**Bachelorarbeit**

# Measure Concentration for Symplectic Groups

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## Contents

## 1 Introduction

If we have a finite group of matrices, then we can equip it with the normalized rank metric and the normalized Haar measure to obtain a *metric measure space*. For some sequences of matrix groups of ever larger matrices there is a well defined limit. Carderi and Thom showed in [**?**] that the closure of the limit of $\mathrm{SL}_n(q)$ is *extremely amenable*. The goal of this thesis is to generalize this result to limits of other matrix group families, namely symplectic, unitary, and orthogonal matrices. The general strategy will be the following: given a family $(G_n)_{n\in\mathbb{N}}$ of (mm) matrix groups we first find an upper bound for the *concentration function* of $G_n$ using a consequence of Azema's inequality [**?**]. As the upper bounds converge to zero we conclude that $(G_n)_{n\in\mathbb{N}}$ is a *Lévy family*, making the closure of their limit a *Lévy group*. Finally, we know from [**?**] that every Lévy group is extremely amenable.

This thesis is structured as follows. In Section 2 we will give a short introduction on how to see matrix groups as metric measure spaces and how to define a limit of a sequence of matrix groups. Furthermore we will introduce the notion of extreme amenability and its connection to Levy groups. In Section 3 we will briefly look at conditional expectation to show Azema's inequality. This will allow us to connect the length of a metric measure space with the measure concentration function. This connection is used in Section 4 to show that the closure of the limit of $\mathrm{SL}_n$ is extremely amenable. To generalize this result we prove Witt's lemma, which says that isometries can be extended, in Section 5. In Section 6 we generalize the result from Section 4 to symplectic, unitary, and orthogonal groups. Finally, in Section 7 a Ramsey theoretic result from [**?**] about $\mathrm{SL}_n(q)$ is generalized to symplectic, unitary, and orthogonal groups.

## 2 Limits of matrix groups and extreme amenability

Let $\mathrm{GL}_n(q)$ be the general linear group over the $q$ element field $\mathbb{F}_q$ and let $G$ be a subgroup of $\mathrm{GL}_n(q)$. We can equip $G$ with the (normalized) *rank-metric* $d(g,h) := \frac{1}{n}\,\mathrm{r}(g-h)$. Since all matrices in $G$ have full rank, this metric is bi-invariant, i.e. $d(kg,kh) = d(g,h) = d(gk,hk)$ for all $g,h,k \in G$. Let $G_n \leq \mathrm{GL}_{2^n}(q)$ be a family of subgroups, such that $\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in G_{n+1}$ for all $g \in G_n$. Note that the map

$$\varphi_n \colon G_n \mapsto G_{n+1}, \text{ where } \varphi_n(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

is an isometric homomorphism for all $n \in \mathbb{N}$. Hence we can define the inductive limit of $(G_n)_{n\in\mathbb{N}}$. We denote the metric completion of this limit by $\mathrm{clim}_{n\to\infty} G_n$.

**Lemma 1.** *The group* $\mathrm{clim}_{n\to\infty} G_n$ *is a topological group.*

*Proof.* The bi-invariance of $d$ is preserved by the limit and the completion. ... □

Now that we have a topology on $G := \operatorname{clim}_{n\to\infty} G_n$ we can ask whether it is *extremely amenable*, i.e. every continuous action of $G$ on a compact topological space admits a fixed point. It is hard to show this directly, but we know that every Lévy group is extremely amenable. Hence we will show that for suitable $(G_n)_{n\in\mathbb{N}}$ the limit $G$ will be a Lévy group.

Before we can define Lévy groups we need the following definition.

**Definition 2.** A *metric measure space* (mm-space) $\boldsymbol{X}$ is a triple $(X, d, \mu)$, where $d$ is a metric on the set $X$ and $\mu$ is a measure on the Borel $\sigma$-algebra induced by $d$. We will always assume that $\mu(X) = 1$. For any set $A \subseteq X$ denote the *$r$ − neighborhood* of $A$, i.e. $\{x \in X \mid \exists y \in A.\, d(x,y) < r\}$, by $N_r(A)$. The *measure concentration function* of $\boldsymbol{X}$ is defined as

Florian sagt: "definition ugly, use $d_A$?"

$$\alpha_{\boldsymbol{X}}(r) = \sup\{1 - \mu(N_r(A)) \mid A \subseteq X, \mu(A) \geq \frac{1}{2}\}.$$

A family of mm-spaces $\boldsymbol{X}_n$ with diameter 1 is called a *Lévy family* if

$$\alpha_{\boldsymbol{X}_n}(r) \to 0$$

for all $r > 0$.

A topological space $\boldsymbol{X}$ is a *Polish space* if it is homeomorphic to a complete metric space that has a countable dense subset.

Now we can come back to groups.

**Definition 3.** A *Polish group* $G$ is a topological group where the underlying topological space is a Polish space. A *Lévy group* is a group $G$ equipped with a metric $d$, where

- $G$ with the topology induced by $d$ is a Polish group and
- there is a sequence $(G_n)_{n\in\mathbb{N}}$ of compact subgroups, such that $(G_n, d|_{G_n}, \mu_n)_{n\in\mathbb{N}}$ is a Lévy family. Here $\mu_n$ is the normalized Haar measure of $G_n$.

Florian sagt: "lim $G_n$ dense in $G$?"

Note that the normalized Haar measure of $G_n$ is just the normalized counting measure. The following theorem from [**?**] gives the desired connection to extreme amenability.

**Theorem 4.** *Every Lévy group is extremely amenable.*

To apply this theorem to our setting we need the following lemma.

**Lemma 5.** *Let $G_n \leq \mathrm{GL}_{2^n}(q)$ and $G = \operatorname{clim}_{n\to\infty} G_n$. Then $G$ is a Polish group.*

*Proof.* By Lemma 1 $G$ is already a topological group and by definition it is also a complete metric space. Furthermore, every $G_n$ is finite. Hence the inductive limit of the $G_n$ is a countable dense subset of $G$. □

Whether $G$ is also a Lévy group depends on the particular choice of $(G_n)_{n\in\mathbb{N}}$. To show that for certain sequences $G$ will be a Lévy group, we will bound $\alpha_{G_n}(r)$. The next section develops the methods necessary to obtain this upper bound.

## 3 An upper bound for the measure concentration function

In this section we will prove Azema's inequality and as a consequence, we will obtain an upper bound for the measure concentration function. As the next results rely heavily on stochastic methods we will briefly introduce the necessary notions. Since the $G_n$ are all finite and equipped with the normalized counting measure we will only consider *probability spaces* $(X, \Sigma, \mu)$, where $X$ is finite, $\Sigma$ is a $\sigma$-algebra over $X$, and $\mu(A) = |A|/|X|$ for $A \subseteq X$. Most of the statements in this section hold in a more general setting **[?]**. Note that $\Sigma$ has a very nice representation.

**Lemma 6.** *Let $\Sigma$ be a $\sigma$-algebra over a finite set $X$, then $\Sigma$ is the smallest $\sigma$-algebra containing the partition $A_1, \ldots, A_n$, where the $A_i$'s are the minimal nonempty sets in $\Sigma$.*

*Proof.* First we show that $A_1, \ldots, A_n$ is a partition of $X$. Since $A_i \cap A_j \in \Sigma$ we conclude, by minimality of $A_i$ and $A_j$, that either $i = j$ or $A_i \cap A_j = \emptyset$. Clearly, every element of $X$ is contained in a one of the $A_i$.

