**Bachelorarbeit**

# Measure Concentration for Symplectic Groups


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# 1 Introduction

The objects studied in this thesis are groups of matrices over some finite field. Given such group, we can equip it with the normalized rank metric and the normalized Haar measure to obtain a *metric measure space*. Observe that, since matrix groups over finite fields are again finite, the normalized Haar measure is just the normalized counting measure. For some sequences of matrix groups of increasing size there is a well defined limit. For example, this is the case for the special linear group of $(n \times n)$-matrices $\mathrm{SL}_n$. Carderi and Thom showed in [3] that a suitable limit of $\mathrm{SL}_n$ is, as a topological group, *extremely amenable*. The goal of this thesis is to generalize this result to limits of other families of matrix groups, namely groups of *symplectic*, *unitary*, and *orthogonal* matrices. These matrices can be seen as *isometries*, i.e. bijective linear maps from a vector space into itself preserving a symplectic, unitary, or orthogonal form. The general strategy to prove extreme amenability for limits of families of these groups will be as follows: Given such a family $(G_n)_{n\in\mathbb{N}}$ of finite matrix groups considered as metric measure spaces, we apply a consequence of Azuma's inequality [2] to obtain an upper bound for the *measure concentration function* of $G_n$ in terms of the *length* of $G_n$. As the upper bounds converge to zero we conclude that $(G_n)_{n\in\mathbb{N}}$ is a *Lévy family*, making their limit a *Lévy group*. Finally, we know from [4] that every Lévy group is extremely amenable.

This thesis is structured as follows. In Section 2, we will give a short introduction on how to view matrix groups as metric measure spaces and how to define a limit of a sequence of matrix groups. Furthermore we will introduce the notion of extreme amenability and its connection to Lévy groups. In Section 3, we will briefly introduce the notion of *conditional expectation* to show Azuma's inequality. Next we will introduce an important invariant of metric measure spaces, namely their *length*. Azuma's inequality will allow us to connect the length of a metric measure space with its measure concentration function. This connection is used in Section 4 to show that the limit of $\mathrm{SL}_n$ is extremely amenable. To generalize this result we give a proof of Witt's lemma, which says that isometries between subspaces can be extended to isometries on the whole space, in Section 5. In Section 6, we generalize the result from Section 4 to symplectic, unitary, and orthogonal groups. Finally, in Section 7, a Ramsey theoretic result from [3] about $\mathrm{SL}_n$ is generalized to symplectic, unitary, and orthogonal groups.

# 2 Preliminaries

Let $q$ be a prime power and $\mathbb{F}_q$ be the unique $q$ element field. Denote the general linear group over $\mathbb{F}_q$ by $\mathrm{GL}_n(q)$. We can equip $\mathrm{GL}_n(q)$ with the (normalized) *rank-metric* $d(g,h) := \frac{1}{n}\,\mathrm{r}(g-h)$, where $\mathrm{r}(g)$ is the rank of $g$ or equivalently if we view $g$ as a linear map from a vector space into itself then $\mathrm{r}(g)$ is the dimension of the image of $g$.

**Lemma 2.1.** *The metric $d$ is* bi-invariant, *i.e. for all $g, h, k \in \mathrm{GL}_n(q)$ we have*

$$d(kg, kh) = d(g, h) = d(gk, hk).$$

*Proof.* Let $g, h, k \in \mathrm{GL}_n(q)$. Note that $k$ has full rank. Therefore

$$n \cdot d(kg, kh) = \mathrm{r}(kg - kh) = \mathrm{r}(k(g - h)) = \mathrm{r}(g - h) = n \cdot d(g, h).$$

The other equality follows similarly. □

Let $G_n \leq \mathrm{GL}_{2^n}(q)$ be a family of subgroups, such that $\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in G_{n+1}$ for all $g \in G_n$. Denote the normalized rank-metric of $G_n$ by $d_n$.

**Lemma 2.2.** *For all $n \in \mathbb{N}$ the map*

$$\varphi_n \colon G_n \mapsto G_{n+1}, \text{ where } \varphi_n(g) = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

*is an isometric embedding.*

*Proof.* Let $n \in \mathbb{N}$ and $g, h \in G_n$. Then

$$\begin{aligned} d(g, h) &= \frac{1}{n} \mathrm{r}(g - h) \\ &= \frac{1}{2n} (\mathrm{r}(g - h) + \mathrm{r}(g - h)) \\ &= \frac{1}{2n} \mathrm{r}(\begin{pmatrix} g - h & 0 \\ 0 & g - h \end{pmatrix}) \\ &= d(\begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}, \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}). \end{aligned}$$

Clearly, $\varphi_n$ is also an injective homomorphism. □

Define the function $\varphi \colon \bigsqcup_{n \in \mathbb{N}} G_n \to \bigsqcup_{n \in \mathbb{N}} G_n$ with $\varphi|_{G_n} := \varphi_n$.

**Definition 2.3.** Let $\sim$ be the equivalence relation on $\bigsqcup_{n \in \mathbb{N}} G_n$, defined by $g \sim h$ iff there are $m, n \in \mathbb{N}$ such that $\varphi^n(g) = \varphi^m(h)$. Then the *limit* of $(G_n)_{n \in \mathbb{N}}$ is defined as

$$\lim_{n \in \mathbb{N}} G_n := \left( \bigsqcup_{n \in \mathbb{N}} G_n \right) / \sim .$$

**Lemma 2.4.** *The sequence of metric groups $(G_n, d_n)_{n \in \mathbb{N}}$ induces a group structure and a bi-invariant metric $d$ on $G = \lim G_n$.*

*Proof.* Note that $\psi_n \colon G_n \to G$, $\psi_n(g) = [g]$, is injective and

$$G = \bigcup_{n \in \mathbb{N}} \psi_n(G_n).$$

For $[g], [h] \in G$ we can assume w.l.o.g. that $g, h \in G_n$ for some $n \in \mathbb{N}$. Hence we define $[g] \cdot [h] := [gh]$ and $d([g],[h]) := d_n(g,h)$. Both functions are well defined since the following diagram commutes for all $n \leq m$.

$$\begin{tikzcd} G_m \arrow[r, hook, "\psi_m"] & G \\ G_n \arrow[u, hook, "\varphi^{m-n}|_{G_n}"] \arrow[ur, hook, "\psi_n"'] & \end{tikzcd}$$

Note that now the $\psi_n$'s are isometric embeddings. Hence $d$ inherits all desired properties from the $d_n$'s. $\square$

Furthermore the metric and the group structure of $G$ interact nicely.

**Definition 2.5.** A group $G$ equipped with a topology is a *topological group* if the maps $G \times G \to G$, $(g,h) \mapsto gh$ and $G \to G$, $g \mapsto g^{-1}$ are continuous. Here we use the product topology on $G \times G$.

**Lemma 2.6.** *Let $(G_n)_{n \in \mathbb{N}}$, $G = \lim G_n$, and $d$ be as before. Then $G$ with the topology induced by $d$ is a topological group.*

*Proof.* Denote the neutral element of $G$ by $e$. First we show that the inverse is continuous. Let $\varepsilon > 0$ and $g, h \in G$ with $d(g,h) < \varepsilon$. Then, by bi-invariance of $d$,

$$d(g^{-1}, h^{-1}) = d(e, gh^{-1}) = d(h,g) < \varepsilon.$$

We use $d_\Sigma((g,h),(g',h')) := d(g,g') + d(h,h')$ as metric on $G \times G$. Let $\varepsilon > 0$ and $g, g', h, h' \in G$ with $d_\Sigma((g,h),(g',h')) < \varepsilon$. Then

$$d(gh, g'h') = d(g'^{-1}g, h'h^{-1}) \leq d(g'^{-1}g, e) + d(e, h'h^{-1}) = d(g,g') + d(h,h') < \varepsilon.$$

This yields the desired result. $\square$

The group we are interested in is the metric completion of $\lim G_n$.

**Lemma 2.7.** *Let $G$ be a topological group with bi-invariant metric $d$. Then there is a unique metric space $(\bar{G}, \bar{d})$ containing $G$ such that $\bar{G}$ is complete and $G$ is dense in $\bar{G}$. Furthermore $\bar{G}$, with the group structure induced by $G$, is a topological group and $\bar{d}$ is still bi-invariant.*

*Proof.* Consider the set $G_C$ of Cauchy sequences in $G$. Define $\bar{G} := G_C / \sim$, where two Cauchy sequences $(g_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ are equivalent if $\lim d(g_n, h_n) = 0$. Furthermore define $\bar{d}([(g_n)_{n\in\mathbb{N}}], [(h_n)_{n\in\mathbb{N}}]) := \lim d(g_n, h_n)$. It is well known that $(\bar{G}, \bar{d})$ is the unique metric completion of $(G, d)$.

