Chapter 29
Ampere's Law

In this chapter our main focus will be on Ampere's law, a general theorem that allows us to calculate the magnetic fields of simple current distributions in much the same way that Gauss' law allowed us to calculate the electric field of simple charge distributions.

As we use them, Gauss' and Ampere's laws are integral theorems. With Gauss' law we related the total flux out through a closed surface to $1/\varepsilon_0$ times the net charge inside the surface. In general, to calculate the total flux through a surface we have to perform what is called a **surface integral***. Ampere's law will relate the integral of the magnetic field around a closed path to the total current flowing through that path. This integral around a closed path is called a* **line integral***.*

Until now we have concentrated on examples that did not require us to say much about integration. But as we discuss Ampere's law in this chapter and the remaining Maxwell equations in the next few chapters, it will be convenient to draw upon the formalism of the surface and line integral. Therefore we will take a short break to discuss the mathematical concepts involved in these integrals.

THE SURFACE INTEGRAL

In our discussion of Gauss' law near the end of Chapter 24, we defined the flux Φ of a fluid in a flow tube as the *amount of water per second flowing past the cross-sectional area of the tube* as shown in Figure (1). This is equal to the velocity v times the cross-sectional area $A_\perp$ of the tube as given in Equation (24-46)

$$\Phi = vA_\perp \qquad \textit{flux in a flow tube} \qquad (24\text{-}46)$$

As seen in Figure (2), if we slice the flow tube by a plane that is not normal to the flow tube, the area A of the intersection of the tube and plane is larger than the cross-sectional area $A_\perp$. The relationship is $A_\perp = A \cos \theta$ where θ is the angle between $\vec{v}$ and $\vec{A}$ (see Equation 24-45). Defining the vector $\vec{A}$ as having a magnitude A and direction normal to the plane, we have

$$\vec{v} \cdot \vec{A} = vA \cos \theta = vA_\perp$$

and the formula for the flux in the flow tube is

$$\Phi = \vec{v} \cdot \vec{A} \qquad (24\text{-}47a)$$

In Chapter 24 we considered only problems where $A_\perp$ was something simple like a sphere around a point source, or a cylinder around a line source, and we could easily write a formula for the total flux. We now wish to consider how we should calculate, at least in principle, the flux in a more complex flow like the stream shown in Figure (3).

To give our flux calculation a sense of reality, suppose that we wish to catch all the salmon swimming up a stream to spawn. As shown in Figure (3), we place a net
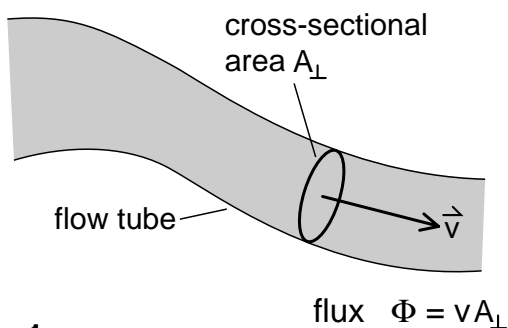
that goes completely across the stream, from bank to bank, from the surface to the bottom. The total flux Φ_T of water flowing through this net is therefore equal to the total current in the stream.

To calculate the total flux Φ_T, we break the stream flow up into a number of small flow tubes bounded by stream lines as shown in (3). Focusing our attention on the i th flow tube, we see that the tube intersects an area $d\vec{A}_i$ of the fish net. The flux through the fish net due to the i th tube is

$$d\Phi_i = \vec{v}_i \cdot d\vec{A}_i \qquad (1)$$

where $\vec{v}_i$ is the velocity of the water at the intersection of the tube and the net. The total flux or current of water in the stream is simply the sum of the fluxes in each flow tube, which can be written

$$\Phi_T = \sum_{\substack{\text{all flow} \\ \text{tubes}}} \Phi_i = \sum_i \vec{v}_i \cdot d\vec{A}_i \qquad (2)$$

where the $d\vec{A}_i$ are just those areas on the fish net marked out by the flow tubes.

If we go to infinitesimal sized flow tubes, the sum in Equation (2) becomes an integral which can be written as

$$\boxed{\Phi_T = \int_{\substack{\text{area of} \\ \text{fish net}}} \vec{v} \cdot d\vec{A}} \qquad \textit{surface integral} \qquad (3)$$

plane slicing
the flow tube

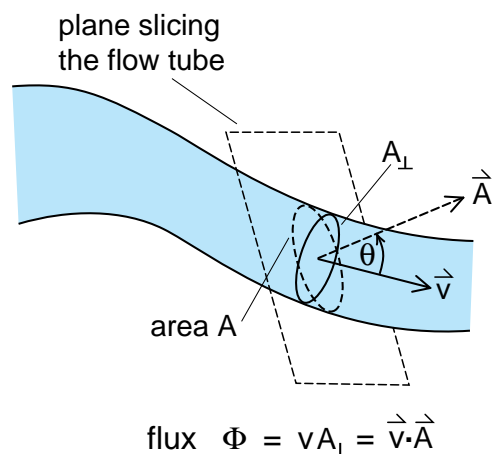flux $\Phi = vA_\perp = \vec{v} \cdot \vec{A}$

Figure 2
If we have an area A that is not normal to the stream, then the cross-sectional area is $A_\perp = A \cos \theta$, and the flux is $\Phi = vA_\perp = vA \cos \theta$, which can also be written $\Phi = \vec{v} \cdot \vec{A}$.

cross-sectional
area $A_\perp$

flow tube

flux $\Phi = vA_\perp$

Figure 1
The flux of water Φ through a flow tube is the amount of water per second flowing past a cross-sectional area $A_\perp$.

where the $d\vec{A}$'s are infinitesimal pieces of area on the fish net and our sum or integration extends over the entire submerged area of the net. Because we are integrating over an area or surface in Equation (3), this integral is called a *surface integral*.

Think of Equation (3), not as an integral you "do", like $\int x^2 dx = x^3/3$, but more as a formal statement of the steps we went through to calculate the total flux Φ_T.

Suppose, for example, someone came up to you and asked how you would calculate the total current in the stream. If you were a mathematician you might answer, "I would calculate the integral

$$\Phi_T = \int_S \vec{v} \cdot d\vec{A} \tag{3a}$$

where S is a surface cutting the stream."

If you were a physicist, you might answer, "Throw a fish net across the stream, making sure that there are no gaps that the fish can get through. (This defines the mathematician's surface S). Then measure the flux of water through each hole in the net (these are the $\vec{v} \cdot d\vec{A}$'s of Equation 3a), and then add them up to get the total flux (do the integral)."

Basically, the mathematician's statement in Equation (3a) is short hand notation for all the steps that the physicist would carry out.
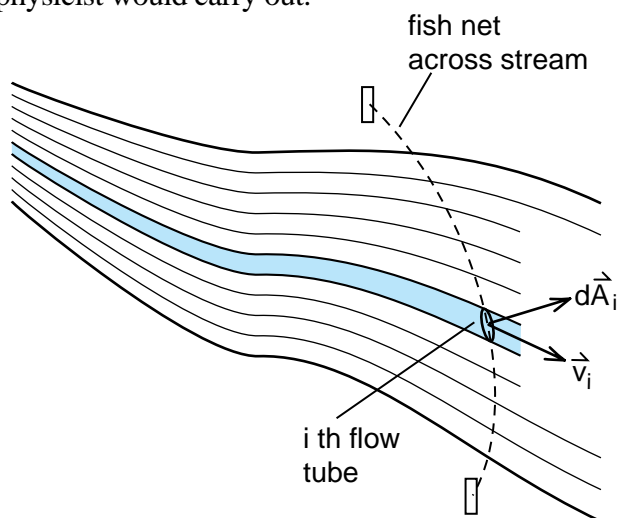

Figure 3
To calculate the flux of water through a fish net, we can first calculate the flux of water through each hole in the net, and then add up the fluxes to get the total flux.

Gauss' Law

The statement of Gauss' law applied to electric fields in Chapter (24) was that the total electric flux Φ_T out through a closed surface was equal to $1/\varepsilon_0$ times the total charge Q_{in} inside the surface. Our surface integral of Equation (3) allows us to give a more formal (at least more mathematical sounding) statement of Gauss' law.

