

Inhabitation in $\lambda 2$

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A $\lambda 2$ type is inhabited in $\lambda 2$ iff there is a closed $\lambda 2$ term of this type. The inhabitation problem in $\lambda 2$ is to determine whether a given $\lambda 2$ type is inhabited. This work gives a formal proof for the fact that the inhabitation problem in $\lambda 2$ is undecidable.

1 Introduction

We will only consider the explicitly typed $\lambda 2$ calculus (Church style), so whenever we speak of $\lambda 2$ terms we know that the type information is given explicitly. Let us take a look at the problem. It is clear that there are closed $\lambda 2$ terms to which no $\lambda 2$ type can be assigned (e.g. to the term $(\lambda x : \forall \alpha \alpha. xx)(\lambda x : \forall \alpha \alpha. xx)$ no type can be assigned). But there are also $\lambda 2$ types which can not be assigned to any closed $\lambda 2$ term. We say that these types are empty. For instance the $\lambda 2$ type $\forall \alpha \alpha$ is empty.

The constructions presented here are mainly based on [2] but the proofs go much more into detail.

2 Basic Definitions

2.1 Conventions

For variable names we will use the following conventions.

$\lambda 2$ types: $t, t', t'', t_1, t_2, \dots, s, s_1, s_2, \dots$

$\lambda 2$ terms: $M, M', M_1, M_2, \dots, N, N', N_1, N_2, \dots$

first-order terms: t, t_1, t_2, \dots

first-order formulas: $\varphi, \varphi_1, \varphi_2, \psi, \psi'$

type-variables: $p, p_1, p_2, \alpha, a, \alpha_1, a_1, \alpha_2, a_2, \dots, \beta, b, \beta_1, b_1, \beta_2, b_2, \dots$

value-variables: x, y, z, x_1, x_2, \dots

predicate-symbols: P, P^1, P^2, \dots

P-variables: $\alpha, a, \alpha_1, a_1, \alpha_2, a_2, \dots, \beta, b, \beta_1, b_1, \beta_2, b_2, \dots$

P-formulas: $A, A', B, B', A_1, A_2, \dots$

states: $Q, Q', \hat{Q}, Q_f, Q_0, Q_1, Q_2, \dots$

If possible we will use Greek letters for bound type-variables and Latin letters for free type-variables.

2.2 λ -calculus $\lambda 2$

In the following let $\mathcal{V}_T = \{\alpha, a, \beta, b, \dots\}$ be a countably infinite set (of type-variables) and $\mathcal{V}_V = \{x, x_1, x_2, \dots\}$ be a countably infinite set (of value-variables).

Definition 1. The set of all $\lambda 2$ types over \mathcal{V}_T , denoted by $T_{\lambda 2}$, is the smallest set T satisfying the following conditions:

- $\mathcal{V}_T \subseteq T$,
- if $t_1, t_2 \in T$ then $(t_1 \rightarrow t_2) \in T$, and
- if $t \in T$ and $\alpha \in \mathcal{V}_T$ then $\forall \alpha. t \in T$.

The set of all $\lambda 2$ terms over \mathcal{V}_T and \mathcal{V}_V , denoted by $\Lambda_{T_{\lambda 2}}$, is the smallest set Λ_T satisfying the following conditions:

- $\mathcal{V}_V \subseteq \Lambda_T$,
- if $M_1, M_2 \in \Lambda_T$ then $M_1 M_2 \in \Lambda_T$,
- if $x \in \mathcal{V}_V, t \in T_{\lambda 2}$, and $M \in \Lambda_T$ then $\lambda x : t. M \in \Lambda_T$,
- if $\alpha \in \mathcal{V}_T$ and $M \in \Lambda_T$ then $\Lambda \alpha. M \in \Lambda_T$, and
- if $M \in \Lambda_T$ and $t \in T_{\lambda 2}$ then $M t \in \Lambda_T$.

If we have a type of the form $(t_1 \rightarrow (t_2 \rightarrow (\dots \rightarrow (t_{n-1} \rightarrow t_n) \dots)))$ we will often omit the brackets and just write $(t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t_{n-1} \rightarrow t_n)$ or $t_1 \rightarrow t_2 \rightarrow \dots \rightarrow t_{n-1} \rightarrow t_n$ instead.

Definition 2. Let $M, N \in \Lambda_{T_{\lambda 2}}$ and $x \in \mathcal{V}_V$. The substitution of x by N in M , denoted by $M[x := N]$ is defined as follows:

$$M[x := N] = \begin{cases} N & \text{if } M = x \\ y & \text{if } M = y \text{ and } y \neq x \\ (M_1[x := N])(M_2[x := N]) & \text{if } M = M_1 M_2 \\ \lambda x : t. M' & \text{if } M = \lambda x : t. M' \\ \lambda y : t. (M'[x := N]) & \text{if } M = \lambda y : t. M' \text{ and } y \neq x \\ \Lambda \alpha. (M'[x := N]) & \text{if } M = \Lambda \alpha. M' \\ (M'[x := N]) t & \text{if } M = M' t \end{cases}$$

Let $t, t' \in T_{\lambda_2}$ and $a \in \mathcal{V}_T$. The substitution of a by t in t' , denoted by $t[a := t']$ is defined as follows:

$$t[a := t'] = \begin{cases} t' & \text{if } t = a \\ b & \text{if } t = b \text{ and } b \neq a \\ (t_1[a := t'] \rightarrow t_2[a := t']) & \text{if } t = t_1 \rightarrow t_2 \\ \forall a.t'' & \text{if } t = \forall a.t'' \\ \forall \beta.(t''[a := t']) & \text{if } t = \forall \beta.t'' \text{ and } \beta \neq a \end{cases}$$

Let $M \in \Lambda_{T_{\lambda_2}}$, $a \in \mathcal{V}_T$, and $t \in T_{\lambda_2}$. The substitution of a by t in M , denoted by $M[a := t]$ is defined as follows:

$$M[a := t] = \begin{cases} x & \text{if } M = x \\ (M_1[a := t])(M_2[a := t]) & \text{if } M = M_1 M_2 \\ \lambda x : t'[a := t].(M'[a := t]) & \text{if } M = \lambda x : t'.M' \\ M & \text{if } M = \Lambda a.M' \\ \Lambda \beta.(M'[a := t]) & \text{if } M = \Lambda \beta.M' \text{ and } \beta \neq a \\ (M'[a := t]) t[a := t] & \text{if } M = M' t \end{cases}$$

In the following we will often abbreviate $(\dots (M[a_n := b_n]) \dots)[a_1 := b_1]$ to $M[\vec{a} := \vec{b}]$ where $\vec{a} = a_1 \dots a_n$ and $\vec{b} = b_1 \dots b_n$.

Definition 3. Let $M \in \Lambda_{T_{\lambda_2}}$. The set of free variables of M , denoted by $FV(M)$, is defined inductively as follows:

$$FV(M) = \begin{cases} \{x\} & \text{if } M = x \\ FV(M_1) \cup FV(M_2) & \text{if } M = M_1 M_2 \\ FV(M') \setminus \{x\} & \text{if } M = \lambda x : t.M' \\ FV(M') & \text{if } M = \Lambda \alpha.M' \\ FV(M') & \text{if } M = M' t \end{cases}$$

The set of bound variables of M , denoted by $BV(M)$, is defined as follows:

$$BV(M) = \begin{cases} \emptyset & \text{if } M = x \\ BV(M_1) \cup BV(M_2) & \text{if } M = M_1 M_2 \\ BV(M') \cup \{x\} & \text{if } M = \lambda x : t.M' \\ BV(M') & \text{if } M = \Lambda \alpha.M' \\ BV(M') & \text{if } M = M' t \end{cases}$$

Definition 4. Let $t \in T_{\lambda_2}$. The set of free type-variables of t , denoted by $FV(t)$, is defined inductively as follows:

$$FV(t) = \begin{cases} \{a\} & \text{if } t = a \\ FV(t_1) \cup FV(t_2) & \text{if } t = t_1 \rightarrow t_2 \\ FV(t') \setminus \{\alpha\} & \text{if } t = \forall \alpha.t' \end{cases}$$

The set of bound type-variables of t , denoted by $BV(t)$, is defined inductively as follows:

$$BV(t) = \begin{cases} \emptyset & \text{if } t = a \\ BV(t_1) \cup BV(t_2) & \text{if } t = t_1 \rightarrow t_2 \\ BV(t') \cup \{\alpha\} & \text{if } t = \forall \alpha. t' \end{cases}$$

Now we can lift this definition to terms.

Definition 5. Let $M \in \Lambda_{T_{\lambda_2}}$. The set of free type-variables of M , denoted by $FTV(M)$, is the union of all sets of free type-variables of types occurring in M .

The set of bound type-variables of M , denoted by $BTV(M)$, is the union of all sets of bound type-variables of types occurring in M .

Definition 6. The β -reduction, denoted by \rightarrow_β , is a binary relation on $\Lambda_{T_{\lambda_2}}$. For all $M, N \in \Lambda_{T_{\lambda_2}}$, $x \in \mathcal{V}_V$, $t \in T_{\lambda_2}$, and $\alpha \in \mathcal{V}_T$ if $BV(M) \cap FV(N) = \emptyset$ then $(\lambda x : t. M)N \rightarrow_\beta M[x := N]$ and from $BTV(M) \cap FTV(N) = \emptyset$ it follows that $(\Lambda \alpha. M)t \rightarrow_\beta M[\alpha := t]$.

The α_1 -conversion, denoted by \rightarrow_{α_1} , is a binary relation on $\Lambda_{T_{\lambda_2}}$. For all $M \in \Lambda_{T_{\lambda_2}}$, $x, x' \in \mathcal{V}_V$, $t \in T_{\lambda_2}$, and $\alpha, \beta \in \mathcal{V}_T$ if $x' \notin FV(M) \cup BV(M)$ then $\lambda x : t. M \rightarrow_{\alpha_1} \lambda x' : t. (M[x := x'])$ and from $\beta \notin FTV(M) \cup BTV(M)$ it follows that $\Lambda \alpha. M \rightarrow_{\alpha_1} \Lambda \beta. (M[\alpha := \beta])$.

The α_2 -conversion, denoted by \rightarrow_{α_2} , is a binary relation on T_{λ_2} . For all $t \in T_{\lambda_2}$, and $\alpha, \beta \in \mathcal{V}_T$ if $\beta \notin FV(t) \cup BV(t)$ then $\forall \alpha. t \rightarrow_{\alpha_2} \forall \beta. (t[\alpha := \beta])$.

