Bachelorarbeit

Topological Entropy of Formal Languages

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Contents

1 Martingale Technique

Definition 1. A *martingale* is a family $(f_i, \mathcal{F}_i)_{i \in \{0,\dots,n\}}$ such that

- f_i is integrable for all $i \in \{0, \dots, n\}$,
- f_i is $\mathcal{F}_i$ measurable for all $i \in \{0, \dots, n\}$, and
- $f_i = \mathbb{E}[f_{i+1} | \mathcal{F}_i]$ for all $i \in \{0, \dots, n-1\}$.

Lemma 2. *For all $x \in \mathbb{R}$*

$$e^x \leq x + e^{x^2}.$$

Lemma 3 (Azema's inequality)**.**

$$\mu(\{x \in X \mid |f(x) - \mathbb{E}(f)| \geq c\}) \leq 2 \exp\left(-\frac{c^2}{4 \sum_{i=1}^{n} \|d_i\|_\infty^2}\right)$$

$$\mu(\{|f - \mathbb{E}(f)| \geq c\}) \leq 2 \exp\left(-\frac{c^2}{4 \sum_{i=1}^{n} \|d_i\|_\infty^2}\right)$$

Definition 4. Let (X, d, μ) be an mm-space.

Theorem 5. *If an mm-space (X, d, μ) has length l, then the concentration function of X satisfies*

$$\alpha_X(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 l^2}\right).$$

Theorem 6. *Let G be a compact group with a bi-invariant metric d, and let*

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

be a chain of subgroups. Denote the diameter of G_i / G_{i-1} with respect to the factor metric by a_i. Then the concentration function of the mm-space (G, d, μ), where μ is the normalized Haar measure, satisfies

$$\alpha_X(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 \sum_{i=1}^{n} a_i^2}\right).$$

Theorem 7. *The normalized counting measure on the groups $\mathrm{SL}_{2^n}(q)$ concentrates with respect to the rank-metric, i.e. for all $r > 0$*

$$\lim_{n \to \infty} \alpha_{\mathrm{SL}_{2^n}}(r) = 0.$$

Florian sagt: "could be that this only holds for finite X as conditions in definition of length are just almost surely"

Lemma 8. *Let (X, d, μ) be an mm-space with diameter d and*

$$\Omega_0 = \{X\} \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

with $a_1, \dots, a_n$ as in Definition 4. Then

$$\sum_{i=1}^{n} a_i \geq d.$$

Proof. Let $x, y \in X$, with $x \neq y$, we show $d(x,y) \leq \sum_{i=1}^{n} a_i$. Let i_0 be the smallest number such that $[x]_{i_0} \neq [y]_{i_0}$. Since $[x]_0 = X = [y]_0$ we know that i_0 is at least 1. Therefore $[x]_{i_0-1} = [y]_{i_0-1}$ and there is an isomorphism $\varphi_{i_0} \colon [x]_{i_0} \to [y]_{i_0}$ such that $d(\varphi_{i_0}(x), y) \leq a_{i_0}$. Let $x_{i_0} = \varphi_{i_0}(x)$, then

Florian sagt: "here is the a.s. problem"

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

If $x_{i_0} = y$, then we are done. Otherwise let i_1 be the smallest number such that $[x_{i_0}]_{i_1} \neq [y]_{i_1}$. Then let $\varphi_{i_1} \colon [x_{i_0}]_{i_1} \to [y]_{i_1}$ be an isomorphism such that $d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$. Define $x_{i_1} = \varphi_{i_1}(x_{i_0})$. Proceeding in this fashion yields elements $x_{i_0}, \dots, x_{i_k}$ such that $x_{i_k} = y$ and

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \dots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \dots + a_{i_k} \leq \sum_{i=1}^{n} a_i.$$

□

Lemma 9. *Let (X, d, μ) be an mm-space with diameter 1 and $\Delta = \min d$. Then the length of X is at least $\Delta^{\frac{1}{2}}$.*

Definition 10. The *symplectic group* of degree $2n$ over a field q, denoted by $\mathrm{Sp}(2n, q)$, is the subgroup of $\mathrm{SL}(2n, q)$ containing all matrices A such that

$$A^T \Omega A = \Omega, \text{ where } \Omega = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Lemma 11. *Let $g \colon V \to V$ be an isomorphism, $V = U \oplus U'$, and $g(U') \subseteq U'$. Then the map*

$$\begin{aligned} g' \colon V &\to V \\ v &\mapsto \begin{cases} g(v) - \pi_{U'}(g(v)) & \text{if } v \notin U' \\ v & \text{if } v \in U' \end{cases} \end{aligned}$$

i.e. $g' = \pi_U \circ g - \pi_U \circ g \circ 1_{U'} + 1_{U'}$, is an isomorphism and $d(g, g') \leq \frac{1}{n} \cdot \dim U'$.

