Sample problems for the 8th lecture (May 3)

1. Give three channel transition matrices $[P_{Y|X}(y|x) = p_{x,y}]$:

$$\mathbb{Q}_1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \\ a_3 & a_1 & a_2 \end{bmatrix}, \quad \mathbb{Q}_2 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_1 \\ a_1 & a_3 & a_2 \end{bmatrix}, \text{ and } \quad \mathbb{Q}_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

   (a) Is $\mathbb{Q}_1$ a symmetric channel? Is it a weakly symmetric channel? Is it a quasi-symmetric channel? Is it a T-symmetric channel?

   Hint: A DMC is *symmetric* if rows of $\mathbb{Q}$ are permutations of each other and columns of $\mathbb{Q}$ are permutations of each other. A DMC is *weakly-symmetric* if rows of $\mathbb{Q}$ are permutations of each other and all the column sums are equal. A DMC is *quasi-symmetric* if $\mathbb{Q}$ can be partitioned along its columns into weakly-symmetric sub-matrices.

   (b) Assign a set of values of a_1, a_2 and a_3 such that $\mathbb{Q}_2$ is T-symmetric but not quasi-symmetric.

   Hint: A channel is T-symmetric if

$$T(x) = \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log_2 \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_{Y|X}(y|x')}$$

   is a constant function of x.

   (c) Find the channel capacity of $\mathbb{Q}_3$.

   Hint: The channel capacity C satisfies

$$\begin{cases} I(x;Y) = C & \forall x \in \mathcal{X} \text{ with } P_{X^*}(x) > 0; \\ I(x;Y) \le C & \forall x \in \mathcal{X} \text{ with } P_{X^*}(x) = 0, \end{cases}$$

   where

$$I(x;Y) = \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \log_2 \frac{P_{Y|X}(y|x)}{P_Y(y)}.$$

Solution.

   (a) Rows of $\mathbb{Q}_1$ are permutations of each other and columns of $\mathbb{Q}_1$ are permutations of each other. Hence, it is a symmetric channel. A symmetric channel must be weakly symmetric, quasi-symmetric and T-symmetric.

(b) First, we note that $a_1 + a_2 + a_3 = 1$. Denote (without loss of generality) that $\mathcal{X} = \mathcal{Y} = \{0, 1, 2\}$. Then,

$$\begin{cases} \sum_{x'\in\{0,1,2\}} P_{Y|X}(0|x') = 2a_1 + a_2 \\ \sum_{x'\in\{0,1,2\}} P_{Y|X}(1|x') = a_2 + 2a_3 \\ \sum_{x'\in\{0,1,2\}} P_{Y|X}(2|x') = a_1 + a_2 + a_3 = 1 \end{cases}$$

$$\begin{aligned} T(0) &= \sum_{y\in\{0,1,2\}} P_{Y|X}(y|0) \log_2 \frac{P_{Y|X}(y|0)}{\sum_{x'\in\{0,1,2\}} P_{Y|X}(y|x')} \\ &= a_1 \log_2 \frac{a_1}{2a_1 + a_2} + a_2 \log_2 \frac{a_2}{a_2 + 2a_3} + a_3 \log_2(a_3) \end{aligned}$$

$$\begin{aligned} T(1) &= \sum_{y\in\{0,1,2\}} P_{Y|X}(y|1) \log_2 \frac{P_{Y|X}(y|1)}{\sum_{x'\in\{0,1,2\}} P_{Y|X}(y|x')} \\ &= a_2 \log_2 \frac{a_2}{2a_1 + a_2} + a_3 \log_2 \frac{a_3}{a_2 + 2a_3} + a_1 \log_2(a_1) \end{aligned}$$

$$\begin{aligned} T(2) &= \sum_{y\in\{0,1,2\}} P_{Y|X}(y|2) \log_2 \frac{P_{Y|X}(y|2)}{\sum_{x'\in\{0,1,2\}} P_{Y|X}(y|x')} \\ &= a_1 \log_2 \frac{a_1}{2a_1 + a_2} + a_3 \log_2 \frac{a_3}{a_2 + 2a_3} + a_2 \log_2(a_2) \end{aligned}$$

Thus, $T(0) = T(1) = T(2)$ holds for any $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 = 1$.

Regarding the quasi-symmetry, if we partition the columns into two sets, then one of the sets should contain only one column. As a result, if we set $a_1 = a_2$ or ~~$a_2 = a_3$~~ or $a_1 = a_3$, $\mathbb{Q}_2$ becomes quasi-symmetric.

