

Day 7

虚位移原理.

(Constrained Force).

(Applied Force).

将速度分解为外来的力和约束力. 而其余的力则称为主动力. 将主动力记为 $\vec{F}_{(ai)}$, 约束力记为 $\vec{N}_{(ai)}$. 常义法类的系为所有质量所受到的和加速度的乘积. 则 $\sum_{ai} \vec{N}_{(ai)} \cdot \vec{\delta x}_{(ai)} = 0$.
 从牛顿力学出发, 我们有 $\vec{F}_{(ai)} + \vec{M}_{(ai)} = m_{(ai)} \vec{x}_{(ai)}$. 从而有 $\sum_{ai} (\vec{F}_{(ai)} + \vec{N}_{(ai)} - m_{(ai)} \vec{x}_{(ai)}) \cdot \vec{\delta x}_{(ai)} = 0 \Rightarrow \sum_{ai} (\vec{F}_{(ai)} - m_{(ai)} \vec{x}_{(ai)}) \cdot \vec{\delta x}_{(ai)} = 0$. 在平衡条件下, $\sum_{ai} \vec{F}_{(ai)} \cdot \vec{\delta x}_{(ai)} = 0$.
 $(\vec{N}_{(ai)} = \text{该质点受到的和加速度相等的场})$.

下面从 d'Alembert's Principle 推出拉格朗日方程. 由于质点的运动并不独立, 不可直接得 $\vec{F}_{(ai)} = m_{(ai)} \vec{x}_{(ai)}$. 故应用广义坐标, 将上式拆成 n 个独立方程: $\vec{\delta x}_{(ai)} = \frac{\partial \vec{x}_{(ai)}}{\partial q^a} \vec{\delta q}^a$

$$\Rightarrow \sum_{ai} (\vec{F}_{(ai)} - m_{(ai)} \vec{x}_{(ai)}) \cdot \frac{\partial \vec{x}_{(ai)}}{\partial q^a} \cdot \vec{\delta q}^a = 0 \Rightarrow \sum_{ai} m_{(ai)} \vec{x}_{(ai)} \cdot \frac{\partial \vec{x}_{(ai)}}{\partial q^a} = \sum_{ai} \vec{F}_{(ai)} \cdot \frac{\partial \vec{x}_{(ai)}}{\partial q^a}, a=1, \dots, n. \quad \text{故布, 为这 } n \text{ 个方程中任一可以用广义坐标表示.}$$

$$\vec{x}_{(ai)} = \frac{\partial \vec{x}_{(ai)}}{\partial q^a} \cdot q^a + \frac{\partial \vec{x}_{(ai)}}{\partial t} \cdot \frac{\partial q^a}{\partial t} \Rightarrow \vec{\dot{x}}_{(ai)} = \frac{\partial \vec{x}_{(ai)}}{\partial q^a} \cdot q^a + \sum_{ai} \frac{\partial \vec{x}_{(ai)}}{\partial q^a} \cdot \frac{\partial q^a}{\partial t} \quad \text{将 } \vec{x}_{(ai)} \text{ 分解为}$$

$$\text{而将时间定义为广义力 } \vec{Q}_a = \sum_{ai} \vec{F}_{(ai)} \cdot \frac{\partial \vec{x}_{(ai)}}{\partial q^a}$$

$$\begin{aligned} \text{(对质点求导)} &= \frac{d}{dt} \left(\sum_{ai} m_{(ai)} \vec{x}_{(ai)} \cdot \frac{\partial \vec{x}_{(ai)}}{\partial q^a} \right) - \sum_{ai} m_{(ai)} \cdot \vec{x}_{(ai)} \cdot \frac{d}{dt} \left(\frac{\partial \vec{x}_{(ai)}}{\partial q^a} \right) \\ \text{(差分形式)} &= \frac{d}{dt} \cdot \frac{\partial}{\partial q^a} \left(\frac{1}{2} \sum_{ai} m_{(ai)} \vec{x}_{(ai)}^2 \right) - \frac{\partial}{\partial q^a} \left(\frac{1}{2} \sum_{ai} m_{(ai)} \vec{x}_{(ai)}^2 \right) \\ &= \frac{d}{dt} \left(\frac{\partial T}{\partial q^a} \right) - \frac{\partial T}{\partial q^a}. \end{aligned}$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial T}{\partial q^a} \right) - \frac{\partial T}{\partial q^a} = \vec{Q}_a. \quad \text{对于所有力为保守力的情况}, \vec{F}_a = -\frac{\partial V}{\partial \vec{x}_{(ai)}}.$$

$$\text{且 } \nabla_{\vec{x}_{(ai)}} \vec{Q}_a = -\frac{\partial V}{\partial \vec{x}_{(ai)}} \cdot \frac{\partial \vec{x}_{(ai)}}{\partial q^a} = -\frac{\partial V}{\partial q^a}. \quad \text{立刻得到标准的拉格朗日方程}$$

→ Jourdain's Principle: 虚功方程: $\sum_{ai} [\vec{F}_{(ai)} - m_{(ai)} \vec{x}_{(ai)}] \cdot \vec{\delta x}_{(ai)} = 0$.

→ 仅讨论单质点的离散微小位移: $S = \frac{1}{2m_{(ai)}} (\vec{F}_{(ai)} - m_{(ai)} \vec{x}_{(ai)})^2$ 最小.