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# 1 Introduction

## 2 Basic Definitions

### 2.1 $\lambda$ -calculus $\lambda_2$

In the following let  $\mathcal{V}_T = \{\alpha, a, \beta, b, \dots\}$  be a countable set (of type-variables) and  $\mathcal{V}_V = \{x_1, x_2, \dots\}$  be a countable set (of value-variables).

**Definition 1.** The set of all  $\lambda_2$  types over  $\mathcal{V}_T$ , denoted by  $T_{\lambda_2}$ , is the smallest set  $T$  satisfying the following conditions:

- $\mathcal{V}_T \subseteq T$ ,
- if  $t_1, t_2 \in T$  then  $(t_1 \rightarrow t_2) \in T$ , and
- if  $t \in T$  and  $\alpha \in \mathcal{V}_T$  then  $\forall \alpha. t \in T$ .

The set of all  $\lambda_2$  terms over  $\mathcal{V}_T$  and  $\mathcal{V}_V$ , denoted by  $\Lambda_{T_{\lambda_2}}$ , is the smallest set  $\Lambda_T$  satisfying the following conditions:

- $\mathcal{V}_V \subseteq \Lambda_T$ ,
- if  $e_1, e_2 \in \Lambda_T$  then  $e_1 e_2 \in \Lambda_T$ ,
- if  $x \in \mathcal{V}_V$ ,  $t \in T_{\lambda_2}$ , and  $e \in \Lambda_T$  then  $\lambda x : t. e \in \Lambda_T$ ,
- if  $\alpha \in \mathcal{V}_T$  and  $e \in \Lambda_T$  then  $\Lambda \alpha. e \in \Lambda_T$ , and
- if  $e \in \Lambda_T$  and  $t \in T_{\lambda_2}$  then  $e t \in \Lambda_T$ .

**Definition 2.** Let  $e \in \Lambda_{T_{\lambda_2}}$ . The free variables of  $e$ , denoted by  $FV(e)$ , are defined inductively as follows:

$$FV(e) = \begin{cases} \{x\} & \text{if } e = x \\ FV(e_1) \cup FV(e_2) & \text{if } e = e_1 e_2 \\ FV(e') \setminus \{x\} & \text{if } e = \lambda x : t. e' \\ FV(e') & \text{if } e = \Lambda \alpha. e' \\ FV(e') & \text{if } e = e' t \end{cases}$$

Or is this definition better?

**Definition 3.** Let  $e \in \Lambda_{T_{\lambda_2}}$ . The free variables of  $e$ , denoted by  $FV(e)$ , are defined inductively as follows:

$$\begin{aligned} FV(y) &= \{x\} \\ FV(e_1 e_2) &= FV(e_1) \cup FV(e_2) \\ FV(\lambda x : t. e') &= FV(e') \setminus \{x\} \\ FV(\Lambda \alpha. e') &= FV(e') \\ FV(e' t) &= FV(e') \end{aligned}$$

**Definition 4.** Let  $\mathcal{V} = \{x_1, \dots, x_n\}$  be a finite subset of  $\mathcal{V}_T$  and  $t_1, \dots, t_n \in \Lambda_{T_{\lambda 2}}$ . A  $\lambda 2$ -basis  $\Gamma = \{(x_1 : t_1), \dots, (x_n : t_n)\}$  is a mapping from  $\mathcal{V}$  to  $T_{\lambda 2}$ . If the kind of basis is clear from the context we abbreviate  $\lambda 2$ -basis to basis.

The free variables of a basis  $\Gamma$ , denoted by  $FV(\Gamma)$ , are  $\bigcup\{FV(t) \mid (x : t) \in \Gamma\}$ .

For a basis  $\Gamma$ ,  $x \in \mathcal{V}_V \setminus \text{dom}(\Gamma)$ , and  $t \in T_{\lambda 2}$  we will abbreviate  $\Gamma \cup \{(x : t)\}$  to  $\Gamma, x : t$ .

**Definition 5.** Let  $e$  be in  $\Lambda_{T_{\lambda 2}}$ ,  $t$  in  $T_{\lambda 2}$ , and  $\Gamma$  be a basis. A statement  $e : t$  is derivable from  $\Gamma$ , denoted by  $\Gamma \vdash e : t$ , if  $e : t$  can be produced using the following rules.

(Axiom)	$\Gamma, x : t \vdash x : t$	
( $\lambda$ -Introduction)	$\frac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x : t_1. e : t_1 \rightarrow t_2}$	
( $\lambda$ -Elimination)	$\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
( $\forall$ -Introduction)	$\frac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. t}$	$\alpha \notin FV(\Gamma)$
( $\forall$ -Elimination)	$\frac{\Gamma \vdash e : \forall \alpha. t}{\Gamma \vdash e t' : t[\alpha := t']}$	$t' \in T_{\lambda 2}$

**Definition 6.** The inhabitation problem for  $\lambda 2$ , denoted by **INHAB**, is defined as follows. Given a  $\lambda 2$  type  $t$ .

Is there a  $\lambda 2$  term  $M$  such that  $\emptyset \vdash M : t$ ?

But we can rephrase this problem so that it becomes more general: Given a basis  $\Gamma$  and a  $\lambda 2$  type  $t$ .

Is there a  $\lambda 2$  term  $M$  such that  $\Gamma \vdash M : t$ ?

Obviously the second version is a special case of the first one. For the other direction consider a basis  $\Gamma = \{(x_1 : t_1), \dots, (x_n : t_n)\}$  and a  $\lambda 2$  type  $t$ . Clearly, for every term  $M$ ,  $\Gamma \vdash M : t$  holds iff  $\emptyset \vdash \lambda x_1 : t_1. \dots \lambda x_n : t_n. M : t_1 \rightarrow \dots \rightarrow t_n \rightarrow t$ .

## 2.2 first-order logic

**Definition 7.** A ranked set is a tuple  $(\Sigma, rk)$ , where  $\Sigma$  is a countable set and  $rk : \Sigma \rightarrow \mathbb{N}$  is a function that maps every symbol from  $\Sigma$  to a natural number (its rank).