Suppose we have a collection of charged particles as shown in Figure (4), which are completely surrounded by a closed surface S. *(Think of the closed surface as being the surface of an inflated balloon. There cannot be any holes in the surface or air would escape.)* The total flux of the field $\vec{E}$ out through the surface S is formally given by the surface integral

$$\Phi_T = \int_S \vec{E} \cdot d\vec{A} \tag{4}$$

where the $d\vec{A}$ are small pieces of the surface, and $\vec{E}$ is the electric field vector at each $d\vec{A}$.
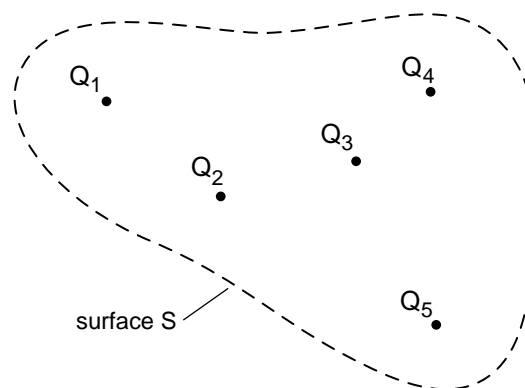

Figure 4
Closed surface S completely surrounding a collection of charges. The flux of $\vec{E}$ out through the closed surface is equal to $1/\varepsilon_0$ times the total charge inside.

The total charge Q_{in} inside the surface S is obtained by adding up all the charges we find inside. Any charges outside do not count. *(We have to have a completely closed surface so that we can decide whether a charge is inside or not.)* Then equating the total flux Φ_T to Q_{in}/ε_0 we get the ***integral equation***

$$\int_{\substack{closed \\ surface\,S}} \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$$

formal statement of Gauss' law   (5)

There is nothing really new in Equation (5) that we did not say back in Chapter (24). What we now have is a convenient short hand notation for all the steps we discussed earlier.

We will now use Equation (5) to calculate the electric field of a point charge. Although we have done this same calculation before, we will do it again to remind us of the steps we actually go through to apply Equation (5). A formal equation like this becomes real or useful only when we have an explicit example to remind us how it is used. ***When you memorize such an equation, also memorize an example to go with it.***

In Figure (5), we have a point charge +Q that produces a radial electric field $\vec{E}$ as shown. To apply Gauss' law we draw a spherical surface S of radius r around the charge. For this surface we have

$$\Phi_T = \int_S \vec{E}\cdot d\vec{A} = EA_\perp = E(r)4\pi r^2$$

Inside the surface the total charge is Q. Thus Gauss' law, Equation (5), gives

$$\int_S \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\varepsilon_0}$$

$$E(r)4\pi r^2 = \frac{Q}{\varepsilon_0}$$

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \tag{6}$$

The only thing that is new here is the use of the notation $\int_S \vec{E}\cdot d\vec{A}$ for total flux Φ_T. When we actually wish to calculate Φ_T, we look for a surface that is perpendicular to $\vec{E}$ so that we can use the simple formula $EA_\perp$.

If the charge distribution were complex, more like Figure (4), we could calculate $\int_S \vec{E}\cdot d\vec{A}$ by casting a fish net all around the charges and evaluating $\vec{E}_i\cdot d\vec{A}_i$ for each hole in the net. The formal expression of the surface integral at least gives us a procedure we can follow if we are desperate.
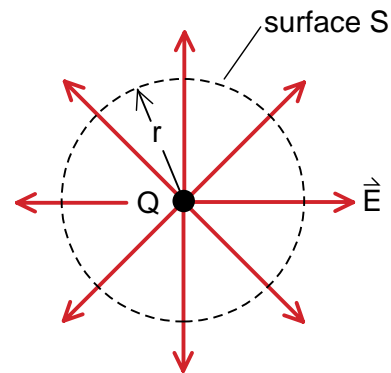

Figure 5
For the electric field of a point charge, we know immediately that the total flux Φ_T out through the spherical surface is the area $4\pi r^2$ times the strength $\vec{E}(r)$ of the field. Thus

$$\Phi_T \equiv \int_S \vec{E}\cdot d\vec{A} = \vec{E}(r)\times 4\pi r^2$$

THE LINE INTEGRAL

Another formal concept which we will use extensively in the remaining chapters on electromagnetic theory is the line integral. You have already been exposed to the idea in earlier discussions of the concept of work. If we exert a force $\vec{F}$ on a particle while the particle moves from Point (1) to Point (2) as shown in Figure (6), then the work we do is given by the integral

$$\text{Work W} = \int_1^2 \vec{F} \cdot d\vec{x} \qquad (7)$$

where we are integrating along the path in Figure (6).

Equation (7) is short hand notation for many steps. What it really says is to draw the path taken by the particle in going from Point (1) to (2), break the path up into lots of little steps $d\vec{x}_i$, calculate the work dW_i we do during each step, $dW_i = \vec{F}_i \cdot d\vec{x}_i$, and then add up all the dW_i to get the total work W.

$$dW_i = \vec{F}_i \cdot d\vec{x}_i$$

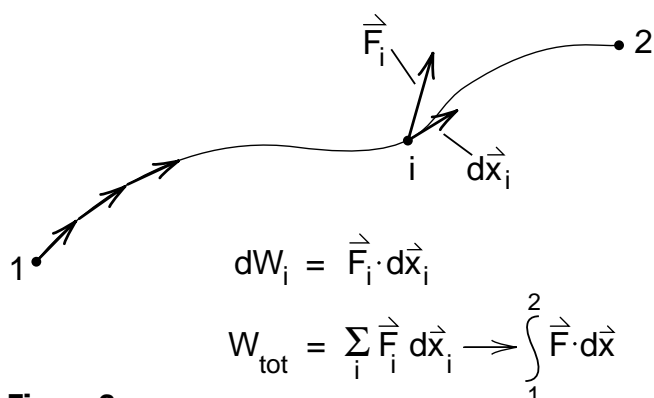$$W_{tot} = \sum_i \vec{F}_i \, d\vec{x}_i \longrightarrow \int_1^2 \vec{F} \cdot d\vec{x}$$

Figure 6
To calculate the total work done moving a particle from point (1) to point (2) along a path , first break the path up into many short displacements $d\vec{x}$. The work dW_i is $dW_i = \vec{F}_i \cdot d\vec{x}_i$. The total work W is the sum of all the dW_i.

The first thing we have to worry about in discussing Equation (7), is what path the particle takes in going from Point (1) to Point (2). If we are moving an eraser over a blackboard, the longer the path, the more work we do. In this case, we cannot do the line integral until the path has been specified.

On the other hand, if we are carrying the particle around the room, exerting a force $\vec{F} = -F_g$ that just overcomes the gravitational force, then the work we do is stored as gravitational potential energy. The change in potential energy, and therefore the line integral of Equation (7), depends only on the end points (1) and (2) and not on the path we take. When the line integral of a force does not depend upon the path, we say that the force is ***conservative***.

A formal statement that a force is conservative is that the line integrals are equal for any two paths -- for example, path (a) and path (b) in Figure (7).

$$\int_{1(\text{path a})}^2 \vec{F} \cdot d\vec{x} = \int_{1(\text{path b})}^2 \vec{F} \cdot d\vec{x} \qquad (8)$$
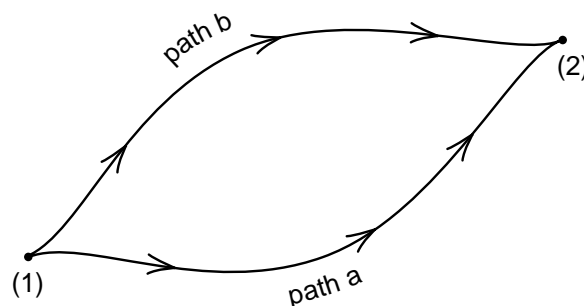

Figure 7
If the work done in carrying the particle from point (1) to point (2) does not depend upon which path we take, we say that the force is conservative.

