Bachelorarbeit

Measure Concentration for Symplectic Groups

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Contents

Preface

If we have a finite group of matrices, then we can equip it with the rank metric and the normalized Haar measure to obtain a *metric measure space*. There is a well defined limit of ... Carderi and Thom showed in [**?**] that the limit of SL_n is *extremely amenable*. The goal of this thesis is to generalize this result to limits of other matrix group families, namely unitary, symplectic, and orthogonal matrices. The general strategy will be the following: given a family $(G_n)_{n\in\mathbb{N}}$ of (mm) matrix groups we first find an upper bound for the *concentration function* of G_n using a consequence of Azemas inequality [**?**]. As the upper bounds converge to zero we conclude that $(G_n)_{n\in\mathbb{N}}$ is a *Lévy family*, making their limit a *Lévy group*. Finally, we know from [**?**] that every Lévy group is extremely amenable.

1 Introduction

Define limit of G_n

Examples of matrices in the limit

structure of thesis:

1. Azema
2. Thoms proof (matrices as automorphisms but without form)
3. want to generalize this so we need a form Hence extending the automorphism becomes harder so use Witts lemma
4. generalized version of the proof
5. application coloring theorem

2 Limits of matrix groups and extreme amenability

Let $\mathrm{GL}_n(q)$ be the general linear group over the q element field $\mathbb{F}_q$ and let G be a subgroup of $\mathrm{GL}_n(q)$. We can equip G with the (normalized) *rank-metric* $d(g,h) := \frac{1}{n}\,\mathrm{r}(g-h)$. Since all matrices in G have full rank, this metric is bi-invariant, i.e. $d(kg,kh) = d(g,h) = d(gk,hk)$ for all $g,h,k \in G$. Let $G_n \leq \mathrm{GL}_{2^n}(q)$ be a family of subgroups, such that $\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in G_{n+1}$ for all $g \in G_n$. Note that the map

$$\varphi_n\colon G_n \mapsto G_{n+1}, \text{ where } \varphi_n(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

is an isometric homomorphism for all $n \in \mathbb{N}$. Hence we can define the inductive limit of $(G_n)_{n\in\mathbb{N}}$. We denote the metric completion of this limit by $\mathrm{clim}_{n\to\infty} G_n$.

Lemma 1. *The group* $\mathrm{clim}_{n\to\infty} G_n$ *is a topological group.*

Proof. The bi-invariance of d is preserved by the limit and the completion. ... $\square$

Now that we have a topology on $G := \operatorname{clim}_{n\to\infty} G_n$ we can ask whether it is *extremely amenable*, i.e. every continuous action of G on a compact topological space admits a fixed point. It is hard to show this directly, but we know that every Lévy group is extremely amenable. Hence we will show that for suitable $(G_n)_{n\in\mathbb{N}}$ the limit G will be a Lévy group.

Before we can define Lévy groups we need the following definition.

Definition 2. A *metric measure space* (mm-space) $\boldsymbol{X}$ is a triple (X, d, μ), where d is a metric on the set X and μ is a measure on the Borel σ-algebra induced by d. We will always assume that $\mu(X) = 1$. For any set $A \subseteq X$ denote the *r − neighborhood* of A, i.e. $\{x \in X \mid \exists y \in A.\, d(x, y) < r\}$, by $N_r(A)$. The *measure concentration function* of $\boldsymbol{X}$ is defined as

Florian sagt: "definition ugly, use d_A?"

$$\alpha_{\boldsymbol{X}}(r) = \sup\{1 - \mu(N_r(A)) \mid A \subseteq X, \mu(A) \geq \frac{1}{2}\}.$$

A family of mm-spaces $\boldsymbol{X}_n$ with diameter 1 is called a *Lévy family* if

$$\alpha_{\boldsymbol{X}_n}(r) \to 0$$

for all $r > 0$.

A topological space $\boldsymbol{X}$ is a *Polish space* if it is homeomorphic to a complete metric space that has a countable dense subset.

Now we can come back to groups.

Definition 3. A *Polish group* G is a topological group where the underlying topological space is a Polish space. A *Lévy group* is a group G equipped with a metric d, where

- G with the topology induced by d is a Polish group and
- there is a sequence $(G_n)_{n\in\mathbb{N}}$ of compact subgroups, such that $(G_n, d|_{G_n}, \mu_n)_{n\in\mathbb{N}}$ is a Lévy family. Here μ_n is the normalized Haar measure of G_n.