Lemma 12. *[what we still need (add conditions for ω if necessary)] Let $\omega \colon V \times V \to k$ be a bilinear map, U, U' subspaces of V, and $h \colon U \to U'$ an isomorphism that preserves ω. Then h can be extended to an isomorphism on V which also preserves ω.*

Proof. w.l.o.g. $\dim U + 1 = \dim V$? □

Lemma 13. *Let $V = U \oplus U'$, ω a bilinear map, G be the group of automorphisms of (V, ω) and $G' \leq G$ the subgroup fixing U'. Then the diameter of G/G_i is at most $\frac{3 \cdot \dim U'}{n}$.*

Florian sagt: "…additional conditions"

Florian sagt: "adapt this"

Proof. Let $g \in G$, we show that there are $g' \in G$ and $g'' \in G'$ such that $g'(U') \subseteq U'$, $g'|_{U'} = 1_{U'}$, and

$$d(g, g'') \leq d(g, g') + d(g', g'') \leq \frac{2 \dim U'}{n} + \frac{\dim U'}{n}.$$

By Lemma 12 we can extend the map $g^{-1}|_{gU'}$ to a map h' on $V' = \langle U', gU' \rangle$. Now define $g' = (1_{V''} \oplus h')g$, where $V = V'' \oplus V'$ and apply Lemma to g' to obtain g''.

$$\begin{aligned} \operatorname{im} g - g' &= \operatorname{im} g - (1_{V''} \oplus h')g \\ &= \operatorname{im}(1_{V''} \oplus 1_{V'} - 1_{V''} \oplus h') \\ &= \operatorname{im}(1_{V'} - h') \\ &\subseteq V' \end{aligned}$$

$$d(g, g') = \frac{1}{n} \dim \operatorname{im} g - g' \leq \frac{\dim V'}{n}$$

$\square$

2 Limits of other Matrix group families are Levy groups too

When studying matrices it is often useful to look at the corresponding linear maps of a suitable vector space. In the case of orthogonal, symplectic, or unitary matrices these are linear maps from the vector space to itself preserving an orthogonal, symplectic, or unitary form respectively. Formally, the symplectic group $\mathrm{Sp}_n(q)$ is isomorphic to $\mathrm{Aut}(V, \omega)$, where V is an n-dimensional $F(q)$ vector space and ω is a symplectic form.

As we have to handle only finite dimensional vector spaces here a lot of nice theorems hold. . . .

Let V be an n dimensional vector space.

Lemma 14. *For all $U \leq V$ there is an $U' \leq V$ such that $U \oplus U' = V$.*

Let ω be a bilinear form on V.

Lemma 15. *Let $U \leq V$. Then $\dim U^\perp \geq \dim V - \dim U$.*

Lemma 16. *There exists a $U \leq V$ with $\dim U \leq 2$ such that $V = U \oplus U^\perp$.*

Proof. Let $e \in V \setminus \{0\}$. By Lemma 15 $\dim e^\perp \geq n - 1$. Since ω is non degenerate $\omega(e, .) \neq 0$ and therefore $e^\perp \neq V$. Hence $\dim e^\perp = n - 1$.

If $e \notin e^\perp$, then $V = \langle e \rangle \oplus e^\perp$. and $\langle e \rangle$ is the desired U.

If $e \in e^\perp$, then extend e to a basis $e, b_2, \ldots, b_{n-1}$ of $e^\perp$ and consider the subspace $U := \langle b_2, \ldots, b_{n-1} \rangle^\perp$. □

Lemma 17. *Let $U \leq V$ and $f \colon V \to V$ be an isometry such that $f|_U = 1_U$. Then $f(U^\perp) \subseteq U^\perp$.*

Proof. It suffices to show that $f(u') \perp u$ for all $u \in U$ and $u' \in U^\perp$.

$$\begin{aligned} \omega(f(u'), u) &= \omega(f(u'), f(u)) \\ &= \omega(u', u) \\ &= 0 \end{aligned}$$

This concludes the proof. □

Lemma 18. *For all $W \leq V$ there is a $W' \leq W^\perp$ such that $W \cap W' = 0$ and*

$$\dim W' \geq \dim V - 2 \dim W.$$

Proof. By Lemma 14 there is a W' such that

$$W^\perp = (W^\perp \cap W) \oplus W'.$$

Clearly, $W \cap W' = 0$ and

$$\dim W' = \dim W^\perp - \dim(W^\perp \cap W) \geq \dim W^\perp - \dim W.$$

Whats left is to show that $\dim W^{\perp} \geq \dim V - \dim W$. Let $b_1, \ldots, b_{\dim W}$ be a basis of W. Then $W^{\perp}$ is equal to the kernel of the linear map

$$V \to F_q^{\dim W} \qquad\qquad v \mapsto \begin{pmatrix} \omega(b_1, v) \\ \vdots \\ \omega(b_{\dim W}, v) \end{pmatrix}.$$

