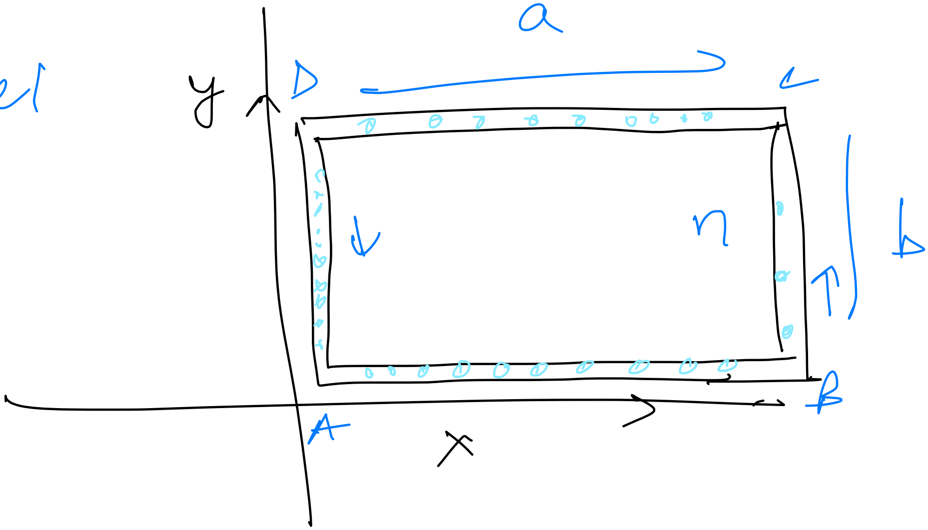
De-coherence - Prelims solutions

Q1

Case 1



Since electric field is along +ve x, particles in BC branch will travel faster (as $\vec{E}$ field will do work)

But current must remain same in entire loop ∴ Density of particles will be less in BC

It can be seen that velocity of particle at any point on AB branch will be opposite to corresponding branch in CD. Also particles average velocity across BC will be same at all points on BC as $E_j = 0$

Let current be I

$I = n_{BC} v_{BC} q$ — i) [n_{BC} = number of particles per unit length in BC)

$I = n_{DA} v_{DA} q$ — ii)

Momenta of particles in AB and CD cancel.

$$\vec{P}_{Total} = \vec{P}_{DA} + \vec{P}_{BC}$$

$$= \gamma_{DA} m (n_{DA} \cdot b) v_{DA} (-\hat{i})$$

$$+ \gamma_{BC} m n_{BC} b v_{BC} (+\hat{i})$$

$$= \frac{mI}{q} \cdot b (\gamma_{BC} - \gamma_{DA}) \hat{i} \quad \text{— iii)}$$

(from (i) and ii)

But change in energy = Work done by E field

$$= (\gamma_{BC} - \gamma_{DA}) mc^2 = q \int_0^a \vec{E} \cdot d\vec{l}$$

$$= q E_0 a \quad \text{— iv)}$$

Put $(\gamma_{BC} - \gamma_{DA})$ in (iii)

$\Rightarrow \vec{P} = \varepsilon_0 m I a b \hat{i}$

Similarly for case 2 we get

$\vec{P} = \varepsilon_0 m I a b \hat{j}$

Correct answer $\longrightarrow$ B

Q2

Speed of light in medium = 13 /s

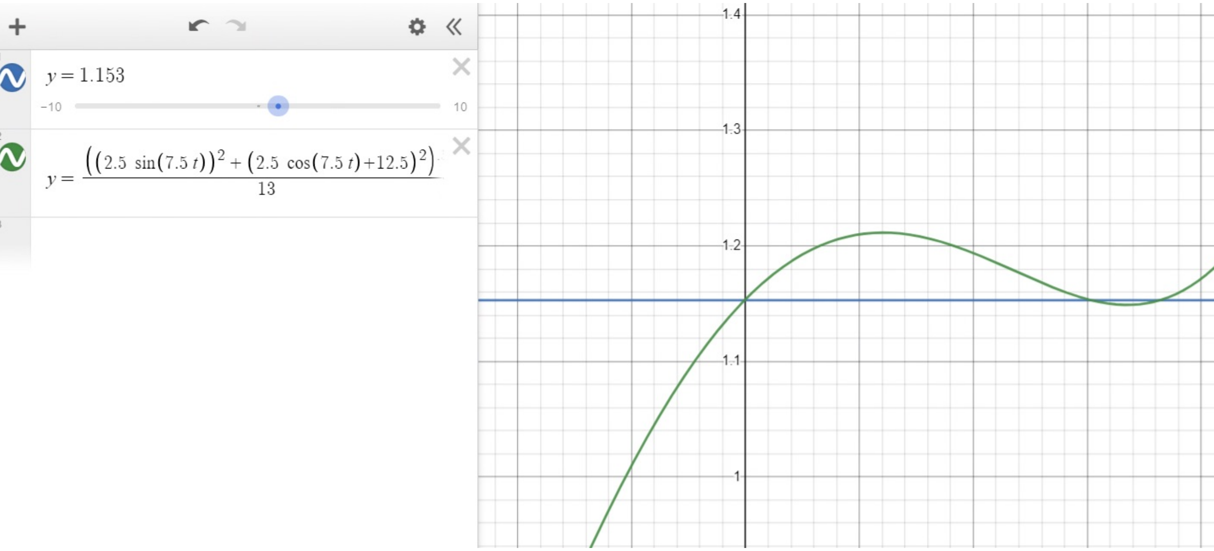
observer will see the particle at t if light from particle emitted at other point t' reaches observer at t. (t>t') (t= 1.538

For that to happen

$$\frac{|\vec{r}(t')|}{c/n} = t - t'$$

$$\frac{\sqrt{\left\{(2.5\sin(7.5t'))^2 + (2.5\cos(7.5t')+2.5)^2\right\}}}{13} + t' = 1.1538$$

Plot LHS on Desmos to check that it intersects y=1.1538 line thrice.



