FORSIDE MAT121

For obligatorisk inlevering i matematikkurset

MAT121, våren 2013

KAND. NR:

(kandidatsnummeret ditt finner du på semesterkortet. Det er **ikke** nummeret på studentkortet!)

Oppgaven:

- Kandidatnummeret skal stå øverst i hjørne på alle sidene du leverer.
- Navnet ditt skal ikke stå på oppgaven.
- Oppgaven skal stiftes sammen (ikke bruk binders).
- **Oppgaven uten denne forsiden blir ikke evaluert!**
- Skriv stort og tydelig.
- Besvarelsen blir returnert.

Frist for innlevering:

- **Torsdag 11 april før kl. 14.00.**
- Lever gjerne oppgaven før innleveringsfristen.

Sted for innlevering:

- Skranken i Inforsenteret for realfagstudenter, Realfagbygget.

Veiledning til den obligatoriske oppgaven

- Settet består av 6 oppgaver. Alle skal besvares.
- Besvarelsen må være så detaljert at det er klart hvordan beregningene er utført. Spørsmål om oppgavene kan eventuelt behandles av gruppelederne under regneøvelsene.
- Der er tilllat å diskutere oppgavene i grupper, men **hver student må formulere sin egen skriftlige besvarelse.** Om to besvarelser er mistenkelig like, kan det betraktes som fusk.
- For å lette arbeidet til alle er oppgavene på norsk og på engelsk. Besvarelsen kan leveres på norsk eller på engelsk.
- Lykke til!

Exercise 1.

a. Let $A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix}$ and $\overrightarrow{b} = \begin{pmatrix} 5 \\ 3 \\ 0 \\ 7 \end{pmatrix}$. Write the reduced echelon form for the augmented matrix $B = [A, \overrightarrow{b}]$ and show pivot columns and pivot positions. Solve the equation $A\overrightarrow{x} = \overrightarrow{b}$.

b. Find a basis and the dimension of $\mathrm{Col}(B)$, $\mathrm{Row}(B)$ and $\mathrm{Null}(B)$. Calculate the rank of B.

c. Is the matrix A invertible? If yes, find A^{-1} by using the Gauss elimination procedure.

Exercise 2.

a. Let $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -1 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}$, $\overrightarrow{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find

$$\overrightarrow{y} = B\overrightarrow{x}, \quad C = AB, \quad A\overrightarrow{y}, \quad C\overrightarrow{x}, \quad C^T B^T, \quad (A^T + B)C.$$

b. Can we calculate A^2, A^{-1}? Why?

c. Find the inverse matrix to $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix}$. Use the formula for A^{-1} with co-factors.

Exercise 3.

a. Find the volume of the parallelepiped spanned by $(-1,0,2)$, $(3,-1,3)$, $(4,0,-1)$.

b. Find the solution of the system

$$\begin{cases} 2x_1 + x_2 + x_3 = 4 \\ -x_1 + \qquad 2x_3 = 2 \\ 3x_1 + x_2 + 3x_3 = -2 \end{cases}$$

by making use of the Cramer rule.

c. Let S be the parallelogram spanned by vectors

$$\overrightarrow{a} = (-3,4), \quad \overrightarrow{b} = (2,5),$$

and

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 6 \end{pmatrix}.$$

Find the area of the image of S under the linear transformation $\overrightarrow{x} \mapsto A\overrightarrow{x}$.

Exercise 4.

a. Find a linearly independent subset of the set:

$$\begin{pmatrix} 4 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ -5 \\ 2 \end{pmatrix}.$$

Explain the answer.

b. Assume that H_1 and H_2 are subspaces of the vector space $\mathbb{R}^n$.
   1. Show that $H = H_1 \cap H_2$ is also a subspace of the vector space $\mathbb{R}^n$. Remember that the intersection $H_1 \cap H_2$ consists of the vectors that belong both to H_1 and H_2.
   2. Show by construction of the example that $R = H_1 \cup H_2$ is not always a subspace of $\mathbb{R}^n$. Remember that the union $H_1 \cup H_2$ consists of the vectors that belong to at least one of the sets H_1 and H_2.

c. Let $\alpha = \{\alpha_1, \alpha_2, \alpha_3\}$ and $\beta = \{\beta_1, \beta_2, \beta_3\}$ be bases for a vector space V and assume that

$$\beta_1 = 2\alpha_1 - \alpha_2 + \alpha_3, \quad \beta_2 = 3\alpha_2 + \alpha_3, \quad \beta_3 = -3\alpha_1 + 2\alpha_3.$$

1. Find the change-of-coordinate matrix from the basis β to the basis α.
2. Write the vector $x = \beta_1 - 2\beta_2 + 2\beta_3$ in the basis α.
3. Write the vector $y = \alpha_2 - 3\alpha_3$ in the basis β.

Exercise 5.

a. Find the eigenvalues and the eigenvectors for the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$

b. Find a diagonal matrix similar to the matrix

$$B = \begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix}.$$

c. Let $T\colon \mathbb{P}_2 \to \mathbb{P}_3$ be a transformation (that acts from the vector space of polynomials of the order at most 2 to the vector space of polynomials of the order at most 3) such that

$$\mathbb{P}_2 \ni p(x) \quad \mapsto \quad (x+10)p(x) \in \mathbb{P}_3.$$

1. Find the image of $p(x) = 5 + x - 3x^2$.
2. Show that T is a linear map.

Exercise 6.

a. Show that if A and B are invertible, then the inverse to $(AB)^T$ is equal to

$$(A^{-1})^T(B^{-1})^T.$$

b. Let $\overrightarrow{e_1}, \dots, \overrightarrow{e_n}$ be the standard basis of $\mathbb{R}^n$ and $\overrightarrow{v_1}, \dots, \overrightarrow{v_n}$ is a basis for an n-dimensional vector space V. The linear map $\phi\colon \mathbb{R}^n \to V$ such as

$$\phi(\overrightarrow{e_1}) = \overrightarrow{v_1}, \quad \dots, \quad \phi(\overrightarrow{e_n}) = \overrightarrow{v_n}$$

is called the coordinate isomorphism. Show that the coordinate isomorphism is a bijective linear map from $\mathbb{R}^n$ to V.

c. Let $\mathcal{T}$ be a triangle with vertices

$$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2), \quad P_3 = (x_3, y_3).$$

Show that the area of the triangle is equal to

$$\frac{1}{2}\left|\det M\right|, \quad \text{where} \quad M = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}.$$