Bachelorarbeit

Measure Concentration for Symplectic Groups

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Contents

1 Introduction

The objects studied in this thesis are groups of matrices over some finite field. Given such group, we can equip it with the normalized rank metric and the normalized Haar measure to obtain a *metric measure space*. Observe that, since matrix groups over finite fields are again finite, the normalized Haar measure is just the normalized counting measure. For some sequences of matrix groups of increasing size there is a well defined limit. For example, this is the case for the special linear group of $(n \times n)$-matrices SL_n. Carderi and Thom showed in [2] that a suitable limit of SL_n is, as a topological group, *extremely amenable*. The goal of this thesis is to generalize this result to limits of other families of matrix groups, namely groups of *symplectic*, *unitary*, and *orthogonal matrices*. These matrices can be seen as *Isometries*, i.e. bijective linear maps from a vector space into itself preserving a symplectic, unitary, or orthogonal form. The general strategy to prove extreme amenability for limits of families of these groups will be as follows: given such a family $(G_n)_{n\in\mathbb{N}}$ of finite matrix groups considered as metric measure spaces. We apply a consequence of Azema's inequality [4] to obtain an upper bound for the *measure concentration function* of G_n in terms of the *length* of G_n. As the upper bounds converge to zero we conclude that $(G_n)_{n\in\mathbb{N}}$ is a *Lévy family*, making their limit a *Lévy group*. Finally, we know from [3] that every Lévy group is extremely amenable.

This thesis is structured as follows. In Section 2, we will give a short introduction on how to view matrix groups as metric measure spaces and how to define a limit of a sequence of matrix groups. Furthermore we will introduce the notion of extreme amenability and its connection to Lévy groups. In Section 3, we will briefly introduce the notion of *conditional expectation* to show Azema's inequality. Next we will introduce an important invariant of metric measure spaces, namely their *length*. Azema's inequality will allow us to connect the length of a metric measure space with their measure concentration function. This connection is used in Section 4 to show that the limit of SL_n is extremely amenable. To generalize this result we give a proof of Witt's lemma, which says that isometries between subspaces can be extended to the whole space, in Section 5. In Section 6, we generalize the result from Section 4 to symplectic, unitary, and orthogonal groups. Finally, in Section 7, a Ramsey theoretic result from [2] about SL_n is generalized to symplectic, unitary, and orthogonal groups.

2 Preliminaries

Let q be a prime power and $\mathbb{F}_q$ be the unique q element field. Denote the the general linear group over $\mathbb{F}_q$ by $\mathrm{GL}_n(q)$. We can equip $\mathrm{GL}_n(q)$ with the (normalized) *rank-metric* $d(g,h) := \frac{1}{n}\,\mathrm{r}(g-h)$, where $r(g)$ is the rank of g or equivalently the dimension of the image of g.

Lemma 1. *The metric d is* bi-invariant, *i.e. for all $g, h, k \in \mathrm{GL}_n(q)$ we have*

$$d(kg, kh) = d(g, h) = d(gk, hk).$$

Proof. Let $g, h, k \in \mathrm{GL}_n(q)$. Note that k has full rank. Therefore

$$n \cdot d(kg, kh) = \mathrm{r}(kg - kh) = \mathrm{r}(k(g - h)) = \mathrm{r}(g - h) = n \cdot d(g, h).$$

The other equality follows similarly. □

Let $G_n \leq \mathrm{GL}_{2^n}(q)$ be a family of subgroups, such that $\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in G_{n+1}$ for all $g \in G_n$. Denote the normalized rank-metric of G_n by d_n.

Lemma 2. *For all $n \in \mathbb{N}$ the map*

$$\varphi_n \colon G_n \mapsto G_{n+1}, \text{ where } \varphi_n(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

is an isometric embedding.

Proof. Let $n \in \mathbb{N}$ and $g, h \in G_n$. Then

$$\begin{aligned} d(g, h) &= \frac{1}{n}\,\mathrm{r}(g - h) \\ &= \frac{1}{2n}(\mathrm{r}(g - h) + \mathrm{r}(g - h)) \\ &= \frac{1}{2n}\,\mathrm{r}\begin{pmatrix} g - h & 0 \\ 0 & g - h \end{pmatrix} \\ &= d(\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}). \end{aligned}$$

Clearly, φ_n is also a homomorphism. □

Let $\varphi \colon \bigsqcup_{n \in \mathbb{N}} G_n \to \bigsqcup_{n \in \mathbb{N}} G_n$ with $\varphi|_{G_n} = \varphi_n$.

Definition 3. Let $\sim$ be the equivalence relation on $\bigsqcup_{n \in \mathbb{N}} G_n$, defined by $g \sim h$ iff there are $m, n \in \mathbb{N}$ such that $\varphi^n(g) = \varphi^m(h)$. Then the *limit* of $(G_n)_{n \in \mathbb{N}}$ is defined as

$$\lim_{n \in \mathbb{N}} G_n := \left(\bigsqcup_{n \in \mathbb{N}} G_n \right) / \sim .$$

Lemma 4. *The sequence of metric groups $(G_n, d_n)_{n \in \mathbb{N}}$ induces a group structure and a bi-invariant metric d on $G = \lim G_n$.*

Proof. Note that $\psi_n \colon G_n \to G$, $\psi(g) = [g]$, is injective and

$$G = \bigcup_{n \in \mathbb{N}} \psi_n(G_n).$$

For $[g], [h] \in G$ we can assume w.l.o.g. that $g, h \in G_n$ for some $n \in \mathbb{N}$. Hence we define $[g] \cdot [h] := [gh]$ and $d([g], [h]) := d_n(g, h)$. This is well defined since the following diagram commutes for all $n \leq m$.

$$\begin{tikzcd} G_m \arrow[r, hook, "\psi_n"] & G \\ G_n \arrow[u, hook, "\varphi_{n,m}"] \arrow[ur, hook, "\psi_n"'] & \end{tikzcd}$$

Here $\varphi_{n,m} := \varphi^{m-n}|_{G_n}$. Note that now the ψ_n are isometric embeddings. Hence d inherits all desired properties from the d_n's. $\square$

Furthermore the metric and the group structure of G interact nicely.

Definition 5. A group G equipped with a topology is a *topological group* if the maps $G \times G \to G$, $(g, h) \mapsto gh$ and $G \to G$, $g \mapsto g^{-1}$ are continuous. Here we use the product topology on $G \times G$.

Lemma 6. *Let $(G_n)_{n \in \mathbb{N}}$, $G = \lim G_n$, and d be as before. Then G with the topology induced by d is a topological group.*

Proof. Denote the neutral element of G by e. First we show that the inverse is continuous. Let $\varepsilon > 0$ and $g, h \in G$ with $d(g, h) < \varepsilon$. Then, by bi-invariance of d,

$$d(g^{-1}, h^{-1}) = d(e, gh^{-1}) = d(h, g) < \varepsilon.$$

We use $d_\Sigma((g, h), (g', h')) := d(g, g') + d(h, h')$ as metric on $G \times G$. Let $\varepsilon > 0$ and $g, g', h, h' \in G$ with $d_\Sigma((g, h), (g', h')) < \varepsilon$. Then

$$d(gh, g'h') = d(gg'^{-1}, h^{-1}h') \leq d(g, g'^{-1}, e) + d(e, h^{-1}h') = d(g, g') + d(h, h') < \varepsilon.$$

This yields the desired result. $\square$

The group we are interested in is the metric completion of $\lim G_n$.

Lemma 7. *Let G be a topological group with bi-invariant metric d. Then there a unique metric space $(\bar{G}, \bar{d})$ containing G such that $\bar{G}$ is complete and G is dense in $\bar{G}$. Furthermore $\bar{G}$, with the group structure induced by G, is a topological group and $\bar{d}$ is still bi-invariant.*

Proof. Consider the set G_C of Cauchy sequences in G. Define $\bar{G} := G_C/\sim$, where two sequences $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are equivalent if $\lim d(g_n, h_n) = 0$. Furthermore $\bar{d}([(g_n)_{n\in\mathbb{N}}], [(h_n)_{n\in\mathbb{N}}]) := \lim d(g_n, h_n)$. It is well known that $(\bar{G}, \bar{d})$ is the unique metric completion of (G, d).

