Introduction to Functional Analysis

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Preface

这是笔者为西安交通大学 23 级数学强基的泛函分析讨论班所准备的讲义[i]. 参考书是 Haim Brezis 的 *Functional Analysis, Sobolev Spaces and Partial Differential Equations*.

文中多处引用曾经在数学分析讲义[ii]中收获的知识，一千零一页脑残粉不解释.

笔者才疏学浅，错漏不足之处犹待读者批评指正.

本讲义适合于已经学习过**数学分析**、**高等代数**和**抽象代数**课程的数学系本科低年级学生. 编写时参照以下原则:

- 依照参考书顺序进行编排.
- 书上给出定义的，沿用书上的记号. 补充定义的，依参考资料或个人习惯给出记号.
- 全部向量空间均视为 $\mathbb{R}$-**vector space**.
- 以反证法完成证明的，以 † 作结；其余证明以 □ 作结.

特别鸣谢：

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[i] 及相应笔记.

[ii] 参看 《数学分析讲义》-于品.

Abbreviations

i.e. 换而言之

l.s.c. lower semi-continuous function

n.v.s. normed vector space

t.v.s. topological vector space

WLOG 不失一般性

this is an example. $\Longrightarrow$

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1 Hahn-Banach thms

1.1 Extension of Linear Functionals

1.1.1 Zorn's lemma

Def 1.1. Set Ω ordered by $(\Omega, \leqslant)$ is called a **partial ordered set** (poset) iff:

- $x \leqslant x, \forall x \in \Omega.$
- $x \leqslant y \wedge y \leqslant z \Rightarrow x \leqslant z, \forall x, y, z \in \Omega.$
- $x \leqslant y \wedge y \leqslant x \Rightarrow x = y, \forall x, y \in \Omega.$

Rmk (maximum and upper bound).

- $m \in \Omega$ is called maximal element iff $m \in \Omega$ and $\forall x \in \Omega, m \leqslant x \Rightarrow m = x$.
- For $\Omega' \subseteq \Omega$, $m \in \Omega$ is called an upper bound of Ω' iff $\forall x \in \Omega', x \leqslant m$.

e.g (maximal ele maybe not unique). Let $\Omega = \{2, 3, 5, 6, 10\}$ with order defined by **exactly divide**. Then 6 and 10 are both maximall eles.

Def 1.2. A poset Ω ordered by $(\Omega, \leqslant)$ is called a **totally ordered set** (toset) iff:

- $x \leqslant y \vee y \leqslant x, \ \forall x, y \in \Omega.$

Def 1.3 (inductive set). P is called an **inductive set** iff for all toset $Q \subseteq P$, Q has an upper bound $q \in P$.

Thm 1.1 (Zorn's lemma). Every nonempty ordered set that is inductive has a maximal element.

1.1.2 Helly, Hahn-Banach thm analytic form

Point. *A linear functional g on G controlled by a Minkowski functional p on E can be extended to the full E still be controlled.*

Def 1.4 (functional). E, vector space over $\mathbb{R}$. Then $\forall F \subseteq E$ as E's subspace, $f\colon F \to \mathbb{R}$ is called an **functional** on E. Moreover, if f satisfying:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \qquad \forall x, y \in E(\text{or } F), \ \forall \alpha, \beta \in \mathbb{R}$$

then f is called **linear functional**.

Rmk (Minkowski functional).

A functional $p\colon E \to \mathbb{R}$ is called a **Minkowski functional**[i] iff:

i. $p(\lambda x) = \lambda p(x), \ \forall x \in E, \lambda \in \mathbb{R}_{>0}$.
ii. $p(x + y) \leqslant p(x) + p(y), \ \forall x, y \in E$.

We call the property i. **positive-homogeneity** and the property ii. **subadditivity**.

[i] Also **sublinear functional**.

Thm 1.2 (Hahn-Banach)**.** For a linear subspace $G \subseteq E$, if:

- p is a Minkowski functional on E.
- g is a linear functional on G.
- $g(x) \leqslant p(x), \ \forall x \in G$[i].

then exists linear functional f on E, s.t. $[g = f]|_G \wedge [f \leqslant p]|_E$.

Point. *Extend the Dom of g step-by-step and contain all the properties we need.*

Proof. Define:

$$P = \left\{ h \colon \mathrm{Dom}\,(h) \to \mathbb{R} \;\middle|\; \begin{array}{l} \mathrm{Dom}\,(h) \ \text{ is a subspace of } E, h \text{ is linear.} \\ G \subseteq \mathrm{Dom}\,(h)\,, \ [h = g]|_G, \ [h \leqslant p]|_{\mathrm{Dom}(h)} \end{array} \right\}$$

ordered by

$$(h_1 \leqslant h_2) \iff (\mathrm{Dom}\,(h_1) \subseteq \mathrm{Dom}\,(h_2)) \wedge [h_1 = h_2]|_{\mathrm{Dom}(h_1)}$$

step 1. Claim: P has a maximum.

$\forall Q \subseteq P$ is toset, denote $Q = \{\mathrm{Dom}\,(h_i)\}_{i \in I}$. Then define

$$h \colon \mathrm{Dom}\,(h) = \bigcup_{i \in I} \mathrm{Dom}\,(h_i)\,, \ [h = h_i]|_{\mathrm{Dom}(h_i)}$$

Obviously $h \in P \implies h$ is an upper bound of Q in P.

So P is inductive, and by Zorn's Lemma: $\exists\ f = \max\limits_{h \in (P, \leqslant)} h \in P$.

step 2. Claim: $\mathrm{Dom}\,(f) = E$.

Proof by contradiction:

If $\mathrm{Dom}\,(f) \subsetneq E$, then $\exists x_0 \in E \setminus \mathrm{Dom}\,(f)$. Let h:

$$\mathrm{Dom}\,(h) = \mathrm{Dom}\,(f) \oplus \mathbb{R}x_0, \ [h = f]|_{\mathrm{Dom}(f)}, \ h(x + tx_0) = f(x) + t\alpha \ (\forall x \in \mathrm{Dom}\,(f)\,, t \in \mathbb{R}).$$

Want choose $\alpha \in \mathbb{R}$, s.t. $h \in P \implies f < h$ induces a contradiction.

$$\begin{aligned}
&\iff \forall x \in \mathrm{Dom}\,(f)\,, t \in \mathbb{R}, \ f(x) + t\alpha \leqslant p(x + tx_0). \\
&\iff \begin{cases} f(x) + \alpha \leqslant p(x + x_0) \\ f(x) - \alpha \leqslant p(x - x_0) \end{cases} \quad \forall x \in \mathrm{Dom}\,(f)\,. \\
&\iff \forall x, y \in \mathrm{Dom}\,(f)\,, \ f(y) - p(y - x_0) \leqslant \alpha \leqslant p(x + x_0) - f(x). \\
&\iff f(x) + f(y) = f(x + y) \leqslant p(x + y) \leqslant p(x + x_0) + p(y - x_0).
\end{aligned}$$

$\implies \alpha$ exists. Then $h \in P$ and $f \leqslant h$. And actually $f \neq h$, so $f < h$. †

1.1.3 n.v.s., metric space, Banach space

Point. *In further subsections vector space E is always supposed to be n.v.s.*

Def 1.5 (n.v.s.)**.** If vector space X is armed with $\|\cdot\| \colon X \to \mathbb{R}_{\geqslant 0}$, s.t.