Note that right now we are not able to reduce terms within a context (e.g there is no $M \in \Lambda_{T_{\lambda_2}}$ such that $\lambda x : t. (\lambda y : t. y)x \rightarrow_\beta M$).

Definition 7. So, for a binary relation \rightarrow on $\Lambda_{T_{\lambda_2}}$ we define the closure of \rightarrow under term contexts, denoted by $\text{cl}_T(\rightarrow)$, as the smallest binary relation \Rightarrow on $\Lambda_{T_{\lambda_2}}$ containing \rightarrow such that for all $N, M, M' \in \Lambda_{T_{\lambda_2}}$, $x \in \mathcal{V}_V$, $t \in T_{\lambda_2}$, and $\alpha \in \mathcal{V}_T$. If $M \Rightarrow M'$ then

$$\begin{array}{lll} MN \Rightarrow M'N & \lambda x : t. M \Rightarrow \lambda x : t. M' & Mt \Rightarrow M't \\ NM \Rightarrow NM' & \Lambda \alpha. M \Rightarrow \Lambda \alpha. M' & \end{array}$$

For a binary relation \rightarrow on T_{λ_2} we define the closure of \rightarrow under type contexts, denoted by $\text{cl}_Y(\rightarrow)$, as the smallest binary relation \Rightarrow on T_{λ_2} containing \rightarrow such that for all $s, t, t' \in T_{\lambda_2}$, and $\alpha \in \mathcal{V}_T$. If $t \Rightarrow t'$ then

$$\forall \alpha. t \Rightarrow \forall \alpha. t' \quad t \rightarrow s \Rightarrow t' \rightarrow s \quad s \rightarrow t \Rightarrow s \rightarrow t'$$

Definition 8.

We define \Rightarrow_β as $\text{cl}_T(\rightarrow_\beta)$.

And we define \Rightarrow_α as union of $\text{cl}_T(\rightarrow_{\alpha_1})$ and $\text{cl}_T(\Rightarrow_{\alpha_2})$ where

$$\Rightarrow_{\alpha_2} := \{(Mt, Mt'), (\lambda x : t. M, \lambda x : t'. M) \mid M \in \Lambda_{T_{\lambda_2}}, x \in \mathcal{V}_V, (t, t') \in \text{cl}_Y(\rightarrow_{\alpha_2})\}$$

Finally we define \Rightarrow_λ as $\Rightarrow_\alpha^* \circ \Rightarrow_\beta$.

Definition 9. Let $M \in \Lambda_{T_{\lambda_2}}$. The term M is in normal form if there is no $N \in \Lambda_{T_{\lambda_2}}$ such that $M \Rightarrow_{\lambda} N$.

M is weakly normalizing if there exists an $N \in \Lambda_{T_{\lambda_2}}$ such that N is in normal form and $M \Rightarrow_{\lambda} N$.

The term M is called strongly normalizing if there is no infinite chain $M \Rightarrow_{\lambda} M_1 \Rightarrow_{\lambda} M_2 \dots$.

Definition 10. Let $\mathcal{V} = \{x_1, \dots, x_n\}$ be a finite subset of \mathcal{V}_V such that $x_i \neq x_j$ for $1 \leq i < j \leq n$ and $t_1, \dots, t_n \in T_{\lambda_2}$. A **$\lambda 2$ -basis** $\Gamma = \{(x_1 : t_1), \dots, (x_n : t_n)\}$ is a mapping from \mathcal{V} to T_{λ_2} . If the kind of basis is clear from the context we abbreviate **$\lambda 2$ -basis** to **basis**.

The free variables of a basis Γ , denoted by $FV(\Gamma)$, are $\bigcup \{FV(t) \mid (x : t) \in \Gamma\}$.

For a basis Γ and another basis Σ such that $\text{dom}(\Gamma) \cap \text{dom}(\Sigma) = \emptyset$, $x \in \mathcal{V}_V \setminus \text{dom}(\Gamma)$, and $t \in T_{\lambda_2}$ we will abbreviate $\Gamma \cup \{(x : t)\}$ to $\Gamma, x : t$ and $\Gamma \cup \Sigma$ to Γ, Σ .

Definition 11. Let M be in $\Lambda_{T_{\lambda_2}}$, t in T_{λ_2} , and Γ be a basis. A statement $M : t$ is derivable from Γ , denoted by $\Gamma \vdash M : t$, if $M : t$ can be produced using the following rules.

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ -Introduction)	$\frac{\Gamma, x : t_1 \vdash M : t_2}{\Gamma \vdash \lambda x : t_1. M : t_1 \rightarrow t_2}$	
(λ -Elimination)	$\frac{\Gamma \vdash M_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash M_2 : t_1}{\Gamma \vdash M_1 M_2 : t_2}$	
(\forall -Introduction)	$\frac{\Gamma \vdash M : t}{\Gamma \vdash \Lambda \alpha. M : \forall \alpha. t}$	$\alpha \notin FV(\Gamma)$
(\forall -Elimination)	$\frac{\Gamma \vdash M : \forall \alpha. t}{\Gamma \vdash M t' : t[\alpha := t']}$	

Definition 12. A term $M \in \Lambda_{T_{\lambda_2}}$ is well typed if there exists a basis Γ and a type $t \in T_{\lambda_2}$ such that $\Gamma \vdash M : t$ holds.

The following two theorems are well known (for formal proofs see [1]).

Theorem 13. Let M, M' be in $\Lambda_{T_{\lambda_2}}$ and $M \Rightarrow_{\alpha}^* M'$ or $M \Rightarrow_{\beta}^* M'$, t in T_{λ_2} , and Γ be a basis. If $\Gamma \vdash M : t$ then $\Gamma \vdash M' : t$.

Theorem 14. All well typed **$\lambda 2$** terms are strongly normalizing.

Definition 15. The inhabitation problem for **$\lambda 2$** , denoted by **INHAB**, is defined as follows. Given a **$\lambda 2$** type t .

Is there a **$\lambda 2$** term M such that $\emptyset \vdash M : t$?

But we can rephrase this problem so that it becomes more general: Given a basis Γ and a $\lambda\mathbf{2}$ type t .

Is there a $\lambda\mathbf{2}$ term M such that $\Gamma \vdash M : t$?

Obviously the second version is a special case of the first one. For the other direction consider a basis $\Gamma = \{(x_1 : t_1), \dots, (x_n : t_n)\}$ and a $\lambda\mathbf{2}$ type t . Clearly, for every term M , $\Gamma \vdash M : t$ holds iff $\emptyset \vdash \lambda x_1 : t_1 \dots \lambda x_n : t_n. M : t_1 \rightarrow \dots \rightarrow t_n \rightarrow t$.

2.3 First-order logic

Definition 16. A ranked set is a tuple (Σ, rk) , where Σ is a countable set and $rk : \Sigma \rightarrow \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk) . The set of all elements in Σ with a certain rank k , denoted by $\Sigma^{(k)}$, is defined as $\Sigma^{(k)} := rk^{-1}(k)$.

For the remainder of this subsection let $\mathcal{V} = \{y, y_1, y_2, \dots\}$ be a countable set (of variables), \mathcal{F} a ranked set (of function symbols), and \mathcal{P} a ranked set (of predicate symbols).

Definition 17. The set of terms over \mathcal{V} and \mathcal{F} , denoted by $\mathcal{T}_{(\mathcal{V}, \mathcal{F})}$, is the smallest set \mathcal{T} satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and
- for every $k \in \mathbb{N}$, if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \dots, t_k) \in \mathcal{T}$.

The set of first-order formulas over \mathcal{V} , \mathcal{F} , and \mathcal{P} , denoted by $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$, is the smallest set \mathcal{L} satisfying the following conditions:

- for every $k \in \mathbb{N}$, if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}_{(\mathcal{V}, \mathcal{F})}$ then $P(t_1, t_2, \dots, t_k) \in \mathcal{L}$.
- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi \in \mathcal{L}$, and
- if $y \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists y.\varphi$, $\forall y.\varphi \in \mathcal{L}$.

We introduce an additional binary operation \rightarrow on formulas, where for some $\varphi, \psi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ the formula $(\varphi \rightarrow \psi)$ is defined as $(\neg\varphi \vee \psi)$, if we have a formula of the form $(\varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots \rightarrow (\varphi_{n-1} \rightarrow \varphi_n) \dots)))$ we will often omit the brackets and just write $(\varphi_1 \rightarrow \varphi_2 \rightarrow \dots \rightarrow \varphi_{n-1} \rightarrow \varphi_n)$ or $\varphi_1 \rightarrow \varphi_2 \rightarrow \dots \rightarrow \varphi_{n-1} \rightarrow \varphi_n$ instead.

For nullary relation symbols P we will abbreviate $P()$ to P . If a formula φ is of the form $Qy.\psi$ (where $Q \in \{\exists, \forall\}$, $y \in \mathcal{V}$, and $\psi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$) we often drop the dot and write $Qy\psi$ instead. If a formula φ has multiple variables bound by the same quantifier (i.e. $\varphi = Qy_1.Qy_2 \dots Qy_n.\psi$ for $Q \in \{\exists, \forall\}$, some $n \in \mathbb{N}$, $y_1, y_2, \dots, y_n \in \mathcal{V}$, and $\psi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$) we abbreviate φ to $Qy_1y_2 \dots y_n.\psi$ or to $Q\vec{y}.\psi$ where $\vec{y} = y_1y_2 \dots y_n$.

Definition 18. The set of variables of a term $t \in \mathcal{T}_{(\mathcal{V}, \mathcal{F})}$, denoted by $V(t)$, is defined by:

$$V(t) = \begin{cases} \{y\} & \text{if } t = y \\ V(t_1) \cup V(t_2) \cup \dots \cup V(t_k) & \text{if } t = f(t_1, t_2, \dots, t_k) \end{cases}$$

The set of free variables of a formula $\varphi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$, denoted by $\text{FV}(\varphi)$, is defined as follows:

$$\text{FV}(\varphi) = \begin{cases} \text{V}(t_1) \cup \text{V}(t_2) \cup \dots \cup \text{V}(t_k) & \text{if } \varphi = P(t_1, t_2, \dots, t_k) \\ \text{FV}(\psi) & \text{if } \varphi = \neg\psi \\ \text{FV}(\varphi_1) \cup \text{FV}(\varphi_2) & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \text{ or } \varphi = (\varphi_1 \vee \varphi_2) \\ \text{FV}(\psi) \setminus \{y\} & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \end{cases}$$

Definition 19. The set of subformulas of a formula $\varphi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$, denoted by $\text{SUB}(\varphi)$, is defined as follows:

$$\text{SUB}(\varphi) = \begin{cases} \{\varphi\} & \text{if } \varphi = P(t_1, t_2, \dots, t_k) \\ \{\varphi\} \cup \text{SUB}(\psi) & \text{if } \varphi = \neg\psi \\ \{\varphi\} \cup \text{SUB}(\varphi_1) \cup \text{SUB}(\varphi_2) & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \text{ or } \varphi = (\varphi_1 \vee \varphi_2) \\ \{\varphi\} \cup \text{SUB}(\psi) & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \end{cases}$$

Definition 20. We say that a formula $\varphi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ contains no dummy quantifiers if for all $\psi \in \text{SUB}(\varphi)$ of the form $\psi = \forall y.\psi'$ or $\psi = \exists y.\psi'$ for some $y \in \mathcal{V}$ and some $\psi' \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ we have that $y \in \text{FV}(\psi')$.