If the function  $rk$  is understood we will just write  $\Sigma$  instead of  $(\Sigma, rk)$ . The set of all elements in  $\Sigma$  with a certain rank  $k$ , denoted by  $\Sigma^{(k)}$ , is defined as  $\Sigma^{(k)} := rk^{-1}(k)$ .

For the remainder of this subsection let  $\mathcal{V} = \{y_1, y_2, \dots\}$  be a countable set (of variables),  $\mathcal{F}$  a ranked set (of function symbols), and  $\mathcal{P}$  a ranked set (of predicate symbols).

**Definition 8.** The set of terms over  $\mathcal{V}$  and  $\mathcal{F}$ , denoted by  $\mathcal{T}_{(\mathcal{V}, \mathcal{F})}$ , is the smallest set  $\mathcal{T}$  satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$ , and
- for every  $k \in \mathbb{N}$  if  $f \in \mathcal{F}^{(k)}$  and  $t_1, t_2, \dots, t_k \in \mathcal{T}$  then  $f(t_1, t_2, \dots, t_k) \in \mathcal{T}$ .

The set of first-order formulas over  $\mathcal{V}$ ,  $\mathcal{F}$ , and  $\mathcal{P}$ , denoted by  $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ , is the smallest set  $\mathcal{L}$  satisfying the following conditions:

- for every  $k \in \mathbb{N}$  if  $P \in \mathcal{P}^{(k)}$  and  $t_1, t_2, \dots, t_k \in \mathcal{T}_{(\mathcal{V}, \mathcal{F})}$  then  $P(t_1, t_2, \dots, t_k) \in \mathcal{L}$ .
- If  $\varphi, \psi \in \mathcal{L}$  then  $(\varphi \wedge \psi)$ ,  $(\varphi \vee \psi)$ ,  $\neg \varphi \in \mathcal{L}$ , and
- if  $y \in \mathcal{V}$  and  $\varphi \in \mathcal{L}$  then  $\exists y.\varphi$ ,  $\forall y.\varphi \in \mathcal{L}$ .

We introduce an additional binary operation  $\rightarrow$  on formulas, where for some  $\varphi, \psi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$  the formula  $(\varphi \rightarrow \psi)$  is defined as  $(\neg \varphi \vee \psi)$ . For nullary relation symbols  $P$  we will abbreviate  $P()$  to  $P$ . If a formula  $\varphi$  is of the form  $Qx.(\psi)$  (where  $Q \in \{\exists, \forall\}$ ,  $x \in \mathcal{V}$ , and  $(\psi) \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ ) we often drop the dot and write  $Qx(\psi)$  instead. If a formula  $\varphi$  has multiple variables binded by the same quantifier (i.e.  $\varphi = Qx_1.Qx_2 \dots Qx_n.\psi$  for  $Q \in \{\exists, \forall\}$ , some  $n \in \mathbb{N}$ ,  $x_1, x_2, \dots, x_n \in \mathcal{V}$ , and  $\psi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ ) we abbreviate  $\varphi$  to  $Qx_1x_2 \dots x_n.\psi$  or to  $Q\vec{x}.\psi$  where  $\vec{x} = (x_1, x_2, \dots, x_n)^\top$ .

**Definition 9.** The variables of a term  $t \in \mathcal{T}_{(\mathcal{V}, \mathcal{F})}$ , denoted by  $V(t)$ , are defined by:

$$V(t) = \begin{cases} \{y\} & \text{if } t = y \\ V(t_1) \cup V(t_2) \cup \dots \cup V(t_k) & \text{if } t = f(t_1, t_2, \dots, t_k) \end{cases}$$

The free variables of a formula  $\varphi \in \mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ , denoted by  $FV(\varphi)$ , are defined as follows:

$$FV(\varphi) = \begin{cases} V(t_1) \cup V(t_2) \cup \dots \cup V(t_k) & \text{if } \varphi = P(t_1, t_2, \dots, t_k) \\ FV(\psi) & \text{if } \varphi = \neg \psi \\ FV(\varphi_1) \cup FV(\varphi_2) & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \text{ or } \varphi = (\varphi_1 \vee \varphi_2) \\ FV(\psi) \setminus \{y\} & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \end{cases}$$

**Definition 10.** Let  $y$  be in  $\mathcal{V}$  and  $t, t' \in \mathcal{T}_{(\mathcal{V}, \mathcal{F})}$ . The substitution of  $y$  by  $t'$  in  $t$ , denoted by  $t[y := t']$ , is defined as follows:

$$t[y := t'] = \begin{cases} t' & \text{if } t = y \\ z & \text{if } t = z \text{ and } z \neq y \\ f(t_1[x := t'], \dots, t_k[x := t']) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Now we can lift this definition to formulas, let  $\varphi$  be in  $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ . The substitution of  $y$  by  $t'$  in  $\varphi$ , denoted by  $\varphi[y := t']$ , is defined as follows:

$$\varphi[x := t'] = \begin{cases} P(t_1[y := t'], \dots, t_k[y := t']) & \text{if } \varphi = P(t_1, \dots, t_k) \\ \neg(\psi[y := t']) & \text{if } \varphi = \neg\psi \\ \varphi_1[y := t'] \circ \varphi_2[y := t'] & \text{if } \varphi = (\varphi_1 \circ \varphi_2), \circ \in \{\wedge, \vee\} \\ \varphi & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \\ Qz.(\psi[y := t']) & \text{if } \varphi = Qz.\psi, Q \in \{\forall, \exists\} \text{ and } z \neq y \end{cases}$$

Now we come to the semantics of first-order formulas.

**Definition 11.** An interpretation  $I$  over  $(\mathcal{V}, \mathcal{F}, \mathcal{P})$  is a triple  $(\Delta, \cdot^I, \omega)$  where  $\Delta$  is a nonempty set (which we call domain),  $\cdot^I$  is a function such that  $f^I: \Delta^k \rightarrow \Delta$  is a function for every  $k \in \mathbb{N}$ ,  $f \in \mathcal{F}^{(k)}$  and  $P^I \subseteq \Delta^k$  is a relation for every  $k \in \mathbb{N}$ ,  $P \in \mathcal{P}^{(k)}$  and  $\omega$  is a function from  $\mathcal{V}$  to  $\Delta$ .

