2.2 控制系统的传递函数

2.2.1 传递函数定义

2.2.2 典型环节传递函数

2.2.3 举例说明建立传递函数的方法

2.2.1 传递函数定义

零初始条件下，系统输出量的拉氏变换与输入量的拉氏变换之比。

$$G(s)=\frac{Y(s)}{U(s)}$$

$$\begin{aligned}&\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\ldots+a_1\frac{dy(t)}{dt}+a_0y(t)\\&=b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}}+\ldots+b_1\frac{du(t)}{dt}+b_0u(t)\end{aligned}\qquad(1)$$

First step: $y(0)=0 \quad y'(0)=0$

Second step: $L[\frac{df(t)}{dt}]=sF(s)-f(0)$ $\Rightarrow$ $\boxed{L[\frac{df(t)}{dt}]=sF(s)}$

$$L[\frac{d^n f(t)}{dt^n}]=s^n\cdot F(s)-s^{n-1}f(0)-s^{n-2}f^{(1)}(0)-\ldots-f^{(n-1)}(0)$$

$$\Downarrow$$

$$\boxed{L[\frac{d^n f(t)}{dt^n}]=s^n\cdot F(s)}$$

$$\frac{d^n y(t)}{dt^n}+a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}}+\ldots+a_1\frac{dy(t)}{dt}+a_0y(t)=b_m\frac{d^m u(t)}{dt^m}+b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}}+\ldots+b_1\frac{du(t)}{dt}+b_0u(t)$$

$$\Downarrow$$

$$\boxed{(s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0)Y(s)=(b_ms^m+b_{m-1}s^{m-1}+\ldots+b_1s+b_0)U(s)}$$

Third step:

$$\boxed{(s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0)Y(s) = (b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0)U(s)}$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0} \qquad (3)$$

传递函数性质:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0}$$

The rational function: the transfer function is called to be **strictly proper** if the order of the denominator polynomial is greater than that of the numerator polynomial (i.e., $n>m$). if $n=m$, the transfer function is called **proper**.

$$n \geq m$$

$$G(s) = L[g(t)]$$

$$R(s) = L[\delta(t)] = 1 \qquad Y(s) = G(s) \cdot R(s) \qquad R(s) = 1 \qquad Y(s) = G(s)$$

$$g(t) = y(t) = L^{-1}[Y(s)] = L^{-1}[G(s)]$$

$$G(s)=\frac{b_m s^m + b_{m-1}s^{m-1}+\ldots+b_0}{s^n + a_{n-1}s^{n-1}+\ldots+a_0}=\frac{N(s)}{D(s)}$$

$$G(s)=K^*\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

$$G(s)=\frac{K(\tau_1 s+1)(\tau_2^2 s^2+2\zeta\tau_2 s+1)\cdots(\tau_i s+1)}{(T_1 s+1)(T_2^2 s^2+2\zeta T_2 s+1)\cdots(T_j s+1)}$$

The roots of the numerator $-z_1, -z_2, \cdots, -z_m$ are called the finite **zeros** of the system.

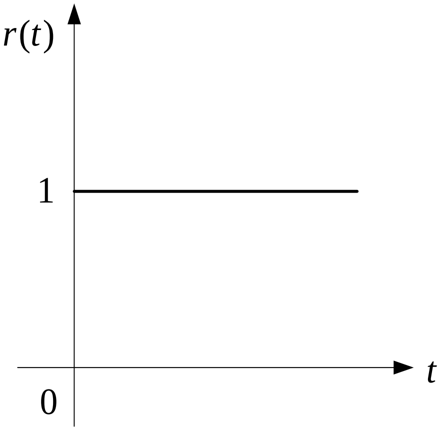
The roots of the denominator, $-p_1, -p_2, \ldots, -p_n$ are called the **poles** of the system. The poles are locations in the s-plane where the magnitude of the transfer function becomes infinite.

