Contents

1 Introduction

2 Basic Definitions

We will denote the set $\{1, \dots, n\}$ by $[n]$.

2.1 λ-calculus $\lambda 2$

$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
In the following let $\mathcal{V}_T = \{\alpha, \beta, \dots\}$ be a countable set (of type-variables) and $\mathcal{V}_V = \{x_1, x_2, \dots\}$ be a countable set (of value-variables).

Definition 1. The set of all $\lambda 2$ types over $\mathcal{V}_T$, denoted by $\mathrm{T}_{\lambda 2}$, is the smallest set T satisfying the following conditions:

- $\mathcal{V}_T \subseteq \mathrm{T}$,
- if $t_1, t_2 \in \mathrm{T}$ then $t_1 \to t_2 \in \mathrm{T}$, and
- if $t \in \mathrm{T}$ and $\alpha \in \mathcal{V}_T$ then $\forall \alpha . t \in \mathrm{T}$.

Definition 2. The set of all $\lambda 2$ terms over $\mathcal{V}_T$ and $\mathcal{V}_V$, denoted by $\Lambda_{\mathrm{T}_{\lambda 2}}$, is the smallest set Λ_T satisfying the following conditions:

- $\mathcal{V}_V \subseteq \Lambda_T$,
- if $e_1, e_2 \in \Lambda_T$ then $e_1 e_2 \in \Lambda_T$,
- if $x \in \mathcal{V}_V$, $t \in \mathrm{T}_{\lambda 2}$, and $e \in \Lambda_T$ then $\lambda x : t.e \in \Lambda_T$,
- if $\alpha \in \mathcal{V}_T$ and $e \in \Lambda_T$ then $\Lambda \alpha . e \in \Lambda_T$, and
- if $e \in \Lambda_T$ and $t \in \mathrm{T}_{\lambda 2}$ then $e\,t \in \Lambda_T$.

Definition 3. Let $e \in \Lambda_{\mathrm{T}_{\lambda 2}}$. The free variables of e, denoted by $\mathrm{FV}(e)$, are defined inductively as follows:

$$\mathrm{FV}(e) = \begin{cases} \{x\} & \text{if } e = x \\ \mathrm{FV}(e_1) \cup \mathrm{FV}(e_2) & \text{if } e = e_1 e_2 \\ \mathrm{FV}(e') \setminus \{x\} & \text{if } e = \lambda x : t.e' \\ \mathrm{FV}(e') & \text{if } e = \Lambda \alpha . e' \\ \mathrm{FV}(e') & \text{if } e = e'\,t \end{cases}$$

Or is this definition better?

Definition 4. Let $e \in \Lambda_{\mathrm{T}_{\lambda 2}}$. The <u>free variables of e</u>, denoted by $\mathrm{FV}(e)$, are defined inductively as follows:

$$\begin{aligned}
\mathrm{FV}(y) &= \{x\} \\
\mathrm{FV}(e_1 e_2) &= \mathrm{FV}(e_1) \cup \mathrm{FV}(e_2) \\
\mathrm{FV}(\lambda x : t.e') &= \mathrm{FV}(e') \setminus \{x\} \\
\mathrm{FV}(\Lambda \alpha.e') &= \mathrm{FV}(e') \\
\mathrm{FV}(e'\, t) &= \mathrm{FV}(e')
\end{aligned}$$

Definition 5. A <u>basis</u> is a finite subset of $\mathcal{V}_V \times \Lambda_{\mathrm{T}_{\lambda 2}}$

$\lambda 2$ deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ-Introduction)	$\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2}$	
(λ-Elimination)	$\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha.e : \forall \alpha.t}$	$\alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash e : \forall \alpha.t}{\Gamma \vdash e\, t' : t\,[\alpha := t']}$	$t' \in \mathrm{T}_{\lambda 2}$

2.2 first-order logic

Definition 6. A <u>ranked set</u> is a tuple (Σ, rk), where Σ is a countable set and $rk : \Sigma \to \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk). The set of all elements with a certain rank k in Σ, denoted by $\Sigma^{(k)}$, is defined by $\Sigma^{(k)} := rk^{-1}(k)$. In the following we will write $\Sigma = \{P^{(0)}, Q^{(3)}\}$ to say that $\Sigma = \{P, Q\}$, $rk(P) = 0$, and $rk(Q) = 3$.

In the following let $\mathcal{V} = \{x_0, x_1, \dots\}$ be a countable set (of variables), $\mathcal{F}$ a ranked set (of function symbols), and $\mathcal{P}$ a ranked set (of predicate symbols).

Definition 7. The set of <u>terms over $(\mathcal{V}, \mathcal{F})$</u>, denoted by $\mathcal{T}_{(\mathcal{V},\mathcal{F})}$, is the smallest set $\mathcal{T}$ satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and

- for every $k \in \mathbb{N}$ if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \dots, t_k) \in \mathcal{T}$.

