Exercise 1

a

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 2 & 1 & 3 \end{pmatrix} \; \vec{b} = \begin{pmatrix} 5 \\ 3 \\ 0 \\ 7 \end{pmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 3 & 7 \end{bmatrix}$$

Add $-1 * row1$ to $row4$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Add $-2 * row2$ to $row1$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Add $2 * row2$ to $row3$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 3 & 3 & 6 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Divide $row3$ by 3

$$\begin{bmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Add $2 * row3$ to $row1$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Add $-1 * row3$ to $row2$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix}$$

Add $-1 * row3$ to $row4$

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivot columns are 1,2 and 3.
Pivot positions are $B_{1,1}$, $B_{2,2}$, $B_{3,3}$.

Solving $A\vec{x} = \vec{b}$. We look at the reduced echelon form of B and that gives us this system:

$$\begin{aligned} x_1 \quad & +2x_4 = 3 \\ x_2 \quad & = 1 \\ & x_3 + x_4 = 2 \end{aligned}$$

Gives us solution:

$x_1 = 3 - 2x_4$

$x_2 = 1$

$x_3 = 2 - x_4$

$x_4 = free$

b

Basis and dimension for Col(B)

Basis: $\{\{1,0,0,0\},\{0,1,0,0\},\{0,0,1,0\}\}$
Dimension: by definition of dimension(p. 193 in our book) dim(Col(B)) =# of vectors in basis = 3

Basis and dimension for Row(B)

Basis: $\{\{1,0,0,2,3\},\{0,1,0,0,1\},\{0,0,1,1,2\}\}$
Dimension: by definition of dimension(p. 193 in our book) dim(Row(B)) = dim(Col(B)) = 3

Basis and dimension for Null(B)

Basis: $\{\{-2,0,-1,1\},\{-3,-1,-2,0,1\}\}$
Solve $Bx = 0$, we already have reduced echelon form of A by B so just add

column in B with all 0s

$$\begin{bmatrix} 1 & 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -2x_4 - 3x_5 \\ -x_5 \\ -x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix}$$

Decomposition: $\begin{bmatrix} -2x_4 - 3x_5 \\ -x_5 \\ -x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Dimension: 2

Rank of B

By The Rank Theorem (p.194 in our book)

$$\begin{aligned} rank(B) + dim(Null(B)) &= n \\ rank(B) &= n - dim(Null(B)) \\ rank(B) &= 5 - 2 \\ rank(B) &= 3 \end{aligned}$$

C

A matrix A is invertible if its row-reduced echelon form is I_4, we already know the row-reduced echelon form of B so row-reduced echelon form of A is just B without the last column. This is because while row-reducing B you also row-reduce A.

RREF of A:

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We see that $A \neq I_4$ so A is not invertible.

Exercise 2

a

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix}, B = \begin{pmatrix} 2 & -1 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}, \vec{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

y=B$\vec{x}$

$$B\vec{x} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 + (-2) \\ -1 + 0 \\ 3 - 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

C=AB

$$AB = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1*2 + (-1)*(-1) + 2*3 & 1*(-1) + (-1)*0 + 2*(-1) \\ 0*2 + 1*(-1) + 3*3 & 0*(-1) + 1*0 + 3*(-1) \end{pmatrix} =$$

$$\begin{pmatrix} 2+1+6 & -1+0-2 \\ 0-1+9 & 0+0-3 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ 8 & -3 \end{pmatrix}$$

$$C = \begin{pmatrix} 9 & -3 \\ 8 & -3 \end{pmatrix}$$

A$\vec{y}$

$$A\vec{y} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 + (-1)*(-1) + 2*1 \\ 0 + (-1)*1 + 3*1 \end{pmatrix} = \begin{pmatrix} 0+1+2 \\ 0-1+3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$A\vec{y} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

C$\vec{x}$

$$C\vec{x} = \begin{pmatrix} 9 & -3 \\ 8 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 + (-3)*2 \\ 8 + (-3)*2 \end{pmatrix} = \begin{pmatrix} 9-6 \\ 8-6 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$C\vec{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$\mathbf{C}^T\mathbf{B}^T$

$\mathrm{C}^T = \begin{pmatrix} 9 & 8 \\ -3 & -3 \end{pmatrix}$, $\mathrm{B}^T = \begin{pmatrix} 2 & -1 & 3 \\ -1 & 0 & -1 \end{pmatrix}$