The group operation can be extended to $\bar{G}$ as follows:

$$[(g_n)_{n\in\mathbb{N}}] \cdot [(h_n)_{n\in\mathbb{N}}] := [(g_n h_n)_{n\in\mathbb{N}}].$$

It is clear from the definition that $\bar{d}$ is also bi-invariant. Next we show that $\bar{G}$ is still a topological group. The proof is very similar to the one of Lemma 2.6. Let $\varepsilon > 0$ and $[(g_n)_{n\in\mathbb{N}}], [(h_n)_{n\in\mathbb{N}}] \in \bar{G}$ with $\bar{d}([(g_n)_{n\in\mathbb{N}}], [(h_n)_{n\in\mathbb{N}}]) < \varepsilon$. Observe that $[(g_n)_{n\in\mathbb{N}}]^{-1} = [(g_n^{-1})_{n\in\mathbb{N}}]$. Therefore

$$\bar{d}([(g_n)_{n\in\mathbb{N}}]^{-1}, [(h_n)_{n\in\mathbb{N}}]^{-1}) = \lim_{n\to\infty} d(g_n^{-1}, h_n^{-1}) = \lim_{n\to\infty} d(g_n, h_n) < \varepsilon.$$

Analogously, we obtain that the group operation is continuous. □

**Definition 2.8.** Let $(G_n)_{n\in\mathbb{N}}$, $G = \lim G_n$, and $d$ be as before. Define the *closure of the limit of* $(G_n)_{n\in\mathbb{N}}$, denoted by $\operatorname{clim} G_n$, as the metric completion of $(G, d)$.

By the previous lemma $\operatorname{clim} G_n$ is a topological group and a complete metric space with a bi-invariant metric. Additionally $\operatorname{clim} G_n$ has another nice property. For this we need to introduce the well established notion of Polish spaces.

**Definition 2.9.** A topological space $(X, \tau)$ is a *Polish space* if there is a metric $d$ on $X$ that induces the topology $\tau$ such that $(X, d)$ is complete and has a countable dense subset. A topological group is a *Polish group* if the underlying topological space is a Polish space.

**Lemma 2.10.** *We have that* $\operatorname{clim} G_n$*, seen as a topological group, is a Polish group.*

*Proof.* Obviously, $\lim G_n$ is countable and dense in $\operatorname{clim} G_n$. By definition $\operatorname{clim} G_n$ is also a complete metric space. □

**Definition 2.11.** A topological group $G$ is *extremely amenable* if every continuous action of $G$ on a compact topological space admits a fixed point.

The goal of this thesis is to show that for certain sequences $(G_n)_{n\in\mathbb{N}}$ we have that $\operatorname{clim} G_n$ is extremely amenable. It is hard to show this directly, but we know from [4] that every Lévy group (see Definition 2.13) is extremely amenable. Hence we will show that $\operatorname{clim} G_n$ is a Lévy group instead.

But before we can talk about Lévy groups we need some more definitions. For an $\varepsilon > 0$, a metric space $(X, d)$, and an $A \subseteq X$, we define the *$\varepsilon$-neighborhood* of $A$ to be

$$N_\varepsilon(A) := \{x \in X \mid \exists y \in A.\, d(x, y) < \varepsilon\}.$$

Note that $N_\varepsilon(A)$ is always an open set.

**Definition 2.12.** A *metric measure space* (mm-space) $\boldsymbol{X}$ is a triple $(X, d, \mu)$, where $d$ is a metric on the set $X$ and $\mu$ is a measure on the Borel-$\sigma$-algebra induced by $d$. We will always assume that $\mu(X) = 1$, i.e. that $\mu$ is a probability measure. The *measure concentration function* $\alpha_{\boldsymbol{X}} \colon (0, \infty) \to [0, \frac{1}{2}]$ of $\boldsymbol{X}$ is defined as

$$\alpha_{\boldsymbol{X}}(\varepsilon) = \sup \left\{ 1 - \mu(N_\varepsilon(A)) \mid A \subseteq X \text{ measurable}, \mu(A) \geq \frac{1}{2} \right\}.$$

A family of mm-spaces $(\boldsymbol{X}_n)_{n \in \mathbb{N}}$ with diameter 1 is called a *Lévy family* if

$$\alpha_{\boldsymbol{X}_n}(\varepsilon) \to 0 \ \text{ as } n \to \infty$$

for all $\varepsilon > 0$.

Now we can come back to groups.

**Definition 2.13.** A *Lévy group* is a group $G$ equipped with a metric $d$, where

- the group $G$ with the topology induced by $d$ is a Polish group and
- there is an increasing sequence $(G_n)_{n \in \mathbb{N}}$ of compact subgroups, such that $\bigcup_{n \in \mathbb{N}} G_n$ is dense in $G$ and $(G_n, d|_{G_n}, \mu_n)_{n \in \mathbb{N}}$ is a Lévy family. Here $\mu_n$ is the normalized Haar measure of $G_n$.

The following theorem gives us the desired connection to extreme amenability.

**Theorem 2.14** (Theorem **3.1.3** in [4]). *Every Lévy group is extremely amenable.*

Whether $G = \operatorname{clim} G_n$ is a Lévy group depends on the particular choice of the sequence $(G_n)_{n \in \mathbb{N}}$. We have already seen in Lemma 2.10 that $G$ is always a Polish group. But the second condition is not so easy to prove. To show that for certain sequences, $G$ is a Lévy group, we will use $(G_n)_{n \in \mathbb{N}}$ as increasing sequence (or $(\psi_n(G_n))_{n \in \mathbb{N}}$ to be precise) and show that it is a Lévy family. Note that $G_n$ is finite and therefore compact. Furthermore the normalized Haar measure $\mu_n$ on $G_n$ is just the normalized counting measure. The plan is to bound $\alpha_{(G_n, d_n, \mu_n)}(\varepsilon)$ in terms of $n$ and $\varepsilon$ and show that this bound converges to 0 as $n$ tends to infinity. The next section develops the methods necessary to obtain this upper bound.

## 3 Azuma's Inequality and Measure Concentration

In this section we will prove Azuma's inequality and as a consequence, we will obtain an upper bound for the measure concentration function in terms of an invariant of mm-spaces called the length. For groups we will then bound the length using an increasing sequence of subgroups.

As the next results rely heavily on stochastic methods we will briefly introduce the necessary notions. Since the $G_n$ are all finite and equipped with the normalized counting measure we will only consider *probability spaces* $(X, \Sigma, \mu)$, where $X$

is finite, $\Sigma$ is a $\sigma$-algebra over $X$, and $\mu(A) = |A|/|X|$ for $A \subseteq X$. In this section we will roughly follow Section 3.2 from [4]. Most of the statements presented in this section hold in a more general setting.

Note that in our case $\Sigma$ has a very nice representation.

**Lemma 3.1.** *Let $\Sigma$ be a $\sigma$-algebra over a finite set $X$ and $A_1, \ldots, A_n$ be the minimal nonempty sets in $\Sigma$. Then $A_1, \ldots, A_n$ is a partition of $X$ and $\Sigma$ is the smallest $\sigma$-algebra containing $A_1, \ldots, A_n$.*

*Proof.* First we show that $A_1, \ldots, A_n$ is a partition of $X$. Since $A_i \cap A_j \in \Sigma$ we conclude, by minimality of $A_i$ and $A_j$, that either $i = j$ or $A_i \cap A_j = \emptyset$. Clearly, every element of $X$ is contained in one of the $A_i$.

For $A \in \Sigma$ we have, again by minimality, that $A \cap A_i$ is either $A_i$ or $\emptyset$. Therefore $A$ can be written as a union of $A_i$'s. □

Note that it follows from the proof that any $A \in \Sigma$ can be written as $\bigcup_{i \in I} A_i$ for a suitable set $I \subseteq \{1, \ldots, n\}$. This lemma allows us to use partitions and $\sigma$-algebras interchangeably. We will denote the partition corresponding to $\Sigma$ by $A_1, \ldots, A_n$ and for $\Sigma'$ we will use $B_1, \ldots, B_m$.

**Lemma 3.2.** *Let $f\colon X \to \mathbb{R}$ be a $\Sigma$-measurable (measurable, for short) function. Then there are $f_1, \ldots, f_n \in \mathbb{R}$ such that $f = \sum_{i=1}^{n} f_i \cdot \mathbb{1}_{A_i}$.*

*Proof.* For all $c \in \mathbb{R}$ the set $\{x \in X \mid f(x) = c\}$ is in $\Sigma$ and therefore a union of $A_i$'s. This already yields the lemma. □

The next definition is simplified a lot by only considering finite $X$.

**Definition 3.3.** Let $(X, \Sigma, \mu)$ be a finite probability space, $f\colon X \to \mathbb{R}$ be a measurable function, and $\Sigma'$ be a sub-$\sigma$-algebra of $\Sigma$. Then the *conditional expectation* of $f$ with respect to $\Sigma'$ is defined as

$$\mathbb{E}(f \mid \Sigma') := \sum_{i=1}^{m} \mathbb{E}(f \mid B_i) \cdot \mathbb{1}_{B_i}.$$

One often thinks of $\Sigma'$ as the available information, a finer partition means more information. The conditional expectation $\mathbb{E}(f \mid \Sigma')$ is the best approximation of $f$ given only the information from $\Sigma'$. With this intuition the statements from the following lemma are not surprising.