Hence, we must set $a_1 \neq a_2$ ~~and $a_2 \neq a_3$~~ and $a_1 \neq a_3$ in order to make $\mathbb{Q}_2$ T-symmetric but not quasi-symmetric. For example, $a_1 = 0.2$, $a_2 = 0.3$ and $a_3 = 0.5$.

(c) For notational convenience, we set $P_X(x) = p_x$ for $x \in \{0, 1, 2\}$. Then, there are four possibilities, i.e.,

$$\begin{cases} \text{Case 1: } I(0;Y) = I(1;Y) = I(2;Y) = C \\ \text{Case 2: } I(1;Y) = I(2;Y) = C \geq I(0;Y) \\ \text{Case 3: } I(0;Y) = I(2;Y) = C \geq I(1;Y) \\ \text{Case 4: } I(0;Y) = I(1;Y) = C \geq I(2;Y) \end{cases}$$

Note that it is not possible to have, e.g., $I(0;Y) = C$, $I(1;Y) \leq C$ and $I(2;Y) \leq C$ because with $p_0 = 1$, $p_1 = 0$ and $p_2 = 0$, the capacity would be zero.

Case 1:

$$I(0;Y) = \sum_{y\in\mathcal{Y}} P_{Y|X}(y|0)\log_2\frac{P_{Y|X}(y|0)}{P_Y(y)} = \frac{1}{2}\log_2\frac{\frac{1}{2}}{P_Y(0)} + \frac{1}{2}\log_2\frac{\frac{1}{2}}{P_Y(1)}$$

$$I(1;Y) = \sum_{y\in\mathcal{Y}} P_{Y|X}(y|0)\log_2\frac{P_{Y|X}(y|0)}{P_Y(y)} = \frac{1}{2}\log_2\frac{\frac{1}{2}}{P_Y(0)} + \frac{1}{2}\log_2\frac{\frac{1}{2}}{P_Y(1)}$$

$$I(2;Y) = \sum_{y\in\mathcal{Y}} P_{Y|X}(y|2)\log_2\frac{P_{Y|X}(y|2)}{P_Y(y)} = \frac{1}{2}\log_2\frac{\frac{1}{2}}{P_Y(1)} + \frac{1}{2}\log_2\frac{\frac{1}{2}}{P_Y(2)}$$

Thus, $I(0;Y) = I(1;Y)$ holds straightforwardly, regardless of the distribution of Y. Solving $I(0;Y) = I(2;Y)$ yields $P_Y(0) = P_Y(2)$. Noting that

$$\begin{cases} P_Y(0) = \sum_{x\in\mathcal{X}} P_X(x)P_{Y|X}(0|x) = \frac{1}{2}p_0 + \frac{1}{2}p_1 \\ P_Y(1) = \sum_{x\in\mathcal{X}} P_X(x)P_{Y|X}(1|x) = \frac{1}{2}p_0 + \frac{1}{2}p_1 + \frac{1}{2}p_2 = \frac{1}{2} \\ P_Y(2) = \sum_{x\in\mathcal{X}} P_X(x)P_{Y|X}(2|x) = \frac{1}{2}p_2 \end{cases}$$

we obtain that $P_Y(0) = P_Y(2)$ implies $p_0 + p_1 = p_2 = \frac{1}{2}$, and $C = \frac{1}{2}$.

Case 2&3: From the derivation in Case 1, we can ignore these two cases since $I(0;Y) = I(1;Y)$.

Case 4: Let $p_2 = 0$. We have $p_0 + p_1 = 1$, which implies $P_Y(0) = P_Y(1) = \frac{1}{2}$ and $P_Y(2) = 0$. This choice will gives $C = 0$.

In summary, only Case 1 works; hence, the capacity of this channel is 1/2 bits per channel use with the optimal input being $(p_0, p_1, \frac{1}{2})$.

