Sample problems for the second lecture (Mar. 8)

1. Let $a_n = (-1)^n \cdot (1 + \frac{1}{n})$ and $\epsilon = 0.1$.

   (a) Find
   $$b_n := \sup_{k \geq n} a_k = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$
   Is $\{b_n\}$ a **monotonic** sequence?

   (b) Find the value of $\limsup_{n\to\infty} a_n$.

   (c) Show that $a_n < \limsup_{m\to\infty} a_m + \epsilon$ for **sufficiently large** n.

   (d) Show that $a_n > \limsup_{m\to\infty} a_m - \epsilon$ for **infinitely many** n.

   (e) Find
   $$c_n := \inf_{k \geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$
   Is $\{c_n\}$ a **monotonic** sequence?

   (f) Find the value of $\liminf_{n\to\infty} a_n$.

   (g) Show that $a_n > \liminf_{m\to\infty} a_m - \epsilon$ for **sufficiently large** n.

   (h) Show that $a_n < \liminf_{m\to\infty} a_m + \epsilon$ for **infinitely many** n.

Solution.

(a) $\{b_n\} = \left\{\sup\limits_{k\geq n} a_k\right\}$ is a monotonically non-increasing sequence by its definition. With $a_n = (-1)^n \cdot (1 + \frac{1}{n})$, we obtain

$$\begin{aligned} b_n &:= \sup_{k\geq n} a_k = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\} \\ &= \begin{cases} 1 + \frac{1}{n}, & n \text{ even} \\ 1 + \frac{1}{n+1}, & n \text{ odd} \end{cases} \\ &= 1 + \frac{1}{2\lceil \frac{n}{2} \rceil}. \end{aligned}$$

This confirms that $\{b_n\}$ is monotonically non-increasing (not strictly decreasing).

(b)

$$\begin{aligned} \limsup_{n\to\infty} a_n &= \lim_{n\to\infty} \left(\sup_{k\geq n} a_n\right) \quad \text{(By definition of limsup)} \\ &= \lim_{n\to\infty} b_n \\ &\quad \text{(The limit always exists by the monotone convergence} \\ &\quad \text{ theorem over } \mathbb{R}\cup\{\pm\infty\}.) \\ &= 1. \end{aligned}$$

(c)

$$a_n < \limsup_{m\to\infty} a_m + \epsilon \Leftrightarrow (-1)^n \cdot \left(1 + \frac{1}{n}\right) < 1 + 0.1$$

which is true for **all** $n \geq 11$, i.e., for sufficiently large n.

(d)

$$a_n > \limsup_{m\to\infty} a_m - \epsilon \Leftrightarrow (-1)^n \cdot \left(1 + \frac{1}{n}\right) > 1 - 0.1 = 0.9.$$

Hence, for a given (positive) integer K, there exists **one** $N > K$ such that $a_n > 0.9$ (e.g., $N = 2\lceil\frac{K+1}{2}\rceil$).

(e) $\{b_n\} = \left\{\inf\limits_{k\geq n} a_k\right\}$ is a monotonically non-decreasing sequence by its definition. With $a_n = (-1)^n \cdot (1 + \frac{1}{n})$, we obtain

$$\begin{aligned} c_n &:= \sup_{k\geq n} a_k = \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} \\ &= \begin{cases} -\left(1+\frac{1}{n}\right), & n \text{ odd} \\ -\left(1+\frac{1}{n+1}\right), & n \text{ even} \end{cases} \\ &= -\left(1 + \frac{1}{2\lfloor\frac{n}{2}\rfloor + 1}\right). \end{aligned}$$

This confirms that $\{c_n\}$ is monotonically non-decreasing (not strictly increasing).

(f)

$$\begin{aligned} \liminf_{n\to\infty} a_n &= \lim_{n\to\infty}\left(\inf_{k\geq n} a_n\right) \quad \text{(By definition of limsup)} \\ &= \lim_{n\to\infty} c_n \\ &\quad \text{(The limit always exists by the monotone convergence} \\ &\quad \text{ theorem over } \mathbb{R}\cup\{\pm\infty\}.) \\ &= -1. \end{aligned}$$

(g)

$$a_n > \liminf_{m\to\infty} a_m - \epsilon \Leftrightarrow (-1)^n \cdot \left(1 + \frac{1}{n}\right) > -1 - 0.1$$

which is true for **all** $n \geq 10$, i.e., for sufficiently large n.

