Indian Institute of Science
Pravega X 2023

Enumeration - Prelims

July 29, 2023

cognavi

Timing: 10:00 AM to 3:00 PM | Max mark: 80

Instructions

- This paper consists of **15** computational problems, each carrying a weightage of 3 marks and **5** subjective problems, each carrying a weightage of 7 marks. The maximum obtainable score is hence 80.
- Submit your responses to all problems via the google form link mentioned in the email. Only one participant per team is to submit the form, and they are to submit it not more than once. However, you are allowed to edit the form responses.
- The duration of the exam is **five** hours, with an extra **fifteen** minutes at the end for scanning and submitting. The Google form starts accepting responses from **10 AM IST** and closes at **3:15 PM IST** sharp.
- The answers to all computational problems are positive integers atmost 10^{10}, and you may fill in your answers directly through the form.
- For each subjective problem, you should submit a **scanned version** of one or more pages consisting of a proof (or attempt at a proof) at the problem.
  - Please write all solutions clearly and legibly.
  - You should submit only pages that you wish to have graded.
  - **Each problem must be submitted as a separate PDF.**
  - The submission should be written in English sentences and read as natural proofs following usual mathematical conventions. Avoid submitting equations with no accompanying explanation, two-column proofs, etc., and write clearly. The graders may deduct points for sufficiently poorly written explanations.

  - Passages which should not be graded must be crossed or struck out. This can increase your score as graders may deduct points for false statements.
  - Leave a 1-inch margin on all pages.
  - Separate problems should be on separate pages.

- The only aids allowed are **writing utensils (pencils, pens, and eraser, including colored pencils and pens), ruler, compass, and paper**. In particular, protractors, calculators, electronic devices of any kind, textbooks, notes, music players, magic crystal balls, etc. are NOT permitted. In particular you are not allowed to use online tools such as GeoGebra, WolframAlpha, so on.

  As a corollary, **solutions must be handwritten (not typeset)**. If you have some physical handicap that makes handwriting impossible, please contact the organisers for an exception.

- You may communicate only with your partner (if any) during the duration of the exam. In particular you are not allowed to seek help from someone that is not on your team. Do **NOT** engage in malpractice. Any evidence of such will result in an instant disqualification. Any form of plagiarism if detected, will result in disqualifcation.

- We first establish an **Computational score cutoff**, which will be announced a few days before the final results are published. **ONLY submissions above or at the cutoff score will have their subjective solutions graded.** In practice, this score cutoff tends to be very lenient as compared to the overall cutoff.

- Participants are first ranked based on the total score. Ties are then broken by scores on the **subjective** section.

- For any queries, contact our coordinators. The easiest way to reach us is either via email or discord.

  - *Kazi Aryan Amin*: **aryanamin@iisc.ac.in**
  - *Rahul Adhikari*: **rahuladhikar@iisc.ac.in**

- If you're unable to submit the final answers via the Google form due to techincal difficulties, email us your answers with details of your team using your registered email ID.

- In case of any dispute, the decision of the organisers will be final and binding on all parties.

Computational Problems

1. Let x and y be positive real numbers satisfying $x + y = 1$. The maximum possible value of $x^4y + xy^4$ can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$. **(3)**

2. Anna writes a sequence of positive integers $(a_1, a_2, \cdots, a_n)$, such that

$$a_1 + a_2 + \cdots + a_n = 2023$$

Suppose the sequence Anna picks is equally likely to be any sequence satisfying the above condition. Find the expected value of n.

Note: The *expected value* of n is the average value of n across all possibilities. **(3)**

3. For a positive integer n relatively prime to 2023, we define its *order* modulo 2023, to be the smallest positive integer d such that 2023 divides $n^d - 1$. Find the number of integers n relatively prime to 2023 such that $1 \leq n \leq 2023$ and the *order* of n modulo 2023 is 136. **(3)**

4. Let $S = \displaystyle\sum_{i=0}^{50}\sum_{j=0}^{50}(-1)^{i+j}\binom{100}{i+j}$. Given that S is a positive integer, find the highest exponent of 2 dividing S. **(3)**

5. Let $ABCD$ be a *bicentric* quadrilateral, that is, it has both an incircle and a circumcircle. Suppose the incircle of $ABCD$ is tangent to BC at X. If $AD = 5$, $BX = 4$ and $CX = 6$, then the area of $ABCD$ can be expressed as $a\sqrt{b}$ where a, b are positive integers and b isn't divisible by the square of any prime. Compute ab. **(3)**

6. An ice cream parlour with infinitely many identical stalls allows one customer to enter every minute. Upon entry, a customer goes to an empty stall and stays there till their order is delivered, and then leaves the parlor instantly. The probability that it takes exactly n minutes to prepare their order is $\frac{1}{n(n+1)}$. Let T denote the expected amount of time after the first customer walks in, for *someone's* order to be delivered. Find $\lfloor 10T \rfloor$.

Note : $\lfloor x \rfloor$ is the smallest integer not exceeding x. **(3)**

7. Let O denote the circumcenter of ΔABC with $AB = \sqrt{3}$ and $AC = 2$. Suppose the circumcircle of $\triangle BOC$ intersects AB and AC again at X, Y, respectively. If XY is tangent to the circumcircle of triangle ΔABC, then BC^2 can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, d are positive integers such that $\gcd(a, b, d) = 1$ and c isn't divisible by the square of any prime. Compute $a + b + c + d$. **(3)**

8. Find the number of ways of placing 4 knights on a 4×4 board such that for every knight, there is an unique knight attacking it.