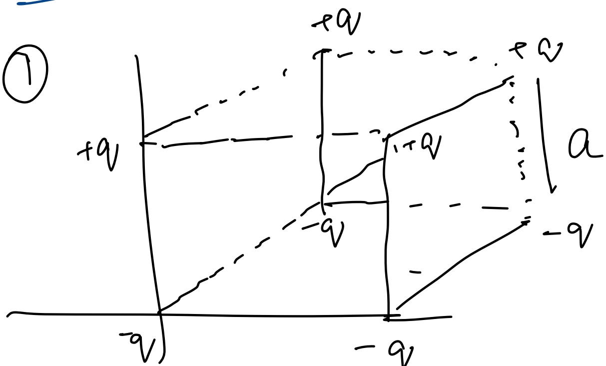
Plotted using desmos

B is correct as particle is always at finite distance from observer

Ans → B , C

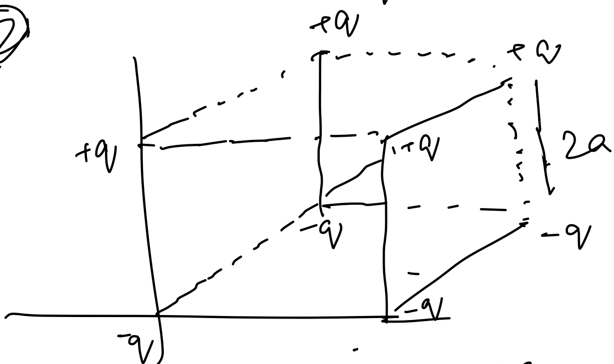
Q3

1



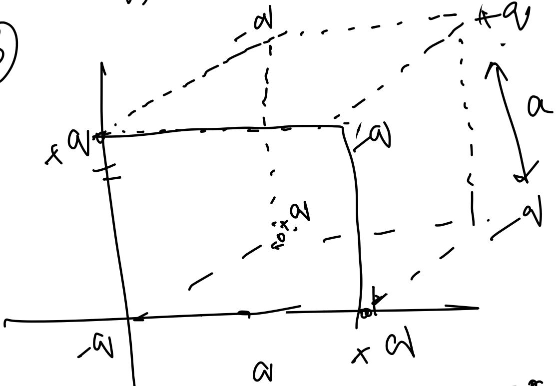
→ dipole

2



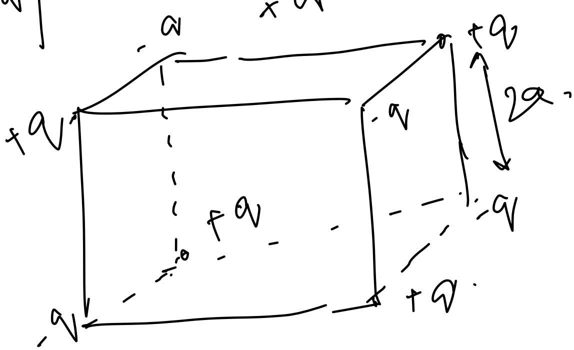
→ dipole

3



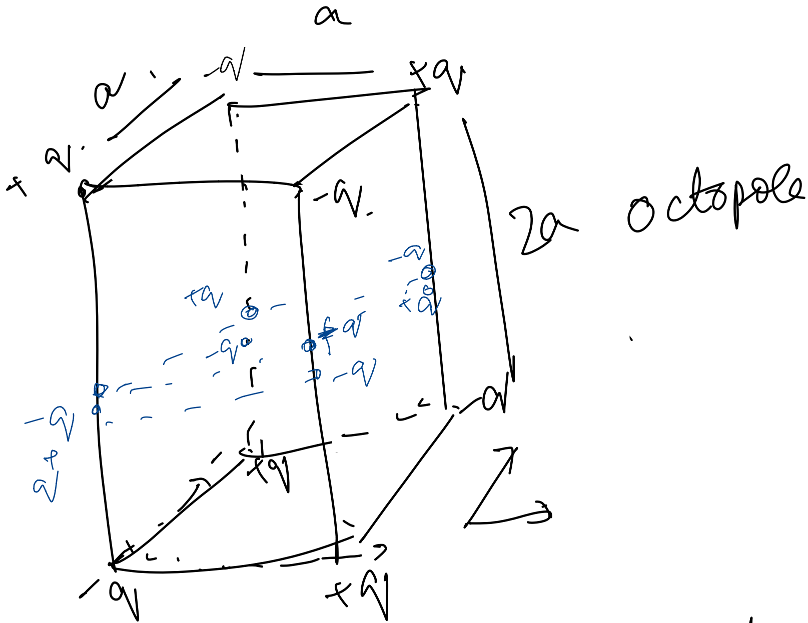
octopole

4



octopole

⑤



If $V(r, \theta, \phi) \neq 0$ for given (θ, ϕ)
then for ideal dipole

(A) Configuration in 5 can be considered as combination of two configurations in 3
$\because$ as $r \to \infty$, the contribution of two sections in 5 become almost equal
$\therefore$ ratio approaches $1/2$

(B) For dipole

$$V \propto \frac{P}{r^2} f(\theta, \phi)$$

Provided ϕ and θ are same and non zero in both cases :

$$V_1 \propto \frac{P_1}{r^2} \qquad V_2 \propto \frac{P_2}{(2r)^2} \qquad (P_2 = 2P_1)$$

$$\therefore \frac{V_1}{V_2} = 2$$

(C) For octopole

$$V \propto \frac{(\text{dimensions of octopole})^3}{r^4} f(\theta, \phi)$$

Provided θ, ϕ and r are fixed

$$V \propto (\text{dimensions})^3$$

$$\therefore \frac{V_3}{V_4} = \frac{1}{2^3} = \frac{1}{8}$$

D → False (as 1 is dipole and $V \propto \frac{1}{r^2}$

and 5 is octopole $V \propto \frac{1}{r^5}$)

$\therefore$ Correct answer ABC

Q 4)

Say wire has linear charge density λ_0 in S_0 frame. Note that charge is invariant under frame change.

Due to length contraction linear charge density increases to $\lambda_0 \gamma_S$ in other frames.

Electric field $\propto \lambda / r$ (r remains same as it is $\perp$ to direction of motion)

$E_{S_0} : E_S : E_{S'}$

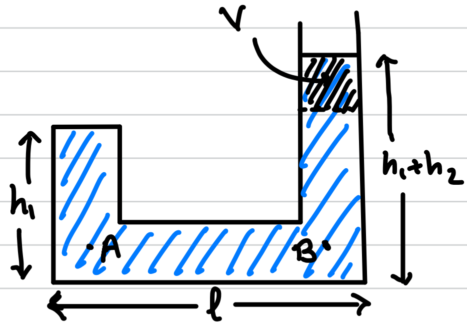
$= 1 : \frac{1}{\sqrt{1-(3c/5c)^2}} : \frac{1}{\sqrt{1-(4c/5c)^2}} = 1 : \frac{1}{\frac{4}{5}} : \frac{1}{\frac{3}{5}}$

$B \propto I/r = \frac{\lambda v}{r} = \lambda_0 \gamma v$

$\therefore B_S : B_{S'} = \frac{v_S \lambda_S}{v_{S'} \lambda_{S'}} = \frac{\frac{3c}{5} \cdot \frac{1}{4}}{\frac{4c}{5} \cdot \frac{1}{3}} = 9:16$

$\therefore$ Correct answer $\to$ C

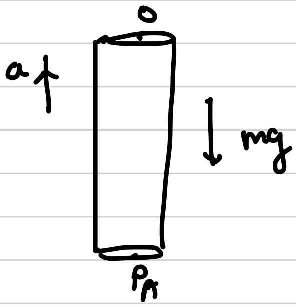
Q5) The seal can withstand force of $\rho g A_0 h_2$. $\therefore$ Height of water column in the other tube is thus h_1+h_2



