Sample problems for the 10th lecture (May 17)

1. Let $Y_g = X_g + Z_g$ and $Y = X_g + Z$. Assume that X_g and Z_g are independent, and X_g and Z are also independent. In addition, Z and Z_g are zero-mean random variables with the same variance, and both X_g and Z_g are Gaussian distributed.

   (a) Prove that

$$\begin{aligned}
&\int_{\mathcal{X}} \int_{\mathcal{Y}} f_{X_g}(x) f_{Z_g}(y-x) \log_2 \frac{f_{Z_g}(y-x)}{f_{Y_g}(y)} dy dx \\
&= \int_{\mathcal{X}} \int_{\mathcal{Y}} f_{X_g}(x) f_{Z}(y-x) \log_2 \frac{f_{Z_g}(y-x)}{f_{Y_g}(y)} dy dx.
\end{aligned}$$

   (b) $I(f_{X_g}, f_{Y|X}) \geq I(f_{X_g}, f_{Y_g|X_g})$.
   Hint: Use (a) directly.

Solution.

(a) Noting that $Z_g \sim \mathcal{N}(0, \sigma^2_{Z_g})$ and $Y_g \sim \mathcal{N}(\mu_{X_g}, \sigma^2_{Y_g})$, we derive

$$
\begin{aligned}
&\int_{\mathcal{X}}\int_{\mathcal{Y}} f_{X_g}(x) f_{Z_g}(y-x)\log_2\frac{f_{Z_g}(y-x)}{f_{Y_g}(y)}dydx \\
&= \int_{\mathcal{X}}\int_{\mathcal{Y}} f_{X_g}(x) f_{Z_g}(y-x)\log_2\frac{\frac{1}{\sqrt{2\pi\sigma^2_{Z_g}}}\exp\left(-\frac{(y-x)^2}{2\sigma^2_{Z_g}}\right)}{\frac{1}{\sqrt{2\pi\sigma^2_{Y_g}}}\exp\left(-\frac{(y-\mu_{X_g})^2}{2\sigma^2_{Y_g}}\right)}dydx \\
&= \int_{\mathcal{X}}\int_{\mathcal{Y}} f_{X_g}(x) f_{Z_g}(y-x)\left(\frac{1}{2}\log_2\left(\frac{\sigma^2_{Y_g}}{\sigma^2_{Z_g}}\right)+\log_2(e)\left(\frac{(y-\mu_{X_g})^2}{2\sigma^2_{Y_g}}-\frac{(y-x)^2}{2\sigma^2_{Z_g}}\right)\right)dydx \\
&= \frac{1}{2}\log_2\left(\frac{\sigma^2_{Y_g}}{\sigma^2_{Z_g}}\right)+\log_2(e)\left(\frac{E[(Y_g-\mu_{X_g})^2]}{2\sigma^2_{Y_g}}-\frac{E[(Y_g-X_g)^2]}{2\sigma^2_{Z_g}}\right) \\
&= \frac{1}{2}\log_2\left(\frac{\sigma^2_{Y_g}}{\sigma^2_{Z_g}}\right)+\log_2(e)\left(\underbrace{\frac{E[(Y-\mu_{X_g})^2]}{2\sigma^2_{Y_g}}}_{\substack{\text{Variances of } Y_g-\mu_{X_g}=X_g-\mu_{X_g}+Z_g \\ \text{and } Y-\mu_{X_g}=X_g-\mu_{X_g}+Z \text{ are the same.}}}-\underbrace{\frac{E[(Y-X_g)^2]}{2\sigma^2_{Z_g}}}_{\substack{\text{Variances of } Y_g-X_g=Z_g \\ \text{and } Y-X_g=Z \text{are the same.}}}\right) \\
&= \int_{\mathcal{X}}\int_{\mathcal{Y}} f_{X_g}(x) f_{Z}(y-x)\left(\frac{1}{2}\log_2\left(\frac{\sigma^2_{Y_g}}{\sigma^2_{Z_g}}\right)+\log_2(e)\left(\frac{(y-\mu_{X_g})^2}{2\sigma^2_{Y_g}}-\frac{(y-x)^2}{2\sigma^2_{Z_g}}\right)\right)dydx \\
&= \int_{\mathcal{X}}\int_{\mathcal{Y}} f_{X_g}(x) f_{Z}(y-x)\log_2\frac{\frac{1}{\sqrt{2\pi\sigma^2_{Z_g}}}\exp\left(-\frac{(y-x)^2}{2\sigma^2_{Z_g}}\right)}{\frac{1}{\sqrt{2\pi\sigma^2_{Y_g}}}\exp\left(-\frac{(y-\mu_{X_g})^2}{2\sigma^2_{Y_g}}\right)}dydx \\
&= \int_{\mathcal{X}}\int_{\mathcal{Y}} f_{X_g}(x) f_{Z}(y-x)\log_2\frac{f_{Z_g}(y-x)}{f_{Y_g}(y)}dydx
\end{aligned}
$$

(b) See Slide 5-56.