Let  $I = (\Delta, \cdot^I, \omega)$  be an interpretation,  $x \in \mathcal{V}$ , and  $d \in \Delta$  the interpretation  $I[x \mapsto d]$  is defined as  $(\Delta, \cdot^I, \omega[x \mapsto d])$  where

$$(\omega[x \mapsto d])(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{otherwise.} \end{cases}$$

**Definition 12.** Let  $I = (\Delta, \cdot^I, \omega)$  be an interpretation and  $t$  a term. The interpretation of  $t$  under  $I$ , denoted by  $t^I$ , is defined as follows:

$$t^I = \begin{cases} \omega(x) & \text{if } t = x \\ f^I(t_1^I, \dots, t_k^I) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Let  $\varphi$  be a formula. The interpretation of  $\varphi$  under  $I$ , denoted by  $\varphi^I$ , is defined recursively as follows:

$$\varphi^I = \begin{cases} \top & \text{if } \varphi = P(t_1, \dots, t_k) \text{ and } (t_1^I, \dots, t_k^I) \in P^I \\ \perp & \text{if } \varphi = P(t_1, \dots, t_k) \text{ and } (t_1^I, \dots, t_k^I) \notin P^I \\ \text{not } \psi^I & \text{if } \varphi = \neg\psi \\ \varphi_1^I \text{ and } \varphi_2^I & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \\ \varphi_1^I \text{ or } \varphi_2^I & \text{if } \varphi = (\varphi_1 \vee \varphi_2) \\ \text{exists } d \in \Delta \psi^I[x \mapsto d] & \text{if } \varphi = \exists x.\psi \\ \text{forall } d \in \Delta \psi^I[x \mapsto d] & \text{if } \varphi = \forall x.\psi \end{cases}$$

The interpretation  $I$  is a model of  $\varphi$ , denoted by  $I \models \varphi$ , if  $\varphi^I = \top$ .

When we define an interpretation  $I$  and we have a nullary predicate symbol  $P$  we write  $P^I = \top$  instead of  $P^I = \{()\}$  and  $P^I = \perp$  for  $P^I = \emptyset$  (this works because  $P()^I = \top$  iff  $() \in P^I$ ).

**Definition 13.** Let  $\Gamma$  be a finite set of first-order formulas.

We say that an interpretation  $I$  is a model of  $\Gamma$ , denoted by  $I \models \Gamma$ , if  $I \models \psi$  for every  $\psi$  in  $\Gamma$ .

The formula  $\varphi$  is a semantic consequence of  $\Gamma$ , denoted by  $\Gamma \vdash \varphi$ , if every model of  $\Gamma$  is also a model of  $\varphi$ .

The free variables of  $\Gamma$ , denoted by  $\text{FV}(\Gamma)$ , are  $\bigcup \{\text{FV}(\varphi) \mid \varphi \in \Gamma\}$ .

### 2.3 two-counter automaton

We will use a version of two-counter automaton which only has two types of transitions. First it can increment a register and second it can decrement a register or jump if the register is already zero. Formally:

**Definition 14.** A deterministic two-counter automaton is a 4-tuple  $M = (\mathcal{Q}, Q_0, Q_f, R)$ ,

- where  $\mathcal{Q}$  is a finite set (of states),
- $Q_0$  is in  $\mathcal{Q}$  (the initial state),
- $Q_f$  is in  $\mathcal{Q}$  (the final state), and
- $R$  is a function from  $\mathcal{Q} \setminus \{Q_f\}$  to  $\mathcal{R}_{\mathcal{Q}}$ ,  
where  $\mathcal{R}_{\mathcal{Q}} = \{+(i, Q') \mid i \in \{1, 2\}, Q' \in \mathcal{Q}\} \cup \{-(i, Q_1, Q_2) \mid i \in \{1, 2\}, Q_1, Q_2 \in \mathcal{Q}\}$

A configuration  $C$  of our automaton is a triple  $\langle Q, m, n \rangle$ , where  $Q \in \mathcal{Q}$  and  $m, n \in \mathbb{N}$ . Let  $r$  be in  $R(\mathcal{Q} \setminus \{Q_f\})$ , then  $\Rightarrow_M^r$  is a binary relation on the configurations of  $M$  such that two configurations  $\langle Q, m, n \rangle, \langle \hat{Q}, \hat{m}, \hat{n} \rangle$  of  $M$  are in the relation if all of the following conditions hold:

- $Q \neq Q_f, r = R(Q)$ ,
- if  $r = +(1, Q')$  for some  $Q' \in \mathcal{Q}$  then  $\hat{Q} = Q', \hat{m} = m + 1$ , and  $\hat{n} = n$ ,
- if  $r = +(2, Q')$  for some  $Q' \in \mathcal{Q}$  then  $\hat{Q} = Q', \hat{m} = m$ , and  $\hat{n} = n + 1$ ,
- if  $r = -(1, Q_1, Q_2)$  for some  $Q_1, Q_2 \in \mathcal{Q}$  then
  - if  $m = 0$  then  $\hat{Q} = Q_2, \hat{m} = 0$ , and  $\hat{n} = n$ ,
  - if  $m \geq 1$  then  $\hat{Q} = Q_1, \hat{m} = m - 1$ , and  $\hat{n} = n$ ,
- if  $r = -(2, Q_1, Q_2)$  for some  $Q_1, Q_2 \in \mathcal{Q}$  then
  - if  $n = 0$  then  $\hat{Q} = Q_2, \hat{m} = m$ , and  $\hat{n} = 0$ ,
  - if  $n \geq 1$  then  $\hat{Q} = Q_1, \hat{m} = m$ , and  $\hat{n} = n - 1$ .