The set of first-order formulas over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$, denoted by $\mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$, is the smallest set $\mathcal{L}$ satisfying the following conditions:

- for every $k \in \mathbb{N}$ if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}_{(\mathcal{V},\mathcal{F})}$ then $P(t_1, t_2, \dots, t_k) \in \mathcal{L}$.
- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi \in \mathcal{L}$, and
- if $x \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists x\varphi$, $\forall x\varphi \in \mathcal{L}$.

We introduce an additional binary operation $\to$ on formulas, where for some φ, $\psi \in \mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$ the formula $(\varphi \to \psi)$ is defined as $(\neg\varphi \vee \psi)$. For nullary relation symbols P we will abbreviate $P()$ to P.

Definition 8. The variables of a term $t \in \mathcal{T}_{(\mathcal{V},\mathcal{F})}$, denoted by $\mathrm{V}(t)$, are defined by:

$$\mathrm{V}(t) = \begin{cases} \{x\} & \text{if } t = x \\ \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \dots \cup \mathrm{V}(t_k) & \text{if } t = f(t_1, t_2, \dots, t_k) \end{cases}$$

The free variables of a formula $\varphi \in \mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$, denoted by $\mathrm{FV}(\varphi)$, are defined as follows:

$$\mathrm{FV}(\varphi) = \begin{cases} \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \dots \cup \mathrm{V}(t_k) & \text{if } \varphi = P(t_1, t_2, \dots, t_k) \\ \mathrm{FV}(\psi) & \text{if } \varphi = \neg\psi \\ \mathrm{FV}(\varphi_1) \cup \mathrm{FV}(\varphi_2) & \text{if } \varphi = \varphi_1 \circ \varphi_2,\ \circ \in \{\wedge, \vee\} \\ \mathrm{FV}(\psi) \setminus \{x\} & \text{if } \varphi = Qx\psi,\ Q \in \{\forall, \exists\} \end{cases}$$

Definition 9. Let x be in $\mathcal{V}$ and $t, t' \in \mathcal{T}_{(\mathcal{V},\mathcal{F})}$. The substitution of x by t' in t, denoted by $t\,[x := t']$, is defined as follows:

$$t\left[x := t'\right] = \begin{cases} t' & \text{if } t = x \\ y & \text{if } t = y \text{ and } y \neq x \\ f(t_1\,[x := t'], \dots, t_k\,[x := t']) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Now we can lift this definition to formulas, let φ be in $\mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$. The substitution of x by t' in φ, denoted by $\varphi\,[x := t']$, is defined as follows:

$$\varphi\left[x := t'\right] = \begin{cases} P(t_1\,[x := t'], \dots, t_k\,[x := t']) & \text{if } \varphi = P(t_1, \dots, t_k) \\ \psi\,[x := t'] & \text{if } \varphi = \neg\psi \\ \varphi_1\,[x := t'] \circ \varphi_2\,[x := t'] & \text{if } \varphi = (\varphi_1 \circ \varphi_2),\ \circ \in \{\wedge, \vee\} \\ \varphi & \text{if } \varphi = Qx\psi,\ Q \in \{\forall, \exists\} \\ Qy(\psi\,[x := t']) & \text{if } \varphi = Qy\psi,\ Q \in \{\forall, \exists\} \text{ and } y \neq x \end{cases}$$

Now we come to the semantics of first-order formulas.

Definition 10. An interpretation I over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$ is a triple $(\Delta, \cdot^I, \omega)$ where

where	Δ	is a nonempty set (which we call domain),
	$\cdot^I$	is a function such that
		$f^I : \Delta^k \to \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$ and
		$P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $f \in \mathcal{P}^{(k)}$
	ω	is a function from $\mathcal{V}$ to Δ.

Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation, $x \in \mathcal{V}$, and $d \in \Delta$ the interpretation $I\,[x \to d]$ is defined as $(\Delta, \cdot^I, \omega\,[x \to d])$ where

$$(\omega\,[x \to d])(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{otherwise.} \end{cases}$$

Definition 11. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and t a term the interpretation of t under I, denoted by t^I, is defined as follows:

$$t^I = \begin{cases} \omega(x) & \text{if } t = x \\ f^I(t_1^I, \ldots, t_k^I) & \text{if } t = f(t_1, \ldots, t_k) \end{cases}$$

Definition 12. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and φ a formula the interpretation of φ under I, denoted by φ^I, is defined recursively as follows:

$$\varphi^I = \begin{cases} \top & \text{if } \varphi = P(t_1, \ldots, t_k) \text{ and } (t_1^I, \ldots, t_k^I) \in P^I \\ \bot & \text{if } \varphi = P(t_1, \ldots, t_k) \text{ and } (t_1^I, \ldots, t_k^I) \notin P^I \\ \text{not } \psi^I & \text{if } \varphi = \neg\psi \\ \varphi_1^I \text{ and } \varphi_2^I & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \\ \varphi_1^I \text{ or } \varphi_2^I & \text{if } \varphi = (\varphi_1 \vee \varphi_2) \\ \text{exists } d \in \Delta\ \psi^{I[x \to d]} & \text{if } \varphi = \exists x \psi \\ \text{forall } d \in \Delta\ \psi^{I[x \to d]} & \text{if } \varphi = \forall x \psi \end{cases}$$

The interpretation I is a model of φ, denoted by $I \models \varphi$, if $\varphi^I = \top$.