Florian sagt: "lim G_n dense in G?"

Note that the normalized Haar measure of G_n is just the normalized counting measure. The following theorem from [**?**] gives the desired connection to extreme amenability.

Theorem 4. *Every Lévy group is extremely amenable.*

To apply this theorem to our setting we need the following lemma.

Lemma 5. *Let $G_n \leq \mathrm{GL}_{2^n}(q)$ and $G = \operatorname{clim}_{n\to\infty} G_n$. Then G is a Polish group.*

Proof. By Lemma 1 G is already a topological group and by definition it is also a complete metric space. Furthermore, every G_n is finite. Hence the inductive limit of the G_n is a countable dense subset of G. □

Whether G is also a Lévy group depends on the particular choice of $(G_n)_{n\in\mathbb{N}}$. To show that for certain sequences G will be a Lévy group, we will bound $\alpha_{G_n}(r)$. The next section develops the methods necessary to obtain this upper bound.

3 An upper bound for the measure concentration function

In this section we will prove Azema's inequality and as a consequence, we will obtain an upper bound for the measure concentration function. As the next results rely heavily on stochastic methods we will briefly introduce the necessary notions. Since the G_n are all finite and equipped with the normalized counting measure we will only consider *probability spaces* (X, Σ, μ), where X is finite, Σ is a σ-algebra over X, and $\mu(A) = |A|/|X|$ for $A \subseteq X$. For a more general approach see [**?**]. Note that Σ has a very nice representation.

Lemma 6. *Let Σ be a σ-algebra over a finite set X, then Σ is the smallest σ-algebra containing the partition $A_1, \ldots, A_n$, where the A_i's are the minimal nonempty sets in Σ.*

Proof. First we show that $A_1, \ldots, A_n$ is a partition of X. Since $A_i \cap A_j \in \Sigma$ we conclude, by minimality of A_i and A_j, that either $i = j$ or $A_i \cap A_j = \emptyset$. Clearly, every element of X is contained in a one of the A_i.

For $A \in \Sigma$ we have, again by minimality, that $A \cap A_i$ is either A_i or $\emptyset$. Therefore A can be written as a union of A_i's. □

Note that it follows from the proof that any $A \in \Sigma$ can be written as $\bigcup_{i \in I} A_i$ for a suitable I. This lemma allows us to use partitions and σ-algebras interchangeably. We will denote the partition corresponding to Σ by $A_1, \ldots, A_n$, for Σ' we will use $A'_1, \ldots, A'_{n'}$, etc. The next definition is simplified a lot by only considering finite X.

Definition 7. Let (X, Σ, μ) be a probability space, $f \colon X \to \mathbb{R}$ be a measurable function, and Σ' be a sub-σ-algebra of Σ. Then the *conditional expectation* of f with respect to Σ' is defined as

$$\mathbb{E}(f \mid \Sigma') := \sum_{i=1}^{n'} \mathbb{E}(f \mid A'_i) \cdot \mathbb{1}_{A'_i}.$$

One often thinks of Σ' as the available information, a finer partition means more information. The conditional expectation $\mathbb{E}(f \mid \Sigma')$ is the best approximation of f given only the information from Σ'. With this intuition the statements from the following lemma are not surprising.

Lemma 8. *Let (X, Σ, μ) be a probability space, $f, g \colon X \to \mathbb{R}$ be measurable functions, $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ be sub-σ-algebras. Then*

i) if $f \leq g$, then $\mathbb{E}(f \mid \Sigma') \leq \mathbb{E}(g \mid \Sigma')$,

ii) for any Σ'-measurable function $h \colon X \to \mathbb{R}$ we have $\mathbb{E}(hf \mid \Sigma') = h \cdot \mathbb{E}(f \mid \Sigma')$,

iii) also $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') = \mathbb{E}(f \mid \Sigma'') = \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma')$.