Definition 21. Let y be in \mathcal{V} and $t, t' \in \mathcal{T}_{(\mathcal{V}, \mathcal{F})}$. The substitution of y by t' in t , denoted by $t[y := t']$, is defined as follows:

$$t[y := t'] = \begin{cases} t' & \text{if } t = y \\ z & \text{if } t = z \text{ and } z \neq y \\ f(t_1[y := t'], \dots, t_k[y := t']) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Now we can lift this definition to formulas, let φ be in $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$. The substitution of y by t' in φ , denoted by $\varphi[y := t']$, is defined as follows:

$$\varphi[y := t'] = \begin{cases} P(t_1[y := t'], \dots, t_k[y := t']) & \text{if } \varphi = P(t_1, \dots, t_k) \\ \neg(\psi[y := t']) & \text{if } \varphi = \neg\psi \\ \varphi_1[y := t'] \circ \varphi_2[y := t'] & \text{if } \varphi = (\varphi_1 \circ \varphi_2), \circ \in \{\wedge, \vee\} \\ \varphi & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \\ Qz.(\psi[y := t']) & \text{if } \varphi = Qz.\psi, Q \in \{\forall, \exists\} \text{ and } z \neq y \end{cases}$$

Now we come to the semantics of first-order formulas.

Definition 22. An interpretation I over \mathcal{V} , \mathcal{F} , and \mathcal{P} is a triple $I = (\Delta, \cdot^I, \omega)$, where

- Δ is a nonempty set (which we call domain),
- \cdot^I is a function such that
 - $f^I: \Delta^k \rightarrow \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$ and
 - $P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $P \in \mathcal{P}^{(k)}$
- ω is a function from \mathcal{V} to Δ .

Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation, $y \in \mathcal{V}$, and $d \in \Delta$ the interpretation $I[y \mapsto d]$ is defined as $(\Delta, \cdot^I, \omega[y \mapsto d])$ where

$$(\omega[y \mapsto d])(z) = \begin{cases} d & \text{if } z = y \\ \omega(y) & \text{otherwise.} \end{cases}$$

Definition 23. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and t a term. The interpretation of t under I , denoted by t^I , is defined as follows:

$$t^I = \begin{cases} \omega(y) & \text{if } t = y \\ f^I(t_1^I, \dots, t_k^I) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Let φ be a formula. The interpretation of φ under I , denoted by φ^I , is defined recursively as follows:

$$\varphi^I = \begin{cases} \top & \text{if } \varphi = P(t_1, \dots, t_k) \text{ and } (t_1^I, \dots, t_k^I) \in P^I \\ \perp & \text{if } \varphi = P(t_1, \dots, t_k) \text{ and } (t_1^I, \dots, t_k^I) \notin P^I \\ \text{not } \psi^I & \text{if } \varphi = \neg\psi \\ \varphi_1^I \text{ and } \varphi_2^I & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \\ \varphi_1^I \text{ or } \varphi_2^I & \text{if } \varphi = (\varphi_1 \vee \varphi_2) \\ \text{exists } d \in \Delta : \psi^I[y \mapsto d] & \text{if } \varphi = \exists y. \psi \\ \text{forall } d \in \Delta : \psi^I[y \mapsto d] & \text{if } \varphi = \forall y. \psi \end{cases}$$

The interpretation I is a model of φ , denoted by $I \models \varphi$, if $\varphi^I = \top$.

When we define an interpretation I and we have a nullary predicate symbol P we write $P^I = \top$ instead of $P^I = \{()\}$ and $P^I = \perp$ for $P^I = \emptyset$ (this works because $P()^I = \top$ iff $() \in P^I$).

Definition 24. Let Γ be a finite set of first-order formulas.

We say that an interpretation I is a model of Γ , denoted by $I \models \Gamma$, if $I \models \psi$ for every ψ in Γ .

The formula φ is a semantic consequence of Γ , denoted by $\Gamma \vdash \varphi$, if every model of Γ is also a model of φ .

The free variables of Γ , denoted by $\text{FV}(\Gamma)$, are $\bigcup \{\text{FV}(\varphi) \mid \varphi \in \Gamma\}$.

2.4 Two-counter automaton

We will use a version of two-counter automaton which only has two types of transitions. First it can increment a register and second it can decrement a register or jump if the register is already zero. Formally:

Definition 25. A deterministic two-counter automaton is a 4-tuple $M = (\mathcal{Q}, Q_0, Q_f, R)$,

- where
- \mathcal{Q} is a finite set (of states),
 - Q_0 is in \mathcal{Q} (the initial state),
 - Q_f is in \mathcal{Q} (the final state), and
 - R is a function from $\mathcal{Q} \setminus \{Q_f\}$ to $\mathcal{R}_{\mathcal{Q}}$,
where $\mathcal{R}_{\mathcal{Q}} = \{+(i, Q') \mid i \in \{1, 2\}, Q' \in \mathcal{Q}\} \cup \{-(i, Q_1, Q_2) \mid i \in \{1, 2\}, Q_1, Q_2 \in \mathcal{Q}\}$

A configuration C of our automaton is a triple $C = \langle Q, m, n \rangle$, where $Q \in \mathcal{Q}$ and $m, n \in \mathbb{N}$. Let r be in $R(\mathcal{Q} \setminus \{Q_f\})$, then \Rightarrow_M^r is a binary relation on the configurations of M such that two configurations $\langle Q, m, n \rangle, \langle \hat{Q}, \hat{m}, \hat{n} \rangle$ of M are in the relation if all of the following conditions hold:

- $Q \neq Q_f, r = R(Q)$,
- if $r = +(1, Q')$ for some $Q' \in \mathcal{Q}$ then $\hat{Q} = Q', \hat{m} = m + 1$, and $\hat{n} = n$,
- if $r = +(2, Q')$ for some $Q' \in \mathcal{Q}$ then $\hat{Q} = Q', \hat{m} = m$, and $\hat{n} = n + 1$,
- if $r = -(1, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then
 - if $m = 0$ then $\hat{Q} = Q_2, \hat{m} = 0$, and $\hat{n} = n$,
 - if $m \geq 1$ then $\hat{Q} = Q_1, \hat{m} = m - 1$, and $\hat{n} = n$,
- if $r = -(2, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then
 - if $n = 0$ then $\hat{Q} = Q_2, \hat{m} = m$, and $\hat{n} = 0$,
 - if $n \geq 1$ then $\hat{Q} = Q_1, \hat{m} = m$, and $\hat{n} = n - 1$.

The transition relation of M , denoted by \Rightarrow_M , is defined as $\bigcup_{r \in R(\mathcal{Q} \setminus \{Q_f\})} \Rightarrow_M^r$.

Let m, n be in \mathbb{N} , we say that M terminates on input (m, n) if there exist $\hat{m}, \hat{n} \in \mathbb{N}$ such that $\langle Q_0, m, n \rangle \Rightarrow_M^* \langle Q_f, \hat{m}, \hat{n} \rangle$ (It follows that there exists an $i \in \mathbb{N}$ and configurations D_1, \dots, D_i of M such that $\langle Q_0, m, n \rangle = D_1 \Rightarrow_M \dots \Rightarrow_M D_i = \langle Q_f, \hat{m}, \hat{n} \rangle$, we call this chain a computation with length $i - 1$).

Definition 26. The halting problem for two-counter automaton, denoted by **HALT**, is defined as follows. Given a two-counter automaton M .

Does M terminate on input $(0, 0)$?

It is well known that **HALT** is undecidable.

3 System P

3.1 Definitions

In the following let $\mathcal{V}_P = \{\alpha, a, \beta, b, \dots\}$ be a countably infinite subset of \mathcal{V}_T (of variables). Let $\mathcal{P}_P = \{P, Q, \dots\}$ be a set (of predicate symbols) and \mathcal{P} a ranked set such that $\mathcal{P}^{(0)} = \{\mathbf{false}\}$, $\mathcal{P}^{(2)} = \mathcal{P}_P$, and $\mathcal{P}^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over \mathcal{V}_P, \emptyset , and \mathcal{P} is an

atomic formula if $\varphi = \mathbf{false}$ or $\varphi = P(a, b)$ for some $P \in \mathcal{P}_P$ and $a, b \in \mathcal{V}_P$.

universal formula if $\varphi = \forall \vec{\alpha}(A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n)$ for some $\vec{\alpha} = \alpha_1 \dots \alpha_m$ where $\alpha_1, \dots, \alpha_m \in \mathcal{V}_P$, some $n \in \mathbb{N}$ and where A_i is an atomic formula for $i \in \{1, \dots, n\}$, $A_i \neq \mathbf{false}$ for $i \in \{1, \dots, n-1\}$ and for each $\alpha \in \text{FV}(A_n) \cap \text{BV}(\varphi)$ there exists an $i \in \{1, \dots, n-1\}$ such that $\alpha \in \text{FV}(A_i)$.

existential formula if there is a $\vec{\alpha} = \alpha_1 \dots \alpha_m$ where $\alpha_1, \dots, \alpha_m \in \mathcal{V}_P$, an $n \in \mathbb{N}^+$, atomic formulas $A_i \neq \mathbf{false}$ for $i \in \{1, \dots, n\}$, $\beta \in \mathcal{V}_P$, such that for each $\alpha \in (\text{FV}(A_n) \cap \text{BV}(\varphi)) \setminus \{\beta\}$ there exists an $i \in \{1, \dots, n-1\}$ such that $\alpha \in \text{FV}(A_i)$ and $\varphi = \forall \vec{\alpha}(A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n-1} \rightarrow \forall \beta(A_n \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$.

The set of formulas of System **P** (= set of **P**-formulas) over \mathcal{V}_P and \mathcal{P}_P is the set of all first-order formulas in $\mathcal{L}_{(\mathcal{V}_P, \emptyset, \mathcal{P})}$ that are either an atomic, universal or existential formula. In what follows we assume all **P**-formulas to contain no dummy quantifiers.