When we define an interpretation I and we have a nullary predicate symbol P we write $P^I = \top$ instead of $P^I = \{()\}$ and $P^I = \bot$ for $P^I = \emptyset$ (this works because $P()^I = \top$ iff $() \in P^I$).

Definition 13. Let Γ be a finite set of first-oder formulas.

We say that an interpretation I is a model of Γ if $I \models \psi$ for every ψ in Γ.

The formula φ is a semantic consequence of Γ, denoted by $\Gamma \vdash \varphi$, if every model of Γ is also a model of φ.

The free variables of Γ, denoted by $\mathrm{FV}(\Gamma)$, are $\bigcup\{\mathrm{FV}(\varphi) \mid \varphi \in \Gamma\}$.

2.3 two-counter automaton

We will use a version of two-counter automaton which only has two types of transitions. First it can increment a register and second it can try to decrement a register and jump if the register is already zero. Formally:

Definition 14. A deterministic two-counter automaton is a 4-tuple $M = (\mathcal{Q}, Q_0, Q_f, R)$,

where $\mathcal{Q}$ is a finite set (of states),
Q_0 is in $\mathcal{Q}$ (the initial state),
Q_f is in $\mathcal{Q}$ (the final state), and
R is a function from $\mathcal{Q} \setminus \{Q_f\}$ to $\mathcal{R}_{\mathcal{Q}}$,
where $\mathcal{R}_{\mathcal{Q}} = \{+(i, Q') \mid i \in \{1,2\}, Q' \in \mathcal{Q}\}$
$\cup\{-(i, Q_1, Q_2) \mid i \in \{1,2\}, Q_1, Q_2 \in \mathcal{Q}\}$

A configuration C of our automaton is a triple $\langle Q, m, n\rangle$, where $Q \in \mathcal{Q}$ and $m, n \in \mathbb{N}$. Let r be in $R(\mathcal{Q} \setminus \{Q_f\})$, then $\Rightarrow^r_M$ is a binary relation on the configurations of M such that two configurations $\langle Q, m, n\rangle$, $\langle Q', m', n'\rangle$ of M are in the in the relation if all of the following conditions hold:

- $Q \neq Q_f$, $r = R(Q)$,
- if $r = +(1, P)$ for some $P \in \mathcal{Q}$ then $Q' = P$, $m' = m + 1$, and $n' = n$,
- if $r = +(2, P)$ for some $P \in \mathcal{Q}$ then $Q' = P$, $m' = m$, and $n' = n + 1$,
- if $r = -(1, P_1, P_2)$ for some $P_1, P_2 \in \mathcal{Q}$ then
  if $m = 0$ then $Q' = P_2$, $m' = 0$, and $n' = n$,
  if $m \geq 1$ then $Q' = P_1$, $m' = m - 1$, and $n' = n$,
- if $r = -(2, P_1, P_2)$ for some $P_1, P_2 \in \mathcal{Q}$ then
  if $n = 0$ then $Q' = P_2$, $m' = m$, and $n' = 0$,
  if $n \geq 1$ then $Q' = P_1$, $m' = m$, and $n' = n - 1$.

The transition relation of M, denoted by $\Rightarrow_M$, is defined as $\bigcup_{r \in R(Q\setminus\{Q_f\})} \Rightarrow^r_M$. We denote the transitive reflexive closure of $\Rightarrow_M$ by $\Rightarrow^*_M$

Let m, n be in $\mathbb{N}$, we say that M terminates on input (m, n) if there exist $m', n' \in \mathbb{N}$ such that $\langle Q_0, m, n\rangle \Rightarrow^*_M \langle Q_f, m', n'\rangle$.

Definition 15. The halting problem for two-counter automaton, denoted by **HALT**, is defined as follows. Given a two-counter automaton M.

Does M terminate on input $(0, 0)$.

It is well known that **HALT** is undecidable.

3 System P

3.1 Definitions

In the following let $\mathcal{V}_P = \{\alpha, a, \beta, b, \dots\}$ be a countably infinite set (of variables). Let $\mathcal{P}_P = \{P, Q, \dots\}$ be a set (of predicate symbols) and $\mathcal{P}$ a ranked set such that $\mathcal{P}^{(0)} = \{\textbf{false}\}$, $\mathcal{P}^{(2)} = \mathcal{P}_P$, and $\mathcal{P}^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over $(\mathcal{V}_P, \emptyset, \mathcal{P})$ is an

atomic formula if $\varphi = \textbf{false}$ or $\varphi = P(a, b)$ for some $P \in \mathcal{P}_P$ and $a, b \in \mathcal{V}_P$.

universal formula if $\varphi = \forall \overrightarrow{\alpha}(A_1 \to A_2 \to \cdots \to A_n)$ where A_i is an atomic formula for $i \in [n]$, $A_i \neq \textbf{false}$ for $i \in [n-1]$ and for each $\alpha \in \mathrm{FV}(\varphi) \cap \mathrm{FV}(A_n)$ there exists an $i \in [n-1]$ such that $\alpha \in \mathrm{FV}(A_i)$.

existential formula if there exits $n \geq 0$, atomic formulas $A_i \neq \textbf{false}$ for $i \in [n]$ such that $\varphi = \forall \overrightarrow{\alpha}(A_1 \to A_2 \to \cdots \to A_{n-1} \to \forall \beta (A_n \to \textbf{false}) \to \textbf{false})$.