Let us write Equation (8) in the form

$$\int\limits_{1(\text{path a})}^{2} \vec{F}\cdot d\vec{x} \;-\; \int\limits_{1(\text{path b})}^{2} \vec{F}\cdot d\vec{x} \;=\; 0$$

Now take the minus sign inside the integral over Path (b) so that we have a sum of $\vec{F}_i\cdot\left(-d\vec{x}_i\right)$

$$\int\limits_{1(\text{path a})}^{2} \vec{F}\cdot d\vec{x} \;+\; \int\limits_{1(\text{path b})}^{2} \vec{F}\cdot\left(-d\vec{x}\right) \;=\; 0 \tag{9}$$

For the path (b) integral, we have reversed the direction of each step. The sum of the reversed steps is the same as going back, from point (2) to point (1) as illustrated in Figure (8). Thus Equation (9) becomes

$$\int\limits_{1(\text{path a})}^{2} \vec{F}\cdot d\vec{x} \;+\; \int\limits_{2(\text{path b})}^{1} \vec{F}\cdot d\vec{x} \;=\; 0 \tag{10}$$

If Equation (10) applies for any Path (a) and (b), then the force $\vec{F}$ is conservative.
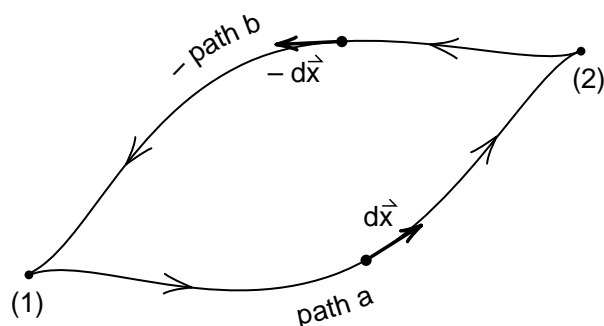

Figure 8
If we take path b backwards, i.e. go from point (2) to point (1), the $d\vec{x}_i$ on path b are reversed and the integral along path b changes sign. If the line integral from (1) to (2) does not depend upon the path, then the line integral for any return trip must be the negative of the integral for the trip out, and the sum of the two integrals must be zero.

Equation (10) does not really depend upon the Points (1) and (2). More generally, it says that if you go out and then come back to your starting point, and the sum of all your $\vec{F}_i\cdot d\vec{x}_i$ is zero, then the force is conservative. This special case of a line integral that comes back to the starting point as in Figure (9) is called ***the line integral around a closed path***, and is denoted by an integral sign with a circle in the center

$$\oint \vec{F}\cdot d\vec{x} \;\equiv\; \begin{cases} \text{the line integral} \\ \text{around a closed path} \\ \text{as in Figure 9} \end{cases} \tag{11}$$

With the notation of Equation (11), we can formally define a conservative force $\vec{F}$ as one for which

$$\boxed{\;\oint\limits_{\substack{\text{for any}\\ \text{closed path}}} \vec{F}\cdot d\vec{x} \;=\; 0\;}\qquad \begin{array}{l}\textit{\textbf{definition of a}}\\ \textit{\textbf{conservative force}}\end{array} \tag{12}$$

This line integral around a closed path will turn out to be an extremely useful mathematical tool. We have already seen that it distinguishes a conservative force like gravity, where $\oint \vec{F}\cdot d\vec{x} = 0$, from a non conservative force like friction on a blackboard eraser, for which $\oint \vec{F}\cdot d\vec{x} \neq 0$. In another case, namely Ampere's law to be discussed next, the line integral of the magnetic field around a closed path tells us something about the currents that flow through the path.
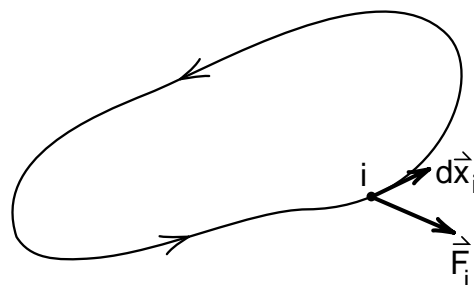

Figure 9
For a conservative force, the line integral $\oint \vec{F}\cdot d\vec{x}$, that goes completely around a closed path, must be zero. It is not necessary to specify where the calculation starts.

AMPERE'S LAW

Figure (10), which is similar to Figure (28-14), is a sketch of the magnetic field produced by a current in a straight wire. In this figure the current i is directed up and out of the paper, and the magnetic field lines travel in counter clockwise circles as shown. We saw from equation (28-18) that the strength of the magnetic field is given by

$$\left|\vec{B}\right| = \frac{\mu_0 i}{2\pi r} \tag{28-18}$$

In Figure (11) we have drawn a circular path of radius r around the wire and broken the path into a series of steps indicated by short vectors $d\vec{\ell}_i$. We drew the stick figure to emphasize the idea that this is really a path and that $d\vec{\ell}_i$ shows the length and direction of the i th step.

For each of the steps, calculate the dot product $\vec{B}_i \cdot d\vec{\ell}_i$ where $\vec{B}_i$ is the magnetic field at that step, and then add up the $\vec{B}_i \cdot d\vec{\ell}_i$ for all the steps around the path to get

$$\sum_{\substack{\text{all steps} \\ \text{around path}}} \vec{B}_i \cdot d\vec{\ell}_i \rightarrow \oint \vec{B} \cdot d\vec{\ell} \tag{13}$$

The result is the line integral of $\vec{B}$ around the closed path.

Why bother calculating this line integral? Let us put in the value for $\left|\vec{B}\right|$ given by Equation (28-18) and see why. We happen to have chosen a path where each step $d\vec{\ell}_i$ is parallel to $\vec{B}$ at that point, so that

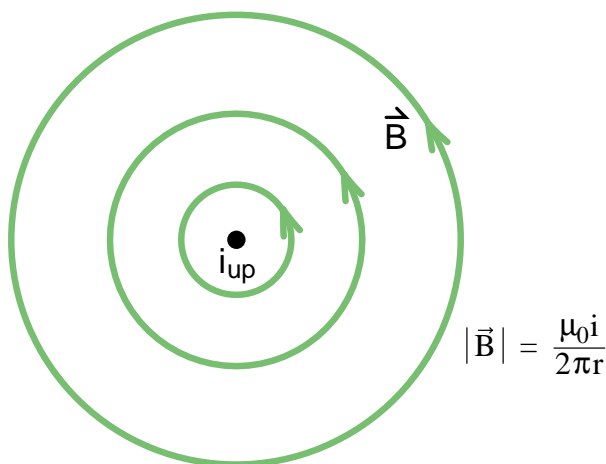$$\vec{B}_i \cdot d\vec{\ell}_i = B_i d\ell_i \tag{14}$$

Figure 10
Circular magnetic field of a wire.

$$\left|\vec{B}\right| = \frac{\mu_0 i}{2\pi r}$$

and Equation (13) becomes

$$\oint \vec{B} \cdot d\vec{\ell} = \oint B d\ell \tag{15}$$

In addition, our path has a constant radius r, so that $B = \mu_0 i / 2\pi r$ is constant all around the path. We can take this constant outside the integral in Equation (15) to get

$$\oint \vec{B} \cdot d\vec{\ell} = B \oint d\ell \tag{16}$$

Next we note that $\oint d\ell$ is just the sum of the lengths of our steps around the circle; i.e., it is just the circumference $2\pi r$ of the circle, and we get

$$\oint \vec{B} \cdot d\vec{\ell} = B \oint d\ell = B \times 2\pi r \tag{17}$$

Finally substituting the value of B from Equation (28-18) we get

$$\boxed{\oint \vec{B} \cdot d\vec{\ell} = \frac{\mu_0 i}{2\pi r} \times 2\pi r = \mu_0 i} \tag{18}$$

$$\sum_i \vec{B}_i \cdot d\vec{\ell}_i \rightarrow \oint \vec{B} \cdot d\vec{\ell} = B(r) \times 2\pi r$$
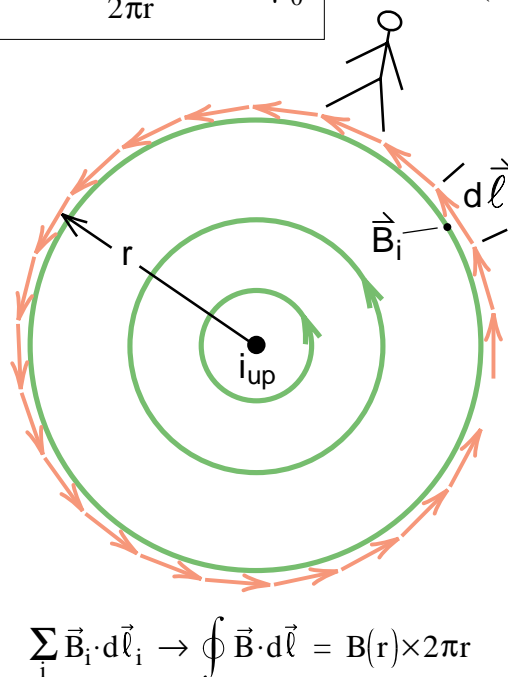
Figure 11
Circular path of radius r around the wire. As we walk around the path, each step represents a displacement $d\vec{\ell}$. To calculate the line integral $\oint \vec{B} \cdot d\hat{\ell}$, we take the dot product of $d\hat{\ell}$ with B at each interval and add them up as we go around the entire path. In this case the result is simply B(r) times the circumference $2\pi r$ of the path.