Definition 27. A finite set of **P**-formulas Γ is called **P**-basis, or basis if it is clear from the context whether a **P**-basis or a **$\lambda 2$** -basis is meant.

For a **P**-basis Γ , another **P**-basis Σ , and a **P**-formula A we will abbreviate $\Gamma \cup \{A\}$ to Γ, A and $\Gamma \cup \Sigma$ to Γ, Σ (c.f. **$\lambda 2$** -basis).

Definition 28. Let A be a **P**-formula, and Γ be a basis. The formula A is a semantic consequence of Γ , denoted by $\Gamma \vdash A$, if A can be produced using the following deduction rules.

(Axiom)	$\Gamma, A \vdash A$	
(\rightarrow -Introduction)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
(\rightarrow -Elimination)	$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
(\forall -Introduction)	$\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin \text{FV}(\Gamma)$
(\forall -Elimination)	$\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$	$b \in \mathcal{V}_P$

We define a more general consequence relation in which we demand that **false** is interpreted with \perp . In this relation existential formulas will behave like the name suggests. Formally:

Definition 29. Let Γ be a basis. The **P**-formula A is a semantic consequence with falsity of Γ , denoted by $\Gamma \vdash_f A$, if for every interpretation I

$$I \models \Gamma \text{ and } \mathbf{false}^I = \perp \text{ implies } I \models A.$$

This allows us to add the following deduction rule.

$$(\exists\text{-Introduction}) \quad \frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, A' := \forall\alpha(A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \vdash_f B} \quad a \notin FV(\Gamma, A', B)$$

Proof. Let $I = (\Delta, \cdot^I, \omega)$ be a model of $\Gamma, A' := \forall\alpha(A \rightarrow \mathbf{false}) \rightarrow \mathbf{false}$ with $\mathbf{false}^I = \perp$ and $a \in \mathcal{V}_P$ a variable such that $a \notin FV(\Gamma, A', B)$.

$$\begin{aligned} I \models \Gamma, \forall\alpha(A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} &\Rightarrow I \models \forall\alpha(A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \\ &\Rightarrow \text{not } (\forall\alpha(A \rightarrow \mathbf{false}))^I \text{ or } \mathbf{false}^I \\ &\Rightarrow \text{not } (\forall\alpha(A \rightarrow \mathbf{false}))^I \text{ or } \perp \\ &\Rightarrow \text{not } (\forall\alpha(A \rightarrow \mathbf{false}))^I \\ &\Rightarrow \text{not } (\text{forall } d \in \Delta: (A \rightarrow \mathbf{false})^{I[\alpha \mapsto d]}) \\ &\Rightarrow \text{exists } d \in \Delta: \text{not } (\text{not } A^{I[\alpha \mapsto d]} \text{ or } \mathbf{false}^{I[\alpha \mapsto d]}) \\ &\Rightarrow \text{exists } d \in \Delta: \text{not } (\text{not } A^{I[\alpha \mapsto d]} \text{ or } \perp) \\ &\Rightarrow \text{exists } d \in \Delta: \text{not } (\text{not } A^{I[\alpha \mapsto d]}) \\ &\Rightarrow \text{exists } d \in \Delta: A^{I[\alpha \mapsto d]} \end{aligned}$$

Together with $a \notin FV(\Gamma, A')$, it follows that $I[a \mapsto d]$ is a model of $\Gamma, A[\alpha := a]$. Which implies $I[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B . \square

Definition 30. The consistency problem, denoted by **CONS**, is defined as follows. Given a set of **P**-formulas Γ .

Does $\Gamma \vdash \mathbf{false}$ not hold?

3.2 Consistency in System **P** is undecidable

We will show that $\mathbf{HALT} \leq \mathbf{CONS}$ then the undecidability of **CONS** directly follows from the undecidability of **HALT**. For a given two-counter automaton M we will effectively construct a **P**-basis Γ_M such that

$$M \text{ terminates on input } (0, 0) \quad \text{iff} \quad \Gamma_M \vdash \mathbf{false} \text{ holds in System } \mathbf{P}.$$

Let $M = (\mathcal{Q}, Q_0, Q_f, R)$ be a two-counter automaton, w.l.o.g. $S, P, R_1, R_2, E, D, G \notin \mathcal{Q}$. In the following we will consider **P**-formulas over \mathcal{V}_P and \mathcal{P}_P , where $\mathcal{P}_P = \mathcal{Q} \uplus \{S, P, R_1, R_2, E, D, G\}$. We will abbreviate $P(a, a)$ to $P(a)$, note that this way we can use binary predicate symbols as unary ones.

The intended informal meaning for these new relation symbols is the following:

- The meaning of $Q(a)$ is “ a represents a configuration and Q is the state of this configuration”.

- For $i \in \{1, 2\}$, $R_i(a, m)$ denotes that “the value of register i in the configuration represented by a is represented by m ” (we call m anchor of a for register i).
- With $S(a, b)$ we state that “ b is a successor of a ”.
- The meaning of $P(a, b)$ is “ b is a predecessor of a ”.
- And $E(a)$ marks “ a as the end of chain”.
- With $D(a)$ we state that “ a is not the end of a chain”.
- Finally $G(a)$ has no actual meaning, it holds for all elements representing a configuration or a number. But we just need it for the existential formulas.

For a configuration $C = \langle Q, m, n \rangle$ of M we define a set of **P**-formulas Γ_C . It contains the following formulas:

- $Q(a), G(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \dots, m\}$
- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \dots, n\}$
- $D(a_i), D(b_j), G(a_i), G(b_j)$ for $i \in \{0, \dots, m-1\}$ and $j \in \{0, \dots, n-1\}$
- $E(a_m), E(b_n), G(a_m), G(b_n)$

Next we need sets of **P**-formulas for all possible transitions. For every $Q \in \mathcal{Q} \setminus \{Q_f\}$ and $r \in \mathcal{R}_{\mathcal{Q}}$ we define $\Gamma_{Q,r}$. If $r = +(1, Q')$ for some $Q' \in \mathcal{Q}$ then $\Gamma_{Q,+(1,Q')}$ contains the following formulas:

- $\forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$
change of state
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))$
increment register 1
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$
do not change the value register 2

If $r = -(1, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then $\Gamma_{Q,-(1,Q_1,Q_2)}$ contains the following formulas:

- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$
jump to Q_2 if register 1 is zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow R_1(\beta, \gamma))$
if register 1 is zero it stays zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$
change state to Q_1 if register 1 is greater zero

- $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow R_1(\beta, \delta))$
decrement register 1 if possible
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$
do not change register 2 in both cases

For $r = +(2, Q')$ for some $Q' \in \mathcal{Q}$ or $r = -(2, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ the sets $\Gamma_{Q,r}$ are defined analogously.

We also need a set Γ_1 to ensure that our representation works correctly. The following formula are in Γ_1 :

- $\forall \alpha \beta (S(\alpha, \beta) \rightarrow G(\beta))$
- $\forall \alpha (D(\alpha) \rightarrow G(\alpha))$
- $\forall \alpha \beta (P(\alpha, \beta) \rightarrow D(\alpha))$
no predecessor is the end of a chain
- $\forall \alpha (G(\alpha) \rightarrow \forall \beta (R_1(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$
every element that represents a configuration has a value for register 1
- $\forall \alpha (G(\alpha) \rightarrow \forall \beta (R_2(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$
every element that represents a configuration has a value for register 2
- $\forall \alpha (G(\alpha) \rightarrow \forall \beta (S(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$
every element that represents a configuration has a successor

Note that in the last three formulas the only task of $G(\alpha)$ is to make these formulas existential formulas (e.g. $\forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$ is not an existential formula).

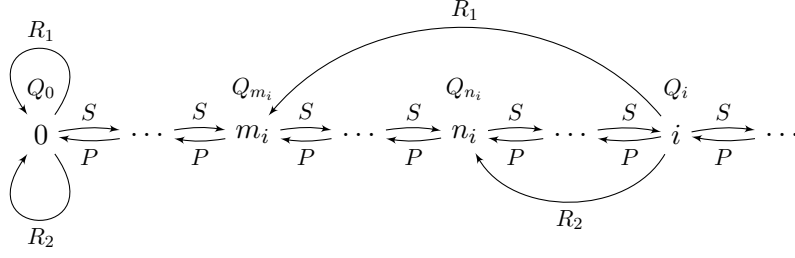
We define $\Gamma_{\overline{M}}$ as $\bigcup_{Q \in \mathcal{Q} \setminus \{Q_f\}} \Gamma_{Q,R(Q)} \cup \{\forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})\} \cup \Gamma_1$. We have added the formula $\forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})$ to be able to deduce **false** if our automaton terminates. Finally we can define Γ_M as $\Gamma_{C_0} \cup \Gamma_{\overline{M}}$, where $C_0 = \langle Q_0, 0, 0 \rangle$ is the initial configuration.

Lemma 31.

$$\Gamma_M \vdash \mathbf{false} \text{ holds in System } \mathbf{P} \quad \implies \quad M \text{ terminates on input } (0, 0)$$

Proof. Assume M does not terminate it follows that there is an infinite chain $C_0 \Rightarrow_M C_1 \Rightarrow_M C_2 \Rightarrow_M \dots$ ($C_i = \langle Q_i, m_i, n_i \rangle$ for $i \in \mathbb{N}$). Now we construct a model of Γ_M which interprets **false** with \perp this contradicts $\Gamma_M \vdash \mathbf{false}$.

To illustrate the idea we will use a graphical notation for an interpretation I . By $d_1 \xrightarrow{R} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{P}{d}$ to say that $(d, d) \in P^I$ for predicate symbols that are used as unary predicate symbols. As domain for our interpretation we will use the natural numbers. Every number will have two tasks: firstly it will represent itself as a possible value for register 1 or 2 and secondly every number i will also represent the i^{th} configuration of our infinite computation. Now the idea for our model of Γ_M looks like this:



We have $0 \in E^I$, all other numbers are in D^I , and all numbers are in G^I . Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I, \omega)$.

$$\begin{aligned}
P^I &= \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I &= \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I &= \{(i, n_i) \mid i \in \mathbb{N}\} \\
S^I &= \{(i, i+1) \mid i \in \mathbb{N}\} & D^I &= \{(i, i) \mid i \in \mathbb{N}^+\} & E^I &= \{(0, 0)\} \\
Q^I &= \{(i, i) \mid i \in \mathbb{N}, Q = Q_i\} \text{ for every } Q \in \mathcal{Q} & \text{false}^I &= \perp \\
G^I &= \mathbb{N}
\end{aligned}$$

$$a^I = 0$$

$$a_0^I = 0$$

$$b_0^I = 0$$

Since there are no free variables in Γ_M we can just set $\omega(x) = 0$ for every $x \in \mathcal{V}_P$. It is easy to see that I is indeed a model of Γ_M . \square

We proof the other direction by induction on the length of the computation. But to be able to use the induction hypothesis we need a slightly more general statement (this is why we defined $\Gamma_{\overline{M}}$ and not just Γ_M right away).

