ON TRANSFORMATIONS BETWEEN SYMMETRIC SPACES.

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Abstract. We consider the concept proposed by [1].

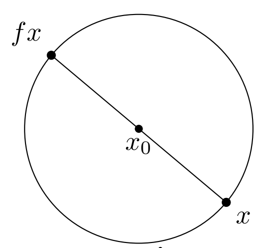
1. Isometries of Symmetric Spaces.

Definition 1.1. Let X be a set. We say that (X,d) is a **metric space** if $d : X^2 \longrightarrow \mathbb{R}$ satisfies:

(1) $d(x,y) = 0 \iff x = y$.
(2) $d(x,y) = d(y,x)$, $\forall x, y \in X$.
(3) $d(x,y) \leqslant d(x,z) + d(z,y)$, $\forall x, y, z \in X$.

Definition 1.2. Let (X,d) and (Y,d') be metric spaces. We say that a function $f : X \longrightarrow Y$ is an **isometry** if $d(x,y) = d'(fx, fy)$, $\forall x, y \in X$. In that case, X and Y are **isometric**.

Definition 1.3. Let X be a metric space.



We say that X **is** x_0-**symmetric with the map** $f : X \longrightarrow X$ if

(1) f is an isometry.
(2) $d(x, x_0) = d(fx, x_0) = \dfrac{1}{2} d(x, fx)$, $\forall x \in X$.

In that case, if there is no ambiguity, we say that X is a x_0-**symmetric space**.

Every isometry is biyective, and its inverse is also an isometry. The composition of isometries is an isometry.

Theorem 1.4. *If (X,d) is x_0-symmetric with f, then X is x_0-symmetric with f^{-1}.*

Theorem 1.5. *Let $(X,d), (Y,d')$ be metric spaces. Let X be a x_0-symmetric space with f. Let $\varphi : X \longrightarrow Y$ be an isometry. Then Y is a φx_0-symmetric space with $g = \varphi \circ f \circ \varphi^{-1}$.*

Proof. (1) $\boxed{g \text{ is an isometry}}$

$$d'(gx, gy) = d'(\varphi f \varphi^{-1} x\ ,\ \varphi f \varphi^{-1} y) = d'(x,y).$$

(2) $\boxed{d'(x, \varphi x_0) \overset{(a)}{=} d'(gx, \varphi x_0) \overset{(b)}{=} \dfrac{1}{2} d'(x, gx),\ \forall x \in Y}$.

2010 *Mathematics Subject Classification.* Primary .
Key words and phrases. Metric spaces, symmetric spaces, transformations, mappings.

(a)

$$\begin{aligned} d'(gx, \varphi x_0) &= d'(\varphi f \varphi x, \varphi x_0) \\ &= d(f\varphi^{-1}x, x_0) \\ &= d(\varphi^{-1}x, x_0) \\ &= d'(x, \varphi x_0). \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{2}d'(x, gx) &= \frac{1}{2}d'(x, \varphi f \varphi^{-1}x) \\ &= \frac{1}{2}d(\varphi^{-1}x, f\varphi^{-1}x) \\ &= d(f\varphi^{-1}x, x_0) \\ &= d'(\varphi f \varphi^{-1}x, \varphi x_0) = d'(gx, \varphi x_0). \end{aligned}$$

$\square$

Example 1.6. If (X, d) is symmetric to x_0 with f and g, then no necessarily X is symmetric to x_0 with $f \circ g$. Consider $X = \mathbb{R}$ with $d(x, y) = |x - y|$, and $f = g : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = -x$. Then

(1) f is an isometry.

(2) $d(x, 0) = d(fx, 0) = \dfrac{1}{2}d(x, fx)$.

But $f \circ f = I$, the identity map, it's not a symmetric function to the space, since $\dfrac{1}{2}d(x, f \circ fx) = 0$, $\forall x \in X$.