There are several points we want to make about Equation (18). First we made the calculation easy by choosing a circular path that was parallel to $\vec{B}$ all the way around. This allowed us to replace the dot product $\vec{B} \cdot d\vec{\ell}$ by a numerical product $Bd\ell$, pull the constant B outside the integral, and get an answer almost by inspection. This should be reminiscent of our work with Gauss' law where we chose surfaces that made it easy to solve the problem.

The second point is that we get an exceptionally simple answer for the line integral of $\vec{B}$ around the wire, namely

$$\boxed{\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i} \qquad \begin{array}{l} \textit{Ampere's} \\ \textit{Law} \end{array} \qquad (18a)$$

The line integral depends only on the current i through the path and not on the radius r of the circular path.

What about more general paths that go around the wire? To find out, we have to do a slightly harder calculation, but the answer is interesting enough to justify the effort.

In Figure (12) we have constructed a closed path made up of three arc sections of lengths $r_1\theta_1$, $r_2\theta_2$, and $r_3\theta_3$ connected by radial sections as shown. These arcs are sections of circles of radii r_1, r_2 and r_3, respectively. We wish to calculate $\oint \vec{B} \cdot d\vec{\ell}$ for this path and see how the answer compares with what we got for the circular path.

The first thing to note as we go around our new path is that in all the *radial* sections, $\vec{B}$ and $d\vec{\ell}$ are perpendicular to each other, so that $\vec{B} \cdot d\vec{\ell} = 0$. The radial sections do not contribute to our line integral and all we have to do is add up the contributions from the three arc segments. These are easy to calculate because $\vec{B} \cdot d\vec{\ell} = Bd\ell$ and B is constant over each arc, so that the integral of $\vec{B} \cdot d\vec{\ell}$ over an arc segment is just the value of B times the length $r\theta$ of the arc. We get

$$\int_{\text{arc 1}} \vec{B} \cdot d\ell = B_1 r_1 \theta_1 = \frac{\mu_0 i}{2\pi r_1} r_1 \theta_1 = \frac{\mu_0 i \theta_1}{2\pi}$$

$$\int_{\text{arc 2}} \vec{B} \cdot d\ell = B_2 r_2 \theta_2 = \frac{\mu_0 i}{2\pi r_2} r_2 \theta_2 = \frac{\mu_0 i \theta_2}{2\pi}$$

$$\int_{\text{arc 3}} \vec{B} \cdot d\ell = B_3 r_3 \theta_3 = \frac{\mu_0 i}{2\pi r_3} r_3 \theta_3 = \frac{\mu_0 i \theta_3}{2\pi}$$

Adding the contribution from each arc segment we get the line integral around the closed path

$$\oint \vec{B} \cdot d\vec{\ell} = \frac{\mu_0 i \theta_1}{2\pi} + \frac{\mu_0 i \theta_2}{2\pi} + \frac{\mu_0 i \theta_3}{2\pi}$$

$$= \frac{\mu_0 i}{2\pi} (\theta_1 + \theta_2 + \theta_3)$$

But $(\theta_1 + \theta_2 + \theta_3)$ is the sum of the angles around the circle, and is therefore equal to 2π. Thus we get for the path of Figure (12)
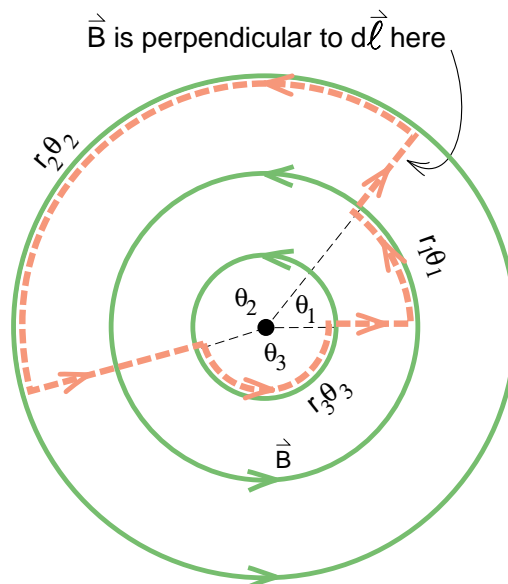

Figure 12
A somewhat arbitrary path around the wire is made of arc sections connected by radial sections. Since B is perpendicular to $d\dot{\ell}$ in the radial sections, the radial sections do not contribute to the $\oint \vec{B} \cdot d\dot{\ell}$ for this path. In the arcs, the length of the arc increases with r, but B decreases as 1/r, so that the contribution of the arc does not depend upon how far out it is. As a result the $\oint \vec{B} \cdot d\dot{\ell}$ is the same for this path as for a circular path centered on the wire.

$$\oint \vec{B} \cdot d\vec{\ell} = \frac{\mu_0 i}{2\pi}(2\pi) = \mu_0 i \qquad (19)$$

which is the same answer we got for the circular path.

The result in Equation (19) did not depend upon how many line segments we used, because each arc contributed an angle θ, and if the path goes all the way around, the angles always add up to 2π. In Figure (13) we have imitated a smooth path (the dotted line) by a path consisting of many arc sections. The more arcs we use the closer the imitation. We can come arbitrarily close to the desired path using paths whose integral $\oint \vec{B} \cdot d\vec{\ell}$ is $\mu_0 i$. In this sense we have proved that Equation (19) applies to any closed path around the wire.

It is another story if the path does not go around the wire. In Figure (14) we have such a path made up of two arc and two radial segments as shown. As before, we can ignore the radial segments because $\vec{B}$ and $d\vec{\ell}$ are perpendicular and $\vec{B} \cdot d\vec{\ell} = 0$.
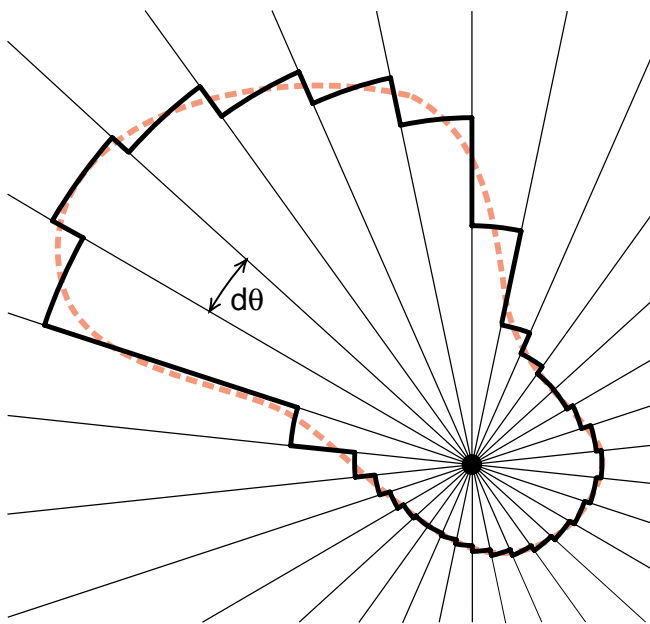

Figure 13
We can approximate an arbitrary path (dotted line) by a series of connected radial and arc sections. The smaller the angle $d\theta$ marking the arc sections, the better the approximation. In calculating $\vec{B} \cdot d\vec{\ell}$, the radial sections do not count, and we can bring all the arc sections back to a single circle centered on the wire. As a result, $\oint \vec{B} \cdot d\vec{\ell}$ does not depend upon the shape of the path, as long as the path goes around the wire.

