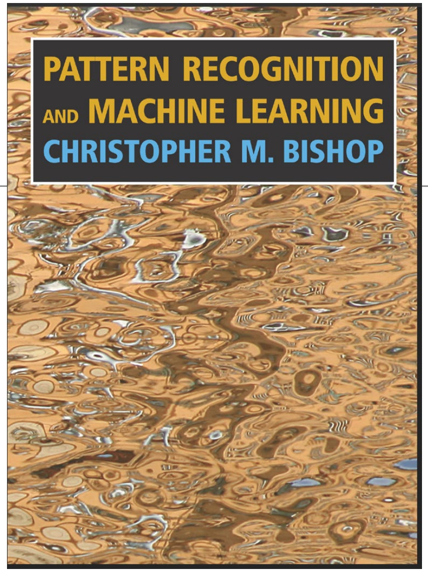
Unit 12: LDPC Codes

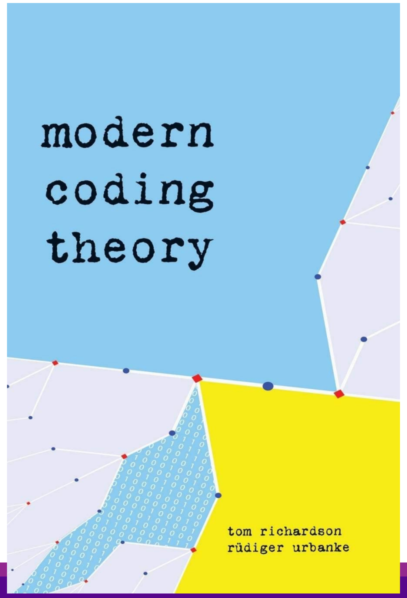
EL-GY 6013: DIGITAL COMMUNICATIONS

PROF. SUNDEEP RANGAN

References

- Extensive tutorial presentation:
  - An Introduction to Low-Density Parity-Check Codes, Paul Siegel
  - https://cmrr-star.ucsd.edu/static/presentations/ldpc_tutorial.pdf
- Bishop, Pattern Recognition and ML
  - General book on ML from 2006
  - Excellent chapter (Chapter 8) on graphical models
- Modern Coding Theory, Richardson and Urbanke
  - From 2008
  - By two pioneers in the field
  - Extensive math
- Richardson, Tom, and Shrinivas Kudekar. "Design of low-density parity check codes for 5G new radio." IEEE Communications Magazine 56.3 (2018): 28-34.





Outline

- LDPC Codes: Basics and Motivation
- LDPC Encoding
- Graphical Models
- Inference via Belief Propagation
- LDPC Decoding
- 5G LDPC codes

Generator Matrices

- Consider (n, k) binary linear code
  - k information bits $\boldsymbol{b} = (b_1, \dots, b_k)$
  - n coded bits $\boldsymbol{c} = (c_1, \dots, c_n)$
- Described by a generator matrix $\boldsymbol{c} = \boldsymbol{bG}$
- Example: $(7,4)$ Hamming Code: $\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$

Parity Check Matrix

- Parity check matrix: An $n \times (n-k)$ matrix $\boldsymbol{H}$ such that $\boldsymbol{GH} = \boldsymbol{0}$
- Properties:
  - For any codeword, **c**, **cH**=**0**. Why?
  - Conversely, if **G** and **H** are rank k, then **cH**=**0** implies that **c** is a codeword. [Need some more linear algebra to prove this]
- For a binary systematic code: $\boldsymbol{G} = [\boldsymbol{I}_k \mid \boldsymbol{P}] \Rightarrow \boldsymbol{H} = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{I}_{n-k} \end{bmatrix}$
- Example: Hamming code $\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

LDPC Codes

- Low-Density Parity Check Codes
- Based on three key ideas
- Linear codes where the parity check matrix is sparse
  - Number of "ones" grows linearly in n
  - Number of "ones" per row is roughly constant
- Randomness in the construction
  - Random placement of "ones"
- Iterative, message-passing decoder
  - Simple "local" decoding at nodes
  - Iterative exchange of information (message-passing)
  - Based on belief propagation from

First LDPC codes

- Gallager, R. G., Low-Density Parity-Check Codes, M.I.T. Press, Cambridge, Mass: 1963.
- Regular (n, j, k) code
  - n = codeword length
  - Number of 1's in each row of the parity check matrix
  - Number of 1's in each column of the parity check matrix
  - Location of 1's chosen randomly
- Rate of the code $1 - \frac{j}{k}$
- Minimum distance tends to grow linearly with n

Gallager's Original Regular Code

$(n,j,k) = (20,3,4)$

- First $\boldsymbol{n/k} = 5$ rows have $\boldsymbol{k} = 4$ 1's each, descending.
- Next $\boldsymbol{j}$-**1**=2 submatrices of size $\boldsymbol{n/k}$ **x** $\boldsymbol{n}$ =5 x 20 obtained by applying randomly chosen column permutation to first submatrix.
- Result: $\boldsymbol{jn/k}$ x $\boldsymbol{n}$ = 15 x 20 parity check matrix for a $\boldsymbol{(n,j,k)} = (20,3,4)$ LDPC code.

$$
\begin{array}{c|cccccccccccccccccccc|}
\hline
 & 1&1&1&1& & & & & & & & & & & & & & & & \\
 & & & & &1&1&1&1& & & & & & & & & & & & \\
 & & & & & & & & &1&1&1&1& & & & & & & & \\
 & & & & & & & & & & & & &1&1&1&1& & & & \\
 & & & & & & & & & & & & & & & & &1&1&1&1\\
\hdashline
\pi_1 &1& & & &1& & & &1& & & &1& & & & & & & \\
 & &1& & & &1& & & &1& & & & & & &1& & & \\
 & & &1& & & &1& & & & & & &1& & & &1& & \\
 & & & &1& & & & & & &1& & & &1& & & &1& \\
 & & & & & & & &1& & & &1& & & &1& & & &1\\
\hdashline
\pi_2 &1& & & & &1& & & & & &1& & & & & &1& & \\
 & &1& & & & &1& & & &1& & & & &1& & & & \\
 & & &1& & & & &1& & & & &1& & & & & &1& \\
 & & & &1& & & & &1& & & & &1& & &1& & & \\
 & & & & &1& & & & &1& & & & &1& & & & &1\\
\hline
\end{array}
$$

From Siegel, An Introduction to Low-Density Parity-Check Codes

Performance of Regular Codes on BSC

Gallager, 1963

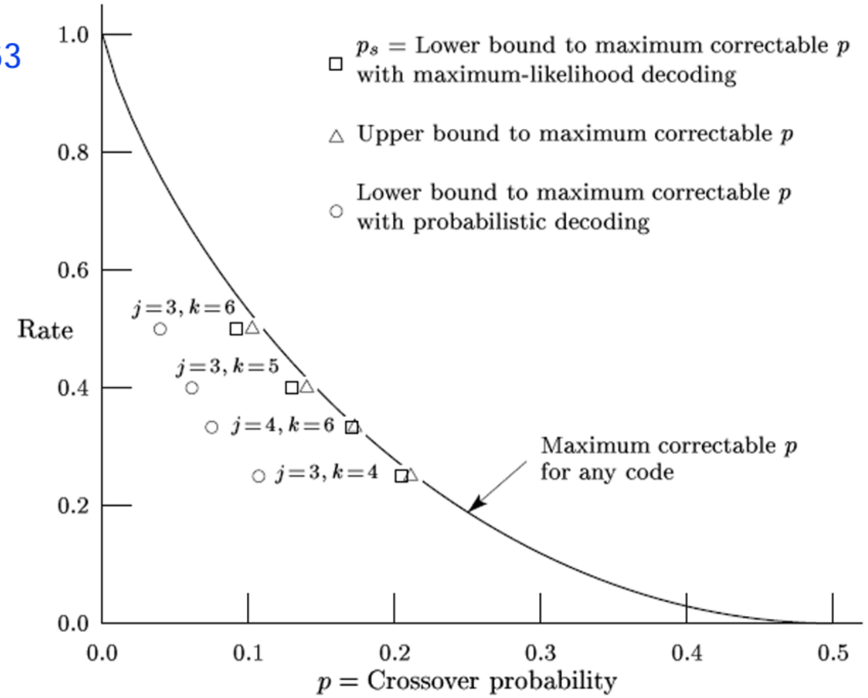
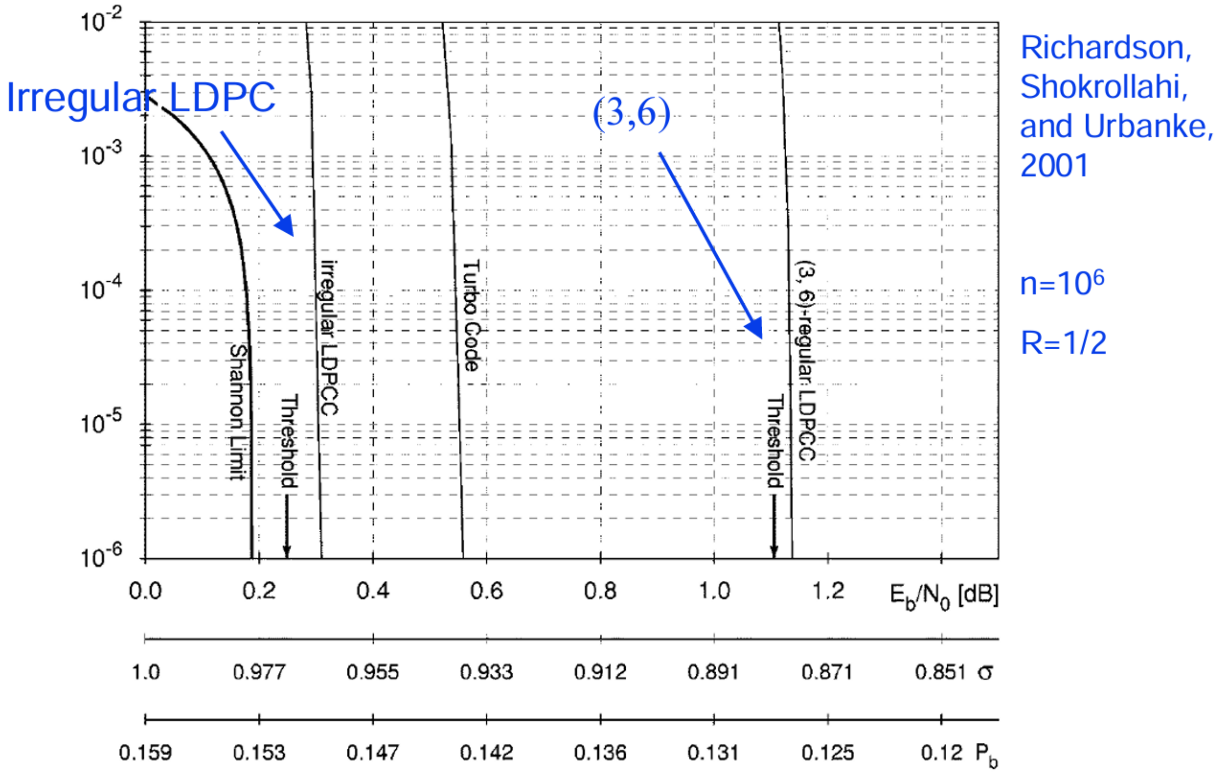


Figure 3.5: Error-correcting properties of (n, j, k) codes on BSC as function of rate for large n.

