

# 计算流体力学第二次作业

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## 1 数理算法原理

### 1.1 对于一阶导数 $\frac{\partial u}{\partial x}$ 的差分格式

#### 1.1.1 采用两个网格点的一阶格式

利用泰勒展开式，对于  $u_{i+1}$ ，我们有：

$$u_{i+1} = u_i + \Delta x \frac{\partial u}{\partial x} + O(\Delta x^2) \quad (1)$$

在等式两侧同时除以  $\Delta x$ ，我们可以得到：

$$\frac{u_{i+1} - u_i}{\Delta x} = \frac{\partial u}{\partial x} + O(\Delta x) \quad (2)$$

由此可知，差分格式为：

$$\frac{\partial u}{\partial x} = \frac{u_{i+1} - u_i}{\Delta x} \quad (3)$$

将右侧展开后观察阶数：

$$\begin{aligned} \frac{u_{i+1} - u_i}{\Delta x} &= \frac{u_i + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + O(\Delta x^3) - u_i}{\Delta x} \\ &= \frac{\Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + O(\Delta x^3)}{\Delta x} \\ &= \frac{\partial u}{\partial x} + \frac{\Delta x}{2} \frac{\partial^2 u}{\partial x^2} + O(\Delta x^2) \end{aligned} \quad (4)$$

由于截断误差为一阶，所以，该格式为一阶格式。

#### 1.1.2 采用四个网格点的三阶格式

采用待定系数法，我们采用四个网格点来构造差分格式，设这一格式为：

$$\frac{\partial u}{\partial x} = a_1 u_{i-1} + a_2 u_i + a_3 u_{i+1} + a_4 u_{i+2} \quad (5)$$

将  $u_{i-1}$ 、 $u_{i+1}$ 、 $u_{i+2}$  分别用泰勒展开式展开，得到：

$$\begin{aligned} u_{i-1} &= u_i - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \\ u_{i+1} &= u_i + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \\ u_{i+2} &= u_i + 2\Delta x \frac{\partial u}{\partial x} + 2(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + \frac{4(\Delta x)^3}{3} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \end{aligned} \quad (6)$$

带入 (4) 式中, 得到:

$$\begin{aligned}
 \frac{\partial u}{\partial x} &= a_1 \left( u_i - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \right) + a_2 u_i \\
 &+ a_3 \left( u_i + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \right) \\
 &+ a_4 \left( u_i + 2\Delta x \frac{\partial u}{\partial x} + 2(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + \frac{4(\Delta x)^3}{3} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \right) \\
 &= (a_1 + a_2 + a_3 + a_4)u_i + (-a_1 + a_3 + 2a_4) \Delta x \frac{\partial u}{\partial x} + \left( \frac{a_1}{2} + \frac{a_2}{2} + a_3 + 2a_4 \right) (\Delta x)^2 \frac{\partial^2 u}{\partial x^2} \\
 &+ \left( -\frac{a_1}{6} + \frac{a_2}{6} + \frac{a_3}{6} + \frac{4a_4}{3} \right) (\Delta x)^3 \frac{\partial^3 u}{\partial x^3} + O(\Delta x^3)
 \end{aligned} \tag{7}$$

比较两侧系数得到一下方程组:

$$\begin{cases} a_1 + a_2 + a_3 + a_4 &= 0 \\ \Delta x(-a_1 + a_3 + 2a_4) &= 1 \\ \frac{a_1}{2} + \frac{a_2}{2} + a_3 + 2a_4 &= 0 \\ -\frac{a_1}{6} + \frac{a_2}{6} + \frac{a_3}{6} + \frac{4a_4}{3} &= 0 \end{cases} \tag{8}$$

解得:

$$\begin{cases} a_1 &= -\frac{1}{3\Delta x} \\ a_2 &= -\frac{1}{2\Delta x} \\ a_3 &= \frac{1}{\Delta x} \\ a_4 &= -\frac{1}{6\Delta x} \end{cases} \tag{9}$$

因此, 最终的四网格点差分格式为:

$$\frac{\partial u}{\partial x} = \frac{-2u_{i-1} - 3u_i + 6u_{i+1} - u_{i+2}}{6\Delta x} \tag{10}$$