**Lemma 3.4.** *Let $(X, \Sigma, \mu)$ be a finite probability space, $f, g\colon X \to \mathbb{R}$ be measurable functions, $\Sigma'' \subseteq \Sigma' \subseteq \Sigma$ be sub-$\sigma$-algebras. Then*

*i) if $f \leq g$, then $\mathbb{E}(f \mid \Sigma') \leq \mathbb{E}(g \mid \Sigma')$,*

*ii) for any $\Sigma'$-measurable function $h\colon X \to \mathbb{R}$ we have $\mathbb{E}(hf \mid \Sigma') = h \cdot \mathbb{E}(f \mid \Sigma')$,*

*iii) also $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') = \mathbb{E}(f \mid \Sigma'') = \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma')$.*

*Proof.* To i): If $f \leq g$, then

$$\mathbb{E}(f \mid \Sigma') = \sum_{i=1}^{m} \mathbb{E}(f \mid B_i) \cdot \mathbb{1}_{B_i} \leq \sum_{i=1}^{m} \mathbb{E}(g \mid B_i) \cdot \mathbb{1}_{B_i} = \mathbb{E}(g \mid \Sigma').$$

To ii): Let $h\colon X \to \mathbb{R}$ be $\Sigma'$-measurable function, then, by Lemma 3.2, $h$ is of the form $\sum_{i=1}^{m} h_i \cdot \mathbb{1}_{B_i}$. Now

$$\begin{aligned}
\mathbb{E}(hf \mid \Sigma') &= \sum_{i=1}^{m} \mathbb{E}(hf \mid B_i) \cdot \mathbb{1}_{B_i} \\
&= \sum_{i=1}^{m} h_i \cdot \mathbb{E}(f \mid B_i) \cdot \mathbb{1}_{B_i} \\
&= h \cdot \mathbb{E}(f \mid \Sigma').
\end{aligned}$$

To iii): Let $C_1, \ldots, C_k$ be the partition corresponding to $\Sigma''$. Furthermore note that $\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid B) = \mathbb{E}(f \mid B)$ for all $B \in \Sigma'$. Hence

$$\begin{aligned}
\mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid \Sigma'') &= \sum_{i=1}^{k} \mathbb{E}(\mathbb{E}(f \mid \Sigma') \mid C_i) \cdot \mathbb{1}_{C_i} \\
&= \sum_{i=1}^{k} \mathbb{E}(f \mid C_i) \cdot \mathbb{1}_{C_i} && (C_i \in \Sigma') \\
&= \mathbb{E}(f \mid \Sigma'') \\
&= \sum_{j=1}^{k} \mathbb{E}(f \mid C_j) \cdot \mathbb{1}_{C_j} \cdot \sum_{i=1}^{m} \mathbb{1}_{B_i} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{k} \mathbb{E}(\mathbb{E}(f \mid C_j) \mid B_i) \cdot \mathbb{1}_{C_j} \cdot \mathbb{1}_{B_i} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{k} \mathbb{E}(\mathbb{E}(f \mid C_j) \cdot \mathbb{1}_{C_j} \mid B_i) \cdot \mathbb{1}_{B_i} && \text{(by ii)} \\
&= \mathbb{E}(\mathbb{E}(f \mid \Sigma'') \mid \Sigma').
\end{aligned}$$

This concludes the proof. □

The following lemma might not seem very interesting, but changing the exponent from $x$ to $x^2$ is the crucial ingredient for the proof of Azuma's inequality.

**Lemma 3.5.** *For all $x \in \mathbb{R}$*

$$e^x \leq x + e^{x^2}.$$

*Proof.* Note that for $x = 0$ both sides are equal to 1. As both sides are differentiable it suffices to show that the derivative of the right hand side is larger than the derivative of the left hand side for all $x \geq 0$ and smaller for all $x \leq 0$. Hence, we want to show

$$e^x \geq 1 + 2xe^{x^2} \text{ for all } x \leq 0 \qquad \text{and} \qquad e^x \leq 1 + 2xe^{x^2} \text{ for all } x \geq 0.$$

As for $x = 0$ both sides are again equal to 1 we can reduce the problem, by similar reasoning, to the question whether

$$e^x \leq 2e^{x^2} + 4x^2e^{x^2} \text{ for all } x \in \mathbb{R}.$$

Showing this inequality comes down to a case study.

- For $x \leq 0$ the left side is bounded by 1, while the right side is larger than 2.
- For $0 < x \leq \ln 2$ note that both sides are increasing. Furthermore the right side is at least 2, while the left side is at most 2.
- For $\ln 2 < x < 1$ both sides are still increasing. Also, $\ln 2 \geq \frac{1}{2}$ and therefore the right side with $x = \ln 2$ evaluates to a number larger then $e$. Whereas the left side is clearly less than $e$.
- For $1 \leq x$ we have $x \leq x^2$ and the inequality holds trivially.

Hence the inequality holds for the second derivatives and therefore also for the original functions. $\square$

Before we will prove Azuma's inequality, let us introduce some useful notation. Whenever there is no danger of confusion we will abbreviate sets of the form

$$\{x \in X \mid \text{Condition}(x) \text{ holds}\} \qquad \text{by} \qquad \{\text{Condition}\}.$$

For example $\{x \in X \mid f(x) = c\}$ becomes $\{f = c\}$. For a function $f\colon X \to \mathbb{R}$ we denote the *sup norm* of $f$ by $\|f\|_\infty := \sup f(X)$.

**Lemma 3.6** (Azuma's inequality)**.** *Let $(X, \Sigma, \mu)$ be a finite probability space, $f\colon X \to \mathbb{R}$ a measurable function, and $\{X, \emptyset\} = \Sigma_0 \subseteq \cdots \subseteq \Sigma_n = \Sigma$ a chain of sub-$\sigma$-algebras. Define $f_0 := \mathbb{E}(f \mid \Sigma_0)$ and $f_i := \mathbb{E}(f \mid \Sigma_i)$, $d_i := f_i - f_{i-1}$ for $i \in \{1, \ldots, n\}$. Then for every $\varepsilon \geq 0$*

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \cdot \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^n \|d_i\|_\infty^2}\right).$$

Note that $(f_i, \Sigma_i)_{0 \leq i \leq n}$ is a discrete martingale. But we will not formally introduce this notion as it is not necessary for the upcoming results.

*Proof.* First, observe that $f_0 = \mathbb{E}(f \mid \{X, \emptyset\}) = \mathbb{E}(f)$ and $f_n = \mathbb{E}(f \mid \Sigma) = f$. Using a simple telescoping sum we obtain

$$f - \mathbb{E}(f) = d_1 + \cdots + d_n.$$

Therefore we have

$$\begin{aligned}
\mu(\{f - \mathbb{E}(f) \geq \varepsilon\}) &= \mu(\{\sum_{i=1}^{n} d_i \geq \varepsilon\}) \\
&= \mu(\{\lambda \cdot \sum_{i=1}^{n} d_i \geq \lambda\varepsilon\}) && \text{(for any } \lambda > 0) \\
&= \mu(\{\exp(\lambda \cdot \sum_{i=1}^{n} d_i) \geq e^{\lambda\varepsilon}\}) \\
&\leq \mathbb{E}(\exp(\lambda \cdot \sum_{i=1}^{n} d_i)) \cdot e^{-\lambda\varepsilon} && (*) \\
&= \mathbb{E}(\mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}} \cdot e^{\lambda d_n} \mid \Sigma_{n-1})) \cdot e^{-\lambda\varepsilon} && \text{(Lemma 3.4 iii))} \\
&= \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}} \cdot \mathbb{E}(e^{\lambda d_n} \mid \Sigma_{n-1})) \cdot e^{-\lambda\varepsilon} && \text{(Lemma 3.4 ii))} \\
&\leq \mathbb{E}(e^{\lambda d_1} \cdot \ldots \cdot e^{\lambda d_{n-1}}) \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda\varepsilon} && (**) \\
&\ \ \vdots \\
&\leq e^{\lambda^2 \cdot \|d_1\|_\infty^2} \cdot \ldots \cdot e^{\lambda^2 \cdot \|d_{n-1}\|_\infty^2} \cdot e^{\lambda^2 \cdot \|d_n\|_\infty^2} \cdot e^{-\lambda\varepsilon} \\
&= \exp(\lambda^2 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2 - \lambda\varepsilon).
\end{aligned}$$

For $(*)$ note that for any measurable function $g \colon X \to \mathbb{R}$ and $c \in \mathbb{R}$ we have

$$\mu(\{e^g \geq e^c\}) = \mathbb{E}(\mathbb{1}_{\{e^g \geq e^c\}}) \leq \mathbb{E}(e^g)e^{-c}.$$

For $(**)$ we need to use Lemma 3.5

$$\begin{aligned}
\mathbb{E}(e^{\lambda d_i} \mid \Sigma_{i-1}) &\leq \mathbb{E}(\lambda d_i \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&= \lambda \cdot \mathbb{E}(f_i - f_{i-1} \mid \Sigma_{i-1}) + \mathbb{E}(e^{\lambda^2 d_i^2} \mid \Sigma_{i-1}) \\
&\leq 0 + e^{\lambda^2 \|d_i\|_\infty^2}. && \text{(Lemma 3.4)}
\end{aligned}$$

Substituting $\frac{\varepsilon}{2 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}$ for $\lambda$, we conclude that

$$\mu(\{f - \mathbb{E}(f) \geq \varepsilon\}) \leq \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

The same calculations with $-d_i$ instead of $d_i$ yield the dual inequality

$$\mu(\{f - \mathbb{E}(f) \leq -\varepsilon\}) \leq \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^{n} \|d_i\|_\infty^2}\right).$$

These two statements obviously give us the desired result. □

Since $\mu$ is the counting measure Azuma's inequality bounds the number of elements for which $f$ differs at least $\varepsilon$ from its mean. This seems at least somewhat connected to the measure concentration function, as there we want to show that for any measurable set $A$ with $\mu(A) \geq \frac{1}{2}$ only a few elements are more than $\varepsilon$ away from $A$. The next goal is to formalize this connection. But to achieve this we first need to introduce the length of an mm-space.