(h)

$$a_n < \liminf_{m\to\infty} a_m + \epsilon \Leftrightarrow (-1)^n \cdot \left(1+\frac{1}{n}\right) < -1 + 0.1 = -0.9.$$

Hence, for a given (positive) integer K, there exists **one** $N > K$ such that $a_n < -0.9$ (e.g., $N = 2\lceil \frac{K+1}{2} \rceil + 1$).

2. Give an example of $\{a_n\}$ and $\{b_n\}$ such that the following strict inequalities hold:

$$\begin{aligned}
\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n &< \liminf_{n\to\infty}(a_n + b_n) \\
&< \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} b_n \\
&< \limsup_{n\to\infty}(a_n + b_n) \\
&< \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.
\end{aligned}$$

Solution. Let $a_n = (-1)^n$. Then, the above inequalities become:

$$\begin{aligned}
-1 + \liminf_{n\to\infty} b_n &< \liminf_{n\to\infty}((-1)^n + b_n) \\
&< 1 + \liminf_{n\to\infty} b_n \\
&< \limsup_{n\to\infty}((-1)^n + b_n) \\
&< 1 + \limsup_{n\to\infty} b_n,
\end{aligned}$$

which are equivalent to:

$$\begin{aligned}
-1 + \liminf_{n\to\infty} b_n &< \min\left\{ \liminf_{m\to\infty}(-1 + b_{2m+1}), \liminf_{m\to\infty}(1 + b_{2m}) \right\} \\
&< 1 + \liminf_{n\to\infty} b_n \\
&< \max\left\{ \limsup_{m\to\infty}(-1 + b_{2m+1}), \limsup_{m\to\infty}(1 + b_{2m}) \right\} \\
&< 1 + \limsup_{n\to\infty} b_n.
\end{aligned}$$

These can be equivalently rewritten as:

$$\begin{aligned}
-1 + \liminf_{n\to\infty} b_n &< \min\left\{ -1 + \liminf_{m\to\infty} b_{2m+1}, 1 + \liminf_{m\to\infty} b_{2m} \right\} && (1) \\
&< 1 + \liminf_{n\to\infty} b_n && (2) \\
&< \max\left\{ -1 + \limsup_{m\to\infty} b_{2m+1}, 1 + \limsup_{m\to\infty} b_{2m} \right\} && (3) \\
&< 1 + \limsup_{n\to\infty} b_n. && (4)
\end{aligned}$$

Since

$$\liminf_{n\to\infty} b_n = \min\Big\{\liminf_{m\to\infty} b_{2m+1}, \liminf_{m\to\infty} b_{2m}\Big\},$$

the validations of (1) and (2) require

$$\underbrace{\liminf_{n\to\infty} b_n}_{=A} = \liminf_{m\to\infty} b_{2m} < \underbrace{\liminf_{m\to\infty} b_{2m+1}}_{=B}$$

and

$$\begin{aligned} -1+A &< \min\Big\{-1+B, 1+A\Big\} = -1+B \\ &< 1+A. \end{aligned}$$

A solution is to let $B = 1 + A$.

Similarly, we note that

$$\limsup_{n\to\infty} b_n = \max\Big\{\limsup_{m\to\infty} b_{2m+1}, \limsup_{m\to\infty} b_{2m}\Big\},$$

for which the validation of (4) requires

$$\underbrace{\limsup_{n\to\infty} b_n}_{=C} = \limsup_{m\to\infty} b_{2m+1} > \underbrace{\limsup_{m\to\infty} b_{2m}}_{=D}$$

and

$$\max\{-1+C, 1+D\} < 1+C.$$

A solution is to let $C = 1 + D$.

As a consequence, any $\{b_n\}$ satisfying

$$\begin{cases} \liminf_{m\to\infty} b_{2m} = A < \limsup_{m\to\infty} b_{2m} = D \\ \liminf_{m\to\infty} b_{2m+1} = 1+A < \limsup_{m\to\infty} b_{2m+1} = 1+D \end{cases}$$

is the solution.

3. Suppose the probability space is given as

$$\begin{aligned} \Omega &= \{\blacktriangle, \blacktriangledown, \square, \blacksquare, \lozenge, \blacklozenge\} \\ \mathcal{F} &= \{\text{all subsets of } \Omega\} = \text{powerset of } \Omega \\ P &= \begin{cases} P(\blacktriangle) = 0.05, P(\blacktriangledown) = 0.1, P(\square) = 0.15, \\ P(\blacksquare) = 0.2, P(\lozenge) = 0.25, P(\blacklozenge) = 0.25 \end{cases} \end{aligned}$$

