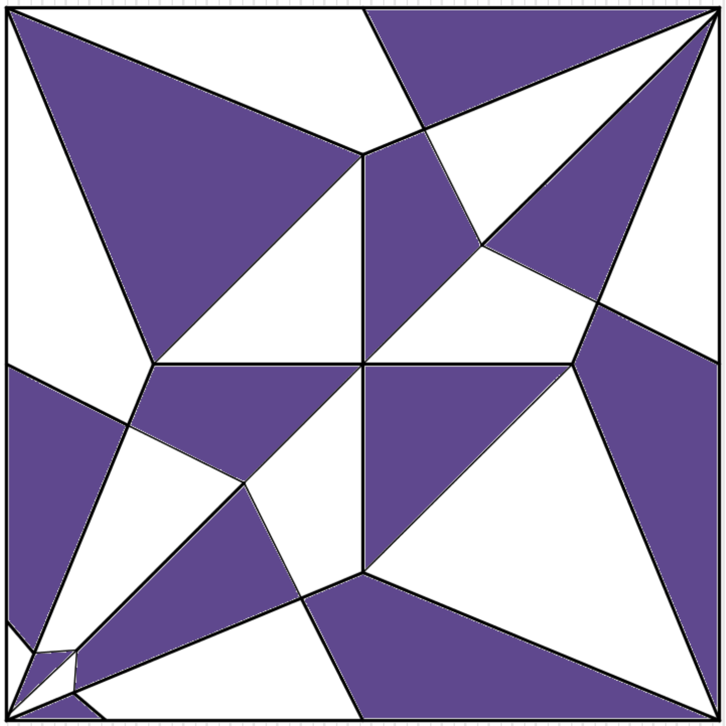
Flat-Foldability of Crease Patterns



- Three criteria for **φ**:

Continuity, Piecewise isometry, Noncrossing

2-Colorable



- Under the mapping **φ**, some faces are flipped while others are only translated and rotated.

Maekawa-Justin Theorem

At any interior vertex, the number of mountain and valley folds differ by two.

Kawasaki-Justin Theorem

At any interior vertex,
a given crease pattern can be folded flat
if and only if
alternating angles sum to 180 degrees.

Layer ordering

- No self-intersections
  - A face cannot penetrate another face
  - A face cannot penetrate a fold
  - A fold cannot penetrate a fold
- Global flat-foldability is hard!
  - NP-complete

Map-folding Problem

Given a rectangle partitioned into
an *m* by *n* grid of squares with
mountain/valley crease assignments,
can the map be folded flat into one square?

Origami design

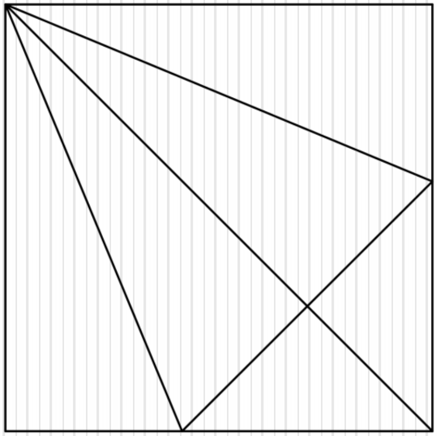
- Classic origami (intuition and trial-and-error):



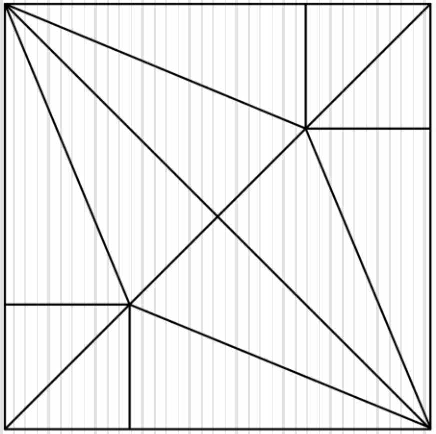
- Origami sekkei (intuition and algorithms): [examples]
- What changed?
  - Appendages were added efficiently
  - Paper usage was optimized