**Definition 3.7.** Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space. The *length* of $\boldsymbol{X}$, denoted by $\operatorname{len}(\boldsymbol{X})$, is the minimum over all $l \in \mathbb{R}_{\geq 0}$ with the following property. There is a refining sequence of partitions

$$\{X\} = \Omega_0 \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\},$$

where for every $i \in \{1, \dots, n\}$ there is an $a_i$ such that $\sum_{i=1}^{n} a_i^2 = l^2$ and for every $A \in \Omega_{i-1}$, $x, y \in A$ (denote the sets from $\Omega_i$ containing $x$ and $y$ by $[x]_i$ and $[y]_i$, respectively) there is an isometry $\varphi\colon [x]_i \to [y]_i$ with

$$d(z, \varphi(z)) \leq a_i \text{ for all } z \in [x]_i.$$

Note that since $\mu$ is the counting measure $\varphi$ is also an isomorphism of mm-spaces. Furthermore $\boldsymbol{X}$ is finite and therefore the infimum becomes a minimum. As this definition is quite hard we will look at some properties and examples of the length of $\boldsymbol{X}$ before proceeding.

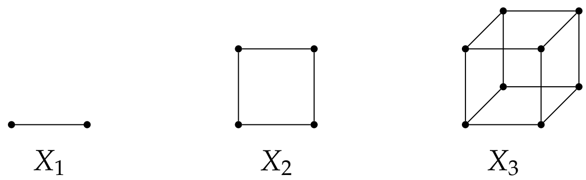
**Lemma 3.8.** *Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space. Then*

$$\operatorname{len}(\boldsymbol{X}) \leq \operatorname{diam}(\boldsymbol{X}).$$

*Proof.* Consider only the two partitions $\{X\} \prec \{\{x\} \mid x \in X\}$. Clearly, $x \to y$ is an isometry between $\{x\}$ and $\{y\}$ and $d(x, y) \leq \operatorname{diam}(\boldsymbol{X})$ for all $x, y \in X$. $\square$

We will see more properties later in Lemma 4.3 and Lemma 4.4.

**Example 3.9.** Let us look at the $n$-dimensional cube $X_n = \{0, 1\}^n$.




We will only consider the following sequence of partitions

$$\Omega_0 \prec \cdots \prec \Omega_n \text{ with } \Omega_i = \{wX_{n-i} \mid w \in \{0, 1\}^i\}.$$

First, we equip $X_n$ with the euclidean metric and rescale it such that the diameter is 1. To bound the length of the resulting space $\boldsymbol{X}_n^E$ consider $[x]_i \neq [y]_i$.

Note that $x$ and $y$ are w.l.o.g. of the form $w0u$ and $w1v$ for some $w \in \{0,1\}^{i-1}$, $u, v \in \{0,1\}^{n-i}$. The isometry $\varphi$ takes an element $w0u'$ in $[x]_i$ and maps it to $w1u'$. The length of a side in $\boldsymbol{X}_n^E$ is $\frac{1}{\sqrt{n}}$, hence $a_i$ is $\frac{1}{\sqrt{n}}$ for every $i$ and the length of $\boldsymbol{X}_n^E$ is bounded by $(\sum_{i=1}^n \frac{1}{\sqrt{n}^2})^{\frac{1}{2}} = 1$. We already got the same upper bound from Lemma 3.8 and the question remains whether $\operatorname{len}(\boldsymbol{X}_n^E)$ is indeed 1.

Secondly, we use the normalized Hamming metric $d_H$, defined by

$$d_H(u_1 \dots u_n, v_1 \dots v_n) = \frac{1}{n} \cdot |\{i \in \{1, \dots, n\} \mid u_i \neq v_i\}|,$$

and obtain the mm-space $\boldsymbol{X}_n^H$ with diameter 1. Now, the n-dimensional cube has side length $\frac{1}{n}$ and therefore the length of $\boldsymbol{X}_n^H$ is bounded by $(\sum_{i=1}^n \frac{1}{n^2})^{\frac{1}{2}} = n^{-\frac{1}{2}}$. In Lemma 4.4 we will show that this is also a lower bound. Therefore $\operatorname{len}(\boldsymbol{X}_n^H) = n^{-\frac{1}{2}}$. We see that here the length of $\boldsymbol{X}_n^H$ converges to 0 as $n$ tends to infinity. We will show that this means that the measure concentration function $\alpha_{\boldsymbol{X}_n^H}(\varepsilon)$ also converges to 0 for any fixed $\varepsilon > 0$.

Now we come back to the connection between Azuma's inequality and the measure concentration function.

**Lemma 3.10.** *Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space of length $l$ and $f \colon X \to \mathbb{R}$ be a 1-Lipschitz function. Then*

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4l^2}\right) \text{ for every } \varepsilon > 0.$$

*Proof.* Let $\Omega_0 \prec \cdots \prec \Omega_n$ be a refining sequence of partitions with $a_1, \dots, a_n$ as in Definition 3.7 such that $\sum_{i=1}^n a_i^2 = l^2$. These partitions correspond to $\sigma$-algebras $\Sigma_0 \subseteq \cdots \subseteq \Sigma_n$. Now we can apply Azuma's inequality to obtain

$$\mu(\{|f - \mathbb{E}(f)| \geq \varepsilon\}) \leq 2 \cdot \exp\left(-\frac{\varepsilon^2}{4 \cdot \sum_{i=1}^n \|d_i\|_\infty^2}\right),$$

where $f_i = \mathbb{E}(f \mid \Sigma_i)$ and $d_i = f_i - f_{i-1}$ as before. Therefore we only need to show that $\|d_i\|_\infty \leq a_i$. Since on any $A \in \Omega_{i-1}$ we have $f_{i-1} = \mathbb{E}(f_i \mid A)$ it suffices to show that for all $A \in \Omega_{i-1}$ it holds that $f_i(x) - f_i(y) \leq a_i$ for all $x, y \in A$. Let $\varphi \colon [x]_i \to [y]_i$ be the isometry from Definition 3.7.

$$\begin{aligned}
f_i(x) - f_i(y) &= \mathbb{E}(f \mid [x]_i) - \mathbb{E}(f \mid [y]_i) \\
&= \mathbb{E}(f \mid [x]_i) - \mathbb{E}(f \circ \varphi \mid [x]_i) \\
&= \mathbb{E}(f - f \circ \varphi \mid [x]_i) \\
&\leq \mathbb{E}(d(\,.\,, \varphi(\,.\,)) \mid [x]_i) && (f \text{ is 1-Lipschitz}) \\
&\leq a_i
\end{aligned}$$

This concludes the proof. $\square$

Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space and $A \subseteq X$ measurable. Observe that $d_A \colon X \to \mathbb{R}$, $d_A(x) := \inf_{y \in A} d(x, y)$ is a 1-Lipschitz function. Using $d_A$ we can rewrite the definition of the measure concentration function

$$\alpha_{\boldsymbol{X}}(\varepsilon) = \sup \left\{ \mu(\{d_A \geq \varepsilon\}) \mid A \subseteq X \text{ measurable}, \mu(A) \geq \frac{1}{2} \right\}.$$

This gives us the desired connection.

**Theorem 3.11.** *If a finite mm-space $\boldsymbol{X} = (X, d, \mu)$ has length $l$, then the measure concentration function of $\boldsymbol{X}$ satisfies*

$$\alpha_{\boldsymbol{X}}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 l^2}\right) \text{ for all } \varepsilon > 0.$$

*Proof.* Let $\varepsilon > 0$ and $A \subseteq X$ be measurable with $\mu(A) \geq \frac{1}{2}$. As mentioned above $d_A$ is 1-Lipschitz and therefore, by Lemma 3.10,

$$\mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4 l^2}\right).$$

Now there are two cases to consider.

The first case is the more interesting one. If $\mathbb{E}(d_A) \leq \varepsilon$, then for any $x$ with $d_A(x) \geq 2\varepsilon$, we know $d_A(x) \geq \varepsilon + \mathbb{E}(d_A)$ and therefore $|d_A(x) - \mathbb{E}(d_A)| \geq \varepsilon$. As a consequence

$$\mu(\{d_A \geq 2\varepsilon\}) \leq \mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4 l^2}\right).$$

Replacing $\varepsilon$ by $\frac{\varepsilon}{2}$ gives the desired inequality.