The transition relation of  $M$ , denoted by  $\Rightarrow_M$ , is defined as  $\bigcup_{r \in R(Q \setminus \{Q_f\})} \Rightarrow_M^r$ . We denote the transitive reflexive closure of  $\Rightarrow_M$  by  $\Rightarrow_M^*$ .

Let  $m, n$  be in  $\mathbb{N}$ , we say that  $M$  terminates on input  $(m, n)$  if there exist  $\hat{m}, \hat{n} \in \mathbb{N}$  such that  $\langle Q_0, m, n \rangle \Rightarrow_M^* \langle Q_f, \hat{m}, \hat{n} \rangle$  (It follows that there exists an  $i \in \mathbb{N}$  and configurations  $D_1, \dots, D_i$  of  $M$  such that  $\langle Q_0, m, n \rangle = D_1 \Rightarrow_M \dots \Rightarrow_M D_i = \langle Q_f, \hat{m}, \hat{n} \rangle$ , we call this chain a computation with length  $i$ ).

**Definition 15.** The halting problem for two-counter automaton, denoted by **HALT**, is defined as follows. Given a two-counter automaton  $M$ .

Does  $M$  terminate on input  $(0, 0)$ ?

It is well known that **HALT** is undecidable.

## 3 System P

### 3.1 Definitions

In the following let  $\mathcal{V}_P = \{\alpha, a, \beta, b, \dots\}$  be a countably infinite subset of  $\mathcal{V}_T$  (of variables). Let  $\mathcal{P}_P = \{P, Q, \dots\}$  be a set (of predicate symbols) and  $\mathcal{P}$  a ranked set such that  $\mathcal{P}^{(0)} = \{\mathbf{false}\}$ ,  $\mathcal{P}^{(2)} = \mathcal{P}_P$ , and  $\mathcal{P}^{(k)} = \emptyset$  for all  $k \in \mathbb{N} \setminus \{0, 2\}$ . A first-order logic formula  $\varphi$  over  $(\mathcal{V}_P, \emptyset, \mathcal{P})$  is an

**atomic formula** if  $\varphi = \mathbf{false}$  or  $\varphi = P(a, b)$  for some  $P \in \mathcal{P}_P$  and  $a, b \in \mathcal{V}_P$ .

**universal formula** if  $\varphi = \forall \vec{\alpha}(A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n)$  for some  $n \in \mathbb{N}$  and where  $A_i$  is an atomic formula for  $i \in \{1, \dots, n\}$ ,  $A_i \neq \mathbf{false}$  for  $i \in \{1, \dots, n-1\}$  and for each  $\alpha \in \text{FV}(\varphi) \cap \text{FV}(A_n)$  there exists an  $i \in \{1, \dots, n-1\}$  such that  $\alpha \in \text{FV}(A_i)$ .

**existential formula** if there exists an  $n \in \mathbb{N}$ , atomic formulas  $A_i \neq \mathbf{false}$  for  $i \in \{1, \dots, n\}$  such that  $\varphi = \forall \vec{\alpha}(A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n-1} \rightarrow \forall \beta(A_n \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$ .

The set of formulas of System **P** (= set of **P**-formulas) over  $(\mathcal{V}_P, \mathcal{P}_P)$  is the set of all first-order formulas over  $(\mathcal{V}_P, \emptyset, \mathcal{P})$  that are either an atomic, universal or existential formula.

**Definition 16.** A finite set of **P**-formulas  $\Gamma$  is called **P**-basis, or basis if it is clear from the context whether a **P**-basis or a  **$\lambda 2$** -basis is meant.

For a **P**-basis  $\Gamma$  and a **P**-formula  $A$  we will abbreviate  $\Gamma \cup \{A\}$  to  $\Gamma, A$  (c.f.  **$\lambda 2$** -basis).

**Definition 17.** Let  $A$  be a **P**-formula, and  $\Gamma$  be a basis. The formula  $A$  is a semantic consequence of  $\Gamma$  if  $A$  can be produced using the following deduction rules.

(Axiom)	$\Gamma, A \vdash A$	
( $\rightarrow$ -Introduction)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
( $\rightarrow$ -Elimination)	$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
( $\forall$ -Introduction)	$\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin FV(\Gamma)$
( $\forall$ -Elimination)	$\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$	$b \in \mathcal{V}_P$

We define a more general consequence relation in which we demand that **false** is interpreted with  $\perp$ . In this relation existential formulas will behave like the name suggests. Formally:

**Definition 18.** Let  $\Gamma$  be a basis. The **P**-formula  $A$  is a sematic consequence with falsity of  $\Gamma$ , denoted by  $\Gamma \vdash_f A$ , if for every interpretation  $I$

$$I \models \Gamma \text{ and } \mathbf{false}^I = \perp \text{ implies } I \models A.$$

This allows us to add the following deduction rule.

( $\exists$ -Introduction)	$\frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \vdash_f B}$	$a \notin FV(\Gamma, A, B)$
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*Proof.* Let  $I = (\Delta, \cdot^I, \omega)$  be a model of  $\Gamma, \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false}$  with  $\mathbf{false}^I = \perp$  and  $a \in \mathcal{V}_P$  a variable such that  $a \notin FV(\Gamma, A, B)$ .

$$\begin{aligned}
I \models \Gamma, \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} &\Rightarrow I \models \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \\
&\Rightarrow (\forall \alpha (A \rightarrow \mathbf{false}))^I \rightarrow \mathbf{false}^I \\
&\Rightarrow (\forall \alpha (A \rightarrow \mathbf{false}))^I \rightarrow \perp \\
&\Rightarrow \neg(\forall \alpha (A \rightarrow \mathbf{false}))^I \\
&\Rightarrow \neg(\forall d \in \Delta: (A \rightarrow \mathbf{false})^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta: \neg(A^{I[\alpha \mapsto d]} \rightarrow \mathbf{false}^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta: \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta: \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta: A^{I[\alpha \mapsto d]}
\end{aligned}$$



Together with  $a \notin FV(\Gamma, A)$ , it follows that  $I[a \mapsto d]$  is a model of  $\Gamma, A[\alpha := a]$ . Which implies  $I[a \mapsto d] \models B$ . Since  $a$  is not free in  $B$  we conclude that  $I$  is also a model of  $B$ .  $\square$

**Definition 19.** The problem to decide whether a given set of **P**-formulas is consistent, denoted by **CONS**, is defined as follows. Given a set of **P**-formulas  $\Gamma$ .