2. Subspaces.

Corollary 2.1. *Let (X, d) be ax_0-symmetric space with f. Let $A \subseteq X$, and $a \in A$. If A is $a-$symmetric space with g, then $f(A)$ is $f(a)-$symmetric with $g' : f(A) \longrightarrow f(A)$, by $g' = f \circ g \circ f^{-1}$.*

Proof. It follows from the fact that A and $f(A)$ are isometric. $\square$

Definition 2.2. Let (X, d) be a metric space, and $A, B \subseteq X$. Let $x \in A$. We define $d(x, A) = \inf_{y \in A} d(x, y)$, and

$$d(A, B) = \inf_{\substack{x \in A \\ y \in B}} d(x, y).$$

It's easily seen that $d(A, B) = \inf_{y \in B} d(y, A)$.

Theorem 2.3. *Let X, d be a x_0-symmetric space with f. Let $A \subseteq X$. Then*

$$d(A, x_0) = d(f(A), x_0) = \frac{1}{2}d(A, f(A)).$$

Proof. (1) $\boxed{d(A, x_0) = d(f(A), x_0)}$ Since f is a biyection, we have

$$d(f(A), x_0) = \inf_{y \in f(A)} d(y, x_0) = \inf_{x \in A} d(fx, x_0) = \inf_{x \in A} d(x, x_0) = d(A, x_0).$$

(2) $\boxed{d(f(A),x_0)=\frac{1}{2}d(A,f(A))}$ Since f is a biyection

$$d(f(A),x_0)=\inf_{x\in A} d(fx,x_0)=\frac{1}{2}\inf_{x\in A} d(x,fx)\geqslant \frac{1}{2}\inf_{\substack{x\in A\\ y\in f(A)}} d(x,y)=\frac{1}{2}d(A,f(A)).$$

And, we notice that

$$d(fx,x_0)=\frac{1}{2}d(x,fx)\leqslant \frac{1}{2}\big[d(x,y)+d(y,fx)\big],\qquad \forall x\in A\ ,\ y\in f(A).$$

Then

$$\begin{aligned} d(f(A),x_0)=\inf_{x\in A} d(fx,x_0) &\leqslant \frac{1}{2}\inf_{\substack{x\in A\\ y\in f(A)}}\big[d(x,y)+d(y,fx)\big]\\ &= \frac{1}{2}\inf_{\substack{x\in A\\ y\in f(A)}} d(x,y)\qquad (\forall y\in f(A)\ ,\ \exists x\in A\ :\ fx=y)\\ &= \frac{1}{2}d(A,f(A)). \end{aligned}$$

So $d(f(A),x_0)=\frac{1}{2}d(A,f(A))$.

□

TODO:

(1) X is bounded $((B(x_0,r)=X))$. Define $d(f,g)=\sup_{x\in X} d(fx,gx)$. What happens if $g=I$.
(2) diameter
(3) orbit of f.
(4) Properties of *the Hausdorff metric* $\delta:\mathscr{P}(X)^2\longrightarrow\mathbb{R}$, defined by

$$\delta(A,B)=\max\{\sup_{x\in A} d(x,B)\ ,\ \sup_{y\in B} d(A,y)\}.$$

3. Completion.

Definition 3.1. Let (X,d) be a metric space, and $A\subseteq X$. We define $cerr A=\{x\in X\ :\ d(x,A)=0\}$. We say that A **is dense on** X, if $cerr A=X$.

Definition 3.2. We say that a metric space (X,d) is **complete** if every Cauchy sequence of X converges on X.

Definition 3.3. Let (X,d) be a metric space. We say that a metric space (Y,d') is **a completion of** (X,d) if

(1) Y is complete.
(2) $\exists\varphi:X\longrightarrow Y$ isometry.
(3) $\varphi(X)\subseteq Y$ is dense.

Theorem 3.4. *Every completion of a symmetric space is symmetric.*

4. Mappings Between Two Symmetric Spaces.

5. Pairs of Symmetric Spaces.

6. Path Metric Spaces.

$$dil(f) = \sup_{x \neq y} \frac{d(fx, fy)}{d(x, y)}.$$

$$dil_x(f) = \lim_{\varepsilon \to 0} dil(f\big|_{B(x,\varepsilon)}).$$

The lentgh of a function $f : A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$:

$$\ell(f) = \int_A dil_x(f) dx.$$

$$\beta(x,t) = \inf_P \left\{ t^{-1} \sup_{y \in E \cap B(x,t)} d(y, P) \right\}.$$

References

[1] Yuming Feng, *Symmetric metric space*, Italian Journal of Pure and Applied Mathematics (2017), 542–545.

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