- $\|x\| = 0 \iff x = 0$.
- $\|x + y\| \leqslant \|x\| + \|y\|, \ \forall x, y \in X$.
- $\|\alpha x\| = |\alpha| \cdot \|x\|, \ \forall \alpha \in \mathbb{R}, x \in X$.

which is called **norm**. Then $(X, \|\cdot\|)$ is called a **normed vector space** (Also **n.v.s.** for short).

[i] Denote "$\mathbf{P}(x)$ is True for $\forall x \in \mathbf{Q}$" as $[\mathbf{P}(x)]|_{\mathbf{Q}}$, or simply $[\mathbf{P}]|_{\mathbf{Q}}$.

Rmk (semi-norm).

Minkowski functional is called a **semi-norm** iff $p(\lambda x) = |\lambda| p(x), \ \forall \lambda \in \mathbb{R}, x \in X$.

Def 1.6 (metric space). If nonempty set X is armed with $d\colon X \times X \to \mathbb{R}_{\geqslant 0}$, s.t.

- $d(x, y) = 0 \iff x = y, \ \forall x, y \in X$.
- $d(x, z) \leqslant d(x, y) + x(y, z), \ \forall x, y, z \in X$.
- $d(x, y) = d(y, z), \ \forall x, y \in X$.

which is called **metric**[i]. Then (X, d) is called a **metric space**[ii].

Def 1.7 (Cauchy seq on metric space). For metric space (X, d) and $\{x_n\} \subseteq X$, if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall n, m \geqslant N \ [d(x_n, x_m < \varepsilon)],$$

then call $\{x_n\}$ a **Cauchy seq** in (X, d).

Prop 1.3.

- A norm can induce a metric. i.e. $d(x, y) = \|x - y\|$.
- A metric can induce a topology over (X, d) with open sets $U \coloneqq \bigcup_{\alpha \in \mathcal{A}} \mathbb{B}(x_\alpha, r_\alpha) \subseteq X$.

Rmk. But generally a metric can't induce a norm.

Def 1.8 (density). Fixed metric space (X, d) and $Y \subseteq X$. If

$$\forall x \in X, \varepsilon \in \mathbb{R}_{>0}, \ \exists y \in Y, \text{ s.t. } d(x, y) < \varepsilon,$$

then Y is **dense** in X.

Obviously the definition is compatible with density defined on the topology induced by metric.

Def 1.9 (complete metric space). (X, d), metric space with property:

$$\text{For all Cauchy seq } \{x_n\}, \ \exists x \in X, \text{ s.t. } \lim_{n \to \infty} d(x_n, x) = 0$$

is called **complete metric space**.

Rmk (Banach space). n.v.s. $(X, \|\cdot\|)$ is called a **Banach space** iff (X, d) is a complete metric space while d is induced by $\|\cdot\|$.

e.g (non-complete metric space). Let $C_C(\mathbb{R}) = \{f \in C(\mathbb{R}) \colon \operatorname{supp}(f) \text{ is compact}\}$. Consider $(C_C(\mathbb{R}), \|\cdot\|_\infty)$, for $f \in C_C(\mathbb{R})$:

$$\|f\|_\infty \coloneqq \operatorname*{ess\,sup}_{x \in \mathbb{R}} |f(x)| = \inf\{M > 0 \colon |f| \leqslant M \text{ a.e. on } E\}.$$

[i] Also **distance**.

[ii] Also **distance space**.

Actually, $\|f\|_\infty \xlongequal{f\in C(\mathbb{R})} \sup\limits_{x\in\mathbb{R}}|f(x)| \xlongequal{\operatorname{supp}(f)\text{ compact}} \max\limits_{x\in\mathbb{R}}|f(x)| \implies \forall$ Cauchy seq $\{f_n\}\in C_C(\mathbb{R})$:

$$\|f_n - f_m\|_\infty < \varepsilon \to 0 \iff \max_{x\in\mathbb{R}}|f_n - f_m| \to 0 \implies f_n \rightrightarrows f \in C(\mathbb{R})$$

But we can find some $\{f_n\}\in C_C(\mathbb{R})$ s.t. $\operatorname{supp}(f)$ is not compact, which means $f\notin C_C(\mathbb{R})$.

e.g (cut-off function[i]). Let:

$$\phi(x) = \begin{cases} e^{-\frac{1}{x^2}} & ,x>0 \\ 0 & ,x\leqslant 0 \end{cases},\ \chi_{a,b}(x) = \frac{\phi(b-|x|)}{\phi(b-|x|)+\phi(|x|-a)} \qquad (0<a<b)$$

Obviously $\chi_{a,b}\in C^\infty(\mathbb{R})$. But let $a=\frac{1}{2}, b=n$, then $\operatorname{supp}(\chi_{a,b}) = (-n,n) \xrightarrow{n\to\infty} \mathbb{R}$. So $(C_C(\mathbb{R}), \|\cdot\|_\infty)$ is not complete.

You can find further materials in section 1.1.5*, where we'll give $(C_C(\mathbb{R}), \|\cdot\|_\infty)$ a completion.

1.1.4 Applications: dual space of n.v.s.

Def 1.10 (operator). $(X,\|\cdot\|_X),(Y,\|\cdot\|_Y)$ two n.v.s. Then map $T\colon X\to Y$ is called an operator from X to Y.

- linear: $\forall\alpha,\beta\in\mathbb{R}, x_1,x_2\in X,\ T(\alpha x_1+\beta x_2) = \alpha T(x_1)+\beta T(x_2)$.
- continuous: $x_n \xrightarrow{n\to\infty} x\in X \implies T(x_n) \xrightarrow{n\to\infty} T(x)$.
- bounded: $\exists M>0$, s.t. $\|T(x)\|_Y \leqslant M\|x\|_X,\ \forall x\in X$.

Prop 1.4. If $T\colon X\to Y$ is linear operatior, then T is bounded iff T is continuous.

Proof.

step 1. Suppose that T is bounded.

Let $\{x_n\}\subseteq X,\ x_n\to x\in X$. Then

$$\|T(x_n)-T(x)\|_Y = \|T(x_n-x)\|_Y \leqslant M\|x_n-x\|_X \to 0$$

means $T(x_n)\to T(x)$. □

step 2. Suppose that T is continuous.

Proof by contradiction:

If T is unbounded, then $\exists\{x_n\}\subseteq X$ s.t. $\|T(x_n)\|_Y > n\|x_n\|_X$. Let $\{\widetilde{x_n}\}\subseteq X$, $\widetilde{x_n} = \frac{x_n}{n\|x_n\|_X} \implies \|\widetilde{x_n}\|_X = \frac{1}{n}\to 0$. But $\|T(\widetilde{x_n})\|_Y > 1,\ \forall n\in\mathbb{N}$. †

Rmk. Let $(Y,\|\cdot\|_Y) = (R,|\cdot|)$, then linear/continuous/bounded operator becomes linear/-continuous/bounded functional.

Def 1.11 (dual space). n.v.s. $(E,\|\cdot\|)$. Then we denote by E^* the **dual space** of E, that is the space of all continuous linear functionals on E. Denote $\langle f,x\rangle \coloneqq f(x)$ as **scalar product** for the duality E^*, E.

[i] Also truncated function.