The group operation can be extended to $\bar{G}$ as follows:

$$[(g_n)_{n\in\mathbb{N}}] \cdot [(h_n)_{n\in\mathbb{N}}] := [(g_n h_n)_{n\in\mathbb{N}}].$$

It is clear from the definition that $\bar{d}$ is also bi-invariant. Next we show that $\bar{G}$ is still a topological group. The proof is very similar to the one of Lemma 6. Let $\varepsilon > 0$ and $[(g_n)_{n\in\mathbb{N}}], [(h_n)_{n\in\mathbb{N}}] \in \bar{G}$ with $\bar{d}([(g_n)_{n\in\mathbb{N}}], [(h_n)_{n\in\mathbb{N}}]) < \varepsilon$. Then

$$\bar{d}([(g_n)_{n\in\mathbb{N}}]^{-1}, [(h_n)_{n\in\mathbb{N}}]^{-1}) = \lim_{n\to\infty} d(g_n^{-1}, h_n^{-1}) = \lim_{n\to\infty} d(g_n, h_n) < \varepsilon.$$

Analogously, we obtain that the group operation is continuous. □

Definition 8. Let $(G_n)_{n\in\mathbb{N}}$, $G = \lim G_n$, and d be as before. Define the *closure of the limit of* $(G_n)_{n\in\mathbb{N}}$, denoted by $\operatorname{clim} G_n$, as the metric completion of (G, d).

By the previous lemma $\operatorname{clim} G_n$ is a topological group and a complete metric space with bi-invariant metric. And $\operatorname{clim} G_n$ has another nice property. For this we need to introduce the well established notion of Polish spaces. A topological space (X, τ) is a *Polish space* if there is a metric d on X that induces the topology τ such that (X, d) is complete and has a countable dense subset. A topological group is a *Polish group* if the underlying topological space is a Polish space.

Lemma 9. *We have that* $\operatorname{clim} G_n$*, seen as a topological group, is a Polish group.*

Proof. Obviously, $\lim G_n$ is countable and dense in $\operatorname{clim} G_n$. By definition $\operatorname{clim} G_n$ is also a complete metric space. □

Definition 10. A topological group G is *extremely amenable* if every continuous action of G on a compact topological space admits a fixed point.

The goal of this thesis is to show that for certain sequences $(G_n)_{n\in\mathbb{N}}$ we have that $\operatorname{clim} G_n$ is extremely amenable. It is hard to show this directly, but we know from [3] that every Lévy group (see Definition 12) is extremely amenable. Hence we will show that $\operatorname{clim} G_n$ is a Lévy group instead.

Before we can talk about Lévy groups we need some more definitions. For an $\varepsilon > 0$ and a metric space (X, d) with an $A \subseteq X$, we define the *ε-neighborhood* of A to be

$$N_\varepsilon(A) := \{x \in X \mid \exists y \in A.\, d(x, y) < \varepsilon\}.$$

Definition 11. A *metric measure space* (mm-space) $\boldsymbol{X}$ is a triple (X, d, μ), where d is a metric on the set X and μ is a measure on the Borel σ-algebra induced by d. We will always assume that $\mu(X) = 1$. For any set $A \subseteq X$ denote the *ε-neighborhood* of A, i.e. $\{x \in X \mid \exists y \in A.\ d(x,y) < \varepsilon\}$, by $N_\varepsilon(A)$.The *measure concentration function* of $\boldsymbol{X}$ is defined as

$$\alpha_{\boldsymbol{X}}(\varepsilon) = \sup\{1 - \mu(N_\varepsilon(A)) \mid A \subseteq X, \mu(A) \geq \frac{1}{2}\}.$$

A family of mm-spaces $\boldsymbol{X}_n$ with diameter 1 is called a *Lévy family* if

$$\alpha_{\boldsymbol{X}_n}(\varepsilon) \to 0$$

for all $\varepsilon > 0$.

A topological space $\boldsymbol{X}$ is a *Polish space* if the topology is induced by a metric such that $\boldsymbol{X}$ as a metric space is complete and has a countable dense subset.

Now we can come back to groups.

Definition 12. A *Polish group* G is a topological group where the underlying topological space is a Polish space. A *Lévy group* is a group G equipped with a metric d, where

- G with the topology induced by d is a Polish group and
- there is a sequence $(G_n)_{n\in\mathbb{N}}$ of compact subgroups, such that $(G_n, d|_{G_n}, \mu_n)_{n\in\mathbb{N}}$ is a Lévy family. Here μ_n is the normalized Haar measure of G_n.

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Note that, since G_n is finite, the normalized Haar measure of G_n is just the normalized counting measure. The following theorem from [3, Theorem **4.1.3**] gives the desired connection to extreme amenability.

Theorem 13. *Every Lévy group is extremely amenable.*

To apply this theorem to our setting we need the following lemma.

Lemma 14. *Let $G_n \leq \mathrm{GL}_{2^n}(q)$ with the normalized rank metric d_n and $G = \mathrm{clim}_{n\to\infty} G_n$. Then G equipped with the metric induced by d_n is a Polish group.*

Proof. By Lemma **??** G is already a topological group and by definition it is also a complete metric space. Furthermore, every G_n is finite. Hence the inductive limit of the G_n is a countable dense subset of G. $\square$

Whether G is also a Lévy group depends on the particular choice of $(G_n)_{n\in\mathbb{N}}$. To show that for certain sequences G will be a Lévy group, we will bound $\alpha_{G_n}(\varepsilon)$ in terms of n and ε and show that this bound converges to 0 as n tends to infinity. The next section develops the methods necessary to obtain this upper bound.

3 Azema's Inequality and Measure Concentration

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In this section we will prove Azema's inequality and as a consequence, we will obtain an upper bound for the measure concentration function. As the next results rely heavily on stochastic methods we will briefly introduce the necessary notions. Since the G_n are all finite and equipped with the normalized counting measure we will only consider *probability spaces* (X, Σ, μ), where X is finite, Σ is a σ-algebra over X, and $\mu(A) = |A|/|X|$ for $A \subseteq X$. In this section we will roughly follow Section 3.2 from [4]. Most of the statements presented in this section hold in a more general setting. Note that Σ has a very nice representation.

Lemma 15. *Let Σ be a σ-algebra over a finite set X and $A_1, \ldots, A_n$ be the minimal nonempty sets in Σ. Then $A_1, \ldots, A_n$ is a partition of X and Σ is the smallest σ-algebra containing $A_1, \ldots, A_n$.*

Proof. First we show that $A_1, \ldots, A_n$ is a partition of X. Since $A_i \cap A_j \in \Sigma$ we conclude, by minimality of A_i and A_j, that either $i = j$ or $A_i \cap A_j = \emptyset$. Clearly, every element of X is contained in one of the A_i.

For $A \in \Sigma$ we have, again by minimality, that $A \cap A_i$ is either A_i or $\emptyset$. Therefore A can be written as a union of A_i's. $\square$

Note that it follows from the proof that any $A \in \Sigma$ can be written as $\bigcup_{i \in I} A_i$ for a suitable $I \subseteq \{1, \ldots, n\}$. This lemma allows us to use partitions and σ-algebras interchangeably. We will denote the partition corresponding to Σ by $A_1, \ldots, A_n$, for Σ' we will use $A'_1, \ldots, A'_{n'}$, etc. The next definition is simplified a lot by only considering finite X.

Definition 16. Let (X, Σ, μ) be a finite probability space, $f\colon X \to \mathbb{R}$ be a measurable function, and Σ' be a sub-σ-algebra of Σ. Then the *conditional expectation* of f with respect to Σ' is defined as

$$\mathbb{E}(f \mid \Sigma') := \sum_{i=1}^{n'} \mathbb{E}(f \mid A'_i) \cdot \mathbb{1}_{A'_i}.$$

One often thinks of Σ' as the available information, a finer partition means more information. The conditional expectation $\mathbb{E}(f \mid \Sigma')$ is the best approximation of f given only the information from Σ'. With this intuition the statements from the following lemma are not surprising.