Lemma 32. *Let $C = \langle Q, m, n \rangle$ be a configuration of M . If a final configuration (i.e. a configuration $\langle Q_f, \hat{m}, \hat{n} \rangle$ for some $\hat{m}, \hat{n} \in \mathbb{N}$) is reachable from C then $\Gamma_C \cup \Gamma_{\overline{M}} \vdash \mathbf{false}$.*

Proof. By induction on the length i of the computation.

Induction Base: $i = 0$

Since a final configuration is reachable in 0 steps C must be this final configuration. So $C = \langle Q_f, m, n \rangle$ for some $m, n \in \mathbb{N}$. Hence, $Q_f(a)$ is in Γ_C for some $a \in \mathcal{V}_P$ and $\forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})$ is in $\Gamma_{\overline{M}}$, we can easily deduce **false**.

$$\frac{\frac{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a) \rightarrow \mathbf{false}} \quad \Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a)}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \mathbf{false}}$$

Induction Step: $i = i' + 1$

Since $I \models \mathbf{false}$ holds trivially if I interprets **false** with \top we only need to consider models of $\Gamma_C \cup \Gamma_{\overline{M}}$ that interpret **false** with \perp (note that there are no such models if M terminates which is exactly what we want to proof). As result of this observation we can use the \exists -Introduction rule.

From the fact that a final configuration is reachable from C in i steps we can deduce that there exists a configuration $D = \langle \hat{Q}, \hat{m}, \hat{n} \rangle$ such that $C \Rightarrow_M^r D$ for some $r \in \mathcal{R}_Q$ and a final configuration is reachable from D in i' steps. We also know that $C = \langle Q, m, n \rangle$ for some $Q \in \mathcal{Q} \setminus \{Q_f\}$ and some $m, n \in \mathbb{N}$. The set Γ_C contains the formulas:

$$\begin{aligned} &R_1(a, a_0), P(a_{i-1}, a_i), G(a_{i-1}), \text{ and } D(a_{i-1}) \text{ for } i \in \{1, \dots, m\}, \\ &R_2(a, b_0), P(b_{i-1}, b_i), G(b_{i-1}), \text{ and } D(b_{i-1}) \text{ for } i \in \{1, \dots, n\}, \\ &Q(a), E(a_m), E(b_n), G(a), G(a_m), \text{ and } G(b_n). \end{aligned}$$

And Γ_D contains the formulas:

$$\begin{aligned} &R_1(\hat{a}, \hat{a}_0), P(\hat{a}_{i-1}, \hat{a}_i), G(\hat{a}_{i-1}), \text{ and } D(\hat{a}_{i-1}) \text{ for } i \in \{1, \dots, \hat{m}\}, \\ &R_2(\hat{a}, \hat{b}_0), P(\hat{b}_{i-1}, \hat{b}_i), G(\hat{b}_{i-1}), \text{ and } D(\hat{b}_{i-1}) \text{ for } i \in \{1, \dots, \hat{n}\}, \\ &\hat{Q}(\hat{a}), E(\hat{a}_{\hat{m}}), E(\hat{b}_{\hat{n}}), G(\hat{a}), G(\hat{a}_{\hat{m}}), \text{ and } G(\hat{b}_{\hat{n}}). \end{aligned}$$

The basic idea is to deduce Γ_D from $\Gamma_C \cup \Gamma_{\overline{M}}$ and then apply the induction hypothesis to $\Gamma_D \cup \Gamma_{\overline{M}}$.

$$\frac{\text{Induction Hypothesis} \quad \frac{\Gamma_C \cup \Gamma_{\overline{M}} \cup \Gamma_D \vdash_f \mathbf{false} \quad \Gamma_C \cup \Gamma_{\overline{M}} \vdash_f \Gamma_D}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash_f \mathbf{false}}}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash_f \mathbf{false}}$$

We achieve this by looking at the four possible cases for the type of the rule r . We will only consider the cases $r = +(1, Q')$ and $r = -(1, Q_1, Q_2)$, because the two remaining cases $r = +(2, Q')$ and $r = -(2, Q_1, Q_2)$ follow by exchanging the roles of register 1 and register 2 in the first two cases.

In every case we need a new free variable representing the configuration D . Also the value in register 2 does not change, because in both cases we are only concerned with register 1. In the following tableau proofs we will abbreviate **false** by **f** and we will drop $\Gamma_C \cup \Gamma_{\overline{M}}$ and only write new formulas on the left side of \vdash_f .

We first introduce a new variable representing the new configuration D (let $b \in \mathcal{V}_P \setminus \text{FV}(\Gamma_C)$, note that $\text{FV}(\Gamma_{\overline{M}}) = \emptyset$).

$$\begin{array}{c} \vdots \\ \hline \frac{S(a, b) \vdash_f \mathbf{f}}{\forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f} \vdash_f \mathbf{f}} \quad \frac{\vdash_f \forall \alpha (G(\alpha) \rightarrow \forall \beta (S(\alpha, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f})}{\vdash_f G(a) \rightarrow (\forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f})} \quad \vdash_f G(a) \\ \hline \frac{\vdash_f (\forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}) \rightarrow \mathbf{f} \quad \vdash_f \forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}}{\vdash_f \mathbf{f}} \end{array}$$

For the new variable b we have to deduce $G(b)$. Again we will just drop $S(a, b)$ on the left side for comprehensibility.

$$\begin{array}{c}
\vdots \\
\hline
G(b) \vdash_f \mathbf{f} \\
\hline
\vdash_f G(b) \rightarrow \mathbf{f}
\end{array}
\quad
\begin{array}{c}
\vdash_f \forall \alpha \beta (S(\alpha, \beta) \rightarrow G(\beta)) \\
\hline
\vdash_f S(a, b) \rightarrow G(b)
\end{array}
\quad
\vdash_f S(a, b)$$

$$\vdash_f G(b)$$

$$\vdash_f \mathbf{f}$$

Since register 2 should not change we need $R_2(b, b_0)$.

$$\begin{array}{c}
\vdots \\
\hline
R_2(b, b_0) \vdash_f \mathbf{f} \\
\hline
\vdash_f R_2(b, b_0) \rightarrow \mathbf{f}
\end{array}
\quad
\begin{array}{c}
\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma)) \\
\hline
\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0)
\end{array}
\quad
\vdash_f Q(a)$$

$$\vdash_f S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0)$$

$$\vdash_f R_2(a, b_0) \rightarrow R_2(b, b_0)$$

$$\vdash_f R_2(b, b_0)$$

$$\vdash_f \mathbf{f}$$

For the case that $\mathbf{r} = +(\mathbf{1}, Q')$, we have that $\hat{Q} = Q'$, $\hat{m} = m + 1$, and $\hat{n} = n$. So we need to increment register 1 and ensure that the state of the configuration represented by b is Q' .

$$\begin{array}{c}
\vdots \\
\hline
Q'(b) \vdash_f \mathbf{f} \\
\hline
\vdash_f Q'(b) \rightarrow \mathbf{f}
\end{array}
\quad
\begin{array}{c}
\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta)) \\
\hline
\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)
\end{array}
\quad
\vdash_f Q(a)$$

$$\vdash_f S(a, b) \rightarrow Q'(b)$$

$$\vdash_f Q'(b)$$

$$\vdash_f \mathbf{f}$$

To increment register 1 we need a new free variable as anchor of b for register 1 (let $d \in \mathcal{V}_P \setminus \text{FV}(\Gamma_C)$ and $d \neq b$).

$$\begin{array}{c}
\vdots \\
\hline
R_1(b, d) \vdash_f \mathbf{f} \\
\hline
\vdash_f (\forall \beta (R_1(b, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}
\end{array}
\quad
\begin{array}{c}
\vdash_f \forall \alpha (G(\alpha) \rightarrow \forall \beta (R_1(\alpha, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}) \\
\hline
\vdash_f G(a) \rightarrow \forall \beta (R_1(b, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}
\end{array}
\quad
\vdash_f G(a)$$

$$\vdash_f \forall \beta (R_1(b, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}$$

$$\vdash_f \mathbf{f}$$

Now we need to connect d with a_0 (the anchor of a for register 1).

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow P(d, a_0) \quad \vdash_f Q(a)} \\
\frac{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow P(d, a_0) \quad \vdash_f S(a, b)}{\vdash_f R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow P(d, a_0) \quad \vdash_f R_1(a, a_0)} \\
\vdots \\
\frac{P(d, a_0) \vdash_f \mathbf{f}}{\vdash_f P(d, a_0) \rightarrow \mathbf{f}} \quad \frac{\vdash_f R_1(b, d) \rightarrow P(d, a_0) \quad \vdash_f R_1(b, d)}{\vdash_f P(d, a_0)} \\
\hline
\vdash_f \mathbf{f}
\end{array}$$

We have to make sure that we do not get an artificial zero. We achieve this by deducing $D(d)$.

$$\begin{array}{c}
\vdots \\
\frac{D(d) \vdash_f \mathbf{f}}{\vdash_f D(d) \rightarrow \mathbf{f}} \quad \frac{\vdash_f \forall \alpha \beta (P(\alpha, \beta) \rightarrow D(\alpha))}{\vdash_f P(d, a_0) \rightarrow D(d)} \quad \vdash_f P(d, a_0) \\
\hline
\vdash_f D(d)
\end{array}$$

Now we can easily deduce $G(d)$.

$$\begin{array}{c}
\vdots \\
\frac{G(d) \vdash_f \mathbf{f}}{\vdash_f G(d) \rightarrow \mathbf{f}} \quad \frac{\vdash_f \forall \alpha (D(\alpha) \rightarrow G(\alpha))}{\vdash_f D(d) \rightarrow G(d)} \quad \vdash_f D(d) \\
\hline
\vdash_f G(d)
\end{array}$$

Now we have already deduced Γ_D . To see why we define $\hat{a} := b$, $\hat{b}_i := b_i$ for $i \in \{0, \dots, n\}$, $\hat{a}_0 := d$, and $\hat{a}_{i+1} := a_i$ for $i \in \{0, \dots, m\}$. It follows that $\Gamma_D \subseteq (\Gamma_C \cup \{S(a, b), G(b), Q'(b), R_2(b, b_0), R_1(b, d), P(d, a_0), D(d), G(d)\})$. Hence we can deduce **false** by induction hypothesis.