Now the statement follows from the rank-nullity theorem. □

Florian sagt: "maybe $g\colon U_1 \to U_2$ and $h\colon W_1 \to W_2$ better"

Lemma 19. *Let $g_1\colon U_1 \to W_1$ and $g_2\colon U_2 \to W_2$ be isometries such that $U_1 \perp U_2$, $U_1 \cap U_2 = 0$, $W_1 \perp W_2$, and $W_1 \cap W_2 = 0$. Then $g_1 \oplus g_2\colon U_1 \oplus U_2 \to W_1 \oplus W_2$ is also an isomtry.*

Proof. Obviously, $g_1 \oplus g_2$ is again a bijective linear map. Consider $v_1 + v_2, u_1 + u_2 \in U_1 \oplus U_2$

$$\begin{aligned}
\omega(v_1 + v_2, u_1 + u_2) &= \omega(v_1, u_1) + \omega(v_1, u_2) + \omega(v_2, u_1) + \omega(v_2, u_2) \\
&= \omega(v_1, u_1) + 0 + 0 + \omega(v_2, u_2) && (U_1 \perp U_2) \\
&= \omega(g_1(v_1), g_1(u_1)) + \omega(g_2(v_2), g_2(u_2)) \\
&= \omega(g_1(v_1), g_1(u_1)) + \omega(g_1(v_1), g_2(u_2)) \\
&\quad + \omega(g_2(v_2), g_1(u_1)) + \omega(g_2(v_2), g_2(u_2)) && (W_1 \perp W_2) \\
&= \omega(g_1 \oplus g_2(v_1 + v_2), g_1 \oplus g_2(u_1 + u_2))
\end{aligned}$$

Hence $g_1 \oplus g_2$ preserves ω. □

[other useful theorems]

Theorem 20 (Witt)**.** *Let V be an orthogonal, symplectic, or unitary space. Let U and W be subspaces of V and suppose $\alpha\colon U \to W$ is an isometry. Then α extends to an isometry of V.*

Lemma 21. *Let G be an orthogonal, symplectic, or unitary group. ...*

Florian sagt: "maybe choose the basis in a clever way"

Florian sagt: "not jet, we still need $hg \in H$, i.e. $hg(\langle e_1, \ldots, e_{n-1} \rangle) \subseteq \langle e_1, \ldots, e_{n-1} \rangle$"

Proof. $G = \mathrm{Aut}(V, \omega)$ for some vector space V with bilinear form ω. Let $\{e_1, \ldots, e_n\}$ be a basis of V and $H = \mathrm{Aut}(\langle e_1, \ldots, e_{n-1} \rangle, \omega)$. Our aim is to find for any $g \in G$ an $g' \in H$ such that $d(g, g') \leq \frac{4}{n}$. The idea is to find a map $h \in H$ that behaves like the inverse of g on ge_n and like the identity an most of the rest. Then hg is the desired g'.

Let $g \in G$ and define $W = \langle e_n, ge_n \rangle$. By Lemma 18 there is a W' such that $\dim W' \geq n - 4$ and $W' \leq W^{\perp}$. Consider the map

$$g^{-1}|_{\langle ge_n \rangle} \oplus 1_{W'}\colon \langle ge_n \rangle \oplus W' \to \langle e_n \rangle \oplus W'$$

as $g^{-1}|_{\langle ge_n \rangle}$ and $1_{W'}$ are isometries and $W \perp W'$ Lemma 19 implies that the above map is also an isomtry. By Witt's lemma this isometry can be extended to an isometry $h\colon V \to V$. □

3 Fun

Consider an n-dimensional cube with 2^k nodes on each edge. Then its diameter $\nabla_{n,k}$ and length $L_{n,k}$ are

$$\nabla_{n,k} = \sqrt{(2^k - 1) \cdot n} \qquad\qquad L_{n,k} = \sqrt{\sum_{i=0}^{k-1} 2^{2i} \cdot n}.$$

Henceforth

$$\lim_{n \to \infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{L_{1,k}}{\nabla_{1,k}} \qquad \text{and} \qquad \lim_{k \to \infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{1}{\sqrt{3}}.$$

ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Topological Entropy of Formal Languages" selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum Unterschrift