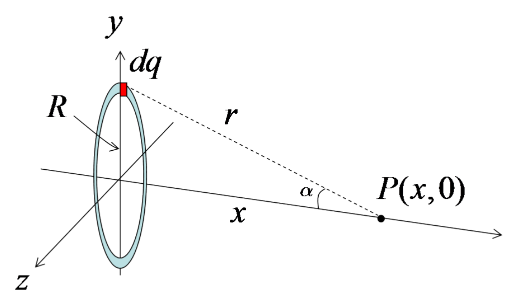
#2 Electric Field

1 Electric field of a ring of charge

A ring-shaped conductor with radius R carries a total charge Q uniformly distributed around it. Find the electric field at a point P that lies on the axis of the ring at a distance x from its center.



(The ring lies on the yz plane.)
We first calculate the linear charge density,

$$\lambda = \frac{Q}{2\pi R}.$$

Then dq is

$$dq = \lambda\, ds = \frac{Q}{2\pi R} R d\theta = \frac{Q}{2\pi} d\theta.$$

The distance between dq and P is r, which is

$$r = \sqrt{x^2 + R^2},$$

and, from the figure,

$$\cos\alpha = \frac{x}{r} = \frac{x}{\sqrt{x^2+a^2}}, \qquad \sin\alpha = \frac{R}{r} = \frac{R}{\sqrt{x^2+R^2}}.$$

Note that both r and the angle α are identical for any point ds on the circle so that they do not depend on the integration variable θ. From the geometry of the system, we can see that the resulting electric field points the positive x-direction. (The other component of $d\vec{E}$ lies on the yz plane and is canceled out.) Then

$$d\vec{E} = dE\cos\alpha\,\hat{i}_x,$$

where

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{x^2+R^2}.$$

Since

$$E_x = \int d\vec{E}_x,$$

we get

$$\begin{aligned} E_x &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{dq}{x^2+R^2}\cos\alpha \\ &= \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\frac{Q}{2\pi}d\theta\frac{1}{x^2+R^2}\frac{x}{\sqrt{x^2+R^2}} \\ &= \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{x}{(x^2+R^2)^{3/2}}\int_0^{2\pi}d\theta \\ &= \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+R^2)^{3/2}}. \end{aligned}$$

Therefore,

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+R^2)^{3/2}}\,\hat{i}_x.$$

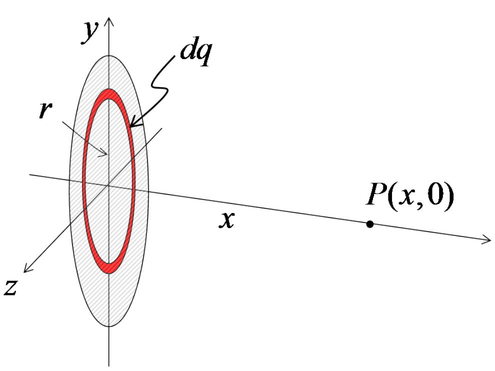
Therefore, when P is at the center of the ring, $E = 0$. When P is much farther from the ring ($x \gg R$), we get

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+R^2)^{3/2}} \approx \frac{1}{4\pi\epsilon_0}\frac{Qx}{x^3} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2},$$

which means that the ring would appear like a point charge of Q.

2 Electric field of a uniformly charged disk

Find the electric field caused by a disk of radius R with a uniform positive surface charge density σ at a point along the axis of disk a distance x from its center. Assume that x is positive.



