Stereographic Projection and the Color Cube

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1 Preamble

In this paper, the reader will explore a way to stereographically project the color cube on to a 2d space through changing basis, spherizing the cube, as well as exploring stereographic projection itself.

2 Introduction for the Color Cube

Let us define a color as a composition or tuple of red, green, and blue(RGB) values such that they can range from [-1,1]. A value of -1 means no units of color is added and 1 means the maximum amount of colors is added. For example, the color red is (1,0,0); the color green is (0,1,0); and the color blue is (0,0,1). One way to display all colors is to display it through a color cube.

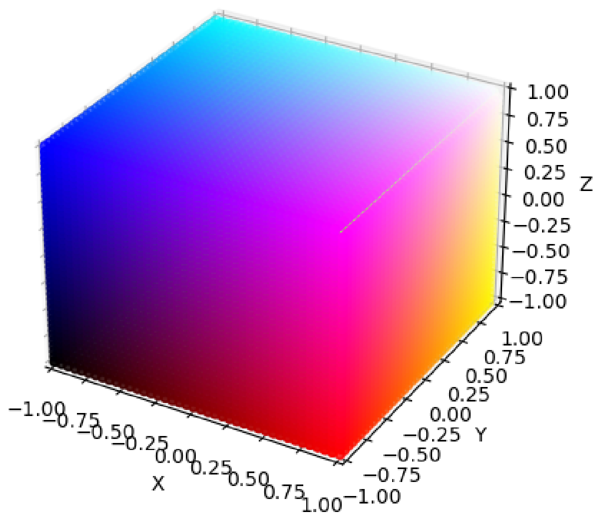

Figure 1: The Color Cube

2.1 Interior of the Color Cube

The following figure is the interior of the color cube.
The figure has staggered points, one should imagine the points as on the interior of the space encapsulated by the faces of figure 1.

2.2 Notes about the Color Cube

From the two above figures, some facts about the color cube to keep in mind:

There are eight vertices of the color cube. They are when one(or more) of the values of RGB is -1 or 1. When full or null intensity of **all** colors, they are white$(1,1,1)$ or black$(-1,-1,-1)$. When there exists only one of RGB in full

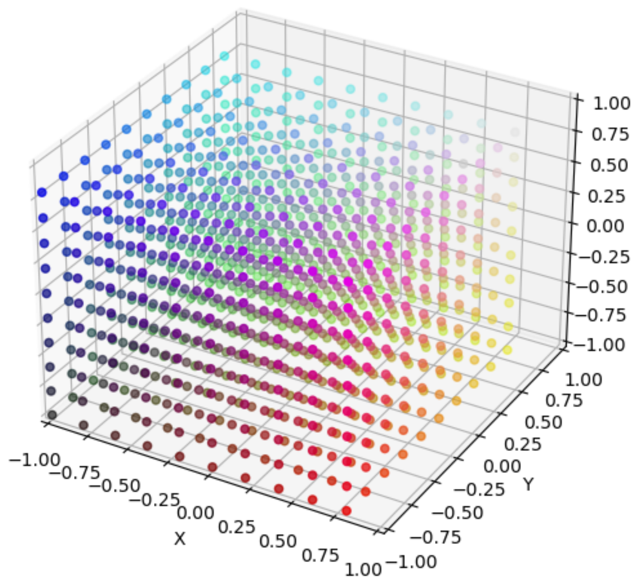

Figure 2: Interior of the Color Cube

intensity or red,green,blue. When there is a composition of two colors in full intesity. These colors are yellow$(1, 1, -1)$, cyan$(-1, 1, 1)$, and magenta$(1, -1, 1)$. We can use these vertices as a sign for transformations of the cube.

3 Linear Transformations of the Cube

In this section we will find a set of basis that transforms RGB into a space with a vector representing the amount of ”color” is added. One can view this as a brightness factor or how black to grey to white a color is. Please refer to the figure 3 for a visual representation.

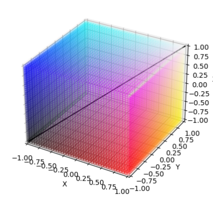

Figure 3: The black line is the brightness of the color running from values of black to grey to white. We hope to find a set of basis that includes said vector.

3.1 Why transform the cube?

In this section we will transform the cube such that one of the axis is the grey scale. Readers can skip to figure 4 for a sneak peak of what that would look like. We will be able to better visually interpret the stereographic projection.