On the outer segment we are going in the same direction as $\vec{B}$ so that $\vec{B} \cdot d\vec{\ell}$ is positive and we get

$$\int_{\text{arc 1}} \vec{B} \cdot d\vec{\ell} = B_1(r_1\theta) = \frac{\mu_0 i}{2\pi r_1} r_1 \theta = \frac{\mu_0 i}{2\pi}\theta$$

On the inner arc we are coming back around in a direction opposite to $\vec{B}$, the quantity $\vec{B} \cdot d\vec{\ell}$ is negative, and we get

$$\int_{\text{arc 2}} \vec{B} \cdot d\vec{\ell} = -B_2(r_2\theta) = \frac{-\mu_0 i}{2\pi r_2} r_2 \theta = \frac{-\mu_0 i}{2\pi}\theta$$

Adding up the two contributions from the two arcs, we get

$$\oint_{\substack{\text{Path of} \\ \text{Figure 14}}} \vec{B} \cdot d\vec{\ell} = \left(\frac{\mu_0 i\theta}{2\pi}\right)_{\text{arc 1}} + \left(\frac{-\mu_0 i\theta}{2\pi}\right)_{\text{arc 2}} = 0 \quad (20)$$

For this closed path which does not go around the current, we get $\oint \vec{B} \cdot d\vec{\ell} = 0$. This result is not changed if we add more arcs and radial segments to the path. As long as the path does not go around the current, we get zero for $\oint \vec{B} \cdot d\vec{\ell}$.
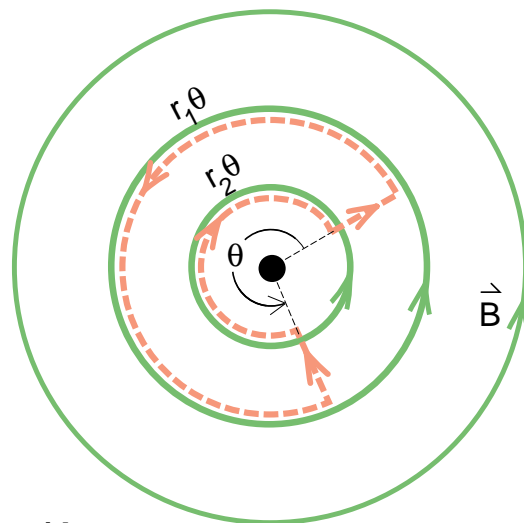

Figure 14
In this example, where the path does not go around the wire, the sections labeled $r_1\theta$ and $r_2\theta$ contribute equal and opposite amounts to the line integral $\oint \vec{B} \cdot d\vec{\ell}$. As a result $\oint \vec{B} \cdot d\vec{\ell}$ is zero for this, or any path that does not go around the wire.

Several Wires

It is relatively straightforward to generalize our results to the case where we have several wires as in Figures (15 a,b). Here we have three currents i_1, i_2, and i_3 each alone producing a magnetic field $\vec{B}_1$, $\vec{B}_2$ and $\vec{B}_3$ respectively.

The first step is to show that the net field $\vec{B}$ at any point is the vector sum of the fields of the individual wires. We can do this by considering the force on a test particle of charge q moving with a velocity $\vec{v}$ as shown in Figure (15a). Our earlier results tell us that the current i_1 exerts a force

$$\vec{F}_1 = q\vec{v} \times \vec{B}_1$$

Similarly i_2 and i_3 exert forces

$$\vec{F}_2 = q\vec{v} \times \vec{B}_2$$

$$\vec{F}_3 = q\vec{v} \times \vec{B}_3$$

Newton's second law required us to take the vector sum of the individual forces to get the total force $\vec{F}$ acting on an object

$$\begin{aligned} \vec{F} &= \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \\ &= q\vec{v} \times (\vec{B}_1 + \vec{B}_2 + \vec{B}_3) \end{aligned} \tag{21}$$
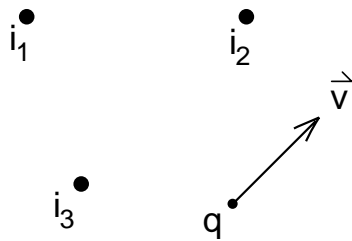
If we write this total force in the form

Figure 15a
A charge q moving in the vicinity of three currents i_1, i_2 and i_3. If the magnetic field $\vec{B}$ at the charge is the vector sum of the fields $\vec{B}_1$, $\vec{B}_2$ and $\vec{B}_3$ of the three wires, then the net magnetic force $\vec{F}_B$ on q is given by
$$\begin{aligned} \vec{F}_B = q\vec{v} \times \vec{B} &= q\vec{v} \times (\vec{B}_1 + \vec{B}_2 + \vec{B}_3) \\ &= q\vec{v} \times \vec{B}_1 + q\vec{v} \times \vec{B}_2 + q\vec{v} \times \vec{B}_3 \\ &= \vec{F}_{B1} + \vec{F}_{B2} + \vec{F}_{B3} \end{aligned}$$
and we get the desired result that the net force on q is the vector sum of the forces exerted by each wire.

$$\vec{F}_T = q\vec{v} \times \vec{B} \tag{22}$$

where $\vec{B}$ is the effective field acting on the test particle, then Equations (21) and (22) give

$$\boxed{\vec{B} = \vec{B}_1 + \vec{B}_2 + \vec{B}_3} \tag{23}$$

The fact that magnetic fields add vectorially is a consequence of the vector addition of forces and our use of the magnetic force law to define $\vec{B}$.

With Equation (23) we can now calculate $\oint \vec{B} \cdot d\vec{\ell}$ for the field of several wires. Let us draw a path around two of the wires, as shown in Figure (15b). For this path, we get

$$\oint_{\substack{\text{Closed path} \\ \text{of Fig. 29–15}}} \vec{B} \cdot d\vec{\ell} = \oint (\vec{B}_1 + \vec{B}_2 + \vec{B}_3) \cdot d\vec{\ell}$$

$$= \oint \vec{B}_1 \cdot d\vec{\ell} + \oint \vec{B}_2 \cdot d\vec{\ell} + \oint \vec{B}_3 \cdot d\vec{\ell}$$

$$\tag{24}$$

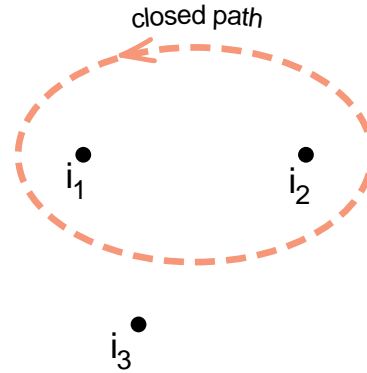Since the closed path goes around currents i_1 and i_2, we get from Equation (19)

Figure 15b
Calculating $\oint \vec{B} \cdot d\vec{\ell}$ for a path that goes around two of the wires.

$$\oint \vec{B}_1 \cdot d\vec{\ell} = \mu_0 i_1$$

$$\oint \vec{B}_2 \cdot d\vec{\ell} = \mu_0 i_2$$

Since the path misses i_3, we get

$$\oint \vec{B}_3 \cdot d\vec{\ell} = 0$$

and Equation (24) gives

$$\underset{\substack{\text{Closed path} \\ \text{of Fig. 15}}}{\oint \vec{B} \cdot d\vec{\ell}} = \mu_0(i_1 + i_2) = \mu_0 \times \begin{pmatrix} \text{current} \\ \text{enclosed} \\ \text{by path} \end{pmatrix} \quad (25)$$

Equation (25) tells us that $\oint \vec{B} \cdot d\vec{\ell}$ around a closed path is equal to μ_0 times the total current $i = i_1 + i_2$ encircled by the path. This has the flavor of Gauss' law which said that the total flux or surface integral of $\vec{E}$ out through a closed surface was $1/\varepsilon_0$ times the total charge Q_{in} inside the surface. Just as charge outside the closed surface did not contribute to the surface integral of $\vec{E}$, currents outside the closed path do not contribute to the line integral of $\vec{B}$.

We derived Equation (25) for the case that all our currents were in parallel straight wires. It turns out that it does not matter if the wires are straight, bent, or form a hideous tangle. As a general rule, if we construct a closed path, then the line integral of $\vec{B}$ around the closed path is μ_0 times the net current $i_{enclosed}$ flowing through the path

$$\underset{\substack{\text{any closed} \\ \text{path}}}{\oint \vec{B} \cdot d\vec{\ell}} = \mu_0 i_{enclosed} \qquad \begin{array}{c} \textbf{\textit{(26)}} \\[4pt] \textbf{\textit{Ampere's}} \\ \textbf{\textit{Law}} \end{array}$$

This extremely powerful and general theorem is known as **_Ampere's law_**.

So far in this chapter we have focused on mathematical concepts. Let us now work out some practical applications of Ampere's law to get a feeling for how the law is used.

Field of a Straight Wire

Our first application of Ampere's law will be to calculate the magnetic field of a straight wire. We will use this trivial example to illustrate the steps used in applying Ampere's law.