The set of formulas of System $\mathbf{P}$ (= set of $\mathbf{P}$-formulas) over $(\mathcal{V}_P, \mathcal{P}_P)$ is the set of all first-order formulas over $(\mathcal{V}_P, \emptyset, \mathcal{P})$ that are either an atomic, universal or existential formula.
Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
($\to$ -Introduction)	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}$	
($\to$ -Elimination)	$\dfrac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B\,[\alpha := b]}$	$b \in \mathcal{V}_P$

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta$....
If we interpret **false** with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

($\exists$-Introduction)	$\dfrac{\Gamma, A\,[\alpha := a] \vdash_{\mathrm{f}} B}{\Gamma, \forall \alpha (A \to \textbf{false}) \to \textbf{false} \vdash_{\mathrm{f}} B}$	$a \notin FV(\Gamma, A, B)$

Proof. Let $I = (\Delta, \cdot^I, \omega)$ be a model of $\Gamma, \forall\alpha(A \to \mathbf{false}) \to \mathbf{false}$ with $\mathbf{false}^I = \bot$ and $a \in \mathcal{V}_P$ a variable such that $a \notin FV(\Gamma, A, B)$.

$$\begin{aligned}
I \models \Gamma, \forall\alpha(A \to \mathbf{false}) \to \mathbf{false} &\Rightarrow I \models \forall\alpha(A \to \mathbf{false}) \to \mathbf{false} \\
&\Rightarrow (\forall\alpha(A \to \mathbf{false}))^I \to \mathbf{false}^I \\
&\Rightarrow (\forall\alpha(A \to \mathbf{false}))^I \to \bot \\
&\Rightarrow \neg(\forall\alpha(A \to \mathbf{false}))^I \\
&\Rightarrow \neg(\forall d \in \Delta : (A \to \mathbf{false})^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha\mapsto d]} \to \mathbf{false}^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha\mapsto d]} \to \bot) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha\mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. □

Definition 16. The problem to decide whether a given set of **P**-formulas is consistent, denoted by **CONS**, is defined as follows. Given a set of **P**-formulas Γ.

Does $\Gamma \vdash \mathbf{false}$ not hold.

3.2 CONS is undecidable

We will show that **HALT** $\leq$ **CONS** then the undecidability of **CONS** directly follows from the undecidability of **HALT**. For a given two-counter automaton M we will effectively construct a set of **P**-formulas Γ_M such that

M terminates on input $(0, 0)$ iff $\Gamma_M \vdash \mathbf{false}$ holds in system P.

Let $M = (\mathcal{Q}, Q_0, Q_f, R)$ be a two-counter automaton, w.l.o.g. $S, P, R_1, R_2, E, D \notin \mathcal{Q}$. In the following we will consider **P**-formulas over $(\mathcal{V}_P, \mathcal{P}_P)$, where $\mathcal{P}_P = \mathcal{Q} \uplus \{S, P, R_1, R_2, E, D\}$. We will abbreviate $P(a, a)$ to $P(a)$, note that this way we can use binary predicate symbols as unary ones.

Intuitively $Q(a)$ stands for "a is in state Q", $R_i(a, m)$ stands for "in a the value of register i is m" for $i \in \{1, 2\}$, $S(a, b)$ states that "b is a successor of a", $P(a, b)$ states that "b is a predecessor of a", $E(a)$ marks "a as the end of chain", and $D(a)$ states that "a is not the end of a chain".

For a configuration $C = \langle Q, m, n\rangle$ of M we define a set of **P**-formulas Γ_C. It contains the following formulas:

- $Q(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$

- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$
- $D(a), D(a_i), D(b_j)$ for $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$
- $E(a_m), E(b_n)$

Next we need sets of **P**-formulas for all possible transitions. For every $Q \in \mathcal{Q} \setminus \{Q_f\}$ and $r \in \mathcal{R}_\mathcal{Q}$ we define $\Gamma_{Q,r}$. If $r = +(1, Q')$ for some $Q' \in \mathcal{Q}$ then $\Gamma_{Q,+(1,Q')}$ contains the following formulas:

- $\forall \alpha\beta (Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall \alpha\beta\gamma\delta (Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to R_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall \alpha\beta\delta (Q(\alpha) \to S(\alpha, \beta) \to R_1(\beta, \delta) \to D(\delta))$
  prevent zero in register 1
- $\forall \alpha\beta\gamma (Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change the value register 2

If $r = -(1, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then $\Gamma_{Q,-(1,Q_1,Q_2)}$ contains the following formulas:

- $\forall \alpha\beta\gamma (Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to Q_2(\beta))$
  jump to Q_2 if register 1 is zero
- $\forall \alpha\beta\gamma (Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to R_1(\beta, \gamma))$
  if register 1 is zero it stays zero
- $\forall \alpha\beta\gamma (Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to Q_1(\beta))$
  change state to Q_1 if register 1 is greater zero
- $\forall \alpha\beta\gamma\delta (Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to P(\gamma, \delta) \to R_1(\beta, \delta))$
  decrement register 1
- $\forall \alpha\beta\gamma (Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2 in both cases

For $r = +(2, Q')$ for some $Q' \in \mathcal{Q}$ or $r = -(2, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ the sets $\Gamma_{Q,r}$ are defined analogously.