3.2 The Algebra

For this to work, we will try to first convert the mapping of the basis from $e_1 = e_{\text{red}} = (1, 0, 0)$,$e_2 = e_{\text{green}} = (0, 1, 0)$,$e_3 = e_{\text{blue}} = (0, 0, 1)$ to one with the basis of $e_1' = e_{\text{brightness}} = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$. There are infinitely many orthonormal basis e_2', e_3' to choose from. But it must satisfy the condition that

$$e_i \cdot e_j = 0 \text{ for any } i, j = 1, 2, 3$$

The following computation will find a two other basis.

$$(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) \cdot (a, b, c) = 0 \to \frac{1}{\sqrt{3}}(a + b + c) = 0$$

or that $a + b + c = 0$ must be true given e_2, e_3. Let $e_2 = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0)$. To find e_3 we know that $e_2 \cdot e_3 = 0$ or

$$\frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 0 \to a = b$$

e_3 then must abide by $a + b + c = 0, a = b$. Let $c = 1$, then from a series of algebraic calculations, a normal vector is $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1)$ and an orthonormal vector is $(-\frac{1}{2\sqrt{2}}, -\frac{1}{2\sqrt{2}}, \frac{1}{\sqrt{2}})$.
The resulting can be described by the following transformation:

$$T_{\text{Rotate}}(\underline{x} | \underline{x} \in R^3) = \begin{bmatrix} -\frac{1}{2\sqrt{2}} & -\frac{1}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \underline{x}$$

$$= (\frac{2b - (r + g)}{\sqrt{2}}, \frac{r - g}{\sqrt{2}}, \frac{r + b + g}{\sqrt{3}})$$

3.3 Visualization

The following visualizes the described transformation.

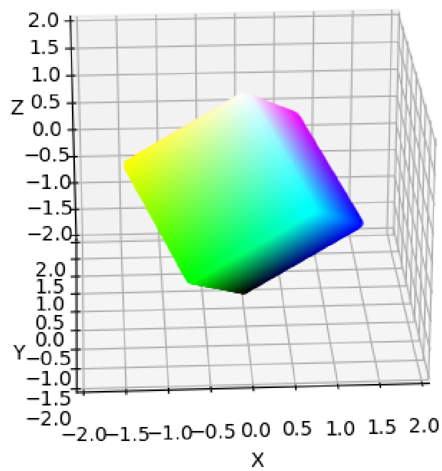

Figure 4: The Color cube now is tipped at the black vertex

3.4 Notes

The way the matrix was chosen above mattered a lot. depending on the way the matrix is chosen or the ordering of the basis we found, could induce the cube to rotate in different ways. For example, here is a different basis that is represent by having $(b_1, b_2, b_3) \rightarrow (b_3, b_2, b_1)$. The black vertex is now according to the x rather than the z.

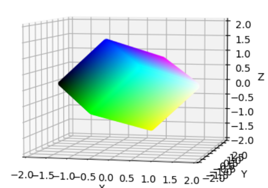

Figure 5: Brightness scaled by the z values

Another interesting fact is the transformation described is a linear transformation. This transformation will not deform the cube. The distance between all the points remain the same.[1]

4 Stereographic Projection

4.1 Preparations

Stereographic projection is not defined well for non-spherical objects. Recall that the equation for stereographic projection is

$$\varphi(x,y,z) = \begin{cases} \frac{x}{1-z} + \frac{y}{1-z}i & \text{if } z \neq 1 \\ \infty & z = 1 \end{cases} \quad \text{where } (x,y,z) \in \text{surface of unit sphere}$$

We cannot simply apply φ to the surface of cube because there will exist some points that will be mapped to the same point. For example, one can view figure 4 and notice a whole line of faces can be mapped to the same point. This is the motivating reason for transforming the color cube into a color sphere.
One attempted way was to use the inverse projection to generate colors. **Note this method should work, but I haven't made it work yet**

4.2 Spherize Function

Let us define the spherize function as

$$T_{\text{sphereize}}(\underline{c} \in \text{surface of the square}) = (\frac{c_1}{|c|}, \frac{c_2}{|c|}, \frac{c_3}{|c|})$$

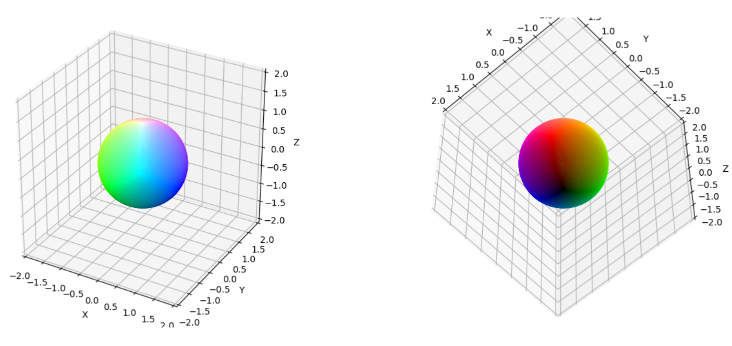

Figure 6: $T_{\text{spherize}}(T_{\text{Rotate}}(x))$

[1] Does this make this in the group of isometries in 3d space?