2. *An alternative form of the entropy-power inequality:* Show that the entropy-power inequality in (5.7.2) can be written as

$$h(Z_1 + Z_2) \geq h(Y_1 + Y_2)$$

where Z_1 and Z_2 are two independent continuous random variables, and Y_1 and Y_2 are two independent Gaussian random variables such that

$$h(Y_1) = h(Z_1) \quad \text{and} \quad h(Y_2) = h(Z_2).$$

Solution. First note that since $h(Z_1) = h(Y_1)$ and Y_1 is Gaussian, we have

$$h(Z_1) = h(Y_1) = \frac{1}{2}\log_2(2\pi e \mathrm{Var}(Y_1))$$

and thus

$$2\pi e \mathrm{Var}(Y_1) = 2^{2h(Z_1)}.$$

Similarly, we have

$$2\pi e \mathrm{Var}(Y_2) = 2^{2h(Z_2)}.$$

Furthermore note that since Y_1 and Y_2 are independent Gaussians, we have that

$$h(Y_1 + Y_2) = \frac{1}{2}\log_2\left[2\pi e\left(\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)\right)\right],$$

which is equivalent to

$$2^{h(Y_1+Y_2)} = 2\pi e\left(\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)\right).$$

Now the entropy-power inequality in (5.7.2) gives

$$2^{2h(Z_1+Z_2)} \geq 2^{2h(Z_1)} + 2^{2h(Z_2)}.$$

Thus

$$\begin{aligned} 2^{2h(Z_1+Z_2)} &\geq 2\pi e \mathrm{Var}(Y_1) + 2\pi e \mathrm{Var}(Y_2) \\ &= 2\pi e\left(\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2)\right) \\ &= 2^{h(Y_1+Y_2)}. \end{aligned}$$

Therefore

$$h(Z_1 + Z_2) \geq h(Y_1 + Y_2).$$

Note that equality holds when Z_1 and Z_2 are Gaussians.

3. Suppose the capacity of k parallel channels, where the ith channel uses power P_i, follows a logarithm law, i.e.,

$$C(P_1, \ldots, P_k) = \sum_{i=1}^{k} \frac{1}{2}\log_2\left(1 + \frac{P_i}{\sigma_i^2}\right),$$

where σ_i^2 is the noise variance of channel i. Determine the optimal $\{P_i^*\}_{i=1}^k$ that maximize $C(P_1, \ldots, P_k)$ subject to $\sum_{i=1}^k P_i = P$.

Solution. By using the Lagrange multipliers technique and verifying the KKT condition, the maximizer $(P_1, \ldots, P_k)$ of

$$\max \left\{ \sum_{i=1}^{k} \frac{1}{2} \log_2 \left(1 + \frac{P_i}{\sigma_i^2}\right) + \sum_{i=1}^{k} \lambda_i P_i - \nu \left(\sum_{i=1}^{k} P_i - P \right) \right\}$$

can be found by taking the derivative of the above equation (with respect to P_i) and setting it to zero, which yields

$$\lambda_i = \begin{cases} -\dfrac{1}{2\ln(2)} \dfrac{1}{P_i + \sigma_i^2} + \nu = 0, & \text{if } P_i > 0; \\ -\dfrac{1}{2\ln(2)} \dfrac{1}{P_i + \sigma_i^2} + \nu \geq 0, & \text{if } P_i = 0. \end{cases}$$

Hence,

$$\begin{cases} P_i = \theta - \sigma_i^2, & \text{if } P_i > 0; \\ P_i \geq \theta - \sigma_i^2, & \text{if } P_i = 0, \end{cases} \qquad (\text{equivalently, } P_i = \max\{0, \theta - \sigma_i^2\}),$$

where $\theta := \log_2(e)/(2\nu)$ is chosen to satisfy $\sum_{i=1}^{k} P_i = P$. □

4. Lemma 5.42 states the following.

   Lemma 5.42 For a discrete-time continuous-alphabet memoryless additive-noise channel with input power constraint P and noise variance σ^2, its capacity satisfies

$$\frac{1}{2} \log_2 \frac{P + \sigma^2}{Z_e} \geq C(P) \geq \frac{1}{2} \log_2 \frac{P + \sigma^2}{\sigma^2}.$$

   The proof of the upper bound follows from

$$I(X;Y) = h(Y) - h(Z) \leq \frac{1}{2} \log_2[2\pi e(P + \sigma^2)] - \frac{1}{2} \log_2[2\pi e Z_e].$$

   What is the distribution of Y that equates the upper bound? Justify your answer.

   Solution. Equality of the upper bound holds iff

$$h(Y) = \frac{1}{2} \log_2[2\pi e(P + \sigma^2)],$$

   which is valid iff Y is Gaussian distributed with variance $P + \sigma^2$.