For $A \in \Sigma$ we have, again by minimality, that $A \cap A_i$ is either $A_i$ or $\emptyset$. Therefore $A$ can be written as a union of $A_i$'s. □

Note that it follows from the proof that any $A \in \Sigma$ can be written as $\bigcup_{i \in I} A_i$ for a suitable $I$. This lemma allows us to use partitions and $\sigma$-algebras interchangeably. We will denote the partition corresponding to $\Sigma$ by $A_1, \ldots, A_n$, for $\Sigma'$ we will use $A'_1, \ldots, A'_{n'}$, etc. The next definition is simplified a lot by only considering finite $X$.

**Definition 7.** Let $(X, \Sigma, \mu)$ be a probability space, $f \colon X \to \mathbb{R}$ be a measurable function, and $\Sigma'$ be a sub-$\sigma$-algebra of $\Sigma$. Then the *conditional expectation* of $f$ with respect to $\Sigma'$ is defined as

$$\mathbb{E}(f \mid \Sigma') := \sum_{i=1}^{n'} \mathbb{E}(f \mid A'_i) \cdot \mathbb{1}_{A'_i}.$$

One often thinks of $\Sigma'$ as the available information, a finer partition means more information. The conditional expectation $\mathbb{E}(f \mid \Sigma')$ is the best approximation of $f$ given only the information from $\Sigma'$. With this intuition the statements from the following lemma are not surprising.

**Lemma 8.** *Let $(X, \Sigma, \mu)$ be a probability space, $f, g \colon X \to \mathbb{R}$ be measurable functions, $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ be sub-$\sigma$-algebras. Then*

*i) if $f \leq g$, then $\mathbb{E}(f \mid \Sigma') \leq \mathbb{E}(g \mid \Sigma')$,*

*ii) for any $\Sigma'$-measurable function $h \colon X \to \mathbb{R}$ we have $\mathbb{E}(hf \mid \Sigma') = h \cdot \mathbb{E}(f \mid \Sigma')$,*

*iii) also $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') = \mathbb{E}(f \mid \Sigma'') = \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma')$.*

*Proof.* To i): If $f \leq g$, then

$$\mathbb{E}(f \mid \Sigma') = \sum_{i=1}^{n} \mathbb{E}(f \mid A_i') \cdot \mathbb{1}_{A_i'} \leq \sum_{i=1}^{n} \mathbb{E}(g \mid A_i') \cdot \mathbb{1}_{A_i'} = \mathbb{E}(g \mid \Sigma').$$

To ii): Let $h\colon X \to \mathbb{R}$ be $\Sigma'$-measurable function, then $h = \sum_{i=1}^{n'} h_i \mathbb{1}_{A_i'}$. Now

$$\begin{aligned}
\mathbb{E}(hf \mid \Sigma') &= \sum_{i=1}^{n'} \mathbb{E}(hf \mid A_i')\mathbb{1}_{A_i'} \\
&= \sum_{i=1}^{n'} h_i \mathbb{E}(f \mid A_i')\mathbb{1}_{A_i'} \\
&= h \cdot \mathbb{E}(f \mid \Sigma').
\end{aligned}$$

To iii): Note that $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid A') = \mathbb{E}(f \mid A')$ for all $A' \in \Sigma'$.

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') &= \sum_{i=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid A_i'') \cdot \mathbb{1}_{A_i''} \\
&= \sum_{i=1}^{n''} \mathbb{E}(f \mid A_i'') \cdot \mathbb{1}_{A_i''} && (A_i'' \in \Sigma') \\
&= \mathbb{E}(f \mid \Sigma'') \\
&= \sum_{j=1}^{n''} \mathbb{E}(f \mid A_j'') \cdot \mathbb{1}_{A_j''} \cdot \sum_{i=1}^{n'} \mathbb{1}_{A_i'} \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid A_j'') \mid A_i') \cdot \mathbb{1}_{A_j''} \cdot \mathbb{1}_{A_i'} \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid A_j'') \cdot \mathbb{1}_{A_j''} \mid A_i') \cdot \mathbb{1}_{A_i'} && \text{(by ii)} \\
&= \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma')
\end{aligned}$$

This concludes the proof. □

The following lemma might not seem very interesting, but changing the exponent from $x$ to $x^2$ is the very foundation for Azema's inequality.

**Lemma 9.** *For all* $x \in \mathbb{R}$

$$e^x \leq x + e^{x^2}.$$

*Proof.* Note that for $x = 0$ both sides are equal to 1. As both sides are differentiable it suffices to show that the derivative of the right hand side is larger than the derivative of the left hand side for all $x \geq 0$ and smaller for all $x \leq 0$. Hence, we want to show

$$e^x \geq 1 + 2xe^{x^2} \text{ for all } x \leq 0 \qquad \text{and} \qquad e^x \leq 1 + 2xe^{x^2} \text{ for all } x \geq 0.$$

As for $x = 0$ both sides are again equal to 1 we can reduce the problem, by similar reasoning, to the question whether

$$e^x \leq 2e^{x^2} + 4x^2e^{x^2} \text{ for all } x \in \mathbb{R}.$$

- For $x = 0$ the terms reduce to $1 \leq 2$.
- For $x < 0$ the left hand side is bounded by 1, while the right hand side is still larger that 2.
- For $1 \leq x$ we have $x \leq x^2$ and the inequality holds trivially.
- For $0 < x < 1$ note that the both sides are increasing. Hence the inequality holds for all $x$ with $e^x \leq 2$. Finally, $\ln 2 \geq \frac{1}{2}$ and therefore the right hand side with $x = \ln 2$ evaluates to a number larger then $e$.

□

Before we will prove Azema's inequality let us introduce some useful notation. Whenever there is no danger of confusion we will abbreviate sets of the form

$$\{x \in X \mid \text{Condition}(x) \text{ holds}\} \qquad \text{by} \qquad \{\text{Condition}\}.$$

For example $\{x \in X \mid f(x) = c\}$ becomes $\{f = c\}$.

**Lemma 10.** *[Azema's inequality] Let $(X, \Sigma, \mu)$ be a probability space, $f\colon X \to \mathbb{R}$ a measurable function, and $\{X\} = \Sigma_0 \subseteq \cdots \subseteq \Sigma_n = \Sigma$ a chain of sub-$\sigma$-algebras. Define $f_i := \mathbb{E}(f \mid \Sigma_i)$ and $d_i := f_i - f_{i-1}$. Then for every $\varepsilon \geq 0$*

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \cdot \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^n \|d_i\|_\infty^2}\right).$$

Note that $(f_i, \Sigma_i)_{0 \leq i \leq n}$ is a discrete martingale.

