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$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
$\lambda 2$ deduction Rules

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(λ-Introduction)	$\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2}$	
(λ-Elimination)	$\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.t}$	$\alpha \notin FV(\Gamma)$
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2 System P

2.1 Definitions

$FV(\Gamma) = \bigcup\{FV(A) \mid A \in \Gamma\}$
Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
($\to$ -Introduction)	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$	
($\to$ -Elimination)	$\dfrac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash B}{\Gamma \vdash \forall\alpha B}$	$\alpha \notin FV(\Gamma)$
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An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta \ldots$
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

$$\text{(}\exists\text{-Elimination)} \qquad \frac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false} \vdash_f B} \qquad a \notin FV(\Gamma, A, B)$$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false}$ with $\mathit{false}^I = \bot$.

$$\begin{aligned}
I \models \Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false} &\Rightarrow I \models \forall\alpha(A \to \mathit{false}) \to \mathit{false} \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \mathit{false}^I \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \bot \\
&\Rightarrow \neg(\forall\alpha(A \to \mathit{false}))^I \\
&\Rightarrow \neg(\forall a \in \Delta : (A \to \mathit{false})^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \to \mathit{false}^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \to \bot) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. □

2.2 Provability in System P is undecidable

Γ_C :

- $Q(a)$
- $P_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$
- $P_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$
- $D(a), D(a_i), D(b_j)$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$
- $E(a_m), E(b_n)$

  $+(Q, 1, Q')$:

- $\forall\alpha\beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to P_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to D(\gamma))$
  prevent zero

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to P_2(\alpha,\gamma) \to P_2(\beta,\gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2)$:

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to P_1(\alpha,\gamma) \to E(\gamma) \to Q_2(\beta))$
  jump on zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to P_1(\alpha,\gamma) \to E(\gamma) \to P_1(\beta,\gamma)$
  register 1 stays zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to P_1(\alpha,\gamma) \to D(\gamma) \to Q_1(\beta))$
  change state if register 1 is greater zero
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to P_1(\alpha,\gamma) \to D(\gamma) \to P(\gamma,\delta) \to P_1(\beta,\delta))$
  decrement register 1
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to P_2(\alpha,\gamma) \to P_2(\beta,\gamma))$
  do not change register 2

Lemma 1.

$$M \textit{ terminates on input } (0,0) \qquad \textit{iff} \qquad \Gamma_M \vdash \mathrm{false} \textit{ holds in system } P.$$

Claim 2. *If a final state is reachable from C then* $\Gamma_C \cup \Gamma \vdash \mathrm{false}$.

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by *f*.
Induction Base trivial ...
Induction Step
$C \to^r_M D$
We need to make a case distinction on the rule r.
Case $r = +(Q, 1, Q')$
Basic idea:

$$\dfrac{\dfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \qquad \dfrac{}{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models$ *false* holds trivially if I interprets *false* with $\top$ we only need to consider models (note that there are none if M terminates which is exactly what we want to proof) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\bot$ (so we can use our new deduction rule).
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.
We first introduce the new variables needed for Γ_D (let $b, d \in V_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{S(a,b), D(b) \vdash_f f}{S(a,b) \vdash_f D(b) \to f} \qquad \dfrac{\dfrac{S(a,b) \vdash_f \forall\alpha\beta S(\alpha,\beta) \to D(\beta)}{S(a,b) \vdash_f S(a,b) \to D(b)}}{S(a,b) \vdash_f D(b)}}{S(a,b) \vdash_f f}}{\vdash_f (\forall\beta(S(a,\beta) \to f) \to f) \to f} \qquad \dfrac{\vdash_f \forall\alpha(\forall\beta(S(\alpha,\beta) \to f) \to f)}{\vdash_f \forall\beta(S(a,\beta) \to f) \to f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

The formula $P_1(b,d)$ can be acquired in a similar way.
Now we create Γ_D

$$\frac{\dfrac{Q'(b) \vdash f}{\vdash_f Q'(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

Alternative tableau with tikz:

$$\frac{\dfrac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

Starting from $Q'(b) \vdash_f$ *false* we can deduce:

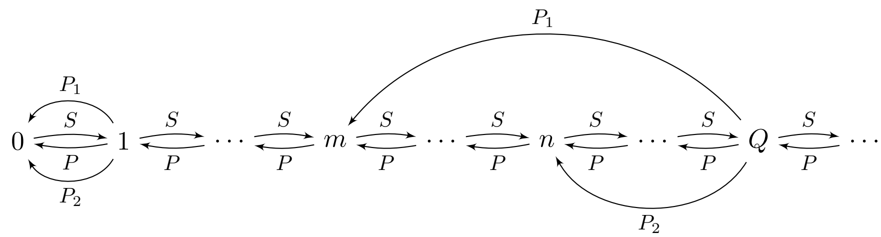
$$\frac{\dfrac{P(d,a_0) \vdash_f f}{\vdash_f P(d,a_0) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to P_1(\alpha,\gamma) \to P_1(\beta,\delta) \to P(\delta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to P_1(a,a_0) \to P_1(b,d) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to P_1(a,a_0) \to P_1(b,d) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f P_1(a,a_0) \to P_1(b,d) \to Q'(b)} \quad \vdash_f P_1(a,a_0)}{\vdash_f P_1(b,d) \to Q'(b)} \quad \vdash_f P_1(b,d)}{\vdash_f P(d,a_0)}}{\vdash_f f}$$

$P_2(b,b_0)$ can be deduced in the same way.
Now we have Γ_C (Since $P(a_{i-1},a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.
Case $r = -(Q,1,Q_1,Q_2)$ $\square$

Claim 3.

$$\Gamma_M \vdash \text{false } \textit{holds in system } P \qquad \Longrightarrow \qquad M \textit{ terminates on input } (0,0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_0 \Rightarrow_M C_1 \Rightarrow_M C_3 \Rightarrow_M \ldots.$ $(C_i = \langle Q_i, m_i, n_i\rangle)$ Now we construct a model of Γ_M which interprets *false* with $\perp$ this contradicts $\Gamma_M \vdash$ *false*.
The idea looks like this:



Formal definition:
$I = (\mathbb{N}, \cdot^I)$

$$\begin{array}{lll} P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & P_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & P_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I = \{i \in \mathbb{N} \mid Q = Q_i\} & D^I = \mathbb{N} \setminus \{0\} & E^I = \{0\} \\ S^I = \{(i, i+1) \mid i \in \mathbb{N}\} & & \end{array}$$

$\square$