

**Bachelorarbeit**

**Topological Entropy of  
Formal Languages**

**Florian Starke**

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Technische Universität Dresden  
Fakultät Mathematik  
Institut für Geometrie

Betreuender Hochschullehrer: Prof. Dr. Andreas Thom



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# 1 Martingale Technique

**Definition 1.** A *martingale* is a family  $(f_i, \mathcal{F}_i)_{i \in \{0, \dots, n\}}$  such that

- $f_i$  is integrable for all  $i \in \{0, \dots, n\}$ ,
- $f_i$  is  $\mathcal{F}_i$  measurable for all  $i \in \{0, \dots, n\}$ , and
- $f_i = \mathbb{E}[f_{i+1} | \mathcal{F}_i]$  for all  $i \in \{0, \dots, n-1\}$ .

**Lemma 2.** For all  $x \in \mathbb{R}$

$$e^x \leq x + e^{x^2}.$$

**Lemma 3** (Azema's inequality).

$$\mu(\{x \in X \mid |f(x) - \mathbb{E}(f)| \geq c\}) \leq 2 \exp \left( -\frac{c^2}{4 \sum_{i=1}^n \|d_i\|_\infty^2} \right)$$

$$\mu(\{|f - \mathbb{E}(f)| \geq c\}) \leq 2 \exp \left( -\frac{c^2}{4 \sum_{i=1}^n \|d_i\|_\infty^2} \right)$$

**Definition 4.** Let  $(X, d, \mu)$  be an mm-space.

**Theorem 5.** If an mm-space  $(X, d, \mu)$  has length  $l$ , then the concentration function of  $X$  satisfies

$$\alpha_X(\varepsilon) \leq 2 \exp \left( -\frac{\varepsilon^2}{16l^2} \right).$$

**Theorem 6.** Let  $G$  be a compact group with a bi-invariant metric  $d$ , and let

$$\{e\} = G_0 < G_1 < \dots < G_n = G$$

be a chain of subgroups. Denote the diameter of  $G_i / G_{i-1}$  with respect to the factor metric by  $a_i$ . Then the concentration function of the mm-space  $(G, d, \mu)$ , where  $\mu$  is the normalized Haar measure, satisfies

$$\alpha_X(\varepsilon) \leq 2 \exp \left( -\frac{\varepsilon^2}{16 \sum_{i=1}^n a_i^2} \right).$$

**Theorem 7.** The normalized counting measure on the groups  $\text{SL}_{2^n}(q)$  concentrates with respect to the rank-metric, i.e. for all  $r > 0$

$$\lim_{n \rightarrow \infty} \alpha_{\text{SL}_{2^n}}(r) = 0.$$

**Lemma 8.** Let  $(X, d, \mu)$  be an mm-space with diameter  $d$  and

$$\Omega_0 = \{X\} \prec \dots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

with  $a_1, \dots, a_n$  as in Definition 4. Then

$$\sum_{i=1}^n a_i \geq d.$$

Florian sagt:  
"could be that  
this only holds  
for finite  $X$  as  
conditions in  
definition of  
length are just  
almost surely"

*Proof.* Let  $x, y \in X$ , with  $x \neq y$ , we show  $d(x, y) \leq \sum_{i=1}^n a_i$ . Let  $i_0$  be the smallest number such that  $[x]_{i_0} \neq [y]_{i_0}$ . Since  $[x]_0 = X = [y]_0$  we know that  $i_0$  is at least 1. Therefore  $[x]_{i_0-1} = [y]_{i_0-1}$  and there is an isomorphism  $\varphi_{i_0}: [x]_{i_0} \rightarrow [y]_{i_0}$  such that  $d(\varphi_{i_0}(x), y) \leq a_{i_0}$ . Let  $x_{i_0} = \varphi_{i_0}(x)$ , then

$$d(x, y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

Florian sagt:  
"here is the a.s.  
problem"

If  $x_{i_0} = y$ , then we are done. Otherwise let  $i_1$  be the smallest number such that  $[x_{i_0}]_{i_1} \neq [y]_{i_1}$ . Then let  $\varphi_{i_1}: [x_{i_0}]_{i_1} \rightarrow [y]_{i_1}$  be an isomorphism such that  $d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$ . Define  $x_{i_1} = \varphi_{i_1}(x_{i_0})$ . Proceeding in this fashion yields elements  $x_{i_0}, \dots, x_{i_k}$  such that  $x_{i_k} = y$  and

$$d(x, y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \dots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \dots + a_{i_k} \leq \sum_{i=1}^n a_i.$$

□

**Lemma 9.** Let  $(X, d, \mu)$  be an mm-space with diameter 1 and  $\Delta = \min d$ . Then the length of  $X$  is at least  $\Delta^{\frac{1}{2}}$ .