2.2.2 典型环节传递函数

1）比例环节（又称放大环节）

2）惯性环节

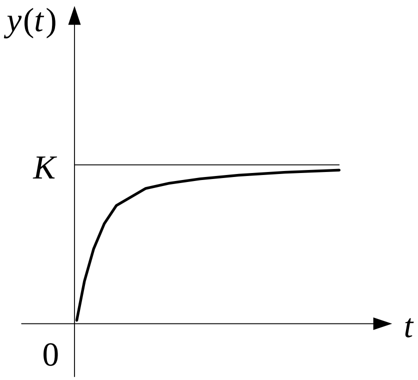
$$r(t)=1(t) \qquad y(t)=K\left(1-\mathrm{e}^{-t/T}\right)$$



$$G(s)=\frac{Y(s)}{R(s)}=\frac{K}{Ts+1}$$

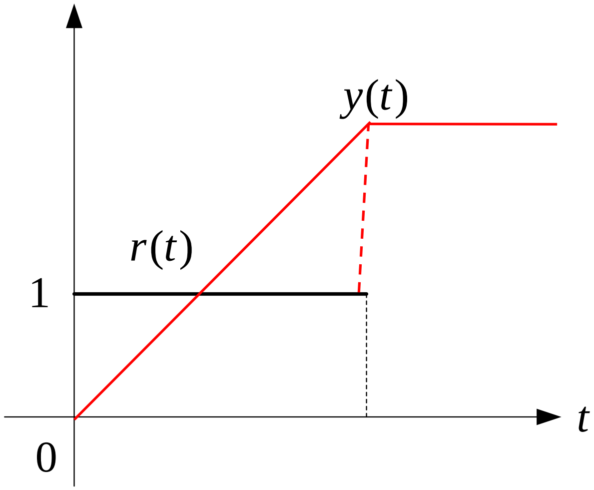
?

