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# 1 Introduction

$FV(\Gamma) = \bigcup \{FV(t) \mid (x : t) \in \Gamma\}$   
 $\lambda 2$  deduction Rules

|                            |   |                            |
|----------------------------|---|----------------------------|
| (Axiom)                    | $\Gamma, x : t \vdash x : t$  |                            |
| ( $\lambda$ -Introduction) | $\frac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x. e : t_1 \rightarrow t_2}$                   |                            |
| ( $\lambda$ -Elimination)  | $\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$ |                            |
| ( $\forall$ -Introduction) | $\frac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. t}$                           | $\alpha \notin FV(\Gamma)$ |
| ( $\forall$ -Elimination)  | $\frac{\Gamma \vdash e : \forall \alpha. t}{\Gamma \vdash e t' : t[\alpha := t']}$                          |                            |

## 1.1 Basic Definitions

We will denote the set  $\{1, \dots, n\}$  by  $[n]$ .

# 2 System P

## 2.1 Definitions

Let  $V_P = \{\alpha, \beta, \dots\}$  be a countably infinite set (of variables) and  $R_P = \{false^{(0)}, P^{(2)}, Q^{(2)}, \dots\}$  a ranked alphabet (of relation symbols). A first-order logic formula  $\varphi$  is an

**atomic formula** if  $\varphi = false$  or  $\varphi = P(\alpha, \beta)$  for some  $P \in R_P$  and  $\alpha, \beta \in V_P$ .

**universal formula** if  $\varphi = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n)$  where  $A_i$  is an atomic formula for  $i \in [n]$ ,  $A_i \neq false$  for  $i \in [n-1]$  and for each  $\alpha \in FV(\varphi) \cap FV(A_n)$  there exists an  $i \in [n-1]$  such that  $\alpha \in FV(A_i)$ .

**existential formula** if there exists  $n \geq 0$ , atomic formulas  $A_i \neq false$  for  $i \in [n]$  such that  $\varphi = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n-1} \rightarrow \forall \beta (A_n \rightarrow false) \rightarrow false)$ .

The set of formulas of System **P** over  $V_P$  and  $R_P$  is the set of all first order formulas over the same "alphabet" that are either an atomic, universal or existential formula.

$FV(\Gamma) = \bigcup \{FV(A) \mid A \in \Gamma\}$   
Deduction Rules

|                                |   |                            |
|--------------------------------|---|----------------------------|
| (Axiom)                        | $\Gamma, A \vdash A$  |                            |
| ( $\rightarrow$ -Introduction) | $\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$                    |                            |
| ( $\rightarrow$ -Elimination)  | $\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$ |                            |
| ( $\forall$ -Introduction)     | $\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$                      | $\alpha \notin FV(\Gamma)$ |
| ( $\forall$ -Elimination)      | $\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$         |                            |

An Interpretation  $I$  of a P formula is a tuple  $I = (\Delta, \cdot^I)$  where  $\Delta$  is a set (called domain),  $P^I \subseteq \Delta^k$  and  $\alpha^I \in \Delta \dots$

If we interpret *false* with the logical constant false ( $\perp$ ) (denoted by  $\vdash_f$ ) we can add a new deduction rule.

|                           |   |                             |
|---------------------------|---|-----------------------------|
| ( $\exists$ -Elimination) | $\frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, \forall \alpha (A \rightarrow false) \rightarrow false \vdash_f B}$ | $a \notin FV(\Gamma, A, B)$ |
|---------------------------|---|-----------------------------|

*Proof.* Let  $I = (\Delta, \cdot^I)$  be a model of  $\Gamma, \forall \alpha (A \rightarrow false) \rightarrow false$  with  $false^I = \perp$ .

$$\begin{aligned}
I \models \Gamma, \forall \alpha (A \rightarrow false) \rightarrow false &\Rightarrow I \models \forall \alpha (A \rightarrow false) \rightarrow false \\
&\Rightarrow (\forall \alpha (A \rightarrow false))^I \rightarrow false^I \\
&\Rightarrow (\forall \alpha (A \rightarrow false))^I \rightarrow \perp \\
&\Rightarrow \neg(\forall \alpha (A \rightarrow false))^I \\
&\Rightarrow \neg(\forall a \in \Delta : (A \rightarrow false)^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow false^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}$$

Together with  $a \notin FV(\Gamma, A)$ , it follows that  $I[a \mapsto d]$  is a model of  $\Gamma, A[\alpha := a]$ . Which implies  $I[a \mapsto d] \models B$ . Since  $a$  is not free in  $B$  we conclude that  $I$  is also a model of  $B$ .  $\square$

## 2.2 Provability in System P is undecidable

$\Gamma_C :$

- $Q(a)$
  - $P_1(a, a_0), P(a_{i-1}, a_i)$  for  $i \in \{1, \dots, m\}$
  - $P_2(a, b_0), P(b_{i-1}, b_i)$  for  $i \in \{1, \dots, n\}$
  - $D(a), D(a_i), D(b_j)$  for  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$
  - $E(a_m), E(b_n)$
- $+(Q, 1, Q') :$
- $\forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$   
change of state
  - $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow P_1(\beta, \delta) \rightarrow P(\delta, \gamma))$   
increment register 1
  - $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma))$   
prevent zero
  - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$   
do not change register 2
- $-(Q, 1, Q_1, Q_2) :$
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$   
jump on zero
  - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow P_1(\beta, \gamma))$   
register 1 stays zero
  - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$   
change state if register 1 is greater zero
  - $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow P_1(\beta, \delta))$   
decrement register 1
  - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$   
do not change register 2

**Lemma 1.**

$M$  terminates on input  $(0, 0)$  iff  $\Gamma_M \vdash \text{false}$  holds in system  $P$ .