*Proof.* First, observe that $f_0 = \mathbb{E}(f \mid \{X\}) = \mathbb{E}(f)$ and $f_n = \mathbb{E}(f \mid \Sigma) = f$. Using a simple telescoping sum we obtain $f - \mathbb{E}(f) = d_1 + \cdots + d_n$. Therefore

$$\begin{aligned}
\mu(\{f - \mathbb{E}(f) \geq \varepsilon\}) &= \mu(\{\sum_{i=1}^{n} d_i \geq \varepsilon\}) \\
&= \mu(\{\lambda \cdot \sum_{i=1}^{n} d_i \geq \lambda\varepsilon\}) && \text{(for } \lambda > 0\text{)} \\
&= \mu(\{e^{\lambda \cdot \sum_{i=1}^{n} d_i - \lambda\varepsilon} \geq 1\}) \\
&\leq \mathbb{E}(e^{\lambda \cdot \sum_{i=1}^{n} d_i}) \cdot e^{-\lambda\varepsilon} && (*) \\
&= \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}} \cdot \mathbb{E}(e^{\lambda d_n} \mid \Sigma_{n-1})) \cdot e^{-\lambda\varepsilon} && \text{(Lemma 8)} \\
&\leq \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}}) \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda\varepsilon} && (**) \\
&\vdots \\
&\leq e^{\lambda^2 \cdot \|d_1\|_\infty^2} \cdot \ldots \cdot e^{\lambda^2 \cdot \|d_{n-1}\|_\infty^2} \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda\varepsilon} \\
&= e^{\lambda^2 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2 - \lambda\varepsilon}.
\end{aligned}$$

For $(*)$ note that for any measurable function $g\colon X \to \mathbb{R}$ with $g \geq 0$ we have

$$\mu(\{g \geq 1\}) = \mathbb{E}(\mathbb{1}_{\{g \geq 1\}}) \leq \mathbb{E}(g).$$

For $(**)$ we need to use Lemma 9

$$\begin{aligned}
\mathbb{E}(e^{\lambda d_i} \mid \Sigma_{i-1}) &\leq \mathbb{E}(\lambda d_i \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&= \lambda \cdot \mathbb{E}(f_i - f_{i-1} \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&\leq 0 + e^{\lambda^2 \|d_i\|_\infty^2}. && \text{(Lemma 8)}
\end{aligned}$$

Substituting $-\frac{\varepsilon^2}{\sum_{i=1}^{n} \|d_i\|_\infty^2}$ for $\lambda$ we conclude that

$$\mu(\{f - \mathbb{E}(f) \geq \varepsilon\}) \leq \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

The same calculations with $-d_i$ instead of $d_i$ yield the dual inequality

$$\mu(\{f - \mathbb{E}(f) \leq -\varepsilon\}) \leq \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

These two statements obviously give us the desired result. $\square$

Since $\mu$ is the counting measure Azema's inequality bounds the number of elements for which $f$ differs more than $\varepsilon$ from its mean. This seems at least somewhat connected to the measure concentration function, as there we want to show that for any set $A$ with $\mu(A) \geq \frac{1}{2}$ only a few elements are more than $\varepsilon$ away from $A$. The next goal is to formalize this connection To achieve this we first need to introduce a new property of mm-spaces.

**Definition 11.** Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space. The *length* of $\boldsymbol{X}$ is the minimum over all $l$ with the following property. There is a refining sequence of partitions

$$\{X\} = \Omega_0 \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\},$$

where for every $i \in \{1, \ldots, n\}$ there is an $a_i$ such that $\sum_{i=1}^n a_i^2 = l^2$ and for every $A \in \Omega_{i-1}$, $x, y \in A$ there is an isomorphism (of metric spaces) $\phi \colon [x]_i \to [y]_i$ with

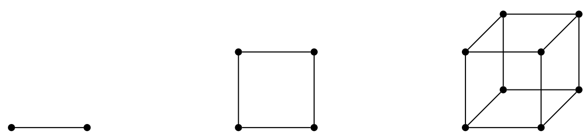
$$d(z, \phi(z)) \leq a_i \text{ for all } z \in [x]_i.$$

Note that since $\mu$ is the counting measure $\phi$ is also an isomorphism of mm-spaces. As this definition is quite hard we will look at some properties and examples of the length of $\boldsymbol{X}$ before proceeding.

**Lemma 12.** *Let $\boldsymbol{X}$ be a finite mm-space. Then the length of $\boldsymbol{X}$ is at most the diameter of $\boldsymbol{X}$.*

*Proof.* Consider only the two partitions $\{X\} \prec \{\{x\} \mid x \in X\}$. □

**Example 13.** Let us look at the $n$-dimensional cube $X_n = \{0, 1\}^n$.



$X_1$ $\qquad$ $X_2$ $\qquad$ $X_3$

We will only consider the following sequence of partitions

$$\Omega_0 \prec \cdots \prec \Omega_n \text{ with } \Omega_i = \{wX_{n-i} \mid w \in \{0,1\}^i\}.$$

First, we equip $X_n$ with the euclidean metric and rescale it such that the diameter is 1. To bound the length of the resulting space $\boldsymbol{X}_n^E$ consider $[x]_i \neq [y]_i$. Note that $x$ and $y$ are w.l.o.g. of the form $w0u$ and $w1v$ for some $w \in \{0,1\}^{i-1}$, $u, v \in \{0,1\}^{n-i}$. The isomorphism $\phi$ takes an element $w0u'$ in $[x]_i$ and maps it to $w1u'$. The length of a side in $\boldsymbol{X}_n^E$ is $\frac{1}{\sqrt{n}}$, hence every $a_i$ is $\frac{1}{\sqrt{n}}$ for every $i$ and the length of $\boldsymbol{X}_n^E$ is bounded by $(\sum_{i=1}^n \frac{1}{\sqrt{n}^2})^{\frac{1}{2}} = 1$.

Secondly, we use the hemming metric and obtain the mm-space $\boldsymbol{X}_n^H$ with diameter 1. It has side length $\frac{1}{n}$ and therefore the length of $\boldsymbol{X}_n^H$ is bounded by $(\sum_{i=1}^n \frac{1}{n^2})^{\frac{1}{2}} = n^{-\frac{1}{2}}$. We see that here the length of $\boldsymbol{X}_n^H$ converges to 0 as $n$ tends to infinity. We will show that this means that the measure concentration function $\alpha_{\boldsymbol{X}_n^H}(\varepsilon)$ also goes to 0 for any fixed $\varepsilon > 0$.

**Lemma 14.** *Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space of length $l$ and $f \colon X \to \mathbb{R}$ be a 1-Lipschitz function. Then*

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4l^2}\right) \textit{ for every } \varepsilon > 0.$$

*Proof.* Let $\Omega_0 \prec \cdots \prec \Omega_n$ be a refining sequence of partitions with $a_1, \ldots, a_n$ as in Definition 11 such that $\sum_{i=1}^n a_i^2 = l^2$. These partitions correspond to $\sigma$-algebras $\Sigma_0 \subseteq \cdots \subseteq \Sigma_n$. Now we can apply Azema's inequality to obtain

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \cdot \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^n \|d_i\|_\infty^2}\right),$$

where $f_i = \mathbb{E}(f \mid \Sigma_i)$ and $d_i = f_i - f_{i-1}$ as before. Therefore we only need to show that $\|d_i\|_\infty \leq a_i$. Since on any $A \in \Omega_{i-1}$ we have $f_{i-1} = \mathbb{E}(f_i \mid A)$ it suffices to show that for all $A \in \Omega_{i-1}$ it holds that $f_i(x) - f_i(y) \leq a_i$ for all $x, y \in A$. Let $\phi\colon [x]_i \to [y]_i$ be the isomorphism from Definition 11.

$$\begin{aligned}
f_i(x) - f_i(y) &= \mathbb{E}(f \mid [x]_i) - \mathbb{E}(f \mid [y]_i) \\
&= \mathbb{E}(f \mid [x]_i) - \mathbb{E}(f \circ \phi \mid [x]_i) \\
&= \mathbb{E}(f - f \circ \phi \mid [x]_i) \\
&\leq \mathbb{E}(d(\,.\,, \phi(\,.\,)) \mid [x]_i) && (f \text{ is 1-Lipschitz}) \\
&\leq a_i
\end{aligned}$$

This concludes the proof. □

Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space and $A \subseteq X$ measurable. Observe that $d_A\colon X \to \mathbb{R}$, $d_A(x) := \inf_{y \in A} d(x, y)$ is a 1-Lipschitz function. Using this we can rewrite the definition of the measure concentration function

$$\alpha_{\boldsymbol{X}}(\varepsilon) = \sup\{\mu(\{d_A \geq \varepsilon\}) \mid \mu(A) \geq \frac{1}{2}\}.$$

This gives us the desired connection.