Classic bases

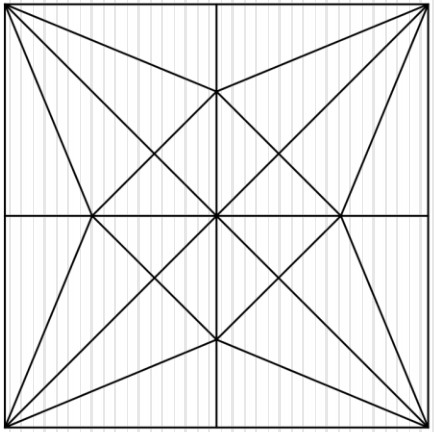
- Kite base



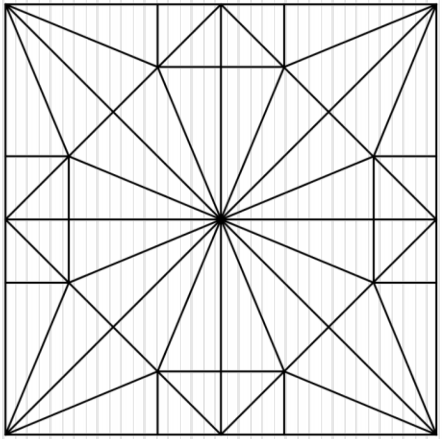
- Fish base



- Bird base

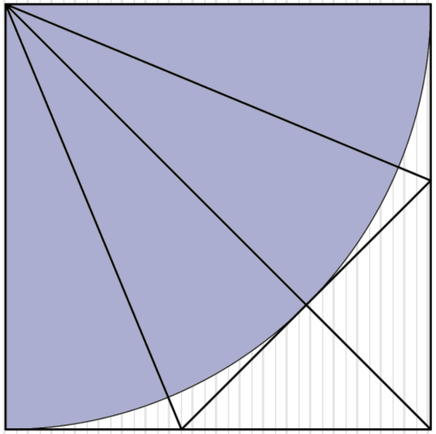


- Frog base

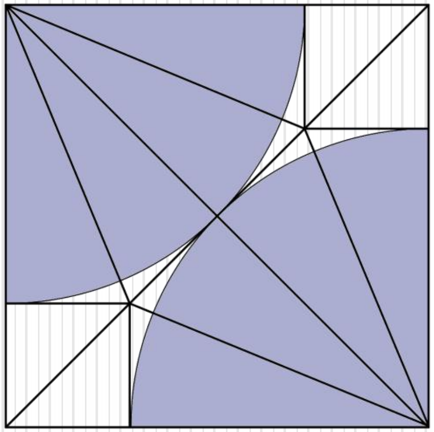


Classic bases

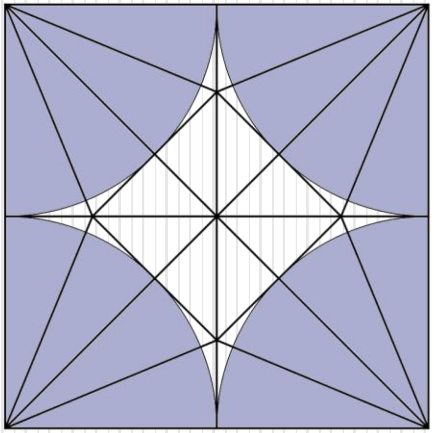
- Kite base



