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1 Introduction

$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
The set of all $\lambda 2$ types $\Lambda_{\lambda 2}$ is the smallest set Λ satisfying the following conditions:

- $\mathcal{V}_T \subseteq \Lambda$,
- if $t_1, t_2 \in \Lambda$ then $t_1 \to t_2 \in \Lambda$, and
- if $t \in \Lambda$ and $\alpha \in \mathcal{V}_T$ then $\forall \alpha.t \in \Lambda$.

$\lambda 2$ deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ-Introduction)	$\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2}$	
(λ-Elimination)	$\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.t}$	$\alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash e : \forall\alpha.t}{\Gamma \vdash et' : t\,[\alpha := t']}$	

2 Basic Definitions

We will denote the set $\{1, \ldots, n\}$ by $[n]$.

Definition 1. A ranked set is a tuple (Σ, rk), where Σ is a countable set and $rk : \Sigma \to \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk). The set of all elements with a certain rank k in Σ, denoted by $\Sigma^{(k)}$, is defined by $\Sigma^{(k)} := rk^{-1}(k)$. In the following we will write $\Sigma = \{P^{(0)}, Q^{(3)}\}$ to say that $\Sigma = \{P, Q\}$, $rk(P) = 0$, and $rk(Q) = 3$.

First-order logic

Definition 2. Let $\mathcal{V} = \{x_0, x_1, \ldots\}$ be a countable set (of variables), $\mathcal{F} = \{\}$ a ranked set (of function symbols), and $\mathcal{P} = \{\}$ a ranked set (of predicate symbols). Then the set of terms over $(\mathcal{V}, \mathcal{F})$, denoted by $\mathcal{T}_{(\mathcal{V},\mathcal{F})}$, is the smallest set $\mathcal{T}$ satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and
- for every $k \in \mathbb{N}$ if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \ldots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \ldots, t_k) \in \mathcal{T}$.

The set of first-order formulas over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$, denoted by $\mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$, is the smallest set $\mathcal{L}$ satisfying the following conditions:

- for every $k \in \mathbb{N}$ if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \ldots, t_k \in \mathcal{T}$ then $P(t_1, t_2, \ldots, t_k) \in \mathcal{L}$.
- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), \neg\varphi \in \mathcal{L}$, and
- if $x \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists x\varphi, \forall x\varphi \in \mathcal{L}$.

To define the free variables of a formula we first need to define variables of a term.

Definition 3. The variables of a term t, denoted by $\mathrm{V}(t)$, are defined by:

$$\mathrm{V}(t) = \begin{cases} \{x\} & \text{if } t = x \\ \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \cdots \cup \mathrm{V}(t_k) & \text{if } t = f(t_1, t_2, \ldots, t_k) \end{cases}$$

Definition 4. The free variables of a formula φ, denoted by $\mathrm{FV}(\varphi)$, are defined as follows:

$$\mathrm{FV}(\varphi) = \begin{cases} \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \cdots \cup \mathrm{V}(t_k) & \text{if } \varphi = P(t_1, t_2, \ldots, t_k) \\ \mathrm{FV}(\varphi_1) \cup \mathrm{FV}(\varphi_2) & \text{if } \varphi = \varphi_1 \circ \varphi_2,\ \circ \in \{\wedge, \vee, \rightarrow\} \\ \mathrm{FV}(\psi) \setminus \{x\} & \text{if } \varphi = Qx\psi,\ Q \in \{\forall, \exists\} \end{cases}$$

Now we come to the semantics of first-order formulas.

Definition 5. An interpretation I over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$ is a triple $(\Delta, \cdot^I, \omega)$ where

where	Δ	is a set (which we call domain),
	$\cdot^I$	is a function such that
		$f^I : \Delta^k \rightarrow \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$ and
		$P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $f \in \mathcal{P}^{(k)}$
	ω	is a function from $\mathcal{V}$ to Δ.

Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation, $x \in \mathcal{V}$, and $d \in \Delta$ the interpretation $I\,[x \rightarrow d]$ is defined as $(\Delta, \cdot^I, \omega\,[x \rightarrow d])$ where

$$(\omega\,[x \rightarrow d])(y) = \begin{cases} d & \text{if } y = x \\ \omega(y) & \text{otherwise.} \end{cases}$$

Definition 6. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and t a term the interpretation of t under I, denoted by t^I, is defined as follows:

$$t^I = \begin{cases} \omega(x) & \text{if } t = x \\ f^I(t_1^I, \ldots, t_k^I) & \text{if } t = f(t_1, \ldots, t_k) \end{cases}$$

Definition 7. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and φ a formula the interpretation of φ under I, denoted by φ^I, is defined recursively as follows:

$$\varphi^I = \begin{cases} \top & \text{if } \varphi = P(t_1, \ldots, t_k) \text{ and } (t_1^I, \ldots, t_k^I) \in P^I \\ \bot & \text{if } \varphi = P(t_1, \ldots, t_k) \text{ and } (t_1^I, \ldots, t_k^I) \notin P^I \\ \text{not } \psi^I & \text{if } \varphi = \neg\psi \\ \varphi_1^I \text{ and } \varphi_2^I & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \\ \varphi_1^I \text{ or } \varphi_2^I & \text{if } \varphi = (\varphi_1 \vee \varphi_2) \\ \text{exists } d \in \Delta \ \psi^{I[x \to d]} & \text{if } \varphi = \exists x \psi \\ \text{forall } d \in \Delta \ \psi^{I[x \to d]} & \text{if } \varphi = \forall x \psi \end{cases}$$

The interpretation I is a model of φ, denoted by $I \models \varphi$, if $\varphi^I = \top$.

3 System P

3.1 Definitions

Let $\mathcal{V}_P = \{\alpha, \beta, \ldots\}$ be a countably infinite set (of variables) and $\mathcal{P}_P = \{\mathit{false}^{(0)}, P^{(2)}, Q^{(2)}, \ldots\}$ a ranked set (of predicate symbols) such that $\mathcal{P}_P^{(0)} = \{\mathit{false}\}$, $\mathcal{P}_P^{(2)} = \{P, Q, \ldots\}$ is a countable infinite set, and $\mathcal{P}_P^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ is an

atomic formula if $\varphi = \mathit{false}$ or $\varphi = P(\alpha, \beta)$ for some $P \in \mathcal{P}_P$ and $\alpha, \beta \in \mathcal{V}_P$.

universal formula if $\varphi = \forall \overrightarrow{\alpha}(A_1 \to A_2 \to \cdots \to A_n)$ where A_i is an atomic formula for $i \in [n]$, $A_i \neq \mathit{false}$ for $i \in [n-1]$ and for each $\alpha \in \mathrm{FV}(\varphi) \cap \mathrm{FV}(A_n)$ there exists an $i \in [n-1]$ such that $\alpha \in \mathrm{FV}(A_i)$.

existential formula if there exits $n \geq 0$, atomic formulas $A_i \neq \mathit{false}$ for $i \in [n]$ such that $\varphi = \forall \overrightarrow{\alpha}(A_1 \to A_2 \to \cdots \to A_{n-1} \to \forall \beta(A_n \to \mathit{false}) \to \mathit{false})$.

The set of formulas of System **P** over $(\mathcal{V}_P, \mathcal{P}_P)$ is the set of all first-order formulas over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ that are either an atomic, universal or existential formula.
$\mathrm{FV}(\Gamma) = \bigcup\{\mathrm{FV}(A) \mid A \in \Gamma\}$
Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
($\rightarrow$ -Introduction)	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
($\rightarrow$ -Elimination)	$\dfrac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B\,[\alpha := b]}$	

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta \ldots$
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

($\exists$-Introduction)	$\dfrac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false} \vdash_f B}$	$a \notin FV(\Gamma, A, B)$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false}$ with $\mathit{false}^I = \perp$.