**Theorem 15.** *If a finite mm-space $\boldsymbol{X} = (X, d, \mu)$ has length $l$, then the measure concentration function of $\boldsymbol{X}$ satisfies*

$$\alpha_{\boldsymbol{X}}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 l^2}\right) \textit{ for all } \varepsilon > 0.$$

*Proof.* Let $\varepsilon > 0$ and $A \subseteq X$ be measurable with $\mu(A) \geq \frac{1}{2}$. As mentioned above $d_A$ is 1-Lipschitz and therefore, by Lemma 14,

$$\mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4 l^2}\right).$$

Now there are two cases to consider, the first case is the more interesting one.

If $\mathbb{E}(d_A) \leq \varepsilon$, then for any $x$ with $d_A(x) \geq 2\varepsilon$ we know $d_A(x) \geq \varepsilon + \mathbb{E}(d_A)$ and therefore $|d_A(x) - \mathbb{E}(d_A)| \geq \varepsilon$. As a consequence

$$\mu(\{d_A \geq 2\varepsilon\}) \leq \mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4 l^2}\right).$$

Replacing $\varepsilon$ by $\frac{\varepsilon}{2}$ gives the desired inequality.

If $\mathbb{E}(d_A) > \varepsilon$, then $A \subseteq \{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}$. Consequently,

$$\mu(\{d_A \geq \varepsilon\}) \leq \mu(X \setminus A) \leq \frac{1}{2} \leq \mu(A) \leq \mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2\exp\left(-\frac{\varepsilon^2}{4l^2}\right).$$

This proves the theorem. □

Note that in the second case the upper bound is at least $\frac{1}{2}$, which means that if $l$ is large enough then we are in the first case and the expected distance to a set with at least half measure is less then $\varepsilon$. In Section 7 we will se a slight modification of this lemma. But for now our goal is to apply Theorem 4 to groups and as it turns out we can bound the length of a group using sequences of subgroups. Before we can write down the corollary we need to make a quick excursion to factor metrics.

**Definition 16.** Let $(X, d)$ be a metric space and let $\sim$ be an equivalence relation on $X$. Then

$$d_\sim([x], [y]) = \inf\{d(p_1, q_1) + \cdots + d(p_n, q_n) \mid q_i \sim p_{i+1}, x \sim p_1, q_n \sim y\}$$

defines a pseudometric on $X/\sim$.

In case that $X$ is a group with bi-invariant metric this definition simplifies.

**Lemma 17.** *Let $G$ be a finite group with bi invariant metric $d$ and $H$ a (not necessarily normal) subgroup of $G$. Then the factor metric $d_H$ on $G/H = \{gH \mid g \in G\}$ is a proper metric and satisfies $d_H(gH, g'H) = \inf\{d(g, g'h) \mid h \in H\}$.*

*Proof.* Let $x, y \in G$. We show that for any path $p_1, q_1, \ldots, p_n, q_n$ as in the definition there are $x \sim p$ and $q \sim y$ such that $d(p, q) \leq d(p_1, q_1) + \cdots + d(p_n, q_n)$. It suffices to show this for $n = 2$. By definition $p_1, q_1, p_2, q_2$ are of the form $g, g', g'h, g''$ form some $g, g', g'' \in G$ and $h \in H$. Since $d$ is bi-invariant

$$d(gh, g'') \leq d(gh, g'h) + d(g'h, g'') = d(g, g') + d(g'h, g'').$$

Furthermore we are given that $G$ is finite. Hence the infimum becomes a minimum and $d_H([x], [y]) = 0$ only if $[x] = [y]$. □

Equipped with this knowledge we can formulate the final statement for this section.

**Corollary 18.** *Let $G$ be a finite group with a bi-invariant metric $d$, and let*

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

*be a chain of subgroups. Denote the diameter of $G_i/G_{i-1}$ with respect to the factor metric by $a_i$. Then the measure concentration function of the mm-space $(G, d, \mu)$, where $\mu$ is the normalized counting measure, satisfies*

$$\alpha_X(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16 \cdot \sum_{i=1}^n a_i^2}\right).$$

*Proof.* We show that the length $l$ of $(G, d, \mu)$ is bounded by $\left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}}$ and apply Theorem 15. Define the sequence of partitions $\Omega_i := \{gG_i \mid g \in G\}$

$$\begin{aligned} \{\{g\} \mid g \in G\} = \Omega_0 \succ \Omega_1 \succ \cdots \succ \Omega_n = \{G\} \\ \{e\} = G_0 < G_1 < \cdots < G_n \ = G. \end{aligned}$$

Take $A \in \Omega_{i+1}$ and $g, g' \in A$. Since the distance of $gG_i$ and $g'G_i$ with respect to the factor metric is at most $a_i$ there is an $h' \in G_i$ such that $d(g, g'h') \leq a_i$. Hence the map

$$\begin{aligned} \phi\colon gG_i &\to g'G_i \\ gh &\mapsto g'h'h \end{aligned}$$

is, by bi-invariance of $d$, an isomorphism of metric spaces with $d(gh, g'h'h) = d(g, g'h') \leq a_i$ for all $gh \in gG_i$. Therefore $\left(\sum_{i=1}^{n} a_i^2\right)$ is an upper bound for $l^2$. □

Carderi and Thom used this result to show that the limit of $\mathrm{SL}_{2^n}(q)$ is extremely amenable [**?**]. We will recreate this proof in the next section.

## 4 The limit of $\mathrm{SL}_{2^n}(q)$ is extremely amenable

When studying matrices it is often useful to look at the corresponding linear maps of a suitable vector space. In the case of $\mathrm{SL}_n(q)$ an $n$ dimensional $\mathbb{F}_q$ vector space $V$ suffices. Fixing a basis $e_1, \ldots, e_n$ gives us an embedding from $\mathrm{SL}_n(q)$ into $\mathrm{Aut}(V)$. Next we will apply the methods from the previous section to show that $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ is extremely amenable.

**Theorem 19.** *The normalized counting measure on the groups $\mathrm{SL}_n(q)$ concentrates with respect to the normalized rank-metric, i.e. for all $\varepsilon > 0$*

$$\lim_{n \to \infty} \alpha_{(\mathrm{SL}_n(q), d, \mu)}(\varepsilon) = 0.$$

Florian sagt: "Boldsymbol or tuple"

Florian sagt: "proof for SL and GL te same?"