Rmk (dual norm)**.** The **dual norm** on E^* is denoted by:

$$\|f\|_{E^*} = \sup_{\|x\|\leqslant 1, x\in E} |\langle f, x\rangle| = \sup_{\|x\|\leqslant 1, x\in E} \langle f, x\rangle = \sup_{x\neq 0, x\in E} \frac{\langle f, x\rangle}{\|x\|} = \sup_{\|x\|=1, x\in E} \langle f, x\rangle.$$

The basic properties can be checked easily, so it is well-defined for all linear functionals on E[i].

Otherwise, we'll always write $\|\cdot\|$ for dual norm rather than $\|\cdot\|_{E^*}$, since x is element in E and f is a linear functional on E, which can be distinguished easily.

Lem 1.1.

$$\boxed{|\langle f, x\rangle| \leqslant \|f\| \|x\| \quad \forall x \in E, f \in E^*}$$

Proof. If $x = 0$, obvious. Or $\dfrac{|\langle f, x\rangle|}{\|x\|} \leqslant \sup_{x\neq 0, x\in E} \dfrac{\langle f, x\rangle}{\|x\|} = \|f\|$. □

Rmk. From the lemma above, we know that f is bounded iff $\exists M$, s.t. $\|f\| \leqslant M$.

Obviously armed with dual norm, $(E^*, \|\cdot\|_{E^*})$ is a n.v.s. Actually, we have a stronger proposition:

Prop 1.5. Dual space is Banach space.

Proof. We've already known that dual space is n.v.s. Now let's prove that E^* is complete:

step 1. Cauchy seq $\{f_n\} \subseteq E^*$ is convergent.

$\{f_n\} \subseteq E^*$ is Cauchy seq $\implies \forall \varepsilon > 0, \exists N \in \mathbb{N}$, s.t. $\forall m, n > N$, $[\|f_n - f_m\| < \varepsilon]$. So for fixed $\varepsilon > 0$, $\forall m, n > N, x \in E$:

$$|f_n(x) - f_m(x)| = |\langle f_n - f_m, x\rangle| \leqslant \|f_n - f_m\| \|x\| < \varepsilon \|x\|, \tag{$\star$}$$

then $\{f_n(x)\}$ is Cauchy seq in $\mathbb{R}$. Let $f(x) := \lim_{n\to\infty} f_n(x)$, $\forall x \in E$. So for fixed n and $m \to \infty$ in ($\star$), then take supreme on both side, we'll get $\|f_n - f\| \to 0$, i.e. $f_n \to f$.

step 2. $f \in E^*$.

$\lim, f_n$ is linear $\implies$ f is linear. So we only need to prove that f is bounded. We know that $f_n \in E^*$ is bounded. So

$$\exists M, \text{ s.t. } |\langle f_n, x\rangle| \leqslant M \|x\|, \ \forall x \in E, n \in \mathbb{N}.$$

Let $n \to \infty$, then $|\langle f, x\rangle| \leqslant M \|x\|$, $\forall x \in E \implies f$ is bounded $\implies f$ is continuous $\implies f \in E^*$. □

Coro 1.6 (norm preserving extension)**.** Let $G \subseteq E$ be a linear subspace. If $g\colon G \to \mathbb{R}$ is a continuous linear functional (i.e. $g \in G^*$), then $\exists f \in E^*$ that extends g s.t. $\|f\| = \|g\|$.

Proof. For $g \in E^*$ which is bounded, $\|g\|$ can be seen as a constant in $\mathbb{R}$, so actually $p(x) = \|g\| \|x\|$ is stronger than Minkowski functional. On the other hand we have $g(x) \leqslant |\langle g, x\rangle| \leqslant p$ by Lem 1.1, meaning that g is controlled by p.

Use Thm 1.2 with p to get conclusion immediately. □

Rmk.

$$\begin{array}{ccc} G & \subseteq & E \ni x \\ \downarrow & & \downarrow \\ G^* & \supseteq & E^* \ni f \end{array}$$

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[i] As we know that $f \in E^*$ are all continuous, so $[\|f\| < +\infty]|_{E^*}$.

Coro 1.7. For every $x_0 \in E$, there exists $f_0 \in E^*$ s.t. $\|f_0\| = \|x_0\| \wedge \langle f_0, x_0 \rangle = \|x_0\|^2$.

Proof. Use Coro 1.6 with $G = \mathbb{R}x_0$ and $g(tx_0) = t\|x_0\|^2$, so $\|g\| = \langle g, \frac{x_0}{\|x_0\|} \rangle = \|x_0\|$. □

Rmk.

- There's enough eles in E^* can distinguish x_1 from x_2 by the meaning of norms.
- The ele f_0 given above is in general not unique. However, if E^* is strictly convex, then f_0 is unique.
- The (multivalued) map $x_0 \mapsto F(x_0)$ with F defined below is called the **duality map** from E into E^*:

$$F(x_0) = \{f_0 \in E^* \colon \|f_0\| = \|x_0\| \wedge \langle f_0, x_0 \rangle = \|x_0\|^2\}.$$

Def 1.12 (bidual space)**.** See bidual space of E.

Coro 1.8. $\forall x \in E$, we have:

$$\|x\| = \sup_{\|f\| \leqslant 1, f \in E^*} \langle f, x \rangle = \max_{\|f\| \leqslant 1, f \in E^*} \langle f, x \rangle$$

Proof. Because the canonical injection J from E to E^{**} is an isometry, so for all $x \in E$:

$$\|x\| = \|J_x\| = \sup_{\|f\| \leqslant 1, f \in E^*} \langle J_x, f \rangle = \sup_{\|f\| \leqslant 1, f \in E^*} \langle f, x \rangle \xlongequal{(x_0, f_0)} \max_{\|f\| \leqslant 1, f \in E^*} \langle f, x \rangle.$$

1.1.5 * Further: completion of metric space

Point. *In this section we'll give* $(C_C(\mathbb{R}), \|\cdot\|_\infty)$ *a completion.*

From our reference we know that:

Thm 1.9. (X, d) is a metric space. Then exists complete metric space $(\overline{X}, \overline{d})$ and isometric embedding $\iota \colon (X, d) \to (\overline{X}, \overline{d})$, s.t.

1. $\iota(X)$ is dense in $\overline{X}$.
2. For any complete metric space (Y, d_Y) and uniformly continuous map $\phi \colon X \to Y$,

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "\phi"'] & \overline{X} \arrow[d, dashed, "\psi"] \\ & Y \end{tikzcd}$$

   exists unique $\psi \colon \overline{X} \to Y$ (an expansion of continuous map) s.t. $\phi = \psi \circ \iota$.

then $(\overline{X}, \overline{d})$ is called **a completion of** (X, d) and it is unique under the meaning of isometric isomorphism.