The acceleration is caused by gravitational force on the volume V indicated in digram [$\because$ other forces cancel out]

$$\therefore \quad a = F/m = \frac{\rho g A h_2}{\rho A(2h_1+h_2+l)} = \frac{h_2 g}{2h_1+h_2+l}$$

$\therefore$ Pressure at A



The FBD for the left column

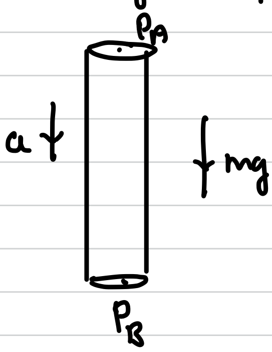
$$P_A(A) - mg = ma$$

$$\Rightarrow \quad P_A(A) = \rho A(h_1)(a+g)$$

$$= P_A = \rho h_1\left(g\left(1+\frac{h_2}{2h_1+h_2+l}\right)\right)$$

$$P_A = \rho h_1 g\left(\frac{2h_1+2h_2+l}{2h_1+h_2+l}\right)$$

Similarly, the pressure at B



The FBD for the right column

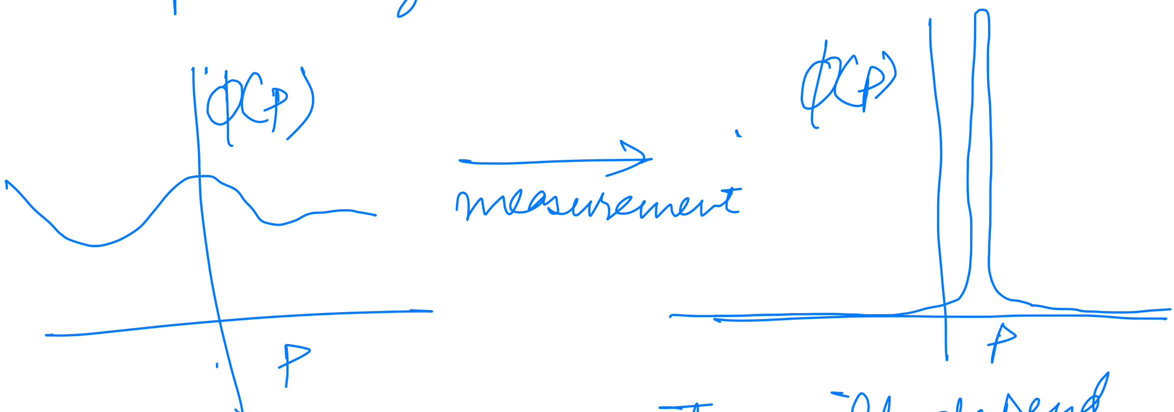
$$mg - P_B A = ma$$

$$\therefore \quad P_B A = m(g-a)$$

$$= \rho(h_1+h_2)Ag\left(1-\frac{h_2}{2h_1+h_2+l}\right)$$

$$P_B = \rho(h_1+h_2)g\left(\frac{2h_1+l}{2h_1+h_2+l}\right)$$

Q6 — Note that since for a free particle energy eigenstate is not normalisable, any momentum measurement would lead to collapse of state into a state which has a finite spread around eigenvalue. An example is given below



The height and width will depend upon details of the experiment

Thus resultant state is not an eigenstate of momentum.

The eigenspace for momentum and Hamiltonian operators are same for free particle

$\therefore$ Resultant state is not energy eigenstate and it evolves with time as per Schrodinger equation

correct answer $\Rightarrow$ B, C

1

Q7

An electron is a fermion with spin $\frac{\hbar}{2}$ $\therefore$ It can never have a state with $\langle S \rangle = 0$

Schrodinger equation

Write any general spin state as $c_+(t)|+\rangle + c_-(t)|-\rangle$

$H = -\mu_z B_0$ ($\because \vec{B}$ is along z)

$= -\frac{g\hbar}{2}\sigma_z B_0$ $\left(\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right)$

Use schrodinger eq.

$$i\hbar \frac{\partial c_+(t)}{\partial t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\hbar \frac{\partial c_-(t)}{\partial t}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{g\hbar B_0}{2}\begin{pmatrix} c_+(t) \\ -c_-(t) \end{pmatrix}$$

Solving above equations give

$c_+(t) = \exp\left(i\frac{gB_0}{2}t\right)$

$$c_-(t) = \exp\left(\frac{-i g B_0 t}{2}\right)$$

$\therefore$ state vector rotates with frequency $\frac{g B_0}{2}$.

$$\underline{\langle S_x \rangle}_{\hbar/2} = \begin{pmatrix} c_+^* & c_-^* \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$$

$$= \begin{pmatrix} c_+^* & c_-^* \end{pmatrix} \begin{pmatrix} c_- \\ c_+ \end{pmatrix}$$

$$= c_+^* c_- + c_-^* c_+$$

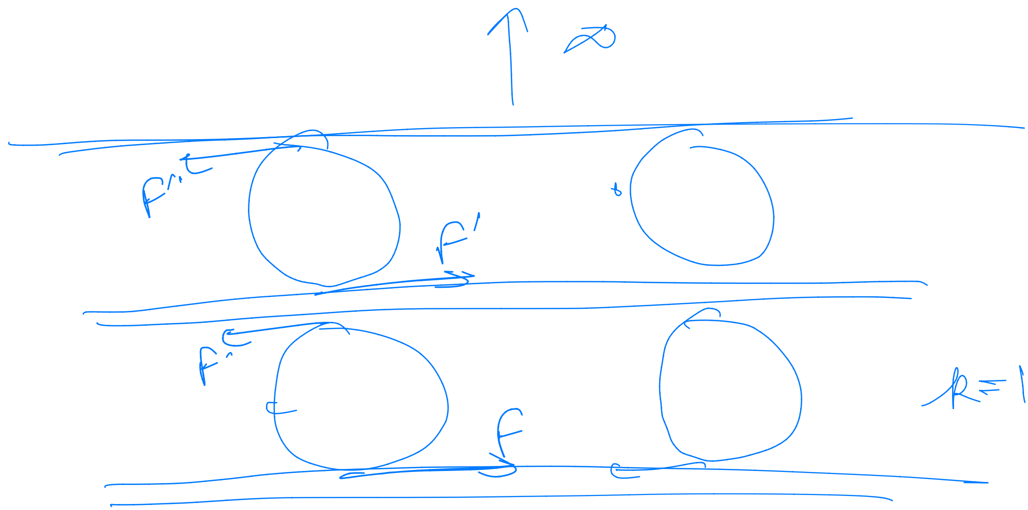
$$= 2\,\mathrm{Real}\,(c_+^* c_-)$$

$$= 2 \cos(g B_0 t)$$

$\therefore$ S_x rotates with twice the frequency as that of state vector

<u>Ans</u> <u>B, C</u>

Q8 Consider that force applied by bottom planck on 1 cylinder horizontally is F. Suppose second planck moves with acceleration a_1