*Proof.* We will apply Corollary 18 to a sequence of subgroups which also shows that the length of $\mathrm{SL}_n(q)$ is bounded by $3n^{-\frac{1}{2}}$. Let $e_1, \ldots, e_n$ be a basis of an $n$ dimensional $\mathbb{F}_q$ vector space $V$. Look at the sequence

$$\mathrm{SL}_0(q) < \mathrm{SL}_1(q) < \cdots < \mathrm{SL}_n(q),$$

where $\mathrm{SL}_{i-1}(q)$ becomes a subgroup of $\mathrm{SL}_i(q)$ via the embedding $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$.

Next we want to bound the diameter of $\mathrm{SL}_i(q) / \mathrm{SL}_{i-1}(q)$ by $\frac{3}{n}$. By Lemma 17 it suffices to show that for any $g, g' \in \mathrm{SL}_i(q)$ there is an $h \in \mathrm{SL}_{i-1}(q)$ such that $d(g, g'h) \leq \frac{3}{n}$. Since $d$ is bi-invariant we can assume w.l.o.g. that $g' = 1_V$. Our goal is now to find a $g' \in \mathrm{SL}_i(q)$ that is the identity on $e_i$.

Take a closer look at $ge_i$. If $e_i$ is an eigenvector of $g$ with eigenvalue $\lambda$, then $\lambda \neq 0$ and $g$ is of the form $\begin{pmatrix} A & 0 \\ c^\perp & \lambda \end{pmatrix}$. Define $h' := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $g' := \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix} \cdot g$. By construction $g' \in \mathrm{SL}_i(q)$ and it is of the form $\begin{pmatrix} A' & 0 \\ c'^\perp & 1 \end{pmatrix}$. Since $\det g' = 1$ we have that $\det A' = 1$ and therefore $A' \in \mathrm{SL}_{i-1}(q)$ making it a suitable candidate for $h$. Using the triangle inequality we obtain

$$\begin{aligned} d(g,h) &\leq d(g,g') + d(g',h) \\ &= d(I_i, \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix}) + \frac{1}{n}\,\mathrm{r}(\begin{pmatrix} 0 & 0 \\ -c'^\perp & 0 \end{pmatrix}) \\ &\leq \frac{2}{n} + \frac{1}{n} \end{aligned}$$

as desired.

If $e_i$ is not an eigenvector of $g$, then we can make a change of basis of $\langle e_1, \ldots, e_{i-1} \rangle$ such that $ge_i = e_{i-1} + \lambda e_i$. Henceforth we can assume w.l.o.g. that $g$ is of the form $\begin{pmatrix} & A & & 0 \\ & & & 1 \\ c^\perp & & c_{i-1} & \lambda \end{pmatrix}$. Define $h' := \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$ and as before $g' := \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix} \cdot g$. Now we can apply the argument from above to get an $h \in \mathrm{SL}_{i-1}(q)$ such that $d(g,h) \leq \frac{3}{n}$. Applying Corollary 18 we obtain

$$\alpha_{(\mathrm{SL}_n(q),d,\mu)}(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16 \cdot \sum_{i=1}^{n} \frac{9}{n^2}}\right) = 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 9}\right),$$

which tends to 0 as $n$ goes to infinity.

□

From this theorem the main result of this section easily follows.

**Corollary 20.** *The Polish group* $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ *is extremely amenable.*

*Proof.* Theorem 19 implies that $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ is a Levy group and is therefore extremely amenable, by Theorem 4. □

As a byproduct we found an upper bound for the length of $\mathrm{SL}_n(q)$. We now ask how good this upper bound is. Therefore we our next goal is to determining also a lower bound. This part is not essential to the rest of the thesis but still interesting.

**Lemma 21.** *Let* $(X, d, \mu)$ *be a finite mm-space with diameter* $\Delta$ *and*

$$\Omega_0 = \{X\} \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

*with* $a_1, \ldots, a_n$ *as in Definition 11. Then*

$$\sum_{i=1}^{n} a_i \geq \Delta.$$

*Proof.* Let $x, y \in X$, with $x \neq y$, we show $d(x,y) \leq \sum_{i=1}^{n} a_i$. Let $i_0$ be the smallest number such that $[x]_{i_0} \neq [y]_{i_0}$. Since $[x]_0 = X = [y]_0$ we know that $i_0$ is at least 1. Therefore $[x]_{i_0-1} = [y]_{i_0-1}$ and there is an isomorphism $\varphi_{i_0} \colon [x]_{i_0} \to [y]_{i_0}$ such that $d(\varphi_{i_0}(x), y) \leq a_{i_0}$. Let $x_{i_0} = \varphi_{i_0}(x)$, then

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

If $x_{i_0} = y$, then we are done. Otherwise let $i_1$ be the smallest number such that $[x_{i_0}]_{i_1} \neq [y]_{i_1}$. Then let $\varphi_{i_1} \colon [x_{i_0}]_{i_1} \to [y]_{i_1}$ be an isomorphism such that $d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$. Define $x_{i_1} = \varphi_{i_1}(x_{i_0})$. Proceeding in this fashion yields elements $x_{i_0}, \ldots, x_{i_k}$ such that $x_{i_k} = y$ and

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \cdots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \cdots + a_{i_k} \leq \sum_{i=1}^{n} a_i.$$

From this the claim immediately follows. ◻

**Lemma 22.** *Let $(X, d, \mu)$ be a finite mm-space with diameter $\Delta$ and $\delta = \min_{x \neq x} d(x,y)$. Then the length of $X$ is at least $(\Delta \cdot \delta)^{\frac{1}{2}}$.*

*Proof.* We show by induction on $n$ that for any nonnegative $a_1, \ldots, a_n$ with $\delta = \min_{1 \leq i \leq n} a_i$ we have

$$\sum_{i=1}^{n} a_i \geq \Delta \implies \sum_{i=1}^{n} a_i^2 \geq \Delta \cdot \delta.$$

For $n = 1$: Note that $\Delta \geq \delta$. Hence

$$\sum_{i=1}^{n} a_i^2 = a_1^2 = \delta^2 \geq \Delta \cdot \delta.$$

For $n > 1$: Assume w.l.o.g. that $a_1 \leq \cdots \leq a_n$. Then

$$\begin{aligned}
& \sum_{i=1}^{n} a_i \geq \Delta \\
\implies & \sum_{i=2}^{n} a_i \geq \Delta - \delta && (a_1 = \delta) \\
\overset{\text{I.H.}}{\implies} & \sum_{i=2}^{n} a_i^2 \geq (\Delta - \delta) \cdot \delta' && (\text{with } \delta' = a_2) \\
\implies & \sum_{i=2}^{n} a_i^2 \geq \Delta \cdot \delta - \delta^2 && (\delta \leq \delta') \\
\implies & \sum_{i=1}^{n} a_i^2 \geq \Delta \cdot \delta. && (a_1^2 = \delta^2)
\end{aligned}$$

From this claim together with Lemma 21 the lemma immediately follows. ◻

Using this we can give an interval for the length of $\mathrm{SL}_n(q)$.