$\mathrm{C}^T\mathrm{B}^T = \begin{pmatrix} 9*2+8*(-1) & 9*(-1)+0 & 9*3+8*(-1) \\ -3*2+(-3)*(-1) & (-3)*(-1)+0 & (-3)*3+(-3)*(-1) \end{pmatrix} = \begin{pmatrix} 18-8 & -9 & 27-8 \\ -6+3 & 3 & -9+3 \end{pmatrix} = \begin{pmatrix} 10 & -9 & 19 \\ -3 & 3 & -6 \end{pmatrix}$

$\mathrm{C}^T\mathrm{B}^T = \begin{pmatrix} 10 & -9 & 19 \\ -3 & 3 & -6 \end{pmatrix}$

$(\mathbf{A}^T+\mathbf{B})\mathbf{C}$

$\mathrm{A}^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix}$

$\mathrm{A}^T C = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 9 & -3 \\ 8 & -3 \end{pmatrix} = \begin{pmatrix} 9+0 & -3+0 \\ -9+8 & 3-3 \\ 18+24 & -6-9 \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ -1 & 0 \\ 42 & -15 \end{pmatrix}$

We notice that $(C^TB^T)^T = BC$ by transpose properties $(A^T)^T = A$ and $(AB)^T = B^TA^T$.

$(C^TB^T)^T = BC = \begin{pmatrix} 10 & -9 & 19 \\ -3 & 3 & -6 \end{pmatrix}^T = \begin{pmatrix} 10 & -3 \\ -9 & 3 \\ 19 & -6 \end{pmatrix}$

$\mathrm{A}^T\mathrm{C} + \mathrm{BC} = \begin{pmatrix} 9 & -3 \\ -1 & 0 \\ 42 & -15 \end{pmatrix} + \begin{pmatrix} 10 & -3 \\ -9 & 3 \\ 19 & -6 \end{pmatrix} = \begin{pmatrix} 19 & -6 \\ -10 & 3 \\ 61 & -21 \end{pmatrix}$

b

A^2?

We cannot calculate A^2 because to multiply two matrices A with dimensions $m \times k$ the second matrix B needs to have dimensions $k \times n$ by theorem. In this exercise $B = A$ and since A is not a square matrix it will not meet the requirements.

A^{-1}?

We cannot calculate A^{-1} because A is not square, therefore $detA$ is not defined and by definition of determinant (p.203 in our book) it does not have an inverse

c

We denote the matrix as A.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 2 & 3 & 3 \end{pmatrix}$$

We apply formula for inverse of $n \times n$ matrix:

$$A^{-1} = \frac{1}{det(A)} adj(A)$$

To find $adj(A)$ we need to find transpose of the cofactor-matrix:

Cofactor-matrix C:

$$A_{11} = \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix}, A_{12} = \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}, A_{13} = \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$A_{21} = \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix}, A_{22} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}, A_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$A_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, A_{32} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}, A_{33} = \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$C = \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = +0, \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} = -(-5), \begin{vmatrix} -1 & 1 \\ 2 & 3 \end{vmatrix} = +(-5) \\ \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = -(-3), \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = +(-1), \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1 \\ \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = +(-1), \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3, \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = +2 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 5 & -5 \\ 3 & -1 & -1 \\ -1 & -3 & 2 \end{pmatrix}$$

We calculate $\det A$ next by multiplying each element at index i in row A_1 with the element in index i in row C_1 and summing them together:

$$\det A = (1 * 0) + (1 * 5) + (2 * -5) = 0 + 5 - 10 = -5$$

$$\text{adj(A)} = \text{C}^T \ \text{C}^T = \begin{pmatrix} 0 & 3 & -1 \\ 5 & -1 & -3 \\ -5 & -1 & 2 \end{pmatrix}$$

$$\text{A}^{-1} = \frac{1}{-5} \begin{pmatrix} 0 & 3 & -1 \\ 5 & -1 & -3 \\ -5 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{3}{-5} & \frac{1}{5} \\ -1 & \frac{1}{5} & \frac{3}{5} \\ 1 & \frac{1}{5} & \frac{2}{-5} \end{pmatrix}$$

$$\text{A}^{-1} = \begin{pmatrix} 0 & \frac{3}{-5} & \frac{1}{5} \\ -1 & \frac{1}{5} & \frac{3}{5} \\ 1 & \frac{1}{5} & \frac{2}{-5} \end{pmatrix}$$

Exercise 3

a

To find the volume of the parallelepiped we need to find the scalar triple product (theorem 9, p. 221 in our book). We can do this by computing the absolute value of the determinant

$\begin{vmatrix} -1 & 0 & 2 \\ 3 & -1 & 3 \\ 4 & 0 & -1 \end{vmatrix} = -1 \begin{vmatrix} -1 & 3 \\ 0 & -1 \end{vmatrix} - 0 \begin{vmatrix} 3 & 3 \\ 4 & -1 \end{vmatrix} + 2 \begin{vmatrix} 3 & -1 \\ 4 & 0 \end{vmatrix} = -1 + 0 + (2*4) = |7| = 7$

The volume of the parallelpiped is 7.

b

Let us view the system as Ax=b

$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix}$

$A_1(b) = \begin{pmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{pmatrix}, \quad A_2(b) = \begin{pmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{pmatrix}, \quad A_3(b) = \begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix}$

Next we need to find determinants.