Give a random process $X_1, X_2, X_3, \ldots$, each of which is defined over the above probability space. Let

$$\begin{array}{rclrclrclrcl}
X_3(\blacktriangle) & = & 1; & X_3(\blacksquare) & = & 2; & X_7(\blacktriangle) & = & 1; & X_7(\blacksquare) & = & 2 \\
X_3(\blacktriangledown) & = & 2; & X_3(\Diamond) & = & 1; & X_7(\blacktriangledown) & = & 1; & X_7(\Diamond) & = & 2 \\
X_3(\square) & = & 1; & X_3(\blacklozenge) & = & 2; & X_7(\square) & = & 1; & X_7(\blacklozenge) & = & 2
\end{array}$$

Then the distributions of X_3 and X_7, as well as the joint distribution between X_3 and X_7, are well defined (by simply providing the mappings sperately) as indicated in the following.

(a) Find $\Pr[X_3 = 1]$.

(b) Find $\Pr[X_7 = 2]$.

(c) Find $\Pr[X_3 = 1 \text{ and } X_7 = 2]$.

Solution.

(a)

$$\begin{aligned}
\Pr[X_3 = 1] &= P(\{\omega \in \Omega : X_3(\omega) = 1\}) \quad \text{(By the function mapping of } X_3\text{)} \\
&\quad \text{(We can deal with "=" because } X_3 \text{ is a real-valued function mapping.)} \\
&= P(\{\blacktriangle, \square, \blacklozenge\}) \quad \text{(Back to the probability measure } P\text{)} \\
&= P(\{\blacktriangle\} \cup \{\square\} \cup \{\Diamond\}) \\
&= P(\{\blacktriangle\}) + P(\{\square\}) + P(\{\Diamond\}) \quad \text{(Axiom 3 of probability measure: Countable additivity for disjoint sets)} \\
&= 0.05 + 0.15 + 0.25 \\
&= 0.45.
\end{aligned}$$

(b)

$$\begin{aligned}
\Pr[X_7 = 2] &= P(\{\omega \in \Omega : X_7(\omega) = 2\}) \quad \text{(By the function mapping of } X_7\text{)} \\
&\quad \text{(We can deal with "=" because } X_7 \text{ is a real-valued function mapping.)} \\
&= P(\{\blacksquare, \Diamond, \blacklozenge\}) \quad \text{(Back to the probability measure } P\text{)} \\
&= P(\{\blacksquare\} \cup \{\Diamond\} \cup \{\blacklozenge\}) \\
&= P(\{\blacksquare\}) + P(\{\Diamond\}) + P(\{\blacklozenge\}) \quad \text{(Axiom 3 of probability measure: Countable additivity for disjoint sets)} \\
&= 0.2 + 0.25 + 0.25 \\
&= 0.7.
\end{aligned}$$

(c)

$$\begin{aligned}
\Pr[X_3 = 1 \text{ and } X_7 = 2] &= P(\{\omega \in \Omega : X_3(\omega) = 1 \text{ and } X_7(\omega) = 2\}) \\
&= P(\{\omega \in \Omega : X_3(\omega) = 1\} \cap \{\omega \in \Omega : X_7(\omega) = 2\}) \\
&= P(\{\blacktriangle, \square, \blacklozenge\} \cap \{\blacksquare, \lozenge, \blacklozenge\}) \\
&= P(\{\blacklozenge\}) \\
&= 0.25.
\end{aligned}$$

4. An event E is **ergodic** if $\mathbb{T}^{-1}(E) = E$. A random process is ergodic if all ergodic events are either with probability one or with probability zero.

   (a) For events over two-sided sequences, the inverse mapping $\mathbb{T}^{-1}$ can be defined as:

   $$\mathbb{T}^{-1}(\boldsymbol{x}) = \mathbb{T}^{-1}(\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) = \ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots,$$

   with $y_n = x_{n-1}$. Give the ergodic set E of two-sided sequences over $\{0,1\}^\infty$, which contain

   $$\ldots, 0, 1, 0, 1, 0, 1, \ldots.$$

   List all possible values of

   $$\lim_{n\to\infty} \frac{x_{-n} + \cdots + x_{-2} + x_{-1} + x_0 + x_1 + x_2 + \cdots + x_n}{2n+1}$$

   for $\boldsymbol{x} \in E$. Is

   $$\cdots = \mathbb{T}^{-2}(E) = \mathbb{T}^{-1}(E) = E = \mathbb{T}(E) = \mathbb{T}^2(E) = \cdots$$

   i.e., the set stays the same as time varies.