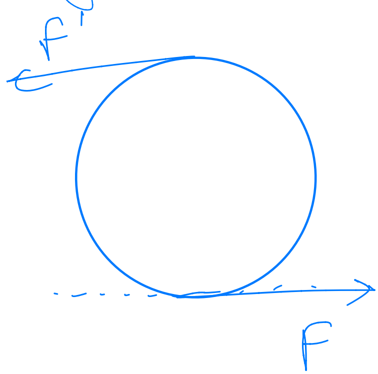
Note that if we delete the first layer then system is exactly identical as before except that a_0 has been replaced by a_1. Response of system depends linearly on a_1 ($\because$ m and a_1 are the only given parameters and dimensional analysis tells us $F \propto m a_1$)

$$\therefore \quad F' = \frac{F a_1}{a_0}$$

and $$F'' = F' \frac{a_1}{a_0} = F\left(\frac{a_1}{a_0}\right)^2$$

Now consider F. B. D of cylinder in first layer and apply condition of no rolling

$$I = \frac{m r^2}{\alpha}$$



$$\frac{F - F'}{m} + \frac{(F + F')\alpha r^2}{m r^2} = a_0$$

$$\Rightarrow F(\alpha + 1) + F'(\alpha - 1) = m a_0 \quad \text{--- (1)}$$

Condition of no rolling on first layer

$$\frac{F' - F''}{m} - \frac{(F + F')\alpha r^2}{m r^2} = a_1$$

$$\Rightarrow F(1-\alpha) - F'(\alpha+1) = ma_1 \quad \text{ii)}$$

case (i) $\alpha = 1$

Put in (i) & (ii)

$\Rightarrow$ $2F = ma_0$ &

$-2F' = ma_1$

But $F' = \dfrac{F a_1}{a_0}$

$$\Rightarrow -2F\frac{a_1}{a_0} = ma_1.$$

If $a_1 \neq 0$, this would give $a_0 = 0$

$\therefore$ $a_1 = 0$

case (ii) $\alpha = 2$

$3F + F' = ma_0$

$-F - 3F' = ma_1.$

$2F - 2F' = m(a_0 + a_1)$

& $2F + 2F' = \dfrac{m(a_0 - a_1)}{2}$

$$\Rightarrow 4F = m\left(\frac{3a_0}{2} + \frac{a_1}{2}\right)$$

$$4F' = m\left(\frac{-a_0}{2} - \frac{3a_1}{2}\right)$$

$$\frac{F'}{F} = \frac{a_y}{a_x} = \frac{-\left(\frac{a_0}{2} + \frac{3a_1}{2}\right)}{\frac{3a_0}{2} + \frac{a_1}{2}}$$

$$\Rightarrow \frac{3a_0 a_1}{\cancel{2}} + \frac{(a_1)^2}{\cancel{2}} = -\frac{(a_0)^2}{\cancel{2}} - \frac{3a_1 a_0}{\cancel{2}}$$

$$(a_1)^2 + (a_0)^2 + 6a_1 a_0 = 0$$

$$\left(\frac{a_1}{a_0}\right)^2 + \frac{6a_1}{a_0} + 1 = 0$$

$$\frac{a_1}{a_0} = \frac{-6 \pm \sqrt{32}}{2} = (-3 \pm 2\sqrt{2})$$

$$\therefore a_1 = (-3 + 2\sqrt{2})\, a_0$$

(If we take the other solution then magnitude of acceleration increases at every step and becomes ∞ at $k \to \infty$)

$$\therefore F' = \frac{+m}{8}\left(-a_0 - 3(-3+2\sqrt{2})\, a_0\right.$$

$$= \frac{m}{8}(8 - 6\sqrt{2})\, a_0 = \left(m\left(1 - \frac{3}{4}\sqrt{2}\right)a_0\right.$$

∴ Ans $\underline{\underline{A \quad C}}$

Q97

NOTES

Solⁿ:

① $\tan(\theta) = v$

$$\Rightarrow \frac{d\theta}{d\tau} = \frac{dv}{d\tau} \cdot \frac{1}{1+v^2}$$

② $\chi = \frac{m}{\gamma}$

$$\Rightarrow \frac{u\chi^2}{2m^2-\chi^2} = \frac{u(1-v^2)}{(1+v^2)}$$

③ $U^\alpha = (\gamma, \gamma v)$

$a^\alpha = \frac{dU^\alpha}{d\tau}$ = four-acceleration

Let $a = \frac{d(\gamma v)}{dt} =$

$a' = $ (proper acceleration) $= \gamma^3 \frac{dv}{d\tau}$
?

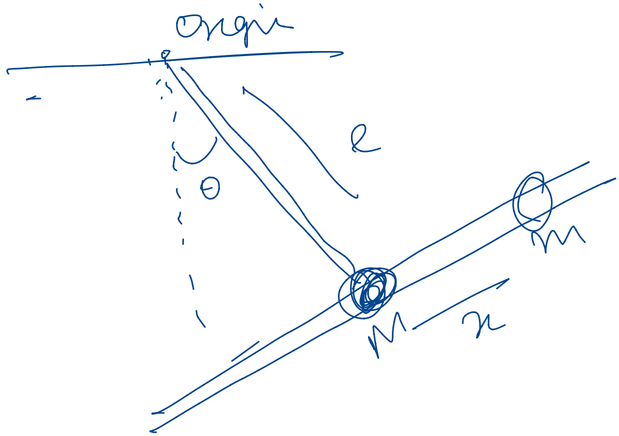
$$a = \frac{1}{\gamma} \cdot \frac{d(\gamma v)}{d\tau} = \frac{1}{\gamma} \cdot a'$$

$$a = \gamma^2 \frac{dv}{d\tau}$$

③ in ① and ②

$$\Rightarrow \boxed{u = a} = \frac{d(\gamma v)}{dt}$$

Q 10



The coordinates of m are $(\ell\sin\theta + x\cos\theta,$
$-\ell\cos\theta + x\sin\theta)$

$\therefore$ Lagrangian can be written as

$$L = \frac{1}{2}\left(m\left\{(\ell\dot\theta)^2 + (\dot x)^2 + 2\ell\dot\theta\dot x + x^2\dot\theta\right\} + M\ell^2(\dot\theta)^2\right) - \left(-Mg\ell\cos\theta - mg(\ell\cos\theta - x\sin\theta)\right)$$

Use Lagrange's Equation of Motion $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) = \frac{\partial L}{\partial\theta}$ & $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot x}\right) = \frac{\partial L}{\partial x}$

and use small angle approximation.