Does  $\Gamma \vdash \mathbf{false}$  not hold?

### 3.2 CONS is undecidable

We will show that **HALT**  $\leq$  **CONS** then the undecidability of **CONS** directly follows from the undecidability of **HALT**. For a given two-counter automaton  $M$  we will effectively construct a **P**-basis  $\Gamma_M$  such that

$$M \text{ terminates on input } (0, 0) \quad \text{iff} \quad \Gamma_M \vdash \mathbf{false} \text{ holds in system } P.$$

Let  $M = (\mathcal{Q}, Q_0, Q_f, R)$  be a two-counter automaton, w.l.o.g.  $S, P, R_1, R_2, E, D \notin \mathcal{Q}$ . In the following we will consider **P**-formulas over  $(\mathcal{V}_P, \mathcal{P}_P)$ , where  $\mathcal{P}_P = \mathcal{Q} \uplus \{S, P, R_1, R_2, E, D\}$ . We will abbreviate  $P(a, a)$  to  $P(a)$ , note that this way we can use binary predicate symbols as unary ones.

Intuitively  $Q(a)$  stands for “ $a$  is in state  $Q$ ”,  $R_i(a, m)$  stands for “in  $a$  the value of register  $i$  is  $m$ ” for  $i \in \{1, 2\}$ ,  $S(a, b)$  states that “ $b$  is a successor of  $a$ ”,  $P(a, b)$  states that “ $b$  is a predecessor of  $a$ ”,  $E(a)$  marks “ $a$  as the end of chain”, and  $D(a)$  states that “ $a$  is not the end of a chain”.

For a configuration  $C = \langle Q, m, n \rangle$  of  $M$  we define a set of **P**-formulas  $\Gamma_C$ . It contains the following formulas:

- $Q(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$  for  $i \in \{1, \dots, m\}$
- $R_2(a, b_0), P(b_{i-1}, b_i)$  for  $i \in \{1, \dots, n\}$
- $D(a), D(a_i), D(b_j)$  for  $i \in \{0, \dots, m-1\}$  and  $j \in \{0, \dots, n-1\}$
- $E(a_m), E(b_n)$

Next we need sets of **P**-formulas for all possible transitions. For every  $Q \in \mathcal{Q} \setminus \{Q_f\}$  and  $r \in \mathcal{R}_Q$  we define  $\Gamma_{Q,r}$ . If  $r = +(1, Q')$  for some  $Q' \in \mathcal{Q}$  then  $\Gamma_{Q,+(1,Q')}$  contains the following formulas:

- $\forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$   
change of state
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))$   
increment register 1

- $\forall \alpha \beta \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\beta, \delta) \rightarrow D(\delta))$   
prevent zero in register 1
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$   
do not change the value register 2

If  $r = -(1, Q_1, Q_2)$  for some  $Q_1, Q_2 \in \mathcal{Q}$  then  $\Gamma_{Q, -(1, Q_1, Q_2)}$  contains the following formulas:

- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$   
jump to  $Q_2$  if register 1 is zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow R_1(\beta, \gamma))$   
if register 1 is zero it stays zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$   
change state to  $Q_1$  if register 1 is greater zero
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow R_1(\beta, \delta))$   
decrement register 1
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$   
do not change register 2 in both cases

For  $r = +(2, Q')$  for some  $Q' \in \mathcal{Q}$  or  $r = -(2, Q_1, Q_2)$  for some  $Q_1, Q_2 \in \mathcal{Q}$  the sets  $\Gamma_{Q, r}$  are defined analogously.

We also need a set  $\Gamma_1$  to ensure that our representation works correctly. The following formula are in  $\Gamma_1$ :

- $\forall \alpha \beta (S(\alpha, \beta) \rightarrow D(\beta))$   
no successor is the end of a chain
- $\forall \alpha (D(\alpha) \rightarrow \forall \beta (R_1(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$   
every element that represents a configuration has a value for register 1
- $\forall \alpha (D(\alpha) \rightarrow \forall \beta (R_2(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$   
every element that represents a configuration has a value for register 2
- $\forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow \mathbf{false}) \rightarrow \mathbf{false})$   
every element has a successor

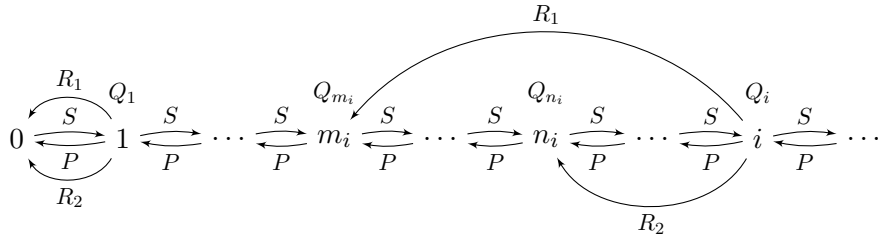
We define  $\Gamma_{\overline{M}}$  as  $\bigcup_{Q \in \mathcal{Q} \setminus \{Q_f\}} \Gamma_{Q, R(Q)} \cup \{\forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})\} \cup \Gamma_1$ . Finally we can define  $\Gamma_M$  as  $\Gamma_{C_1} \cup \Gamma_{\overline{M}}$ , where  $C_1 = \langle Q_0, 0, 0 \rangle$  is the initial configuration.