Lemma 17. *Let (X, Σ, μ) be a finite probability space, $f, g\colon X \to \mathbb{R}$ be measurable functions, $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ be sub-σ-algebras. Then*

i) if $f \leq g$, then $\mathbb{E}(f \mid \Sigma') \leq \mathbb{E}(g \mid \Sigma')$,

ii) for any Σ'-measurable function $h\colon X \to \mathbb{R}$ we have $\mathbb{E}(hf \mid \Sigma') = h \cdot \mathbb{E}(f \mid \Sigma')$,

iii) also $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') = \mathbb{E}(f \mid \Sigma'') = \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma')$.

Proof. To i): If $f \leq g$, then

$$\mathbb{E}(f \mid \Sigma') = \sum_{i=1}^{n} \mathbb{E}(f \mid A'_i) \cdot \mathbb{1}_{A'_i} \leq \sum_{i=1}^{n} \mathbb{E}(g \mid A'_i) \cdot \mathbb{1}_{A'_i} = \mathbb{E}(g \mid \Sigma').$$

To ii): Let $h\colon X \to \mathbb{R}$ be Σ'-measurable function, then $h = \sum_{i=1}^{n'} h_i \mathbb{1}_{A'_i}$. Now

$$\begin{aligned}
\mathbb{E}(hf \mid \Sigma') &= \sum_{i=1}^{n'} \mathbb{E}(hf \mid A'_i) \mathbb{1}_{A'_i} \\
&= \sum_{i=1}^{n'} h_i \mathbb{E}(f \mid A'_i) \mathbb{1}_{A'_i} \\
&= h \cdot \mathbb{E}(f \mid \Sigma').
\end{aligned}$$

To iii): Note that $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid A') = \mathbb{E}(f \mid A')$ for all $A' \in \Sigma'$. Hence

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') &= \sum_{i=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid A''_i) \cdot \mathbb{1}_{A''_i} \\
&= \sum_{i=1}^{n''} \mathbb{E}(f \mid A''_i) \cdot \mathbb{1}_{A''_i} && (A''_i \in \Sigma') \\
&= \mathbb{E}(f \mid \Sigma'') \\
&= \sum_{j=1}^{n''} \mathbb{E}(f \mid A''_j) \cdot \mathbb{1}_{A''_j} \cdot \sum_{i=1}^{n'} \mathbb{1}_{A'_i} \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid A''_j) \mid A'_i) \cdot \mathbb{1}_{A''_j} \cdot \mathbb{1}_{A'_i} \\
&= \sum_{i=1}^{n'} \sum_{j=1}^{n''} \mathbb{E}(\mathbb{E}(f \mid A''_j) \cdot \mathbb{1}_{A''_j} \mid A'_i) \cdot \mathbb{1}_{A'_i} && \text{(by ii)} \\
&= \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma').
\end{aligned}$$

This concludes the proof. □

The following lemma might not seem very interesting, but changing the exponent from x to x^2 is the very foundation for Azema's inequality.

Lemma 18. *For all $x \in \mathbb{R}$*

$$e^x \leq x + e^{x^2}.$$

Proof. Note that for $x = 0$ both sides are equal to 1. As both sides are differentiable it suffices to show that the derivative of the right hand side is larger than the derivative of the left hand side for all $x \geq 0$ and smaller for all $x \leq 0$. Hence, we want to show

$$e^x \geq 1 + 2xe^{x^2} \text{ for all } x \leq 0 \qquad \text{and} \qquad e^x \leq 1 + 2xe^{x^2} \text{ for all } x \geq 0.$$

As for $x = 0$ both sides are again equal to 1 we can reduce the problem, by similar reasoning, to the question whether

$$e^x \leq 2e^{x^2} + 4x^2 e^{x^2} \text{ for all } x \in \mathbb{R}.$$

- For $x = 0$ the terms reduce to $1 \leq 2$.
- For $x < 0$ the left hand side is bounded by 1, while the right hand side is still larger that 2.
- For $1 \leq x$ we have $x \leq x^2$ and the inequality holds trivially.
- For $0 < x < 1$ note that the both sides are increasing. Hence the inequality holds for all x with $e^x \leq 2$. Finally, $\ln 2 \geq \frac{1}{2}$ and therefore the right hand side with $x = \ln 2$ evaluates to a number larger then e.

$\square$

Before we will prove Azema's inequality let us introduce some useful notation. Whenever there is no danger of confusion we will abbreviate sets of the form

$$\{x \in X \mid \text{Condition}(x) \text{ holds}\} \qquad \text{by} \qquad \{\text{Condition}\}.$$

For example $\{x \in X \mid f(x) = c\}$ becomes $\{f = c\}$.

Lemma 19. *[Azema's inequality] Let (X, Σ, μ) be a finite probability space, $f\colon X \to \mathbb{R}$ a measurable function, and $\{X\} = \Sigma_0 \subseteq \cdots \subseteq \Sigma_n = \Sigma$ a chain of sub-σ-algebras. Define $f_i := \mathbb{E}(f \mid \Sigma_i)$ and $d_i := f_i - f_{i-1}$. Then for every $\varepsilon \geq 0$*

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \cdot \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^n \|d_i\|_\infty^2}\right).$$

Note that $(f_i, \Sigma_i)_{0 \leq i \leq n}$ is a discrete martingale.

Proof. First, observe that $f_0 = \mathbb{E}(f \mid \{X\}) = \mathbb{E}(f)$ and $f_n = \mathbb{E}(f \mid \Sigma) = f$. Using a simple telescoping sum we obtain $f - \mathbb{E}(f) = d_1 + \cdots + d_n$. Therefore

$$\begin{aligned}
\mu(\{f - \mathbb{E}(f) \geq \varepsilon\}) &= \mu(\{\sum_{i=1}^{n} d_i \geq \varepsilon\}) \\
&= \mu(\{\lambda \cdot \sum_{i=1}^{n} d_i \geq \lambda\varepsilon\}) && \text{(for } \lambda > 0\text{)} \\
&= \mu(\{e^{\lambda \cdot \sum_{i=1}^{n} d_i - \lambda\varepsilon} \geq 1\}) \\
&\leq \mathbb{E}(e^{\lambda \cdot \sum_{i=1}^{n} d_i}) \cdot e^{-\lambda\varepsilon} && (*) \\
&= \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}} \cdot \mathbb{E}(e^{\lambda d_n} \mid \Sigma_{n-1})) \cdot e^{-\lambda\varepsilon} && \text{(Lemma 17)} \\
&\leq \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}}) \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda\varepsilon} && (**) \\
&\;\;\vdots \\
&\leq e^{\lambda^2 \cdot \|d_1\|_\infty^2} \cdot \ldots \cdot e^{\lambda^2 \cdot \|d_{n-1}\|_\infty^2} \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda\varepsilon} \\
&= e^{\lambda^2 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2 - \lambda\varepsilon}.
\end{aligned}$$

For $(*)$ note that for any measurable function $g\colon X \to \mathbb{R}$ with $g \geq 0$ we have

$$\mu(\{g \geq 1\}) = \mathbb{E}(\mathbb{1}_{\{g \geq 1\}}) \leq \mathbb{E}(g).$$

For $(**)$ we need to use Lemma 18

$$\begin{aligned}
\mathbb{E}(e^{\lambda d_i} \mid \Sigma_{i-1}) &\leq \mathbb{E}(\lambda d_i \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&= \lambda \cdot \mathbb{E}(f_i - f_{i-1} \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&\leq 0 + e^{\lambda^2 \|d_i\|_\infty^2}. && \text{(Lemma 17)}
\end{aligned}$$

Substituting $-\frac{\varepsilon^2}{\sum_{i=1}^{n} \|d_i\|_\infty^2}$ for λ we conclude that

$$\mu(\{f - \mathbb{E}(f) \geq \varepsilon\}) \leq \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

The same calculations with $-d_i$ instead of d_i yield the dual inequality

$$\mu(\{f - \mathbb{E}(f) \leq -\varepsilon\}) \leq \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

These two statements obviously give us the desired result. □

Since μ is the counting measure Azema's inequality bounds the number of elements for which f differs at least ε from its mean. This seems at least somewhat connected to the measure concentration function, as there we want to show that for any set A with $\mu(A) \geq \frac{1}{2}$ only a few elements are more than ε away from A. The next goal is to formalize this connection. But to achieve this we first need to introduce a new property of mm-spaces.