First we sketch the situation as in Figure (16), and then write down Ampere's law to remind us of the law we are using

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i_{enclosed}$$

Next we choose a closed path that makes the line integral as simple as possible. Generally the path should either be along $\vec{B}$ so that $\vec{B} \cdot d\vec{\ell} = B d\ell$, or perpendicular so that $\vec{B} \cdot d\vec{\ell} = 0$. The circular path of Figure (16) gives $\vec{B} \cdot d\vec{\ell} = B d\ell$ with B constant, thus

$$\oint \vec{B} \cdot d\vec{\ell} = \oint B d\ell = B \oint d\ell = B * 2\pi r = \mu_0 i$$

The result is

$$\boxed{B = \frac{\mu_0 i}{2\pi r}}$$

which we expected. When you memorize Ampere's law, **_memorize an example like this to go with it_**.

circular path of radius r
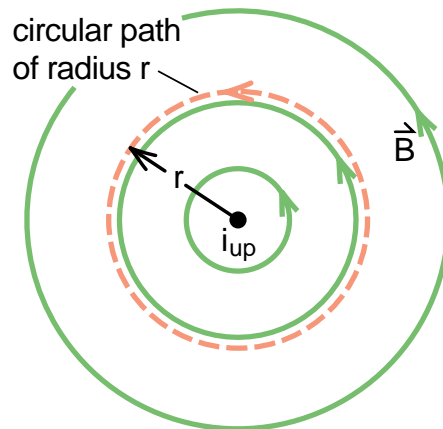
$\vec{B}$

r

i_{up}

Figure 16
Using Ampere's law to calculate the magnetic field of a wire. We have $\oint \vec{B} \cdot d\vec{\ell} = B \times 2\pi r$ around the path. Thus Ampere's law $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i$ gives $B = \mu_0 i / 2\pi r$.

Exercise 1

Each of the indicated eight conductors in Figure (17) carries 2.0A of current into (dark) or out of (white) the page. Two paths are indicated for the line integral $\oint \vec{B} \cdot d\vec{\ell}$. What is the value of the integral for (a) the dotted path? (b) the dashed path?
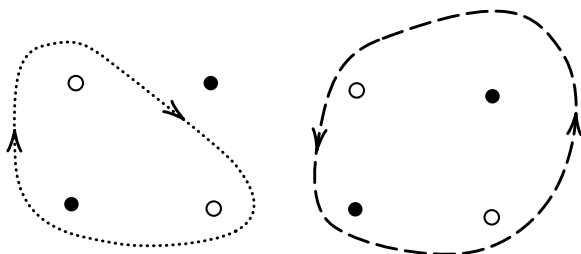
Figure 17

Exercise 2

Eight wires cut the page perpendicularly at the points shown in Figure (18). A wire labeled with the integer k (k = 1, 2..., 8) bears the current ki_0. For those with odd k, the current is up, out of the page; for those with even k it is down, into the page. Evaluate $\oint \vec{B} \cdot d\vec{\ell}$ along the closed path shown, in the direction shown.
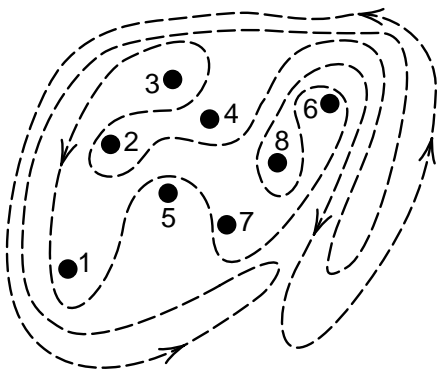
Figure 18

Exercise 3

Show that a uniform magnetic field B cannot drop abruptly to zero as one moves at right angles to it, as suggested by the horizontal arrow through point a in Figure (19). (Hint: Apply Ampere's law to the rectangular path shown by the dashed lines.) In actual magnets "fringing" of the lines of B always occurs, which means that B approaches zero gradually.
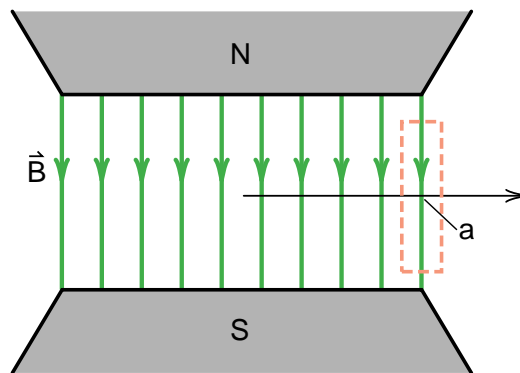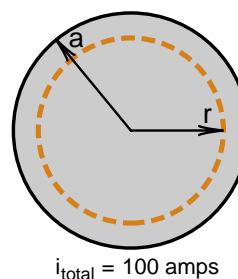

Figure 19

Exercise 4

Figure (20) shows a cross section of a long cylindrical conductor of radius a, carrying a uniformly distributed current i. Assume a = 2.0 cm, i = 100A, and sketch a plot of B(r) over the range 0 < r < 4 cm.

i_{total} = 100 amps

Figure 20

(The above are some choice problems from Halliday and Resnick.)

Exercise 5

Figure (21) shows a cross section of a hollow cylindrical conductor of radii a and b, carrying a uniformly distributed current i.

a) Show that B(r) for the range b < r < a is given by

$$B(r) = \frac{\mu_0 i}{2\pi r}\left(\frac{r^2 - b^2}{a^2 - b^2}\right)$$

b) Test this formula for the special cases of r = a, r = b, and r = 0.

c) Assume a = 2.0 cm, b = 1.8 cm, and i = 100 A. What is the value of B at r = a? (Give your answer in tesla and gauss.)
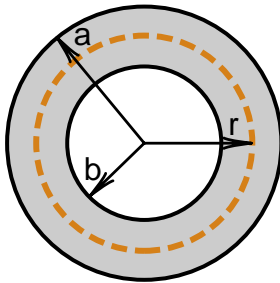

Figure 21

Exercise 6

Figure (22) shows a cross section of a long conductor of a type called a coaxial cable. Its radii (a, b, c) are shown in the figure. Equal but opposite currents i exist in the two conductors. Derive expressions for B(r) in the ranges
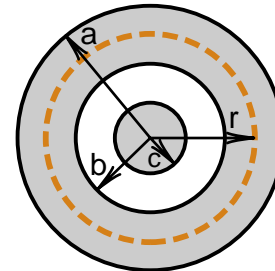
a) r < c,

b) c < r < b,

c) b < r < a, and

d) r > a.

e) Test these expressions for all the special cases that occur to you.

coaxial cable

Figure 22

Exercise 6 is a model of a ***coaxial cable***, where the current goes one way on the inner conductor and back the other way on the outside shield. If we draw any circuit outside the cable, there is no net current through the circuit, thus there is no magnetic field outside. As a result, coaxial cables confine all magnetic fields to the inside of the cable. This is important in many electronics applications where you do not want fields to radiate out from your wires. The cables we use in the lab, the ones with the so called BNC connectors, are coaxial cables, as are the cables that carry cable television.

FIELD OF A SOLENOID

As with Gauss' law, Ampere's law is most useful when we already know the field structure and wish to calculate the strength of the field. The classic example to which ampere's law is applied is the calculation of the magnetic field of a long straight solenoid.

A long solenoid is a coil of wire in which the length L of the coil is considerably larger than the diameter d of the individual turns. The shape of the field produced when a current i flows through the coil was illustrated in Figure (28-21) and is sketched here in Figure (23). Iron filings gave us the shape of the field and Ampere's law will tell us the strength.

The important and useful feature of a solenoid is that we have a nearly uniform magnetic field inside the coil and nearly zero field outside. The longer the solenoid, relative to the diameter d, the more uniform the field $\vec{B}$ inside and the more nearly it is zero outside.
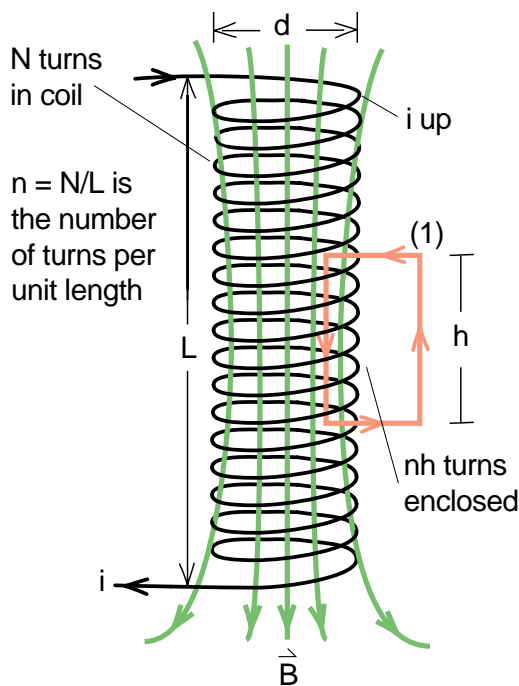

Figure 23
Calculating the magnetic field of a long solenoid. Around the path starting at point (1) we have $\oint \vec{B} \cdot d\vec{\ell} = 0 + Bh + 0 + 0$. The amount of current enclosed by the path is $i_{tot} = (nh)i$ where n is the number of turns per unit length. Thus Ampere's law $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i_{tot}$ gives $Bh = \mu_0 nhi$ or $B = \mu_0 ni$.