Enter Irregular Codes



- Irregular codes
  - Began work in 2001
  - Showed significant improvement
- On BSC channel at infinite lengths:
  - Theoretically hits capacity
- At practical code lengths:
  - Very close to capacity

Outline

- ❑LDPC Codes: Basics and Motivation
- ❑LDPC Encoding
- ❑Graphical Models
- ❑Inference via Belief Propagation
- ❑LDPC Decoding
- ❑5G LDPC codes

Encoding with a Parity Check Matrix

- Given a parity check matrix $\boldsymbol{H}$ how do we encode
- Parity check matrix can be used to tell if $\boldsymbol{c}$ is a codeword
- But does not say how to find a codeword from information bits
- If $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{P} \\ \boldsymbol{I}_{n-k} \end{bmatrix}$ then we can obtain generator matrix: $\boldsymbol{G} = [\boldsymbol{I}_k \;\; | \;\; \boldsymbol{P}\,]$
- Then, we can use $\boldsymbol{G}$: $\quad \boldsymbol{c} = \boldsymbol{bG}$
- But LDPC parity check matrices are not in general systematic

Encoding via Equation Solving

- Partition code word: $\boldsymbol{c} = [\boxed{c_1, \dots, c_{n-k}}, \boxed{c_{n-k+1}, \dots, c_n}]$

  Parity Info

- Set last k bits as information bits: $c_{n-k+\ell} = b_\ell$ for $\ell = 1, \dots, k$
- Write each check node equation:

$$0 = \sum_{\ell=1}^{n-k} c_\ell H_{\ell k} + \sum_{\ell=n-k}^{n} c_\ell H_{\ell k} \Rightarrow \sum_{\ell=n-k}^{n} \boxed{c_\ell H_{\ell k}} = \sum_{\ell=1}^{n-k} \boxed{c_\ell H_{\ell k}}$$

  - Solve $n-k$ unknowns with $n-k$ equations
  - Equations are linear, but over the binary field

Gaussian Elimination

- Equation solving can be performed via Gaussian elimination
- Example: Solve: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
  - Add R1 to R2 and R3: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$
  - Exchange R2 and R3: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$
  - $c_3 = 0$
  - $c_2 = 1 + c_3 = 1$
  - $c_1 = c_2 + c_3 = 1$
  - Solution $c = [1,1,0]$

MATLAB

- MATLAB can also perform binary arithmetic
- Define arrays with elements in GF(2)
  - The binary field
- Solve equations just as if they were real numbers

```
% Create the parity check matrix
H = [1,1,1; 1, 1, 0; 1, 0, 1; 0, 1, 1; 1 0 0; 0 1 0; 0 0 1];
H = gf(H);

% Get dimensions
[n,m] = size(H);
k = n-m;

% Generate random bits
b = randi([0,1], 1, k);
b = gf(b);

% Solve c(1:m)H(1:m,:) = b*H(m+1:n,:)
z = b*H(m+1:n,:);
c = z / H(1:m,:);

% Add information bits
c = [c b];
```

Complexity

- In general, we partition the parity check matrix: $\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}^{(1)} \\ \boldsymbol{H}^{(2)} \end{bmatrix}$
- We solve: $\begin{bmatrix} \boldsymbol{c}^{(1)} & \boldsymbol{c}^{(2)} \end{bmatrix} \begin{bmatrix} \boldsymbol{H}^{(1)} \\ \boldsymbol{H}^{(2)} \end{bmatrix} = 0 \Rightarrow \boldsymbol{c}^{(1)}\boldsymbol{H}^{(1)} = \boldsymbol{c}^{(2)}\boldsymbol{H}^{(2)}$
- Since we place information bits in last code bits: $\boldsymbol{c}^{(2)} = \boldsymbol{b}$
- Therefore, we must solve: $\boldsymbol{c}^{(1)} = \boldsymbol{b}\boldsymbol{H}^{(2)}\left(\boldsymbol{H}^{(1)}\right)^{-1}$
- Complexity: If $\boldsymbol{H}$ has d elements per column:
  - Multiplication: $\boldsymbol{b}\boldsymbol{H}^{(2)}$ has complexity $O(kd)$ ✓
  - $\left(\boldsymbol{H}^{(1)}\right)^{-1}$ can be computed once and stored ✓
  - But $\left(\boldsymbol{H}^{(1)}\right)^{-1}$ is not, in general, sparse.
  - Multiplication by $\left(\boldsymbol{H}^{(1)}\right)^{-1}$ has complexity is $O((n-k)^2)$ ✗

Reducing Complexity

- For most LDPC codes, complexity can be reduced
- Find a “mostly” sparse upper triangular $H^{(1)}$
  - Use column permutations with a greedy algorithm
  - Remaining small number of rows that are dense
- Overall complexity:
  - Pre-processing $O\left(n^{\frac{3}{2}}\right)$
  - Encoding: $O(n)$

Outline

- ❑LDPC Codes: Basics and Motivation
- ❑LDPC Encoding
- ❑Graphical Models
- ❑Inference via Belief Propagation
- ❑LDPC Decoding
- ❑5G LDPC codes

MAP and Marginal Estimation

- Given a posterior distribution:

$$p(\boldsymbol{x}|\boldsymbol{y}), \qquad \boldsymbol{x} = (x_1, \ldots, x_N), \qquad \boldsymbol{y} = (y_1, \ldots, y_M)$$

  - N unknowns: $\boldsymbol{x} = (x_1, \ldots, x_N)$
  - M observed variables: $\boldsymbol{y} = (y_1, \ldots, y_M)$
- Two common estimation problems:
- MAP: Find $\widehat{\boldsymbol{x}} = \arg\max p(\boldsymbol{x}|\boldsymbol{y})$
  - Finds the most likely unknown vector
- Marginal: Find marginal distribution $p(x_i|\boldsymbol{y})$ for some i
  - Finds the posterior probability of a particular variable

Complexity is Generally Exponential

- Brute force estimation is generally exponentially complex
- Suppose N unknowns $\boldsymbol{x} = (x_1, \dots, x_N)$, L choices for each x_i
- MAP: Find $\widehat{\boldsymbol{x}} = \arg\max p(\boldsymbol{x}|\boldsymbol{y})$
  - Search over vectors $\boldsymbol{x} = (x_1, \dots, x_N)$
  - Search over L^N vectors $\boldsymbol{x}$
- Marginal: For any possible value a for x_i:

$$p(x_i = a|y) = \sum_{\boldsymbol{x}\,:\,x_i=a} p(\boldsymbol{x}|\boldsymbol{y})$$

  - Sum is over all vectors $\boldsymbol{x}$ with $x_i = a$
  - For each a, search over L^{N-1} vectors $\boldsymbol{x}$

Need structure for tractable estimation

Factorizable Distributions

- Assume posterior distribution is factorizable:

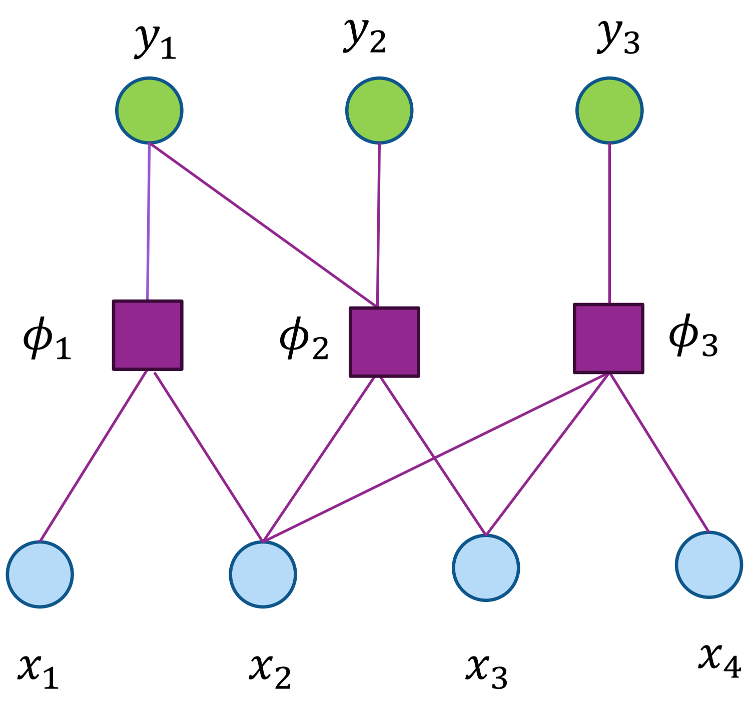
$$p(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{Z(\boldsymbol{y})}\phi_1(\boldsymbol{x},\boldsymbol{y})\cdots\phi_K(\boldsymbol{x},\boldsymbol{y})$$

- Each term $\phi_k(\boldsymbol{x},\boldsymbol{y})$ is a factor
  - Assume depends on only a small number, d, of components x_i
- Normalization term $Z(y) = \sum_x \prod_k \phi_k(\boldsymbol{x},\boldsymbol{y})$
  - Ensures posterior density normalizes to one
  - Generally, we will not have to explicitly compute $Z(\boldsymbol{y})$
- Key idea: Break hard estimation problem to local problems in each factor
  - If d is small, local estimation problems are tractable

Factor Graph

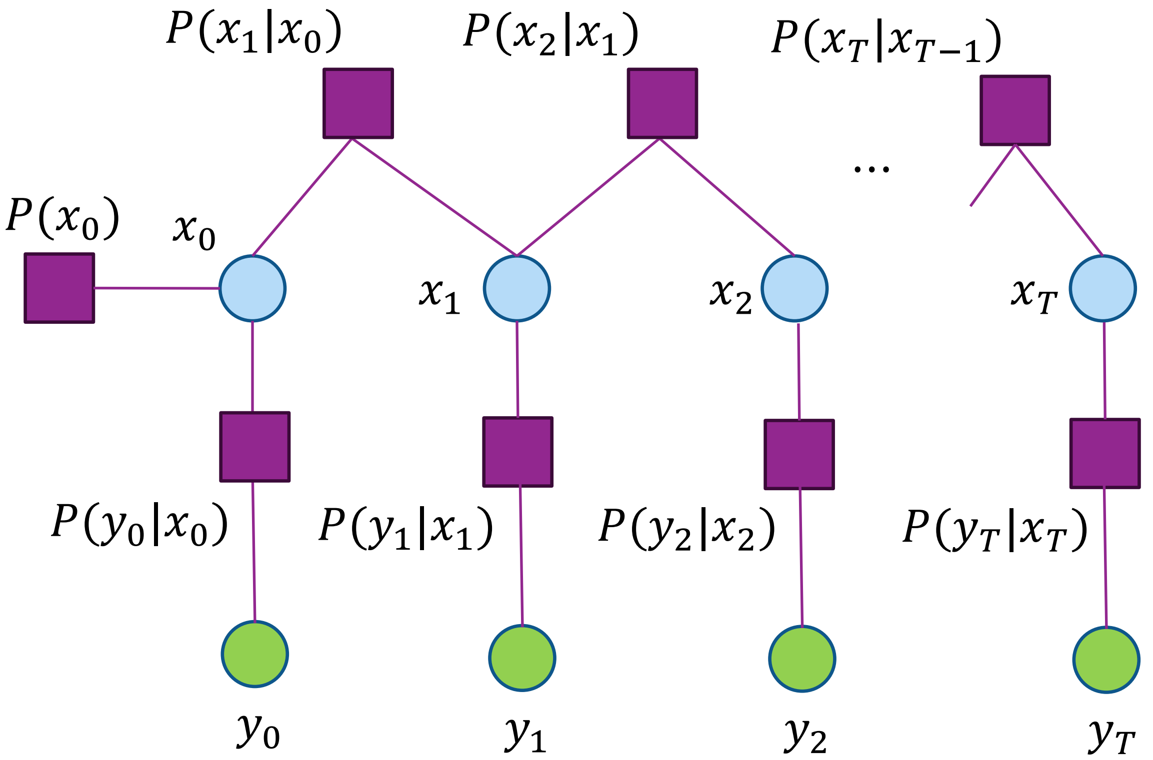
- Represent statistical relations graphically
- Factor graph:
  - Unknown variable nodes
  - Observed or known variable nodes
  - Check or factor nodes
- Edges between variable and factor nodes
  - When variable is part of factor
- Example:

$$p(x_1, \dots, x_4 | y_1, \dots, y_3) = \frac{1}{Z(\mathbf{y})}$$
$$\times\ \phi_1(x_1, x_2, y_1)\phi_2(x_2, x_3, y_1, y_2)\ \phi_3(x_2, x_3, x_4, y_3)$$