$$T\frac{\mathrm{d}y(t)}{\mathrm{d}t}+y(t)=Kr(t)$$



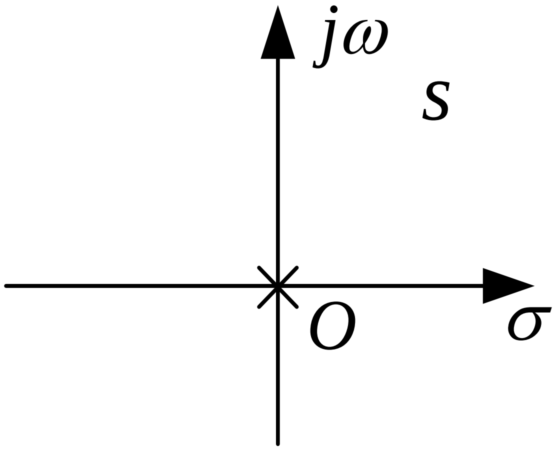
3）积分环节

$$y(t)=\frac{1}{T}\int_0^t r(\tau)\mathrm{d}\tau$$



$$r(t)=1(t) \qquad y(t)=t/T$$

$$\boxed{G(s)=\frac{1}{Ts}=\frac{K}{s}}$$



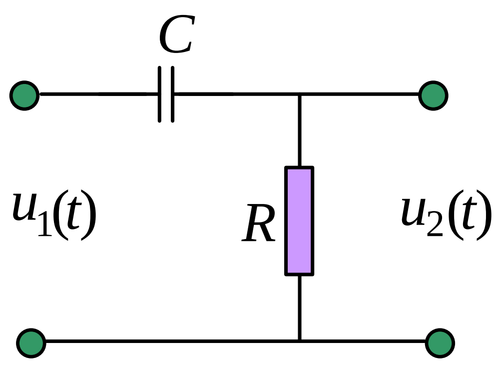
4）微分环节

理想的纯微分环节

$$y(t)=\tau\frac{\mathrm{d}r(t)}{\mathrm{d}t}$$

$$r(t)=1(t) \qquad \boxed{y(t)=\tau\cdot\delta(t)}$$

$$G(s)=\tau s$$

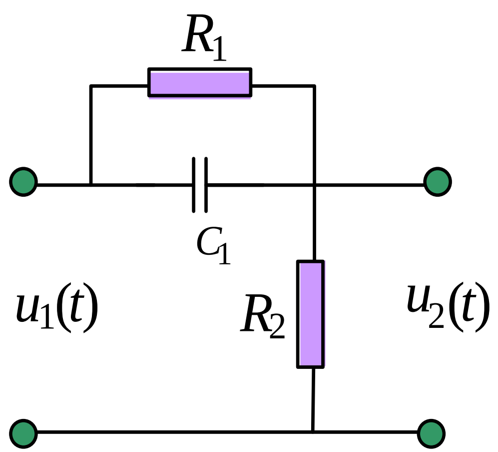


理想的一阶和二阶微分环节的传递函数分别为

$$G(s)=1+\tau s \qquad G(s)=1+2\zeta\tau s+\tau^2 s^2$$

实际物理系统

$$G(s)=\frac{RCs}{RCs+1} \qquad G(s)=\frac{K(\tau_1 s+1)}{\tau_2 s+1}$$



5）振荡环节

$$T^2\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2}+2\zeta T\frac{\mathrm{d}y(t)}{\mathrm{d}t}+y(t)=r(t)$$

$$G(s)=\frac{1}{T^2s^2+2\zeta Ts+1}=\frac{\omega_\mathrm{n}^2}{s^2+2\zeta\omega_\mathrm{n}s+\omega_\mathrm{n}^2}$$

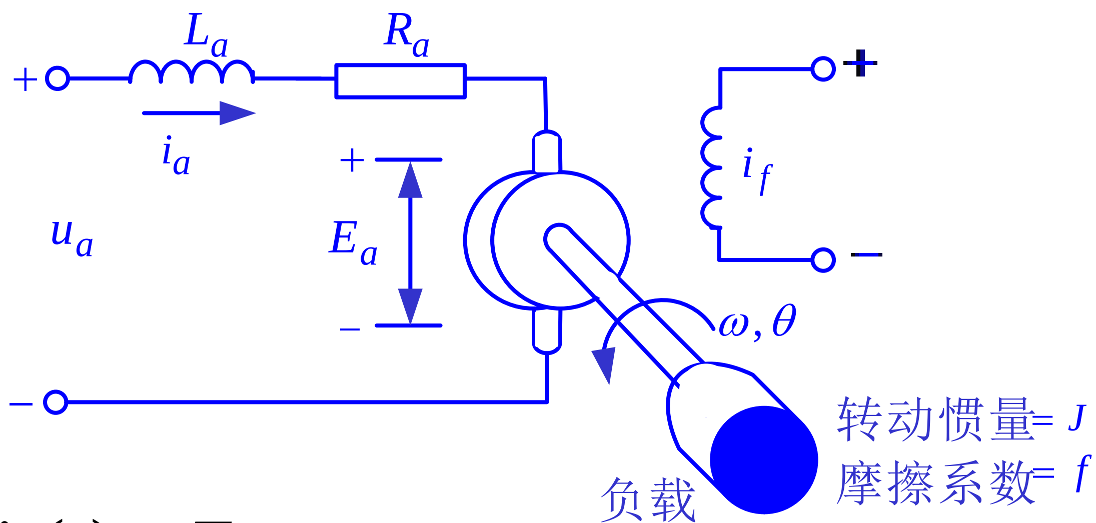
6）滞后环节

$$y(t)=r(t-\tau)$$

$$G(s)=\mathrm{e}^{-\tau s}$$

2.2.3 举例说明建立传递函数的方法

1．重新考虑电枢控制式直流电动机，试求以电枢控制电压$u_a(t)$ 为输入量，电动机转角θ 为输出量的传递函数。



$$E_a = C_e\omega(t)$$

$$M(t) = C_m i_a(t)$$

$$u_a(t) = L_a\frac{di_a(t)}{dt} + R_a i_a(t) + E_a$$

$$J\frac{d\omega(t)}{dt} + f\omega(t) = M(t) - M_c(t) \qquad \omega(t) = \frac{d\theta(t)}{dt}$$

$$u_a(t) = L_a \frac{\mathrm{d}i_a(t)}{\mathrm{d}t} + R_a i_a(t) + E_a$$

$$E_a = C_e \omega(t)$$

$$M(t) = C_m i_a(t)$$

$$J\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} + f\omega(t) = M(t) - M_c(t)$$