Proof. To i): If $f \leq g$, then

$$\mathbb{E}(f \mid \Sigma') = \sum_{i=1}^{n} \mathbb{E}(f \mid A'_i) \cdot \mathbb{1}_{A'_i} \leq \sum_{i=1}^{n} \mathbb{E}(g \mid A'_i) \cdot \mathbb{1}_{A'_i} = \mathbb{E}(g \mid \Sigma').$$

To ii): Let $h \colon X \to \mathbb{R}$ be Σ'-measurable function, then $h = \sum_{i=1}^{n'} h_i \mathbb{1}_{A'_i}$. Now

$$\begin{aligned}
\mathbb{E}(hf \mid \Sigma') &= \sum_{i=1}^{n'} \mathbb{E}(hf \mid A'_i) \mathbb{1}_{A'_i} \\
&= \sum_{i=1}^{n'} h_i \mathbb{E}(f \mid A'_i) \mathbb{1}_{A'_i} \\
&= h \cdot \mathbb{E}(f \mid \Sigma').
\end{aligned}$$

To iii): Note that $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid A') = \mathbb{E}(f \mid A')$ for all $A' \in \Sigma'$.

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') &= \sum_{i=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid A''_i) \cdot \mathbb{1}_{A''_i} \\
&= \sum_{i=1}^{n''} \mathbb{E}(f \mid A''_i) \cdot \mathbb{1}_{A''_i} && (A''_i \in \Sigma') \\
&= \mathbb{E}(f \mid \Sigma'') \\
&= \sum_{j=1}^{n''} \mathbb{E}(f \mid A''_j) \cdot \mathbb{1}_{A''_j} \cdot \sum_{i=1}^{n'} \mathbb{1}_{A'_i} \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid A''_j) \mid A'_i) \cdot \mathbb{1}_{A''_j} \cdot \mathbb{1}_{A'_i} \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid A''_j) \cdot \mathbb{1}_{A''_j} \mid A'_i) \cdot \mathbb{1}_{A'_i} && \text{(by ii)} \\
&= \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma')
\end{aligned}$$

This concludes the proof. □

The following lemma might not seem very interesting, but changing the exponent from x to x^2 is the very foundation for Azema's inequality.

Lemma 9. *For all $x \in \mathbb{R}$*

$$e^x \leq x + e^{x^2}.$$

Proof. Note that for $x = 0$ both sides are equal to 1. As both sides are differentiable it suffices to show that the derivative of the right hand side is larger than the derivative of the left hand side for all $x \geq 0$ and smaller for all $x \leq 0$. Hence, we want to show

$$e^x \geq 1 + 2xe^{x^2} \text{ for all } x \leq 0 \qquad \text{and} \qquad e^x \leq 1 + 2xe^{x^2} \text{ for all } x \geq 0.$$

As for $x = 0$ both sides are again equal to 1 we can reduce the problem, by similar reasoning, to the question whether

$$e^x \leq 2e^{x^2} + 4x^2 e^{x^2} \text{ for all } x \in \mathbb{R}.$$

- For $x = 0$ the terms reduce to $1 \leq 2$.
- For $x < 0$ the left hand side is bounded by 1, while the right hand side is still larger that 2.
- For $1 \leq x$ we have $x \leq x^2$ and the inequality holds trivially.
- For $0 < x < 1$ note that the both sides are increasing. Hence the inequality holds for all x with $e^x \leq 2$. Finally, $\ln 2 \geq \frac{1}{2}$ and therefore the right hand side with $x = \ln 2$ evaluates to a number larger then e.

$\square$

Before we will prove Azema's inequality let us introduce some useful notation. Whenever there is no danger of confusion we will abbreviate sets of the form

$$\{x \in X \mid \text{Condition}(x) \text{ holds}\} \qquad \text{by} \qquad \{\text{Condition}\}.$$

For example $\{x \in X \mid f(x) = c\}$ becomes $\{f = c\}$.

Lemma 10 (Azema's inequality)**.** *Let (X, Σ, μ) be a probability space, $f\colon X \to \mathbb{R}$ a measurable function, and $\{X\} = \Sigma_0 \subseteq \cdots \subseteq \Sigma_n = \Sigma$ a chain of sub-σ-algebras. Define $f_i := \mathbb{E}(f \mid \Sigma_i)$ and $d_i := f_i - f_{i-1}$. Then for every $c \geq 0$*

$$\mu(\{|f - \mathbb{E}(f)| \geq c\}) \leq 2 \cdot \exp\left(-\frac{c^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

Note that $(f_i, \Sigma_i)_{0 \leq i \leq n}$ is a discrete martingale.