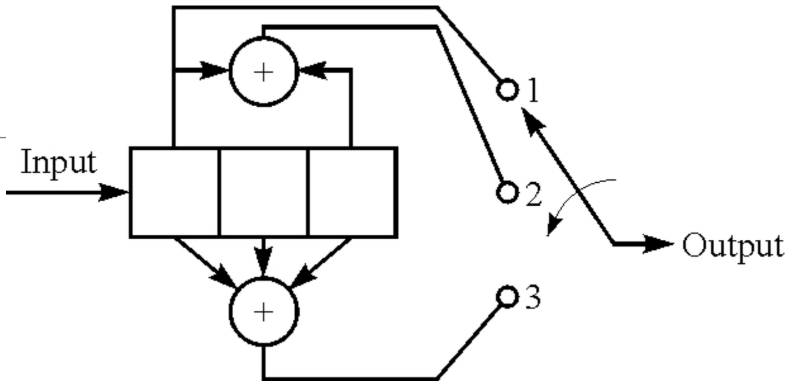


Example: Hidden Markov Chain

- Markov chain:
  - $p(\boldsymbol{x}) = p(x_0)p(x_1|x_0)\cdots p(x_T|x_{T-1})$
- Observations
  - $p(\boldsymbol{y}|\boldsymbol{x}) = p(y_0|x_0)\cdots p(y_T|x_T)$
- Hidden Markov chain problem:
  - Estimate state sequence $\boldsymbol{x} = (x_0, \ldots, x_T)$
  - Use observations $\boldsymbol{y} = (y_0, \ldots, y_{T-1})$
- Posterior: $p(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{p(\boldsymbol{y})}p(\boldsymbol{x})p(\boldsymbol{y}|\boldsymbol{x})$



Convolutional Codes



- Example of an HMM
- Consider $1/k$ convolutional code with constraint length K
- State is $x_t = (b_{t-1}, \ldots, b_{t-K+1})$ [Last $K-1$ bits]
- Transition probability:
  - Given current state x_t, can transition to one of two new states x_{t+1}

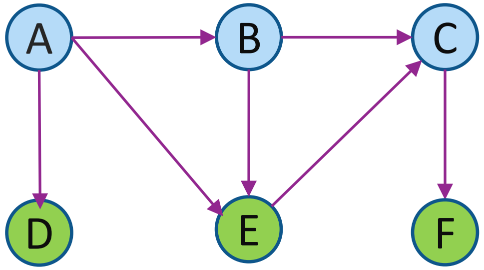
$$P(x_{t+1}|x_t = (b_{t-1}, \ldots, b_{t-K+1})) = \begin{cases} 1/2 & x_{t+1} = (?, b_{t-1}, \ldots, b_{t-K+2}) \\ 0 & else \end{cases}$$

- Output is $y_t = (c_{t1}, \ldots, c_{tk})$
- Assume bitwise channel $P(r_{ti}|c_{ti})$
- Proabability is factorizable:

$$P(r|x) = \prod_{t=0}^{T-1} \prod_{i=1}^{k} P(r_{ti}|x_t) \times \prod_{t=0}^{T-1} P(x_{t+1}|x_t)$$

Bayes Net Representation

- Bayes Net: A directed graph
- Represents conditional probabilities
  - Child is conditioned on parent(s)
  - Example to the right:
    $P(A, \dots, F) = P(A)P(D|A)P(B|A)P(E|A,B)P(C|B,E)P(F|C)$
- Can write any Bayes Net as a factor graph
  - Remove directional arrows
  - Add factor node on leaf nodes and edges
- If a factor graph has no cycles:
  it can be represented as Bayes Net



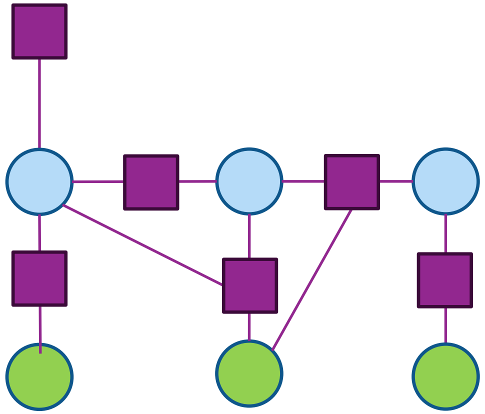


Plate Notation

- Simplify graphical representation
- Write repeated nodes in a “plates”
- Example:
  - HMM with parameter θ
  - $P(x,\theta) = P(x_0)P(\theta)\prod P(x_{t+1}|x_t,\theta)P(y_t|x_t,\theta)$
  - Put update maps in a plate

Standard Representation

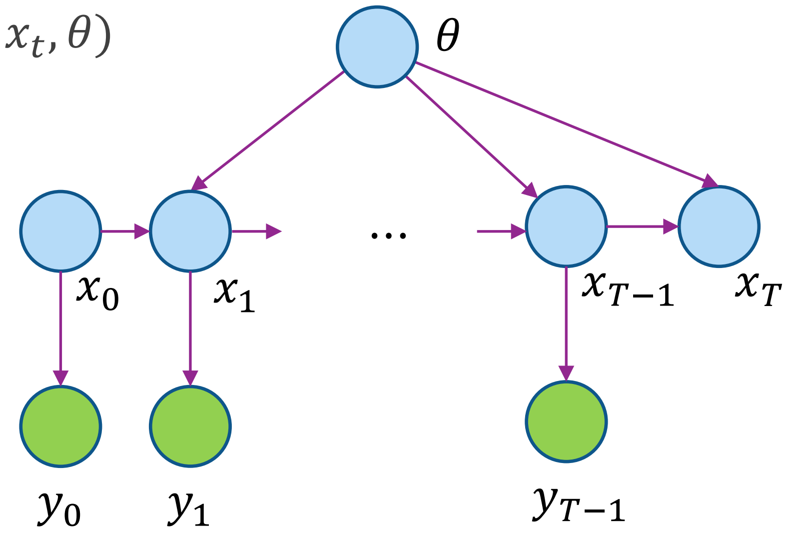
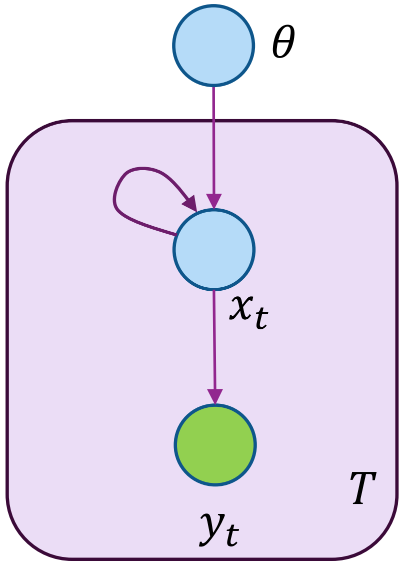
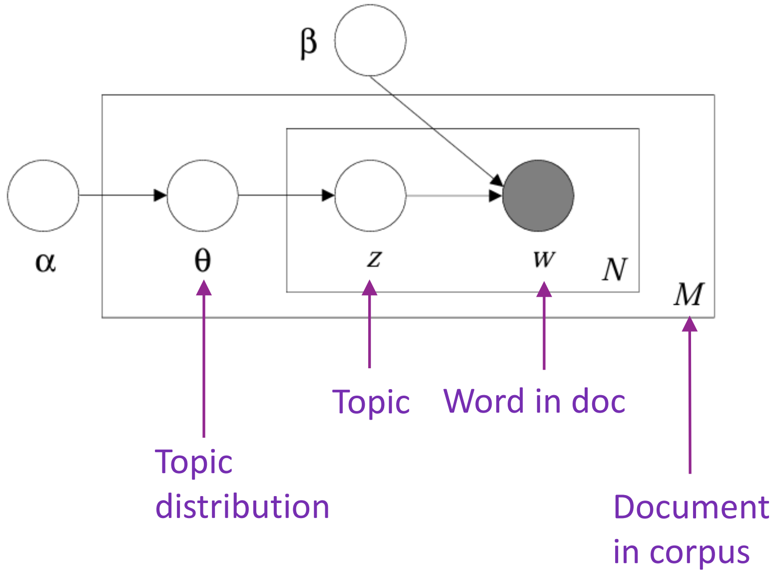


Plate Representation



Examples in ML

- Graphical model descriptions were used in many ML fields
  - Document modeling
  - Image segmentation
  - …
- Allows “explainable” models
  - See example to right
- Not used much any more
- Modern deep NN methods outperform graphical models



Blei, David M., Andrew Y. Ng, and Michael I. Jordan. "Latent dirichlet allocation." *Journal of machine Learning research* 3.Jan (2003): 993-1022.

Conditional Independence

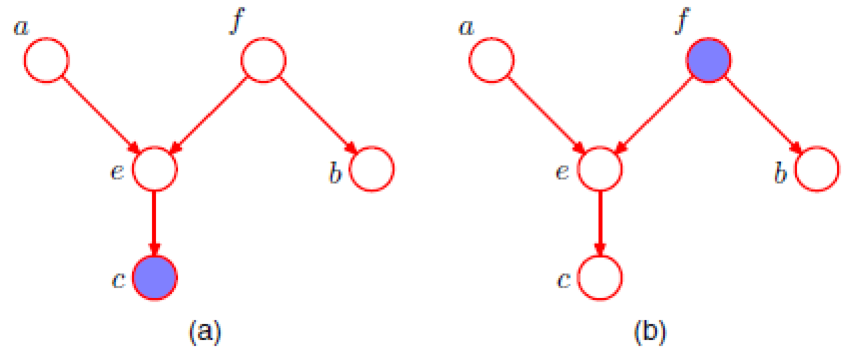
- Let X, Y and Z be random variables
- X and Y are independent if: $P(X, Y) = P(X)P(Y)$
- X and Y are conditionally independent on Z if:

$$P(X, Y|Z) = P(X|Z)P(X|Z)$$

- Written $X \perp Y \mid \mathrm{Z}$
- Note:
  - Conditional independence does not ⇒ Independence
  - Independence does not ⇒ Conditional independence

D-Separation

- Simple test for a Bayes Net
- A set of nodes A and B are d-separated by node C :
- For all paths from A to B every node must be:
  - Arrows must tail-tail or head-tail
  - Or, head-head, then the node and its descendants are not in C
- If A and B are d-separated by C, then
  A and B are conditionally independent of C



False: $a \perp b \mid c$

True: $a \perp b \mid c$

Example: Time-Varying Noise

- Model SNR is a Markov chain:
  - SNR is time-varying
  - $s_t \in \{0, 5, 10\}$ dB. Three possible values
  - Transition probability for s_t: $P = \begin{bmatrix} 1-p & p & 0 \\ p/2 & 1-p & p/2 \\ 0 & p & 1-p \end{bmatrix}$
  - SNR changes with probability $p = 0.1$ in each time step
- Transmission:

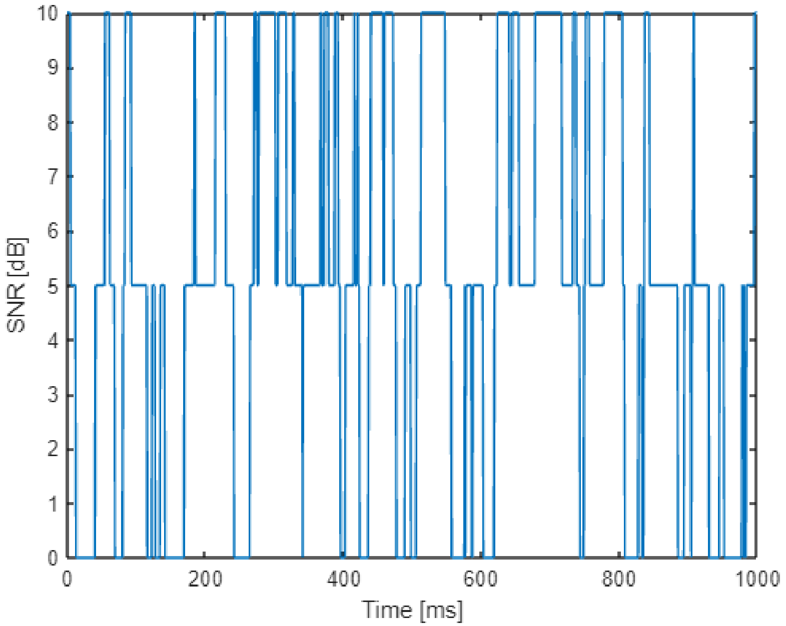
$$r_t = x_t + w_t, \qquad w_t \sim CN\left(0{,}10^{-0.1 s_t}\right)$$