**Corollary 23.** *Consider $(\mathrm{SL}_n(q), d, \mu)$, where $d$ is the normalized rank-metric and $\mu$ is the normalized counting measure. Then the length $l$ of this mm-space satisfies*

$$n^{-\frac{1}{2}} \leq l \leq 3n^{-\frac{1}{2}}.$$

*Proof.* The diameter of $\mathrm{SL}_n(q)$ is equal to 1 and for any $g \neq g' \in \mathrm{SL}_n(q)$ we have $d(g, g') \geq \frac{1}{n}$. □

The next goal is to show that the limit of symplectic groups is also extremely amenable. Theses groups can be seen as automorphism groups of a vector space together with a symplectic form. The proof will be similar to the one for the special linear groups but extending the partial inverse $h'$ becomes much harder. This is why in the next section we will prove Witt's Lemma which does exactly what we need, i.e. extending isometries.

## 5 Witts Lemma

In this section we will prove Witt's Lemma and explore the structure of symplectic spaces. Witt's Lemma states that an isometry between subspaces of a finite dimensional vector space can always be extended to an isomtry on the whole space. We will roughly follow the proof in [**?**]. Since we are mainly interested in symplectic spaces we will only show Witt's Lemma for those but it also holds for unitary and orthogonal spaces.

**Definition 24.** A bilinear form $\omega$ on $V$ is called *symplectic* if $\omega$ is nondegenerate, i.e. $\omega(x, \,.\,) \neq 0_V$ for all $x \in V \setminus \{0\}$, and $\omega(x,y) = -\omega(y,x)$ for all $x, y \in V$. Then we say $(V, \omega)$ is a *symplectic space*.

A finite group $G$ is called *symplectic* if there is a symplectic space $(V, \omega)$ such that $G \cong \mathrm{Aut}(V, \omega)$.

A subspace $U \leq V$ is *nondegenerate* if $\omega$ restricted to $U$ is nondegenerate (iff $U \cap U^{\perp} = \{0\}$).

Throughout this section let $(V, \omega)$ be a finite dimensional symplectic $\mathbb{F}_q$ vector space. Note that $\omega(x,y) = 0$ iff $\omega(y,x) = 0$, also $\omega(x,x) = 0$ for all $x \in V$. We will start of with some technical lemmas.

Florian sagt: "make short-version for each section?"

**Lemma 25.** *For a subspace $U \leq V$ we have*

$$\dim U^{\perp} = \dim V - \dim U.$$

*Proof.* Let $u_1, \ldots, u_m$ be a basis of $U$ and consider the linear map

$$g \colon V \to \mathbb{F}_q \qquad \text{with} \qquad g(v) = \begin{pmatrix} \omega(v, u_1) \\ \vdots \\ \omega(v, u_m) \end{pmatrix}.$$

By definition the kernel of $g$ is $U^{\perp}$ and since $\omega$ is nondegenerate $g$ is also surjective. Therefore the claim follows from the Rank-Nullity Theorem. $\square$

The following lemma is an immediate consequence of the previous one.

**Lemma 26.** *Let $U \leq V$. Then*

*i) $U$ is nondegenerate iff $V = U \oplus U^{\perp}$ and*

*ii) $U^{\perp\perp} = U$.*

*Proof.* If $U$ is nondegenerate then $U \cap U^{\perp} = \{0\}$. By Lemma 25

$$U \oplus U^{\perp} = \langle U, U^{\perp} \rangle = V.$$

The other direction is clear from the definition

For *ii*) note that $U \subseteq U^{\perp\perp}$. By Lemma 25 they also have the same dimension and are therefore equal. $\square$

Next we want to better understand the structure of symplectic spaces. A symplectic space is a *hyperbolic plane* if it is 2-dimensional.

**Lemma 27.** *There is only one hyperbolic plane up to isomorphism.*

*Proof.* Let $(V, \omega)$, $(V', \omega')$ be hyperbolic planes with basis $r, s$ and $r', s'$, respectively. Then $r \mapsto r', s \mapsto \lambda s'$ with $\lambda = \omega'(r', s')^{-1} \cdot \omega(r, s)$ is an isomorphism. $\square$

Next we will strengthen this result and show that any symplectic space is the direct sum of hyperbolic planes.

**Theorem 28.** *Let $r \in V \setminus \{0\}$. Then there is a hyperbolic plane $U \leq V$ containing $r$ such that $V = U \oplus U^\perp$. Furthermore if $W \leq V$ with $W \perp r$ and $r \notin W$, then there is a $U$ as before that also fulfills $W \perp U$.*

*Proof.* Since $\omega(r, r) = 0$ we know that $r \in r^\perp$. Let $H \leq r^\perp$ containing $W$ such that $\langle r \rangle \oplus H = r^\perp$. Then, by Lemma 25, $\dim r^\perp = \dim V - 1$, $\dim H^\perp = 2$, and $r \in H^\perp$. In particular $\dim V > 1$. Our goal is to show that

$$V = H^\perp \oplus H,$$

so $H^\perp$ would be a suitable choice for $U$. It suffices to show that $H^\perp$ is nondegenerate, as then the claim follows from Lemma 26. Let $s \in H^\perp$ such that $H^\perp = \langle r, s \rangle$. Now $H^\perp$ is nondegenerate iff $\omega(r, s) \neq 0$.

Assume $\omega(r, s) = 0$. Then $r \in s^\perp$ and $H \subseteq s^\perp$. Since, by construction, $r \notin H$ we have $r^\perp = \langle r, H \rangle = s^\perp$. Hence $r$ and $s$ are linearly dependent contradicting that $r, s$ is a basis of $H^\perp$.

$\square$

Using Theorem 28 we can describe the structure of symplectic spaces.

**Corollary 29.** *Let $(V, \omega)$ be a symplectic space. Then $V$ is of even dimension $2n$ and there are hyperbolic planes $U_1, \dots, U_n \leq V$ such that $V = U_1 \oplus \cdots \oplus U_n$ and $U_i \perp U_j$ for $i \neq j$. In particular for any $n$ there is exactly one symplectic space of dimension $2n$.*

*Proof.* We use induction on $n = \dim V$.

If $n = 1$ then $V = \langle v \rangle$ for $v \in V \setminus \{0\}$. But $\omega(v, v) = 0$ and therefore $\omega$ is degenerate which is a contradiction.

If $n = 2$ then the claim follows from Lemma 27.

If $n > 2$ then by Theorem 28 there is a hyperbolic plane $U_1 \leq V$ such that $V = U_1 \oplus U_1^\perp$. Then, by induction hypothesis, $n$ is even and $U^\perp = U_2 \oplus \cdots \oplus U_m$ for some hyperbolic planes $U_2, \dots, U_m \leq U^\perp$ with $m = \frac{n}{2}$ and $U_i \perp U_j$ for $i \neq j$. $\square$

With these powerful tools we can easily prove Witt's Lemma. Let $\alpha \colon U \to W$ be an isometry between subspaces $U, W \leq V$.

**Lemma 30.** *There are subspaces $U' \geq U$ and $W' \geq W$ with $U', W'$ nondegenerate such that $\alpha$ can be extended to an isometry $\tilde{\alpha} \colon U' \to W'$.*

*Proof.* We show this claim using induction on $n = \dim(U \cap U^\perp)$.

If $n = 0$ then $U$ itself is nondegenerate and we are done.