(3)

9. Let P_1, P_2, P_3 be three monic quadratic polynomials, which are pairwise unequal and have no double root. Suppose for every $i = 1, 2, 3$, there exists an unique real number $a_i \neq 1$ such that the equation

$$P_i(x) = a_i P_{i+1}(x)$$

has exactly one solution x_i (where $P_4 \equiv P_1$). If $x_1 = 3, a_1 = \frac{-2}{3}$ and $a_2 = \frac{3}{5}$, find the sum of all possible values of $x_2 + x_3$. **(3)**

10. Circles ω_1 and ω_2 with centres O_1 and O_2, and radii $13, 15$ respectively, intersect at points X, Y. Points P, Q lie on ω_1, ω_2 respectively such that P, Y, Q are collinear in that order. Suppose PO_1 and QO_2 intersect at T. If $XT \parallel PQ$, and $O_1O_2 = 14$, then the circumradius of $\triangle XPQ$ can be expressed as $\frac{a}{b}$, for relatively prime positive integers a, b. Compute $a + b$. **(3)**

11. Consider the polynomial :

$$P(x) = \prod_{1 \leq a \leq 101} (x - a^5)$$

Let $Q(x)$ be the remainder obtained when $P(x)$ is divided by $x^3 - 1$. Find the remainder when $Q(2)$ is divided by 101. **(3)**

12. Let A, B be points on an eliipse $\mathcal{E}$ with focii F_1, F_2. Suppose AF_2 intersects BF_1 at C and AF_1 intersects BF_2 at D. The tangents to $\mathcal{E}$ at A, B intersect at P. Suppose C, D, F_1, F_2 lie on a circle. If $PF_1 = 5$, $F_1F_2 = 7$ and $PF_2 = 8$, then CD can be expressed as $\frac{a}{b}$ for relatively prime positive integers a, b. Compute $a + b$. **(3)**

13. Let a, b, c, d be real numbers such that $abcd = -1$ and the following equations hold :

$$|d(a-b)(b-c)(c-a)| = 1$$
$$|a(b-c)(c-d)(d-b)| = 2$$
$$|b(c-a)(a-d)(d-c)| = 6$$

Find the sum of all possible values of $|c(d-a)(a-b)(b-d)|$. **(3)**

14. A positive integer n is said to be a *quadratic residue modulo 97*, if there exists an integer m, such that $m^2 - n$ is divisible by 97 and $97 \nmid n$. Alice randomly picks a triple (a, b, c) of *quadratic residues modulo 97*, where $1 \leq a, b, c \leq 97$ and her choice of triple is equally likely among all possible choices. If the probability that $a + b + c$ is **NOT** a *quadratic residue modulo 97* can be expressed as $\frac{x}{y}$, for relatively prime positive integers x, y, then compute the value of $x + y$. **(3)**

15. Let N denote the number of upright paths from $(0, 0)$ to $(10, 10)$ which intersect the line $x = y$ at exactly 5 points other than the start and end points. If N can be expressed as $2^a \times b$, where b is an odd positive integer, compute $a + b$.

    Note : A upright path is one where we only take steps towards right or upwards. For example the following is an upright path from $(0, 0)$ to $(2, 2)$:

    $$(0,0) \to (1,0) \to (2,0) \to (2,1) \to (2,2)$$

    whereas the following is not (as it has a leftward step):

    $$(0,0) \to (1,0) \to (2,0) \to (3,0) \to (3,1) \to (3,2) \to (2,2)$$

    (3)

Subjective Problems

1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the following equation holds for all $x, y \in \mathbb{R}$:

$$f(yf(x)) + f(xy) = 2yf(x)$$

(7)

2. Find all pairs of positive integers (a, b) such that for all positive integers $n > 2023^{2023}$, the number $n^2 + an + b$ has a divisor $d > 1$, such that $n \mid d - 1$. **(7)**

3. Let ω denote the circumcircle of triangle ΔABC. Suppose the internal and external bisectors of $\angle BAC$, intersect $\overline{BC}$ and ω again at K, L respectively. Points X, Y lie on ω such that $LK = LX = LY$. Prove that $\overline{XY}$, and the line through K perpendicular to $\overline{BC}$ meet on the A-median. **(7)**

4. 2002 people stand in a line. Each person either always tells the truth, or always lies. Starting from the back, Alice asks 1995 people :

   How many liars are are standing in front of you?

   and records their answers. Prove Alice can pick a subset of natural numbers $\mathcal{S}$ such that the sum of elements of $\mathcal{S}$ is less than 2023 and she can guarantee that number of truthful people is in $\mathcal{S}$. **(7)**

5. The A-excircle of ΔABC is tangent to $\overline{BC}$ at D. The line $\overline{AD}$ intersects the incircle of ΔABC at points Y, Z such that $AZ > AY$. The line through Z parallel to the external angle bisector of $\angle BAC$ meets $\overline{BC}$ at X. Prove that $\overline{XY}$ passes through the midpoint of arc $\widehat{BAC}$ in the circumcircle of ΔABC. **(7)**

Best wishes