Bachelorarbeit

Topological Entropy of Formal Languages

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October 5, 2017

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1 Martingale Technique

Definition 1. A *martingale* is a family $(f_i, \mathcal{F}_i)_{i\in\{0,\ldots,n\}}$ such that

- f_i is integrable for all $i \in \{0, \ldots, n\}$,
- f_i is $\mathcal{F}_i$ measurable for all $i \in \{0, \ldots, n\}$, and
- $f_i = \mathbb{E}[f_{i+1}|\mathcal{F}_i]$ for all $i \in \{0, \ldots, n-1\}$.

Lemma 2. *For all $x \in \mathbb{R}$*

$$e^x \leq x + e^{x^2}.$$

Lemma 3 (Azema's inequality)**.**

$$\mu(\{x \in X \mid |f(x) - \mathbb{E}(f)| \geq c\}) \leq 2\exp\left(-\frac{c^2}{4\sum_{i=1}^n \|d_i\|_\infty^2}\right)$$

$$\mu(\{|f - \mathbb{E}(f)| \geq c\}) \leq 2\exp\left(-\frac{c^2}{4\sum_{i=1}^n \|d_i\|_\infty^2}\right)$$

Definition 4. Let (X, d, μ) be an mm-space.

Theorem 5. *If an mm-space (X, d, μ) has length l, then the concentration function of X satisfies*

$$\alpha_X(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16l^2}\right).$$

Theorem 6. *Let G be a compact group with a bi-invariant metric d, and let*

$$\{e\} = G_0 < G_1 < \cdots < G_n = G$$

be a chain of subgroups. Denote the diameter of G_i/G_{i-1} with respect to the factor metric by a_i. Then the concentration function of the mm-space (G, d, μ), where μ is the normalized Haar measure, satisfies

$$\alpha_X(\varepsilon) \leq 2\exp\left(-\frac{\varepsilon^2}{16\sum_{i=1}^n a_i^2}\right).$$

Theorem 7. *The normalized counting measure on the groups $\mathrm{SL}_{2^n}(q)$ concentrates with respect to the rank-metric, i.e. for all $r > 0$*

$$\lim_{n\to\infty} \alpha_{\mathrm{SL}_{2^n}}(r) = 0.$$

Lemma 8. *Let (X, d, μ) be an mm-space with diameter d and*

Florian sagt: "could be that this only holds for finite X as conditions in definition of length are just almost surely"

$$\Omega_0 = \{X\} \prec \cdots \prec \Omega_n = \{\{x\} \mid x \in X\}$$

with $a_1, \ldots, a_n$ as in Definition 4. Then

$$\sum_{i=1}^n a_i \geq d.$$

Proof. Let $x, y \in X$, with $x \neq y$, we show $d(x,y) \leq \sum_{i=1}^{n} a_i$. Let i_0 be the smallest number such that $[x]_{i_0} \neq [y]_{i_0}$. Since $[x]_0 = X = [y]_0$ we know that i_0 is at least 1. Therefore $[x]_{i_0-1} = [y]_{i_0-1}$ and there is an isomorphism $\varphi_{i_0} \colon [x]_{i_0} \to [y]_{i_0}$ such that $d(\varphi_{i_0}(x), y) \leq a_{i_0}$. Let $x_{i_0} = \varphi_{i_0}(x)$, then

Florian sagt: "here is the a.s. problem"

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, y).$$

If $x_{i_0} = y$, then we are done. Otherwise let i_1 be the smallest number such that $[x_{i_0}]_{i_1} \neq [y]_{i_1}$. Then let $\varphi_{i_1} \colon [x_{i_0}]_{i_1} \to [y]_{i_1}$ be an isomorphism such that $d(\varphi_{i_1}(x_{i_0}), y) \leq a_{i_1}$. Define $x_{i_1} = \varphi_{i_1}(x_{i_0})$. Proceeding in this fashion yields elements $x_{i_0}, \ldots, x_{i_k}$ such that $x_{i_k} = y$ and

$$d(x,y) \leq d(x, x_{i_0}) + d(x_{i_0}, x_{i_1}) + \cdots + d(x_{i_{k-1}}, x_{i_k}) \leq a_{i_0} + \cdots + a_{i_k} \leq \sum_{i=1}^{n} a_i.$$

