Sample problems from the first lecture (Feb. 22)

1. One provides the following two code designs for event probabilities $\{1/2, 1/4, 1/8, 1/8\}$:

$$\text{Code 1}\left\{\begin{array}{rcl} \text{event one} & : & 0 \\ \text{event two} & : & 1 \\ \text{event three} & : & 00 \\ \text{event four} & : & 11 \end{array}\right. \qquad \text{Code 2}\left\{\begin{array}{rcl} \text{event one} & : & 0 \\ \text{event two} & : & 10 \\ \text{event three} & : & 110 \\ \text{event four} & : & 111 \end{array}\right.$$

$$\begin{array}{cl} & \text{Average codeword length} \\ = & (1/2)\times 1 \text{ bit} + (1/4)\times 1 \text{ bit} \\ + & (1/8)\times 2 \text{ bits} + (1/8)\times 2 \text{ bits} \\ = & 5/4 \text{ bits per event} \end{array} \qquad \begin{array}{cl} & \text{Average codeword length} \\ = & (1/2)\times 1 \text{ bit} + (1/4)\times 2 \text{ bits} \\ + & (1/8)\times 3 \text{ bits} + (1/8)\times 3 \text{ bits} \\ = & 7/4 \text{ bits per event} \end{array}$$

The entropy of the source is

$$\frac{1}{2}\log_2\frac{1}{1/2} + \frac{1}{4}\log_2\frac{1}{1/4} + \frac{1}{8}\log_2\frac{1}{1/8} + \frac{1}{8}\log_2\frac{1}{1/8} = \frac{7}{4} \text{ bits.}$$

Although Code 1 has a smaller average codeword length than Code 2, and even gives an average codeword length less than entropy, please name one potential problem in its practice?

Solution. When Code 1 is used repetitively in a consecutive manner, e.g.,

event one, event three, event two, event three, event four, ..., etc

a punctuation mechanism, such as comma, must be added as

$$0, 00, 1, 00, 11, \ldots$$

Without the punctuation mechanism, it is not possible to recover the source from $00010011\ldots$. As a result, the required average codeword length shall become

$$\frac{5}{4} \text{ bits} + 1 \text{ comma,}$$

and hence Code 1 is more compact than Code 2 only when the length of a comma is smaller than $\frac{1}{2}$ bits.

2. What are the three axioms raised by Shannon for the measurement of information?

Solution.

i) Monotonicity in event probability

ii) Additivity for independent events

iii) Continuity in event probability

3. Verify the following equation:

$$\lim_{t \downarrow 0} \frac{1}{t} \log_2 \left(\sum_{i=1}^{L} p_i \, 2^{t \cdot \ell_i} \right) \quad = \quad \sum_{i=1}^{L} p_i \cdot \ell_i$$

Do we need the condition that $\sum_{i=1}^{L} p_i = 1$ for the validity of the above equation?

Solution.

$$\begin{aligned}
& \lim_{t \downarrow 0} \frac{\log_2 \left(\sum_{i=1}^{L} p_i \, 2^{t \cdot \ell_i} \right)}{t} \\
&= \lim_{t \downarrow 0} \frac{\ln \left(\sum_{i=1}^{L} p_i \, e^{t \cdot \ell_i \cdot \ln(2)} \right)}{t \ln(2)} && (1) \\
&= \lim_{t \downarrow 0} \frac{\frac{\left(\sum_{i=1}^{L} p_i \, e^{t \cdot \ell_i \cdot \ln(2)} \right)'}{\left(\sum_{i=1}^{L} p_i \, e^{t \cdot \ell_i \cdot \ln(2)} \right)}}{\ln(2)} \quad \text{By L'Hospital's Rule} && (2) \\
&= \lim_{t \downarrow 0} \frac{\sum_{i=1}^{L} p_i \, \ell_i \, \ln(2) \, e^{t \cdot \ell_i \cdot \ln(2)}}{\ln(2) \sum_{i=1}^{L} p_i \, e^{t \cdot \ell_i \cdot \ln(2)}} && (3) \\
&= \frac{\sum_{i=1}^{L} p_i \, \ell_i \, \ln(2)}{\ln(2) \sum_{i=1}^{L} p_i} && (4) \\
&= \sum_{i=1}^{L} p_i \, \ell_i . && (5)
\end{aligned}$$

In order to apply L'Hospital's Rule, we do need the condition that $\sum_{i=1}^{L} p_i = 1$.

4. If a channel block code maps 3 information bits to 7 channel code bits, what is the code rate of this channel block code? If the channel capacity is 0.5 information bits per channel usage, is the code rate a reliable transmission rate?

Solution. The code rate is $3/7$. Since it is less than $1/2$, it is a reliable transmission code rate.