We also need a set Γ_1 to ensure that our representation works correctly. The following formula are in Γ_1:

- $\forall \alpha\beta (S(\alpha, \beta) \to D(\beta))$
  no successor is the end of a chain
- $\forall \alpha (D(\alpha) \to \forall \beta (R_1(\alpha, \beta) \to \mathbf{false}) \to \mathbf{false})$
  every element that represents a configuration has a value for register 1

- $\forall\alpha(D(\alpha) \to \forall\beta(R_2(\alpha,\beta) \to \mathbf{false}) \to \mathbf{false})$
  every element that represents a configuration has a value for register 2
- $\forall\alpha(\forall\beta(S(\alpha,\beta) \to \mathbf{false}) \to \mathbf{false})$
  every element has a successor

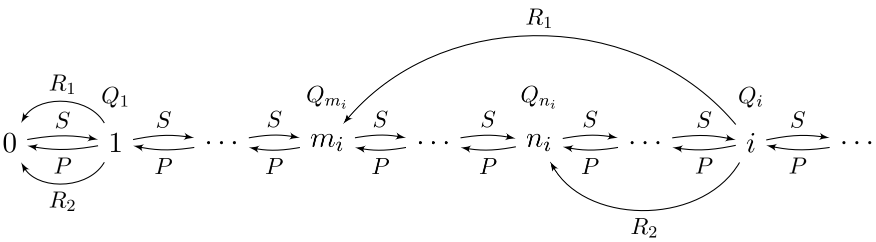
We define $\Gamma_{\overline{M}}$ as $\bigcup_{Q\in\mathcal{Q}\setminus\{Q_f\}} \Gamma_{Q,R(Q)} \cup \{\forall\alpha(Q_f(\alpha) \to \mathbf{false})\} \cup \Gamma_1$. Finally we can define Γ_M as $\Gamma_{C_1} \cup \Gamma_{\overline{M}}$, where $C_1 = \langle Q_0, 0, 0\rangle$ is the initial configuration.

Claim 17.

$$\Gamma_M \vdash \textit{\textbf{false}} \textit{ holds in system } P \qquad \Longrightarrow \qquad M \textit{ terminates on input } (0,0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \ldots$ ($C_i = \langle Q_i, m_i, n_i\rangle$ for $i \in \mathbb{N}^+$). Now we construct a model of Γ_M which interprets **false** with $\perp$ this contradicts $\Gamma_M \vdash$ **false**.

To illustrate the idea we will use a graphical notation for an interpretation I. By $d_1 \xrightarrow{\mathrm{R}} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{\mathrm{P}}{d}$ to say that $(d,d) \in P^I$ for predicate symbols that are used as unary predicate symbols. As domain for our interpretation we will use the natural numbers. Every number will have two tasks: firstly it will represent itself as a possible value for register 1 or 2 and secondly every number i greater than zero will also represent the i^{th} configuration of our infinite computation. Now the idea for our model of Γ_M looks like this:



We have $0 \in E^I$ and all other numbers are in D^I.
Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I, \omega)$.

$$\begin{array}{lll} P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I = \{(i,i) \mid i \in \mathbb{N}^+, Q = Q_i\} & D^I = \{(i,i) \mid i \in \mathbb{N}^+\} & E^I = \{(0,0)\} \\ S^I = \{(i, i+1) \mid i \in \mathbb{N}\} & \mathbf{false}^I = \perp & \end{array}$$

$$a^I = 1 \qquad a_0^I = 0 \qquad b_0^I = 0$$

Since there are no free variables in Γ_M we can just set $\omega(x) = 0$ for every $x \in \mathcal{V}_P$. It is easy to see that I is indeed a model of Γ_M. $\square$

Claim 18. *Let $C = \langle Q, m, n\rangle$ be a configuration of M. If a final configuration (i.e. a configuration $\langle Q_f, m', n'\rangle$ for some $n', m' \in \mathbb{N}$) is reachable from C then $\Gamma_C \cup \Gamma_{\overline{M}} \vdash$ **false**.*

Proof. By induction on the length i of the computation.