Definition 20. Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space. The *length* of $\boldsymbol{X}$ is the minimum over all l with the following property. There is a refining sequence of partitions

$$\{X\} = \Omega_0 \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\},$$

where for every $i \in \{1, \dots, n\}$ there is an a_i such that $\sum_{i=1}^{n} a_i^2 = l^2$ and for every $A \in \Omega_{i-1}$, $x, y \in A$ there is an isomorphism (of metric spaces) $\phi\colon [x]_i \to [y]_i$ with

$$d(z, \phi(z)) \leq a_i \text{ for all } z \in [x]_i.$$

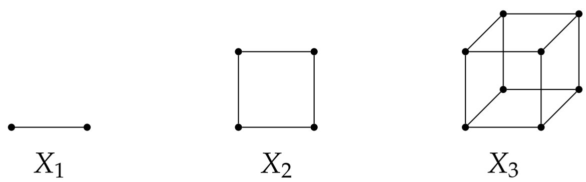
Note that since μ is the counting measure ϕ is also an isomorphism of mm-spaces. As this definition is quite hard we will look at some properties and examples of the length of $\boldsymbol{X}$ before proceeding.

Lemma 21. *Let $\boldsymbol{X}$ be a finite mm-space. Then the length of $\boldsymbol{X}$ is at most the diameter of $\boldsymbol{X}$.*

Proof. Consider only the two partitions $\{X\} \prec \{\{x\} \mid x \in X\}$. Clearly, $x \to y$ is an isomorpism between $\{x\}$ and $\{y\}$ for all $x, y \in X$. $\square$

We will see more properties later in Lemma 30 and Lemma 31.

Example 22. Let us look at the n-dimensional cube $X_n = \{0, 1\}^n$.



We will only consider the following sequence of partitions

$$\Omega_0 \prec \cdots \prec \Omega_n \text{ with } \Omega_i = \{wX_{n-i} \mid w \in \{0, 1\}^i\}.$$

First, we equip X_n with the euclidean metric and rescale it such that the diameter is 1. To bound the length of the resulting space $\boldsymbol{X}_n^E$ consider $[x]_i \neq [y]_i$. Note that x and y are w.l.o.g. of the form $w0u$ and $w1v$ for some $w \in \{0, 1\}^{i-1}$, $u, v \in \{0, 1\}^{n-i}$. The isomorphism ϕ takes an element $w0u'$ in $[x]_i$ and maps it to $w1u'$. The length of a side in $\boldsymbol{X}_n^E$ is $\frac{1}{\sqrt{n}}$, hence a_i is $\frac{1}{\sqrt{n}}$ for every i and the length of $\boldsymbol{X}_n^E$ is bounded by $(\sum_{i=1}^{n} \frac{1}{\sqrt{n}^2})^{\frac{1}{2}} = 1$.

Secondly, we use the normalized hemming metric and obtain the mm-space $\boldsymbol{X}_n^H$ with diameter 1. It has side length $\frac{1}{n}$ and therefore the length of $\boldsymbol{X}_n^H$ is bounded by $(\sum_{i=1}^{n} \frac{1}{n^2})^{\frac{1}{2}} = n^{-\frac{1}{2}}$. We see that here the length of $\boldsymbol{X}_n^H$ converges to 0 as n tends to infinity. We will show that this means that the measure concentration function $\alpha_{\boldsymbol{X}_n^H}(\varepsilon)$ also goes to 0 for any fixed $\varepsilon > 0$.

Now we come back to the connection between Azema's inequality and the measure concentration function.

Lemma 23. *Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space of length l and $f\colon X \to \mathbb{R}$ be a 1-Lipschitz function. Then*

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2\exp\left(-\frac{\varepsilon^2}{4l^2}\right) \text{ for every } \varepsilon > 0.$$

Proof. Let $\Omega_0 \prec \cdots \prec \Omega_n$ be a refining sequence of partitions with $a_1, \ldots, a_n$ as in Definition 20 such that $\sum_{i=1}^n a_i^2 = l^2$. These partitions correspond to σ-algebras $\Sigma_0 \subseteq \cdots \subseteq \Sigma_n$. Now we can apply Azema's inequality to obtain

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \cdot \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^n \|d_i\|_\infty^2}\right),$$

where $f_i = \mathbb{E}(f \mid \Sigma_i)$ and $d_i = f_i - f_{i-1}$ as before. Therefore we only need to show that $\|d_i\|_\infty \leq a_i$. Since on any $A \in \Omega_{i-1}$ we have $f_{i-1} = \mathbb{E}(f_i \mid A)$ it suffices to show that for all $A \in \Omega_{i-1}$ it holds that $f_i(x) - f_i(y) \leq a_i$ for all $x, y \in A$. Let $\phi\colon [x]_i \to [y]_i$ be the isomorphism from Definition 20.

$$\begin{aligned}
f_i(x) - f_i(y) &= \mathbb{E}(f \mid [x]_i) - \mathbb{E}(f \mid [y]_i) \\
&= \mathbb{E}(f \mid [x]_i) - \mathbb{E}(f \circ \phi \mid [x]_i) \\
&= \mathbb{E}(f - f \circ \phi \mid [x]_i) \\
&\leq \mathbb{E}(d(\,.\,, \phi(\,.\,)) \mid [x]_i) && (f \text{ is 1-Lipschitz}) \\
&\leq a_i
\end{aligned}$$

This concludes the proof. □

Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space and $A \subseteq X$ measurable. Observe that $d_A\colon X \to \mathbb{R}$, $d_A(x) := \inf_{y \in A} d(x, y)$ is a 1-Lipschitz function. Using d_A we can rewrite the definition of the measure concentration function

$$\alpha_{\boldsymbol{X}}(\varepsilon) = \sup\{\mu(\{d_A \geq \varepsilon\}) \mid \mu(A) \geq \frac{1}{2}\}.$$

This gives us the desired connection.

Theorem 24. *If a finite mm-space $\boldsymbol{X} = (X, d, \mu)$ has length l, then the measure concentration function of $\boldsymbol{X}$ satisfies*

$$\alpha_{\boldsymbol{X}}(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right) \text{ for all } \varepsilon > 0.$$

Proof. Let $\varepsilon > 0$ and $A \subseteq X$ be measurable with $\mu(A) \geq \frac{1}{2}$. As mentioned above d_A is 1-Lipschitz and therefore, by Lemma 23,

$$\mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2\exp\left(-\frac{\varepsilon^2}{4l^2}\right).$$

Now there are two cases to consider, the first case is the more interesting one.

If $\mathbb{E}(d_A) \leq \varepsilon$, then for any x with $d_A(x) \geq 2\varepsilon$ we know $d_A(x) \geq \varepsilon + \mathbb{E}(d_A)$ and therefore $|d_A(x) - \mathbb{E}(d_A)| \geq \varepsilon$. As a consequence

$$\mu(\{d_A \geq 2\varepsilon\}) \leq \mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2\exp\left(-\frac{\varepsilon^2}{4l^2}\right).$$

Replacing ε by $\frac{\varepsilon}{2}$ gives the desired inequality.

If $\mathbb{E}(d_A) > \varepsilon$, then $A \subseteq \{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}$. Consequently,

$$\mu(\{d_A \geq \varepsilon\}) \leq \mu(X \setminus A) \leq \frac{1}{2} \leq \mu(A) \leq \mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2\exp\left(-\frac{\varepsilon^2}{4l^2}\right).$$

This proves the theorem. □

Note that in the second case the upper bound is at least $\frac{1}{2}$, which means that if l is large enough then we are in the first case and the expected distance to a set with at least half measure is less then ε. In Section 7 we will see a slight modification of this lemma. But for now our goal is to apply Theorem 13 to groups and as it turns out we can bound the length of a group using sequences of subgroups. Before we can write down the corresponding corollary we need to make a quick excursion into factor metrics.