To solve this problem, we use the result of the previous example, i.e. the electric field caused by a ring of charge. First, consider the ring of width dr. Then the integration variable is r and is from $r = 0$ to $r = R$. Since the surface charge density is $\sigma = Q/(\pi R^2)$, the charge dq carried by the ring is

$$dq = \sigma 2\pi r dr.$$

The circumference of the ring is $2\pi r$ and the length in the radial direction is dr, so the area is $2\pi r dr$. You should verify that $\int_0^R dq = Q$. Now, the electric field $d\vec{E}$ due to the ring is pointing the positive x-direction and its magnitude is

$$dE_x = \frac{1}{4\pi\epsilon_0}\frac{dqx}{(x^2+r^2)^{3/2}},$$

from the previous example. Thus,

$$\begin{aligned}
E_x &= \int_{r=0}^{r=R} dE_x = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{dqx}{(x^2+r^2)^{3/2}} \\
&= \frac{1}{4\pi\epsilon_0}\int_0^R \frac{\sigma 2\pi r dr x}{(x^2+r^2)^{3/2}} \\
&= \frac{\sigma x}{2\epsilon_0}\int_0^R \frac{r dr}{(x^2+r^2)^{3/2}} \\
&= \frac{\sigma x}{2\epsilon_0}\left[-\frac{1}{\sqrt{x^2+r^2}}\right]_0^R \\
&= \frac{\sigma x}{2\epsilon_0}\left[-\frac{1}{\sqrt{x^2+R^2}}+\frac{1}{x}\right] \\
&= \frac{\sigma}{2\epsilon_0}\left[1-\frac{x}{\sqrt{x^2+R^2}}\right].
\end{aligned}$$

Again, we can verify that the disk would look like a point charge when x is much larger than R. To see this, use the Taylor expansion

$$(1+a)^n \approx 1+na, \qquad \text{when } a \ll 1.$$

Then, if $x \gg R$, $R/x \ll 1$, and

$$1-\frac{x}{\sqrt{x^2+R^2}} = 1-\frac{x}{x\sqrt{1+R^2/x^2}} = 1-\left(1+\frac{R^2}{x^2}\right)^{-1/2} = 1-\left(1-\frac{1}{2}\frac{R^2}{x^2}\right) = \frac{R^2}{2x^2}.$$

Therefore,

$$E_x \approx \frac{\sigma}{2\epsilon_0}\frac{R^2}{2x^2} = \frac{Q}{\pi R^2 2\epsilon_0}\frac{R^2}{2x^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{x^2},$$

where we have used $\sigma = Q/(\pi R^2)$.

Another interesting limit is when $R \gg x$, i.e., the electric field produced by an infinite sheet of charge. In this case,

$$1-\frac{x}{\sqrt{x^2+R^2}} \approx 1-\frac{x}{R} \approx 1,$$

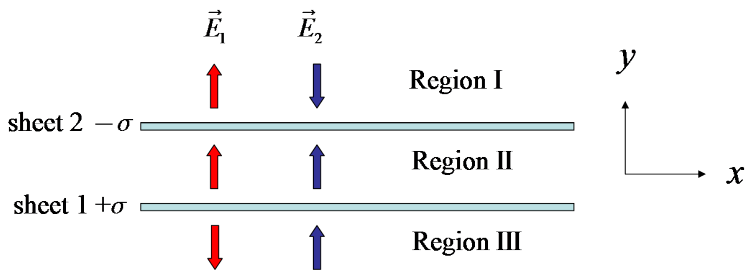
therefore,

$$E_x = \frac{\sigma}{2\epsilon_0}.$$

This shows that the electric field produced by an infinite sheet of charge takes the same value independent of the position x, so is a constant electric field.

3 Electric field of two oppositely charged infinite sheets

Two infinite plane sheets are placed parallel to each other, separated by a distance d. They have uniform charge distribution and the upper plane has surface charge density $-\sigma$ and the lower plane has $+\sigma$. Find the electric fields in the three regions shown in the figure.



From the previous example, we know that the electric field produced by an infinite plane of surface charge density is $E = \sigma/(2\epsilon_0)$. The lower plane has negative charge so the electric field is toward the plane and the upper plane has positive charge so the electric field is from the upper plane. Since these electric field is constant we have electric fields as shown in the figure. Therefore, in Region I, the two electric field is antiparallel with the same magnitude, so there are canceled out. So we have

Region I, above the upper plane $\qquad E = 0,$

Region II, between the planes $\qquad E = \dfrac{\sigma}{\epsilon_0} \quad$ pointing upward,

Region III, below the lower plane $\qquad E = 0.$