Induction Base: $i = 0$

Since a final configuration is reachable in 0 steps C must be this final configuration. So $C = \langle Q_f, m, n\rangle$ for some $n, m \in \mathbb{N}$. Hence, $Q_f(a)$ is in Γ_C for some $a \in \mathcal{V}_P$ and $\forall\alpha(Q_f(\alpha) \to \textbf{false})$ is in $\Gamma_{\overline{M}}$, we can easily deduce false.

$$\dfrac{\dfrac{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \forall\alpha(Q_f(\alpha) \to \textbf{false})}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a) \to \textbf{false}} \qquad \Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a)}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \textbf{false}}$$

Induction Step: $i = i' + 1$

Since $I \models \textbf{false}$ holds trivially if I interprets **false** with $\top$ we only need to consider models of $\Gamma_C \cup \Gamma_{\overline{M}}$ that interpret **false** with $\bot$ (note that there are none if M terminates which is exactly what we want to proof). As result of this observation we can use the $\exists$-Introduction rule.

From the fact that a final configuration is reachable from C in i steps we can deduce that there exists a configuration $D = \langle \widehat{Q}, \widehat{m}, \widehat{n}\rangle$ such that $C \Rightarrow^r_M D$ for some $r \in \mathcal{R}_\mathcal{Q}$ and a final configuration is reachable from D in i' steps. We also know that $C = \langle Q, m, n\rangle$ for some $Q \in \mathcal{Q} \setminus \{Q_f\}$ and some $m, n \in \mathbb{N}$. The set Γ_C contains the formulas:

$R_1(a, a_0)$, $P(a_{i-1}, a_i)$ and $D(a_{i-1})$ for $i \in \{1, \ldots, n\}$,

$R_2(a, b_0)$, $P(b_{i-1}, b_i)$ and $D(b_{i-1})$ for $i \in \{1, \ldots, m\}$,

$Q(a)$, $D(a)$, $E(a_n)$ and $E(b_m)$.

And Γ_D contains the formulas:

$R_1(\widehat{a}, \widehat{a}_0)$, $P(\widehat{a}_{i-1}, \widehat{a}_i)$ and $D(\widehat{a}_{i-1})$ for $i \in \{1, \ldots, \widehat{n}\}$,

$R_2(\widehat{a}, \widehat{b}_0)$, $P(\widehat{b}_{i-1}, \widehat{b}_i)$ and $D(\widehat{b}_{i-1})$ for $i \in \{1, \ldots, \widehat{m}\}$,

$Q(\widehat{a})$, $D(\widehat{a})$, $E(\widehat{a}_{\widehat{n}})$ and $E(\widehat{b}_{\widehat{m}})$.

The basic idea is to deduce Γ_D from $\Gamma_C \cup \Gamma_{\overline{M}}$ and then apply the induction hypothesis to $\Gamma_D \cup \Gamma_{\overline{M}}$.

$$\dfrac{\dfrac{IH}{\Gamma_C \cup \Gamma_{\overline{M}} \cup \Gamma_D \vdash_{\mathrm{f}} \textbf{false}} \qquad \Gamma_C \cup \Gamma_{\overline{M}} \vdash_{\mathrm{f}} \Gamma_D}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash_{\mathrm{f}} \textbf{false}}$$

We achieve this by looking at the four possible cases for the type of the rule r. We will only consider the cases $r = +(1, Q')$ and $r = -(1, Q_1, Q_2)$, the two remaining cases

$r = +(2, Q')$ and $r = -(2, Q_1, Q_2)$ follow by exchanging the roles of register 1 and register 2 in the first two cases.

First we need a new free variable representing the configuration D. Also the value in register 2 does not change, because in both cases we are only concerned with register 1. For the succeeding tableau proofs we will abbreviate **false** by **f** and we will drop $\Gamma_C \cup \Gamma_{\overline{M}}$ and only write new formulas on the left side of $\vdash_{\mathrm{f}}$.

We first introduce a new variable representing D (let $b \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma_{\overline{M}})$).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{S(a,b), D(b) \vdash_{\mathrm{f}} \mathbf{f}}}{S(a,b) \vdash_{\mathrm{f}} D(b) \to \mathbf{f}} \quad \dfrac{\dfrac{S(a,b) \vdash_{\mathrm{f}} \forall\alpha\beta(S(\alpha,\beta) \to D(\beta))}{S(a,b) \vdash_{\mathrm{f}} S(a,b) \to D(b)} \quad S(a,b) \vdash_{\mathrm{f}} S(a,b)}{S(a,b) \vdash_{\mathrm{f}} D(b)}}{S(a,b) \vdash_{\mathrm{f}} \mathbf{f}}}{\dfrac{\forall\beta(S(a,\beta) \to \mathbf{f}) \to \mathbf{f} \vdash_{\mathrm{f}} \mathbf{f}}{\vdash_{\mathrm{f}} (\forall\beta(S(a,\beta) \to \mathbf{f}) \to \mathbf{f}) \to \mathbf{f}}} \quad \dfrac{\vdash_{\mathrm{f}} \forall\alpha(\forall\beta(S(\alpha,\beta) \to \mathbf{f}) \to \mathbf{f})}{\vdash_{\mathrm{f}} \forall\beta(S(a,\beta) \to \mathbf{f}) \to \mathbf{f}}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