If $n > 0$ then let $r \in (U \cap U^\perp) \setminus \{0\}$ and $\tilde{U} \leq V$ such that $\langle r \rangle \oplus \tilde{U} = U$. By Theorem 28 there is a hyperbolic plane $H \leq V$ containing $r$ such that $\tilde{U} \perp H$. Similarly, there is a hyperbolic plane $H' \leq V$ containing $r' := \alpha(r)$ such that $H' \perp \tilde{W}$ with $\tilde{W} = \alpha(\tilde{U})$. Let $r, s$ and $r', s'$ be a basis of $H$ and $H'$, respectively. Note that $\omega(r,s) \neq 0$ and $r \perp U$ imply $s \notin U$. We can assume w.l.o.g. that $\omega(r,s) = \omega(r',s')$. Then we can extend $\alpha$ to $\tilde{\alpha} \colon \langle U, s\rangle \to \langle W, s'\rangle$ by $\tilde{\alpha}(s) = s'$. Note that $\langle U, s \rangle = \langle \tilde{U}, H\rangle$. Since $\tilde{U} \perp H$ we have

$$\dim(\langle \tilde{U}, H\rangle \cap \langle \tilde{U}, H\rangle^\perp) = \dim(\tilde{U} \cap \tilde{U}^\perp) < \dim(U \cap U^\perp).$$

Hence we can apply the induction hypothesis to $\tilde{\alpha}$. □

**Lemma 31.** *If $U$ is nondegenerate. Then $\alpha$ can be extended to an isometry $\tilde{\alpha} \colon V \to V$.*

*Proof.* Since $U$ and $W$ are nondegenerate we can apply Lemma 26 and obtain

$$V = U \oplus U^\perp = W \oplus W^\perp.$$

By Lemma 25 we have $\dim U^\perp = \dim W^\perp$. Hence, by Corollary 29, there is an isometry $\beta \colon U^\perp \to W^\perp$. Finally, $\alpha \oplus \beta \colon V \to V$ is an isometry extending $\alpha$. □

With this preparation we can now come to the main result.

**Corollary 32** (Witt's Lemma)**.** *The map $\alpha$ can be extended to an isometry $\tilde{\alpha} \colon V \to V$.*

*Proof.* Using Lemma 30 extend $\alpha$ to $\tilde{\alpha} \colon U' \to W'$ for some $U' \geq U$, $W' \geq W$ nondegenerate. Now apply Lemma 31 to extend $\tilde{\alpha}$ to $\tilde{\tilde{\alpha}} \colon V \to V$. □

Now that we understand symplectic spaces and can extend isometries we are well equipped for the next section, where will show that $\operatorname{clim} \mathrm{Sp}_{2^n}(q)$ is also extremely amenable.

## 6 Limits of other Matrix group families are Levy groups too

Our goal in this section is to show that $\operatorname{clim} \mathrm{Sp}_{2^n}(q)$ is extremely amenable. The structure of the proof is the same as in Section 4 for special linear groups. We will bound the length of $\mathrm{Sp}_n(q) \cong \mathrm{Aut}(V, \omega)$ by applying Corollary 18 to a chain of subgroups $(G_i)_i$. To bound the diameter of $G_i/G_{i-1}$ we will construct for any $g \in G_i$ an $h' \in G_i$ such that the distance between $g$ and $h'g$ is small and $h'g \in G_{i-1}$. The $h'$ will behave like the inverse of $g$ on a small subspace of $V$ and like the identity on most of the rest. The proof can be generalized to unitary and orthogonal groups.

**Definition 33.** Let $V$ be a finite dimensional $\mathbb{F}_q$ vector space and $\omega$ a nondegenerate map from $V \times V$ to $\mathbb{F}_q$.

Then $(V, \omega)$ is an *orthogonal space* if $\omega$ is bilinear, $\omega(x, y) = \omega(y, x)$ for all $x, y \in V$, and if $q = 2$ then $\omega(x, x) = 0$ for all $x \in V$.

And $(V, \omega)$ is a *unitary space* if there is a $h \in \mathrm{Aut}(\mathbb{F}_q)$ with $h^2 = 1$ such that

$$\begin{aligned}\omega(ax + y, z) &= a\omega(x, z) + \omega(y, z)\\ \omega(x, ay + z) &= h(a)\omega(x, y) + \omega(x, z)\\ \omega(x, y) &= h(\omega(y, x))\end{aligned}$$

for all $x, y, z \in V$ and $a \in \mathbb{F}_q$.

*Orthogonal* and *unitary groups* are the automorphism groups of unitary and orthogonal spaces, respectively.

In the following let $(V, \omega)$ be a symplectic, unitary, or orthogonal space. Note that $\omega$ is nondegenerate and

$$\omega(x, y) = 0 \text{ iff } \omega(y, x) = 0 \text{ for all } x, y \in V.$$

Obviously, Lemmas 25 and 26 from the previous section still hold in unitary and orthogonal spaces. Furthermore, Witt's Lemma also holds in unitary and orthogonal spaces, for a proof see **[?]**

**Theorem 34** (Witt's Lemma)**.** *Let $(V, \omega)$ be a symplectic, unitary, or orthogonal space and $\alpha\colon U \to W$ be an isometry between subspaces $U, W \leq V$. Then $\alpha$ can be extended to an isometry $\tilde{\alpha}\colon V \to V$.*

The next lemma is necessary to construct the chain of subgroups, in the case of symplectic spaces it is a trivial consequence of Theorem 28.

**Lemma 35.** *Then there exists a $U \leq V$ with $\dim U \leq 2$ such that $V = U \oplus U^\perp$.*

*Proof.* Let $r \in V \setminus \{0\}$. By Lemma 25 $\dim r^\perp = n - 1$.

If $r \notin r^\perp$, then $V = \langle r \rangle \oplus r^\perp$ and $\langle r \rangle$ is the desired $U$.

If $r \in r^\perp$, then let $H \leq r^\perp$ such that $\langle r \rangle \oplus H = r^\perp$. Now procces as in the proof of Theorem 28 to show that $H^\perp$ is a suitable $U$.

□

The following lemma shows that isometries interact nicely with the complement.

**Lemma 36.** *Let $U \leq V$ and $\alpha\colon V \to V$ be an isometry such that $\alpha(U) = U$. Then $\alpha(U^\perp) = U^\perp$.*

*Proof.* As $\dim \alpha(U^\perp) = \dim U^\perp$ it suffices to show that $\alpha(u') \perp u$ for all $u \in U$ and $u' \in U^\perp$. Let $v \in U$ with $\alpha(v) = u$. Then

$$\begin{aligned}\omega(\alpha(u'), u) &= \omega(\alpha(u'), \alpha(v))\\ &= \omega(u', v)\\ &= 0.\end{aligned}$$

This concludes the proof. □

The next lemma gives us a large subspace on which $h'$ can be the identity without interfering with the part where it is the inverse of $g$.

**Lemma 37.** *For all $W \leq V$ there is a $W' \leq W^{\perp}$ such that $W \cap W' = 0$ and*

$$\dim W' \geq \dim V - 2\dim W.$$

*Proof.* Let $W' \leq W^{\perp}$ such that

$$W^{\perp} = (W^{\perp} \cap W) \oplus W'.$$

Clearly, $W \cap W' = 0$ and

$$\begin{aligned} \dim W' &= \dim W^{\perp} - \dim(W^{\perp} \cap W) \\ &\geq \dim W^{\perp} - \dim W \\ &= \dim V - \dim W - \dim W. && \text{(Lemma 25)} \end{aligned}$$

This concludes the proof. □

Now we can proof the analogue of Theorem 19 from Section 4.