  - x_t is a QAM constellation

MATLAB code

```
% Transition probability
Ptrans = [1-p p 0; p/2 1-p p/2; 0 p 1-p];
nstate = length(snrLevel');

% Initial distribution
p0 = [0.4,0.3,0.3];

% Get initial state
chanState = zeros(nt+1,1);
state0 = randsample(nstate,1,true,p0);
chanState(1) = state0;

snr = zeros(nt,1);
for t = (1:nt)
    % Get the SNR level based on the current state
    snr(t) = snrLevel(chanState(t));

    % Randomly update state
    p = Ptrans(chanState(t),:);
    chanState(t+1) = randsample(nstate,1,true,p);

end
```

- Markov chain simulation
  - Use randsample function



Outline

- LDPC Codes: Motivation and History
- LDPC Encoding
- Graphical Models
- Inference via Belief Propagation
- LDPC Decoding
- 5G LDPC codes

Belief Propagation

- Given a factorizable distribution: $p(\boldsymbol{x}|\boldsymbol{y}) = \frac{1}{Z(\boldsymbol{y})}\phi_1(\boldsymbol{x},\boldsymbol{y})\cdots\phi_K(\boldsymbol{x},\boldsymbol{y})$
- Belief Propagation: Fast algorithm for:
  - MAP estimation: $\hat{\boldsymbol{x}} = \arg\max_{\mathbf{x}} p(\boldsymbol{x}|\boldsymbol{y})$
  - Marginal distribution: Find $p(x_i|\boldsymbol{y})$
- Generalizes the Viterbi algorithm
- Applies when factor graph is a tree
  - That is, equivalent to a Bayes Net
- BP complexity is $O\left(KL^d\right)$
  - K = number of factors
  - d = max number of unknowns in each factor, L =number of values for each x_i
- Brute-force complexity is $O(L^{dK})$
  - Significant savings if d is small.
  - Want to break systems in many factors with small number of terms in each factor

BP on a Chain

- Drop dependence on observed variables $\boldsymbol{y}$
- Consider a simple factor graph:

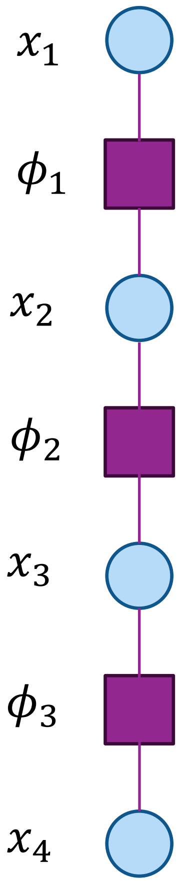
$$p(\boldsymbol{x}) = \frac{1}{Z}\phi_1(x_1)\phi_2(x_1, x_2)\cdots\phi_N(x_{N-1}, x_N)$$

- To compute marginal distribution:

$$p(x_n = j, x_{n-1} = i) = \sum_{x: x_n = j, x_{n-1} = i} p(\boldsymbol{x})$$

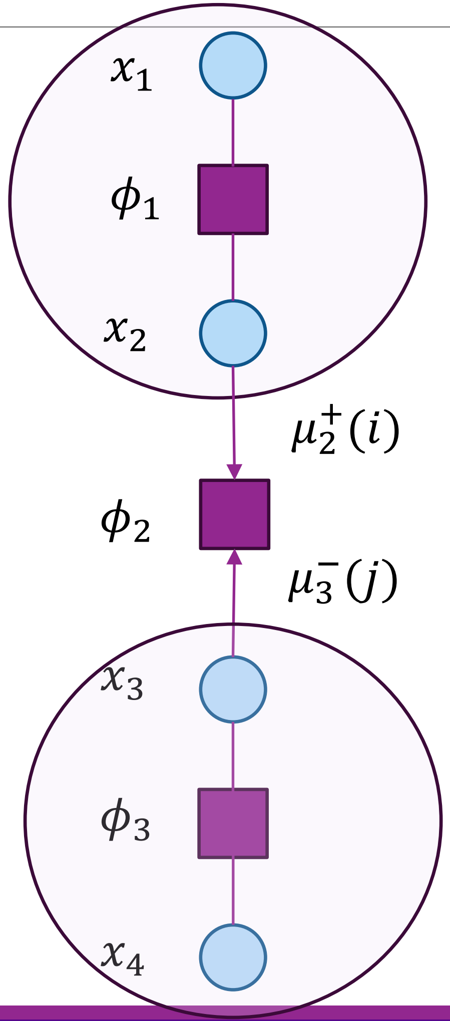
  - Sum is over all x where $x_{n-1} = i, x_n = j$
- Break sum into three terms:

$$p(x_n = i) = \mu_n^+(i)$$



Forward and Reverse Messages

- Define two partial sums:
  - $\mu_n^+(x_n = i) = \sum_{x_1,\ldots,x_{n-1}} \phi_1(x_1) \cdots \phi_{n-1}(x_{n-1}, x_n = i)$
  - $\mu_{n+1}^-(x_{n+1} = j) = \sum_{x_{n+2},\ldots,x_N} \phi_{n+1}(x_{n+1} = j, x_{n+2}) \cdots \phi_{N-1}(x_{N-1}, x_N)$
- Can verify that:
  - $P(x_n, x_{n+1}) = \frac{1}{Z}\mu_n^+(x_n)\mu_{n+1}^-(x_{n+1})\phi(x_n, x_{n+1})$
  - $P(x_n) = \frac{1}{Z}\mu_n^+(x_n)\mu_n^-(x_n)$
- Call:
  - $\mu_n^+(x_n)$ the forward message
  - $\mu_n^-(x_{n+1})$ the reverse message



Recursive Updates

- Forward message:

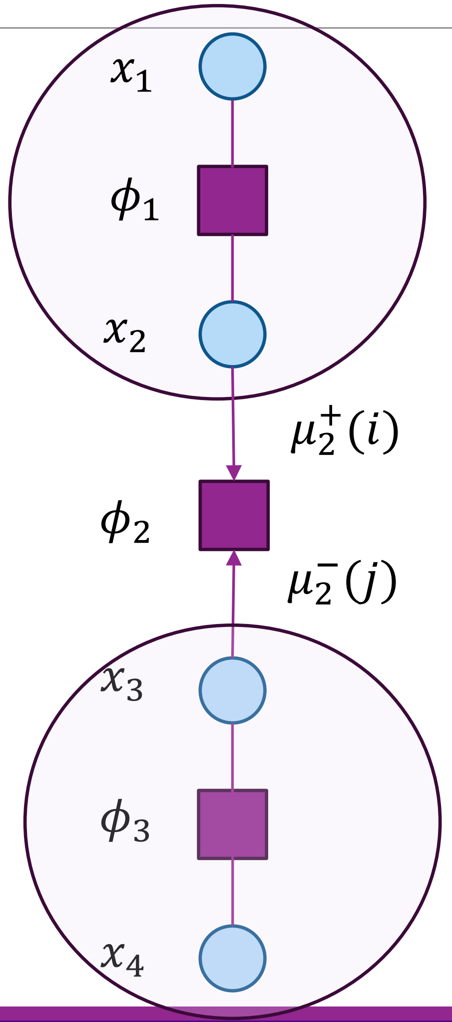
$$\mu_{n+1}^{+}(x_{n+1}) = \sum_{x_1,\ldots,x_n} \phi_1(x_1)\cdots\phi_{n-1}(x_{n-1},x_n)\phi_n(x_n,x_{n+1})$$

$$= \sum_{x_n} \phi_n(x_n,x_{n+1}) \sum_{x_1,\ldots,x_n} \phi_1(x_1)\cdots\phi_{n-1}(x_{n-1},x_n)$$

$$= \sum_{x_n} \phi_n(x_n,x_{n+1})\mu_n^{+}(x_n)$$

- Similarly, for the reverse message:

$$\mu_n^{-}(x_n) = \sum_{x_n} \phi_n(x_n,x_{n+1})\mu_{n+1}^{-}(x_{n+1})$$

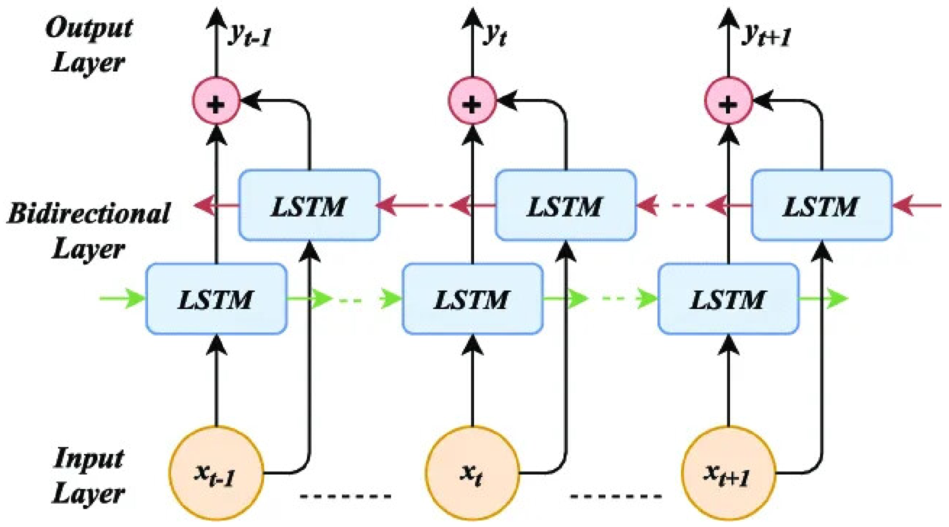


Summary: BP on a Chain

- Compute $\mu_n^+(x_n)$ recursively forward from $n = 1$ to N
- Compute $\mu_n^-(x_n)$ recursively in reverse from $n = N$ to 1
- Compute marginal distributions:
  - $P(x_n, x_{n+1}) = \frac{1}{Z}\mu_n^+(x_n)\mu_{n+1}^-(x_{n+1})\phi(x_n, x_{n+1})$
  - $P(x_n) = \frac{1}{Z}\mu_n^+(x_n)\mu_n^-(x_n)$
- Note that Z can be found since we know $\sum P(x_n) = 1$
- Complexity is $O(NL^2)$
  - Linear in time steps N
- Method is called message passing
  - We pass messages back and forth along the chain