Since register 2 should not change we need $R_2(b, b_0)$. Again we will just drop $S(a,b)$ and $D(b)$ on the left side for comprehensibility.

$$
\dfrac{\dfrac{\dfrac{\vdots}{R_2(b,b_0) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} R_2(b,b_0) \to \mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_2(\alpha,\gamma) \to R_2(\beta,\gamma))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_{\mathrm{f}} R_2(a,b_0)}{\vdash_{\mathrm{f}} R_2(b,b_0)}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

For the case that $\boldsymbol{r = +(1, Q')}$, we have that $\widehat{Q} = Q'$, $\widehat{n} = n + 1$, and $\widehat{m} = m$. So we need to increment register 1 and ensure that the state of b is Q'.

$$
\dfrac{\dfrac{\dfrac{\vdots}{Q'(b) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} Q'(b) \to \mathbf{f}} \quad \dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to Q'(b)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to Q'(b)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} Q'(b)}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

To increment register 1 we need a new free variable as anchor for register 1 (let $d \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma_{\overline{M}})$ and $d \neq b$).

$$\dfrac{\dfrac{\dfrac{\overset{\vdots}{R_1(b,d)\vdash_{\mathrm{f}}\mathbf{f}}}{\forall\beta(R_1(b,\beta)\to\mathbf{f})\to\mathbf{f}\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}}(\forall\beta(R_1(b,\beta)\to\mathbf{f})\to\mathbf{f})\to\mathbf{f}}\qquad\dfrac{\dfrac{\vdash_{\mathrm{f}}\forall\alpha(D(\alpha)\to\forall\beta(R_1(\alpha,\beta)\to\mathbf{f})\to\mathbf{f})}{\vdash_{\mathrm{f}}D(b)\to\forall\beta(R_1(b,\beta)\to\mathbf{f})\to\mathbf{f}}\quad\vdash_{\mathrm{f}}D(b)}{\vdash_{\mathrm{f}}\forall\beta(R_1(b,\beta)\to\mathbf{f})\to\mathbf{f}}}{\vdash_{\mathrm{f}}\mathbf{f}}$$

Now we need to connect d with a_0 (the anchor of a for register 1).

$$\dfrac{\dfrac{\overset{\vdots}{P(d,a_0)\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}}P(d,a_0)\to\mathbf{f}}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}}\forall\alpha\beta\gamma\delta(Q(\alpha)\to S(\alpha,\beta)\to R_1(\alpha,\gamma)\to R_1(\beta,\delta)\to P(\delta,\gamma))}{\vdash_{\mathrm{f}}Q(a)\to S(a,b)\to R_1(a,a_0)\to R_1(b,d)\to Q'(b)}\quad\vdash_{\mathrm{f}}Q(a)}{\vdash_{\mathrm{f}}S(a,b)\to R_1(a,a_0)\to R_1(b,d)\to Q'(b)}\quad\vdash_{\mathrm{f}}S(a,b)}{\vdash_{\mathrm{f}}R_1(a,a_0)\to R_1(b,d)\to Q'(b)}\quad\vdash_{\mathrm{f}}R_1(a,a_0)}{\vdash_{\mathrm{f}}R_1(b,d)\to Q'(b)}\quad\vdash_{\mathrm{f}}R_1(b,d)}{\vdash_{\mathrm{f}}P(d,a_0)}}{\vdash_{\mathrm{f}}\mathbf{f}}$$

At last we have to make sure that we do not get an artificial zero by deducing $D(d)$.

$$\dfrac{\dfrac{\overset{\vdots}{D(d)\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}}D(d)\to\mathbf{f}}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}}\forall\alpha\beta\delta(Q(\alpha)\to S(\alpha,\beta)\to R_1(\beta,\delta)\to D(\delta))}{\vdash_{\mathrm{f}}Q(a)\to S(a,b)\to R_1(b,d)\to D(d)}\quad\vdash_{\mathrm{f}}Q(a)}{\vdash_{\mathrm{f}}S(a,b)\to R_1(b,d)\to D(d)}\quad\vdash_{\mathrm{f}}S(a,b)}{\vdash_{\mathrm{f}}R_1(b,d)\to D(d)}\quad\vdash_{\mathrm{f}}R_1(b,d)}{\vdash_{\mathrm{f}}D(d)}}{\vdash_{\mathrm{f}}\mathbf{f}}$$

Now we already have deduced Γ_D, to see why define $\widehat{a}:=b$, $\widehat{b_i}:=b_i$ for $i\in\{0,\dots,m\}$, $\widehat{a}_0:=d$, and $\widehat{a}_{i+1}:=a_i$ for $i\in\{0,\dots,n\}$. Hence we can deduce **false** by induction hypothesis.