Apply this theorem to our situation and let $(X, d) = (C_C(\mathbb{R}), \|\cdot - \cdot\|_\infty)$:

$$\mathcal{C} = \left\{\{f_n\}_{n \geqslant 1} \subseteq X \colon \{f_n\}_{n \geqslant 1} \text{ is a Cauchy-seq}\right\}$$

$$\{f_n\}_{n \geqslant 1} \sim \{g_n\}_{n \geqslant 1} \iff \lim_{n \to \infty} \|f_n - g_n\|_\infty = 0$$

$$\overline{X} = X/\sim, \ \overline{d}([f], [g]) = \lim_{n \to \infty} \|f_n - g_n\|_\infty$$

And turn the metric to a norm:

$$\begin{aligned}
[f]+[g] &\coloneqq [f+g], && \forall [f],[g]\in\overline{X}\\
\alpha[f] &\coloneqq [\alpha f], && \forall \alpha\in\mathbb{R},[f]\in\overline{X}\\
\|[f]\| &\coloneqq \left\|\lim_{n\to\infty} f_n\right\|_\infty && \forall [f]\in\overline{X}
\end{aligned}$$

Let's check the n.v.s. $(\overline{X},\|\cdot\|)$ is well defined:

Proof.

1. $\|[f]\| = \lim\limits_{n\to\infty}\|f_n\|_\infty = 0 \iff \lim\limits_{n\to\infty}\|f_n-0\|_\infty = 0 \iff [f]=[0]$.
2. $\|[f]+[g]\| = \|[f+g]\| = \left\|\lim\limits_{n\to\infty} f_n + \lim\limits_{n\to\infty} g_n\right\|_\infty \leqslant \left\|\lim\limits_{n\to\infty} f_n\right\|_\infty + \left\|\lim\limits_{n\to\infty} g_n\right\|_\infty = \|[f]\|+\|[g]\|$.
3. $\|[\alpha f]\| = \|\alpha[f]\| = |\alpha|\,\|f\|$.

Prop 1.10. $\phi\colon(\overline{X},\|\cdot\|)\to(C_0(\mathbb{R}),\|\cdot\|_\infty), [f]\mapsto \lim\limits_{n\to\infty} f_n$ is an isometric bijection.

Proof.

step 1. ϕ is well defined:

Obviously $[f]\in\overline{X} \implies f_n \rightrightarrows f = \phi([f])\in C_0(\mathbb{R})$ refers to cut-off function. Furthermore, $\|[f]\| = \left\|\lim\limits_{n\to\infty} f_n\right\|_\infty = \|f\|_\infty = \|\phi([f])\|_\infty$.

step 2. ϕ is an injection:

If $\phi([f]) = \lim\limits_{n\to\infty} f_n = h = \lim\limits_{n\to\infty} g_n = \phi([g])$, then $\begin{cases} f_n\rightrightarrows h\\ g_n\rightrightarrows h\end{cases} \implies \begin{cases}\lim\limits_{n\to\infty}\|f_n-h\|_\infty = 0\\ \lim\limits_{n\to\infty}\|g_n-h\|_\infty=0\end{cases}$. So $\lim\limits_{n\to\infty}\|f_n-g_n\|_\infty \leqslant \lim\limits_{n\to\infty}\|f_n-h\|_\infty + \lim\limits_{n\to\infty}\|g_n-h\|_\infty = 0 \implies [f]=[g]$.

step 3. ϕ is a surjection:

For all $f\in C_0(\mathbb{R})$, let

$$f_n(x) = \begin{cases} f(x), & |x|\leqslant n\\ f(\operatorname{sgn}(x)\cdot n)(n+1-|x|), & n<|x|\leqslant n+1\\ 0, & |x|>n+1\end{cases}.$$

Then $[f] = [\{f_n\}_{n\geqslant 1}]\in\overline{X}$ s.t. $\phi([f]) = f$. $\square$

In summary, $(C_0(\mathbb{R}),\|\cdot\|_\infty)$ is the completion of $(C_C(\mathbb{R}),\|\cdot\|_\infty)$.

1.2 Separation of Convex Sets

Def 1.13 (convex set)**.** X, $\mathbb{R}$-vector space. A subspace $C \subseteq X$ is called a **convex set** iff:

$$\boxed{\forall x, y \in C, \lambda \in [0,1],\ [\lambda x + (1-\lambda)y \in C].}$$

Def 1.14 (t.v.s.)**.** Let $\mathbb{K}$ a topological field (always $\mathbb{R}$ or $\mathbb{C}$). Vector space X over $\mathbb{K}$ armed with a topology satisfying:

- Plus $+\colon X \times X \to X$ is continuous.
- Scalar multiply $\cdot\colon \mathbb{K} \times X \to X$ is continuous.

is called a **topological vector space** (Also **t.v.s.** for short).

In a sense a t.v.s. is a vector space where the convergence process is well-defined.

Rmk (strictly convex)**.** X, t.v.s. A subspace $C \subseteq X$ is called a **strictly convex set** iff:

$$\forall x, y \in C, \lambda \in (0,1),\ [\lambda x + (1-\lambda)y \in C^\circ].$$

where C° is the interior of C.

Def 1.15 (affine hyperplane)**.** An **affine hyperplane** is a subset H of E of the form

$$H = \{x \in E \colon f(x) = \alpha\} \xlongequal{\triangle} [f = \alpha]$$

where $f \neq 0$ is a linear functional and α is a given constant. We call $f = \alpha$ is the equation of H.

There're some examples that linear functional maybe not continuous:

e.g. Let E a n.v.s. with infinite dimension. Claim:

1. E has basis $(e_i)_{i \in I}$, s.t. $\|e_i\| = 1$.
2. Let $f\colon E \to \mathbb{R},\ e_j \mapsto \begin{cases} n & ,j = i_n \\ 0 & ,\text{other} \end{cases}$, where $(e_{i_n})_{n\in\mathbb{N}}$ is a sub-sequence of $(e_i)_{i \in I}$. Then f is linear but not continuous.

Proof.

1. Let $S = \{A \subseteq E \colon A = (a_i)_{i \in I_A}, \{a_i\}_{i \in I_A}$ is linear independent $, \|a_i\| = 1\}$. Then obviously S is nonempty and S is inductive while partial ordered by $\subseteq$:

   For all toset $T \subseteq S$, WLOG $T = \{A_\lambda \colon \lambda \in \Lambda\}$ and $R = \bigcup\limits_{\lambda \in \Lambda} A_\lambda$. So $\forall \{x_i\}_{i=1}^n \subseteq R$, let $x_i \in A_{\lambda_i}$ and WLOG $A_{\lambda_i} \subseteq A_{\lambda_{i+1}}$ as T is a toset. Then $\{x_i\}_{i=1}^n \subseteq A_{\lambda_n}$, which means $\{x_i\}$ is linear independent and $\|x_i\| = 1 \implies R \in S$.

   So for all toset $T \subseteq S$ exists an upper bound $R = \bigcup\limits_{A \in T} A \in S$, which means S is inductive. Then by Zorn's Lemma we know exists $\tilde{A}$ as the maximal ele in $(S, \subseteq)$.

   If exists $x \in E \setminus \{0\}$ can't be represented by linear combination of eles in $\tilde{A}$, then $\tilde{A} \cup \{\frac{x}{\|x\|}\} \in S$ is bigger than $\tilde{A}$ in $(S, \subseteq)$. †

2. Consider $\{v_n\}_{n=1}^{\infty}$, $v_n = e_{i_n}$. Obviously f is linear. Furthermore, $\begin{cases} \|v_n\| = 1 \\ f(v_n) = n \end{cases}$ says $\frac{|f(v_n)|}{\|v_n\|} = n \to \infty \implies f$ is not bounded $\xRightarrow{f \text{ is linear}}$ not continuous. $\square$

Prop 1.11. The hyperplane $H = [f = \alpha]$ is closed iff f is continuous(i.e. $f \in E^*$).