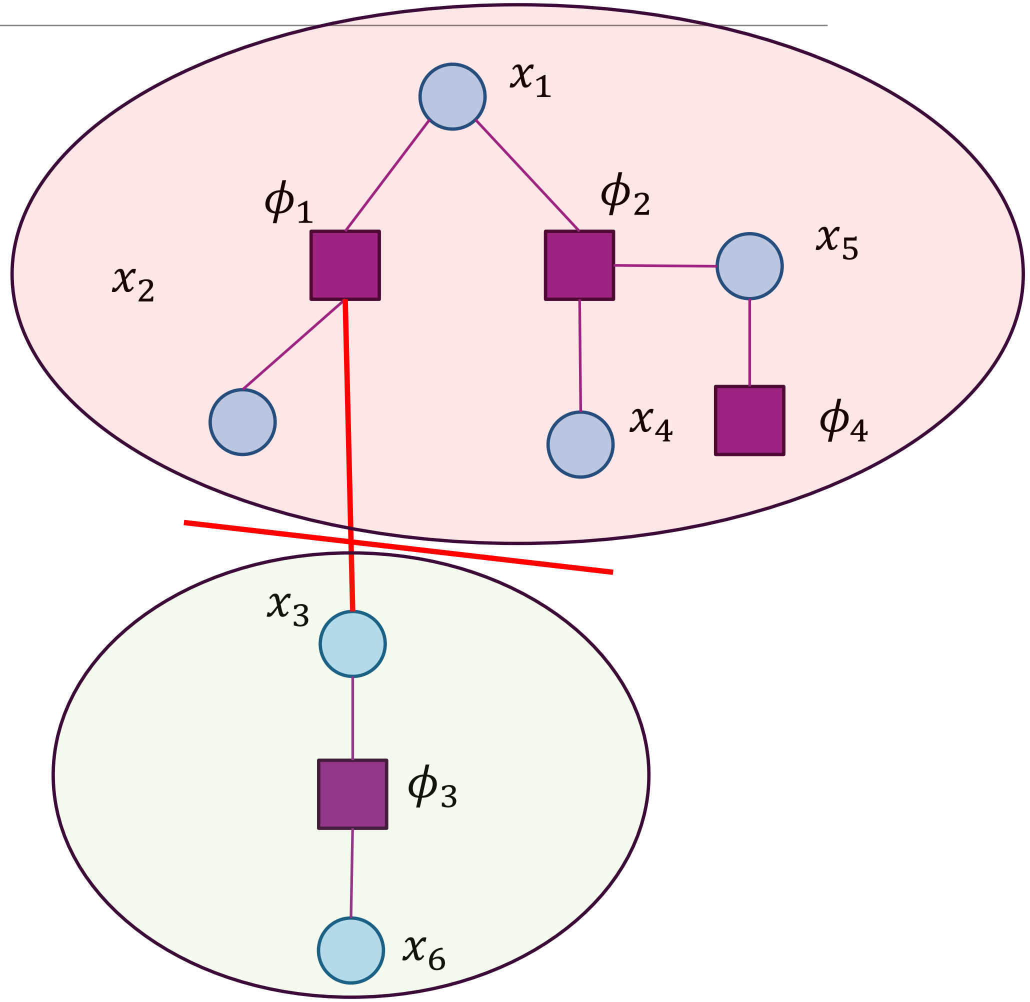
Relation to Bi-LSTMs

- Bi-LSTMs: Widely-used sequence-to-sequence model in ML
  - Input sequence $(x_0, x_1, \ldots, x_{T-1})$
  - Output sequence $(y_0, y_1, \ldots, y_{T-1})$
  - Used in NLP, time-series, signal processing, …
- Model has a forward and backward structure
- If the LSTM cell has sufficiently many states
  - Can model an optimal state estimator
  - x_t is an observation
  - y_t is the estimate of state or some function of a state
- LSTMs are trained with data
  - Do not need a model of the transition probabilities



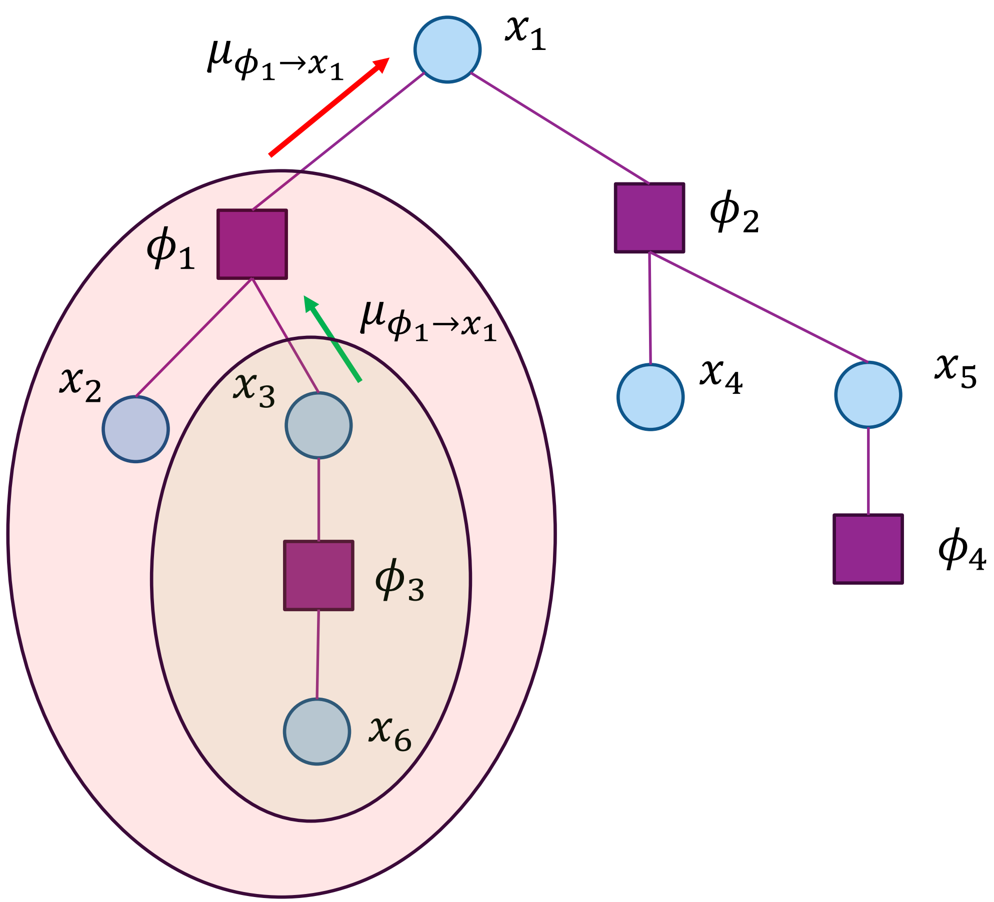
Extensions to a Tree

- BP can be easily extended to trees
- Select any node as the root of the tree
  - Can be a variable or factor node
- Given any factor ϕ_i define:
  - $x^{(i)}$ = set of variables in factor node i
  - Ex: $x^{(1)} = (x_1, x_2, x_3)$, $x^{(3)} = (x_3, x_6)$
- Given any edge (x_j, ϕ_i), define:
  - $C^+(x_j, \phi_i) :=$ set of factors on the "root side" of the edge
  - $C^-(x_j, \phi_i) :=$ set of factors on the "leaf side" of the edge
  - Ex: $C^+(x_3, \phi_1) = \{\phi_1, \phi_2, \phi_4\}$, $C^-(x_3, \phi_1) = \{\phi_3\}$



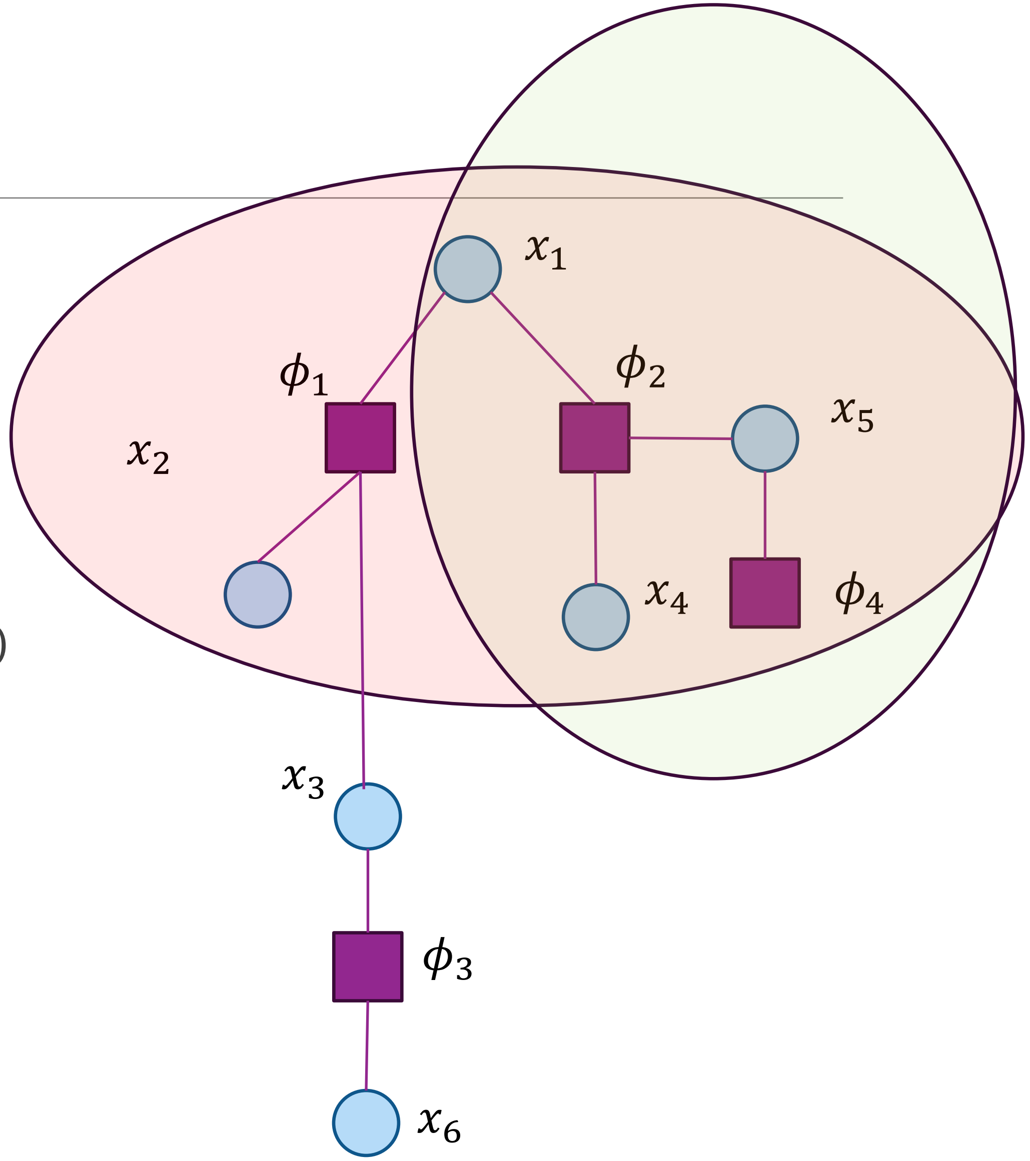
Forward Messages

- Generalize the forward messages on chain:
  - Message towards the tree root
- When x_j is a parent of ϕ_i:
  - $\mu_{\phi_i \to x_j}(a) = \sum_{\mathbf{x}: x_j = a} \prod_{\phi_k \in C^-(\phi_i)} \phi_k(x^{(k)})$
  - Sum is over variables that on the "leaf side" of ϕ_i
  - Ex: $\mu_{\phi_1 \to x_1}(a) = \sum_{x_2, x_3, x_6} \phi_1(x_1 = a, x_2, x_3)\phi_3(x_3, x_6)$
- When ϕ_i is a parent of x_j
  - $\mu_{x_j \to \phi_i}(a) = \sum_{\mathbf{x}: x_j = a} \prod_{\phi_k \in C^-(x_3)} \phi_k(x^{(k)})$
  - Sum is over variables that on leaf side of x_j
  - Ex: $\mu_{x_3 \to \phi_1}(a) = \sum_{x_6} \phi_3(x_3 = a, x_6)$