同理, 将右侧展开后观察阶数可以得知该格式为三阶格式。

## 1.2 对于二阶导数 $\frac{\partial^2 u}{\partial x^2}$ 的差分格式

### 1.2.1 采用三个网格点的一阶格式

利用泰勒展开式, 对于  $u_{i+1}$ , 我们有:

$$u_{i+1} = u_i + \Delta x \frac{\partial u}{\partial x} + \frac{\Delta x^2}{2} \frac{\partial^2 u}{\partial x^2} + O(\Delta x^3) \tag{11}$$

对于  $u_{i+2}$ , 我们有:

$$u_{i+2} = u_i + 2\Delta x \frac{\partial u}{\partial x} + 2(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + O(\Delta x^3) \tag{12}$$

利用 (10) 式和 (11) 式, 消去  $\frac{\partial u}{\partial x}$ , 我们可以得到该差分公式:

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta x)^2} \tag{13}$$

将右侧展开后观察阶数：

$$\begin{aligned}
 \frac{u_{i+2} - 2u_{i+1} + u_i}{(\Delta x)^2} &= \frac{1}{(\Delta x)^2} \cdot [(u_i + 2\Delta x \frac{\partial u}{\partial x} + 2(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + \frac{4(\Delta x)^3}{3} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)) \\
 &\quad - 2(u_i + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)) + u_i] \\
 &= \frac{(\Delta x)^2 \frac{\partial^2 u}{\partial x^2} + (\Delta x)^3 \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)}{(\Delta x)^2} \\
 &= \frac{\partial^2 u}{\partial x^2} + (\Delta x) \frac{\partial^3 u}{\partial x^3} + O(\Delta x^2)
 \end{aligned} \tag{14}$$

因此，该格式为一阶格式。

### 1.2.2 采用三个网格点的二阶格式

同理，首先设定差分格式为：

$$\frac{\partial^2 u}{\partial x^2} = b_1 u_{i-1} + b_2 u_i + b_3 u_{i+1} \tag{15}$$

对于  $u_{i-1}$ 、 $u_{i+1}$ ，我们有：

$$\begin{aligned}
 u_{i-1} &= u_i - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \\
 u_{i+1} &= u_i + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4)
 \end{aligned} \tag{16}$$

带入得到：

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x^2} &= b_1 \left( u_i - \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} - \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \right) \\
 &\quad + b_2 u_i + b_3 \left( u_i + \Delta x \frac{\partial u}{\partial x} + \frac{(\Delta x)^2}{2} \frac{\partial^2 u}{\partial x^2} + \frac{(\Delta x)^3}{6} \frac{\partial^3 u}{\partial x^3} + O(\Delta x^4) \right) \\
 &= (b_1 + b_2 + b_3) u_i + (-b_1 + b_3) \Delta x \frac{\partial u}{\partial x} + \left( \frac{b_1}{2} + \frac{b_3}{2} \right) (\Delta x)^2 \frac{\partial^2 u}{\partial x^2} \\
 &\quad + \left( -\frac{b_1}{6} + \frac{b_3}{6} \right) (\Delta x)^3 \frac{\partial^3 u}{\partial x^3} + O(\Delta x^2)
 \end{aligned} \tag{17}$$

比较两侧系数得到以下方程组：

$$\begin{cases} b_1 + b_2 + b_3 &= 0 \\ -b_1 + b_3 &= 0 \\ (\Delta x)^2 \left( \frac{b_1}{2} + \frac{b_3}{2} \right) &= 1 \\ -\frac{b_1}{6} + \frac{b_3}{6} &= 0 \end{cases} \tag{18}$$

解得：

$$\begin{cases} b_1 &= \frac{1}{\Delta x^2} \\ b_2 &= -2 \frac{1}{\Delta x^2} \\ b_3 &= \frac{1}{\Delta x^2} \end{cases} \tag{19}$$

因此，最终的三网格点差分格式为：

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1} - 2u_i + u_{i+1}}{\Delta x^2} \tag{20}$$

同理，将右侧展开后观察阶数可以得知该格式为二阶格式。

## 2 代码生成与调试

## 3 结果讨论和物理解释