Proof. Obviously if f is continuous, then $H = f^{-1}(\{\alpha\})$ is closed. Now suppose that H is closed.

H is closed $\implies$ H^c is open and nonempty[i)]. Let $x_0 \in H^c \implies f(x_0) \neq \alpha$, so WLOG $f(x_0) < \alpha$, and H^c open $\implies \exists r$, s.t. $\overline{\mathbb{B}(x_0, r)} \subseteq H^c$. Claim:

$$[f(x) < \alpha]|_{\overline{\mathbb{B}(x_0,r)}}$$

Proof by contradiction:

If exists $x_1 \in \overline{\mathbb{B}(x_0, r)} \subseteq H^c$[ii)], then $f(x_1) > \alpha$. Consider

$$T = \{x_t = (1-t)x_0 + tx_1 : t \in [0,1]\} \subseteq \overline{\mathbb{B}(x_0, r)} \subseteq H^c$$

(Because T is convex) So for all $t \in [0,1]$, $f(x_t) \neq \alpha$. But let $t = \frac{f(x_1)-\alpha}{f(x_1)-f(x_0)} \implies f(x_t) = \alpha$. †

Then let's estimate $\|f\| = \sup\limits_{\|x\|=1, x\in E} f(x)$:

We know that $[f(x_0 + rz) < \alpha]|_{\overline{\mathbb{B}(0,1)}} \xRightarrow{f \text{ linear}} [f(z) < \frac{1}{r}(\alpha - f(x_0))]|_{\overline{\mathbb{B}(0,1)}}$. Then take sup both side: $\|f\| \leqslant \frac{1}{r}(\alpha - f(x_0))$. $\square$

Rmk.

- If f is linear, then f is bounded on E iff f is bounded on $\mathbb{B}(0,1)$.
- If $f(x_0) > \alpha$, then $f(-x_0) < -\alpha$, similarly...

Def 1.16 (separate)**.** Let $A, B \subseteq E$. We say $[f = \alpha]$ **separates** A and B if

$$\boxed{[f(x) \leqslant \alpha]|_A \quad \text{and} \quad [f(x) \geqslant \alpha]|_B.}$$

We say that $[f = \alpha]$ **strictly separates** A and B if exists $\varepsilon > 0$ s.t.

$$\boxed{[f(x) \leqslant \alpha - \varepsilon]|_A \quad \text{and} \quad [f(x) \geqslant \alpha + \varepsilon]|_B.}$$

Thm 1.12 (Hahn-Banach, first geometric form)**.** Let $A, B \subseteq E$.

- $A, B \neq \emptyset$ are convex sets such that $A \cap B = \emptyset$.
- One of them is open.

Then there exists a closed hyperplane that **separates** A and B.

Lem 1.2 (Minkowski functional of open convex set)**.** Let $C \subseteq E$ be an open convex set with $0 \in C$. For every $x \in E$, set

$$p(x) = \inf\{\alpha > 0 \colon \alpha^{-1}x \in C\}$$

is a Minkowski functional and satisfies the following properties:

- $\exists M$ s.t. $[0 \leqslant p(x) \leqslant M \|x\|]|_E$.
- $C = \{x \in E \colon p(x) < 1\}$.

---

i) Otherwise $H = E \implies f \equiv \alpha \xRightarrow{\text{linear}} \alpha = 0 \implies f \equiv 0$.

ii) So $f(x_1) \neq \alpha$.

Proof.

1. $\exists M$ s.t. $[0 \leqslant p(x) \leqslant M \|x\|]|_E$:

   C is open $\implies \exists r > 0$ s.t. $\mathbb{B}(0, r) \subseteq C$, so

$$p(x) \leqslant \frac{1}{r} \|x\| \quad \forall x \in E.$$

2. $C = \{x \in E : p(x) < 1\}$:

   Firstly, $C \subseteq \{x \in E : p(x) < 1\}$. If $x \in C$, then $\exists \varepsilon > 0$ s.t. $(1 + \varepsilon)x \in C$ since C is open. So $p(x) \leqslant \frac{1}{1+\varepsilon} < 1$.

   Secondly, $\{x \in E : p(x) < 1\} \subseteq C$. If $p(x) < 1$, then $\exists \alpha < 1$ s.t. $\alpha^{-1}x \in C \implies x = \alpha \cdot (\alpha^{-1}x) + (1-\alpha) \cdot 0 \in C$.

3. p is a Minkowski functional:

   Consider $\forall x \in E, \lambda \in \mathbb{R}_{>0}$:

$$\begin{aligned} p(\lambda x) &= \inf\{\alpha > 0 : \alpha^{-1}(\lambda x) \in C\} \\ &= \inf\{\alpha > 0 : (\frac{\alpha}{\lambda})^{-1} x \in C\} \\ &= \inf\{\lambda\alpha > 0 : \alpha^{-1} x \in C\} \\ &= \lambda \inf\{\alpha > 0 : \alpha^{-1} x \in C\} \\ &= \lambda p(x) \end{aligned}$$

   means that p is positive-homogeneity.

   Then let $x, y \in E$ and $\varepsilon > 0$. From above we know that $\frac{x}{p(x)+\varepsilon}, \frac{y}{p(y)+\varepsilon} \in C$. Thus $\frac{tx}{p(x)+\varepsilon} + \frac{(1-t)y}{p(y)+\varepsilon} \in C$ for all $t \in [0, 1]$. Choosing $t = \frac{p(x)+\varepsilon}{p(x)+p(y)+2\varepsilon}$, we find that $\frac{x+y}{p(x)+p(y)+2\varepsilon} \in C$, which means $p(x + y) < p(x) + p(y) + 2\varepsilon$ for all $\varepsilon > 0$. □

The motivation of considering $\frac{x}{p(x)+\varepsilon}$ and $\frac{y}{p(y)+\varepsilon}$ is geometric: They can approach the boundary of C in their own direction to any degree. As expected, we can find a representative in the direction of $x + y$ on the segment between them.

Lem 1.3 (single point separate)**.** Let $C \subseteq E$ be a nonempty open convex set and let $x_0 in E$ with $x_0 \notin C$. Then there exists $f \in E$ s.t. $[f(x) < f(x_0)]|_C$. In particular, the hyperplane $[f = f(x_0)]$ separates $\{x_0\}$ and C.

Proof. WLOG let's assume that $0 \in C$. Consider the linear subspace $G = \mathbb{R}x_0$ and the linear functional

$$g : G \to \mathbb{R}, \ tx_0 \mapsto t.$$

Let $\alpha = p(tx_0)$ for $\forall t \in \mathbb{R}$. If $t > \alpha \wedge t > 0$, then exists $\varepsilon > 0$, s.t. $t > \varepsilon + \alpha$ and $\frac{tx_0}{\alpha+\varepsilon} \in C$, so $x_0 = (1 - \frac{\alpha+\varepsilon}{t}) \cdot 0 + \frac{\alpha+\varepsilon}{t} x_0 \in C$, there's a contradiction. If $t < 0$ then consider $p((-t)x_0)$ and it'll give out the same contradiction.