   (b) For events over one-sided sequences as considered in most communication systems, the inverse (time-shift) mapping $\mathbb{T}^{-1}$ in general does not exist. As an extension, Shields [P. C. Shields 1991, p. 3] adopted the definition as the following:

   $$\mathbb{T}^{-1}E := \{\boldsymbol{x} \in \mathcal{X}^\infty : \mathbb{T}\boldsymbol{x} \in E\}$$

   which includes all right-shift counterparts of "transient" elements into E (and is consistent with the definition of $\mathbb{T}^{-1}$ for sets over two-sided sequences). Give the ergodic set E of one-sided sequences over $\{0,1\}^\infty$, which contain

$$0,1,0,1,0,1,\ldots.$$

List all possible values of

$$\lim_{n\to\infty}\frac{x_1+x_2+\cdots+x_n}{n}$$

for $\boldsymbol{x}=(x_1,x_2,x_3,\ldots)\in E$. Is

$$\cdots=\mathbb{T}^{-2}(E)=\mathbb{T}^{-1}(E)=E=\mathbb{T}(E)=\mathbb{T}^2(E)=\cdots$$

i.e., the set stays the same as time varies.

Solution.

(a)

$$E=\{\boldsymbol{x}\in\{0,1\}^\infty:(\forall\ n)\,x_n=x_{n+2}\text{ and }x_n\neq x_{n+1}\}$$

and the limit is unique for all elements in E and is equal to $\frac{1}{2}$. The set definitely stays the same as time varies.

(b)

$$E=\bigcup_{k=1}^{\infty}E_k,$$

where

$$E_k=\{\boldsymbol{x}\in\{0,1\}^\infty:(\forall\ n\geq k)\,x_n=x_{n+2}\text{ and }x_n\neq x_{n+1}\}.$$

For example,

$$E_1=\{010101\ldots,101010\ldots\},$$

$$E_2 = \{0010101\ldots,1010101\ldots,0101010\ldots,1101010\ldots\},$$

and

$$\begin{aligned}E_3 = \{&00010101\ldots,01010101\ldots,10010101\ldots,11010101\ldots,\\ &00101010\ldots,01101010\ldots,10101010\ldots,11101010\ldots\}.\end{aligned}$$

For $\boldsymbol{x}\in E_k$ (with k fixed), we obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{x_1+\cdots+x_{k-1}+x_k+\cdots+x_n}{n}\\ &=\lim_{n\to\infty}\left(\underbrace{\frac{x_1+\cdots+x_{k-1}}{n}}_{\text{go to zero as }n\to\infty}+\frac{x_k+\cdots+x_n}{n}\right)\\ &=\lim_{n\to\infty}\left(\frac{\frac{n-k+1}{2}}{n}\right)=\frac{1}{2}.\end{aligned}$$

Thus, the limit is unique for all elements in E and is equal to $\frac{1}{2}$. This set definitely stays the same as time varies (either by "chopping" as $\mathbb{T}$ or by "enlarging" as $\mathbb{T}^{-1}$).

5. Suppose Jack and Mary separately make consecutive observations on the same phenomenon in a one-observation-at-a-time manner, and then compute the time averages based on their own observations. Answer the following two questions for each of the subproblems.

   - Does it guarantee that Jack and Mary will conclude almost the same average value as the number of observations they make grows very large?
   - Does it guarantee that their average value equals the ensemble average (i.e., expected value of the phenomenon) as the number of observations tends to infinity?

   (a) The observations on the phenomenon constitute an ergodic process.

   (b) The observations on the phenomenon constitute a stationary process.

   (c) The observations on the phenomenon constitute a stationary-ergodic process.

   (d) The observations on the phenomenon follow the law of large numbers.

   (e) The observations on the phenomenon constitute an independent and identically distributed (i.i.d.) random process.

   (f) The observations on the phenomenon constitute a memoryless random process.

   (g) The observations on the phenomenon constitute a Markov process.

   Solution.

   (a) Yes for the first question but no for the second; in other words, the time average may not equal to the ensemble average.

   (b) No for both questions; in other words, Jack and Mary may obtain very different averages even if both accurately and professionally do their measurements.

   (c) Yes for both questions.

(d) Yes for both questions. This is exactly the behavior dictated by the law of large numbers, i.e., the time average converges to the ensemble average.

(e) Yes for both questions as an i.i.d. random process is stationary ergodic.

(f) Yes for both questions as "memoryless" is an alternative name for "i.i.d."

(g) Not necessarily for both questions since a Markov process can be neither stationary nor ergodic. Note that an irreducible, homogeneous, aperiodic, finite-state Markov process with stationary initial probability is stationary ergodic.