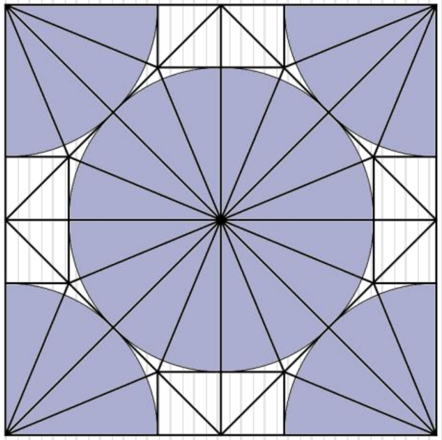
- Fish base



- Bird base



- Frog base



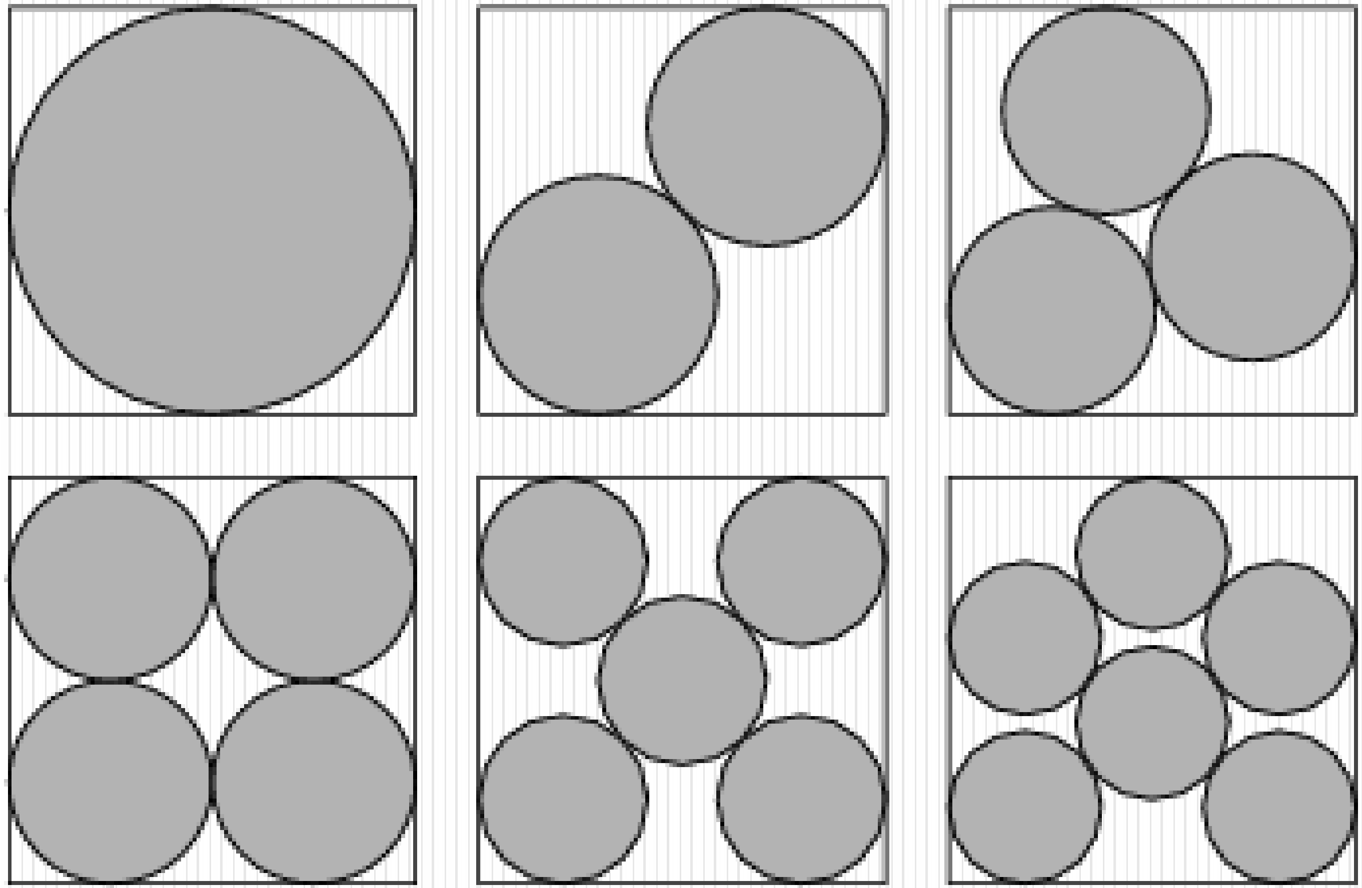
Adding appendages

Suppose we want to design an origami creature with *n* appendages of equal length.

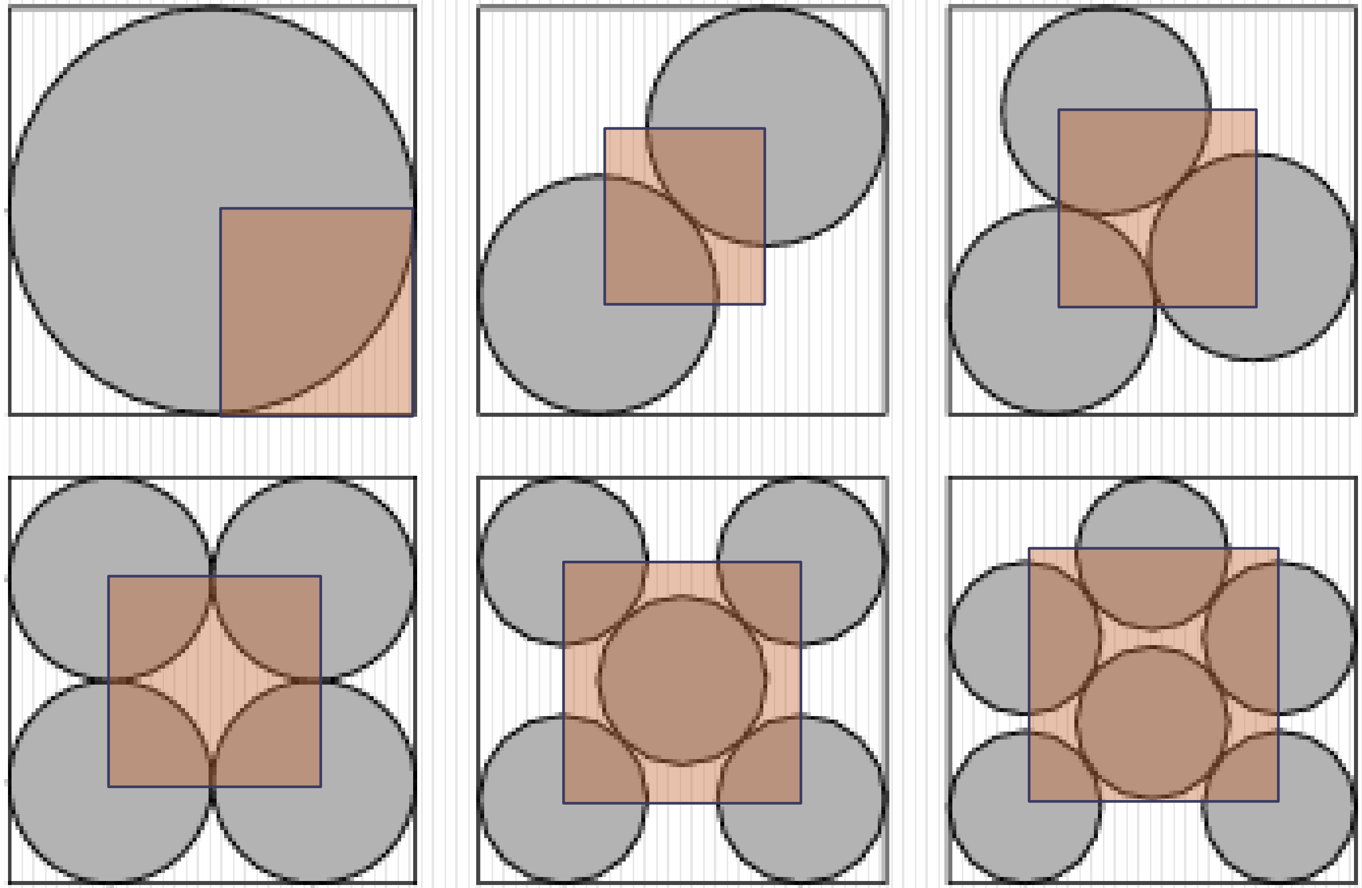
What is the most efficient use of paper?