4.3 Stereographic projection

Applying the stereographic projection from section 4.1 where (x, y, z) represents the points created by the new basis, we will have the final projections.

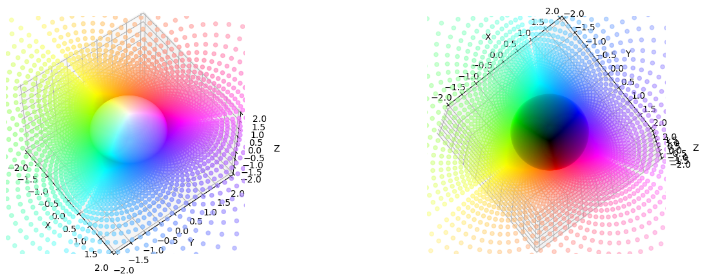

Figure 7: The top and bottom side of the projection

Some notes about this imperfect display:

1. The projection maps to a 2d space. The space displayed here is still 3d. The sphere is still present. The z value of the projection is set to 0.

5 Conclusion and Post-amblings

5.1 Finale

We have shown a possible way to stereographically project the space of the surface of the color cube through generating a basis, spherizing the cube, and applying stereographic projection.

5.2 Improvements

5.2.1 Stereographic Mapping

One improvement to the projection is the way the projection is displayed. The projection maps out the pre computed points on the face to some point in R^2. While mathematically such a process works, this is computationally intensive to generate a space of decent density. The problem is that a viewer views a finite space of $n \times n$ points, but the figures generated by my process causes white spaces. Please refer to figure 7 and look at the corners. As a point moves a way from the origin, the distance between the itself and the nearest mapped point increases. Thus, in order for my process to reduce these white spaces, we would have to increase the amount of computed points on the sphere. Furthermore, some of these points will be mapped outside of the $n \times n$ view.

Given a projection, we would like to calculate the inverse the projection.

$$\varphi^{-1}(z \in \mathbb{C}) \rightarrow Re(\varphi(z)) = u = \frac{a}{1-c} \text{ and } Im(\varphi(z)) = v = \frac{b}{1-c}$$

Then $\varphi^{-1}(u+vi) = s(u, v, -1) + (0, 0, 1) = (su, sv, 1-s)$. Under the projection of the sphere we have the additional fact that $(su^2) + (sv)^2 + (1-s)^2 = 1$. Thus

$$s^2(u^2 + v^2) + (1 - 2s + s^2) = 1$$

$$s^2(u^2 + v^2 + 1) = 2s$$

$$\text{Thus } s = \frac{2}{u^2 + v^2 + 1} \text{ and } u^2 + v^2 = r^2 \rightarrow s = \frac{2}{1 + r^2}$$

We can then substitute s to find the inverse of the projection:

$$\varphi^{-1} = (\frac{2u}{1+r^2}, \frac{2v}{1+r^2}, 1 - \frac{2u}{1+r^2})$$

If we can map the points of the viewers $n \times n$ space, we should be able to map without any density problems.

5.3 Question

The initial research was based on transformation then applied to stereographic projection. A question is given a transformation Q of the projected space, does there exist a transformation P of the cube such that the projection equals?

$$Q(\varphi(T_{\text{rotate}}(\text{color cube}))) = \varphi(P(T_{\text{rotate}}(\text{color cube})))$$

6 The End

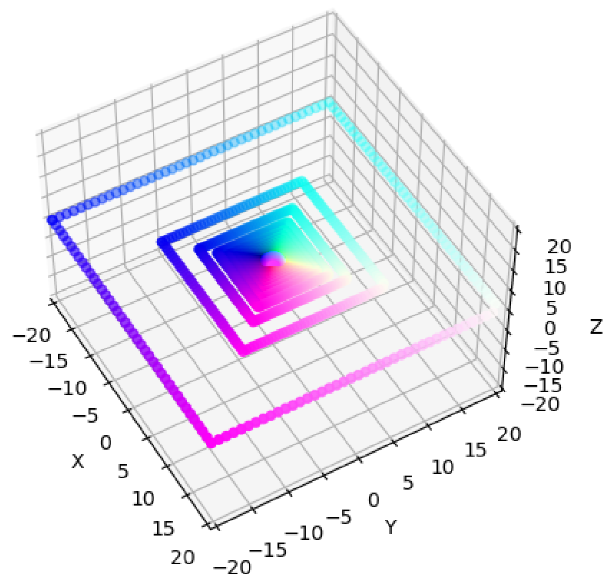

Figure 8: What is this?