**Theorem 38.** *Let $G$ be a symplectic, unitary, or orthogonal group equipped with the rank metric $d$ and of diameter $n$. Then there is a symplectic, unitary, or orthogonal subgroup $H \leq G$ with diameter at most $n-1$ such that the diameter of $G/H$ is at most $8$.*

*Proof.* $G = \mathrm{Aut}(V, \omega)$ for some vector space $V$ with bilinear form $\omega$. Use Lemma 35 to obtain $U \leq V$ such that $V = U \oplus U^{\perp}$ and $\dim U \leq 2$. Define $H = \mathrm{Aut}(U^{\perp}, \omega)$. Our aim is to find for any $g \in G$ an $g' \in H$ such that $d(g, g') \leq 8$. The idea is to find a map $h' \in G$ that behaves like the inverse of $g$ on $gU$ and like the identity on most of the rest. Then $h'g$ is the desired $g'$.

Let $g \in G$ and define $W = \langle U, gU \rangle$. By Lemma 37 there is a $W'$ such that $\dim W' \geq n - 8$, $W' \leq W^{\perp}$, and $W' \cap W = 0$. Consider the map

$$g^{-1}|_{gU} \oplus 1_{W'} \colon gU \oplus W' \to U \oplus W'$$

as $g^{-1}|_{gU}$ and $1_{W'}$ are isometries and $W \perp W'$ we have that the above map is also an isomtry. By Theorem 34 this isometry can be extended to an isometry $h' \colon V \to V$. Furthermore,

$$\begin{aligned} d(g, h'g) &= \dim \mathrm{im}(g - h'g) \\ &\leq 8 + \dim \mathrm{im}(g - h'g)|_{W'} && (\dim W' \geq n-8) \\ &= 8 + \dim \mathrm{im}(g - g)|_{W'} && (h'|_{W'} = 1_{W'}) \\ &= 8. \end{aligned}$$

Finally, we need to show that $h'g \in H$, here the choice of $H$ using Lemma 35 comes into play. By construction of $h'$ we have that $h'g|_U = 1_U$. Therefore we can apply Lemma 36 and get that $h'g(U^{\perp}) = U^{\perp}$. Hence $h'g \in H$ and $d(g, h'g) \leq 8$. □

**Corollary 39.** *Let $G = \mathrm{Aut}(V, \omega)$ be a symplectic, unitary, or orthogonal group equipped with the normalized rank metric $d$ and the normalized counting measure $\mu$, where $V$ is $n$ dimensional. Then the length of $G$ is at most $8n^{-\frac{1}{2}}$ and for all $\varepsilon > 0$*

$$\alpha_{(G,d,\mu)}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right).$$

*Proof.* Applying Theorem 38 multiple times gives us a sequence of subgroups $\{e\} = G_0 \leq \cdots \leq G_m = G$ such that $m \leq n$ and the diameter of $G_i/G_{i-1}$ is at most $\frac{8}{n}$. Now we can use Corollary 18 to obtain the desired upper bound. □

Now we can prove the main result of this thesis.

**Corollary 40.** *Let $(V_0, \omega_0) \subset (V_1, \omega_1) \subset \ldots$ be a sequence of $\mathbb{F}_q$ vector spaces such that $(V_n, \omega_n)$ is a symplectic, unitary, or orthogonal space of dimension $2^n$ and $\omega_{n+1}|_{V_n} = \omega_n$ for all $n \in \mathbb{N}$. Let $G_n = \mathrm{Aut}(V_n, \omega_n)$ equipped with the normalized rank metric $d_n$ and the normalized counting measure $\mu_n$. Then*

$$\lim_{n\to\infty} \alpha_{(G_n,d_n,\mu_n)}(\varepsilon) = 0$$

*for all $\varepsilon > 0$ and $\mathrm{clim}\, G_n$ is extremely amenable.*

*Proof.* Immediate from Corollary 39 and Theorem 4. □

## 7 Application

In this section we will use the upper bound obtained for the length of symplectic, unitary, and orthogonal groups to deduce a Ramsey theoretic result. As in Section 6 the results from this section are already shown in [?] for special linear groups.

The first lemma is very similar to Theorem 15.

**Lemma 41** (Lemma 2.7 in [?])**.** *Let $(X, d, \mu)$ be a finite mm-space with length $l$. Then for $\varepsilon > 0$ and $A \subseteq X$ with $\mu(A) > 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right)$ we have*

$$\mu(N_\varepsilon(A)) \geq 1 - 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

A covering $\mathcal{U}$ of a metric space $(X, d)$ is an *$\varepsilon$-covering* if for every $x \in X$ the $\varepsilon$-neighborhood of $x$ is contained in some $U \in \mathcal{U}$.

**Theorem 42.** *Let $\varepsilon > 0$, $k, m \in \mathbb{N}$. Define $N := 16 \cdot 64\varepsilon^{-2} \cdot \max\{\ln(2k), \ln(2m)\}$ and let $G = \mathrm{Aut}(V, \omega)$, where $(V, \omega)$ is a symplectic, unitary, or orthogonal space of dimension $n > N$, with an $\varepsilon$-cover $\mathcal{U}$ of cardinality at most $m$. Then there is a $U \in \mathcal{U}$ such that for all $F \subseteq G$ satisfying $|F| \leq k$ there is a $g \in G$ with $gF \subseteq U$.*

Intuitively the theorem says that whenever we color $G$ with $m$ colors, where a single element can have multiple colors, such that all elements of $\varepsilon$-balls have at least one color in common, then there is one color $c$ such that for every $F$ with at most $k$ elements there is a $g$ where the elements of $gF$ all have the color $c$.

*Proof.* Look at $G$ as the usual mm-space with normalized rank metric and normalized counting measure. Let $l$ be the length of $G$, observe that, by Corollary 39, $l \leq 8n^{-\frac{1}{2}}$. For $U \in \mathcal{U}$ define $\mathrm{Core}(U) := \{x \in U \mid N_\varepsilon(x) \subseteq U\}$. Since $\mathcal{U}$ is an $\varepsilon$-covering we have $\bigcup_{U \in \mathcal{U}} \mathrm{Core}(U) = G$. Therefore there is a $U \in \mathcal{U}$ such that $\mu(\mathrm{Core}(U)) \geq \frac{1}{m}$. As $n > 16 \cdot 64\varepsilon^{-2} \cdot \ln(2m)$ we have

$$\frac{1}{m} > 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right) \geq 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

Now we can apply Lemma 41 to $\mathrm{Core}(U)$ and obtain

$$\mu(U) \geq \mu(N_\varepsilon(\mathrm{Core}(U))) \geq 1 - 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right) \geq 1 - 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right).$$

Let $F \subseteq G$ with $|F| \leq k$. Note that

$$\{g \in G \mid gF \subseteq U\} = \bigcap_{h \in F} \{g \in G \mid gh \in U\} = \bigcap_{h \in F} Uh^{-1}.$$

Therefore, $\mu(\{g \in G \mid gF \subseteq U\}) \geq 1 - k \cdot 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right)$. By assumption $n > 16 \cdot 64\varepsilon^{-2} \cdot \ln(2k)$, hence $\mu(\{g \in G \mid gF \subseteq U\}) > 0$ and there is a suitable $g$. $\square$

## ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Topological Entropy of Formal Languages" selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum Unterschrift