If $\mathbb{E}(d_A) > \varepsilon$, then $A \subseteq \{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}$. Consequently,

$$\mu(\{d_A \geq \varepsilon\}) \leq \mu(X \setminus A) \leq \frac{1}{2} \leq \mu(A) \leq \mu(\{|d_A - \mathbb{E}(d_A)| \geq \varepsilon\}) \leq 2 \exp\left(-\frac{\varepsilon^2}{4 l^2}\right).$$

This proves the theorem. □

Note that in the second case, the upper bound is at least $\frac{1}{2}$, which means that if $l$ is large enough, then we are in the first case and the expected distance to a set with at least half measure is at most $\varepsilon$. In Section 7, we will see a slight modification of this lemma. But for now, our goal is to apply Theorem 2.14 to groups and as it turns out, we can bound the length of a group using sequences of subgroups. Before we can write down the corresponding corollary, we need to make a quick excursion into factor metrics.

**Definition 3.12.** Let $(X, d)$ be a metric space and let $\sim$ be an equivalence relation on $X$. Then

$$d_\sim([x], [y]) = \inf\{d(p_1, q_1) + \cdots + d(p_n, q_n) \mid q_i \sim p_{i+1}, x \sim p_1, q_n \sim y, n \in \mathbb{N}\}$$

defines a pseudo-metric on $X/\sim$. We call $d_\sim$ the *factor metric* on $X/\sim$.

Note that if $X$ is finite, then the infimum becomes a minimum and $d_\sim$ is a proper metric. In case that $X$ is a group with a bi-invariant metric this definition simplifies.

**Lemma 3.13.** *Let $G$ be a finite group with a bi-invariant metric $d$ and $H$ a (not necessarily normal) subgroup of $G$. Then the factor metric $d_H$ on $G/H = \{gH \mid g \in G\}$ satisfies*

$$d_H(gH, g'H) = \inf\{d(g, g'h) \mid h \in H\}.$$

*Proof.* Let $x, y \in G$. We show that for any path $p_1, q_1, \ldots, p_n, q_n$ as in the definition there are $x \sim p$ and $q \sim y$ such that $d(p,q) \leq d(p_1,q_1) + \cdots + d(p_n,q_n)$. It suffices to show this for $n = 2$. By definition $p_1, q_1, p_2, q_2$ are of the form $g, g', g'h, g''$ for some $g, g', g'' \in G$ and $h \in H$. Since $d$ is bi-invariant

$$d(gh, g'') \leq d(gh, g'h) + d(g'h, g'') = d(g, g') + d(g'h, g'').$$

For $n > 2$ the statement follows by induction. $\square$

Equipped with this knowledge, we can formulate the final statement for this section.

**Corollary 3.14.** *Let $G$ be a finite group with a bi-invariant metric $d$, and let*

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

*be a chain of subgroups. Denote the diameter of $G_i/G_{i-1}$ with respect to the factor metric by $a_i$. Then the length of $G$ is at most $\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ and the measure concentration function of the mm-space $(G, d, \mu)$, where $\mu$ is the normalized counting measure, satisfies*

$$\alpha_{(G,d,\mu)}(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16 \cdot \sum_{i=1}^n a_i^2}\right).$$

*Proof.* We show that the length $l$ of $(G, d, \mu)$ is bounded by $\left(\sum_{i=1}^n a_i^2\right)^{\frac{1}{2}}$ and apply Theorem 3.11. Define the sequence of partitions $\Omega_i := \{gG_i \mid g \in G\}$

$$\begin{aligned} \{\{g\} \mid g \in G\} &= \Omega_0 \succ \Omega_1 \succ \cdots \succ \Omega_n = \{G\} \\ \{e\} &= G_0 < G_1 < \cdots < G_n \ = G. \end{aligned}$$

Take $A \in \Omega_{i+1}$ and $g, g' \in A$. Since the distance of $gG_i$ and $g'G_i$ with respect to the factor metric is at most $a_i$ there is an $h' \in G_i$ such that $d(g, g'h') \leq a_i$. Hence the map

$$\begin{aligned} \varphi\colon gG_i &\to g'G_i \\ gh &\mapsto g'h'h \end{aligned}$$

is, by bi-invariance of $d$, an isomorphism of metric spaces with $d(gh, g'h'h) = d(g, g'h') \leq a_i$ for all $gh \in gG_i$. Therefore $\left(\sum_{i=1}^n a_i^2\right)$ is an upper bound for $l^2$. $\square$

Carderi and Thom used this result to show that the limit of $\mathrm{SL}_{2^n}(q)$ is extremely amenable [3]. We will explain their proof in the next section.

# 4 Special Linear Groups and Extreme Amenability

When studying matrices it is often useful to look at the corresponding linear maps of a suitable vector space. For example fixing a basis $e_1, \dots, e_n$ of an $n$-dimensional $\mathbb{F}_q$-vector space $V$ gives us an embedding from $\mathrm{SL}_n(q)$ into $\mathrm{Aut}(V)$. In the following we will use matrices and their corresponding linear maps interchangeably. Next we will apply the methods from the previous section to show that $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ is a Lévy group and therefore extremely amenable.

**Theorem 4.1.** *The normalized counting measure $\mu_n$ on the groups $\mathrm{SL}_n(q)$ concentrates with respect to the normalized rank-metric $d_n$, i.e. for all $\varepsilon > 0$*

$$\lim_{n\to\infty} \alpha_{(\mathrm{SL}_n(q), d_n, \mu_n)}(\varepsilon) = 0.$$

*Proof.* We will apply Corollary 3.14 to a sequence of subgroups which also shows that the length of $\mathrm{SL}_n(q)$ is bounded by $3n^{-\frac{1}{2}}$. Let $e_1, \dots, e_n$ be a basis of an $n$-dimensional $\mathbb{F}_q$-vector space $V$. Look at the sequence

$$\mathrm{SL}_0(q) < \mathrm{SL}_1(q) < \cdots < \mathrm{SL}_n(q),$$

where the group $\mathrm{SL}_{i-1}(q)$ is considered as a subgroup of $\mathrm{SL}_i(q)$ via the embedding $g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$. Next we want to bound the diameter of $\mathrm{SL}_i(q) / \mathrm{SL}_{i-1}(q)$ by $\frac{3}{n}$. By Lemma 3.13 it suffices to show that for any $g, g' \in \mathrm{SL}_i(q)$ there is an $h \in \mathrm{SL}_{i-1}(q)$ such that $d(g, g'h) \leq \frac{3}{n}$. Since $d$ is bi-invariant we can assume w.l.o.g. that $g'$ is equal to the identity matrix $I_i$. First, our goal is to find a $\tilde{g} \in \mathrm{SL}_i(q)$ that is the identity on $e_i$ and has a small distance to $g$.

Let us take a closer look at $ge_i$. If $e_i$ is an eigenvector of $g$ with eigenvalue $\lambda$, then $\lambda \neq 0$ and $g$ is of the form $\begin{pmatrix} A & 0 \\ c^{\perp} & \lambda \end{pmatrix}$. Define

$$h' := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \qquad \text{and} \qquad \tilde{g} := \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix} \cdot g.$$

By construction $\tilde{g} \in \mathrm{SL}_i(q)$ and it is of the form $\begin{pmatrix} A' & 0 \\ c'^{\perp} & 1 \end{pmatrix}$. Since $\det \tilde{g} = 1$ we have that $\det A' = 1$ and therefore $A' \in \mathrm{SL}_{i-1}(q)$ making it a suitable candidate for $h$. Using the triangle inequality we obtain

$$\begin{aligned} d(g,h) &\leq d(g,\tilde{g}) + d(\tilde{g},h) \\ &= d(I_i, \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix}) + \frac{1}{n} \operatorname{r}(\begin{pmatrix} 0 & 0 \\ -c'^{\perp} & 0 \end{pmatrix}) \\ &\leq \frac{2}{n} + \frac{1}{n} \end{aligned}$$

as desired.

If $e_i$ is not an eigenvector of $g$, then we change the basis of $\langle e_1, \dots, e_{i-1} \rangle$ such that $ge_i = e_{i-1} + \lambda e_i$. Henceforth we can assume w.l.o.g. that $g$ is of the form $\begin{pmatrix} & A & & 0 \\ & & & 1 \\ c^{\perp} & & c_{i-1} & \lambda \end{pmatrix}$. Define

$$h' := \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix} \qquad \text{and as before} \qquad \tilde{g} := \begin{pmatrix} I_{i-2} & 0 \\ 0 & h' \end{pmatrix} \cdot g.$$

Now we can apply the argument from above to obtain an $h \in \mathrm{SL}_{i-1}(q)$ such that $d(g, h) \leq \frac{3}{n}$. Applying Corollary 3.14 we conclude

$$\alpha_{(\mathrm{SL}_n(q), d_n, \mu_n)}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2}{16 \cdot \sum_{i=1}^{n} \frac{9}{n^2}}\right) = 2 \exp\left(-\frac{\varepsilon^2 n}{16 \cdot 9}\right),$$

which converges to 0 as $n$ tends to infinity. $\square$

From this theorem the main result of this section follows easily.