$$\omega(t) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t}$$

$$U_a(s) = L_a s I_a(s) + R_a I_a(s) + E_a$$

$$E_a = C_e \omega(s)$$

$$M(s) = C_m I_a(s)$$

$$Js\omega(s) + f\omega(s) = M(s) - M_c(s)$$

$$\omega(s) = s\theta(s)$$

$$L_a J \frac{d^2\omega(t)}{dt^2} + (L_a f + R_a J)\frac{d\omega(t)}{dt} + (R_a f + C_m C_e)\omega(t) =$$

$$C_m u_a(t) - L_a \frac{dM_c(t)}{dt} - R_a M_c(t)$$

$I_a(s)$、E_a、$M(s)$、$\omega(s)$

$$G(s) = \frac{\theta(s)}{U_a(s)} = \frac{C_m}{s[(L_a s + R_a)(Js + f) + C_e C_m]}$$

$$G(s)=\frac{\theta(s)}{U_a(s)}=\frac{C_m}{s[(L_a s+R_a)(Js+f)+C_e C_m]}$$

电枢时间常数 $\tau_a=L_a/R_a$ 可以忽略不计

$$G(s)=\frac{\theta(s)}{U_a(s)}=\frac{C_m}{s[R_a(Js+f)+C_e C_m]}=\frac{K_1}{s(T_m s+1)}$$

如果电枢电阻和电动机的转动惯量都很小，可忽略不计时

$$G(s)=\frac{\theta(s)}{U_a(s)}=\frac{K_1}{s}$$

可将直流电动机作为积分环节

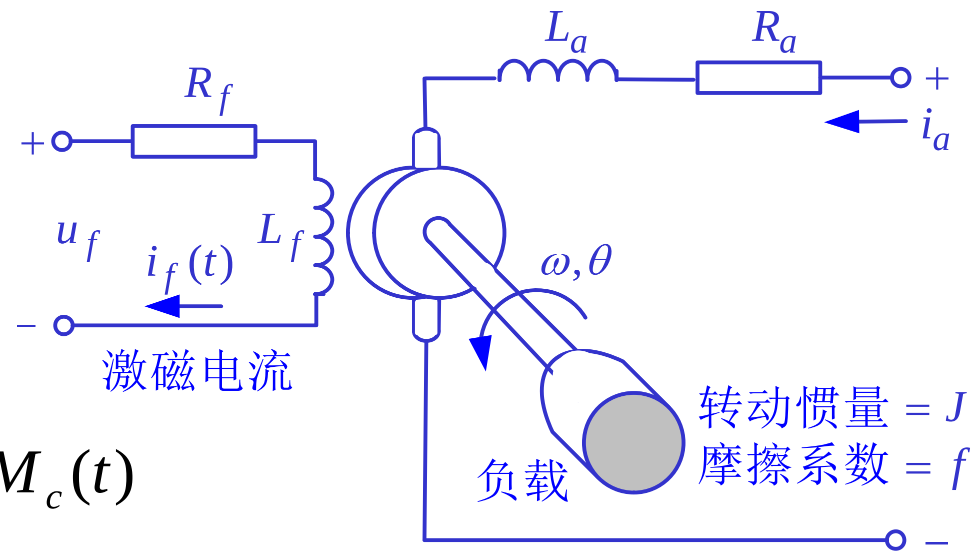
2. 重新考虑磁场控制式直流电动机，试求以激磁电压为输入量，电动机转角为输出量的传递函数。