Proof. First, observe that $f_0 = \mathbb{E}(f \mid \{X\}) = \mathbb{E}(f)$ and $f_n = \mathbb{E}(f \mid \Sigma) = f$. Using a simple telescoping sum we obtain $f - \mathbb{E}(f) = d_1 + \cdots + d_n$. Therefore

$$\begin{aligned}
\mu(\{f - \mathbb{E}(f) \geq c\}) &= \mu(\{\sum_{i=1}^{n} d_i \geq c\}) \\
&= \mu(\{\lambda \cdot \sum_{i=1}^{n} d_i \geq \lambda c\}) && \text{(for } \lambda > 0) \\
&= \mu(\{e^{\lambda \cdot \sum_{i=1}^{n} d_i - \lambda c} \geq 1\}) \\
&\leq \mathbb{E}(e^{\lambda \cdot \sum_{i=1}^{n} d_i}) \cdot e^{-\lambda c} && (*) \\
&= \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}} \cdot \mathbb{E}(e^{\lambda d_n} \mid \Sigma_{n-1})) \cdot e^{-\lambda c} && \text{(Lemma 8)} \\
&\leq \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}}) \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda c} && (**) \\
&\;\;\vdots \\
&\leq e^{\lambda^2 \cdot \|d_1\|_\infty^2} \cdot \ldots \cdot e^{\lambda^2 \cdot \|d_{n-1}\|_\infty^2} \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda c} \\
&= e^{\lambda^2 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2 - \lambda c}.
\end{aligned}$$

For $(*)$ note that for any measurable function $g\colon X \to \mathbb{R}$ with $g \geq 0$ we have

$$\mu(\{g \geq 1\}) = \mathbb{E}(\mathbb{1}_{\{g \geq 1\}}) \leq \mathbb{E}(g).$$

For $(**)$ we need to use Lemma 9

$$\begin{aligned}
\mathbb{E}(e^{\lambda d_i} \mid \Sigma_{i-1}) &\leq \mathbb{E}(\lambda d_i \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&= \lambda \cdot \mathbb{E}(f_i - f_{i-1} \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&\leq 0 + e^{\lambda^2 \|d_i\|_\infty^2}. && \text{(Lemma 8)}
\end{aligned}$$

Substituting $-\frac{c^2}{\sum_{i=1}^{n} \|d_i\|_\infty^2}$ for λ we conclude that

$$\mu(\{f - \mathbb{E}(f) \geq c\}) \leq \exp\left(-\frac{c^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

The same calculations with $-d_i$ instead of d_i yield the dual inequality

$$\mu(\{f - \mathbb{E}(f) \leq -c\}) \leq \exp\left(-\frac{c^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

These two statements obviously give us the desired result. $\square$

Definition 11. Let (X, d, μ) be an mm-space.

Florian sagt: "n dimensional cubes with diameter 1 have length $\frac{1}{\sqrt{3}}$ and if $|X|$ is a prime, then the length of X is equal to its diameter, the length of a circle is the same as its diameter"

Between any two points in X there is almost surely a path $x_0, \ldots, x_n$ such that

$$\sum_{i=1}^{n} d(x_{i-1}, x_i)^2 \leq \operatorname{len}(X)^2$$

(this path probably has some special properties).

Theorem 12. *If an mm-space (X, d, μ) has length l, then the concentration function of X satisfies*

$$\alpha_X(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 l^2}\right).$$

Theorem 13. *Let G be a compact group with a bi-invariant metric d, and let*

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

be a chain of subgroups. Denote the diameter of G_i / G_{i-1} with respect to the factor metric by a_i. Then the concentration function of the mm-space (G, d, μ), where μ is the normalized Haar measure, satisfies

$$\alpha_X(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 \sum_{i=1}^n a_i^2}\right).$$

Theorem 14. *The normalized counting measure on the groups $\mathrm{SL}_{2^n}(q)$ concentrates with respect to the rank-metric, i.e. for all $r > 0$*

$$\lim_{n \to \infty} \alpha_{\mathrm{SL}_{2^n}}(r) = 0.$$

Lemma 15. *Let (X, d, μ) be an mm-space with diameter d and*

$$\Omega_0 = \{X\} \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

with $a_1, \ldots, a_n$ as in Definition 11. Then

$$\sum_{i=1}^n a_i \geq d.$$

Florian sagt: "could be that this only holds for finite X as conditions in definition of length are just almost surely"

