Contents

1 Introduction

$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
$\lambda 2$ deduction Rules

$$
\begin{array}{lll}
(\text{Axiom}) & \Gamma, x : t \vdash x : t & \\[1em]
(\lambda\text{-Introduction}) & \dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2} & \\[1em]
(\lambda\text{-Elimination}) & \dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2} & \\[1em]
(\forall\text{-Introduction}) & \dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.t} & \alpha \notin FV(\Gamma) \\[1em]
(\forall\text{-Elimination}) & \dfrac{\Gamma \vdash e : \forall\alpha.t}{\Gamma \vdash et' : t\,[\alpha := t']} &
\end{array}
$$

2 System P

2.1 Definitions

$FV(\Gamma) = \bigcup\{FV(A) \mid A \in \Gamma\}$
Deduction Rules

$$
\begin{array}{lll}
(\text{Axiom}) & \Gamma, A \vdash A & \\[1em]
(\to\text{-Introduction}) & \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \to B} & \\[1em]
(\to\text{-Elimination}) & \dfrac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B} & \\[1em]
(\forall\text{-Introduction}) & \dfrac{\Gamma \vdash B}{\Gamma \vdash \forall\alpha B} & \alpha \notin FV(\Gamma) \\[1em]
(\forall\text{-Elimination}) & \dfrac{\Gamma \vdash \forall\alpha B}{\Gamma \vdash B\,[\alpha := b]} &
\end{array}
$$

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta \ldots$
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

$$(\exists\text{-Elimination}) \qquad \frac{\Gamma \vdash_f \forall\alpha(A \to \mathit{false}) \to \mathit{false}}{\Gamma \vdash_f A\,[\alpha := a]} \qquad a \notin FV(\Gamma)$$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of Γ with $\mathit{false}^I = \bot$.

$$\begin{aligned}
I \models \Gamma &\Rightarrow I \models \forall\alpha(A \to \mathit{false}) \to \mathit{false} \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \mathit{false}^I \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \bot \\
&\Rightarrow \neg(\forall\alpha(A \to \mathit{false}))^I \\
&\Rightarrow \neg(\forall a \in \Delta : (A \to \mathit{false})^{I[\alpha:=a]}) \\
&\Rightarrow \exists a \in \Delta : \neg(A^{I[\alpha:=a]} \to \mathit{false}^{I[\alpha:=a]}) \\
&\Rightarrow \exists a \in \Delta : \neg(A^{I[\alpha:=a]} \to \bot) \\
&\Rightarrow \exists a \in \Delta : \neg(\neg A^{I[\alpha:=a]}) \\
&\Rightarrow \exists a \in \Delta : A^{I[\alpha:=a]}
\end{aligned}$$

□

$$(\exists\text{-Elimination}) \qquad \frac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false} \vdash_f B} \qquad a \notin FV(\Gamma, A, B)$$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false}$ with $\mathit{false}^I = \bot$.

$$\begin{aligned}
I \models \Gamma, \forall\alpha(A \to \mathit{false}) \to \mathit{false} &\Rightarrow I \models \forall\alpha(A \to \mathit{false}) \to \mathit{false} \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \mathit{false}^I \\
&\Rightarrow (\forall\alpha(A \to \mathit{false}))^I \to \bot \\
&\Rightarrow \neg(\forall\alpha(A \to \mathit{false}))^I \\
&\Rightarrow \neg(\forall a \in \Delta : (A \to \mathit{false})^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha\mapsto d]} \to \mathit{false}^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha\mapsto d]} \to \bot) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha\mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha\mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. □

2.2 Provability in System P is undecidable

Γ_C :

- $Q(a)$
- $P_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$
- $P_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$
- $D(a) \ldots$

$+(Q, 1, Q')$:

- $\forall \alpha \beta (Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to P_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to D(\gamma)$
  prevent zero
- $\forall \alpha \beta \gamma (Q(\alpha) \to S(\alpha, \beta) \to P_2(\alpha, \gamma) \to P_2(\beta, \gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2)$:

- $\forall \alpha \beta \gamma (Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to E(\gamma) \to Q_2(\beta))$
  jump on zero
- $\forall \alpha \beta \gamma (Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to E(\gamma) \to P_1(\beta, \gamma)$
  register 1 stays zero
- $\forall \alpha \beta \gamma (Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to D(\gamma) \to Q_1(\beta))$
  change state if register 1 is greater zero
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \to S(\alpha, \beta) \to P_1(\alpha, \gamma) \to D(\gamma) \to P(\gamma, \delta) \to P_1(\beta, \delta))$
  decrement register 1
- $\forall \alpha \beta \gamma (Q(\alpha) \to S(\alpha, \beta) \to P_2(\alpha, \gamma) \to P_2(\beta, \gamma))$
  do not change register 2

Lemma 1.

$$M \textit{ terminates on input } (0,0) \qquad \textit{iff} \qquad \Gamma_M \vdash \text{false} \textit{ hold in system } P.$$

Claim 2. $\Gamma_C \cup \Gamma \vdash \text{false} \ldots$

Proof. For the tableau proofs we will abbreviate *false* by *f*.
Induction Base trivial ...
Induction Step
$C \to D$
Basic idea:

$$\dfrac{\dfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \qquad \dfrac{}{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models$ *false* holds trivially if I interprets *false* with $\top$ we only need to consider models (there are none if M terminates) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\bot$ (so we can use our new deduction rule)
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side We first introduce the new variables needed for Γ_D (let $b, \in V_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$):

$$\dfrac{\dfrac{\dfrac{\dfrac{S(a,b), D(b) \vdash_f f}{S(a,b) \vdash_f D(b) \to f} \quad \dfrac{\dfrac{S(a,b) \vdash_f \forall \alpha\beta S(\alpha,\beta) \to D(\beta)}{S(a,b) \vdash_f S(a,b) \to D(b)}}{S(a,b) \vdash_f D(b)}}{S(a,b) \vdash_f f}}{\vdash_f (\forall \beta (S(a,\beta) \to f) \to f) \to f} \qquad \dfrac{\vdash_f \forall \alpha (\forall \beta (S(\alpha,\beta) \to f) \to f)}{\vdash_f \forall \beta (S(a,\beta) \to f) \to f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

Now we create Γ_D

$$\dfrac{\dfrac{Q'(b) \vdash f}{\vdash Q'(b) \to f} \quad \dfrac{\dfrac{\dfrac{\dfrac{\vdash \forall \alpha\beta (Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash Q(a) \to S(a,b) \to Q'(b)} \quad \vdash Q(a)}{\vdash S(a,b) \to Q'(b)} \quad \vdash S(a,b)}{\vdash Q'(b)}}{\vdash_f f}$$

$\square$