$$u_f(t) = L_f \frac{di_f(t)}{dt} + R_f i_f(t)$$

$$M(t) = k_m i_f(t)$$

$$J\frac{d\omega(t)}{dt} + f\omega(t) = M(t) - M_c(t)$$

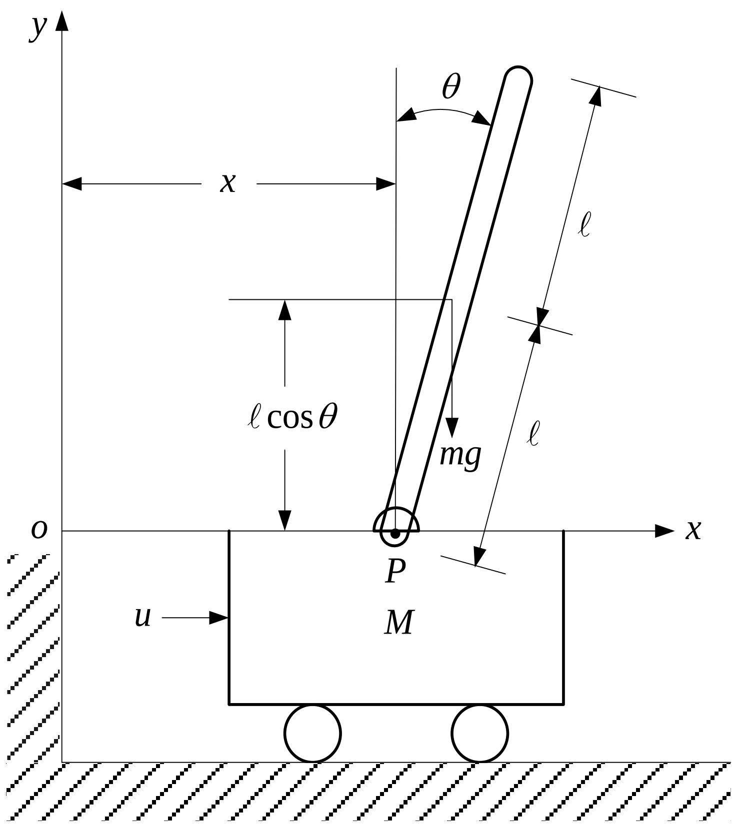
$$\omega(t) = \frac{d\theta(t)}{dt}$$

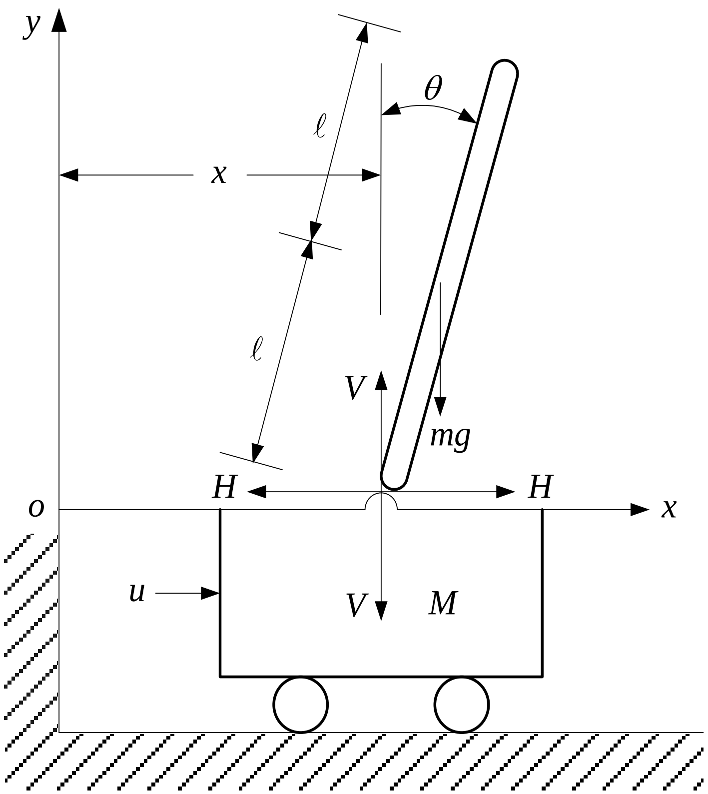


$$T_m = \frac{J}{f} \qquad T_f = \frac{L_f}{R_f}$$

$$G(s) = \frac{\theta(s)}{U_f(s)} = \frac{k_m}{s\left(L_f s + R_f\right)\left(Js + f\right)} = \frac{k_m'}{s\left(T_m s + 1\right)\left(T_f s + 1\right)}$$