The other case, that $\mathbf{r} = -(\mathbf{Q}, \mathbf{1}, \mathbf{Q}_1, \mathbf{Q}_2)$, has to be split into two cases again. If $\mathbf{m} = \mathbf{0}$ then $\hat{Q} = Q_2$, $\hat{m} = 0$, and $\hat{n} = n$. We only need to ensure that the successor state is Q_2 and that register 1 is still zero.

$$\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)} \quad \vdash_f S(a, b)$$

$$\vdots$$

$$\frac{\vdash_f R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)}{\vdash_f R_1(a, a_0)} \quad \vdash_f R_1(a, a_0)$$

$$\frac{Q_2(b) \vdash_f \mathbf{f}}{\vdash_f Q_2(b) \rightarrow \mathbf{f}} \quad \frac{\vdash_f E(a_0) \rightarrow Q_2(b)}{\vdash_f E(a_0)}$$

$$\frac{\vdash_f Q_2(b) \rightarrow \mathbf{f}}{\vdash_f \mathbf{f}}$$

Register 1 stays zero.

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow R_1(\beta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow R_1(b, a_0)} \quad \vdash_f Q(a) \\
\frac{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow R_1(b, a_0)}{\vdash_f S(a, b)} \quad \vdash_f S(a, b) \\
\vdots \\
\frac{R_1(b, a_0) \vdash_f \mathbf{f}}{\vdash_f R_1(b, a_0) \rightarrow \mathbf{f}} \quad \frac{\vdash_f R_1(a, a_0) \rightarrow E(a_0) \rightarrow R_1(b, a_0)}{\vdash_f R_1(a, a_0)} \quad \vdash_f E(a_0) \\
\frac{\vdash_f R_1(b, a_0) \rightarrow \mathbf{f}}{\vdash_f R_1(b, a_0)} \quad \vdash_f E(a_0) \\
\vdash_f \mathbf{f}
\end{array}$$

If we define $\widehat{a} := b$, $\widehat{b}_i := b_i$ for $i \in \{0, \dots, n\}$, and $\widehat{a}_0 := a_0$ then it is clear that we have deduced all formulas required for Γ_D . So we can use the induction hypothesis to deduce **false**.

In the last case $\mathbf{m} > \mathbf{0}$, so $\hat{Q} = Q_1$, $\hat{m} = m - 1$, and $\hat{n} = n$. First we ensure that b is in state Q_1 .

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)} \quad \vdash_f Q(a) \\
\frac{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)}{\vdash_f R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)} \quad \vdash_f S(a, b) \\
\vdots \\
\frac{Q_1(b) \vdash_f \mathbf{f}}{\vdash_f Q_1(b) \rightarrow \mathbf{f}} \quad \frac{\vdash_f D(a_0) \rightarrow Q_1(b) \quad \vdash_f D(a_0)}{\vdash_f Q_1(b)} \\
\hline
\vdash_f \mathbf{f}
\end{array}$$

Now we decrement register 1 by taking a_1 (the predecessor of a_0) as anchor of b for register 1.

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow R_1(\beta, \delta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f Q(a) \\
\frac{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)}{\vdash_f R_1(a, a_0) \rightarrow D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f S(a, b) \\
\vdots \\
\frac{R_1(b, a_1) \vdash_f \mathbf{f}}{\vdash_f R_1(b, a_1) \rightarrow \mathbf{f}} \quad \frac{\vdash_f D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)}{\vdash_f P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f D(a_0) \\
\frac{\vdash_f R_1(b, a_1) \rightarrow \mathbf{f}}{\vdash_f \mathbf{f}} \quad \frac{\vdash_f P(a_0, a_1) \rightarrow R_1(b, a_1)}{\vdash_f R_1(b, a_1)}
\end{array}$$

Again it is obvious that we have deduced $\Gamma_D (\hat{a} := b, \hat{b}_i := b_i \text{ for } i \in \{0, \dots, n\}, \text{ and } \hat{a}_{i-1} := a_i \text{ for } i \in \{1, \dots, m\})$. Hence, by induction hypothesis, we can deduce **false**. \square

Lemma 33.

$$M \text{ terminates on input } (0, 0) \quad \text{iff} \quad \Gamma_M \vdash \mathbf{false} \text{ holds in system } P.$$

Proof. The \Leftarrow direction is proven in Lemma 31. And the \Rightarrow direction is a direct consequence of Lemma 32 with $C = \langle Q_0, 0, 0 \rangle$. \square

Theorem 34. *The consistency problem is undecidable.*

Proof. Since by Lemma 33 for a given two-counter automaton M we can effectively construct a set of **P**-formulas Γ_M such that M terminates on input $(0, 0)$ iff Γ_M is not consistent. It follows that **HALT** \leq **CONS**. Since **HALT** is undecidable we have shown that **CONS** is undecidable too. \square

4 Inhabitation in $\lambda 2$ is undecidable

Now we can show that the inhabitation problem in $\lambda 2$ is undecidable by reducing **CONS** to **INHAB**. Given a **P**-basis Γ we construct a $\lambda 2$ -basis $\bar{\Gamma}$ such that

$$\Gamma \vdash \mathbf{false} \quad \text{iff} \quad \text{There is a } \lambda 2 \text{ term } M \text{ such that } \bar{\Gamma} \vdash M : \mathbf{false}$$

where $\mathbf{false} \in \mathcal{V}_T$. Furthermore for every $P \in \mathcal{P}_P$ we have $p, p_1, p_2 \in \mathcal{V}_T$.

Definition 35. For a **P**-formula A we define the code of A , denoted by \bar{A} , as follows.

If A is an atomic formula then

$$\bar{A} = \begin{cases} \mathbf{false} & \text{if } A = \mathbf{false} \\ (\alpha \rightarrow p_1) \rightarrow (\beta \rightarrow p_2) \rightarrow p & \text{if } A = P(\alpha, \beta) \end{cases}$$

We will abbreviate $(\alpha \rightarrow p_1) \rightarrow (\beta \rightarrow p_2) \rightarrow p$ to $P_{\alpha\beta}$.

If A is a universal formula, it follows that there is an $n \in \mathbb{N}$, atomic formulas A_1, A_2, \dots, A_n , and an $\vec{\alpha} = \bar{\alpha}_1 \dots \bar{\alpha}_m$ for some $m \in \mathbb{N}$ and $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathcal{V}_P$ such that $A = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n)$, then

$$\bar{A} = \forall \vec{\alpha} (\bar{A}_1 \rightarrow \bar{A}_2 \rightarrow \dots \rightarrow \bar{A}_n)$$

If A is an existential formula, it follows that for some $n \in \mathbb{N}^+$, some atomic formulas A_1, \dots, A_n , some $\vec{\alpha} = \bar{\alpha}_1 \dots \bar{\alpha}_m$ for some $m \in \mathbb{N}$ and $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathcal{V}_P$, and some $\beta \in \mathcal{V}_P$ it holds that $A = \forall \vec{\alpha} (A_1 \rightarrow \dots \rightarrow A_{n-1} \rightarrow \forall \beta ((A_n) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$, then

$$\bar{A} = \forall \vec{\alpha} (\bar{A}_1 \rightarrow \dots \rightarrow \forall \beta (\bar{A}_n \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$$

For a **P**-basis Γ we define the code of Γ , denoted by $\bar{\Gamma}$, as $\{(x_A : \bar{A}) \mid A \in \Gamma\}$.

In the following lemma we prove the \Rightarrow direction by constructing a **$\lambda 2$** term M with the required type.

Lemma 36. *Let Γ be a **P**-basis and A a **P**-formula such that $\Gamma \vdash A$. Then there exists a term $M \in \Lambda_{T_{\lambda 2}}$ such that $\bar{\Gamma} \vdash M : \bar{A}$ holds.*

Proof. We proof this by induction on the structure of the proof.

A is produced by the Axiom rule. It follows that $A \in \Gamma$ and therefore $(x_A : \bar{A}) \in \bar{\Gamma}$. Now the term $M := x_A$ fulfills the condition.

A is produced by the \rightarrow -Introduction rule. It follows that $A = A' \rightarrow B'$ for some **P**-formulas A' and B' . We can now apply the induction hypothesis to $\Gamma, A' \vdash B'$ and we get that there exists an $M' \in \Lambda_{T_{\lambda 2}}$ such that $\bar{\Gamma}, \bar{A}' \vdash M' : \bar{B}'$. With the λ -Introduction rule we deduce $\bar{\Gamma} \vdash \lambda x_{A'} : \bar{A}'. M' : \bar{A}' \rightarrow \bar{B}'$. Since A has to be a universal or an existential formula $\bar{A}' \rightarrow \bar{B}' = \bar{A}' \rightarrow \bar{B}'$. So $M := \lambda x_{A'} : \bar{A}'. M'$ has the required type.

A is produced by the \rightarrow -Elimination rule. So there exists a **P**-formula B such that $\Gamma \vdash B \rightarrow A$ and $\Gamma \vdash B$. Now we apply the induction hypothesis and get that there exist $M_1, M_2 \in \Lambda_{T_{\lambda 2}}$ such that $\bar{\Gamma} \vdash M_1 : \bar{B} \rightarrow \bar{A}$ and $\bar{\Gamma} \vdash M_2 : \bar{B}$. Again we have that $\bar{B} \rightarrow \bar{A} = \bar{B} \rightarrow \bar{A}$. It follows that $M := M_1 M_2$ has the type \bar{A} .

A is produced by the \forall -Introduction rule. It follows that $A = \forall \beta B$ for some $\beta \in \mathcal{V}_P \setminus \text{FV}(\Gamma)$ and some **P**-formula B . By applying the induction hypothesis to $\Gamma \vdash B$ we get that there exists an $M' \in \Lambda_{T_{\lambda 2}}$ such that $\bar{\Gamma} \vdash M' : \bar{B}$. We deduce that $M := \Lambda \beta. M'$ has type $\forall \beta. \bar{B} = \overline{\forall \beta B}$ as desired.

A is produced by the \forall -Elimination rule. Then there is a **P**-formula B and variables $\alpha, b \in \mathcal{V}_P$ such that $\Gamma \vdash \forall \alpha B$ and $A = B[\alpha := b]$. The induction hypothesis implies that there exists an $M' \in \Lambda_{T_{\lambda 2}}$ such that $\bar{\Gamma} \vdash M' : \overline{\forall \alpha B}$. Since $\overline{\forall \alpha B} = \forall \alpha. \bar{B}$ the term $M := M' b$ has the type $\bar{B}[\alpha := b] = \bar{B}[\alpha := b]$.