Also if we assume oscillations are periodic with small amplitude of x, we need $\frac{x}{\ell} \ll 1$

After performing above tasks we get

$$\ell\ddot\theta + \ddot x + g\theta = 0 \quad \text{— i)}$$

$$M\ell(\ell\ddot\theta + g\theta) + m\ell(\ell\ddot\theta + \ddot x) + mg\ell\theta + mgx = 0 \quad \text{— ii)}$$

using (i) to simplify ii

$$-M\ell\ddot x + mgx = 0 \quad \text{—— iii)}$$

From iii) it is clear that x will vary exponentially in time $\therefore$ Oscillations are not possible if x were to vary

$\therefore$ $x = 0$ $\forall t$ if motion is periodic

From (i)

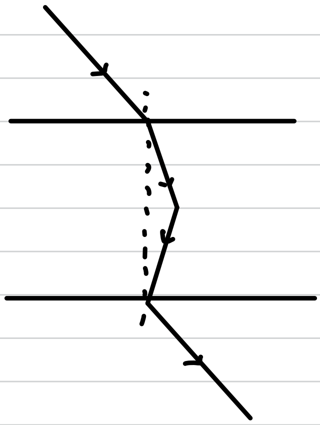
$$l\ddot{\theta} = -g\theta \Rightarrow \ddot{\theta} = -\left(\frac{g}{l}\right)\theta \quad \{SHM: \ddot{\theta} = -\omega^2\theta\}$$

$\therefore$ $\theta(t)$ oscillates with frequency $\omega = \sqrt{\frac{g}{l}}$

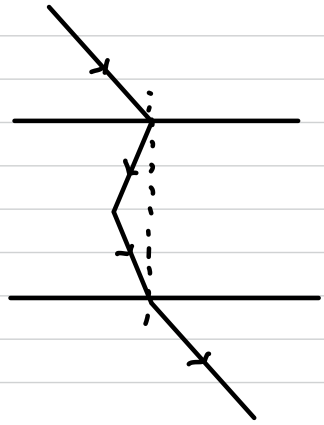
Ans : A

Q11) By observation, (A) and (C) can be ruled out, ∵ light spends unequal amount of time in +ve and -ve refractive indices and thus couldn't have returned to the position right below where it was incident.

(B)



(D)



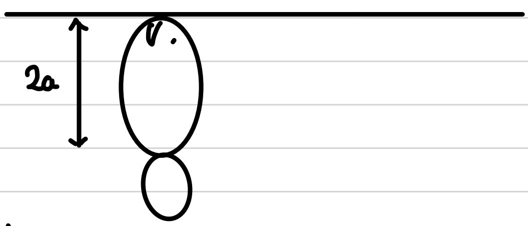
Q12) We use energy conservation. We track the gain in potential of the COM of the fish and the potential energy of water which now occupies the place the fish was.

COM of fish

$$= \frac{M(a) + \frac{M}{8}\left(2a + \frac{a}{2}\right)}{M + M/8}$$

$$= \frac{a + \frac{5a}{16}}{9/8} = \frac{21a(8)}{(16)(9)}$$

$$= \frac{7a}{6}$$



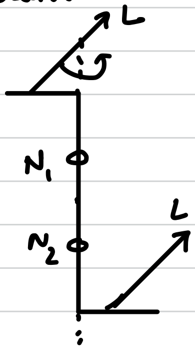
∴ Potential energy lost by water

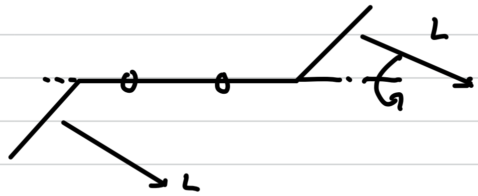
$$= M_w g\left(\frac{7a}{6}\right) = M_f g(h') = \text{energy gained by fish}$$

$$= \frac{M_w}{4} g(h') \Rightarrow h' = \frac{14a}{3}$$

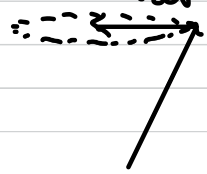
13) Consider the angular momentum vector of the object at that particular instant

The direction of the angular momentum vector at that instant is given below.





We need to see how the angular momentum vector changes to follow this circular path, as shown.



$\therefore \frac{dL}{dt}$ is inward towards the centre = Torque

For such a Torque to be possible, $N_1 > N_2$

Note: This happens because it is not the principal axis.

14) The energy content of light inside the box = $nh\nu$

$\therefore$ The "effective mass" of the box = $\frac{nh\nu}{c^2}$

$\therefore$ acceleration $a = \frac{F}{nh\nu/c^2} = \frac{Fc^2}{nhc/\lambda} = \frac{Fc\lambda}{nh}$

This can be made more rigorous by considering a photon gas which exchanges momentum with the walls of the box.

Consider a solid cone with base radius R and height H placed on an incline of angle θ. ~~Mass varies~~ Mass density and charge density vary with distance from top of cone (denoted by y), as $\rho_m = \rho_0 y/H$ and $\rho_q = \cancel{\rho_0} ky/H$. Find magnitude of magnetic field that must be present in ~~direction~~ space ~~perpendicular~~ along incline to prevent the cone from toppling. The cone rotates with angular velocity ω about its symmetry axis. Ignore friction.

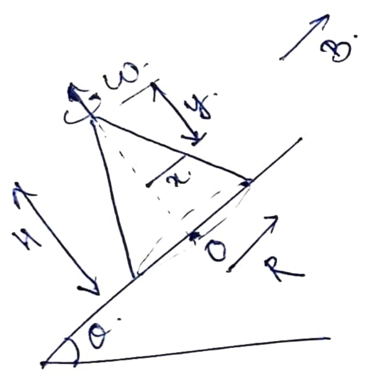
a) $\frac{4}{3}\left(\rho_0 H / k R^2 \omega\right) g \sin\theta$

b) $\frac{6}{5}\left(\rho_0 R / k H^2 \omega\right) g \sin\theta$

c) $\frac{3}{4}\left(\rho_0 H / k \pi R^2 \omega\right) g \sin\theta$

d) $\frac{6}{5}\left(\rho_0 H / k \pi R^2 \omega\right) g \sin\theta$

Ans.



dI

$$\frac{x}{y} = R/H$$

$$dI = \frac{dm\,x^2}{2} = \frac{\rho_m \pi x^2 \cdot x^2 \cdot dy}{2}$$

$$= \frac{\rho_0 y}{H} \cdot \frac{\pi}{2} \cdot x^4 \, dy$$

$$= \frac{\rho_0 \pi}{2H} y \left(\frac{Ry}{H}\right)^4 dy$$

$$= \frac{\rho_0 \pi R^4}{2H^5} y^5 \, dy$$

$$I = \frac{\rho_0 \pi R^4 H^6}{2H^5 \cdot 6} = \frac{\rho_0 \pi R^4 H}{12}$$

Distance of CG from top $= \frac{\int dm \cdot y}{\int dm}$

$$= \frac{\int \rho_m \pi x^2 dy \cdot y}{\int \rho_m \pi x^2 dy}$$

$$= \frac{\int \frac{\rho_0 y}{H} x^2 y \, dy}{\int \frac{\rho_0 y}{H} x^2 dy}$$

$$= \frac{\int_{y=0}^{H} y^2 \left(\frac{R}{H} y\right)^2 dy}{\int_{y=0}^{H} \left(\frac{R}{H} y\right)^2 y \, dy} = \frac{H^5 \cdot 4}{5 \cdot H^4} = \frac{4H}{5}$$