3．重新考虑倒立摆系统，试求这个系统的传递函数。



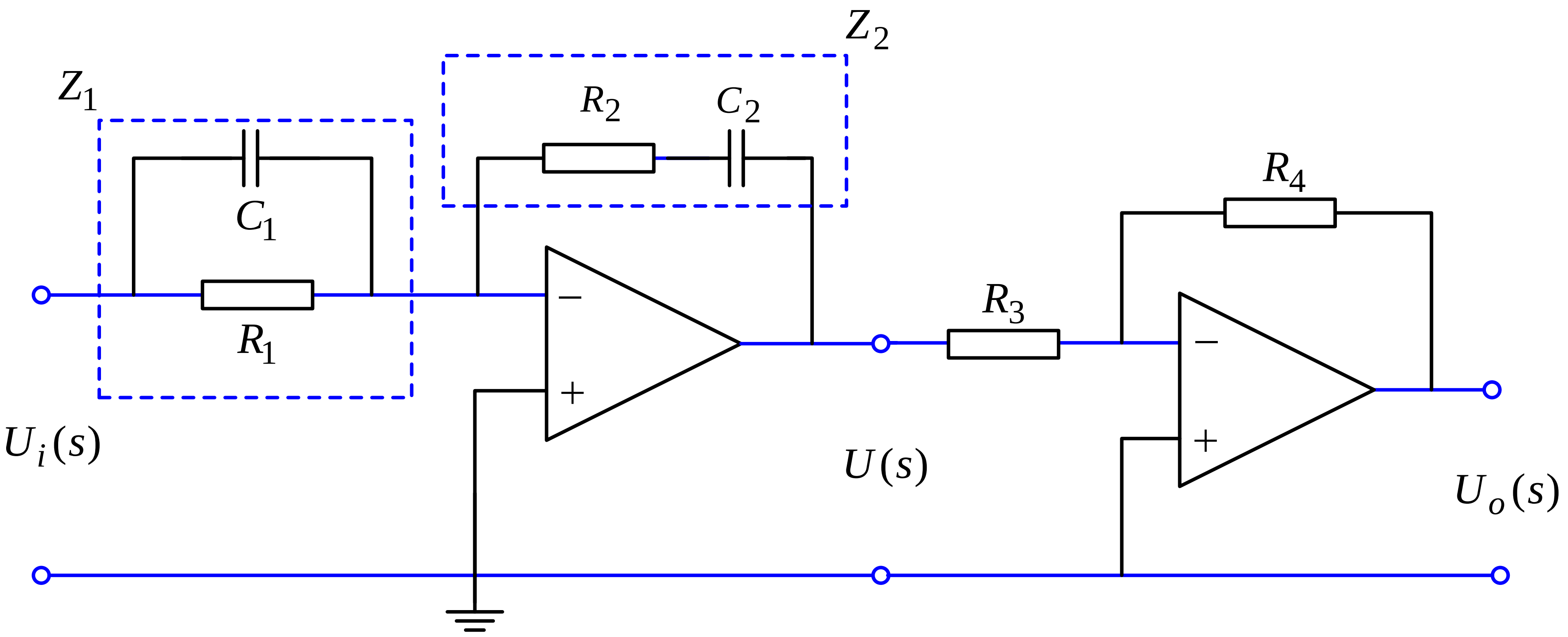


$$(M+m)\ddot{x}+m\ell\ddot{\theta}=u$$

$$\left(I+m\ell^{2}\right)\ddot{\theta}+m\ell\ddot{x}=mg\ell\theta$$

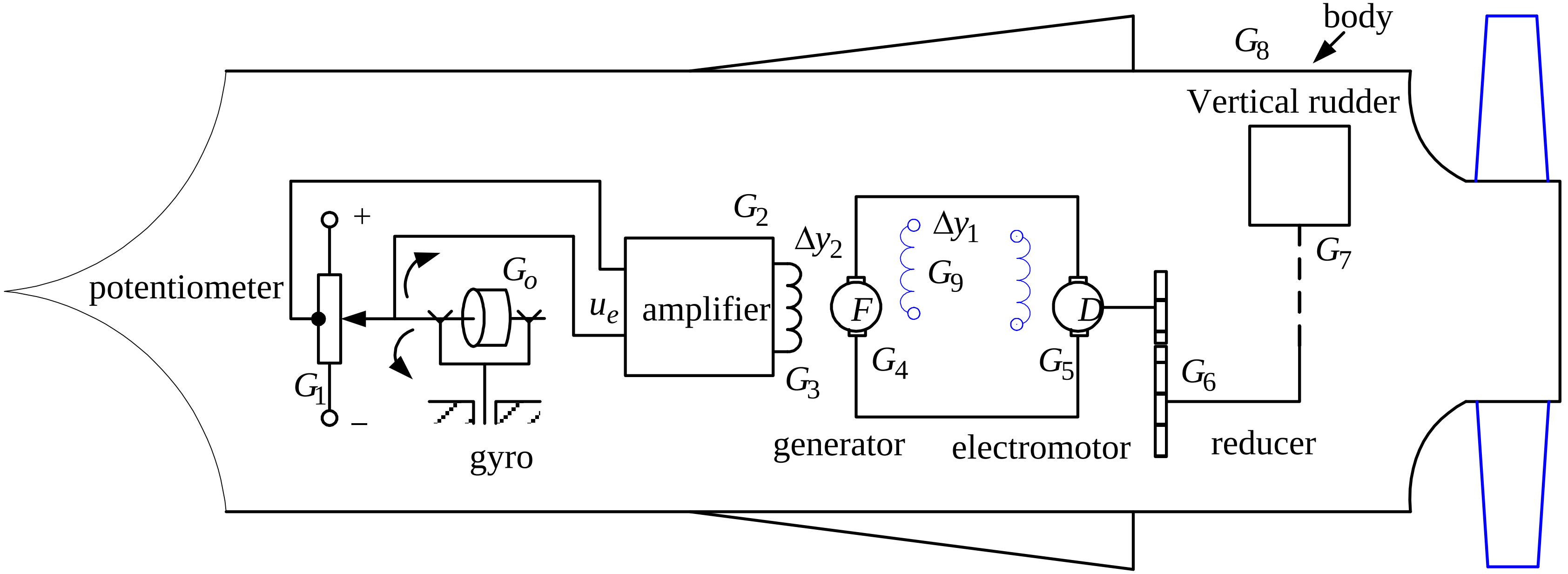