**Corollary 4.2.** *The Polish group* $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ *is extremely amenable.*

*Proof.* Theorem 4.1 implies that $\operatorname{clim} \mathrm{SL}_{2^n}(q)$ is a Lévy group and is therefore extremely amenable, by Theorem 2.14. $\square$

As a byproduct we found an upper bound for the length of $\mathrm{SL}_n(q)$. The natural question to ask is: How good is this upper bound? Therefore our next goal is to also determine a lower bound. This part is not essential to the rest of the thesis, but still interesting.

**Lemma 4.3.** *Let $(X, d, \mu)$ be a finite mm-space with diameter $\Delta$ and*

$$\Omega_0 = \{X\} \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

*with $a_1, \dots, a_n$ as in Definition 3.7. Then*

$$\sum_{i=1}^{n} a_i \geq \Delta.$$

*Proof.* Let $x, y \in X$, with $x \neq y$, we show $d(x, y) \leq \sum_{i=1}^{n} a_i$. Let $i_0$ be the smallest number such that $[x]_{i_0} \neq [y]_{i_0}$. Since $[x]_0 = X = [y]_0$ we know that $i_0$ is at least 1. Therefore $[x]_{i_0-1} = [y]_{i_0-1}$ and there is an isomorphism $\varphi_{i_0} \colon [x]_{i_0} \to [y]_{i_0}$ such that $d(\varphi_{i_0}(x), y) \leq a_{i_0}$. Let $x_{i_0} = \varphi_{i_0}(x)$, then

$$d(x, y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

If $x_{i_0} = y$, then we are done. Otherwise let $i_1$ be the smallest number such that $[x_{i_0}]_{i_1} \neq [y]_{i_1}$. Then let $\varphi_{i_1} \colon [x_{i_0}]_{i_1} \to [y]_{i_1}$ be an isomorphism such that

$d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$. Define $x_{i_1} = \varphi_{i_1}(x_{i_0})$. Proceeding in this fashion yields elements $x_{i_0}, \dots, x_{i_k}$ such that $x_{i_k} = y$ and

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \cdots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \cdots + a_{i_k} \leq \sum_{i=1}^{n} a_i.$$

From this the claim immediately follows. □

**Lemma 4.4.** *Let $\boldsymbol{X} = (X, d, \mu)$ be a finite mm-space with diameter $\Delta$ and $\delta = \min_{x \neq y} d(x,y)$. Then the length of $\boldsymbol{X}$ is at least $(\Delta \cdot \delta)^{\frac{1}{2}}$.*

*Proof.* Let $a_1, \dots, a_n$ as in Definition 3.7 with $\sum_{i=1}^{n} a_i^2 = \operatorname{len}(\boldsymbol{X})^2$. We know from Lemma 4.3 that $\sum_{i=1}^{n} a_i \geq \Delta$. Therefore

$$\operatorname{len}(\boldsymbol{X})^2 = \sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} \delta a_i = \delta \cdot \sum_{i=1}^{n} a_i \geq \delta \cdot \Delta.$$

This concludes the proof. □

Using this we can give an interval for the length of $\mathrm{SL}_n(q)$.

**Corollary 4.5.** *Consider $(\mathrm{SL}_n(q), d, \mu)$, where $d$ is the normalized rank-metric and $\mu$ is the normalized counting measure. Then the length $l$ of this mm-space satisfies*

$$n^{-\frac{1}{2}} \leq l \leq 3n^{-\frac{1}{2}}.$$

*Proof.* The diameter of $\mathrm{SL}_n(q)$ is equal to 1 and for any $g \neq g' \in \mathrm{SL}_n(q)$ we have that $d(g, g') \geq \frac{1}{n}$. □

The next goal is to show that the limit of symplectic groups is also extremely amenable. Theses groups can be seen as automorphism groups of a vector space together with a symplectic form. The proof will be similar to the one for the special linear groups, but extending the partial inverse $h'$ becomes much harder. This is why in the next section we will prove Witt's Lemma which does exactly what we need, i.e. extending isometries.

# 5 Witt's Lemma

In this section we will prove Witt's Lemma and explore the structure of symplectic vector spaces. Witt's Lemma states that an isometry between subspaces of a finite dimensional symplectic, unitary, or orthogonal vector space can always be extended to a linear isometry on the whole space. We will roughly follow the proof in [1]. Since we are mainly interested in symplectic vector spaces we will only show Witt's Lemma for those but it also holds for unitary and orthogonal spaces.

**Definition 5.1.** Let $V$ be an $\mathbb{F}_q$-vector space. A bilinear form $\omega$ on $V$ is called *symplectic* if $\omega$ is nondegenerate, i.e. for all $x \in V \setminus \{0\}$ there is an $y \in V$ such that $\omega(x,y) \neq 0$, $\omega(x,y) = -\omega(y,x)$ for all $x, y \in V$, and if $q = 2$, then $\omega(x,x) = 0$ for all $x \in V$. For $x, y \in V$ with $\omega(x,y) = 0$ we often write $x \perp y$. We call $(V, \omega)$ a *symplectic vector space*.

A finite group $G$ is called *symplectic* if there is a symplectic vector space $(V, \omega)$ such that $G \cong \mathrm{Aut}(V, \omega)$, if $V$ is $n$-dimensional, then we will denote $\mathrm{Aut}(V, \omega)$ by $\mathrm{Sp}_n(q)$.

A subspace $U \leq V$ is *nondegenerate* if $\omega$ restricted to $U$ is nondegenerate.

Throughout this section let $(V, \omega)$ be a finite dimensional symplectic $\mathbb{F}_q$-vector space. Note that $x \perp y$ iff $y \perp x$, also $x \perp x$ for all $x \in V$. We will start of with some technical lemmas.

**Lemma 5.2.** *Let $U \leq V$ be a subspace. Then $U$ is nondegenerate iff $U \cap U^\perp = \{0\}$.*

*Proof.* If $U$ is degenerate, then there is a $u \in U$ such that $u \perp v$ for all $v \in U$. Therefore $u \in U^\perp$. Vice versa if $u \in (U \cap U^\perp) \setminus \{0\}$, then $\omega(u, v) = 0$ for all $v \in U$. □

**Lemma 5.3.** *For a subspace $U \leq V$ we have*

$$\dim U^\perp = \dim V - \dim U.$$

*Proof.* Let $u_1, \ldots, u_m$ be a basis of $U$ and consider the linear map

$$g \colon V \to \mathbb{F}_q^m \qquad \text{with} \qquad g(v) = \begin{pmatrix} \omega(v, u_1) \\ \vdots \\ \omega(v, u_m) \end{pmatrix}.$$

By definition the kernel of $g$ is $U^\perp$ and since $\omega$ is nondegenerate $g$ is also surjective. Therefore the claim follows from the Rank-Nullity Theorem. □

The following lemma is an immediate consequence of the previous one.

**Lemma 5.4.** *Let $U \leq V$. Then*

*i) $U$ is nondegenerate iff $V = U \oplus U^\perp$ and*

*ii) $U^{\perp\perp} = U$.*

*Proof.* To i): If $U$ is nondegenerate, then, by Lemma 5.2, $U \cap U^\perp = \{0\}$. By Lemma 5.3 $\dim U + \dim U^\perp = \dim V$ and therefore

$$U \oplus U^\perp = \langle U, U^\perp \rangle = V.$$

The other direction is clear from the definition.

To ii): Note that $U \subseteq U^{\perp\perp}$. By Lemma 5.3 they also have the same dimension and are therefore equal. □

In the following we want to investigate the structure of symplectic vector spaces. We start with 2-dimensional symplectic vector spaces, which we will call *hyperbolic planes*.

**Lemma 5.5.** *There is only one hyperbolic plane up to isometry.*

*Proof.* Let $(V, \omega)$ and $(V', \omega')$ be hyperbolic planes with basis $r, s$ and $r', s'$, respectively. Then $r \mapsto r', s \mapsto \lambda s'$ with $\lambda = \omega'(r', s')^{-1} \cdot \omega(r, s)$ is an isometry between the two hyperbolic planes. $\square$

Next we will strengthen this result and show that any symplectic vector space is the direct sum of hyperbolic planes.

**Theorem 5.6.** *Let $r \in V \setminus \{0\}$. Then there is a hyperbolic plane $U \leq V$ containing $r$ such that $V = U \oplus U^\perp$. Furthermore, if $W \leq V$ with $W \perp r$ and $r \notin W$, then there is a $U$ as before that also fulfills $W \perp U$.*

*Proof.* Since $\omega(r, r) = 0$ we know that $r \in r^\perp$. Let $H \leq r^\perp$ containing $W$ such that $\langle r \rangle \oplus H = r^\perp$. Then, by Lemma 5.3, $\dim r^\perp = \dim V - 1$, $\dim H^\perp = 2$, and $r \in H^\perp$. In particular $\dim V > 1$. Our goal is to show that

$$V = H^\perp \oplus H,$$

as then $H^\perp$ would be a suitable choice for $U$. It suffices to show that $H^\perp$ is nondegenerate, since then the claim follows from Lemma 5.4. Let $s \in H^\perp$ such that $H^\perp = \langle r, s \rangle$. Now $H^\perp$ is nondegenerate iff $\omega(r, s) \neq 0$.