$\therefore$ Distance of CG from O $= H/5$.

$$M = \frac{\rho_0}{H} \cdot \frac{R^2}{H^2} \cdot \frac{H^4}{4} = \frac{\rho_0 R^2 H}{4}$$

Since mass and charge densities are similar,

$$\vec{\mu} = \frac{q}{2M} \vec{L} \qquad (\vec{\mu} = \text{magnetic moment})$$

$$q = \frac{k R^2 H}{4} \qquad I\omega$$

$$\therefore \vec{\mu} = \frac{k R^2 H \cdot 4}{4 \cdot 2 \cdot \rho_0 R^2 H} \cdot$$

$$= \frac{k}{2\rho_0} \cdot \frac{\rho_0 \pi R^4 H \omega}{12}$$

$$= \frac{k \pi R^4 H \omega}{24}$$

Torque due to gravity (about O) $= H/5 \left(\rho_0 R^2 H / 4\right) \cdot g \sin\theta$

Torque due to $B = \vec{\mu} \times \vec{B} = \frac{k\pi R^4 H \omega}{24} \cdot B$.

$$\therefore \frac{\rho_0 R^2 H^2 g \sin\theta}{20} = \frac{k\pi R^4 H \omega B}{24} \Rightarrow B = \frac{6 \rho_0 H g \sin\theta}{5 k \pi R^2 \omega} \text{ (Ans)}$$

16. We try to model spherical raindrops falling a short distance through mist. (You may neglect air resistance)
The drops will grow will grow in size as they gather more water from the mist. Assume the rate of change of mass is proportional to the current surface area of the raindrop. (i.e., $\frac{dm}{dt} \propto r^2$, where r is the radius of the raindrop)
Which of the following options is correct in that case?

1. Velocity of the COM increases linearly with time
2. Velocity of the COM becomes almost constant after a short time
3. Acceleration of the COM is an increasing function of time
4. Acceleration of the COM is a decreasing function of time

Solution:-

$$m\frac{dv}{dt} = mg - v\frac{dm}{dt} \qquad -(1)$$

Now,

$$\frac{dm}{dt} \propto r^2 \frac{dr}{dt} \propto r^2$$
$$\implies \frac{dr}{dt} = \gamma r^0$$
$$\implies r = \gamma t$$

We need to write $\frac{dm}{dt}$ in terms of $\frac{dr}{dt}$

$$m = \frac{4}{3}\pi r^3 \rho$$

$$\implies \frac{dm}{dt} = 4\pi r^2 \rho \frac{dr}{dt} = 4\pi\rho\gamma^3 t^2$$

From (1),

$$m\frac{dv}{dt} = mg - kvt^2$$

(where $k = 4\pi\rho\gamma^3$)

$$\frac{dv}{dt} = g - \frac{k}{m}vt^2$$
$$\implies \frac{dv}{dt} = g - \frac{k}{\frac{4}{3}\pi\rho\gamma^3 t^3}vt$$
$$\implies \frac{dv}{dt} + 3\frac{v}{t} = g$$
$$\implies t^3(\frac{dv}{dt} + 3\frac{v}{t}) = gt^3$$
$$\implies \frac{d}{dt}(t^3 v) = gt^3$$
$$\implies t^3 v = g\frac{t^4}{4} + c$$

At $t = 0$, we have $v = 0$, and hence $c = 0$.

$$v = \frac{gt}{4}$$

Thus, option(1) is correct.

Q17) $4\pi G\rho = \nabla E_g$

$\Rightarrow \dfrac{1}{r^2}\dfrac{dE_g}{dr} = 4\pi G\rho(r)$ due to radial symmetry.

$\Rightarrow E_{g_a} - E_{g_0} = \dfrac{4\pi G k a^3}{3}$

$E_{g_b} - E_{g_a} = \dfrac{4\pi G k'(b^2-a^2)}{2}$

$E_{g_c} - E_{g_b} = 4\pi G k'' \ln(c/b)$.

$$E_{g_r} = \begin{cases} \dfrac{4\pi G k r^3}{3} & (0 \le r \le a) \\ \dfrac{4\pi G k'(r^2-a^2)}{2} + \dfrac{4\pi G k a^3}{3} & (a < r \le b) \\ 4\pi G k'' \ln(r/b) + \dfrac{4\pi G k'(b^2-a^2)}{2} + \dfrac{4\pi G k a^3}{3} & \end{cases}$$

$\therefore E_g = \dfrac{d^2x}{dt^2}$ since $F = m\dfrac{d^2x}{dt^2} = E_g \cdot m$.

$$\Rightarrow v\frac{dv}{dx} = \begin{cases} \dfrac{4\pi G}{3} x^3 & \text{for } 0 < x < a \\ 4\pi G\left(x^2 + \dfrac{700}{3}\right) & \text{for } a < x < b \\ 4\pi G k'' \ln x - 4\pi G k'' \ln b + 2\pi G k'(b^2-a^2) + \dfrac{4\pi G k a^3}{3} & \text{for } b < x < r_0 \end{cases}$$

$\dfrac{v_b^2}{2} = \left\{ 12\pi G(\ln x - 1) + 6000\pi G - 12\pi G \ln 40 + \dfrac{4000\pi G}{3} \right\} x \Big|_b^{r_0} = \left[2.51\times10^{-5}(\ln x - 1) + 1.53\times10^{-2}\right] x \Big|_b^{r_0} \simeq 0.925$

$\dfrac{v_a^2 - v_b^2}{2} = \dfrac{4\pi G}{3} x^3 + \dfrac{2800\pi G}{3} x \Big|_a^b = 2.794\times10^{-6} x^3 + 1.96\times10^{-3} x \Big|_a^b = 0.234.$

$\dfrac{v_0^2 - v_a^2}{2} = \dfrac{\pi G x^4}{3}\Big|_0^a = 6.98\times10^{-3}.$

$\dfrac{v^2 - v_b^2}{2} = \left(2.51\times10^{-5}(\ln x - 1) + 1.53\times10^{-2}\right) x \Big|_b^r$

$\dfrac{v^2 - v_a^2}{2} = \left(2.794\times10^{-6}\times x^3 + 1.96\times10^{-3}\times x\right)\Big|_a^r$

$\dfrac{v^2 - v_0^2}{2} = 6.98\times10^{-7} r^4.$

$$\Rightarrow \frac{v^2}{2} = \begin{cases} r\ln r \times 2.51\times10^{-5} + 1.53\times10^{-2}\times r - 0.616 + 0.925 & b < r < r_0 \\ 2.794\times10^{-6}\times r^3 + 1.96\times10^{-3}\times r - 2.99\times10^{-3} + 0.925 + 0.234 & a < r < b \\ 6.98\times10^{-7}\times r^4 + 0.925 + 0.234 + 6.98\times10^{-3} & 0 < r < a \end{cases}$$

Approximate values of constant terms have been computed.