$|detA| = 2 \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + 1 \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} = 2*(-2) - (-9) + (-1) = -4 + 9 - 1 = 4$

$|detA_1(b)| = 4 \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} - 1 \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} + 1 \begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} = 4*(-2) - 10 + 2 = -16$

$|detA_2(b)| = 2 \begin{vmatrix} 2 & 2 \\ -2 & 3 \end{vmatrix} - 4 \begin{vmatrix} -1 & 2 \\ 3 & 3 \end{vmatrix} + 1 \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} = 2*10 - 4(-9) + (-4) = 20 + 36 - 4 = 52$

$|detA_3(b)| = 2 \begin{vmatrix} 0 & 2 \\ 1 & -2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 2 \\ 3 & -2 \end{vmatrix} + 4 \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} = 2*(-2) - (-4) + 4*(-1) = -4 + 4 - 4 = -4$

Since $|detA|$ is non-zero the system has a unique solution, by applying Cramer's rule we get solutions:

$$x_1 = \frac{detA_1(b)}{detA} = \frac{-16}{4} = -4$$
$$x_2 = \frac{detA_2(b)}{detA} = \frac{52}{4} = 13$$
$$x_3 = \frac{detA_3(b)}{detA} = \frac{-4}{4} = -1$$

c

$\vec{a} = (-3, 4), \qquad \vec{b} = (2, 5), \qquad A = \begin{pmatrix} 1 & 2 \\ -2 & 6 \end{pmatrix}$

We apply theorem about linear transformations and paralellogram (theorem 10, p.223 in our book).
$Area(T(S)) = |detA|\dot{A}rea(S)$

$|detA| = 6 - (-4) = 10$
Area of parallellogram is computed by absolute value of the determinant from the matrix C formed by $\vec{a}$ and $\vec{b}$. $C = \begin{pmatrix} -3 & 4 \\ 2 & 5 \end{pmatrix}$

$|detC| = -15 - 8 = |-23| = 23$

$10 * 23 = 230$

Area of S is 230

Exercise 4

a

$$\begin{pmatrix} 4 \\ 1 \\ -3 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -2 \\ 3 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \\ -5 \\ 2 \end{pmatrix}$$

Finding a linearly independent set is the same as finding the basis.
Put all the vectors in the set in a matrix A, then row-reduce and see which columns are pivot columns.

$$A = \begin{pmatrix} 4 & -2 & 3 & 1 \\ 1 & 3 & -1 & 2 \\ -3 & 4 & 2 & -5 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Divide row 1 by 4

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 1 & 3 & -1 & 2 \\ -3 & 4 & 2 & -5 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Add -1*row1 to row2

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{7}{2} & -\frac{7}{4} & \frac{7}{4} \\ -3 & 4 & 2 & -5 \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Add 3*row1 to row3

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{7}{2} & -\frac{7}{4} & \frac{7}{4} \\ 0 & \frac{5}{2} & \frac{17}{4} & -\frac{17}{4} \\ 2 & 0 & 0 & 2 \end{pmatrix}$$

Add -2*row1 to row4

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 0 & \frac{7}{2} & -\frac{7}{4} & \frac{7}{4} \\ 0 & \frac{5}{2} & \frac{17}{4} & -\frac{17}{4} \\ 0 & 1 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$$

Divide row2 by $\frac{7}{2}$

$$\begin{pmatrix} 1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} \\ 0 & \frac{5}{2} & \frac{17}{4} & -\frac{17}{4} \\ 0 & 1 & -\frac{3}{2} & \frac{3}{2} \end{pmatrix}$$

Add $-\frac{5}{2}$*row2 to row3

$$\begin{pmatrix}1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4}\\ 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{11}{2} & -\frac{11}{2}\\ 0 & 1 & -\frac{3}{2} & \frac{3}{2}\end{pmatrix}$$

Add -1*row2 to row4

$$\begin{pmatrix}1 & -\frac{1}{2} & \frac{3}{4} & \frac{1}{4}\\ 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{11}{2} & -\frac{11}{2}\\ 0 & 0 & -1 & 1\end{pmatrix}$$