□

In the next two lemmas we will prove the \Leftarrow direction.

Lemma 37. *Let Γ be a \mathbf{P} -basis, $M \in \Lambda_{T_{\lambda 2}}$, $P \in \mathcal{P}_P$, and $s, t \in T_{\lambda 2}$ such that $\bar{\Gamma} \vdash M : P_{st}$ holds. Then $s, t \in \mathcal{V}_P$ (remember that $\mathcal{V}_P \subseteq \mathcal{V}_T$). Furthermore $\Gamma \vdash P(s, t)$ holds.*

Proof. Note that M is a well types $\lambda 2$ term and hence, by Theorem 13, there is a $N \in \Lambda_{T_{\lambda 2}}$ such that N is in normal form and $M \Rightarrow_{\lambda}^* N$. From Theorem 14 it follows that the statement $N : P_{st}$ is derivable from $\bar{\Gamma}$. Therefore we can assume w.l.o.g. that M is in normal form.

We now proof the lemma by structural induction on the term M .

$M = x$ for some $x \in \mathcal{V}_V$.

It follows that $(x : P_{st}) \in \bar{\Gamma}$. Now the definition of $\bar{\Gamma}$ yields that $P(s, t) \in \Gamma$. Therefore $s, t \in \mathcal{V}_P$ and $\Gamma \vdash P(s, t)$ holds trivially.

$M = M_1 M_2$ for some $M_1, M_2 \in \Lambda_{T_{\lambda 2}}$.

Since M is in normal form we have that $M_1 = x N_1 \dots N_k$ for some $x \in \mathcal{V}_V$, $k \in \mathbb{N}$, and some $N_1, \dots, N_k \in \Lambda_{T_{\lambda 2}} \cup T_{\lambda 2}$.

We conclude that $x = x_A$ and $(x : \bar{A}) \in \bar{\Gamma}$ for some universal formula $A = \forall \vec{\alpha} (P^1(\alpha_1, \beta_1) \rightarrow \dots \rightarrow P^n(\alpha_n, \beta_n) \rightarrow P(\alpha, \beta))$ in Γ where $\vec{\alpha} = \bar{\alpha}_1 \dots \bar{\alpha}_m$ for some $m \in \mathbb{N}$ and some $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathcal{V}_P$.

Furthermore $M = x \vec{t} \vec{N}$ for some $\vec{t} = \bar{t}_1 \dots \bar{t}_m$ with $\bar{t}_1, \dots, \bar{t}_m \in T_{\lambda 2}$ and some $\vec{N} = N_1 \dots N_n$ with $N_1, \dots, N_n \in \Lambda_{T_{\lambda 2}}$ and $\bar{\Gamma} \vdash N_i : P_{s_i t_i}^i$ (where $s_i = \alpha_i [\vec{\alpha} := \vec{t}]$ and $t_i = \beta_i [\vec{\alpha} := \vec{t}]$) for $i \in \{1, \dots, n\}$.

$$\frac{\bar{\Gamma} \vdash x : \forall \vec{\alpha} (P_{\alpha_1 \beta_1}^1 \rightarrow \dots \rightarrow P_{\alpha_n \beta_n}^n \rightarrow P_{\alpha \beta})}{\frac{\bar{\Gamma} \vdash x \vec{t} : P_{s_1 t_1}^1 \rightarrow \dots \rightarrow P_{s_n t_n}^n \rightarrow P_{st} \quad \bar{\Gamma} \vdash N_1 : P_{s_1 t_1}^1}{\vdots} \quad \frac{\bar{\Gamma} \vdash x \vec{t} N_1 \dots N_{n-1} : P_{s_n t_n}^n \rightarrow P_{st} \quad \bar{\Gamma} \vdash N_n : P_{s_n t_n}^n}{\bar{\Gamma} \vdash (x \vec{t} N_1 \dots N_{n-1}) N_n : P_{st}}}$$

For $i \in \{1, \dots, n\}$ we can now apply the induction hypothesis to $\bar{\Gamma} \vdash N_i : P_{s_i t_i}^i$ and we get that $s_i, t_i \in \mathcal{V}_P$ and that $\Gamma \vdash P^i(s_i, t_i)$ holds.

If $\alpha = \bar{\alpha}_j$ for some $j \in \{1, \dots, m\}$ then because there are no dummy quantifiers we get that $s = \bar{t}_j$. Furthermore since $\alpha \in \text{FV}(P(\alpha, \beta)) \setminus \text{FV}(A)$ it follows that there exists an $i \in \{1, \dots, n\}$ such that $\alpha \in \text{FV}(P^i(\alpha_i, \beta_i))$, i.e. $\alpha = \alpha_i$ or $\alpha = \beta_i$. It follows that $s = s_i$ or $s = t_i$, in both cases we get that $s \in \mathcal{V}_P$.

If $\alpha \neq \bar{\alpha}_j$ for all $j \in \{1, \dots, m\}$ then $\alpha \in \text{FV}(A)$ and therefore $s = \alpha$ and $s \in \mathcal{V}_P$.

For t we can make a similar argument and get that $t \in \mathcal{V}_P$.

Finally we have to show that $P(s, t)$ is a semantic consequence of Γ .

$$\begin{array}{c}
\frac{\Gamma \vdash \forall \vec{\alpha}(P^1(\alpha_1, \beta_1) \rightarrow \dots \rightarrow P^n(\alpha_n, \beta_n) \rightarrow P(\alpha, \beta))}{\Gamma \vdash P^1(s_1, t_1) \rightarrow \dots \rightarrow P^n(s_n, t_n) \rightarrow P(s, t)} \quad \Gamma \vdash P^1(s_1, t_1) \\
\vdots \\
\frac{\Gamma \vdash P^n(s_n, t_n) \rightarrow P(a, b) \quad \Gamma \vdash P^n(s_n, t_n)}{\Gamma \vdash P(s, t)}
\end{array}$$

$\underline{M = \lambda x : t'. M'}$ for some $M' \in \Lambda_{T_{\lambda_2}}$, some $x \in \mathcal{V}_V$, and some $t' \in T_{\lambda_2}$ (w.l.o.g. $x \notin \text{dom}(\Gamma)$).

It follows that $t' = s \rightarrow p_1$ and $\bar{\Gamma}, x : s \rightarrow p_1 \vdash M' : (t \rightarrow p_2) \rightarrow p$.

If $M' = yx$ for some $y \in \mathcal{V}_V$ then it has to be that $y = x_{P(s,t)}$ and $(y : (s \rightarrow p_1) \rightarrow (t \rightarrow p_2) \rightarrow p) \in \bar{\Gamma}$. It follows that $P(s, t) \in \Gamma$ and therefore $s, t \in \mathcal{V}_P$ and $\Gamma \vdash P(s, t)$.

If $M' = \lambda y : t \rightarrow p_2. zxy$ for some $y, z \in \mathcal{V}_V$ then $z = x_{P(s,t)}$ and therefore $(z : (s \rightarrow p_1) \rightarrow (t \rightarrow p_2) \rightarrow p) \in \bar{\Gamma}$. We get that $P(s, t) \in \Gamma$ and conclude that $s, t \in \mathcal{V}_P$ and $\Gamma \vdash P(s, t)$.

All other cases for M' are impossible because there are no **P**-formulas A such that \bar{A} has the required type.

$\underline{M = \Lambda \gamma. M'}$ for some $M' \in \Lambda_{T_{\lambda_2}}$ and some $\gamma \in \mathcal{V}_T$.

It follows that $\bar{\Gamma} \vdash M : \forall \gamma. t'$ for some $t' \in T_{\lambda_2}$. But this can not be since $P_{st} = (s \rightarrow p_1) \rightarrow (t \rightarrow p_2) \rightarrow p$. Therefore M is not of the form $\Lambda \gamma. M'$ and this case is impossible.

$\underline{M = M' t'}$ for some $M' \in \Lambda_{T_{\lambda_2}}$ and some $t' \in T_{\lambda_2}$.

Since M is in normal form we have that $M' = xM_1 \dots M_n$ for some $x \in \mathcal{V}_V$, $n \in \mathbb{N}$, and some $M_1, \dots, M_n \in \Lambda_{T_{\lambda_2}} \cup T_{\lambda_2}$. Hence, $(x : \bar{A}) \in \bar{\Gamma}$ for some **P**-formula A and $M = M' t'$, we get that this case is impossible because no such A exists.

The only case where the contradiction is not obvious is when A is a universal formula and $M_1, \dots, M_n \in T_{\lambda_2}$. Furthermore because there are no dummy quantifiers $n \leq 1$. So A is of the form $A = \forall \vec{\alpha}(P(\alpha, \beta))$ where $\vec{\alpha} \in \{\alpha\beta, \beta\alpha, \alpha, \beta\}$. But in every case A is not a **P**-formula since there always is a $\gamma \in \text{FV}(P(\alpha, \beta)) \setminus \text{FV}(A)$. □

Lemma 38. *Let Γ be a **P**-basis, $M \in \Lambda_{T_{\lambda_2}}$ such that $\bar{\Gamma} \vdash M : \mathbf{false}$ holds. Then $\Gamma \vdash \mathbf{false}$ holds.*

Proof. By structural induction on the term M . Again we can assume that M is in normal form.

$\underline{M = x}$ for some $y \in \mathcal{V}_V$.

It follows that $(x : \mathbf{false}) \in \bar{\Gamma}$. Now the definition of $\bar{\Gamma}$ yields that $\mathbf{false} \in \Gamma$. Therefore $\Gamma \vdash \mathbf{false}$ holds.

$M = M_1 M_2$ for some $M_1, M_2 \in \Lambda_{T_{\lambda 2}}$.

Because M is in normal form we have that $M_1 = x N_1 \dots N_k$ for some $x \in \mathcal{V}_V$, $k \in \mathbb{N}$, and some $N_1, \dots, N_k \in \Lambda_{T_{\lambda 2}} \cup T_{\lambda 2}$. We know that $x = x_A$ for some $A \in \Gamma$.

Firstly A could be a universal formula. It follows that A is of the form $A = \forall \vec{\alpha} (P^1(\alpha_1, \beta_1) \rightarrow \dots \rightarrow P^n(\alpha_n, \beta_n) \rightarrow \mathbf{false})$ where $\vec{\alpha} = \bar{\alpha}_1 \dots \bar{\alpha}_m$ for some $m \in \mathbb{N}$ and some $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathcal{V}_P$. In this case $M = x \vec{t} \vec{N}$ for some $\vec{t} = \bar{t}_1 \dots \bar{t}_m$ with $\bar{t}_1, \dots, \bar{t}_m \in T_{\lambda 2}$ and some $\vec{N} = N_1 \dots N_n$ with $N_1, \dots, N_n \in \Lambda_{T_{\lambda 2}}$. Now $\Gamma \vdash \mathbf{false}$ can be deduced as in the previous proof.