$v = -\dfrac{dx}{dt} \Rightarrow dt = \dfrac{dx}{-v}$

$$\Rightarrow \int_0^t dt = \underbrace{\int_b^{r_0} \frac{dx}{v}}_{t} + \underbrace{\int_a^b \frac{dx}{v}}_{t} + \underbrace{\int_0^a \frac{dx}{v}}_{t}.$$

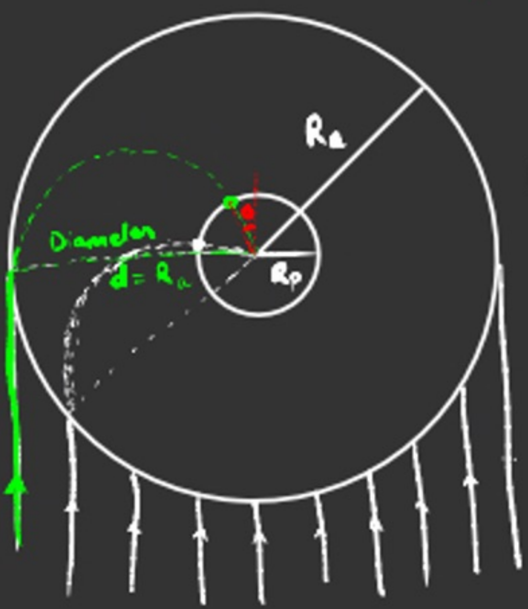
$t_{br_0} \simeq 36.53\,s.$

$t_{ab} \simeq 18.88\,s$

$t_{0a} \simeq 6.54\,s$

$\underline{t_{net} \simeq 62\,s}$

Ques 18

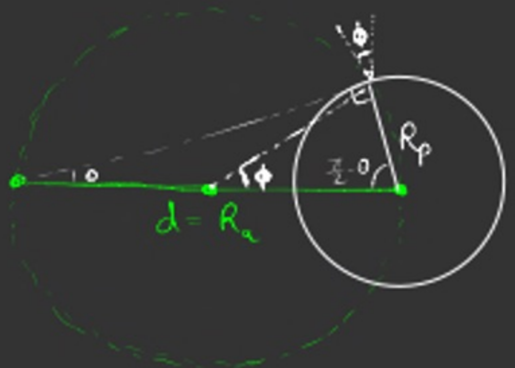


$$n(r) = \left(\frac{r}{R_a}\right)^2$$

Assume period of revolution is much greater than period of rotation.

Star S

Note that the green ray is the last ray of light than anyone on the planet will observe.



$$(\phi = 2\theta)$$

$$\theta = \sin^{-1}\left(\frac{R_p}{R_a}\right)$$

$$\text{Duration of daylight} = \left(\frac{2\pi - 2\theta}{2\pi}\right)T = \left(1 - \frac{\theta}{\pi}\right)T = \left(\frac{\pi - \sin^{-1}\left(\frac{R_p}{R_a}\right)}{\pi}\right)T$$

$$= \left(\frac{\frac{\pi}{2} + \cos^{-1}\left(\frac{R_p}{R_a}\right)}{\pi}\right)T$$

$\boxed{\text{Ans: (iii)}}$

Ques 19)

$$\phi - \theta = \theta = \sin^{-1}\left(\frac{R_p}{R_a}\right)$$

$$Ans = 2\theta = 2\sin^{-1}\left(\frac{R_p}{R_a}\right)$$

$$\boxed{Ans = (ii)}$$

Ques 3(a) ⟩ These problems are very approximate description of a real phenomenon of ferroelectricity.

- $V = 10^7 (V)$

- Assume that the orientation of the dipole is always along the feild line (i.e. it can be only in two en states)

$\uparrow E = 10^7 N/C$

(Let the upper direction be positive)

Electric feild applied on any given molecule will be: $E' = E + \frac{P}{3\epsilon_0}$

$$P_{\substack{\text{created}\\ \text{by } E'}} = P_1 = Np \frac{\left(e^{+pE'/kT} - e^{-pE'/kT} \right)}{e^{+pE'/kT} + e^{-pE'/kT}}$$

$$P_1 = Np \tanh\left(\frac{pE'}{kT}\right)$$

When $E = 0$

$$P_1 = Np \tanh\left(\frac{pP}{3\epsilon_0 kT}\right)$$

$$P_{\text{creating } E'} = P_2 = 3\epsilon_0 E'$$

At $T > T_c$

$$P_1 < P_2 \Rightarrow Np \tanh\left(\frac{pE'}{kT}\right) < 3\epsilon_0 E'$$

At $T = T_c$

$$\left.\frac{\partial P_1}{\partial E'}\right|_{E'=0} = \left.\frac{\partial P_2}{\partial E'}\right|_{E'=0}$$

$$\Rightarrow \frac{Np^2}{kT_c} = 3\epsilon_0$$

$$\Rightarrow \boxed{T_c = \frac{Np^2}{3\epsilon_0 k}}$$

$$T_c \approx 3.43 \text{ K}$$

$\boxed{\text{Ans : (iv)}}$

Ques 3(b) >

$$E'\left(\substack{\text{generated from}\\ \text{given } P \,\&\, E}\right) = E'\left(\substack{\text{got from tanh}\\ \text{formula}}\right)$$

$$E + \frac{P}{3\epsilon_0} = \frac{kT}{p}\tanh^{-1}\left(\frac{P}{Np}\right) \quad \text{— Eq (I)}$$

At $T < T_c$ ($1K < T_c$)

For any sufficiently large positive electric feild applied in the start we get that the water has a net non-zero polarisation in the positive direction (i.e. along the directio of the feild)

At $E = 0$,

$$\frac{P}{3\epsilon_0} = \frac{kT}{p}\tanh^{-1}\left(\frac{P}{Np}\right)$$

Using Desmos,

$$P_{retained} = P_{ret} \approx 0.000205 \quad (\text{in SI units})$$

$\boxed{\text{Ans: (ii)}}$

C.