Proof. Let $x, y \in X$, with $x \neq y$, we show $d(x, y) \leq \sum_{i=1}^n a_i$. Let i_0 be the smallest number such that $[x]_{i_0} \neq [y]_{i_0}$. Since $[x]_0 = X = [y]_0$ we know that i_0 is at least 1. Therefore $[x]_{i_0-1} = [y]_{i_0-1}$ and there is an isomorphism $\varphi_{i_0} \colon [x]_{i_0} \to [y]_{i_0}$ such that $d(\varphi_{i_0}(x), y) \leq a_{i_0}$. Let $x_{i_0} = \varphi_{i_0}(x)$, then

Florian sagt: "here is the a.s. problem"

$$d(x, y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

If $x_{i_0} = y$, then we are done. Otherwise let i_1 be the smallest number such that $[x_{i_0}]_{i_1} \neq [y]_{i_1}$. Then let $\varphi_{i_1} \colon [x_{i_0}]_{i_1} \to [y]_{i_1}$ be an isomorphism such that $d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$. Define $x_{i_1} = \varphi_{i_1}(x_{i_0})$. Proceeding in this fashion yields elements $x_{i_0}, \ldots, x_{i_k}$ such that $x_{i_k} = y$ and

$$d(x, y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \cdots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \cdots + a_{i_k} \leq \sum_{i=1}^n a_i.$$

□

Lemma 16. *Why is this not the case? (it contradicts the main theorem)*

$$\frac{1}{\sqrt{3}}\operatorname{diam}(X) \le \operatorname{len}(X) \le \operatorname{diam}(X)$$

Lemma 17. *Let (X, d, μ) be an mm-space with diameter 1 and $\Delta = \min d$. Then the length of X is at least $\Delta^{\frac{1}{2}}$.*

Definition 18. The *symplectic group* of degree $2n$ over a field q, denoted by $\mathrm{Sp}(2n, q)$, is the subgroup of $\mathrm{SL}(2n, q)$ containing all matrices A such that

$$A^T \Omega A = \Omega, \text{ where } \Omega = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Lemma 19. *Let $g\colon V \to V$ be an isomorphism, $V = U \oplus U'$, and $g(U') \subseteq U'$. Then the map*

$$\begin{aligned} g'\colon V &\to V \\ v &\mapsto \begin{cases} g(v) - \pi_{U'}(g(v)) & \text{if } v \notin U' \\ v & \text{if } v \in U' \end{cases} \end{aligned}$$

i.e. $g' = \pi_U \circ g - \pi_U \circ g \circ 1_{U'} + 1_{U'}$, is an isomorphism and $d(g, g') \le \frac{1}{n} \cdot \dim U'$.

Lemma 20. *[what we still need (add conditions for ω if necessary)] Let $\omega\colon V \times V \to k$ be a bilinear map, U, U' subspaces of V, and $h\colon U \to U'$ an isomorphism that preserves ω. Then h can be extended to an isomorphism on V which also preserves ω.*

Proof. w.l.o.g. $\dim U + 1 = \dim V$? $\square$

Florian sagt: "…additional conditions"

Lemma 21. *Let $V = U \oplus U'$, ω a bilinear map, G be the group of automorphisms of (V, ω) and $G' \le G$ the subgroup fixing U'. Then the diameter of G/G_i is at most $\frac{3 \cdot \dim U'}{n}$.*

Florian sagt: "adapt this"

Proof. Let $g \in G$, we show that there are $g' \in G$ and $g'' \in G'$ such that $g'(U') \subseteq U'$, $g'|_{U'} = 1_{U'}$, and

$$d(g, g'') \le d(g, g') + d(g', g'') \le \frac{2 \dim U'}{n} + \frac{\dim U'}{n}.$$

By Lemma 20 we can extend the map $g^{-1}|_{gU'}$ to a map h' on $V' = \langle U', gU' \rangle$. Now define $g' = (1_{V''} \oplus h')g$, where $V = V'' \oplus V'$ and apply Lemma to g' to obtain g''.