**Claim 2.** If a final state is reachable from  $C$  then  $\Gamma_C \cup \Gamma \vdash \text{false}$ .

*Proof.* By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by  $f$ .

Induction Base trivial ...

Induction Step

$C \rightarrow_M^r D$

We need to make a case distinction on the rule  $r$ .

Case  $r = +(Q, 1, Q')$

Basic idea:

$$\frac{\frac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \quad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since  $I \models \text{false}$  holds trivially if  $I$  interprets *false* with  $\top$  we only need to consider models (note that there are none if  $M$  terminates which is exactly what we want to prove) of  $\Gamma_C \cup \Gamma$  that interpret *false* with  $\perp$  (so we can use our new deduction rule).

We will just drop  $\Gamma_C \cup \Gamma$  and only write new formulas on the left side.

We first introduce the new variables needed for  $\Gamma_D$  (let  $b, d \in V_P \setminus \text{FV}(\Gamma_C \cup \Gamma)$ ):

$$\frac{\frac{\frac{S(a, b), D(b) \vdash_f f}{S(a, b) \vdash_f D(b) \rightarrow f} \quad \frac{\frac{S(a, b) \vdash_f \forall \alpha \beta S(\alpha, \beta) \rightarrow D(\beta)}{S(a, b) \vdash_f S(a, b) \rightarrow D(b)}}{S(a, b) \vdash_f D(b)} \quad \frac{S(a, b) \vdash_f f}{\vdash_f (\forall \beta (S(a, \beta) \rightarrow f) \rightarrow f)} \quad \frac{\vdash_f \forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f \forall \beta (S(a, \beta) \rightarrow f) \rightarrow f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

The formula  $P_1(b, d)$  can be acquired in a similar way.

Now we create  $\Gamma_D$

$$\frac{\frac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\frac{\frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f Q'(b)}$$

Alternative tableau with tikz:

$$\frac{\frac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\frac{\frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f Q'(b)} \quad \vdash_f f$$

Starting from  $Q'(b) \vdash_f \text{false}$  we can deduce:

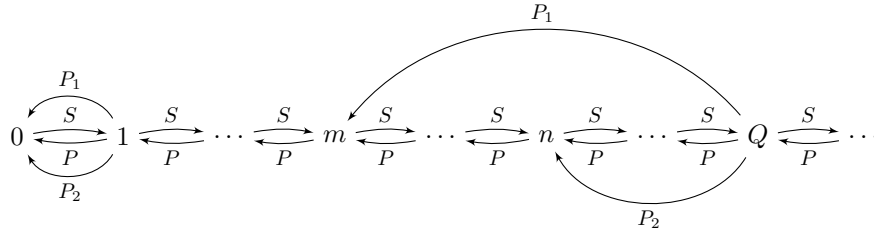
$$\frac{\frac{\frac{\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow P_1(\beta, \delta) \rightarrow P(\delta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow P_1(a, a_0) \rightarrow P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow P_1(a, a_0) \rightarrow P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f P_1(a, a_0) \rightarrow P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f P_1(a, a_0)}{\frac{\frac{P(d, a_0) \vdash_f f}{\vdash_f P(d, a_0) \rightarrow f}}{\vdash_f P(d, a_0) \rightarrow f} \quad \frac{\vdash_f P_1(b, d) \rightarrow Q'(b) \quad \vdash_f P_1(b, d)}{\vdash_f P(d, a_0)}}{\vdash_f f}$$

$P_2(b, b_0)$  can be deduced in the same way.  
 Now we have  $\Gamma_C$  (Since  $P(a_{i-1}, a_i)$  is already in  $\Gamma_D$ ) and can deduce *false* by induction hypothesis.  
Case  $r = -(Q, 1, Q_1, Q_2)$  □

**Claim 3.**

$$\Gamma_M \vdash \text{false holds in system } P \quad \Longrightarrow \quad M \text{ terminates on input } (0,0)$$

*Proof.* Assume  $M$  does not terminate then there is an infinite chain  $C_0 \Rightarrow_M C_1 \Rightarrow_M C_3 \Rightarrow_M \dots$  ( $C_i = \langle Q_i, m_i, n_i \rangle$ ) Now we construct a model of  $\Gamma_M$  which interprets *false* with  $\perp$  this contradicts  $\Gamma_M \vdash \textit{false}$ . The idea looks like this:



Formal definition:  
 $I = (\mathbb{N}, \cdot^I)$

$$\begin{array}{lll} P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & P_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & P_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I = \{i \in \mathbb{N} \mid Q = Q_i\} & D^I = \mathbb{N} \setminus \{0\} & E^I = \{0\} \\ S^I = \{(i, i+1) \mid i \in \mathbb{N}\} & & \end{array}$$