From above it is clear that $g(x) \leqslant p(x)$ for all $x \in G$. It follows from Thm 1.2 that there exists a linear functional f on E that extends g and satisfies $f(x) \leqslant p(x) \leqslant M \|x\|$ for all $x \in E$, so f is continuous (i.e. $f \in E^*$). In particular, we have $f(x_0) = 1$ and $f(x) \leqslant p(x) < 1$ for all $x \in C$. □

Because C is convex, so naturally $x \notin C$ seems to be "x outside of C", which means a functional looks like p will be bounded by p, then we can extend it to the whole space.

Then we can test the gap between A and B, also between $A - B$ and $\{0\}$, which becomes a single point separate:

Proof. Set $C = A - B \coloneqq \{c : c = a - b, a \in A, b \in B\}$, so that C is convex (obviously), and open[i)], and $0 \notin C$ (because $A \cap B = \emptyset$). By Lem 1.3 there's some $f \in E^*$ such that $f(z) < 0$ for all $z \in C$. That is

$$f(x) < f(y) \quad \forall x \in A, \quad \forall y \in B.$$

[i)] $C = \bigcup_{b \in B} (A - \{b\})$ where $A - \{b\}$ is open for all $b \in B$.

Fix a constant α satisfying

$$\sup_{x\in A} f(x) \leqslant \alpha \leqslant \inf_{y\in B} f(y).$$

Clearly, the hyperplane $[f=\alpha]$ separates A and B. $\square$

Thm 1.13 (Hahn-Banach, second geometric form)**.** Let $A \subseteq E$ and $B \subseteq E$ be two nonempty convex subsets such that $A \cap B = \emptyset$. Assume that A is closed and B is compact. Then there exists a closed hyperplane that **strictly separates** A and B.

Proof. Set $C = A - B$, so that C is convex, closed[i)], and $0 \notin C$. Hence, there's some $r > 0$ such that $\mathbb{B}(0,r) \cap C = \emptyset$. By Thm 1.12 there's a closed hyperplane separates $\mathbb{B}(0,r)$ and C. Therefore, there's some $f \in E^*, f \not\equiv 0$, such that

$$f(x-y) \leqslant f(rz) \quad \forall x \in A, \quad \forall y \in B, \quad \forall z \in \mathbb{B}(0,1).$$

Take $\inf\limits_{z\in\mathbb{B}(0,1)} \cdot$ both side, it follows that $f(x-y) \leqslant -r\,\|f\|$. Letting $\varepsilon = \frac{1}{2} r\,\|f\| > 0$, we obtain

$$f(x) + \varepsilon \leqslant f(y) - \varepsilon \quad \forall x \in A, \quad \forall y \in B.$$

Then choosing α such that $\sup\limits_{x\in A} f(x) + \varepsilon \leqslant \alpha \leqslant \inf\limits_{y\in B} f(y) - \varepsilon$, we see that $[f=\alpha]$ separates A and B. $\square$

The secret to transforming separation into strict separation lies in that the closed $C \not\ni 0$ extends $\{0\} \cap C = \emptyset$ to $\mathbb{B}(0,r) \cap C = \emptyset$.

Rmk. If E is finite dimensional space, then it can *always* separate any two nonempty convex set A and B such that $A \cap B = \emptyset$.

Coro 1.14 (dense subspace)**.** Let $F \subseteq E$ be a linear subspace such that $\overline{F} \neq E$. Then there exists some $f \in E^*, f \not\equiv 0$, such that

$$\langle f, x\rangle = 0 \quad \forall x \in F.$$

Proof. Let $x_0 \in E$ with $x_0 \notin \overline{F}$. Using Thm 1.13 with $A = \overline{F}$ and $B = \{x_0\}$, we find a closed hyperplane $[f=\alpha]$ that strictly separates F and $\{x_0\}$. Thus, we have

$$\langle f, x\rangle < \alpha < \langle f, x_0\rangle \quad \forall x \in F.$$

It follows[ii)] that $\langle f, x\rangle = 0 \quad \forall x \in F$. $\square$

The most commonly used form of this Coro is its contrapositive:

Coro 1.14*. If for all $f \in E^*$ such that $[\langle f, x\rangle = 0]|_{x\in F}$, we have $[\langle f, x\rangle = 0]|_{x\in E}$ (i.e. $f \equiv 0$). Then $\overline{F} = E$ (i.e. F is dense in E).

i) Suppose that $A - B \ni a_n - b_n \to c$. Because B is compact, there exists sub-seq $b_{n_i} \to b \in B$, so $a_{n_i} \to c + b = a \in A$ since A is closed. Then $c = a - b \in A - B$, which means C is closed.

ii) Because F is a linear subsepace, so if $\langle f, \tilde{x}\rangle > 0$ for $\tilde{x} \in F$, then exists $\lambda \in \mathbb{R}_{>0}$ such that $\lambda\langle f, \tilde{x}\rangle > \alpha$. Similarly, if $\langle f, \tilde{x}\rangle < 0$ for $\tilde{x} \in F$, then exists $\lambda \in \mathbb{R}_{<0}$ such that $\lambda\langle f, \tilde{x}\rangle > \alpha$.

1.3 The Bidual E^{**} and Orthogonality Relations

Def 1.17 (bidual space of E)**.** n.v.s. $(E, \|\cdot\|)$. Then we denote by E^{**} the **bidual space** of E, that is the space of all continuous linear functionals on E^*. Denote $\langle \xi, f\rangle \coloneqq \xi(f)$ as **scalar product** for the duality E^{**}, E^*.

Rmk (bidual norm)**.** Similarly, the **dual norm** on E^{**} is denoted by:

$$\|\xi\|_{E^{**}} = \sup_{\|f\|\leqslant 1, f\in E^*} |\langle \xi, f\rangle| = \sup_{\|f\|\leqslant 1, f\in E^*} \langle \xi, f\rangle = \sup_{f\neq 0, f\in E^*} \frac{\langle \xi, f\rangle}{\|f\|} = \sup_{\|f\|=1, f\in E^*} \langle \xi, f\rangle.$$

Notice: We need to understand the meaning of norm and scalar product by recognizing *where the eles come from.*

Def 1.18 (canonical injection)**.** The map

$$J\colon E \hookrightarrow E^{**}, x \mapsto J_x, \text{ where } J_x\colon E^* \to \mathbb{R}, f \mapsto \langle f, x\rangle.$$

is called the **canonical injection** from E onto E^{**}. It's well defined, since J_x is always a linear continuous functional on E^*.

Prop 1.15. The canonical injection J is an **isometry**. i.e. $\|J_x\| = \|x\|$ is always true.

Proof.

step 1. Notice that $|\langle J_x, f\rangle| = |\langle f, x\rangle| \leqslant \|f\|\,\|x\|$. Take supreme $\sup\limits_{\|f\|\leqslant 1, f\in E^*}$ · both side $\Longrightarrow$ $\|J_x\| \leqslant \|x\|$.

step 2. For all $x \in E$, set $f_0 \in E^*$ by Coro 1.7. Then for all $x \in E$:

$$\|J_x\| = \sup_{f\neq 0, f\in E^*} \frac{\langle J_x, f\rangle}{\|f\|} \geqslant \frac{\langle J_x, f_0\rangle}{\|f_0\|} = \frac{\langle f_0, x\rangle}{\|f_0\|} = \|x\|$$

□

Also E can be seen embedding in E^{**} as a subspace under the canonical injection J.

Def 1.19.