$\square$

Lemma 9. *Let (X, d, μ) be an mm-space with diameter 1 and $\Delta = \min d$. Then the length of X is at least $\Delta^{\frac{1}{2}}$.*

Definition 10. The *symplectic group* of degree $2n$ over a field q, denoted by $\mathrm{Sp}(2n, q)$, is the subgroup of $\mathrm{SL}(2n, q)$ containing all matrices A such that

$$A^T \Omega A = \Omega, \text{ where } \Omega = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}.$$

Lemma 11. *Let $g \colon V \to V$ be an isomorphism, $V = U \oplus U'$, and $g(U') \subseteq U'$. Then the map*

$$\begin{aligned} g' \colon V &\to V \\ v &\mapsto \begin{cases} g(v) - \pi_{U'}(g(v)) & \text{if } v \notin U' \\ v & \text{if } v \in U' \end{cases} \end{aligned}$$

i.e. $g' = \pi_U \circ g - \pi_U \circ g \circ 1_{U'} + 1_{U'}$, is an isomorphism and $d(g, g') \leq \frac{1}{n} \cdot \dim U'$.

Lemma 12. *[what we still need (add conditions for ω if necessary)] Let $\omega \colon V \times V \to k$ be a bilinear map, U, U' subspaces of V, and $h \colon U \to U'$ an isomorphism that preserves ω. Then h can be extended to an isomorphism on V which also preserves ω.*

Proof. w.l.o.g. $\dim U + 1 = \dim V$? $\square$

Florian sagt: "…additional conditions"

Lemma 13. *Let $V = U \oplus U'$, ω a bilinear map, G be the group of automorphisms of (V, ω) and $G' \leq G$ the subgroup fixing U'. Then the diameter of G/G_i is at most $\frac{3 \cdot \dim U'}{n}$.*

Florian sagt: "adapt this"

Proof. Let $g \in G$, we show that there are $g' \in G$ and $g'' \in G'$ such that $g'(U') \subseteq U'$, $g'|_{U'} = 1_{U'}$, and

$$d(g, g'') \leq d(g, g') + d(g', g'') \leq \frac{2 \dim U'}{n} + \frac{\dim U'}{n}.$$

By Lemma 12 we can extend the map $g^{-1}|_{gU'}$ to a map h' on $V' = \langle U', gU' \rangle$. Now define $g' = (1_{V''} \oplus h')g$, where $V = V'' \oplus V'$ and apply Lemma to g' to obtain g''.

$$\begin{aligned}
\operatorname{im} g - g' &= \operatorname{im} g - (1_{V''} \oplus h')g \\
&= \operatorname{im}(1_{V''} \oplus 1_{V'} - 1_{V''} \oplus h') \\
&= \operatorname{im}(1_{V'} - h') \\
&\subseteq V'
\end{aligned}$$

$$d(g, g') = \frac{1}{n} \dim \operatorname{im} g - g' \leq \frac{\dim V'}{n}$$

$\square$

2 Fun

Consider an n-dimensional cube with 2^k nodes on each edge. Then its diameter $\nabla_{n,k}$ and length $L_{n,k}$ are

$$\nabla_{n,k} = \sqrt{(2^k - 1) \cdot n} \qquad\qquad L_{n,k} = \sqrt{\sum_{i=0}^{k-1} 2^{2i} \cdot n}.$$

Henceforth

$$\lim_{n\to\infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{L_{1,k}}{\nabla_{1,k}} \qquad \text{and} \qquad \lim_{k\to\infty} \frac{L_{n,k}}{\nabla_{n,k}} = \frac{1}{\sqrt{3}}.$$

ERKLÄRUNG

Hiermit erkläre ich, dass ich die am heutigen Tag eingereichte Diplomarbeit zum Thema "Topological Entropy of Formal Languages" selbstständig erarbeitet, verfasst und Zitate kenntlich gemacht habe. Andere als die angegebenen Hilfsmittel wurden von mir nicht benutzt.

Datum Unterschrift