Definition 25. Let (X, d) be a metric space and let $\sim$ be an equivalence relation on X. Then

$$d_\sim([x],[y]) = \inf\{d(p_1, q_1) + \cdots + d(p_n, q_n) \mid q_i \sim p_{i+1}, x \sim p_1, q_n \sim y\}$$

defines a pseudometric on $X/\sim$.

In case that X is a group with bi-invariant metric this definition simplifies.

Lemma 26. *Let G be a finite group with bi-invariant metric d and H a (not necessarily normal) subgroup of G. Then the factor metric d_H on $G/H = \{gH \mid g \in G\}$ is a proper metric and satisfies $d_H(gH, g'H) = \inf\{d(g, g'h) \mid h \in H\}$.*

Proof. Let $x, y \in G$. We show that for any path $p_1, q_1, \ldots, p_n, q_n$ as in the definition there are $x \sim p$ and $q \sim y$ such that $d(p,q) \leq d(p_1, q_1) + \cdots + d(p_n, q_n)$. It suffices to show this for $n = 2$. By definition p_1, q_1, p_2, q_2 are of the form $g, g', g'h, g''$ for some $g, g', g'' \in G$ and $h \in H$. Since d is bi-invariant

$$d(gh, g'') \leq d(gh, g'h) + d(g'h, g'') = d(g, g') + d(g'h, g'').$$

Furthermore we are given that G is finite. Hence the infimum becomes a minimum and $d_H([x],[y]) = 0$ only if $[x] = [y]$. □

Equipped with this knowledge we can formulate the final statement for this section.

Corollary 27. *Let G be a finite group with a bi-invariant metric d, and let*

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

be a chain of subgroups. Denote the diameter of G_i / G_{i-1} with respect to the factor metric by a_i. Then the length of G is at most $\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ and the measure concentration function of the mm-space (G, d, μ), where μ is the normalized counting measure, satisfies

$$\alpha_{(G,d,\mu)}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 \cdot \sum_{i=1}^n a_i^2}\right).$$

Proof. We show that the length l of (G, d, μ) is bounded by $\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ and apply Theorem 24. Define the sequence of partitions $\Omega_i := \{gG_i \mid g \in G\}$

$$\begin{aligned} \{\{g\} \mid g \in G\} = \Omega_0 \succ \Omega_1 \succ \cdots \succ \Omega_n &= \{G\} \\ \{e\} = G_0 < G_1 < \cdots < G_n &= G. \end{aligned}$$

Take $A \in \Omega_{i+1}$ and $g, g' \in A$. Since the distance of gG_i and $g'G_i$ with respect to the factor metric is at most a_i there is an $h' \in G_i$ such that $d(g, g'h') \leq a_i$. Hence the map

$$\begin{aligned} \phi\colon gG_i &\to g'G_i \\ gh &\mapsto g'h'h \end{aligned}$$

is, by bi-invariance of d, an isomorphism of metric spaces with $d(gh, g'h'h) = d(g, g'h') \leq a_i$ for all $gh \in gG_i$. Therefore $\left(\sum_{i=1}^n a_i^2\right)$ is an upper bound for l^2. □

Carderi and Thom used this result to show that the closure of the limit of $\mathrm{SL}_{2^n}(q)$ is extremely amenable [2]. We will recreate this proof in the next section.

4 Special Linear Groups and Extreme Amenability

When studying matrices it is often useful to look at the corresponding linear maps of a suitable vector space. In the case of $\mathrm{SL}_n(q)$ an n dimensional $\mathbb{F}_q$ vector space V suffices. Fixing a basis $e_1, \ldots, e_n$ gives us an embedding from $\mathrm{SL}_n(q)$ into $\mathrm{Aut}(V)$. Next we will apply the methods from the previous section to show that $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ is extremely amenable.

Theorem 28. *The normalized counting measure μ_n on the groups $\mathrm{SL}_n(q)$ concentrates with respect to the normalized rank-metric d_n, i.e. for all $\varepsilon > 0$*

$$\lim_{n \to \infty} \alpha_{(\mathrm{SL}_n(q), d_n, \mu_n)}(\varepsilon) = 0.$$

Proof. We will apply Corollary 27 to a sequence of subgroups which also shows that the length of $\mathrm{SL}_n(q)$ is bounded by $3n^{-\frac{1}{2}}$. Let $e_1, \ldots, e_n$ be a basis of an n dimensional $\mathbb{F}_q$ vector space V. Look at the sequence

Florian sagt: "proof for SL and GL te same?"

$$\mathrm{SL}_0(q) < \mathrm{SL}_1(q) < \cdots < \mathrm{SL}_n(q),$$

where $\mathrm{SL}_{i-1}(q)$ becomes a subgroup of $\mathrm{SL}_i(q)$ via the embedding $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Next we want to bound the diameter of $\mathrm{SL}_i(q) / \mathrm{SL}_{i-1}(q)$ by $\frac{3}{n}$. By Lemma 26 it suffices to show that for any $g, g' \in \mathrm{SL}_i(q)$ there is an $h \in \mathrm{SL}_{i-1}(q)$ such that $d(g, g'h) \leq \frac{3}{n}$. Since d is bi-invariant we can assume w.l.o.g. that g' is equal to the identity matrix I_i. Our goal is now to find a $g' \in \mathrm{SL}_i(q)$ that is the identity on e_i and has a small distance to g.

Let us take a closer look at ge_i. If e_i is an eigenvector of g with eigenvalue λ, then $\lambda \neq 0$ and g is of the form $\begin{pmatrix} A & 0 \\ c^{\perp} & \lambda \end{pmatrix}$. Define $h' := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $g' := \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix} \cdot g$. By construction $g' \in \mathrm{SL}_i(q)$ and it is of the form $\begin{pmatrix} A' & 0 \\ c'^{\perp} & 1 \end{pmatrix}$. Since $\det g' = 1$ we have that $\det A' = 1$ and therefore $A' \in \mathrm{SL}_{i-1}(q)$ making it a suitable candidate for h. Using the triangle inequality we obtain

$$\begin{aligned} d(g,h) &\leq d(g,g') + d(g',h) \\ &= d(I_i, \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix}) + \frac{1}{n} \mathrm{r}(\begin{pmatrix} 0 & 0 \\ -c'^{\perp} & 0 \end{pmatrix}) \\ &\leq \frac{2}{n} + \frac{1}{n} \end{aligned}$$

as desired.

If e_i is not an eigenvector of g, then we can make a change of basis of $\langle e_1, \ldots, e_{i-1} \rangle$ such that $ge_i = e_{i-1} + \lambda e_i$. Henceforth we can assume w.l.o.g. that g is of the form $\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{\Large A}} & 0 \\ & & 1 \\ c^{\perp} & c_{i-1} & \lambda \end{pmatrix}$. Define $h' := \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$ and as before $g' := \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix} \cdot g$. Now we can apply the argument from above to get an $h \in \mathrm{SL}_{i-1}(q)$ such that $d(g,h) \leq \frac{3}{n}$. Applying Corollary 27 we obtain

$$\alpha_{(\mathrm{SL}_n(q), d_n, \mu_n)}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 \cdot \sum_{i=1}^{n} \frac{9}{n^2}} \right) = 2 \exp\left(-\frac{\varepsilon^2 n}{16 \cdot 9} \right),$$

which tends to 0 as n goes to infinity.

$\square$

From this theorem the main result of this section easily follows.

Corollary 29. *The Polish group* $\mathrm{clim}\, \mathrm{SL}_{2^n}(q)$ *is extremely amenable.*

Proof. Theorem 28 implies that $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ is a Levy group and is therefore extremely amenable, by Theorem 13. $\square$

As a byproduct we found an upper bound for the length of $\mathrm{SL}_n(q)$. The natural question to ask is: How good is this upper bound? Therefore our next goal is to also determine a lower bound. This part is not essential to the rest of the thesis but still interesting.