**Claim 20.**

$$\Gamma_M \vdash \mathbf{false} \text{ holds in system } P \quad \Longrightarrow \quad M \text{ terminates on input } (0, 0)$$

*Proof.* Assume  $M$  does not terminate then there is an infinite chain  $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \dots$  ( $C_i = \langle Q_i, m_i, n_i \rangle$  for  $i \in \mathbb{N}^+$ ). Now we construct a model of  $\Gamma_M$  which interprets **false** with  $\perp$  this contradicts  $\Gamma_M \vdash \mathbf{false}$ .

To illustrate the idea we will use a graphical notation for an interpretation  $I$ . By  $d_1 \xrightarrow{R} d_2$  we say that  $(d_1, d_2) \in R^I$ . And we use  $\overset{P}{d}$  to say that  $(d, d) \in P^I$  for predicate symbols that are used as unary predicate symbols. As domain for our interpretation we will use the natural numbers. Every number will have two tasks: firstly it will represent itself as a possible value for register 1 or 2 and secondly every number  $i$  greater than zero will also represent the  $i^{\text{th}}$  configuration of our infinite computation. Now the idea for our model of  $\Gamma_M$  looks like this:



We have  $0 \in E^I$  and all other numbers are in  $D^I$ .

Here is the more formal definition of our model  $I = (\mathbb{N}, \cdot^I, \omega)$ .

$$\begin{aligned} P^I &= \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I &= \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I &= \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I &= \{(i, i) \mid i \in \mathbb{N}^+, Q = Q_i\} & D^I &= \{(i, i) \mid i \in \mathbb{N}^+\} & E^I &= \{(0, 0)\} \\ S^I &= \{(i, i+1) \mid i \in \mathbb{N}\} & \mathbf{false}^I &= \perp \end{aligned}$$

$$a^I = 1$$

$$a_0^I = 0$$

$$b_0^I = 0$$

Since there are no free variables in  $\Gamma_M$  we can just set  $\omega(x) = 0$  for every  $x \in \mathcal{V}_P$ . It is easy to see that  $I$  is indeed a model of  $\Gamma_M$ .  $\square$

We proof the other direction by induction on the length of the computation. But to be able to use the induction hypothesis we need a slightly more general statement.

**Claim 21.** *Let  $C = \langle Q, m, n \rangle$  be a configuration of  $M$ . If a final configuration (i.e. a configuration  $\langle Q_f, \hat{m}, \hat{n} \rangle$  for some  $\hat{n}, \hat{m} \in \mathbb{N}$ ) is reachable from  $C$  then  $\Gamma_C \cup \Gamma_{\overline{M}} \vdash \mathbf{false}$ .*

*Proof.* By induction on the length  $i$  of the computation.

Induction Base:  $i = 0$

Since a final configuration is reachable in 0 steps  $C$  must be this final configuration. So  $C = \langle Q_f, m, n \rangle$  for some  $n, m \in \mathbb{N}$ . Hence,  $Q_f(a)$  is in  $\Gamma_C$  for some  $a \in \mathcal{V}_P$  and  $\forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})$  is in  $\Gamma_{\overline{M}}$ , we can easily deduce false.

$$\frac{\frac{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \forall \alpha (Q_f(\alpha) \rightarrow \mathbf{false})}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a) \rightarrow \mathbf{false}} \quad \Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a)}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \mathbf{false}}$$

Induction Step:  $i = i' + 1$

Since  $I \models \mathbf{false}$  holds trivially if  $I$  interprets **false** with  $\top$  we only need to consider models of  $\Gamma_C \cup \Gamma_{\overline{M}}$  that interpret **false** with  $\perp$  (note that there are no such models if  $M$  terminates which is exactly what we want to proof). As result of this observation we can use the  $\exists$ -Introduction rule.

From the fact that a final configuration is reachable from  $C$  in  $i$  steps we can deduce that there exists a configuration  $D = \langle \hat{Q}, \hat{m}, \hat{n} \rangle$  such that  $C \Rightarrow_M^r D$  for some  $r \in \mathcal{R}_{\mathcal{Q}}$  and a final configuration is reachable from  $D$  in  $i'$  steps. We also know that  $C = \langle Q, m, n \rangle$  for some  $Q \in \mathcal{Q} \setminus \{Q_f\}$  and some  $m, n \in \mathbb{N}$ . The set  $\Gamma_C$  contains the formulas:

$R_1(a, a_0), P(a_{i-1}, a_i)$  and  $D(a_{i-1})$  for  $i \in \{1, \dots, n\}$ ,

$R_2(a, b_0), P(b_{i-1}, b_i)$  and  $D(b_{i-1})$  for  $i \in \{1, \dots, m\}$ ,

$Q(a), D(a), E(a_n)$  and  $E(b_m)$ .

And  $\Gamma_D$  contains the formulas:

$R_1(\hat{a}, \hat{a}_0), P(\hat{a}_{i-1}, \hat{a}_i)$  and  $D(\hat{a}_{i-1})$  for  $i \in \{1, \dots, \hat{n}\}$ ,

$R_2(\hat{a}, \hat{b}_0), P(\hat{b}_{i-1}, \hat{b}_i)$  and  $D(\hat{b}_{i-1})$  for  $i \in \{1, \dots, \hat{m}\}$ ,

$Q(\hat{a}), D(\hat{a}), E(\hat{a}_{\hat{n}})$  and  $E(\hat{b}_{\hat{m}})$ .

The basic idea is to deduce  $\Gamma_D$  from  $\Gamma_C \cup \Gamma_{\overline{M}}$  and then apply the induction hypothesis to  $\Gamma_D \cup \Gamma_{\overline{M}}$ .

$$\frac{\frac{\text{Induction Hypothesis}}{\Gamma_C \cup \Gamma_{\overline{M}} \cup \Gamma_D \vdash_f \mathbf{false}} \quad \Gamma_C \cup \Gamma_{\overline{M}} \vdash_f \Gamma_D}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash_f \mathbf{false}}$$

We achieve this by looking at the four possible cases for the type of the rule  $r$ . We will only consider the cases  $r = +(1, Q')$  and  $r = -(1, Q_1, Q_2)$ , the two remaining cases  $r = +(2, Q')$  and  $r = -(2, Q_1, Q_2)$  follow by exchanging the roles of register 1 and register 2 in the first two cases.