Assume $\omega(r, s) = 0$. Then $r \in s^\perp$ and $H \subseteq s^\perp$. Since, by construction, $r \notin H$ we have $r^\perp = \langle r, H \rangle = s^\perp$. Hence $r$ and $s$ are linearly dependent, contradicting that $r, s$ is a basis of $H^\perp$. $\square$

Using Theorem 5.6 we can describe the structure of symplectic vector spaces.

**Corollary 5.7.** *Let $(V, \omega)$ be a symplectic vector space. Then $V$ is of dimension $2n$ for some $n \in \mathbb{N}$. Furthermore there are hyperbolic planes $U_1, \ldots, U_n \leq V$ such that $V = U_1 \oplus \cdots \oplus U_n$ and $U_i \perp U_j$ for $i \neq j$. In particular for any $n \in \mathbb{N}$ there is exactly one symplectic vector space of dimension $2n$.*

*Proof.* We use induction on $n = \dim V$.

If $n = 1$, then $V = \langle v \rangle$ for $v \in V \setminus \{0\}$. But $\omega(v, v) = 0$ and therefore $\omega$ is degenerate which is a contradiction.

If $n = 2$, then the claim follows from Lemma 5.5.

If $n > 2$, then by Theorem 5.6 there is a hyperbolic plane $U_1 \leq V$ satisfying $V = U_1 \oplus U_1^\perp$. Hence, by induction hypothesis, $n$ is even and $U^\perp = U_2 \oplus \cdots \oplus U_m$ for some hyperbolic planes $U_2, \ldots, U_m \leq U^\perp$ with $m = \frac{n}{2}$. Furthermore $U_i \perp U_j$ for $i \neq j$. $\square$

With these powerful tools we can prove Witt's Lemma easily. Let $\alpha \colon U \to W$ be an isometry between subspaces $U, W \leq V$.

**Lemma 5.8.** *There are subspaces $U' \geq U$ and $W' \geq W$ with $U', W'$ nondegenerate such that $\alpha$ can be extended to an isometry $\tilde{\alpha}\colon U' \to W'$.*

*Proof.* We show this claim using induction on $n = \dim(U \cap U^{\perp})$.

If $n = 0$, then $U$ itself is nondegenerate and we are done.

If $n > 0$, then let $r \in (U \cap U^{\perp}) \setminus \{0\}$ and $\tilde{U} \leq V$ such that $\langle r \rangle \oplus \tilde{U} = U$. By Theorem 5.6 there is a hyperbolic plane $H \leq V$ containing $r$ such that $\tilde{U} \perp H$. Similarly, there is a hyperbolic plane $H' \leq V$ containing $r' := \alpha(r)$ such that $H' \perp \tilde{W}$ with $\tilde{W} = \alpha(\tilde{U})$. Let $r, s$ and $r', s'$ be a basis of $H$ and $H'$, respectively. Note that $\omega(r, s) \neq 0$ and $r \perp U$ imply $s \notin U$. Analogously, $s' \notin W$. We can assume w.l.o.g. that $\omega(r, s) = \omega(r', s')$. Now we can extend $\alpha$ to $\tilde{\alpha}\colon \langle U, s \rangle \to \langle W, s' \rangle$ by defining $\tilde{\alpha}(s) := s'$. Note that $\langle U, s \rangle = \langle \tilde{U}, H \rangle$. Since $\tilde{U} \perp H$ we have

$$\dim(\langle \tilde{U}, H \rangle \cap \langle \tilde{U}, H \rangle^{\perp}) = \dim(\tilde{U} \cap \tilde{U}^{\perp}) < \dim(U \cap U^{\perp}).$$

Hence we can apply the induction hypothesis to $\tilde{\alpha}$. □

**Lemma 5.9.** *If $U$ is nondegenerate. Then $\alpha$ can be extended to an isometry $\tilde{\alpha}\colon V \to V$.*

*Proof.* Since $U$ is nondegenerate we have that $W$ is also nondegenerate and we can apply Lemma 5.4 to obtain

$$V = U \oplus U^{\perp} = W \oplus W^{\perp}.$$

Furthermore Lemma 5.3 implies $\dim U^{\perp} = \dim W^{\perp}$. Hence, by Corollary 5.7, there is an isometry $\beta\colon U^{\perp} \to W^{\perp}$. Finally, $\alpha \oplus \beta\colon V \to V$ is an isometry extending $\alpha$. □

With this preparation we can now come to the main result.

**Corollary 5.10** (Witt's Lemma)**.** *The map $\alpha$ can be extended to an isometry $\tilde{\alpha}\colon V \to V$.*

*Proof.* Using Lemma 5.8, extend $\alpha$ to $\tilde{\alpha}\colon U' \to W'$ for some $U' \geq U$, $W' \geq W$ nondegenerate. Now apply Lemma 5.9 to extend $\tilde{\alpha}$ to $\tilde{\tilde{\alpha}}\colon V \to V$. □

Now that we understand symplectic vector spaces and can extend isometries we are well equipped for the next section, where will show that the limit of symplectic groups is also extremely amenable.

## 6 Symplectic Groups and Extreme Amenability

Our goal in this section is to show that $\operatorname{clim} \mathrm{Sp}_{2^n}(q)$ is extremely amenable. The structure of the proof is the same as in Section 4 with special linear groups. We will bound the length of $\mathrm{Sp}_n(q) \cong \mathrm{Aut}(V, \omega)$ by applying Corollary 3.14 to a sequence of subgroups $(G_i)_{0 \leq i \leq n}$. To bound the diameter of $G_i/G_{i-1}$ we will construct for any $g \in G_i$ an $h' \in G_i$ such that the distance between $g$ and $h'g$ is small and $h'g \in G_{i-1}$. The $h'$ will behave like the inverse of $g$ on a small subspace of $V$ and like the identity on most of the rest. The proof can be generalized to unitary and orthogonal groups. Therefore we will briefly introduce those groups and afterwards show extreme amenability for the limits of symplectic, unitary, and orthogonal groups.

**Definition 6.1.** Let $V$ be a finite dimensional $\mathbb{F}_q$-vector space and $\omega$ a nondegenerate map from $V \times V$ to $\mathbb{F}_q$.

Then $(V, \omega)$ is an *orthogonal vector space* if $\omega$ is bilinear, $\omega(x, y) = \omega(y, x)$ for all $x, y \in V$, and if $q = 2$ then $\omega(x, x) = 0$ for all $x \in V$.

And $(V, \omega)$ is a *unitary vector space* if there is an $h \in \mathrm{Aut}(\mathbb{F}_q)$ with $h^2 = 1$ such that

$$\begin{aligned} \omega(ax + y, z) &= a \cdot \omega(x, z) + \omega(y, z) \\ \omega(x, ay + z) &= h(a) \cdot \omega(x, y) + \omega(x, z) \\ \omega(x, y) &= h(\omega(y, x)) \end{aligned}$$

for all $x, y, z \in V$ and $a \in \mathbb{F}_q$.

*Orthogonal* and *unitary groups* are the automorphism groups of unitary and orthogonal vector spaces, respectively.

In the following let $(V, \omega)$ be a symplectic, unitary, or orthogonal vector space. Note that in any case $\omega$ is nondegenerate and

$$\omega(x, y) = 0 \text{ iff } \omega(y, x) = 0 \text{ for all } x, y \in V.$$

Obviously, Lemmas 5.3 and 5.4 from the previous section still hold in unitary and orthogonal vector spaces. Furthermore, Witt's Lemma also holds in unitary and orthogonal vector spaces, for a proof see [1].

**Theorem 6.2** (Witt's Lemma)**.** *Let $(V, \omega)$ be a symplectic, unitary, or orthogonal vector space and $\alpha \colon U \to W$ be an isometry between subspaces $U, W \leq V$. Then $\alpha$ can be extended to an isometry $\tilde{\alpha} \colon V \to V$.*

The next lemma is necessary to construct the chain of subgroups, in the case of symplectic vector spaces it is a trivial consequence of Theorem 5.6.

**Lemma 6.3.** *There exists a $U \leq V$ with $\dim U \leq 2$ such that $V = U \oplus U^\perp$.*

*Proof.* Let $r \in V \setminus \{0\}$. By Lemma 5.3 $\dim r^\perp = n - 1$.

If $r \notin r^\perp$, then $V = \langle r \rangle \oplus r^\perp$ and $\langle r \rangle$ is the desired $U$.

If $r \in r^\perp$, then let $H \leq r^\perp$ such that $\langle r \rangle \oplus H = r^\perp$. Now proceed as in the proof of Theorem 5.6 to show that $H^\perp$ is a suitable $U$. □

The following lemma shows that isometries interact nicely with complements.

