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1 Introduction

$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
$\lambda 2$ deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ-Introduction)	$\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2}$	
(λ-Elimination)	$\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.t}$	$\alpha \notin FV(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash e : \forall\alpha.t}{\Gamma \vdash et' : t\,[\alpha := t']}$	

1.1 Basic Definitions

We will denote the set $\{1, \dots, n\}$ by $[n]$.

2 System P

2.1 Definitions

Let $V_P = \{\alpha, \beta, \dots\}$ be a countably infinite set (of variables) and $R_P = \{\mathit{false}^{(0)}, P^{(2)}, Q^{(2)}, \dots\}$ a ranked alphabet (of relation symbols). A first-order logic formula φ is an

atomic formula if $\varphi = \mathit{false}$ or $\varphi = P(\alpha, \beta)$ for some $P \in R_P$ and $\alpha, \beta \in V_P$.

universal formula if $\varphi = \forall\overrightarrow{\alpha}(A_1 \to A_2 \to \cdots \to A_n)$ where A_i is an atomic formula for $i \in [n]$, $A_i \neq \mathit{false}$ for $i \in [n-1]$ and for each $\alpha \in FV(\varphi) \cap FV(A_n)$ there exists an $i \in [n-1]$ such that $\alpha \in FV(A_i)$.

existential formula if there exits $n \geq 0$, atomic formulas $A_i \neq \mathit{false}$ for $i \in [n]$ such that $\varphi = \forall\overrightarrow{\alpha}(A_1 \to A_2 \to \cdots \to A_{n-1} \to \forall\beta(A_n \to \mathit{false}) \to \mathit{false})$.

The set of formulas of System **P** over V_P and R_P is the set of all first order formulas over the same "alphabet" that are either an atomic, universal or existential formula.
$FV(\Gamma) = \bigcup\{FV(A) \mid A \in \Gamma\}$
Deduction Rules

$$
\begin{array}{lll}
\text{(Axiom)} & \Gamma, A \vdash A & \\[1em]
(\rightarrow\text{-Introduction}) & \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} & \\[1em]
(\rightarrow\text{-Elimination}) & \dfrac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} & \\[1em]
(\forall\text{-Introduction}) & \dfrac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B} & \alpha \notin FV(\Gamma) \\[1em]
(\forall\text{-Elimination}) & \dfrac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B\,[\alpha := b]} &
\end{array}
$$

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta \ldots$
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

$$
(\exists\text{-Elimination}) \quad \frac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false} \vdash_f B} \quad a \notin FV(\Gamma, A, B)
$$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false}$ with $\mathit{false}^I = \perp$.

$$
\begin{aligned}
I \models \Gamma, \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false} &\Rightarrow I \models \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false} \\
&\Rightarrow (\forall\alpha(A \rightarrow \mathit{false}))^I \rightarrow \mathit{false}^I \\
&\Rightarrow (\forall\alpha(A \rightarrow \mathit{false}))^I \rightarrow \perp \\
&\Rightarrow \neg(\forall\alpha(A \rightarrow \mathit{false}))^I \\
&\Rightarrow \neg(\forall a \in \Delta : (A \rightarrow \mathit{false})^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \mathit{false}^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}
$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. $\square$

2.2 Provability in System P is undecidable

Γ_C :

- $Q(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$
- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$
- $D(a), D(a_i), D(b_j)$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$
- $E(a_m), E(b_n)$

$+(Q, 1, Q')$:

- $\forall\alpha\beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to R_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma))$
  prevent zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2)$:

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to Q_2(\beta))$
  jump on zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to R_1(\beta, \gamma)$
  register 1 stays zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to Q_1(\beta))$
  change state if register 1 is greater zero
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to P(\gamma, \delta) \to R_1(\beta, \delta))$
  decrement register 1
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

Lemma 1.

$$M \textit{ terminates on input } (0,0) \qquad \textit{iff} \qquad \Gamma_M \vdash \text{false} \textit{ holds in system } P.$$