$$\begin{aligned} \operatorname{im} g - g' &= \operatorname{im} g - (1_{V''} \oplus h')g \\ &= \operatorname{im}(1_{V''} \oplus 1_{V'} - 1_{V''} \oplus h') \\ &= \operatorname{im}(1_{V'} - h') \\ &\subseteq V' \end{aligned}$$

$$d(g, g') = \frac{1}{n} \dim \operatorname{im} g - g' \le \frac{\dim V'}{n}$$

$\square$

4 The limit of $\mathrm{SL}_n(q)$ is extremely amenable

5 Witts Lemma

6 Limits of other Matrix group families are Levy groups too

When studying matrices it is often useful to look at the corresponding linear maps of a suitable vector space. In the case of orthogonal, symplectic, or unitary matrices these are linear maps from the vector space to itself preserving an orthogonal, symplectic, or unitary form respectively. Formally, the symplectic group $\mathrm{Sp}_n(q)$ is isomorphic to $\mathrm{Aut}(V, \omega)$, where V is an n-dimensional $F(q)$ vector space and ω is a symplectic form.

As we have to handle only finite dimensional vector spaces here a lot of nice theorems hold. ...

Let V be an n dimensional vector space.

Lemma 22. *For all $U \leq V$ there is an $U' \leq V$ such that $U \oplus U' = V$.*

Let ω be a bilinear form on V.

Lemma 23. *Let $U \leq V$. Then $\dim U^\perp = \dim V - \dim U$.*

Lemma 24. *Let $U \leq V$. Then $U^{\perp^\perp} = U$.*

Lemma 25. *There exists a $U \leq V$ with $\dim U \leq 2$ such that $V = U \oplus U^\perp$.*

Proof. Let $e \in V \setminus \{0\}$. By Lemma 23 $\dim e^\perp = n - 1$.

If $e \notin e^\perp$, then $V = \langle e \rangle \oplus e^\perp$ and $\langle e \rangle$ is the desired U.

If $e \in e^\perp$, then extend e to a basis $e, b_2, \ldots, b_{n-1}$ of $e^\perp$ and consider the 2-dimensional subspace $U := \langle b_2, \ldots, b_{n-1} \rangle^\perp$. Now we have to show that

$$U \cap U^\perp = 0.$$

Take v from the intersection. By Lemma 24 $U^\perp = \langle b_2, \ldots, b_{n-1} \rangle$ and $v \perp b_i$ for all $i \in \{2, \ldots, n-1\}$. Since $\langle b_2, \ldots, b_{n-1} \rangle \leq e^\perp$ we also have $v \perp e$. Hence $v \in e^{\perp^\perp} = \langle e \rangle$ and $v = \lambda e$. Now $e \notin \langle b_2, \ldots, b_{n-1} \rangle$ implies $v = 0$. Henceforth $V = U \oplus U^\perp$. □

Lemma 26. *Let $U \leq V$ and $f\colon V \to V$ be an isometry such that $f|_U = 1_U$. Then $f(U^\perp) = U^\perp$.*

Proof. As $\dim f(U^\perp) = \dim U^\perp$ it suffices to show that $f(u') \perp u$ for all $u \in U$ and $u' \in U^\perp$.

$$\begin{aligned}
\omega(f(u'), u) &= \omega(f(u'), f(u)) \\
&= \omega(u', u) \\
&= 0
\end{aligned}$$

This concludes the proof. □

Lemma 27. *For all $W \leq V$ there is a $W' \leq W^\perp$ such that $W \cap W' = 0$ and*

$$\dim W' \geq \dim V - 2 \dim W.$$

Proof. By Lemma 22 there is a W' such that

$$W^\perp = (W^\perp \cap W) \oplus W'.$$

Clearly, $W \cap W' = 0$ and

$$\dim W' = \dim W^\perp - \dim(W^\perp \cap W) \geq \dim W^\perp - \dim W.$$

Whats left is to show that $\dim W^\perp \geq \dim V - \dim W$. Let $b_1, \ldots, b_{\dim W}$ be a basis of W. Then $W^\perp$ is equal to the kernel of the linear map

$$V \to F_q^{\dim W} \qquad v \mapsto \begin{pmatrix} \omega(b_1, v) \\ \vdots \\ \omega(b_{\dim W}, v) \end{pmatrix}.$$

Now the statement follows from the rank-nullity theorem. $\square$

Lemma 28. *Let $U, W \leq V$ such that $U \perp W$ and $U \cap W = 0$. Then $\langle U, W \rangle \cong U \oplus W$.*

Lemma 29. *Let $g_1 \colon U_1 \to W_1$ and $g_2 \colon U_2 \to W_2$ be isometries such that $U_1 \perp U_2$, $U_1 \cap U_2 = 0$, $W_1 \perp W_2$, and $W_1 \cap W_2 = 0$. Then $g_1 \oplus g_2 \colon U_1 \oplus U_2 \to W_1 \oplus W_2$ is also an isomtry.*