$$\begin{aligned}
I \models \Gamma, \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false} &\Rightarrow I \models \forall\alpha(A \rightarrow \mathit{false}) \rightarrow \mathit{false} \\
&\Rightarrow (\forall\alpha(A \rightarrow \mathit{false}))^I \rightarrow \mathit{false}^I \\
&\Rightarrow (\forall\alpha(A \rightarrow \mathit{false}))^I \rightarrow \perp \\
&\Rightarrow \neg(\forall\alpha(A \rightarrow \mathit{false}))^I \\
&\Rightarrow \neg(\forall d \in \Delta : (A \rightarrow \mathit{false})^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \mathit{false}^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. $\square$

3.2 Provability in System P is undecidable

Γ_C :

- $Q(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$
- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$
- $D(a), D(a_i), D(b_j)$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$
- $E(a_m), E(b_n)$

$+(Q, 1, Q')$:

- $\forall \alpha\beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall \alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to R_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall \alpha\beta\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\beta, \delta) \to D(\delta))$
  prevent zero
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2)$:

- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to Q_2(\beta))$
  jump on zero
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to R_1(\beta, \gamma))$
  register 1 stays zero
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to Q_1(\beta))$
  change state if register 1 is greater zero
- $\forall \alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to P(\gamma, \delta) \to R_1(\beta, \delta))$
  decrement register 1
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

Lemma 8.

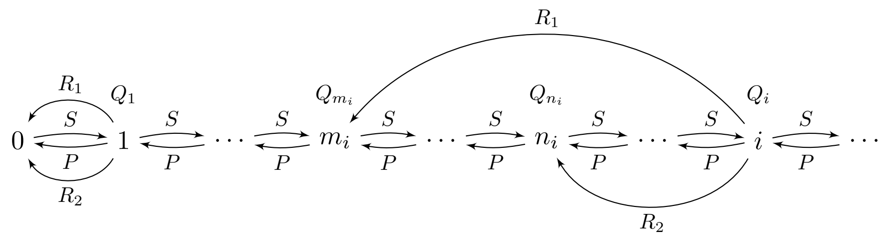
$$M \textit{ terminates on input } (0,0) \qquad \textit{iff} \qquad \Gamma_M \vdash \text{false} \textit{ holds in system } P.$$

Claim 9.

$$\Gamma_M \vdash \text{false} \textit{ holds in system } P \qquad \Longrightarrow \qquad M \textit{ terminates on input } (0,0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \ldots$. ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of Γ_M which interprets *false* with $\perp$ this contradicts $\Gamma_M \vdash$ *false*.

To illustrate the idea we will use a graphical notation for an interpretation I. By $d_1 \xrightarrow{\text{R}} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{\text{P}}{d}$ to say that $d \in P^I$ for unary predicate symbols. Now the idea for our model of Γ_M looks like this:



We have $0 \in E^I$ and all other numbers are in D^I.

Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I)$.

$$\begin{array}{lll} P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I = \{i \in \mathbb{N} \setminus \{0\} \mid Q = Q_i\} & D^I = \mathbb{N} \setminus \{0\} & E^I = \{0\} \\ S^I = \{(i, i+1) \mid i \in \mathbb{N}\} & & \end{array}$$

$$a^I = 1 \qquad a_0^I = 0 \qquad b_0^I = 0$$

□

Claim 10. *If a final state is reachable from C then* $\Gamma_C \cup \Gamma \vdash$ false.

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by f.

Induction Base trivial ...

Induction Step

$C \Rightarrow_M^r D$

We need to make a case distinction on the rule r.

Case $r = +(Q, 1, Q')$

Basic idea:

$$\dfrac{\dfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \qquad \dfrac{}{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models$ *false* holds trivially if I interprets *false* with $\top$ we only need to consider models (note that there are none if M terminates which is exactly what we want to proof) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\perp$ (so we can use