**Definition 10.** The symplectic group of degree  $2n$  over a field  $q$ , denoted by  $\text{Sp}(2n, q)$ , is the subgroup of  $\text{SL}(2n, q)$  containing all matrices  $A$  such that

$$A^T \Omega A = \Omega, \text{ where } \Omega = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

**Lemma 11.** Let  $g: V \rightarrow V$  be an isomorphism,  $V = U \oplus U'$ , and  $g(U') \subseteq U'$ . Then the map

$$g': V \rightarrow V$$

$$v \mapsto \begin{cases} g(v) - \pi_{U'}(g(v)) & \text{if } v \notin U' \\ v & \text{if } v \in U' \end{cases}$$

i.e.  $g' = \pi_U \circ g - \pi_U \circ g \circ 1_{U'} + 1_{U'}$ , is an isomorphism and  $d(g, g') \leq \frac{1}{n} \cdot \dim U'$ .

**Lemma 12.** [what we still need (add conditions for  $\omega$  if necessary)] Let  $\omega: V \times V \rightarrow k$  be a bilinear map,  $U, U'$  subspaces of  $V$ , and  $h: U \rightarrow U'$  an isomorphism that preserves  $\omega$ . Then  $h$  can be extended to an isomorphism on  $V$  which also preserves  $\omega$ .

*Proof.* w.l.o.g.  $\dim U + 1 = \dim V$  ?

□

**Lemma 13.** Let  $V = U \oplus U'$ ,  $\omega$  a bilinear map,  $G$  be the group of automorphisms of  $(V, \omega)$  and  $G' \leq G$  the subgroup fixing  $U'$ . Then the diameter of  $G/G_i$  is at most  $\frac{3 \cdot \dim U'}{n}$ .

Florian sagt:  
"...additional  
conditions"

Florian sagt:  
"adapt this"

*Proof.* Let  $g \in G$ , we show that there are  $g' \in G$  and  $g'' \in G'$  such that  $g'(U') \subseteq U'$ ,  $g'|_{U'} = 1_{U'}$ , and

$$d(g, g'') \leq d(g, g') + d(g', g'') \leq \frac{2 \dim U'}{n} + \frac{\dim U'}{n}.$$

By Lemma 12 we can extend the map  $g^{-1}|_{gU'}$  to a map  $h'$  on  $V' = \langle U', gU' \rangle$ . Now define  $g' = (1_{V''} \oplus h')g$ , where  $V = V'' \oplus V'$  and apply Lemma to  $g'$  to obtain  $g''$ .

$$\begin{aligned} \operatorname{im} g - g' &= \operatorname{im} g - (1_{V''} \oplus h')g \\ &= \operatorname{im}(1_{V''} \oplus 1_{V'} - 1_{V''} \oplus h') \\ &= \operatorname{im}(1_{V'} - h') \\ &\subseteq V' \end{aligned}$$

$$d(g, g') = \frac{1}{n} \dim \operatorname{im} g - g' \leq \frac{\dim V'}{n}$$

□

## 2 Limits of other Matrix group families are Levy groups too

When studying matrices it is often useful to look at the corresponding linear maps of a suitable vector space. In the case of orthogonal, symplectic, or unitary matrices these are linear maps from the vector space to itself preserving an orthogonal, symplectic, or unitary form respectively. Formally, the symplectic group  $\mathrm{Sp}_n(q)$  is isomorphic to  $\mathrm{Aut}(V, \omega)$ , where  $V$  is an  $n$ -dimensional  $F(q)$  vector space and  $\omega$  is a symplectic form.

As we have to handle only finite dimensional vector spaces here a lot of nice theorems hold. ...

Let  $V$  be an  $n$  dimensional vector space.