Lemma 30. *Let (X, d, μ) be a finite mm-space with diameter Δ and*

$$\Omega_0 = \{X\} \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

with $a_1, \ldots, a_n$ as in Definition 20. Then

$$\sum_{i=1}^{n} a_i \geq \Delta.$$

Proof. Let $x, y \in X$, with $x \neq y$, we show $d(x,y) \leq \sum_{i=1}^{n} a_i$. Let i_0 be the smallest number such that $[x]_{i_0} \neq [y]_{i_0}$. Since $[x]_0 = X = [y]_0$ we know that i_0 is at least 1. Therefore $[x]_{i_0-1} = [y]_{i_0-1}$ and there is an isomorphism $\varphi_{i_0} \colon [x]_{i_0} \to [y]_{i_0}$ such that $d(\varphi_{i_0}(x), y) \leq a_{i_0}$. Let $x_{i_0} = \varphi_{i_0}(x)$, then

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

If $x_{i_0} = y$, then we are done. Otherwise let i_1 be the smallest number such that $[x_{i_0}]_{i_1} \neq [y]_{i_1}$. Then let $\varphi_{i_1} \colon [x_{i_0}]_{i_1} \to [y]_{i_1}$ be an isomorphism such that $d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$. Define $x_{i_1} = \varphi_{i_1}(x_{i_0})$. Proceeding in this fashion yields elements $x_{i_0}, \ldots, x_{i_k}$ such that $x_{i_k} = y$ and

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \cdots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \cdots + a_{i_k} \leq \sum_{i=1}^{n} a_i.$$

From this the claim immediately follows. $\square$

Lemma 31. *Let (X, d, μ) be a finite mm-space with diameter Δ and $\delta = \min_{x \neq y} d(x,y)$. Then the length of X is at least $(\Delta \cdot \delta)^{\frac{1}{2}}$.*

Proof. We show by induction on n that for any nonnegative $a_1, \ldots, a_n$ with $\delta = \min_{1 \leq i \leq n} a_i$ we have

$$\sum_{i=1}^{n} a_i \geq \Delta \implies \sum_{i=1}^{n} a_i^2 \geq \Delta \cdot \delta.$$

For $n = 1$: Note that $\Delta \geq \delta$. Hence

$$\sum_{i=1}^{n} a_i^2 = a_1^2 = \delta \cdot \delta \geq \Delta \cdot \delta.$$

For $n > 1$: Assume w.l.o.g. that $a_1 \leq \cdots \leq a_n$. Then

$$\begin{aligned}
\sum_{i=1}^{n} a_i \geq \Delta &\implies \sum_{i=2}^{n} a_i \geq \Delta - \delta && (a_1 = \delta) \\
&\overset{\text{I.H.}}{\implies} \sum_{i=2}^{n} a_i^2 \geq (\Delta - \delta) \cdot \delta' && (\text{with } \delta' = a_2) \\
&\implies \sum_{i=2}^{n} a_i^2 \geq \Delta \cdot \delta - \delta^2 && (\delta \leq \delta') \\
&\implies \sum_{i=1}^{n} a_i^2 \geq \Delta \cdot \delta. && (a_1^2 = \delta^2)
\end{aligned}$$

From this claim together with Lemma 30 we obtain the desired lower bound. □

Using this we can give an interval for the length of $\mathrm{SL}_n(q)$.

Corollary 32. *Consider $(\mathrm{SL}_n(q), d, \mu)$, where d is the normalized rank-metric and μ is the normalized counting measure. Then the length l of this mm-space satisfies*

$$n^{-\frac{1}{2}} \leq l \leq 3n^{-\frac{1}{2}}.$$

Proof. The diameter of $\mathrm{SL}_n(q)$ is equal to 1 and for any $g \neq g' \in \mathrm{SL}_n(q)$ we have that $d(g, g') \geq \frac{1}{n}$. □

The next goal is to show that the closure of the limit of symplectic groups is also extremely amenable. Theses groups can be seen as automorphism groups of a vector space together with a symplectic form. The proof will be similar to the one for the special linear groups but extending the partial inverse h' becomes much harder. This is why in the next section we will prove Witt's Lemma which does exactly what we need, i.e. extending isometries.

5 Witt's Lemma

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In this section we will prove Witt's Lemma and explore the structure of symplectic spaces. Witt's Lemma states that an isometry between subspaces of a finite dimensional symplectic, unitary, or orthogonal space can always be extended to an isomtry on the whole space. We will roughly follow the proof in [1]. Since we are mainly interested in symplectic spaces we will only show Witt's Lemma for those but it also holds for unitary and orthogonal spaces.

Definition 33. Let V be an $\mathbb{F}_q$ vector space. A bilinear form ω on V is called *symplectic* if ω is nondegenerate, i.e. $\omega(x, \,.\,) \neq 0_V$ for all $x \in V \setminus \{0\}$, and $\omega(x, y) = -\omega(y, x)$ for all $x, y \in V$. We call (V, ω) a *symplectic space*.

A finite group G is called *symplectic* if there is a symplectic space (V, ω) such that $G \cong \mathrm{Aut}(V, \omega)$.

A subspace $U \leq V$ is *nondegenerate* if ω restricted to U is nondegenerate (this is the case if and only if $U \cap U^\perp = \{0\}$).

Throughout this section let (V, ω) be a finite dimensional symplectic $\mathbb{F}_q$ vector space. Note that $\omega(x, y) = 0$ iff $\omega(y, x) = 0$, also $\omega(x, x) = 0$ for all $x \in V$. We will start of with some technical lemmas.

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Lemma 34. *For a subspace $U \leq V$ we have*

$$\dim U^\perp = \dim V - \dim U.$$

Proof. Let $u_1, \ldots, u_m$ be a basis of U and consider the linear map

$$g \colon V \to \mathbb{F}_q^m \qquad \text{with} \qquad g(v) = \begin{pmatrix} \omega(v, u_1) \\ \vdots \\ \omega(v, u_m) \end{pmatrix}.$$

By definition the kernel of g is $U^\perp$ and since ω is nondegenerate g is also surjective. Therefore the claim follows from the Rank-Nullity Theorem. □

The following lemma is an immediate consequence of the previous one.

Lemma 35. *Let $U \leq V$. Then*

i) U is nondegenerate iff $V = U \oplus U^\perp$ and

ii) $U^{\perp\perp} = U$.

Proof. If U is nondegenerate then $U \cap U^\perp = \{0\}$. By Lemma 34

$$U \oplus U^\perp = \langle U, U^\perp \rangle = V.$$

The other direction is clear from the definition

For *ii*) note that $U \subseteq U^{\perp\perp}$. By Lemma 34 they also have the same dimension and are therefore equal. □

Now we want to better understand the structure of symplectic spaces. We start with 2-dimensional symplectic spaces, which we will call *hyperbolic planes*.

Lemma 36. *There is only one hyperbolic plane up to isometry.*

Proof. Let (V, ω) and (V', ω') be hyperbolic planes with basis r, s and r', s', respectively. Then $r \mapsto r', s \mapsto \lambda s'$ with $\lambda = \omega'(r', s')^{-1} \cdot \omega(r, s)$ is an isometry between the two hyperbolic planes. $\square$

Next we will strengthen this result and show that any symplectic space is the direct sum of hyperbolic planes.

Theorem 37. *Let $r \in V \setminus \{0\}$. Then there is a hyperbolic plane $U \leq V$ containing r such that $V = U \oplus U^{\perp}$. Furthermore, if $W \leq V$ with $W \perp r$ and $r \notin W$, then there is a U as before that also fulfills $W \perp U$.*

Proof. Since $\omega(r, r) = 0$ we know that $r \in r^{\perp}$. Let $H \leq r^{\perp}$ containing W such that $\langle r \rangle \oplus H = r^{\perp}$. Then, by Lemma 34, $\dim r^{\perp} = \dim V - 1$, $\dim H^{\perp} = 2$, and $r \in H^{\perp}$. In particular $\dim V > 1$. Our goal is to show that

$$V = H^{\perp} \oplus H,$$

then $H^{\perp}$ would be a suitable choice for U. It suffices to show that $H^{\perp}$ is nondegenerate, as then the claim follows from Lemma 35. Let $s \in H^{\perp}$ such that $H^{\perp} = \langle r, s \rangle$. Now $H^{\perp}$ is nondegenerate iff $\omega(r, s) \neq 0$.

Assume $\omega(r, s) = 0$. Then $r \in s^{\perp}$ and $H \subseteq s^{\perp}$. Since, by construction, $r \notin H$ we have $r^{\perp} = \langle r, H \rangle = s^{\perp}$. Hence r and s are linearly dependent contradicting that r, s is a basis of $H^{\perp}$.