**Lemma 6.4.** *Let $U \leq V$ and $\alpha \colon V \to V$ be an isometry such that $\alpha(U) = U$. Then $\alpha(U^\perp) = U^\perp$.*

*Proof.* As $\dim \alpha(U^\perp) = \dim U^\perp$ it suffices to show that $\alpha(u') \perp u$ for all $u \in U$ and $u' \in U^\perp$. Let $v \in U$ with $\alpha(v) = u$. Then

$$\begin{aligned}\omega(\alpha(u'), u) &= \omega(\alpha(u'), \alpha(v)) \\ &= \omega(u', v) \\ &= 0.\end{aligned}$$

This concludes the proof. □

The next lemma gives us a large subspace on which $h'$ can be the identity without interfering with the part where it is the inverse of $g$.

**Lemma 6.5.** *For all $W \leq V$ there is a $W' \leq W^\perp$ such that $W \cap W' = 0$ and*

$$\dim W' \geq \dim V - 2\dim W.$$

*Proof.* Let $W' \leq W^\perp$ such that

$$W^\perp = (W^\perp \cap W) \oplus W'.$$

Clearly, $W \cap W' = 0$ and

$$\begin{aligned}\dim W' &= \dim W^\perp - \dim(W^\perp \cap W) \\ &\geq \dim W^\perp - \dim W \\ &= \dim V - \dim W - \dim W. && \text{(Lemma 5.3)}\end{aligned}$$

This concludes the proof. □

Now we can proof the analogue of Theorem 4.1 from Section 4.

**Theorem 6.6.** *Let $G$ be a symplectic, unitary, or orthogonal group equipped with the rank metric $d$. Let $n$ be the diameter of $G$. Then there is a symplectic, unitary, or orthogonal subgroup $H \leq G$ with diameter at most $n-1$ such that the diameter of $G/H$ with respect to the factor metric is at most $8$.*

*Proof.* We have $G \cong \mathrm{Aut}(V, \omega)$ for some $n$-dimensional vector space $V$ and some $\omega$. Use Lemma 6.3 to obtain $U \leq V$ such that $V = U \oplus U^\perp$ and $\dim U \leq 2$. Define $H = \mathrm{Aut}(U^\perp, \omega)$. Our aim is to find for any $g \in G$ an $\tilde{g} \in H$ such that $d(g, \tilde{g}) \leq 8$. The idea is to find a map $h' \in G$ that behaves like the inverse of $g$ on $gU$ and like the identity on most of the rest. Then $h'g$ is the desired $\tilde{g}$.

Let $g \in G$ and define $W = \langle U, gU \rangle$. By Lemma 6.5, there is a $W'$ such that $\dim W' \geq n - 8$, $W' \leq W^\perp$, and $W' \cap W = 0$. Consider the map

$$g^{-1}|_{gU} \oplus 1_{W'} \colon gU \oplus W' \to U \oplus W'$$

since $g^{-1}|_{gU}$ and $1_{W'}$ are isometries and $W \perp W'$ we have that the above map is also an isometry. By Theorem 6.2, this isometry can be extended to an isometry $h' \colon V \to V$. Furthermore

$$\begin{aligned} d(g, h'g) &= \dim \operatorname{im}(g - h'g) \\ &\leq 8 + \dim \operatorname{im}(g - h'g)|_{W'} && (\dim W' \geq n - 8) \\ &= 8 + \dim \operatorname{im}(g - g)|_{W'} && (h'|_{W'} = 1_{W'}) \\ &= 8. \end{aligned}$$

Finally, we need to show that $h'g \in H$, here the choice of $H$ using Lemma 6.3 comes into play. By construction of $h'$ we have that $h'g|_U = 1_U$. Therefore we can apply Lemma 6.4 and deduce that $h'g(U^{\perp}) = U^{\perp}$. Hence $h'g \in H$ and $d(g, h'g) \leq 8$. □

**Corollary 6.7.** *Let $G = \operatorname{Aut}(V, \omega)$ be a symplectic, unitary, or orthogonal group equipped with the normalized rank metric $d$ and the normalized counting measure $\mu$, where $V$ is $n$-dimensional. Then the length of $G$ is at most $8n^{-\frac{1}{2}}$ and for all $\varepsilon > 0$*

$$\alpha_{(G,d,\mu)}(\varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right).$$

*Proof.* Applying Theorem 6.6 multiple times gives us a sequence of subgroups $\{e\} = G_0 \leq \cdots \leq G_m = G$ such that $m \leq n$ and the diameter of $G_i / G_{i-1}$ is at most $\frac{8}{n}$. Now we can use Corollary 3.14 to obtain the desired upper bound. □

Observe that, as in Section 4, we can apply Lemma 4.4 and obtain $n^{-\frac{1}{2}}$ as a lower bound for the length of $G$. Now we can prove the main result of this thesis.

**Corollary 6.8.** *Let $(V_0, \omega_0) \subset (V_1, \omega_1) \subset \ldots$ be a sequence of $\mathbb{F}_q$ vector spaces such that $(V_n, \omega_n)$ is a symplectic, unitary, or orthogonal vector space of dimension $2^n$ and $\omega_{n+1}|_{V_n} = \omega_n$ for all $n \in \mathbb{N}$. Let $G_n = \operatorname{Aut}(V_n, \omega_n)$ equipped with the normalized rank metric $d_n$ and the normalized counting measure $\mu_n$. Then*

$$\lim_{n \to \infty} \alpha_{(G_n, d_n, \mu_n)}(\varepsilon) = 0$$

*for all $\varepsilon > 0$ and $\operatorname{clim} G_n$ is extremely amenable.*

*Proof.* Immediate from Corollary 6.7 and Theorem 2.14. □

## 7 Symplectic Groups and Ramsey Theory

In this section we will use the upper bound obtained for the length of symplectic, unitary, and orthogonal groups to deduce a Ramsey theoretic result. As in Section 6, the results from this section are already shown in [3] for special linear groups.

The first lemma is very similar to Theorem 3.11.

**Lemma 7.1** (Lemma **2.7** in [3]). *Let $(X, d, \mu)$ be a finite mm-space with length $l$. Then for every $\varepsilon > 0$ and $A \subseteq X$ with $\mu(A) > 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right)$ we have*

$$\mu(N_\varepsilon(A)) \geq 1 - 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

A covering $\mathcal{U}$ of a metric space $(X, d)$ is an *$\varepsilon$-covering* if for every $x \in X$ the $\varepsilon$-neighborhood of $x$ is contained in some $U \in \mathcal{U}$.

**Theorem 7.2.** *Let $\varepsilon > 0$, $k, m \in \mathbb{N}$. Define $N := 16 \cdot 64\varepsilon^{-2} \cdot \max\{\ln(2k), \ln(2m)\}$ and let $G = \operatorname{Aut}(V, \omega)$, where $(V, \omega)$ is a symplectic, unitary, or orthogonal vector space of dimension $n > N$, with an $\varepsilon$-cover $\mathcal{U}$ of cardinality at most $m$. Then there is a $U \in \mathcal{U}$ such that for all $F \subseteq G$ satisfying $|F| \leq k$ there is a $g \in G$ with $gF \subseteq U$.*

Intuitively the theorem says that whenever we color $G$ with $m$ colors, where a single element can have multiple colors, such that all elements of $\varepsilon$-balls have at least one color in common, then there is one color $c$ such that for every $F$, with at most $k$ elements, there is a $g$ where the elements of $gF$ all have the color $c$.

*Proof.* Look at $G$ as the usual mm-space with normalized rank metric and normalized counting measure. Let $l$ be the length of $G$. Observe that, by Corollary 6.7, $l \leq 8n^{-\frac{1}{2}}$. For $U \in \mathcal{U}$ define $\operatorname{Core}(U) := \{x \in U \mid N_\varepsilon(x) \subseteq U\}$. Since $\mathcal{U}$ is an $\varepsilon$-covering we have $\bigcup_{U \in \mathcal{U}} \operatorname{Core}(U) = G$. Therefore there is a $U \in \mathcal{U}$ such that $\mu(\operatorname{Core}(U)) \geq \frac{1}{m}$. As $n > 16 \cdot 64\varepsilon^{-2} \cdot \ln(2m)$ we have

$$\frac{1}{m} > 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right) \geq 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

Now we can apply Lemma 7.1 to $\operatorname{Core}(U)$ and obtain

$$\mu(U) \geq \mu(N_\varepsilon(\operatorname{Core}(U))) \geq 1 - 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right) \geq 1 - 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right).$$

Let $F \subseteq G$ with $|F| \leq k$. Note that

$$\{g \in G \mid gF \subseteq U\} = \bigcap_{h \in F} \{g \in G \mid gh \in U\} = \bigcap_{h \in F} Uh^{-1}.$$

Therefore, $\mu(\{g \in G \mid gF \subseteq U\}) \geq 1 - k \cdot 2\exp\left(-\frac{\varepsilon^2 n}{16 \cdot 64}\right)$. By assumption $n > 16 \cdot 64\varepsilon^{-2} \cdot \ln(2k)$, hence $\mu(\{g \in G \mid gF \subseteq U\}) > 0$ and there is a suitable $g$. $\square$

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## ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Topological Entropy of Formal Languages" selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

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