First we need a new free variable representing the configuration  $D$ . Also the value in register 2 does not change, because in both cases we are only concerned with register 1. For the succeeding tableau proofs we will abbreviate **false** by **f** and we will drop  $\Gamma_C \cup \Gamma_{\overline{M}}$  and only write new formulas on the left side of  $\vdash_f$ .

We first introduce a new variable representing  $D$  (let  $b \in \mathcal{V}_P \setminus \text{FV}(\Gamma_C \cup \Gamma_{\overline{M}})$ ).

$$\begin{array}{c}
\vdots \\
\frac{S(a, b), D(b) \vdash_{\mathbf{f}} \mathbf{f}}{S(a, b) \vdash_{\mathbf{f}} D(b) \rightarrow \mathbf{f}} \quad \frac{S(a, b) \vdash_{\mathbf{f}} \forall \alpha \beta (S(\alpha, \beta) \rightarrow D(\beta))}{S(a, b) \vdash_{\mathbf{f}} S(a, b) \rightarrow D(b)} \quad \frac{S(a, b) \vdash_{\mathbf{f}} S(a, b)}{S(a, b) \vdash_{\mathbf{f}} D(b)} \\
\frac{S(a, b) \vdash_{\mathbf{f}} D(b) \rightarrow \mathbf{f}}{S(a, b) \vdash_{\mathbf{f}} \mathbf{f}} \\
\frac{\forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f} \vdash_{\mathbf{f}} \mathbf{f}}{\vdash_{\mathbf{f}} (\forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}} \quad \frac{\vdash_{\mathbf{f}} \forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f})}{\vdash_{\mathbf{f}} \forall \beta (S(a, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}} \\
\vdash_{\mathbf{f}} \mathbf{f}
\end{array}$$

Since register 2 should not change we need  $R_2(b, b_0)$ . Again we will just drop  $S(a, b)$  and  $D(b)$  on the left side for comprehensibility.

$$\begin{array}{c}
\vdots \\
\frac{R_2(b, b_0) \vdash_{\mathbf{f}} \mathbf{f}}{\vdash_{\mathbf{f}} R_2(b, b_0) \rightarrow \mathbf{f}} \quad \frac{\vdash_{\mathbf{f}} \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))}{\vdash_{\mathbf{f}} Q(a) \rightarrow S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0)} \quad \frac{\vdash_{\mathbf{f}} Q(a)}{\vdash_{\mathbf{f}} S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0)} \quad \frac{\vdash_{\mathbf{f}} S(a, b)}{\vdash_{\mathbf{f}} R_2(a, b_0) \rightarrow R_2(b, b_0)} \quad \frac{\vdash_{\mathbf{f}} R_2(a, b_0)}{\vdash_{\mathbf{f}} R_2(b, b_0)} \\
\vdash_{\mathbf{f}} \mathbf{f}
\end{array}$$

For the case that  $\mathbf{r} = +(\mathbf{1}, Q')$ , we have that  $\widehat{Q} = Q'$ ,  $\widehat{m} = m + 1$ , and  $\widehat{n} = n$ . So we need to increment register 1 and ensure that the state of  $b$  is  $Q'$ .

$$\begin{array}{c}
\vdots \\
\frac{Q'(b) \vdash_{\mathbf{f}} \mathbf{f}}{\vdash_{\mathbf{f}} Q'(b) \rightarrow \mathbf{f}} \quad \frac{\vdash_{\mathbf{f}} \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_{\mathbf{f}} Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \frac{\vdash_{\mathbf{f}} Q(a)}{\vdash_{\mathbf{f}} S(a, b) \rightarrow Q'(b)} \quad \frac{\vdash_{\mathbf{f}} S(a, b)}{\vdash_{\mathbf{f}} Q'(b)} \\
\vdash_{\mathbf{f}} \mathbf{f}
\end{array}$$

To increment register 1 we need a new free variable as anchor for register 1 (let  $d \in \mathcal{V}_P \setminus \text{FV}(\Gamma_C \cup \Gamma_{\overline{M}})$  and  $d \neq b$ ).

$$\begin{array}{c}
\vdots \\
\frac{R_1(b, d) \vdash_{\mathbf{f}} \mathbf{f}}{\vdash_{\mathbf{f}} (\forall \beta (R_1(b, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}} \quad \frac{\vdash_{\mathbf{f}} \forall \alpha (D(\alpha) \rightarrow \forall \beta (R_1(\alpha, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f})}{\vdash_{\mathbf{f}} D(b) \rightarrow \forall \beta (R_1(b, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}} \quad \frac{\vdash_{\mathbf{f}} D(b)}{\vdash_{\mathbf{f}} \forall \beta (R_1(b, \beta) \rightarrow \mathbf{f}) \rightarrow \mathbf{f}} \\
\vdash_{\mathbf{f}} \mathbf{f}
\end{array}$$

Now we need to connect  $d$  with  $a_0$  (the anchor of  $a$  for register 1).

$$\begin{array}{c}
\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b)} \quad \vdash_f S(a, b) \\
\vdots \\
\frac{\frac{P(d, a_0) \vdash_f \mathbf{f}}{\vdash_f P(d, a_0) \rightarrow \mathbf{f}}}{\vdash_f P(d, a_0)} \quad \frac{\frac{\vdash_f R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b)}{\vdash_f R_1(b, d) \rightarrow Q'(b)} \quad \vdash_f R_1(a, a_0)}{\vdash_f R_1(b, d)} \\
\vdash_f \mathbf{f}
\end{array}$$

At last we have to make sure that we do not get an artificial zero. We achieve this by deducing  $D(d)$ .

$$\begin{array}{c}
\frac{\frac{\frac{\vdash_f \forall \alpha \beta \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\beta, \delta) \rightarrow D(\delta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(b, d) \rightarrow D(d)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(b, d) \rightarrow D(d)} \quad \vdash_f S(a, b) \\
\vdots \\
\frac{\frac{D(d) \vdash_f \mathbf{f}}{\vdash_f D(d) \rightarrow \mathbf{f}}}{\vdash_f D(d)} \quad \frac{\vdash_f R_1(b, d) \rightarrow D(d)}{\vdash_f D(d)} \quad \vdash_f R_1(b, d) \\
\vdash_f \mathbf{f}
\end{array}$$

Now we already have deduced  $\Gamma_D$ , to see why define  $\hat{a} := b$ ,  $\hat{b}_i := b_i$  for  $i \in \{0, \dots, m\}$ ,  $\hat{a}_0 := d$ , and  $\hat{a}_{i+1} := a_i$  for  $i \in \{0, \dots, n\}$ . Hence we can deduce **false** by induction hypothesis.