$\square$

Using Theorem 37 we can describe the structure of symplectic spaces.

Corollary 38. *Let (V, ω) be a symplectic space. Then V is of even dimension $2n$ and there are hyperbolic planes $U_1, \ldots, U_n \leq V$ such that $V = U_1 \oplus \cdots \oplus U_n$ and $U_i \perp U_j$ for $i \neq j$. In particular for any n there is exactly one symplectic space of dimension $2n$.*

Proof. We use induction on $n = \dim V$.

If $n = 1$ then $V = \langle v \rangle$ for $v \in V \setminus \{0\}$. But $\omega(v, v) = 0$ and therefore ω is degenerate which is a contradiction.

If $n = 2$ then the claim follows from Lemma 36.

If $n > 2$ then by Theorem 37 there is a hyperbolic plane $U_1 \leq V$ satisfying $V = U_1 \oplus U_1^{\perp}$. Hence, by induction hypothesis, n is even and $U^{\perp} = U_2 \oplus \cdots \oplus U_m$ for some hyperbolic planes $U_2, \ldots, U_m \leq U^{\perp}$ with $m = \frac{n}{2}$ and $U_i \perp U_j$ for $i \neq j$. $\square$

With these powerful tools we can easily prove Witt's Lemma. Let $\alpha \colon U \to W$ be an isometry between subspaces $U, W \leq V$.

Lemma 39. *There are subspaces $U' \geq U$ and $W' \geq W$ with U', W' nondegenerate such that α can be extended to an isometry $\tilde{\alpha}\colon U' \to W'$.*

Proof. We show this claim using induction on $n = \dim(U \cap U^\perp)$.

If $n = 0$ then U itself is nondegenerate and we are done.

If $n > 0$ then let $r \in (U \cap U^\perp) \setminus \{0\}$ and $\tilde{U} \leq V$ such that $\langle r \rangle \oplus \tilde{U} = U$. By Theorem 37 there is a hyperbolic plane $H \leq V$ containing r such that $\tilde{U} \perp H$. Similarly, there is a hyperbolic plane $H' \leq V$ containing $r' := \alpha(r)$ such that $H' \perp \tilde{W}$ with $\tilde{W} = \alpha(\tilde{U})$. Let r, s and r', s' be a basis of H and H', respectively. Note that $\omega(r, s) \neq 0$ and $r \perp U$ imply $s \notin U$. Analogously, $s' \notin W$. We can assume w.l.o.g. that $\omega(r, s) = \omega(r', s')$. Now we can extend α to $\tilde{\alpha}\colon \langle U, s \rangle \to \langle W, s' \rangle$ by defining $\tilde{\alpha}(s) := s'$. Note that $\langle U, s \rangle = \langle \tilde{U}, H \rangle$. Since $\tilde{U} \perp H$ we have

$$\dim(\langle \tilde{U}, H \rangle \cap \langle \tilde{U}, H \rangle^\perp) = \dim(\tilde{U} \cap \tilde{U}^\perp) < \dim(U \cap U^\perp).$$

Hence we can apply the induction hypothesis to $\tilde{\alpha}$. $\square$

Lemma 40. *If U is nondegenerate. Then α can be extended to an isometry $\tilde{\alpha}\colon V \to V$.*

Proof. Since U is nondegenerate we have that W is also nondegenerate and we can apply Lemma 35 and obtain

$$V = U \oplus U^\perp = W \oplus W^\perp.$$

By Lemma 34 we have $\dim U^\perp = \dim W^\perp$. Hence, by Corollary 38, there is an isometry $\beta\colon U^\perp \to W^\perp$. Finally, $\alpha \oplus \beta\colon V \to V$ is an isometry extending α. $\square$

With this preparation we can now come to the main result.

Corollary 41 (Witt's Lemma)**.** *The map α can be extended to an isometry $\tilde{\alpha}\colon V \to V$.*

Proof. Using Lemma 39 extend α to $\tilde{\alpha}\colon U' \to W'$ for some $U' \geq U$, $W' \geq W$ nondegenerate. Now apply Lemma 40 to extend $\tilde{\alpha}$ to $\tilde{\tilde{\alpha}}\colon V \to V$. $\square$

Now that we understand symplectic spaces and can extend isometries we are well equipped for the next section, where will show that the closure of the limit of symplectic groups is also extremely amenable.

6 Symplectic Groups and Extreme Amenability

Our goal in this section is to show that $\operatorname{clim} \mathrm{Sp}_{2^n}(q)$ is extremely amenable. The structure of the proof is the same as in Section 4 for special linear groups. We will bound the length of $\mathrm{Sp}_n(q) \cong \mathrm{Aut}(V, \omega)$ by applying Corollary 27 to a sequence of subgroups $(G_i)_{0 \leq i \leq n}$. To bound the diameter of G_i / G_{i-1} we will construct for any $g \in G_i$ an $h' \in G_i$ such that the distance between g and $h'g$ is small and $h'g \in G_{i-1}$. The h' will behave like the inverse of g on a small subspace of V and like the identity on most of the rest. The proof can be generalized to unitary and orthogonal groups and therefore we will briefly introduce those and afterwards show extreme amenability for the closure of the limits of symplectic, unitary, and orthogonal groups.

Definition 42. Let V be a finite dimensional $\mathbb{F}_q$ vector space and ω a nondegenerate map from $V \times V$ to $\mathbb{F}_q$.

Then (V, ω) is an *orthogonal space* if ω is bilinear, $\omega(x, y) = \omega(y, x)$ for all $x, y \in V$, and if $q = 2$ then $\omega(x, x) = 0$ for all $x \in V$.

And (V, ω) is a *unitary space* if there is an $h \in \operatorname{Aut}(\mathbb{F}_q)$ with $h^2 = 1$ such that

$$\begin{aligned} \omega(ax + y, z) &= a\omega(x, z) + \omega(y, z) \\ \omega(x, ay + z) &= h(a)\omega(x, y) + \omega(x, z) \\ \omega(x, y) &= h(\omega(y, x)) \end{aligned}$$

for all $x, y, z \in V$ and $a \in \mathbb{F}_q$.

Orthogonal and *unitary groups* are the automorphism groups of unitary and orthogonal spaces, respectively.

In the following let (V, ω) be a symplectic, unitary, or orthogonal space. Note that in any case ω is nondegenerate and

$$\omega(x, y) = 0 \text{ iff } \omega(y, x) = 0 \text{ for all } x, y \in V.$$

Obviously, Lemmas 34 and 35 from the previous section still hold in unitary and orthogonal spaces. Furthermore, Witt's Lemma also holds in unitary and orthogonal spaces, for a proof see [1]

Theorem 43 (Witt's Lemma). *Let (V, ω) be a symplectic, unitary, or orthogonal space and $\alpha\colon U \to W$ be an isometry between subspaces $U, W \leq V$. Then α can be extended to an isometry $\tilde{\alpha}\colon V \to V$.*

The next lemma is necessary to construct the chain of subgroups, in the case of symplectic spaces it is a trivial consequence of Theorem 37.

Lemma 44. *There exists a $U \leq V$ with $\dim U \leq 2$ such that $V = U \oplus U^\perp$.*

Proof. Let $r \in V \setminus \{0\}$. By Lemma 34 $\dim r^\perp = n - 1$.

If $r \notin r^\perp$, then $V = \langle r \rangle \oplus r^\perp$ and $\langle r \rangle$ is the desired U.

If $r \in r^\perp$, then let $H \leq r^\perp$ such that $\langle r \rangle \oplus H = r^\perp$. Now proceed as in the proof of Theorem 37 to show that $H^\perp$ is a suitable U.

□

The following lemma shows that isometries interact nicely with complements.