Add $\frac{1}{2}$*row2 to row1

$$\begin{pmatrix}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & \frac{11}{2} & -\frac{11}{2}\\ 0 & 0 & -1 & 1\end{pmatrix}$$

Divide row3 by $\frac{11}{2}$

$$\begin{pmatrix}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 1 & -1\\ 0 & 0 & -1 & 1\end{pmatrix}$$

Add 1*row3 to row4

$$\begin{pmatrix}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & -\frac{1}{2} & \frac{1}{2}\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Add $\frac{1}{2}$*row3 to row2

$$\begin{pmatrix}1 & 0 & \frac{1}{2} & \frac{1}{2}\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Add $-\frac{1}{2}$*row3 to row1

$$\begin{pmatrix}1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Pivot columns at 1, 2 and 3, so a linearly independent subset of the set is:

$$\begin{pmatrix}4\\ 1\\ -3\\ 2\end{pmatrix}, \quad \begin{pmatrix}-2\\ 3\\ 4\\ 0\end{pmatrix}, \quad \begin{pmatrix}3\\ -1\\ 2\\ 0\end{pmatrix}$$

b

1

$H_1 \subseteq \mathbb{R}^n$
$H_2 \subseteq \mathbb{R}^n$
$H = H_1 \cap H_2$

To prove H is a subspace of $\mathbb{R}^n$ it needs to satisfy the three properties for definition of subspace (p. 236 in our book):

(a) **The zero vector of $\mathbb{R}^n$ is in H**
Since the zero vector of $\mathbb{R}^n$ is in H_1 and H_2 it will be in H per definition of intersection

(b) **H is closed under vector addition**
Take two vectors $\vec{u}$ and $\vec{v}$ from H, $\vec{u}, \vec{v} \subset H_1$ and $\vec{u}, \vec{v} \subset H_2$. Since we know H_1 and H_2 are subspaces of $\mathbb{R}^n$ $\vec{u} + \vec{v}$ is closed under vector addition in H_1 and H_2 and therefore closed under vector addition in their intersection.

(c) **H is closed under multiplication by scalars**
Take any vector $\vec{u} \subset H$ and scalar c, all vectors $c\vec{u}$ are in both H_1 and H_2 which are closed by scalar multiplication, therefore their intersection is too.

c

$\alpha = \{\alpha_1, \alpha_2, \alpha_3\}, \quad \beta = \{\beta_1, \beta_2, \beta_3\}$

$\beta_1 = 2\alpha_1 - \alpha_2 + \alpha 3, \quad \beta_2 = 3\alpha_2 + \alpha_3, \quad \beta_3 = -3\alpha_1 + 2\alpha_3$

1

$$\underbrace{\mathrm{P}}_{\alpha \leftarrow \beta} = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ -3 & 0 & 2 \end{pmatrix}$$

Exercise 5

a

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

First we find the eigenvalues, we need to find $|detA|$

$$|detA| = \begin{vmatrix} (1-\lambda) & -1 & 2 \\ 2 & (-2-\lambda) & 4 \\ 0 & 1 & (1-\lambda) \end{vmatrix} = (1-\lambda)\begin{vmatrix} (-2-\lambda) & 4 \\ 1 & (1-\lambda) \end{vmatrix} - (-1)\begin{vmatrix} 2 & 4 \\ 0 & (1-\lambda) \end{vmatrix} + 2\begin{vmatrix} 2 & -2-\lambda \\ 0 & 1 \end{vmatrix}$$

$= (1-\lambda)(\lambda^2+\lambda-6) + (2-2\lambda) + 4$
$= \lambda^2+\lambda-6-\lambda^3-\lambda^2+6\lambda+2-2\lambda+4$
$= -\lambda^3+5\lambda$

The eigenvalues are: $\lambda_1 = 0, \quad \lambda_2 = \sqrt{5}, \quad \lambda_3 = -\sqrt{5}$

To find the eigenvectors we have to solve the equation $(A-\lambda_i I)\vec{v} = 0$ for $i = 1, 2, 3$

We start with $i = 1, \lambda_1 = 0$

$$\begin{pmatrix} (1-0) & -1 & 2 \\ 2 & (-2-0) & 4 \\ 0 & 1 & (1-0) \end{pmatrix} \vec{v}_1 = \vec{0} \rightarrow$$

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow[row-ops]{} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We get the system:
$x = -3z$
$y = -z$
$zfree$

Choose $z = 1$ and we get:
$x = -3, \quad y = -1, \quad z = 1$

Which gives us the eigenvector $\vec{v}_1 = \begin{pmatrix} -3 \\ -1 \\ 1 \end{pmatrix}$

