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# 1 Introduction

$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
$\lambda 2$ deduction Rules

| | | |
|---|---|---|
| (Axiom) | $\Gamma, x : t \vdash x : t$ | |
| ($\lambda$-Introduction) | $\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2}$ | |
| ($\lambda$-Elimination) | $\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$ | |
| ($\forall$-Introduction) | $\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.t}$ | $\alpha \notin FV(\Gamma)$ |
| ($\forall$-Elimination) | $\dfrac{\Gamma \vdash e : \forall\alpha.t}{\Gamma \vdash et' : t\,[\alpha := t']}$ | |

# 2 System P

## 2.1 Definitions

$FV(\Gamma) = \bigcup\{FV(A) \mid A \in \Gamma\}$
Deduction Rules

| | | |
|---|---|---|
| (Axiom) | $\Gamma, A \vdash A$ | |
| ($\to$ -Introduction) | $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$ | |
| ($\to$ -Elimination) | $\dfrac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}$ | |
| ($\forall$-Introduction) | $\dfrac{\Gamma \vdash B}{\Gamma \vdash \forall\alpha B}$ | $\alpha \notin FV(\Gamma)$ |
| ($\forall$-Elimination) | $\dfrac{\Gamma \vdash \forall\alpha B}{\Gamma \vdash B\,[\alpha := b]}$ | |

An Interpretation $I$ of a P formula is a tuple $I = (\Delta, \cdot^I)$ where $\Delta$ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta$...
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

$$(\exists\text{-Elimination}) \qquad \frac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false} \vdash_f B} \qquad a \notin FV(\Gamma, A, B)$$

*Proof.* Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false}$ with $\mathit{false}^I = \bot$.

$$\begin{aligned}
I \models \Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false} &\Rightarrow I \models \forall\alpha(A \to \mathit{false}) \to \mathit{false} \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \mathit{false}^I \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \bot \\
&\Rightarrow \neg(\forall\alpha(A \to \mathit{false}))^I \\
&\Rightarrow \neg(\forall a \in \Delta : (A \to \mathit{false})^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha\mapsto d]} \to \mathit{false}^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha\mapsto d]} \to \bot) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha\mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since $a$ is not free in $B$ we conclude that $I$ is also a model of $B$. □

## 2.2 Provability in System P is undecidable

$\Gamma_C$ :

- $Q(a)$
- $P_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \dots, m\}$
- $P_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \dots, n\}$
- $D(a) \dots$

$+(Q, 1, Q')$ :

- $\forall\alpha\beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to P_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to D(\gamma)$
  prevent zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to P_2(\alpha, \gamma) \to P_2(\beta, \gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2)$ :

- $\forall \alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$
  jump on zero
- $\forall \alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow P_1(\beta, \gamma)$
  register 1 stays zero
- $\forall \alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$
  change state if register 1 is greater zero
- $\forall \alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow P_1(\beta, \delta))$
  decrement register 1
- $\forall \alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$
  do not change register 2

**Lemma 1.**

$$M \textit{ terminates on input } (0,0) \qquad \textit{iff} \qquad \Gamma_M \vdash \text{false} \textit{ holds in system } P.$$

**Claim 2.** $\Gamma_C \cup \Gamma \vdash$ false...

*Proof.* For the tableau proofs we will abbreviate *false* by *f*.
Induction Base trivial ...
Induction Step
$C \rightarrow D$
Basic idea:

$$\dfrac{\dfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \qquad \dfrac{}{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models$ *false* holds trivially if $I$ interprets *false* with $\top$ we only need to consider models (there are none if $M$ terminates) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\bot$ (so we can use our new deduction rule)
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side We first introduce the new variables needed for $\Gamma_D$ (let $b, \in V_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{S(a,b), D(b) \vdash_f f}{S(a,b) \vdash_f D(b) \rightarrow f} \qquad \dfrac{\dfrac{S(a,b) \vdash_f \forall \alpha\beta S(\alpha,\beta) \rightarrow D(\beta)}{S(a,b) \vdash_f S(a,b) \rightarrow D(b)}}{S(a,b) \vdash_f D(b)}}{S(a,b) \vdash_f f}}{\vdash_f (\forall \beta(S(a,\beta) \rightarrow f) \rightarrow f) \rightarrow f} \qquad \dfrac{\vdash_f \forall \alpha(\forall \beta(S(\alpha,\beta) \rightarrow f) \rightarrow f)}{\vdash_f \forall \beta(S(a,\beta) \rightarrow f) \rightarrow f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

Now we create $\Gamma_D$

$$\frac{\dfrac{Q'(b) \vdash f}{\vdash_f Q'(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \qquad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \qquad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

Alternative tableau with tikz:

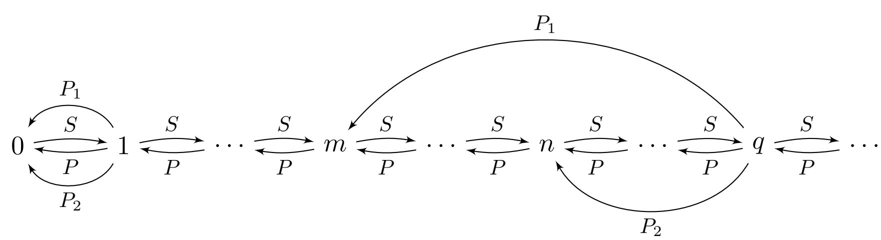
$$\frac{\dfrac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \qquad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \qquad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

$\square$

**Claim 3.**

$$\Gamma_M \vdash \text{false } \textit{holds in system } P \qquad \Longrightarrow \qquad M \textit{ terminates on input } (0,0)$$

*Proof.* Assume $M$ does not terminate then there is an infinite chain $C_0 \Rightarrow_M C_1 \Rightarrow_M C_3 \Rightarrow_M \dots$. ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of $\Gamma_M$ which interprets *false* with $\perp$ this contradicts $\Gamma_M \vdash$ *false*.
The idea looks like this:




Formal definition:
$I = (\mathbb{N}, \cdot^I)$

$$\begin{array}{lll} P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & P_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & P_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q_j^I = \{i \mid Q_j = Q_i, i \in \mathbb{N}\} & D^I = \mathbb{N} \setminus \{0\} & E^I = \{0\} \end{array}$$

$\square$