2. Read Lemma 4.16 and its proof.

Lemma 4.16. The capacity of a weakly-symmetric channel $\mathbb{Q}$ is achieved by a uniform input distribution and is given by

$$C = \log_2|\mathcal{Y}| - H(q_1, q_2, \cdots, q_{|\mathcal{Y}|}) \tag{3}$$

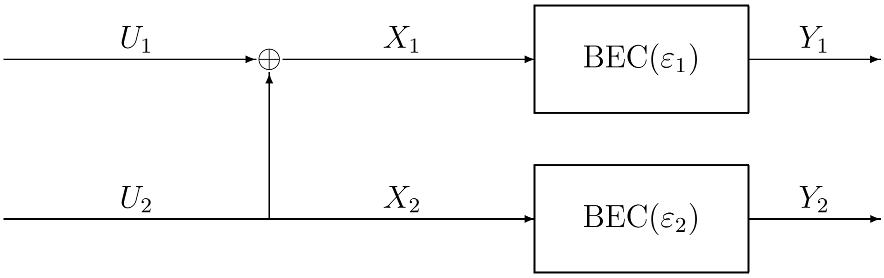
where $(q_1, q_2, \cdots, q_{|\mathcal{Y}|})$ denotes any row of $\mathbb{Q}$ and

$$H(q_1, q_2, \cdots, q_{|\mathcal{Y}|}) := -\sum_{i=1}^{|\mathcal{Y}|} q_i \log_2 q_i$$

is the row entropy.

3. Please memorize the capacity of BSC(ϵ) and BEC(ϵ). The former is $1 - H_b(\epsilon)$ and the latter is $1 - \epsilon$. Which one is larger (equivalently, which channel is noisier)?

4. For the basic transformation below, answer the following questions.



   (a) Determine the erasure probability for U_2 given U_1 is known (i.e., frozen) to the receiver?

   (b) Re-do (a) if both U_1 and U_2 are unknown to the receiver.
   Hint: Here, we assume U_1 and U_2 are both uniform over $\{0, 1\}$ and are independent of each other.

   (c) If we choose to freeze U_2 (i.e., the receiver knows U_2), what will be the erasure probability of U_1?

Solution.

(a) From

$$\mathbb{Q}^+ : \quad U_2 = \begin{cases} Y_1 \oplus U_1, & \text{if } Y_1 \in \{0, 1\} \\ Y_2, & \text{if } Y_2 \in \{0, 1\} \\ \ ?\ , & \text{if } Y_1 = Y_2 = E \end{cases}$$

U_2 is erased iff both Y_1 and Y_2 equal erasure, which occurs with probability $\varepsilon_1 \varepsilon_2$.

(b) From,

$$\mathbb{Q}^+ : \quad U_2 = \begin{cases} Y_1 \oplus U_1, & \text{if } Y_1 \in \{0,1\} \\ Y_2, & \text{if } Y_2 \in \{0,1\} \\ ?\,, & \text{if } Y_1 = Y_2 = E \end{cases}$$

U_2 is not erased iff Y_2 is not easured, which occurs with probability $(1-\varepsilon_2)$. Hence, the erasure probability is still ε_2. In other words, without freezing the previous bit, the channel combining manipulation do not help improving the system performance.

(c) Since U_2 is known and Y_2 has nothing to do with U_1, the channel output Y_2 is irrelevant to the decision making of U_1; hence, from,

$$\mathbb{Q}^- : \quad U_1 = \begin{cases} Y_1 \oplus U_2, & \text{if } Y_1 \in \{0,1\} \\ ?\,, & \text{otherwise} \end{cases}$$

U_1 is erased iff Y_1 equals erasure, which occurs with probability ε_1. No improvement of performance due to the knowledge of U_2.

5. (a) Suppose a stationary ergodic source has entropy rate $H(\mathcal{V}) = 0.9$ bits per source letter, and a DMC has channel capacity $C = 0.1$ bits per channel use. Assume the source encoder takes in m source letters at each encoding manipulation and the channel encoder processes at an n-tuple basis. What is the condition on the reliable transmission bandwidth expansion factor $\rho = m/n$?

(b) If the source generates the source letters at a rate of one source letter in every 10 ms, while the channel can process its transmission at a rate of one channel use per 1 ms, can the source be reliable transmitted over the channel? Justify your asnwer.

Solution.

(a) Accordingly lossless joint source-channel coding theorem,

$$\limsup_{m\to\infty} \frac{m}{n} < \frac{C}{H(\mathcal{V})} = \frac{0.1}{0.9} = \frac{1}{9}.$$

(b) The source entropy rate is 0.9 bits per source letter = 90 bps. The channel capacity is 0.1 bits per channel usage = 100 bps. Since 90 bps < 100 bps, the lossless joint source-channel coding theorem implies the source can be reliably transmitted over the channel.

Hint: The answer in (a) indicates the channel should process at a rate that is nine times faster than the source generating rate (i.e., $n > 9m$). Here, we use some exemplified numbers for better understanding of the theorem.