Now for $i = 2, \lambda_2 = \sqrt{5}$

$$\begin{pmatrix} (1-\sqrt{5}) & -1 & 2 \\ 2 & (-2-\sqrt{5}) & 4 \\ 0 & 1 & (1-\sqrt{5}) \end{pmatrix} \vec{v}_2 = \vec{0} \rightarrow$$

$$\begin{pmatrix} (1-\sqrt{5}) & -1 & 2 \\ 2 & (-2-\sqrt{5}) & 4 \\ 0 & 1 & (1-\sqrt{5}) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now for $i = 3, \lambda_3 = -\sqrt{5}$

$$\begin{pmatrix} (1+\sqrt{5}) & -1 & 2 \\ 2 & (-2+\sqrt{5}) & 4 \\ 0 & 1 & (1+\sqrt{5}) \end{pmatrix} \vec{v}_3 = \vec{0} \rightarrow$$

$$\begin{pmatrix} (1+\sqrt{5}) & -1 & 2 \\ 2 & (-2+\sqrt{5}) & 4 \\ 0 & 1 & (1+\sqrt{5}) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

b

$$B = \begin{pmatrix} 0 & -4 & -6 \\ -1 & 0 & -3 \\ 1 & 2 & 5 \end{pmatrix}$$

We need to find an invertible matrix P and a diagonal matrix D such that $B = PDP^{-1}$

Eigenvalues of B:

$$B - \lambda I_2 = \begin{pmatrix} (0-\lambda) & -4 & -6 \\ -1 & (0-\lambda) & -3 \\ 1 & 2 & (5-\lambda) \end{pmatrix}$$

$$|detB - \lambda I_2| = -\lambda \begin{vmatrix} -\lambda & -3 \\ 2 & (5-\lambda) \end{vmatrix} + 4 \begin{vmatrix} -1 & -3 \\ 1 & (5-\lambda) \end{vmatrix} - 6 \begin{vmatrix} -1 & -\lambda \\ 1 & 2 \end{vmatrix}$$
$$= -\lambda(\lambda^2 - 5\lambda + 6) + 4(\lambda - 2) - 6(-2 + \lambda)$$
$$= -\lambda^3 + 5\lambda^2 - 6\lambda + 4\lambda - 8 + 12 - 6\lambda$$
$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

Characteristic polynomial is $-\lambda^3 + 5\lambda^2 - 8\lambda + 4$

Eigenvalues are:

$\lambda_1 = 2, \quad \lambda_2 = 2, \quad \lambda_3 = 1$

Eigenvectors:

$\lambda_1 = 2$

$$\begin{pmatrix} (0-2) & -4 & -6 \\ -1 & (0-2) & -3 \\ 1 & 2 & (5-2) \end{pmatrix} \vec{v}_1 = \vec{0} \rightarrow$$

$$\begin{pmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \overrightarrow{row - ops}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We get the system
$x = -2y - 3z$
$y free$
$z free$

Choose $z = 1, y = 0$, we get $x = -3$
Choose $z = 0, y = 1$, we get $x = -2$

We get eigenvectors
$\vec{v}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \qquad \vec{v}_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$

Eigenvector for $\lambda_3 = 1$:
$\begin{pmatrix} (0-1) & -4 & -6 \\ -1 & (0-1) & -3 \\ 1 & 2 & (5-1) \end{pmatrix} \vec{v}_3 = \vec{0} \rightarrow$
$\begin{pmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow[row-ops]{} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
We get the system
$x = -2z$
$y = -z$
$z free$

Choose $z = 1$, we get $y = -1, \quad x = -2$

The eigenvector is
$\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

We know there exists a diagonal matrix because there are n distinct eigenvectors, by theorem 6 (p. 339 in our book)
This gives us $P = \begin{pmatrix} -3 & -2 & -2 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

Now construct D from the eigenvalues with respect to the columns we the constructed P from $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Exercise 6

a

$(AB)^T$
$((AB)^T)^{-1} =$
$((AB)^{-1})^T =$
$(B^{-1}A^{-1})^T =$
$(A^{-1})^T(B^{-1})^T$

Alt:
$(AB)^T = B^T A^T$
$(B^T A^T)^{-1} =$
$(A^T)^{-1}(B^T)^{-1} =$
$(A^{-1})^T(B^{-1})^T$

c

T is a triangle with vertices
$P_1 = (x_1, y_1), \quad P_2 = (x_2, y_2), \quad P_3 = (x_3, y_3$

$$|detM| = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} =$$