- $M \subseteq E$, linear subspace, we set $M^\perp = \{f \in E^*\colon \langle f, x\rangle = 0 \forall x \in M\}$.
- $N \subseteq E^*$, linear subspace, we set $N^\perp = \{f \in E\colon \langle f, x\rangle = 0 \forall f \in N\}$.

Rmk. $M^\perp$ and $N^\perp$ are closed linear subspace.

Proof. Consider affine hyperplane in E^*: for fixed $\xi \in E^{**}$,

$$[\langle \xi, f\rangle = \alpha] = \{f \in E^*\colon \langle \xi, f\rangle = \alpha\}.$$

Since $J(E) \subseteq E^{**}$, let $\xi = J_x$ is linear continuous functional. Then[i] for $x \in M, \alpha = 0$:

$$[\langle J_x, f\rangle = 0] = \{f \in E^*\colon \langle f, x\rangle = \langle J_x, f\rangle = 0\}$$

is closed linear subspace. Consequently $M^\perp = \bigcap\limits_{x\in M} [\langle J_x, f\rangle = 0]$ is closed linear subspace.

Similarly $N^\perp$ is closed linear subspace.

[i] By closed hyperplane iff linear continuous functional.

We say that $M^\perp$ (resp. $N^\perp$) is the space **orthogonal to** M (resp. N).

Prop 1.16.

- $M \subseteq E$, linear subspace, then $(M^\perp)^\perp = \overline{M}$.
- $N \subseteq E^*$, linear subspace, then $(N^\perp)^\perp \supseteq \overline{N}$.

Proof.

step 1.

Notice that $(M^\perp)^\perp = \{x \in E\colon \langle f, x\rangle = 0, \forall f \in M^\perp\} \implies$ Clearly $M \subseteq (M^\perp)^\perp$. With $(M^\perp)^\perp$ closed, obviously $\overline{M} \subseteq (M^\perp)^\perp$.

Similarly $\overline{N} \subseteq (N^\perp)^\perp$.

step 2.

Suppose by contradiction: If $(M^\perp)^\perp \not\subseteq \overline{M}$, then $\exists x_0 \in (M^\perp)^\perp, x_0 \notin \overline{M}$. Notice that in n.v.s. E, $\{x_0\}$ is compact and $\overline{M}$ closed. By Thm 1.11, $\exists$ closed hyperplane strictly separates $\{x_0\}$ and $\overline{M}$. i.e.

$$\exists f \in E^*, \alpha \in \mathbb{R}, \text{ s.t. } \langle f, x\rangle < \alpha < \langle f, x_0\rangle, \ \forall x \in M.$$

Notice M is a linear subspace, so if $\tilde{x} \in M$, s.t. $\langle f, \tilde{x}\rangle \neq 0$, WLOG $\langle f, \tilde{x}\rangle > 0$. Then $\forall n \in \mathbb{N}$, $n\tilde{x} \in M \implies \langle f, n\tilde{x}\rangle \to +\infty$. Contradict with $\langle f, x\rangle < \alpha$ for all $x \in M$.

So $\forall x \in M$, $\langle f, x\rangle = 0$, which means $f \in M^\perp$ and $\langle f, x_0\rangle > 0$. But as we supposed, $x_0 \in (M^\perp)^\perp \implies \langle f, x_0\rangle = 0$. †

e.g (Why $(N^\perp)^\perp \not\subseteq \overline{N}$?). Also we can separate $\{f_0\}$ and $\overline{N}$, then $\exists \xi \in E^{**}$, s.t. $\langle \xi, f_0\rangle > 0$. But maybe $\xi \notin J(E)$. Anyway, $(N^\perp)^\perp \not\subseteq \overline{N}$ holds when E is a reflexive space.

We can't give out an example temporarily. Wait for Sec 11.3, ℓ^1 and ℓ^∞ space.

1.4 A Glance of Conjugate Convex Functions

Point. *In this section, $\varphi\colon E \to (-\infty, +\infty]$, let* $\text{Dom}\,(\varphi) = \{x \in E\colon \varphi < +\infty\}$.

Def 1.20 (epigraph)**.**

$$\operatorname{epi}\varphi = \{(x,\lambda) \in E \times \mathbb{R}\colon \varphi(x) \leqslant \lambda\}$$

1.4.1 lower semi-continuous functions and some topology

Point. *We now assume that E is a* ***topological space****.*

Def 1.21 (l.s.c.)**.** φ is called **lower semi-continuous** (Also **l.s.c.** for short) iff:

$$\forall \lambda \in \mathbb{R},\ \text{level set } [\varphi \leqslant \lambda] = \{x \in E\colon \varphi(x) \leqslant \lambda\} \text{ is closed.}$$

Prop 1.17. φ is l.s.c. $\iff$ $\operatorname{epi}\varphi$ is closed.

Proof. We'll prove that φ is l.s.c. $\iff$ $(\operatorname{epi}\varphi)^c$ is open:

$$\begin{aligned}
\varphi \text{ is l.s.c. } &\iff \forall \lambda \in \mathbb{R},\ [\varphi > \lambda] \text{ is open.}\\
&\iff \exists \text{ open set } V \ni x, \forall y \in V,\ [\varphi(y) > \mu].\\
&\iff \text{open set } V \times (-\infty, \mu) \subseteq (\operatorname{epi}\varphi)^c, \text{ while } (x,\lambda) \in V \times (-\infty,\mu).\\
&\iff (\operatorname{epi}\varphi)^c \text{ is open.}
\end{aligned}$$

□

Prop 1.18. φ is l.s.c. $\iff$ $\forall x \in E, \varepsilon > 0,\ \exists$ open set $V \ni x,$ s.t. $\varphi(y) \leqslant \varphi(x) - \varepsilon,\ \forall y \in V$.

Proof.

step 1. $\Rightarrow$

$\forall x \in E, \varepsilon > 0$, let $\lambda = \varphi(x) - \varepsilon$. Obviously $x \in [\varphi > \lambda] \implies \exists$ open set $V \ni x,\ \forall y \in V, [\varphi(y) > \lambda = \varphi(x) - \varepsilon]$.

step 2. $\Leftarrow$

For fixed λ, $\forall x \in [\varphi > \lambda]$, we have $\varphi(x) > \lambda$. Let $\varepsilon = \frac{\varphi(x)-\lambda}{2} \implies \varphi(y) \leqslant \varphi(x) - \varepsilon = \frac{\varphi(x)+\lambda}{2} > \lambda \implies V \subseteq [\varphi > \lambda] \implies [\varphi > \lambda]$ is open. □

Prop 1.19. φ_1, φ_2 is l.s.c., then $\varphi_1 + \varphi_2$ is l.s.c.

Proof. Actually, $\forall x \in E$ and $\varepsilon > 0$, $\exists V_1, V_2 \ni x,$ s.t.

$$\varphi_1(y) \geqslant \varphi_1(x) - \frac{\varepsilon}{2} \text{ on } V_1 \qquad \varphi_2(y) \geqslant \varphi_2(x) - \frac{\varepsilon}{2} \text{ on } V_2.$$

Let $V = V_1 \cap V_2$, then $(\varphi_1 + \varphi_2)(y) \geqslant (\varphi_1 + \varphi_2)(x) - \varepsilon \implies \varphi_1 + \varphi_2$ is l.s.c. □

Prop 1.20. If $\{\varphi_i\}_{i\in I}$ is a family of l.s.c. functions then their *superior envelope* $\varphi = \sup\limits_{i \in I} \varphi_i$ is also l.s.c.