Reverse Messages

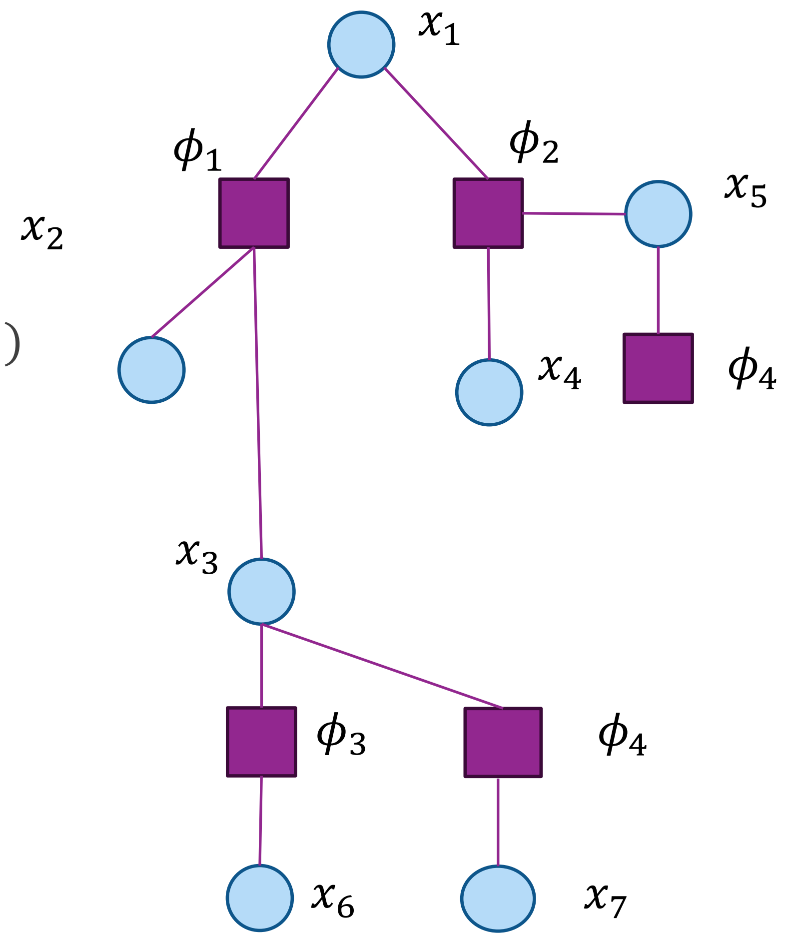
- Generalize the reverse messages on chain:
  - Message towards the leaves of the tree
- When x_j is a child of ϕ_i:
  - $\mu_{\phi_i \to x_j}(a) = \sum_{x:x_j=a} \prod_{\phi_k \in C^+(\phi_i)} \phi_k(x^{(k)})$
  - Sum is over variables on "root" side of ϕ_i
  - Ex: $\mu_{\phi_1 \to x_3}(a) = \sum_{x_1, x_4, x_5} \phi_1(x_1, x_3 = a)\, \phi_2(x_1, x_4) \phi_4(x_5)$
- When ϕ_i is a parent of x_j
  - $\mu_{x_j \to \phi_i}(a) = \sum_{x:x_j=a} \prod_{\phi_k \in C^+(x_3)} \phi_k(x^{(k)})$
  - Sum is over variables on "root side" of x_j
  - Ex: $\mu_{x_1 \to \phi_1}(a) = \sum_{x_4, x_5} \phi_2(x_1 = a, x_4) \phi_4(x_5)$



Recursive Updates

- Messages can be computed recursively
  - Let $\partial(\phi_i)$ = variables connected to ϕ_i
  - $\mu_{\phi_i \to x_j}(a) = \sum_{x^{(i)}:x_j=a} \phi_i\left(x^{(i)}\right) \prod_{x_k \in \partial(\phi_i) - x_j} \mu_{x_k \to \phi_i}(x_k)$
  - $\mu_{x_j \to \phi_i}(a) = \prod_{\phi_k \in \partial(x_j) - \phi_i} \mu_{\phi_k \to x_j}(a)$
  - Ex: $\mu_{\phi_1 \to x_1}(a) = \sum_{x_2, x_3} \phi_1(x_1 = a, x_2, x_3) \mu_{x_2 \to \phi_1}(x_2) \mu_{x_3 \to \phi_1}(x_3)$
  - Ex: $\mu_{x_3 \to \phi_1}(a) = \mu_{\phi_3 \to x_3}(a) \mu_{\phi_4 \to x_3}(a)$

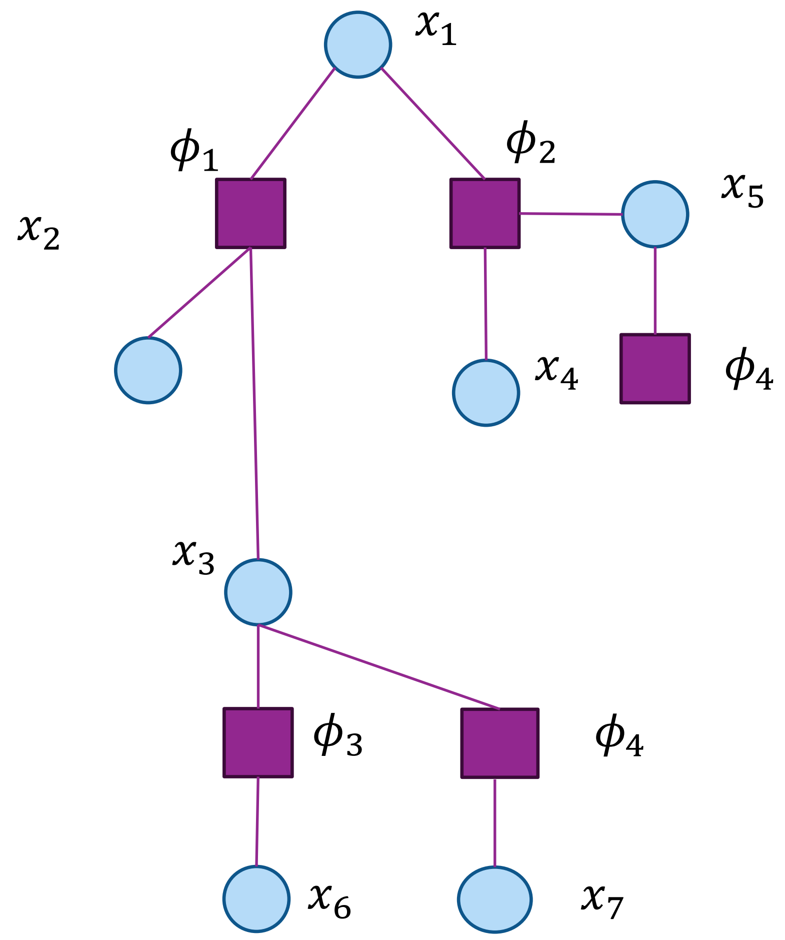
- Initialization at leaf nodes:
  - For variables leaf nodes: $\mu_{x_j \to \phi_i}(a) = 1$ for all a
  - For factor leaf nodes: $\mu_{\phi_i \to x_j}(a) = \phi_1(x_j = a)$ for all a



BP Algorithm Summary

- Called the sum-product algorithm
- Select a root node
  - Many nodes may be possible
- Initial forward messages from leaf nodes
- Recursively compute messages towards root node
- Recursively compute all messages back to leaf nodes
- Final marginal probabilities are:

$$P(x_j) = \frac{1}{Z} \prod_{\phi_k \in \partial(x_j)} \mu_{\phi_k \to x_j}(a)$$



Max-Sum

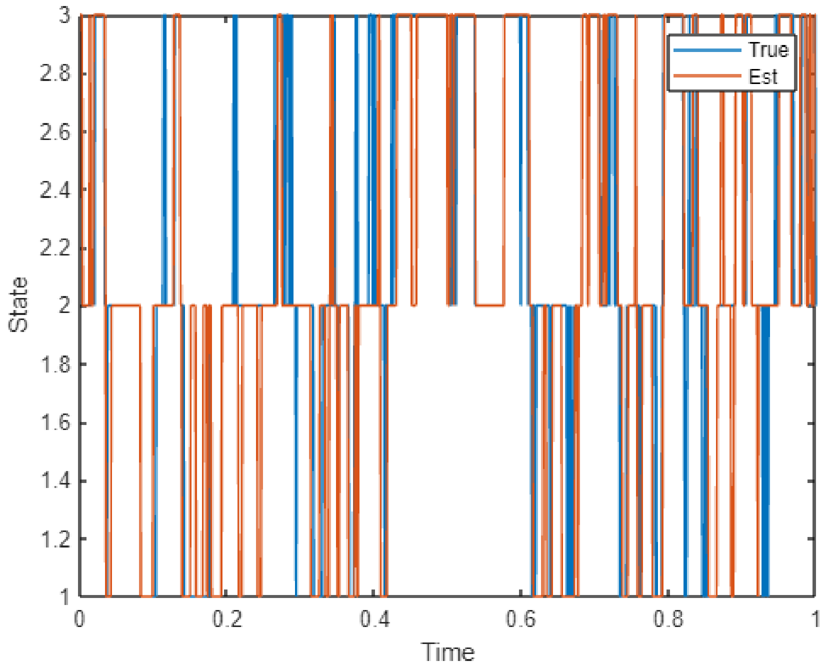
- Sum product computes marginal distributions $P(x_j|\boldsymbol{y})$
- Max-sum finds MAP estimates: $\hat{\boldsymbol{x}} = \arg\max p(\boldsymbol{x}|\boldsymbol{y})$
  - Replace summations with maximum
- Example, suppose in sum-product: $\mu_{\phi_1 \to x_3}(a) = \sum_{x_1,x_4,x_5} \phi_1(x_1, x_3 = a)\, \phi_2(x_1, x_4)\phi_4(x_5)$
- In max-sum update is: $\mu_{\phi_1 \to x_3}(a) = \max_{x_1,x_4,x_5} \phi_1(x_1, x_3 = a)\, \phi_2(x_1, x_4)\phi_4(x_5)$
- In a chain, max-sum obtains the Viterbi algorithm
- See Bishop for more details

Implementation Details

- Often easier to use $\log \mu_{x\to\phi}$ instead of $\mu_{x\to\phi}$.
  - Provides better conditioning
- Can rescale weights
- Constant multiplicative factors will not influence result

HMM Example:

- Return to the time varying noise problem
- Simple to implement in MATLAB




```
% Forward pass
mup = p0;
mup(1,:) = ones(1,nstate);
for t=1:nt

    mup(t,:) = mup(t,:).*pr(t,:);
    mup(t,:) = mup(t,:)./ sum(mup(t,:));
    mup(t+1,:) = mup(t,:)*Ptrans;

end

% Reverse pass
mun = zeros(nt+1,nstate);
mun(nt+1,:) = ones(1,nstate);
for t=nt:-1:1
    mun(t,:) = mun(t+1,:)*Ptrans';
    mun(t,:) = mun(t,:).*pr(t,:);
    mun(t,:) = mun(t,:)./ sum(mun(t,:));

end

% Estimated posterior
phat = mun.*mup;
psum = sum(phat,2);
phat = phat ./ psum;

% Compute most likely estimate
[m, im] = max(phat,[],2);
```

Outline

- LDPC Codes: Motivation and History
- Graphical Models
- Inference via Belief Propagation
- LDPC Encoding
- LDPC Decoding
- 5G LDPC codes