Secondly A could be an existential formula. It follows that A is of the form $A = \forall \vec{\alpha} (P^1(\alpha_1, \beta_1) \rightarrow \dots \rightarrow \forall \beta (P^n(\alpha_n, \beta_n) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$ where $\vec{\alpha} = \bar{\alpha}_1 \dots \bar{\alpha}_m$ for some $m \in \mathbb{N}$ and some $\bar{\alpha}_1, \dots, \bar{\alpha}_m \in \mathcal{V}_P$ (w.l.o.g. $\beta \notin \text{FV}(\bar{\Gamma})$ and $\beta \neq \bar{\alpha}_i$ for all $i \in \{1, \dots, m\}$). Then M has to be of the form $M = x \vec{t} \vec{N} L$ for some $\vec{t} = \bar{t}_1 \dots \bar{t}_m$ with $\bar{t}_1, \dots, \bar{t}_m \in T_{\lambda 2}$, some $\vec{N} = N_1 \dots N_{n-1}$ with $N_1, \dots, N_{n-1} \in \Lambda_{T_{\lambda 2}}$, and some $L \in \Lambda_{T_{\lambda 2}}$. It also has to hold that $\bar{\Gamma} \vdash L : \forall \beta (P^n_{s_n t_n} \rightarrow \mathbf{false})$ and for $i \in \{1, \dots, n-1\}$ that $\bar{\Gamma} \vdash N_i : P^i_{s_i t_i}$ (where $s_i = \alpha_i [\vec{\alpha} := \vec{t}]$ and $t_i = \beta_i [\vec{\alpha} := \vec{t}]$ for $i \in \{1, \dots, n\}$).

$$\begin{array}{c}
\bar{\Gamma} \vdash x : \forall \vec{\alpha} (P^1_{\alpha_1 \beta_1} \rightarrow \dots \rightarrow \forall \beta (P^n_{\alpha_n \beta_n} \rightarrow \mathbf{false}) \rightarrow \mathbf{false}) \\
\hline
\begin{array}{c}
\bar{\Gamma} \vdash x \vec{t} : P^1_{s_1 t_1} \rightarrow \dots \rightarrow \forall \beta (P^n_{s_n t_n} \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \quad \bar{\Gamma} \vdash N_1 : P^1_{s_1 t_1} \\
\vdots \\
\bar{\Gamma} \vdash x \vec{t} \vec{N} : \forall \beta (P^n_{s_n t_n} \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \quad \bar{\Gamma} \vdash L : \forall \beta (P^n_{s_n t_n} \rightarrow \mathbf{false})
\end{array} \\
\hline
\bar{\Gamma} \vdash (x \vec{t} \vec{N}) L : \mathbf{false}
\end{array}$$

For $i \in \{1, \dots, n-1\}$ we can apply Lemma 37 to $\bar{\Gamma} \vdash N_i : P^i_{s_i t_i}$ to get that $s_i, t_i \in \mathcal{V}_P$ and that $\Gamma \vdash P^i(s_i, t_i)$. But to proof that \mathbf{false} is a semantic consequence of Γ we still need $\Gamma \vdash \forall \beta (P^n(s_n, t_n) \rightarrow \mathbf{false})$.

To deduce this we have to take a closer look at L . First note that because either $\alpha_n = s_n$ or there exists an $i \in \{1, \dots, n-1\}$ such that $\alpha_n \in \text{FV}(P^i(\alpha_i, \beta_i))$ which implies that $s_n = s_i$ or $s_n = t_i$. In all cases we get that $s_n \in \mathcal{V}_P$. A similar argument yields $t_n \in \mathcal{V}_P$.

If, for some $y \in \mathcal{V}_V$, $l \in \mathbb{N}$, and $s, t, t', t_1, \dots, t_l \in T_{\lambda 2}$ with $M' := y t_1 \dots t_l$, the term L is equal to y , to $\Lambda \beta. y$, to $\Lambda \beta. M' t'$, or to $M' t'$ then $y = x_A$ for some universal formula $A = \forall \vec{\alpha} (P^n(s, t) \rightarrow \mathbf{false}) \in \Gamma$. It is easy to see that in all three cases we can indeed deduce $\Gamma \vdash \forall \beta (P^n(s_n, t_n) \rightarrow \mathbf{false})$.

If $L = \Lambda \beta. M_1 M_2$ for some $M_1, M_2 \in \Lambda_{T_{\lambda 2}}$ it follows that $M_1 = x_A \vec{t}' N'_1 \dots N'_{n'}$ for some universal formula $A \in \Gamma$, $l \in \mathbb{N}$, some $\vec{t}' = t'_1 \dots t'_l$ where $t'_1, \dots, t'_l \in T_{\lambda 2}$, $n' \in \mathbb{N}$, and some $N'_1, \dots, N'_{n'} \in \Lambda_{T_{\lambda 2}}$. We get that $L = \Lambda \beta. x_A \vec{t}' \vec{N}'$ where $\vec{N}' := N'_1 \dots N'_{n'}$. Hence, $\beta \notin \text{FV}(\bar{\Gamma})$, we can use the \forall -Introduction rule to deduce $\bar{\Gamma} \vdash x_A \vec{t}' \vec{N}' : P^n_{s_n t_n} \rightarrow \mathbf{false}$. Now we can conclude $\Gamma \vdash P^n(s_n, t_n) \rightarrow \mathbf{false}$.

as in the proof of Lemma 37. Since β is also not in $\text{FV}(\Gamma)$ we can use the \forall -Introduction of System **P** to deduce $\Gamma \vdash \forall\beta(P^n(s_n, t_n) \rightarrow \mathbf{false})$ as desired.

If $L = \Lambda\beta.\lambda y : t'.N$ for some $y \in \mathcal{V}_V$, some $t' \in T_{\lambda_2}$, and some $N \in \Lambda_{T_{\lambda_2}}$ then $t' = P^n_{s_n t_n}$. Furthermore:

$$\frac{\frac{\overline{\Gamma}, y : P^n_{s_n t_n} \vdash N : \mathbf{false}}{\overline{\Gamma} \vdash \lambda y : P^n_{s_n t_n}.N : P^n_{s_n t_n} \rightarrow \mathbf{false}}}{\overline{\Gamma} \vdash \Lambda\beta.\lambda y : P^n_{s_n t_n}.N : \forall\beta(P^n_{s_n t_n} \rightarrow \mathbf{false})}$$

Because $s_n, t_n \in \mathcal{V}_P$ we know that $P^n(s_n, t_n)$ is a valid **P**-formula. So we can apply the induction hypothesis to $\overline{\Gamma}, y : P^n_{s_n t_n} \vdash N : \mathbf{false}$ and it follows that $\Gamma, P^n(s_n, t_n) \vdash \mathbf{false}$. Now we can deduce $\Gamma \vdash \forall\beta(P^n(s_n, t_n) \rightarrow \mathbf{false})$.

$$\frac{\frac{\Gamma, P^n(s_n, t_n) \vdash \mathbf{false}}{\Gamma \vdash P^n(s_n, t_n) \rightarrow \mathbf{false}}}{\Gamma \vdash \forall\beta(P^n(s_n, t_n) \rightarrow \mathbf{false})}$$

All other forms for L (i.e. $M_1 M_2$, $\lambda y : t'.M'$, $\Lambda\beta.\Lambda\gamma.M'$, and $M' t'$ with $M' \neq y t_1 \dots t_l$) are impossible.

Now we can show that **false** is a semantic consequence of Γ .

$$\frac{\Gamma \vdash \forall\vec{\alpha}(P^1(\alpha_1, \beta_1) \rightarrow \dots \rightarrow \forall\beta(P^n(\alpha_n, \beta_n) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})}{\frac{\Gamma \vdash P^1(s_1, t_1) \rightarrow \dots \rightarrow \forall\beta(P^n(s_n, t_n) \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \quad \Gamma \vdash P^1(s_1, t_1)}{\vdots}}{\frac{\Gamma \vdash \forall\beta(P^n(s_n, t_n) \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \quad \Gamma \vdash \forall\beta(P^n(s_n, t_n) \rightarrow \mathbf{false})}{\Gamma \vdash \mathbf{false}}}$$

$M = \lambda x : t_1.M'$ for some $M' \in \Lambda_{T_{\lambda_2}}$, some $x \in \mathcal{V}_V \setminus \text{dom}(\Gamma)$, and some $t_1 \in T_{\lambda_2}$. It follows that $\mathbf{false} = t_1 \rightarrow t_2$ for some $t_2 \in T_{\lambda_2}$ which is impossible.

$M = \Lambda\gamma.M'$ for some $M' \in \Lambda_{T_{\lambda_2}}$ and some $\gamma \in \mathcal{V}_T$.

It follows that $\mathbf{false} = \forall\gamma.t'$ for some $t' \in T_{\lambda_2}$. Again is a contradiction and makes this case impossible.

$M = M' t'$ for some $M' \in \Lambda_{T_{\lambda_2}}$ and some $t' \in T_{\lambda_2}$.

Since M is in normal form we have that $M' = x M_1 \dots M_n$ for some $x \in \mathcal{V}_V$, $n \in \mathbb{N}$, and some $M_1, \dots, M_n \in \Lambda_{T_{\lambda_2}} \cup T_{\lambda_2}$. Hence, $(x : \overline{A}) \in \overline{\Gamma}$ for some **P**-formula A and $M = M' t'$, we get that this case is impossible because no such A exists.

□

Lemma 39.

$$\Gamma \vdash \mathbf{false} \quad \text{iff} \quad \text{There is a } \lambda\mathbf{2} \text{ term } M \text{ such that } \bar{\Gamma} \vdash M : \mathbf{false}$$

Proof. The \Leftarrow direction follows from Lemma 38. And the \Rightarrow direction follows from Lemma 36 with $A = \mathbf{false}$. □

Theorem 40. *The inhabitation problem for $\lambda\mathbf{2}$ is undecidable.*

Proof. From Lemma 39 it follows that $\mathbf{CONS} \leq \mathbf{INHAB}$. Since, by Theorem 34, \mathbf{CONS} is undecidable we have shown that \mathbf{INHAB} is undecidable too. □

References

- [1] H.P. Barendregt, Lambda calculi with types, Handbook of Logic in Computer Science, Volume II, 1993.
- [2] P. Urzyczyn, Inhabitation in Typed Lambda-Calculi (A Syntactic Approach), Typed Lambda Calculi and Applications, 1997.