Florian sagt: "maybe $g\colon U_1 \to U_2$ and $h\colon W_1 \to W_2$ better"

Proof. Obviously, $g_1 \oplus g_2$ is again a bijective linear map. Consider $v_1 + v_2, u_1 + u_2 \in U_1 \oplus U_2$

$$\begin{aligned} \omega(v_1 + v_2, u_1 + u_2) &= \omega(v_1, u_1) + \omega(v_1, u_2) + \omega(v_2, u_1) + \omega(v_2, u_2) \\ &= \omega(v_1, u_1) + 0 + 0 + \omega(v_2, u_2) && (U_1 \perp U_2) \\ &= \omega(g_1(v_1), g_1(u_1)) + \omega(g_2(v_2), g_2(u_2)) \\ &= \omega(g_1(v_1), g_1(u_1)) + \omega(g_1(v_1), g_2(u_2)) \\ &\quad + \omega(g_2(v_2), g_1(u_1)) + \omega(g_2(v_2), g_2(u_2)) && (W_1 \perp W_2) \\ &= \omega(g_1 \oplus g_2(v_1 + v_2), g_1 \oplus g_2(u_1 + u_2)) \end{aligned}$$

Hence $g_1 \oplus g_2$ preserves ω. $\square$

[other useful theorems]

Theorem 30 (Witt)**.** *Let V be an orthogonal, symplectic, or unitary space. Let U and W be subspaces of V and suppose $\alpha \colon U \to W$ is an isometry. Then α extends to an isometry of V.*

Lemma 31. *Let G be an orthogonal, symplectic, or unitary group. ...*

Proof. $G = \mathrm{Aut}(V, \omega)$ for some vector space V with bilinear form ω. Use Lemma 25 to obtain $U \leq V$ such that $V = U \oplus U^{\perp}$ and $\dim U \leq 2$. Define $H = \mathrm{Aut}(U^{\perp}, \omega)$. Our aim is to find for any $g \in G$ an $g' \in H$ such that $d(g, g') \leq \frac{8}{n}$. The idea is to find a map $h \in H$ that behaves like the inverse of g on gU and like the identity on most of the rest. Then hg is the desired g'.

Let $g \in G$ and define $W = \langle U, gU \rangle$. By Lemma 27 there is a W' such that $\dim W' \geq n - 8$, $W' \leq W^{\perp}$, and $W' \cap W = 0$. Consider the map

$$g^{-1}|_{gU} \oplus 1_{W'} \colon gU \oplus W' \to U \oplus W'$$

as $g^{-1}|_{gU}$ and $1_{W'}$ are isometries and $W \perp W'$ Lemma 29 implies that the above map is also an isomtry. By Witt's lemma this isometry can be extended to an isometry $h \colon V \to V$.

$$\begin{aligned}
n \cdot d(g, hg) &= \dim \mathrm{im}(g - hg) \\
&\leq 8 + \dim \mathrm{im}(g - hg)|_{W'} && (\dim W' \geq n - 8) \\
&= 8 + \dim \mathrm{im}(g - g)|_{W'} && (h|_{W'} = 1_{W'}) \\
&= 8
\end{aligned}$$

Finally, we need to show that $hg \in H$, here the choice of H using Lemma 25 comes into play. By construction of h we have that $hg|_U = 1_U$. Therefore we can apply Lemma 26 and get that $hg(U^{\perp}) = U^{\perp}$. Hence $hg \in H$ and $d(g, hg) \leq \frac{8}{n}$. □

7 Fun

Consider an n-dimensional cube with 2^k nodes on each edge. Then its diameter $\nabla_{n,k}$ and length $L_{n,k}$ are

$$\nabla_{n,k} = \sqrt{(2^k - 1) \cdot n} \qquad\qquad L_{n,k} = \sqrt{\sum_{i=0}^{k-1} 2^{2i} \cdot n}.$$

Henceforth

$$\lim_{n \to \infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{L_{1,k}}{\nabla_{1,k}} \qquad \text{and} \qquad \lim_{k \to \infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{1}{\sqrt{3}}.$$

ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Topological Entropy of Formal Languages" selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum Unterschrift