The other case, that  $\mathbf{r} = -(\mathbf{Q}, \mathbf{1}, \mathbf{Q}_1, \mathbf{Q}_2)$ , has to be split into two cases again. If  $\mathbf{m} = \mathbf{0}$  then  $\hat{Q} = Q_2$ ,  $\hat{m} = 0$ , and  $\hat{n} = n$ . We only need to ensure that the successor state is  $Q_2$  and that register 1 is still zero.

$$\begin{array}{c}
\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)} \quad \vdash_f S(a, b) \\
\vdots \\
\frac{\frac{Q_2(b) \vdash_f \mathbf{f}}{\vdash_f Q_2(b) \rightarrow \mathbf{f}}}{\vdash_f Q_2(b)} \quad \frac{\frac{\vdash_f R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)}{\vdash_f E(a_0) \rightarrow Q_2(b)} \quad \vdash_f R_1(a, a_0)}{\vdash_f E(a_0)} \\
\vdash_f \mathbf{f}
\end{array}$$

Register 1 stays zero.

$$\begin{array}{c}
\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow R_1(\beta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow R_1(b, a_0)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow R_1(b, a_0)} \quad \vdash_f S(a, b) \\
\vdots \quad \frac{\vdash_f R_1(a, a_0) \rightarrow E(a_0) \rightarrow R_1(b, a_0)}{\vdash_f R_1(a, a_0)} \quad \vdash_f R_1(a, a_0) \\
\frac{R_1(b, a_0) \vdash_f \mathbf{f}}{\vdash_f R_1(b, a_0) \rightarrow \mathbf{f}} \quad \frac{\vdash_f E(a_0) \rightarrow R_1(b, a_0) \quad \vdash_f E(a_0)}{\vdash_f R_1(b, a_0)} \\
\hline
\vdash_f \mathbf{f}
\end{array}$$

If we define  $\hat{a} := b$ ,  $\hat{b}_i := b_i$  for  $i \in \{0, \dots, m\}$ , and  $\hat{a}_0 := a_0$  then it is clear that we have deduced all formulas required for  $\Gamma_D$ . So we can use the induction hypothesis to deduce **false**.

In the last case  $\mathbf{m} > \mathbf{0}$ , so  $\hat{Q} = Q_1$ ,  $\hat{m} = m - 1$ , and  $\hat{n} = n$ . First we ensure that  $b$  is in state  $Q_1$ .

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)} \quad \vdash_f Q(a) \\
\vdots \quad \frac{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)}{\vdash_f R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)} \quad \vdash_f S(a, b) \\
\frac{Q_1(b) \vdash_f \mathbf{f}}{\vdash_f Q_1(b) \rightarrow \mathbf{f}} \quad \frac{\vdash_f R_1(a, a_0) \rightarrow D(a_0) \rightarrow Q_1(b)}{\vdash_f D(a_0) \rightarrow Q_1(b)} \quad \vdash_f R_1(a, a_0) \\
\vdash_f Q_1(b) \rightarrow \mathbf{f} \quad \frac{\vdash_f D(a_0) \rightarrow Q_1(b)}{\vdash_f Q_1(b)} \quad \vdash_f D(a_0) \\
\vdash_f \mathbf{f}
\end{array}$$

Now we decrement register 1 by taking  $a_1$  (the predecessor of  $a_0$ ) as anchor of  $b$  for register 1.

$$\frac{\frac{\frac{\frac{\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow R_1(\beta, \delta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f S(a, b)}{\vdash_f R_1(a, a_0) \rightarrow D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f R_1(a, a_0)}{\vdash_f D(a_0) \rightarrow P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f D(a_0)}{\vdash_f P(a_0, a_1) \rightarrow R_1(b, a_1)} \quad \vdash_f P(a_0, a_1)}{\vdash_f R_1(b, a_1)} \quad \vdash_f \mathbf{f}}{\vdash_f R_1(b, a_1) \rightarrow \mathbf{f}} \quad \vdash_f \mathbf{f}$$

Again it is obvious that we have deduced  $\Gamma_D$  ( $\widehat{a} := b, \widehat{b}_i := b_i$  for  $i \in \{0, \dots, m\}$ , and  $\widehat{a}_{i-1} := a_i$  for  $i \in \{1, \dots, n\}$ ). Hence, by induction hypothesis, we can deduce **false**.  $\square$

**Lemma 22.**

$$M \text{ terminates on input } (0,0) \quad \text{iff} \quad \Gamma_M \vdash \mathbf{false} \text{ holds in system } P.$$

*Proof.* The  $\Leftarrow$  direction is proven in Claim 20. And the  $\Rightarrow$  direction is a direct consequence of Claim 21 with  $C = \langle Q_0, 0, 0 \rangle$ .  $\square$

**Theorem 23.** *CONS is undecidable.*

*Proof.* Since by Lemma 22 for a given two-counter automaton  $M$  we can effectively construct a set of **P**-formulas  $\Gamma_M$  such that  $M$  terminates on input  $(0,0)$  iff  $\Gamma_M$  is not consistent. It follows that **HALT**  $\leq$  **CONS**. Since **HALT** is undecidable we have shown that **CONS** is undecidable too.  $\square$

## References

- [1] H.P. Barendregt, 1993. Lambda Calculi with Types, Handbook of Logic in Computer Science, Volume II, 34-68.