$$G(s)=\frac{\theta(s)}{U(s)}=\frac{-\dfrac{m\ell}{M+m}}{\left[\left(I+m\ell^{2}\right)-\dfrac{m^{2}\ell^{2}}{M+m}\right]s^{2}-mg\ell}$$

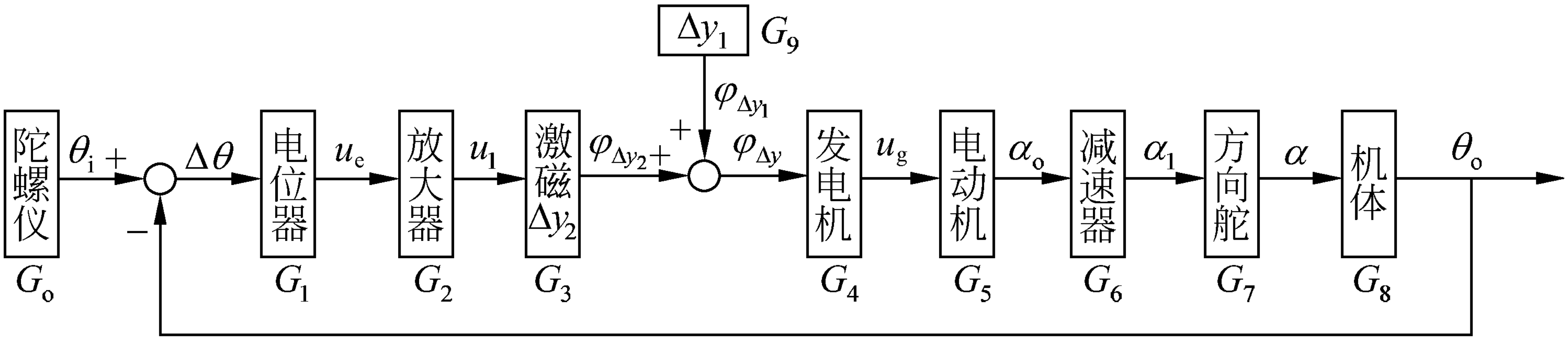
4．比例、积分、微分（亦即PID proportion integral differential）控制器如图所示，试求其传递函数。