our new deduction rule).
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.
We first introduce the new variables needed for Γ_D (let $b, d \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$). Intuitively b will represent the successor state and d will be the anchor for register one.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{\vdots}{S(a,b), D(b) \vdash_f f}
      }{S(a,b) \vdash_f D(b) \to f}
      \qquad
      \dfrac{
        \dfrac{S(a,b) \vdash_f \forall\alpha\beta(S(\alpha,\beta) \to D(\beta))}{S(a,b) \vdash_f S(a,b) \to D(b)}
        \quad
        S(a,b) \vdash_f S(a,b)
      }{S(a,b) \vdash_f D(b)}
    }{S(a,b) \vdash_f f}
  }{
    \dfrac{\forall\beta(S(a,\beta) \to f) \to f \vdash_f f}{\vdash_f (\forall\beta(S(a,\beta) \to f) \to f) \to f}
  }
  \qquad
  \dfrac{\vdash_f \forall\alpha(\forall\beta(S(\alpha,\beta) \to f) \to f)}{\vdash_f \forall\beta(S(a,\beta) \to f) \to f}
}{\vdash_f f}
$$

The formula $R_1(b,d)$ can be acquired in a similar way. Again we will just drop $S(a,b)$ and $D(b)$ on the left side for comprehensibility.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\vdots}{R_1(b,d) \vdash_f f}
    }{\forall\beta(R_1(b,\beta) \to f) \to f \vdash_f f}
  }{\vdash_f (\forall\beta(R_1(b,\beta) \to f) \to f) \to f}
  \qquad
  \dfrac{
    \dfrac{\vdash_f \forall\alpha(D(\alpha) \to \forall\beta(R_1(\alpha,\beta) \to f) \to f)}{\vdash_f D(b) \to \forall\beta(R_1(b,\beta) \to f) \to f}
    \quad
    \vdash_f D(b)
  }{\vdash_f \forall\beta(R_1(b,\beta) \to f) \to f}
}{\vdash_f f}
$$

Now we have all the new free variables we need and we continue by ensuring that these variables fulfill all the formulas in Γ_D.

$$
\dfrac{
  \dfrac{
    \dfrac{\vdots}{Q'(b) \vdash_f f}
  }{\vdash_f Q'(b) \to f}
  \qquad
  \dfrac{
    \dfrac{
      \dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)}
      \quad
      \vdash_f Q(a)
    }{\vdash_f S(a,b) \to Q'(b)}
    \quad
    \vdash_f S(a,b)
  }{\vdash_f Q'(b)}
}{\vdash_f f}
$$

Starting from $Q'(b) \vdash_f \mathit{false}$ we can connect d and a_0.

$$
\dfrac{\dfrac{\dfrac{\vdots}{P(d,a_0) \vdash_f f}}{\vdash_f P(d,a_0) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to R_1(\beta,\delta) \to P(\delta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f R_1(a,a_0)}{\vdash_f R_1(b,d) \to Q'(b)} \quad \vdash_f R_1(b,d)}{\vdash_f P(d,a_0)}}{\vdash_f f}
$$

For register one we still need $D(d)$.

$$
\dfrac{\dfrac{\dfrac{\vdots}{D(d) \vdash_f f}}{\vdash_f D(d) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\beta,\delta) \to D(\delta))}{\vdash_f Q(a) \to S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_f S(a,b)}{\vdash_f R_1(b,d) \to D(d)} \quad \vdash_f R_1(b,d)}{\vdash_f D(d)}}{\vdash_f f}
$$

Since register two should not change we only need $R_2(b,b_0)$.

$$
\dfrac{\dfrac{\dfrac{\vdots}{R_2(b,b_0) \vdash_f f}}{\vdash_f R_2(b,b_0) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_2(\alpha,\gamma) \to R_2(\beta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f S(a,b)}{\vdash_f R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f R_2(a,b_0)}{\vdash_f R_2(b,b_0)}}{\vdash_f f}
$$

Now we have Γ_C (Since $P(a_{i-1},a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.

<u>Case $r = -(Q,1,Q_1,Q_2)$</u> $r1 = 0$

$$
\dfrac{\dfrac{\dfrac{\vdots}{Q_2(b) \vdash_f f}}{\vdash_f Q_2(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to Q_2(\beta))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_f S(a,b)}{\vdash_f R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_f R_1(a,a_0)}{\vdash_f E(a_0) \to Q_2(b)} \quad \vdash_f E(a_0)}{\vdash_f Q_2(b)}}{\vdash_f f}
$$

$\square$