Lemma 45. *Let $U \leq V$ and $\alpha\colon V \to V$ be an isometry such that $\alpha(U) = U$. Then $\alpha(U^\perp) = U^\perp$.*

Proof. As $\dim \alpha(U^\perp) = \dim U^\perp$ it suffices to show that $\alpha(u') \perp u$ for all $u \in U$ and $u' \in U^\perp$. Let $v \in U$ with $\alpha(v) = u$. Then

$$\begin{aligned} \omega(\alpha(u'), u) &= \omega(\alpha(u'), \alpha(v)) \\ &= \omega(u', v) \\ &= 0. \end{aligned}$$

This concludes the proof. □

The next lemma gives us a large subspace on which h' can be the identity without interfering with the part where it is the inverse of g.

Lemma 46. *For all $W \leq V$ there is a $W' \leq W^\perp$ such that $W \cap W' = 0$ and*

$$\dim W' \geq \dim V - 2 \dim W.$$

Proof. Let $W' \leq W^\perp$ such that

$$W^\perp = (W^\perp \cap W) \oplus W'.$$

Clearly, $W \cap W' = 0$ and

$$\begin{aligned} \dim W' &= \dim W^\perp - \dim(W^\perp \cap W) \\ &\geq \dim W^\perp - \dim W \\ &= \dim V - \dim W - \dim W. && \text{(Lemma 34)} \end{aligned}$$

This concludes the proof. □

Now we can proof the analogue of Theorem 28 from Section 4.

Theorem 47. *Let G be a symplectic, unitary, or orthogonal group equipped with the rank metric d and of diameter n. Then there is a symplectic, unitary, or orthogonal subgroup $H \leq G$ with diameter at most $n-1$ such that the diameter of G/H is at most 8.*

Proof. We have $G \cong \mathrm{Aut}(V, \omega)$ for some n-dimensional vector space V and some ω. Use Lemma 44 to obtain $U \leq V$ such that $V = U \oplus U^\perp$ and $\dim U \leq 2$. Define $H = \mathrm{Aut}(U^\perp, \omega)$. Our aim is to find for any $g \in G$ an $g' \in H$ such that $d(g, g') \leq 8$. The idea is to find a map $h' \in G$ that behaves like the inverse of g on gU and like the identity on most of the rest. Then $h'g$ is the desired g'.

Let $g \in G$ and define $W = \langle U, gU \rangle$. By Lemma 46 there is a W' such that $\dim W' \geq n - 8$, $W' \leq W^\perp$, and $W' \cap W = 0$. Consider the map

$$g^{-1}|_{gU} \oplus 1_{W'} \colon gU \oplus W' \to U \oplus W'$$

since $g^{-1}|_{gU}$ and $1_{W'}$ are isometries and $W \perp W'$ we have that the above map is also an isomtry. By Theorem 43 this isometry can be extended to an isometry $h' \colon V \to V$. Furthermore,

$$\begin{aligned} d(g, h'g) &= \dim \operatorname{im}(g - h'g) \\ &\leq 8 + \dim \operatorname{im}(g - h'g)|_{W'} && (\dim W' \geq n - 8) \\ &= 8 + \dim \operatorname{im}(g - g)|_{W'} && (h'|_{W'} = 1_{W'}) \\ &= 8. \end{aligned}$$

Finally, we need to show that $h'g \in H$, here the choice of H using Lemma 44 comes into play. By construction of h' we have that $h'g|_U = 1_U$. Therefore we can apply Lemma 45 and get that $h'g(U^\perp) = U^\perp$. Hence $h'g \in H$ and $d(g, h'g) \leq 8$. □

Corollary 48. *Let $G = \operatorname{Aut}(V, \omega)$ be a symplectic, unitary, or orthogonal group equipped with the normalized rank metric d and the normalized counting measure μ, where V is n dimensional. Then the length of G is at most $8n^{-\frac{1}{2}}$ and for all $\varepsilon > 0$*

$$\alpha_{(G,d,\mu)}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right).$$

Proof. Applying Theorem 47 multiple times gives us a sequence of subgroups $\{e\} = G_0 \leq \cdots \leq G_m = G$ such that $m \leq n$ and the diameter of G_i/G_{i-1} is at most $\frac{8}{n}$. Now we can use Corollary 27 to obtain the desired upper bound. □

Observe that as in Section 4 we can apply Lemma 31 and obtain $n^{-\frac{1}{2}}$ as a lower bound for the length of G. Now we can prove the main result of this thesis.

Corollary 49. *Let $(V_0, \omega_0) \subset (V_1, \omega_1) \subset \ldots$ be a sequence of $\mathbb{F}_q$ vector spaces such that (V_n, ω_n) is a symplectic, unitary, or orthogonal space of dimension 2^n and $\omega_{n+1}|_{V_n} = \omega_n$ for all $n \in \mathbb{N}$. Let $G_n = \operatorname{Aut}(V_n, \omega_n)$ equipped with the normalized rank metric d_n and the normalized counting measure μ_n. Then*

$$\lim_{n \to \infty} \alpha_{(G_n, d_n, \mu_n)}(\varepsilon) = 0$$

for all $\varepsilon > 0$ and $\operatorname{clim} G_n$ is extremely amenable.

Proof. Immediate from Corollary 48 and Theorem 13. □

7 Symplectic Groups and Ramsey Theory

In this section we will use the upper bound obtained for the length of symplectic, unitary, and orthogonal groups to deduce a Ramsey theoretic result. As in Section 6 the results from this section are already shown in [2] for special linear groups.

The first lemma is very similar to Theorem 24.

Lemma 50 (Lemma 2.7 in [2]). *Let (X, d, μ) be a finite mm-space with length l. Then for every $\varepsilon > 0$ and $A \subseteq X$ with $\mu(A) > 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right)$ we have*

$$\mu(N_\varepsilon(A)) \geq 1 - 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

A covering $\mathcal{U}$ of a metric space (X, d) is an *ε-covering* if for every $x \in X$ the ε-neighborhood of x is contained in some $U \in \mathcal{U}$.

Theorem 51. *Let $\varepsilon > 0$, $k, m \in \mathbb{N}$. Define $N := 16 \cdot 64\varepsilon^{-2} \cdot \max\{\ln(2k), \ln(2m)\}$ and let $G = \mathrm{Aut}(V, \omega)$, where (V, ω) is a symplectic, unitary, or orthogonal space of dimension $n > N$, with an ε-cover $\mathcal{U}$ of cardinality at most m. Then there is a $U \in \mathcal{U}$ such that for all $F \subseteq G$ satisfying $|F| \leq k$ there is a $g \in G$ with $gF \subseteq U$.*

Intuitively the theorem says that whenever we color G with m colors, where a single element can have multiple colors, such that all elements of ε-balls have at least one color in common, then there is one color c such that for every F with at most k elements there is a g where the elements of gF all have the color c.

Proof. Look at G as the usual mm-space with normalized rank metric and normalized counting measure. Let l be the length of G. Observe that, by Corollary 48, $l \leq 8n^{-\frac{1}{2}}$. For $U \in \mathcal{U}$ define $\mathrm{Core}(U) := \{x \in U \mid N_\varepsilon(x) \subseteq U\}$. Since $\mathcal{U}$ is an ε-covering we have $\bigcup_{U \in \mathcal{U}} \mathrm{Core}(U) = G$. Therefore there is a $U \in \mathcal{U}$ such that $\mu(\mathrm{Core}(U)) \geq \frac{1}{m}$. As $n > 16 \cdot 64\varepsilon^{-2} \cdot \ln(2m)$ we have

$$\frac{1}{m} > 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right) \geq 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

Now we can apply Lemma 50 to $\mathrm{Core}(U)$ and obtain

$$\mu(U) \geq \mu(N_\varepsilon(\mathrm{Core}(U))) \geq 1 - 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right) \geq 1 - 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right).$$

Let $F \subseteq G$ with $|F| \leq k$. Note that

$$\{g \in G \mid gF \subseteq U\} = \bigcap_{h \in F} \{g \in G \mid gh \in U\} = \bigcap_{h \in F} Uh^{-1}.$$

Therefore, $\mu(\{g \in G \mid gF \subseteq U\}) \geq 1 - k \cdot 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right)$. By assumption $n > 16 \cdot 64\varepsilon^{-2} \cdot \ln(2k)$, hence $\mu(\{g \in G \mid gF \subseteq U\}) > 0$ and there is a suitable g. $\square$

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ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Topological Entropy of Formal Languages" selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

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