That is, how can we make the appendages as long as possible?

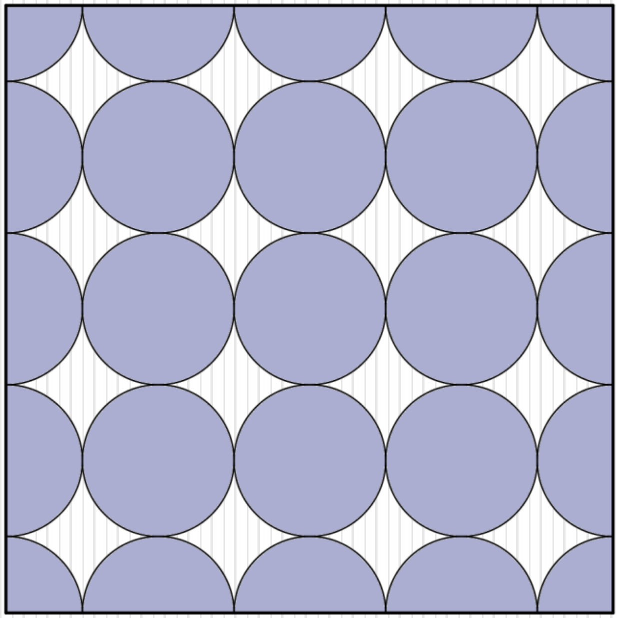
Circle-packing!

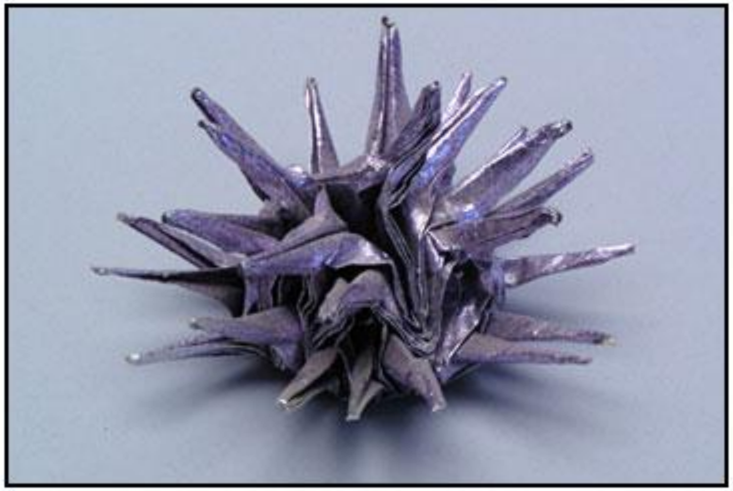


Circle-packing!



n=25 Sea Urchin





- TreeMaker examples

Margulis Napkin Problem

Prove that no matter how one folds a square napkin, the flattened shape can never have a perimeter that exceeds the perimeter of the original square.

Re-cap

- An ancient art modernized by mathematical methods
- Origami is like math: applications may be centuries behind
- Foldability
  - 2-coloring, local vertex conditions, noncrossing
  - Map-folding
- Design
  - Circle-packing
  - TreeMaker
- Flipside: origami methods can be useful in math, too!

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