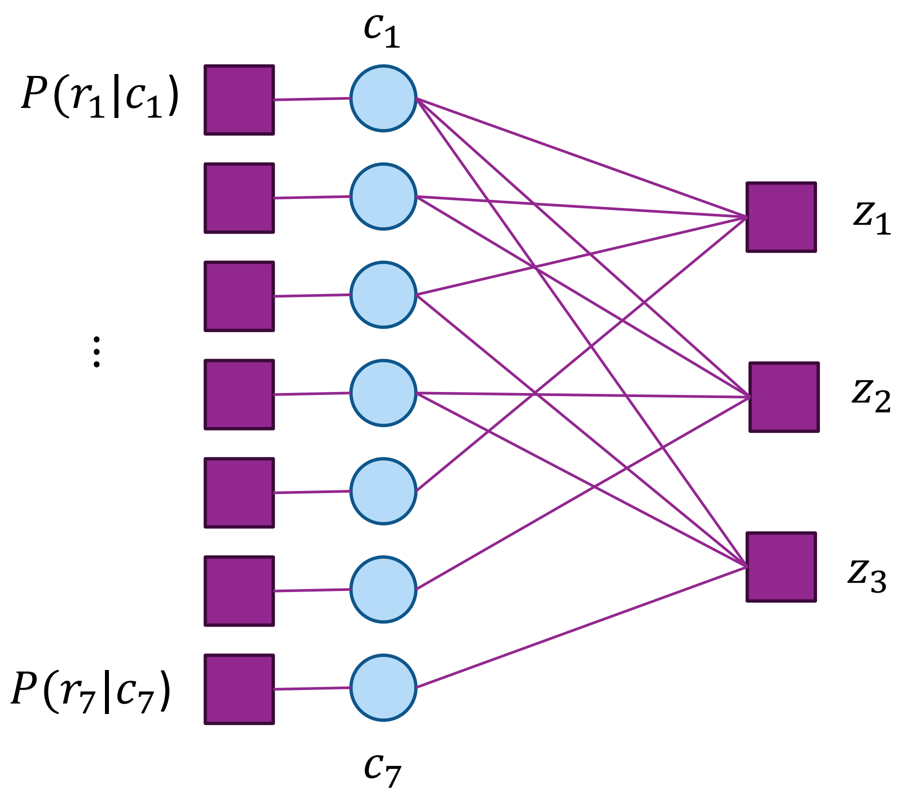
Factorizing the Posterior

- Receive data $\boldsymbol{r}$
  - Assume bit-wise channel $P(\boldsymbol{r}|\boldsymbol{c}) = \prod_{i=1}^{n} P(r_i|c_i)$
- Write parity check equations $\boldsymbol{z} = \boldsymbol{cH}$
  - Let $N_i = \{i \,|H_{ij} = 1\}$
  - Each check node is a linear equation $z_i = \sum_j H_{ij} c_j = \sum_{j \in N_i} c_j$
  - We need $z_i = 0$ for all check nodes i
  - Write a penalty function $\phi_i(z_i) = \begin{cases} 1 & z_i = 0 \\ 0 & z_i = 1 \end{cases}$
  - Penalty $\phi_i(z_i) = \phi_i\left(c^{(i)}\right)$, is a function of the neighbors of z_i: $\mathrm{c}^{(i)} = \left\{c_j, j \in N_i\right\}$
- Posterior probability has a factorizable form:

$$P(c|r) \propto P(r|c)P(c) \propto \prod_{i=1}^{n-k} \phi_i(c^{(i)}) \prod_{j=1}^{n} P(r_j|c_j)$$

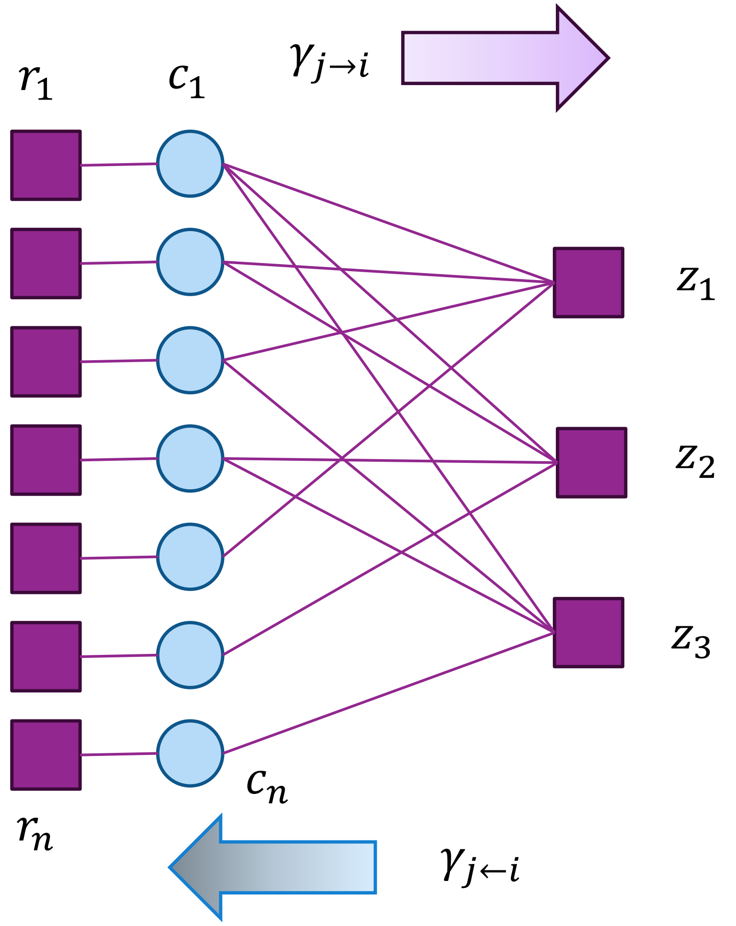
Factor Graph

- Also called the Tanner Graph
- Variable nodes:
  - One for each coded bit c_j
- Factor nodes
  - One for each $P(r_j|c_j)$ [These are often not shown]
  - One for each check node z_i
- Example: Hamming
  - $\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$



Message Passing Decoding with LLRs

- Use LLRs instead of messages
  - Each variable has two values $c_j = 0$ or 1
- Easier to write updates with LLRs
- Variable to check node message:
  - $\gamma_{j\to i} := \log \frac{\mu_{j\to i}(c_j=1)}{\mu_{j\to i}(c_j=0)}$
- Check to variable messages:
  - $\gamma_{j\leftarrow i} := \log \frac{\mu_{j\leftarrow i}(c_j=1)}{\mu_{j\leftarrow i}(c_j=0)}$
- Also define the extrinsic LLRs based on the observed data
  - $\gamma_j^{ext} = \log \frac{P(r_j|c_j=1)}{P(r_j|c_j=0)}$

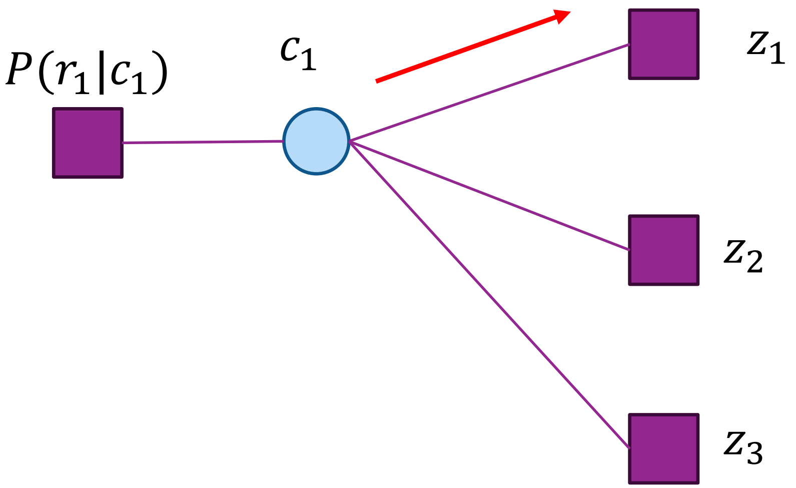


Loopy BP

- In general, factor graph for an LDPC code is not a tree
  - It has "loops"
- Cannot directly apply BP
- Practical iterative method: Use the same update as standard BP in multiple iterations
  - Convergence discussed below
- Initialization: Send $\gamma_{j\to i} = \gamma_j^{ext} = \log\frac{p(r_j|c_j=1)}{p(r_j|c_j=0)}$
  - These are the LLRs provided by the soft symbol demodulator
- Each iteration:
  - Variable to factor node: For each edge (j,i): Send $\gamma_{j\to i}$ from $\gamma_{j\leftarrow \ell}$ for $\ell \neq i$
  - Factor to variable node: For each edge (j,i): Send $\gamma_{j\leftarrow i}$ from $\gamma_{k\to i}$ for $k \neq j$
- Typically use approximately 8 iterations

Variable Node Update

- Consider message from some c_j to z_i
- LLR derivation:
  - $\mu_{j\to i}\left(c_j = 1\right) = P(r_1|c_1 = 1)\prod_{z_k\in N(c_j)-z_i}\mu_{j\leftarrow k}(c_j = 1)$
  - $\mu_{j\to i}\left(c_j = 0\right) = P(r_1|c_1 = 0)\prod_{z_k\in N(c_j)-z_i}\mu_{j\leftarrow k}(c_j = 0)$
  - $\gamma_{j\to i} = \gamma_j^{ext} + \log\left[\prod_{z_k\in N(c_j)-z_i}\frac{\mu_{j\leftarrow k}(c_j=1)}{\mu_{j\leftarrow k}(c_j=0)}\right]$
    $= \gamma_j^{ext} + \sum_{z_k\in N(c_j)-z_i}\gamma_{j\leftarrow k}$
- Example in graph to the right:
  - $\gamma_{1\to 1} = \gamma_1^{ext} + \gamma_{1\leftarrow 2} + \gamma_{1\leftarrow 3}$
  - $\gamma_{1\to 2} = \gamma_1^{ext} + \gamma_{1\leftarrow 1} + \gamma_{1\leftarrow 3}$

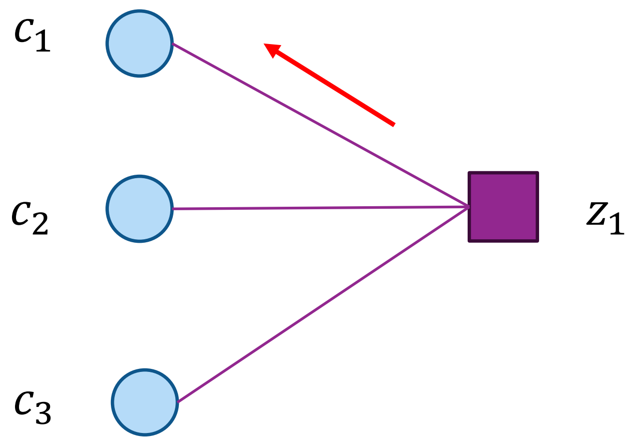


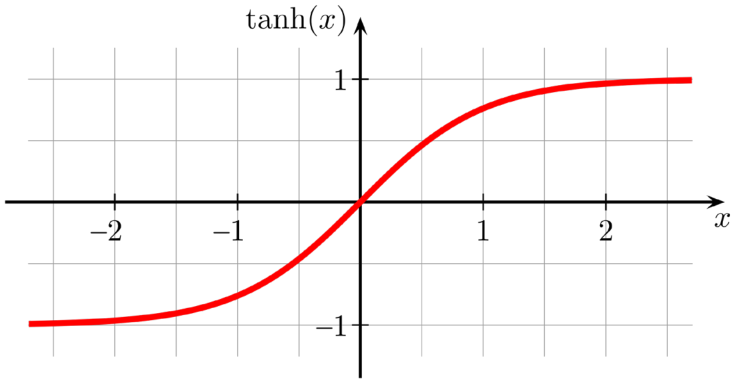
Factor Node Update

- Consider check node: $z_i = c_1 + \cdots + c_d$
- Consider message from z_i to c_j
- Recall: hyperbolic tangent: $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- Theorem:

$$\tanh\left(\frac{\gamma_{j\leftarrow i}}{2}\right) = (-1)^d \prod_{\substack{k=1,\ldots,d \\ k\neq j}} \tanh\left(\frac{\gamma_{k\to i}}{2}\right)$$

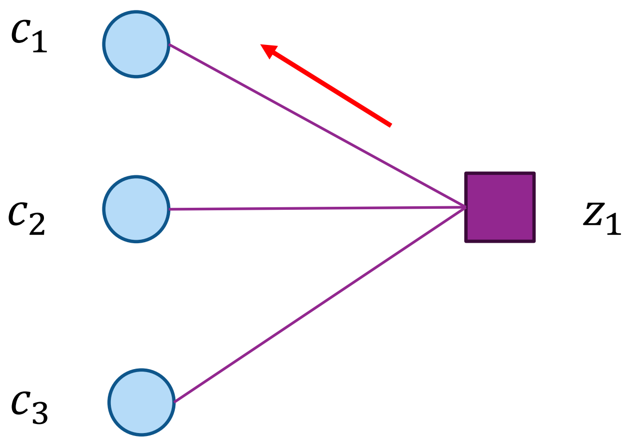
  - Will be proven below
- Allows simple implementation of factor node for arbitrary d
  - Take tanh of all incoming messages $\gamma_{k\to i}$
  - Multiply the tanh terms
  - Take inverse tanh to recover $\gamma_{j\leftarrow i}$