$$G(s)=\frac{U_0(s)}{U_i(s)}=K_p+\frac{K_i}{s}+K_d s$$

5. 导弹航向控制系统。试建立以陀螺仪角度为输入量，导弹机体航向角度为输出量的系统传递函数。

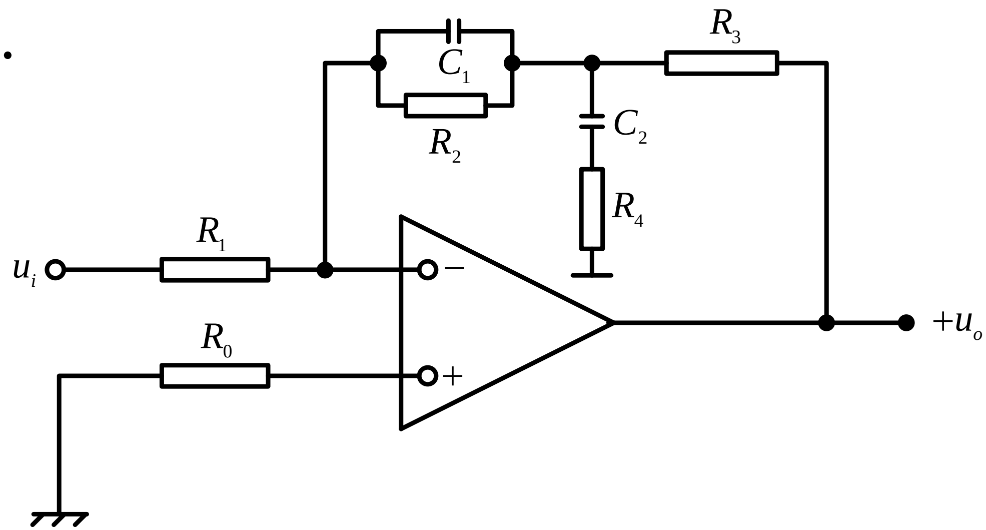




导弹航向控制系统的传递函数为：

$$G(s)=\frac{\theta_o(s)}{\theta_i(s)}=\frac{G'(s)}{1+G'(s)} ? \quad \frac{k}{s(T_m s+1)(T_f s+1)+k}$$

6.



$$Z_1 = \frac{R_2 \cdot \frac{1}{C_1 s}}{R_2 + \frac{1}{C_1 s}} = \frac{R_2}{1 + R_2 C_1 s}$$

$$Z_2 = R_4 + \frac{1}{C_2 s}$$

$$Z = \frac{Z_1 \cdot Z_2}{Z_1 + Z_2} = \frac{R_2\left(1 + R_4 C_2 s\right)}{R_2 C_2 s + \left(1 + R_2 C_1 s\right)\left(1 + R_4 C_2 s\right)}$$

$$\frac{u_i}{R_1} = -\frac{u_o}{R_3 + Z}$$

$$G(s) = \frac{u_o}{u_i} = -\frac{R_3 + Z}{R_1}$$

$$G(s) = -k_1 - \frac{k_2 + k_3 s}{T_1 s^2 + T_2 s + 1}$$

三、传递函数

1. 初始条件非零时系统的传递函数如何求取？

例1：当 $r(t)=1(t)$，$y(t)=1-2e^{-t}+e^{-2t}$ 时，求系统的传递函数$G(s)$。

例2：已知系统初始条件为$y(0)=-1$，$\dot{y}(0)=0$，
在 $t=0$ 给系统加入 $r(t)=1(t)$，响应为
$y(t)=1-4e^{-t}+2e^{-2t}$，求传递函数。

2. 传递函数分解为典型环节，有什么意义？