Case $\boldsymbol{r=-(Q,1,Q_1,Q_2)}$

$\underline{r_1=0}$

$$\dfrac{\dfrac{\dfrac{\vdots}{Q_2(b)\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}}Q_2(b)\to\mathbf{f}}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}}\forall\alpha\beta\gamma(Q(\alpha)\to S(\alpha,\beta)\to R_1(\alpha,\gamma)\to E(\gamma)\to Q_2(\beta))}{\vdash_{\mathrm{f}}Q(a)\to S(a,b)\to R_1(a,a_0)\to E(a_0)\to Q_2(b)}\quad\vdash_{\mathrm{f}}Q(a)}{\vdash_{\mathrm{f}}S(a,b)\to R_1(a,a_0)\to E(a_0)\to Q_2(b)}\quad\vdash_{\mathrm{f}}S(a,b)}{\vdash_{\mathrm{f}}R_1(a,a_0)\to E(a_0)\to Q_2(b)}\quad\vdash_{\mathrm{f}}R_1(a,a_0)}{\vdash_{\mathrm{f}}E(a_0)\to Q_2(b)}\quad\vdash_{\mathrm{f}}E(a_0)}{\vdash_{\mathrm{f}}Q_2(b)}}{\vdash_{\mathrm{f}}\mathbf{f}}$$

r_1 stays zero

$$\dfrac{\dfrac{\dfrac{\vdots}{R_1(b,a_0)\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}} R_1(b,a_0)\to\mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha)\to S(\alpha,\beta)\to R_1(\alpha,\gamma)\to E(\gamma)\to R_1(\beta,\gamma))}{\vdash_{\mathrm{f}} Q(a)\to S(a,b)\to R_1(a,a_0)\to E(a_0)\to R_1(b,a_0)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b)\to R_1(a,a_0)\to E(a_0)\to R_1(b,a_0)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_1(a,a_0)\to E(a_0)\to R_1(b,a_0)} \quad \vdash_{\mathrm{f}} R_1(a,a_0)}{\vdash_{\mathrm{f}} E(a_0)\to R_1(b,a_0)} \quad \vdash_{\mathrm{f}} E(a_0)}{\vdash_{\mathrm{f}} R_1(b,a_0)}}{\vdash_{\mathrm{f}} \mathbf{f}}$$

$\underline{r_1 \geq 1}$
new state Q_1

$$\dfrac{\dfrac{\dfrac{\vdots}{Q_1(b)\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}} Q_1(b)\to\mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha)\to S(\alpha,\beta)\to R_1(\alpha,\gamma)\to D(\gamma)\to Q_1(\beta))}{\vdash_{\mathrm{f}} Q(a)\to S(a,b)\to R_1(a,a_0)\to D(a_0)\to Q_1(b)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b)\to R_1(a,a_0)\to D(a_0)\to Q_1(b)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_1(a,a_0)\to D(a_0)\to Q_1(b)} \quad \vdash_{\mathrm{f}} R_1(a,a_0)}{\vdash_{\mathrm{f}} D(a_0)\to Q_1(b)} \quad \vdash_{\mathrm{f}} D(a_0)}{\vdash_{\mathrm{f}} Q_1(b)}}{\vdash_{\mathrm{f}} \mathbf{f}}$$

decrement r_1

$$\dfrac{\dfrac{\dfrac{\vdots}{R_1(b,a_1)\vdash_{\mathrm{f}}\mathbf{f}}}{\vdash_{\mathrm{f}} R_1(b,a_1)\to\mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma\delta(Q(\alpha)\to S(\alpha,\beta)\to R_1(\alpha,\gamma)\to D(\gamma)\to P(\gamma,\delta)\to R_1(\beta,\delta))}{\vdash_{\mathrm{f}} Q(a)\to S(a,b)\to R_1(a,a_0)\to D(a_0)\to P(a_0,a_1)\to R_1(b,a_1)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b)\to R_1(a,a_0)\to D(a_0)\to P(a_0,a_1)\to R_1(b,a_1)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_1(a,a_0)\to D(a_0)\to P(a_0,a_1)\to R_1(b,a_1)} \quad \vdash_{\mathrm{f}} R_1(a,a_0)}{\vdash_{\mathrm{f}} D(a_0)\to P(a_0,a_1)\to R_1(b,a_1)} \quad \vdash_{\mathrm{f}} D(a_0)}{\vdash_{\mathrm{f}} P(a_0,a_1)\to R_1(b,a_1)} \quad \vdash_{\mathrm{f}} P(a_0,a_1)}{\vdash_{\mathrm{f}} R_1(b,a_1)}}{\vdash_{\mathrm{f}} \mathbf{f}}$$

□

Lemma 19.

*M terminates on input $(0,0)$ iff $\Gamma_M \vdash$ **false** holds in system P.*

Proof. The $\Leftarrow$ directions follows directly from Claim 17. And the $\Rightarrow$ direction is a direct consequence of Claim 18 with $C = \langle Q_0, 0, 0\rangle$. □

Theorem 20. ***CONS** is undecidable.*

Proof. Since by Lemma 19 for a given two-counter automaton M we can effectively construct a set of **P**-formulas Γ_M such that M terminates on input $(0,0)$ iff Γ_M is not consistent. It follows that **HALT** $\leq$ **CONS**. Hence, since **HALT** is undecidable, we have shown that **CONS** is undecidable too. □