Right Hand Rule for Solenoids

The direction of the field inside the solenoid is a bit tricky to figure out. As shown in Figure (24), up near the wires and in between the turns, the field goes in a circle around the wire just as it does for a straight wire. As we go out from the wire the circular patterns merge to create the uniform field in the center of the solenoid.

We see, from Figure (24), that if the current goes around the coil in such a way that the current is up out of the paper on the right side and down into the paper on the left, then the field close to the wires will go in counterclockwise circles on the right and clockwise circles on the left. For both these sets of circles, the field inside the coil points down. As a result the uniform field inside the coil is down as shown.

There is a simple way to remember this result without having to look at the field close to the wires. Curl the fingers of your ***right*** hand in the direction of the flow of the current i in the solenoid, and your thumb will point in the direction of the magnetic field inside the solenoid. We will call this the right hand rule for solenoids.
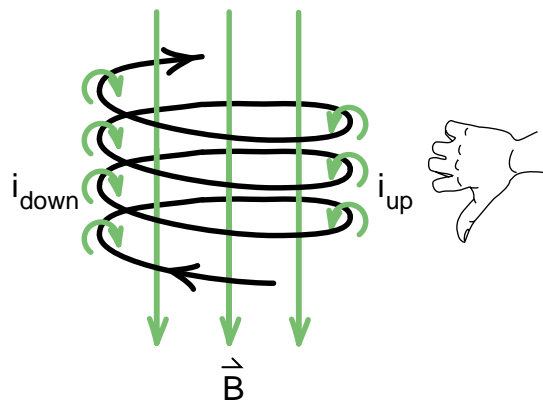

Figure 24
If you know the direction of the current in the wire, you can determine the direction of the magnetic field by looking very close to the wire where the field goes around the wire. You get the same answer if you curl the fingers of your right hand around in the direction the current in the coil is flowing. Your thumb then points in the direction of the field.

Evaluation of the Line Integral

Figure (25) is a detail showing the path we are going to use to evaluate $\oint \vec{B} \cdot d\vec{\ell}$ for the solenoid. This path goes down the solenoid in the direction of $\vec{B}$ (side 1), and out through the coil (side 2), up where $\vec{B} = 0$ (side 3) and back into the coil (side 4).

We can write $\oint \vec{B} \cdot d\vec{\ell}$ as the sum of four terms for the four sides

$$\oint \vec{B} \cdot d\vec{\ell} = \int_{\text{side 1}} \vec{B} \cdot d\vec{\ell} + \int_{\text{side 2}} \vec{B} \cdot d\vec{\ell}$$
$$+ \int_{\text{side 3}} \vec{B} \cdot d\vec{\ell} + \int_{\text{side 4}} \vec{B} \cdot d\vec{\ell}$$

On sides 2 and 4, when the path is inside the coil, $\vec{B}$ and $d\vec{\ell}$ are perpendicular and we get $\vec{B} \cdot d\vec{\ell} = 0$. Outside the coil it is still 0 because there is no field there. Likewise $\vec{B} \cdot d\vec{\ell} = 0$ for side 3 because there is no field there. The only contribution we get is from side 1 inside the coil. If h is the height of our path, then

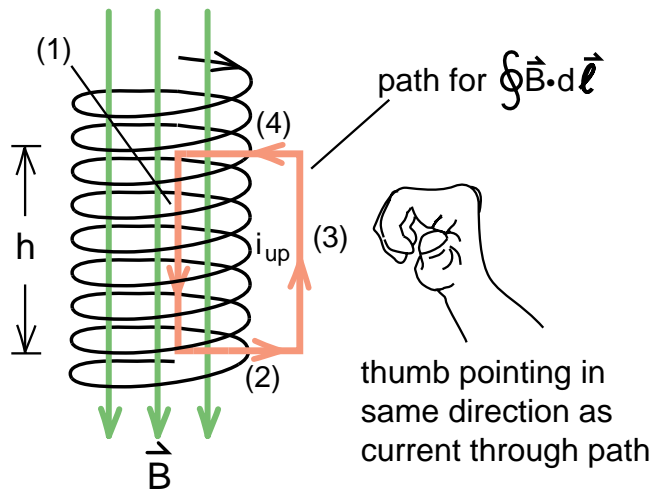$$\oint \vec{B} \cdot d\vec{\ell} = \int_{\text{side 1}} \vec{B} \cdot d\vec{\ell} = Bh \qquad (27)$$

Figure 25
Right hand rule for using Ampere's law. We define the positive direction around the path as the direction you curl the fingers of your right hand when the thumb is pointing in the direction of the current through the path. (As you see, we can come up with a right hand rule for almost anything.)

Calculation of $i_{enclosed}$

From Figure (25) we see that we get a current i up through our path each time another turn comes up through the path. On the left side of the coil the current goes down into the paper, but these downward currents lie outside our path and therefore are not included in our evaluation of $i_{enclosed}$. Only the positive upward currents count, and $i_{enclosed}$ is simply i times the number of turns that go up through the path.

To calculate the number of turns in a height h of the coil, we note that if the coil has a length L and a total of N turns, then the number of turns per unit length n is given by

$$\left.\begin{array}{c}\text{number of turns}\\\text{per unit length}\end{array}\right\} \equiv n = \frac{N}{L} \qquad (28)$$

and in a height h there must be nh turns

$$\left.\begin{array}{c}\text{number of turns}\\\text{in a height h}\end{array}\right\} = nh \qquad (29)$$

With nh turns, each carrying a current i, going up through our path, we see that $i_{enclosed}$ must be

$$i_{enclosed} = inh \qquad (30)$$

Using Ampere's law

We are now ready to apply Ampere's law to evaluate the strength B of the field inside the solenoid. Using Equation (27) for $\oint \vec{B} \cdot d\vec{\ell}$, and Equation (31) for $i_{enclosed}$, we get

$$\int \vec{B} \cdot d\vec{\ell} = \mu_0 i_{enclosed}$$

$$Bh = \mu_0 nih$$

$$\boxed{B = \mu_0 ni} \qquad \begin{array}{l}\textit{magnetic field}\\\textit{inside a solenoid}\end{array} \qquad (31)$$

The uniform magnetic field inside a long solenoid is proportional to the current i in the solenoid, and the number of turns per unit length, n.

Exercise 7

Later in the course, we will so often be using solenoids that you should be able to derive the formula $B = \mu_0 ni$, starting from Ampere's law, without looking at notes. This is a good time to practice. Take a blank sheet of paper, sketch a solenoid of length L with N turns. Then close the text and any notes, and derive the formula for B.

We have mentioned that equations like $\oint \vec{B}_1 \cdot d\vec{\ell} = \mu_0 i_{enclosed}$ are meaningless hen scratching until you know how to use them. The best way to do that is learn worked examples along with the equation. Two good examples for Ampere's law are to be able to calculate the magnetic field inside a wire (Exercise 4), and to be able to derive the magnetic field inside a solenoid. If you can do these two derivations without looking at notes, you should have a fairly good grasp of the law.
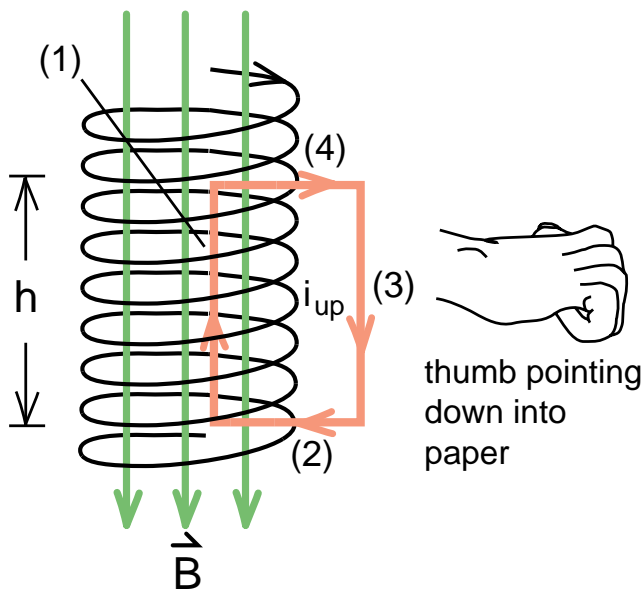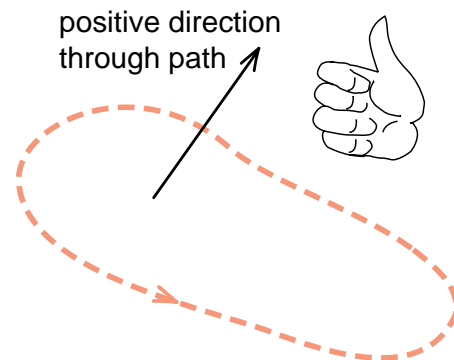

Figure 26
If we go around the wrong way, we just get two minus signs and all the results are the same. Here we went around the path so that our thumb pointed opposite to the direction of the current through the path. As a result the magnetic field in the solenoid points opposite to the direction of the path in the solenoid.