Factor Node Update Proof 1

- Consider check node: $z_i = c_1 + \cdots + c_d = 0$
- Consider message from z_i to c_1
  - Can easily consider messages to other c_j
- Define:
  - $A_0 = \{ (c_2, \dots, c_d) \mid \sum_{k=2}^{d} c_k = 0 \}$
  - $A_1 = \{ (c_2, \dots, c_d) \mid \sum_{k=2}^{d} c_k = 1 \}$
- Example, in message from z_1 to c_1 in graph to right:
  - We want $c_1 + c_2 + c_3 = 0$
  - $A_0 = \{(c_2, c_3) = (0,0), (1,1)\}$
  - $A_1 = \{(c_2, c_3) = (1,0), (0,1)\}$



Factor Node Update Proof 2

- Recall that we can re-scale messages with an arbitrary multiplicative constant
- So, assume that $\mu_{k\to i}(c_k = 1) + \mu_{k\to i}(c_k = 0) = 1$
- Lemma: If $\mu_{k\to i}(c_k = 1) + \mu_{k\to i}(c_k = 0) = 1$ then
$$\mu_{k\to i}(c_k = 1) - \mu_{k\to i}(c_k = 0) = \tanh\left(\frac{\gamma_{k\to i}}{2}\right)$$
- Similarly: $\mu_{k\leftarrow i}(c_k = 1) - \mu_{k\leftarrow i}(c_k = 0) = \tanh\left(\frac{\gamma_{k\leftarrow i}}{2}\right)$
- Proof:
  - $\mu_{k\to i}(c_k = 1) = \frac{e^{\gamma_{k\to i}}}{1+e^{\gamma_{k\to i}}}, \mu_{k\to i}(c_k = 0) = \frac{1}{1+e^{\gamma_{k\to i}}}$
  - Hence, $\mu_{k\to i}(c_k = 1) - \mu_{k\to i}(c_k = 0) = \frac{e^{\gamma_{k\to i}-1}}{1+e^{\gamma_{k\to i}}} = \tanh\left(\frac{\gamma_{k\to i}}{2}\right)$

Factor Node Update Proof 3

- Suppose that $z_i = c_1 + c_2 + \cdots + c_d = 0$ and consider messages from z_i to c_1
- Message are:
  - $c_1 = 0 \Rightarrow (c_2, \ldots, c_d) \in A_0$ hence: $\mu_{1\leftarrow i}(c_1 = 0) = \sum_{(c_2,\ldots,c_d)\in A_0} \mu_{2\rightarrow i}(c_2) \cdots \mu_{d\rightarrow i}(c_d)$
  - $c_1 = 1 \Rightarrow (c_2, \ldots, c_d) \in A_1$ hence: $\mu_{1\leftarrow i}(c_1 = 1) = \sum_{(c_2,\ldots,c_d)\in A_1} \mu_{2\rightarrow i}(c_2) \cdots \mu_{d\rightarrow i}(c_d)$
- Hence difference is:

$$\mu_{1\leftarrow i}(c_1 = 0) - \mu_{1\leftarrow i}(c_1 = 1)$$
$$= \sum_{(c_2,\ldots,c_d)\in A_1} \mu_{2\rightarrow i}(c_2) \cdots \mu_{d\rightarrow i}(c_d) - \sum_{(c_2,\ldots,c_d)\in A_0} \mu_{2\rightarrow i}(c_2) \cdots \mu_{d\rightarrow i}(c_d)$$

Factor Node Update Proof 4

- Note: $(-1)^{c_2+\cdots+c_d} = \begin{cases} 1 & (c_2, \dots, c_d) \in A_0 \\ -1 & (c_2, \dots, c_d) \in A_1 \end{cases}$

- Therefore, we can write the message difference as:

$$\mu_{1\leftarrow i}(c_1 = 0) - \mu_{1\leftarrow i}(c_1 = 1) = \sum_{(c_2,\dots,c_d)} (-1)^{c_2+\cdots+c_d} \mu_{2\rightarrow i}(c_2) \cdots \mu_{d\rightarrow i}(c_d)$$

- This sum is:

$$\mu_{1\leftarrow i}(c_1 = 0) - \mu_{1\leftarrow i}(c_1 = 1) = \prod_{k=2,\dots,d} \left(\mu_{k\rightarrow i}(c_k = 0) - \mu_{k\rightarrow i}(c_k = 1)\right)$$

- From Lemma:

$$\tanh\left(-\frac{\gamma_{1\leftarrow i}}{2}\right) = \prod_{k=2,\dots,d} \tanh\left(-\frac{\gamma_{1\leftarrow i}}{2}\right)$$

LDPC Summary

- Initialization: Send $\gamma_{j\to i} = \gamma_j^{ext} = \log\frac{p(r_j|c_j=1)}{p(r_j|c_j=0)}$
- Check node update:

$$\gamma_{j\leftarrow i} = -2\tanh^{-1}\left(\prod_{k\in V(i)\setminus j} \tanh\left(-\frac{\gamma_{k\to i}}{2}\right)\right)$$

  - $V(i) = \{\, j \mid H_{ji} = 1\}$ = Variable nodes connected to check node i
- Variable node update:

$$\gamma_{j\to i} = \gamma_j^{ext} + \sum_{k\in F(j)\setminus i} \gamma_{j\leftarrow k}$$

  - $F(j) = \{\, i \mid H_{ji} = 1\}$ = Check nodes connected to variable node j

Convergence of Loopy BP

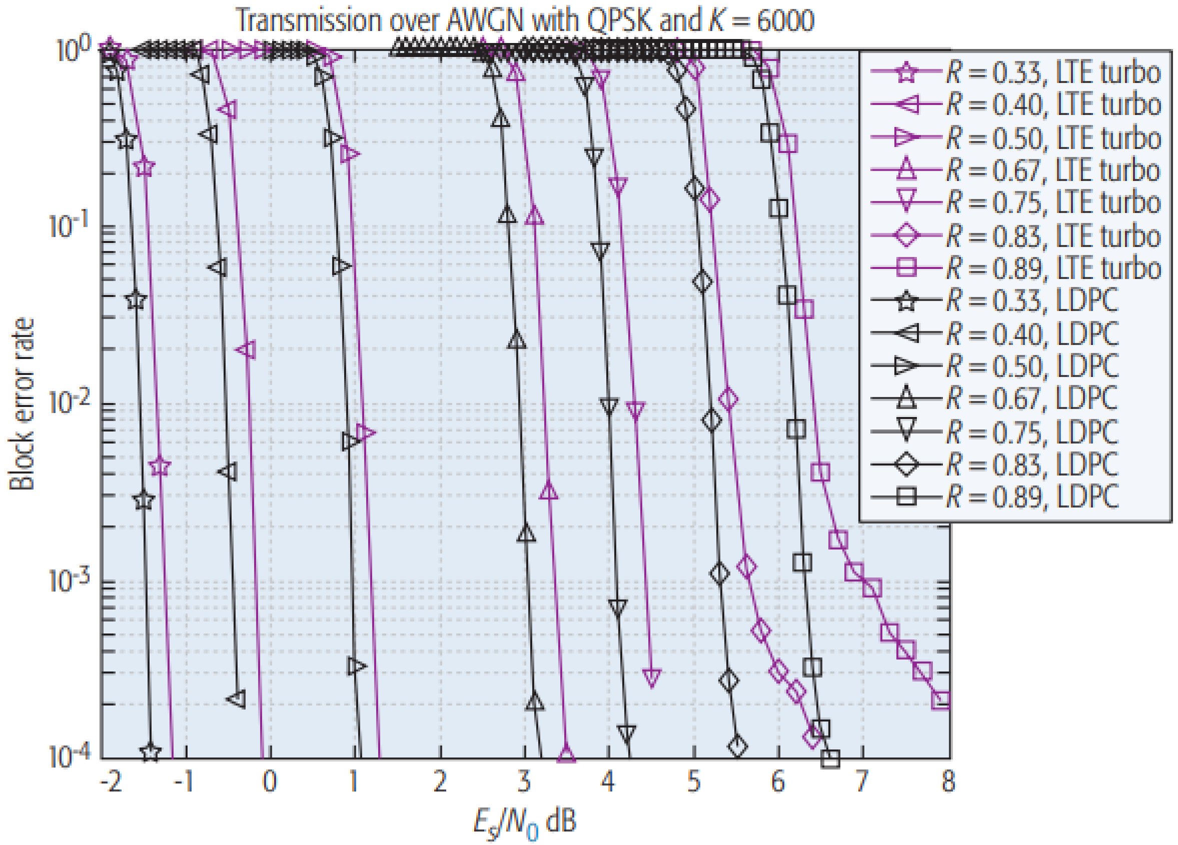
- Two issues of loopy BP
  - Not guaranteed to converge
  - When it converges, the marginal distributions may not be exactly correct
- But for practical LDPC codes:
  - Graphs that are sparse do not have many small loops
  - Tends to provide good convergence
- Significant theoretical work on the convergence
  - EXIT charts
  - Beyond the scope of this class

Outline

- LDPC Codes: Motivation and History
- Graphical Models
- Inference via Belief Propagation
- LDPC Encoding
- LDPC Decoding
- 5G LDPC codes

LDPC vs. Turbo

- Move for Turbo (4G) to LDPC (5G)
- Slight performance improvement
- Complexity is lower



Structure

- Base graph:
  - A small LDPC graph
  - Two base graphs supported in 5G
- Lifting:
  - Used to create longer block lengths
- Number of columns and rows adjustable:
  - To support different rates
  - Support incremental redundancy
- We will discuss IR in the wireless class

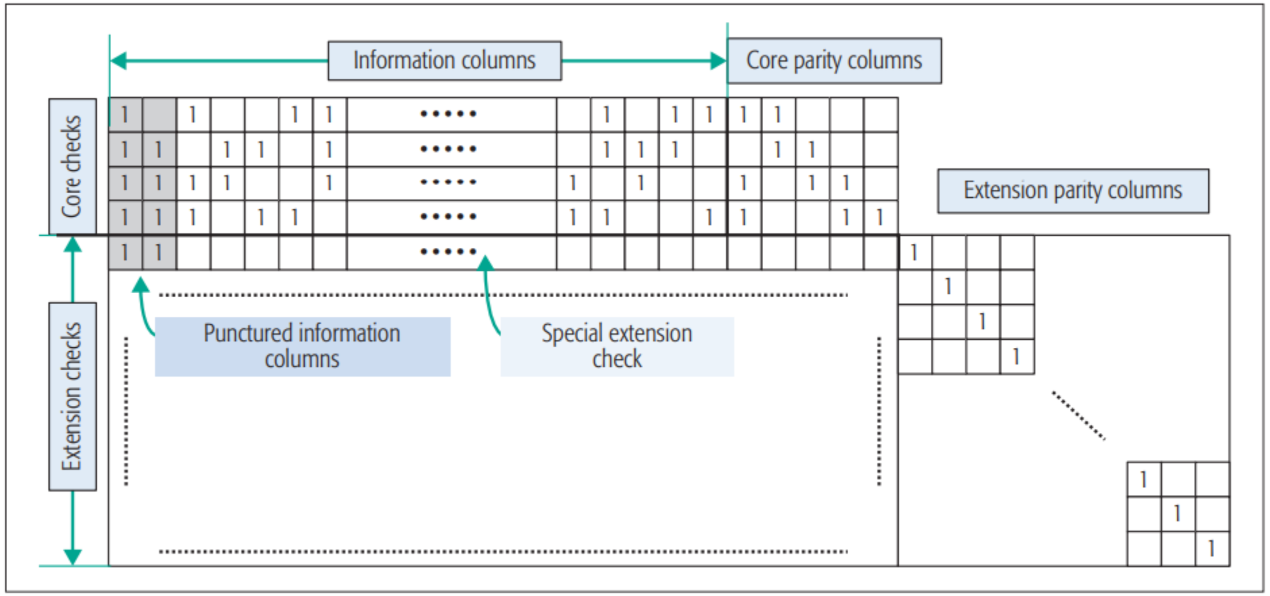


Figure 5. Sketch of base parity check structure for the 5G NR LDPC code.