**Lemma 14.** *For all  $U \leq V$  there is an  $U' \leq V$  such that  $U \oplus U' = V$ .*

Let  $\omega$  be a bilinear form on  $V$ .

**Lemma 15.** *Let  $U \leq V$ . Then  $\dim U^\perp = \dim V - \dim U$ .*

**Lemma 16.** *Let  $U \leq V$ . Then  $U^{\perp\perp} = U$ .*

**Lemma 17.** *There exists a  $U \leq V$  with  $\dim U \leq 2$  such that  $V = U \oplus U^\perp$ .*

*Proof.* Let  $e \in V \setminus \{0\}$ . By Lemma 15  $\dim e^\perp = n - 1$ .

If  $e \notin e^\perp$ , then  $V = \langle e \rangle \oplus e^\perp$  and  $\langle e \rangle$  is the desired  $U$ .

If  $e \in e^\perp$ , then extend  $e$  to a basis  $e, b_2, \dots, b_{n-1}$  of  $e^\perp$  and consider the 2-dimensional subspace  $U := \langle b_2, \dots, b_{n-1} \rangle^\perp$ . Now we have to show that

$$U \cap U^\perp = 0.$$

Take  $v$  from the intersection. By Lemma 16  $U^\perp = \langle b_2, \dots, b_{n-1} \rangle$  and  $v \perp b_i$  for all  $i \in \{2, \dots, n-1\}$ . Since  $\langle b_2, \dots, b_{n-1} \rangle \leq e^\perp$  we also have  $v \perp e$ . Hence  $v \in e^{\perp\perp} = \langle e \rangle$  and  $v = \lambda e$ . Now  $e \notin \langle b_2, \dots, b_{n-1} \rangle$  implies  $v = 0$ . Henceforth  $V = U \oplus U^\perp$ .  $\square$

**Lemma 18.** *Let  $U \leq V$  and  $f: V \rightarrow V$  be an isometry such that  $f|_U = 1_U$ . Then  $f(U^\perp) = U^\perp$ .*

*Proof.* As  $\dim f(U^\perp) = \dim U^\perp$  it suffices to show that  $f(u') \perp u$  for all  $u \in U$  and  $u' \in U^\perp$ .

$$\begin{aligned} \omega(f(u'), u) &= \omega(f(u'), f(u)) \\ &= \omega(u', u) \\ &= 0 \end{aligned}$$

This concludes the proof.  $\square$

**Lemma 19.** *For all  $W \leq V$  there is a  $W' \leq W^\perp$  such that  $W \cap W' = 0$  and*

$$\dim W' \geq \dim V - 2 \dim W.$$

*Proof.* By Lemma 14 there is a  $W'$  such that

$$W^\perp = (W^\perp \cap W) \oplus W'.$$

Clearly,  $W \cap W' = 0$  and

$$\dim W' = \dim W^\perp - \dim(W^\perp \cap W) \geq \dim W^\perp - \dim W.$$

Whats left is to show that  $\dim W^\perp \geq \dim V - \dim W$ . Let  $b_1, \dots, b_{\dim W}$  be a basis of  $W$ . Then  $W^\perp$  is equal to the kernel of the linear map

$$V \rightarrow F_q^{\dim W} \quad v \mapsto \begin{pmatrix} \omega(b_1, v) \\ \vdots \\ \omega(b_{\dim W}, v) \end{pmatrix}.$$

Now the statement follows from the rank-nullity theorem.  $\square$

**Lemma 20.** *Let  $U, W \leq V$  such that  $U \perp W$  and  $U \cap W = 0$ . Then  $\langle U, W \rangle \cong U \oplus W$ .*

**Lemma 21.** *Let  $g_1: U_1 \rightarrow W_1$  and  $g_2: U_2 \rightarrow W_2$  be isometries such that  $U_1 \perp U_2$ ,  $U_1 \cap U_2 = 0$ ,  $W_1 \perp W_2$ , and  $W_1 \cap W_2 = 0$ . Then  $g_1 \oplus g_2: U_1 \oplus U_2 \rightarrow W_1 \oplus W_2$  is also an isometry.*

Florian sagt:  
"maybe  $g: U_1 \rightarrow U_2$  and  $h: W_1 \rightarrow W_2$  better"