One More Right Hand Rule

If we really want to be careful about minus signs (and it is not always necessary), we have to say how the sign of $i_{enclosed}$ is evaluated in Figure (25). If, as in Figure (26) we reversed the direction of our path, then on side (1) $\vec{B} \cdot d\vec{\ell}$ is negative because our path is going in the opposite direction to $\vec{B}$. Thus for this path the complete integral $\oint \vec{B} \cdot d\vec{\ell}$ is negative, and somehow our $i_{enclosed}$ must also be negative, so that we get the same answer we got for Figure (25).

If we curl the fingers of our ***right hand*** in the direction that we go around the path, then in Figure (25) our thumb points up parallel to the current through the path, and in Figure (26) our thumb points down, opposite to the current. If we define the direction indicated by our right hand thumb as the positive direction ***through*** the path, as shown in Figure (27), then the current is going in a positive direction in Figure (25) but in a negative direction in Figure (26). This gives us a negative $i_{enclosed}$ for Figure (26) which goes along with the minus sign we got in the evaluation of $\oint \vec{B} \cdot d\vec{\ell}$.

By now you should be getting the idea of how we define directions in magnetic formula. ***Always use your right hand***. After a while you get so used to using your right hand that you do not have to remember the individual right hand rules.

Figure 27
In general, we use the right hand convention to associate a positive direction around a path to a positive direction through a path.

The Toroid

If we take a long solenoid, bend it in a circle and fit the ends together, we get what is called a toroid shown in Figure (28). The great advantage of a toroid is that there are no end effects. In the straight solenoid the magnetic field at the ends fanned out into space as seen in our iron filing map of Figure (28-23). With the toroid there are no ends. The field is completely confined to the region inside the toroid and there is essentially no field outside. For this reason a toroid is an ideal magnetic field storage device.

It is easy to use Ampere's law to calculate the magnetic field inside the toroid. In Figure (28) we have drawn a path of radius r inside the toroid. Going around this path in the same direction as $\vec{B}$, we immediately get

$$\oint \vec{B} \cdot d\vec{\ell} = B \times 2\pi r \qquad (32)$$

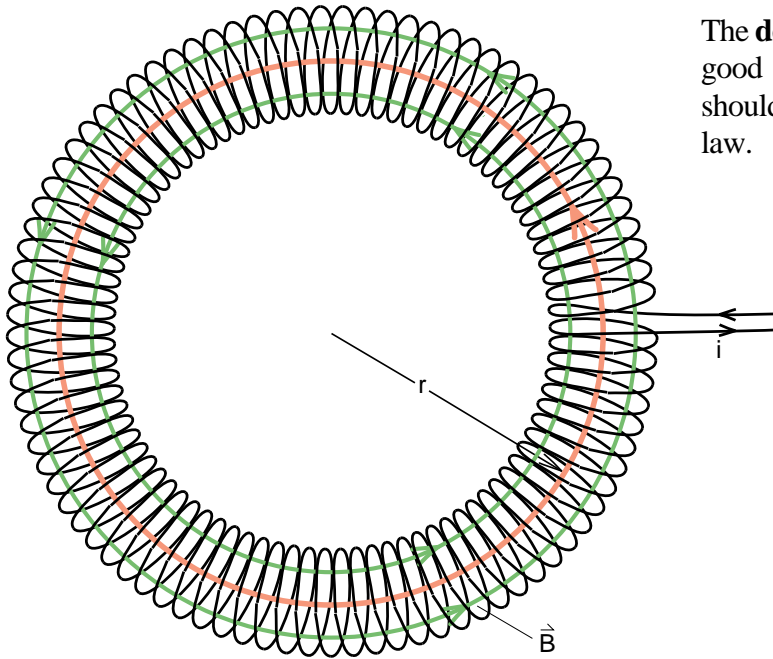because B is constant in magnitude and parallel to $d\vec{\ell}$.

If there are N turns of wire in the toroid, and the wire carries a current i, then all N turns come up through the path on the inside of the solenoid, and $i_{enclosed}$ is given by

$$i_{enclosed} = Ni \qquad (33)$$

Using Equations (32) and (33) in Ampere's law gives

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 i_{enclosed}$$

$$B \times 2\pi r = \mu_0 Ni$$

$$\boxed{B = \frac{\mu_0 Ni}{2\pi r}} \qquad \begin{array}{l} \textit{magnetic field} \\ \textit{of a toroid} \end{array} \qquad (34)$$

Note that $N/2\pi r$ is the number of turns per unit length, n, so that Equation (34) can be written $B = \mu_0 ni$ which is the solenoid formula of Equation (31). To a good approximation the field in a toroid is the same as in the center of a straight solenoid.

The **derivation** of Equation (34) is so easy and such a good illustration of the use of Ampere's law that it should be remembered as an example of Ampere's law.


$$\left.\begin{array}{l} \oint \vec{B} \cdot d\vec{\ell} = B * 2\pi r \\ \mu_0 i_{tot} = \mu_0 Ni \end{array}\right\} \Rightarrow B = \frac{\mu_0 Ni}{2\pi r}$$

Figure 28
When the solenoid is bent into the shape of a toroid, there are no end effects. The magnetic field is confined to the region inside the toroid, and Ampere's law is easily applied. (You should remember this as an example of the use of Ampere's law.)

Exercise 8

Figure (29) shows a 400-turn solenoid that is 47.5 cm long and has a diameter of 2.54 cm. (The 10 turns of wire wrapped around the center are for a later experiment.) Calculate the magnitude of the magnetic field B near the center of the solenoid when the wire carries a current of 3 amperes. (Give your answer in tesla and gauss.)

Exercise 9

Figure (30) shows the toroidal solenoid that we use in several experiments later on. The coil has 696 turns wound on a 2.6 cm diameter plastic rod bent into a circle of radius 21.5 cm. What is the strength of the magnetic field inside the coil when a current of 1 amp is flowing through the wire? (Give your answer in tesla and gauss.)
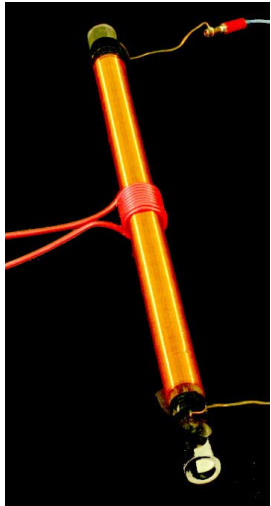
Figure 29
A 400 turn straight solenoid 47.5 cm long,
wound on a 2.54 cm diameter rod.
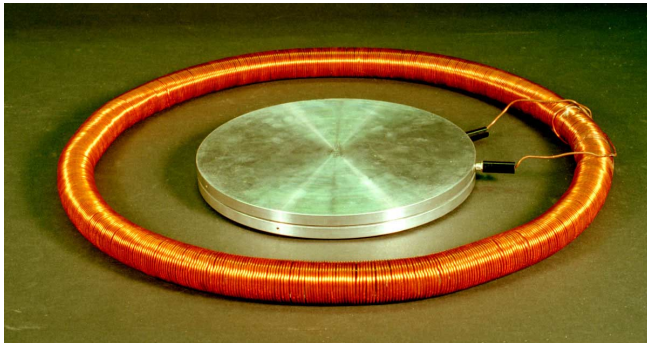
Figure 30
A 696 turn toroidal solenoid wound on
a 2.6 cm diameter plastic rod bent into
a circle of radius 21.5 cm.