$\rho^{(l)}$ and $\rho^{(s)}$ are the densities near the co-existence line. At coexistence, the temperature is the remaining free parameter. Hence, $\mu^{(l)} = \mu^{(s)}$ $P^{(l)} = P^{(s)}$ and $T^{(l)} = T^{(s)} = T$

So, now we rewrite the expressions for the Helmholtz Free Energies as follows:

$$A^{(l)} = \frac{1}{2}\frac{\alpha}{T}\frac{n^2}{V}$$

$$A^{(s)} = \frac{1}{2}\frac{\beta}{T}\frac{n^3}{V^2}$$

Now, we know by Stability conditions:

$$\left(\frac{\partial A^{(l)}}{\partial n}\right)V, T = \left(\frac{\partial A^{(s)}}{\partial n}\right)V, T$$

$$\left(\frac{\partial A^{(l)}}{\partial V}\right)n, T = \left(\frac{\partial A^{(s)}}{\partial V}\right)n, T$$

Hence,

$$\frac{\alpha}{T}\left(\frac{n}{V}\right)^{(l)} = \frac{3}{2}\frac{\beta}{T}\left(\frac{n}{V}\right)^{(s)^2}$$

$$\frac{1}{2}\frac{\alpha}{T}\left(\frac{n}{V}\right)^{(l)^2} = \frac{\beta}{T}\left(\frac{n}{V}\right)^{(s)^3}$$

$$\frac{\alpha}{T}\rho^{(l)} = \frac{3}{2}\frac{\beta}{T}\rho^{(s)^2}$$

$$\frac{1}{2}\frac{\alpha}{T}\rho^{(l)^2} = \frac{\beta}{T}\rho^{(s)^3}$$

Dividing the two equations gives us

$$\frac{1}{2}\rho^{(l)} = \frac{2}{3}\rho^{(s)}$$

Hence

$$\frac{\rho^{(l)}}{\rho^{(s)}} = \frac{4}{3}$$

Q23

For a quantum SHO, the eigenfunctions are alternately even and odd (in x).
The ground state ($E_0 = \frac{\hbar\omega}{2}$) is even.
& first excited state is odd.

Case I admits eigenfunctions which vanish at $x \leq 0$. Now for $x > 0$ all eigenfunctions of SHO satisfy Schrodinger eq. for case I.
But for $x \leq 0$ wave function should vanish. Since wavefunction must be continuous for , case i) admits only odd eigenfunctions of SHO for $x > 0$

∴ Let ψ_n denote nth eigenfunction of SHO. ϕ_m denote mth eigenfunction of case (i)

$$\phi_m(x) = \begin{cases} \psi_{2m}(x) & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Case (i) has energies $\hbar\omega, 2\hbar\omega, 3\hbar\omega$ ....

Case ii) has energies $\frac{\hbar\omega}{2}, \frac{3}{2}\hbar\omega$

use Gibbs principle

$$Z_1 = \sum_{n=1}^{\infty} e^{-\beta \hbar\omega n}$$

$$Z_2 = e^{-\frac{\hbar\omega}{2}} \sum_{n=1}^{\infty} e^{-\beta \hbar\omega n} = e^{-\hbar\omega} Z_1$$

$$\therefore \left|\frac{dZ_2}{d\beta}\right| < \left|\frac{dZ_1}{d\beta}\right|$$

$\therefore$ Average thermal energy is higher in case (i)

Ans: A, C

24. A one-component material can exist in two phases: α and γ. When it is in phase α, it obeys the equation of state:

$$\beta p = a + b\beta\mu,$$

where $\beta = 1/T$, a and b are positive constants. When it is in phase γ:

$$\beta p = c + d\ ln(\beta\mu)$$

where c and d are positive constants. Determine the density change that occurs when the material suffers a phase transformation from phase α to phase γ. (b=3, a=5, c=9, d=2 in SI units)

Solution for 24:
At phase equilibrium:

$$\beta p^{(\alpha)} = \beta p^{(\gamma)}, \quad \beta\mu^{(\alpha)} = \beta\mu^{(\gamma)}, \quad \beta^{(\alpha)} = \beta^{(\gamma)} \text{ (at the transition temperature)}$$

From the first equality,

$$a + b\beta\mu = c + d\ \ln(\beta\mu)$$

Plugging in the given values,

$$5 + 3\beta\mu = 9 + 2\ln(\beta\mu)$$

$$\implies 3\beta\mu - 2\ln(\beta\mu) = 4$$

Solving graphically, we get

$$\beta\mu = 0.176 \text{ or } 1.679$$

From Gibbs-Duhem equation,

$$d\mu = -Sdt + Vdp$$

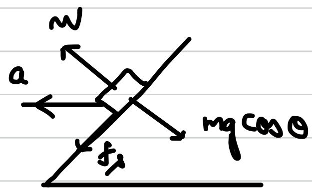
$$\implies \rho = \frac{1}{V} = (\frac{\partial p}{\partial \mu})_T = (\frac{\partial \beta p}{\partial \beta\mu})_T$$

$$\rho^{(\alpha)} = 3$$

$$\rho^{(\gamma)} = \frac{2}{\beta\mu}$$

$$\Delta\rho = \rho^{(\gamma)} - \rho^{(\alpha)} = 8.364 \text{ or } -1.809$$

25) $N\cos\theta = mg + f\sin\theta$
$N\sin\theta + f\cos\theta = \frac{mv^2}{R}$



In limiting case,

$N\cos\theta = mg + \mu_s N \sin\theta$

however, $\because \mu > \cot\theta$
This never occurs.
$\therefore$ Limiting value of static friction is never reached and the car can go infinitely fast.

(Ans) (26)

One needs to actually use a ruler to get the ratio of venus : sun.

lets assume $x_1 : x_2$.

Angular diameter of sun = 32'

Angular diameter of venus $= \frac{32' x_1}{x_2}$

$\therefore$ distance b/w venus & ~~satellite~~ $\approx d\left(\frac{32' x_1}{x_2 \cdot 60'} \times \frac{\pi}{180}\right) = 2R_{venus}$

$$\therefore d = \frac{2R_{venus}}{\left(\frac{32 x_1}{x_2 60'} \times \frac{\pi}{180}\right)} = \frac{12{,}103 \cdot 6}{(0.000387)} \approx 41.60 \times 10^6 \text{ km}.$$

(27)

(Ans) Constellation : Aries, Tauras, Pieces, Triangulam, Perseus, Andromeda, Cassiopia.

Messier objects : Triangulam galaxy, Andromeda galaxy, ~~Pleades~~. Pleiades.

(28)

(Ans) $f_{eyepiece} = 30$mm

FOV = 50°

Absolute mag = 11

App. mag = 3.7

$\boxed{\text{Beehive Cluster}}$

$m - M = 5\log d - 5$

$\frac{11 - 3.7 + 5}{5} = \log d$; $\{d = 288.40$ parsecs$\}$

$\therefore \theta = \frac{2 \times R}{d} = \frac{2 \times 2.29}{288}$ rad $= \frac{2 \times 2.29 \times 180}{288 \times \pi}$

$= 0.8872°$

Q30) Aeries