Claim 2. *If a final state is reachable from C then $\Gamma_C \cup \Gamma \vdash$ false.*

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by *f*.
Induction Base trivial ...
Induction Step
$C \to^r_M D$

We need to make a case distinction on the rule r.
Case $r = +(Q, 1, Q')$
Basic idea:

$$\dfrac{\dfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \qquad \dfrac{}{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models$ *false* holds trivially if I interprets *false* with $\top$ we only need to consider models (note that there are none if M terminates which is exactly what we want to proof) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\perp$ (so we can use our new deduction rule).
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.
We first introduce the new variables needed for Γ_D (let $b, d \in V_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{S(a,b), D(b) \vdash_f f}{S(a,b) \vdash_f D(b) \to f} \qquad \dfrac{\dfrac{S(a,b) \vdash_f \forall\alpha\beta S(\alpha,\beta) \to D(\beta)}{S(a,b) \vdash_f S(a,b) \to D(b)}}{S(a,b) \vdash_f D(b)}}{S(a,b) \vdash_f f}}{\vdash_f (\forall\beta(S(a,\beta) \to f) \to f) \to f} \qquad \dfrac{\vdash_f \forall\alpha(\forall\beta(S(\alpha,\beta) \to f) \to f)}{\vdash_f \forall\beta(S(a,\beta) \to f) \to f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

The formula $R_1(b, d)$ can be acquired in a similar way.
Now we create Γ_D

$$\dfrac{\dfrac{Q'(b) \vdash f}{\vdash_f Q'(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \qquad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \qquad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

Alternative tableau with tikz:

$$\dfrac{\dfrac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \qquad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \qquad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

Starting from $Q'(b) \vdash_f$ *false* we can deduce:

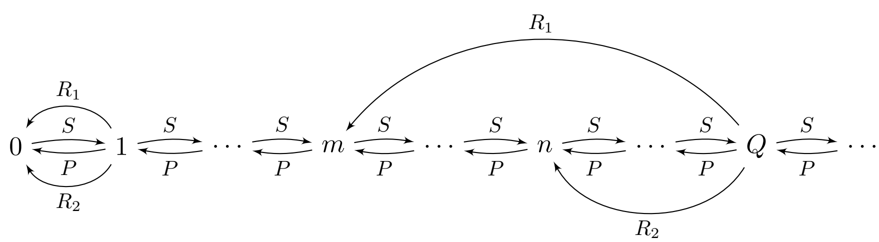
$$
\dfrac{\dfrac{P(d,a_0) \vdash_f f}{\vdash_f P(d,a_0) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to R_1(\beta,\delta) \to P(\delta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f R_1(a,a_0)}{\vdash_f R_1(b,d) \to Q'(b)} \quad \vdash_f R_1(b,d)}{\vdash_f P(d,a_0)}}{\vdash_f f}
$$

$R_2(b, b_0)$ can be deduced in the same way.
Now we have Γ_C (Since $P(a_{i-1}, a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.
Case $r = -(Q, 1, Q_1, Q_2)$ □

Claim 3.

$$\Gamma_M \vdash \text{false } \textit{holds in system } P \qquad \Longrightarrow \qquad M \textit{ terminates on input } (0,0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_0 \Rightarrow_M C_1 \Rightarrow_M C_3 \Rightarrow_M \ldots$ ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of Γ_M which interprets *false* with $\perp$ this contradicts $\Gamma_M \vdash$ *false*.
The idea looks like this:



Formal definition:
$I = (\mathbb{N}, \cdot^I)$

$$
\begin{array}{lll}
P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\
Q^I = \{i \in \mathbb{N} \mid Q = Q_i\} & D^I = \mathbb{N} \setminus \{0\} & E^I = \{0\} \\
S^I = \{(i, i+1) \mid i \in \mathbb{N}\} & &
\end{array}
$$

□