Proof.

step 1. superior envelope:

$$\begin{aligned}\operatorname{epi}\sup_{i\in I}\varphi_i &= \{(x,\lambda)\colon \sup_{i\in I}\varphi_i \leqslant \lambda\}\\ &= \{(x,\lambda)\colon \forall i\in I, \varphi_i \leqslant \lambda\}\\ &= \bigcap_{i\in I}\{(x,\lambda)\colon \varphi_i\leqslant\lambda\}\\ &= \bigcap_{i\in I}\operatorname{epi}\varphi_i\end{aligned}$$

step 2. Then supreme is l.s.c:

φ_i is l.s.c. $\implies$ $\operatorname{epi}\varphi_i$ is closed, so $\operatorname{epi}\sup_{i\in I}\varphi_i = \bigcap_{i\in I}\operatorname{epi}\varphi_i$ is closed, which means $\sup_{i\in I}\varphi_i$ is closed. □

For further propositions, we need some knowledge of topology.

Def 1.22 (Hausdorff property)**.** X, topological space with following property:

$$\forall x_1\neq x_2,\ \exists \text{ open sets } U_1\ni x_1, U_2\ni x_2,\ \text{ s.t. } U_1\cap U_2=\emptyset.$$

Obviously all metric spaces has Hausdorff property with $\mathbb{B}(x_i, \frac{d(x_1,x_2)}{2})$ $(i=1,2)$. Otherwise, Hausdorff property promises that any seq's limit point is unique.

Def 1.23 (converge)**.** For any $\{x_n\}\subseteq X, x\in X$, if:

$$\forall \text{ open set } U\ni x,\ \exists N,\ \text{ s.t. } n>N \implies x_n\in U,$$

then call $x_n\to x$.

e.g. Let $(X,\mathcal{T}_{\text{trivial}})$ a t.v.s, then for fixed seq in X, $\forall x\in X$ is its limit point.

Rmk. $\mathcal{T}_{\text{trivial}}$ is the coarsest topology, where the structure of the open sets in the space is so 'coarse' that the continuity defined by this topology looks like all the points are 'glued together'.

Def 1.24 (seq-compact)**.** t.v.s. X is called **seq-compact** iff:

$$\forall\{x_n\}\subseteq X,\ \exists\{x_{n_k}\} \text{ s.t. } x_{n_k}\xrightarrow{k\to+\infty} x\in X.$$

e.g. Let $(X,\mathcal{T}_{\text{trivial}})$ a t.v.s, then $\forall A\subseteq X$, A is seq-compact but not closed in $(X,\mathcal{T}_{\text{trivial}})$.

Lem 1.4. (X,d) a metric space. Then any seq-compact set $F\subseteq X$ is closed in (X,d).

Proof. Let $x\in F'$, then $\exists\{x_n\}\subseteq F$, s.t. $x_n\to x$. As $F\subseteq X$ is seq-compact, exists a sub-seq of $\{x_n\}$ s.t. x_{k_n} converges to a point in F. While $\{x_{k_n}\}$ is a sub-seq of $\{x_n\}$, we know $x\in F$ immediately, which means F is closed.

Def 1.25. If $\forall\{C_1,\dots,C_n\}\subseteq\mathcal{C}$ as a finite family in X, we have:

$$\bigcap_{i=1}^n C_i\neq\emptyset,$$

then $\mathcal{C}$ is called having **finite intersection property**.

Def 1.26 (compact)**.** If any open covering $\mathcal{A}$ of X has a finite sub-covering, then X is called **compact**.

Thm 1.21. X is compact $\iff$ $\forall \mathcal{C}$ a family of closed set has finite intersection property.

Proof. Fixed $\mathcal{A}$, let $\mathcal{C} = \{X \setminus A \colon A \in \mathcal{A}\}$. Then obviously:

(1). $\mathcal{A}$ is an open family $\iff$ $\mathcal{C}$ is a closed family.

(2). $\mathcal{A}$ is a covering $\iff$ $\bigcap\limits_{C \in \mathcal{C}} C$ is empty.

(3). $\{A_1, \dots, A_n\} \subseteq \mathcal{A}$ covering X $\iff$ $\bigcap\limits_{i=1}^{n} C_i = \bigcap\limits_{i=1}^{n} (X \setminus A_i) = \emptyset$.

Considering the converse proposition, the conclusion is obvious。

Rmk (nested seq thm)**.** For compact space X, if $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_n \supseteq \dots$, where C_n is nonempty closed set in X for any $n \in \mathbb{N}$. Then $\bigcap\limits_{n \in \mathbb{N}} C_n \neq \emptyset$.

Now let's see the propositions:

Prop 1.22. If φ is l.s.c, then for every seq $\{x_n\}$ in E such that $x_n \to x$, we have

$$\boxed{\liminf_{n \to \infty} \varphi(x_n) \geqslant \varphi(x)}$$

and conversely if E is a metric space.

Proof.

step 1. $\Rightarrow$:

Let V_n s.t. $\varphi(y) \geqslant \varphi(x) - \frac{1}{n}$ for $\forall y \in V_n$. Then $\exists \varphi(x_{k_n}) = \inf\limits_{i \geqslant k_n} \varphi(x_i)$, s.t. $x_{k_n} \in V_n$[i], so

$$\liminf_{n \to \infty} \varphi(x_n) = \lim_{n \to \infty} \varphi(x_{k_n}) \geqslant \lim_{n \to \infty} (\varphi(x) - \frac{1}{n}) = \varphi(x).$$

step 2. $\Leftarrow$ with E a metric space:

Let $(x_n, \lambda_n) \in \operatorname{epi} \varphi$, then

$$\varphi(x_n) \leqslant \lambda_n, \ \varphi(x) \leqslant \liminf_{n \to \infty} \varphi(x_n),$$

so $\varphi(x) \leqslant \liminf\limits_{n \to \infty} \lambda_n = \lambda = \lim\limits_{n \to \infty} \lambda_{k_n} \implies (x, \lambda) \in \operatorname{epi} \varphi \implies \operatorname{epi} \varphi$ is seq-compact. Then $\operatorname{epi} \varphi$ is closed while E is metric space.

$\square$

Prop 1.23. E compact, φ l.s.c., then $\inf\limits_{x \in E} \varphi(x)$ is achieved. i.e.

$$\exists a \in E, \text{ s.t. } \varphi(a) = \inf_{x \in E} \varphi(x).$$

Proof. Actually, let $m = \inf\limits_{x \in E} \varphi(x)$. Then for all $n \in \mathbb{Z}_{>0}$, $[\varphi \leqslant m + \frac{1}{n}]$ is nonempty and closed[ii] $\implies$ $[\varphi \leqslant m] = \bigcup\limits_{n > 0} [\varphi \leqslant m + \frac{1}{n}]$ is closed and nonempty. That is, $\exists a \in E$, $\varphi(a) \leqslant m$, i.e. $\varphi(a) = m$. $\square$

1.4.2 convex functions on vector space

Point. *We now assume that E is a* ***vector space****.*

[i] With k_n selected big enough.

[ii] because φ is l.s.c.