*Proof.* Obviously,  $g_1 \oplus g_2$  is again a bijective linear map. Consider  $v_1 + v_2, u_1 + u_2 \in U_1 \oplus U_2$

$$\begin{aligned} \omega(v_1 + v_2, u_1 + u_2) &= \omega(v_1, u_1) + \omega(v_1, u_2) + \omega(v_2, u_1) + \omega(v_2, u_2) \\ &= \omega(v_1, u_1) + 0 + 0 + \omega(v_2, u_2) && (U_1 \perp U_2) \\ &= \omega(g_1(v_1), g_1(u_1)) + \omega(g_2(v_2), g_2(u_2)) \\ &= \omega(g_1(v_1), g_1(u_1)) + \omega(g_1(v_1), g_2(u_2)) \\ &\quad + \omega(g_2(v_2), g_1(u_1)) + \omega(g_2(v_2), g_2(u_2)) && (W_1 \perp W_2) \\ &= \omega(g_1 \oplus g_2(v_1 + v_2), g_1 \oplus g_2(u_1 + u_2)) \end{aligned}$$

Hence  $g_1 \oplus g_2$  preserves  $\omega$ .  $\square$

[other useful theorems]

**Theorem 22 (Witt).** *Let  $V$  be an orthogonal, symplectic, or unitary space. Let  $U$  and  $W$  be subspaces of  $V$  and suppose  $\alpha: U \rightarrow W$  is an isometry. Then  $\alpha$  extends to an isometry of  $V$ .*

**Lemma 23.** *Let  $G$  be an orthogonal, symplectic, or unitary group. ...*

*Proof.*  $G = \text{Aut}(V, \omega)$  for some vector space  $V$  with bilinear form  $\omega$ . Use Lemma 17 to obtain  $U \leq V$  such that  $V = U \oplus U^\perp$  and  $\dim U \leq 2$ . Define  $H = \text{Aut}(U^\perp, \omega)$ . Our aim is to find for any  $g \in G$  an  $g' \in H$  such that  $d(g, g') \leq \frac{4}{n}$ .



The idea is to find a map  $h \in H$  that behaves like the inverse of  $g$  on  $gU$  and like the identity on most of the rest. Then  $hg$  is the desired  $g'$ .

Let  $g \in G$  and define  $W = \langle U, gU \rangle$ . By Lemma 19 there is a  $W'$  such that  $\dim W' \geq n - 8$ ,  $W' \leq W^\perp$ , and  $W' \cap W = 0$ . Consider the map

$$g^{-1}|_{gU} \oplus 1_{W'}: gU \oplus W' \rightarrow U \oplus W'$$

as  $g^{-1}|_{gU}$  and  $1_{W'}$  are isometries and  $W \perp W'$  Lemma 21 implies that the above map is also an isomtry. By Witt's lemma this isometry can be extended to an isometry  $h: V \rightarrow V$ .

$$\begin{aligned} n \cdot d(g, hg) &= \dim \operatorname{im}(g - hg) \\ &\leq 8 + \dim \operatorname{im}(g - hg)|_{W'} && (\dim W' \geq n - 8) \\ &= 8 + \dim \operatorname{im}(g - g)|_{W'} && (h|_{W'} = 1_{W'}) \\ &= 8 \end{aligned}$$

Finally, we need to show that  $hg \in H$ , here the choice of  $H$  using Lemma 17 comes into play. By construction of  $h$  we have that  $hg|_U = 1_U$ . Therefore we can apply Lemma 18 and get that  $hg(U^\perp) = U^\perp$ . Hence  $hg \in H$  and  $d(g, hg) \leq \frac{8}{n}$ .  $\square$

### 3 Fun

Consider an  $n$ -dimensional cube with  $2^k$  nodes on each edge. Then its diameter  $\nabla_{n,k}$  and length  $L_{n,k}$  are

$$\nabla_{n,k} = \sqrt{(2^k - 1) \cdot n} \qquad L_{n,k} = \sqrt{\sum_{i=0}^{k-1} 2^{2i} \cdot n}.$$

Henceforth

$$\lim_{n \rightarrow \infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{L_{1,k}}{\nabla_{1,k}} \qquad \text{and} \qquad \lim_{k \rightarrow \infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{1}{\sqrt{3}}.$$

## ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema “Topological Entropy of Formal Languages” selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum

Unterschrift