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1 Introduction

$FV(\Gamma) = \bigcup\{FV(t) \mid (x : t) \in \Gamma\}$
$\lambda 2$ deduction Rules

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2 Basic Definitions

We will denote the set $\{1, \ldots, n\}$ by $[n]$.

Definition 1. A ranked set is a tuple (Σ, rk), where Σ is a countable set and $rk : \Sigma \to \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk). The set of all elements with a certain rank k in Σ, denoted by $\Sigma^{(k)}$, is defined by $\Sigma^{(k)} := rk^{-1}(k)$. In the following we will write $\Sigma = \{P^{(0)}, Q^{(3)}\}$ to say that $\Sigma = \{P, Q\}$, $rk(P) = 0$, and $rk(Q) = 3$.
First-order logic

Definition 2. Let $\mathcal{V} = \{x_0, x_1, \ldots\}$ be a countable set (of variables), $\mathcal{F} = \{\}$ a ranked set (of function symbols), and $\mathcal{P} = \{\}$ a ranked set (of predicate symbols). Then the set of terms over $(\mathcal{V}, \mathcal{F})$, denoted by $\mathcal{T}_{(\mathcal{V},\mathcal{F})}$, is the smallest set $\mathcal{T}$ satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and
- for every $k \in \mathbb{N}$ if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \ldots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \ldots, t_k) \in \mathcal{T}$.

The set of first-order formulas over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$, denoted by $\mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$, is the smallest set $\mathcal{L}$ satisfying the following conditions:

- for every $k \in \mathbb{N}$ if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \ldots, t_k \in \mathcal{T}$ then $P(t_1, t_2, \ldots, t_k) \in \mathcal{L}$.

- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi), \neg\varphi \in \mathcal{L}$, and
- if $x \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists x\varphi, \forall x\varphi \in \mathcal{L}$.

Definition 3. The free variables of a formula φ, denoted by $\mathrm{FV}(\varphi)$, are defined as follows:

$$\mathrm{FV}(\varphi) = \begin{cases} V(t_1) \cup V(t_2) \cup \cdots \cup V(t_k) & \text{if } \varphi = P(t_1, t_2, \ldots, t_k) \\ \mathrm{FV}(\varphi_1) \cup \mathrm{FV}(\varphi_2) & \text{if } \varphi = \varphi_1 \circ \varphi_2,\ \circ \in \{\wedge, \vee, \to\} \\ \mathrm{FV}(\psi) \setminus \{x\} & \text{if } \varphi = Qx\psi,\ Q \in \{\forall, \exists\} \end{cases}$$

Definition 4. An interpretation I over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$ is a triple $(\Delta, \cdot^I, \omega)$ where Δ is a set (domain),
$\cdot^I$ is a function such that
$f^I : \Delta^k \to \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$
$P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $f \in \mathcal{P}^{(k)}$
ω is a function from $\mathcal{V}$ to Δ

3 System P

3.1 Definitions

Let $\mathcal{V}_P = \{\alpha, \beta, \ldots\}$ be a countably infinite set (of variables) and $\mathcal{P}_P = \{\mathit{false}^{(0)}, P^{(2)}, Q^{(2)}, \ldots\}$ a ranked set (of predicate symbols) such that $\mathcal{P}_P^{(0)} = \{\mathit{false}\}$, $\mathcal{P}_P^{(2)} = \{P, Q, \ldots\}$ is a countable infinite set, and $\mathcal{P}_P^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ is an

atomic formula if $\varphi = \mathit{false}$ or $\varphi = P(\alpha, \beta)$ for some $P \in \mathcal{P}_P$ and $\alpha, \beta \in \mathcal{V}_P$.

universal formula if $\varphi = \forall\vec{\alpha}(A_1 \to A_2 \to \cdots \to A_n)$ where A_i is an atomic formula for $i \in [n]$, $A_i \neq \mathit{false}$ for $i \in [n-1]$ and for each $\alpha \in \mathrm{FV}(\varphi) \cap \mathrm{FV}(A_n)$ there exists an $i \in [n-1]$ such that $\alpha \in \mathrm{FV}(A_i)$.

existential formula if there exits $n \geq 0$, atomic formulas $A_i \neq \mathit{false}$ for $i \in [n]$ such that $\varphi = \forall\vec{\alpha}(A_1 \to A_2 \to \cdots \to A_{n-1} \to \forall\beta(A_n \to \mathit{false}) \to \mathit{false})$.

The set of formulas of System **P** over $(\mathcal{V}_P, \mathcal{P}_P)$ is the set of all first-order formulas over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ that are either an atomic, universal or existential formula.
$\mathrm{FV}(\Gamma) = \bigcup\{\mathrm{FV}(A) \mid A \in \Gamma\}$
Deduction Rules

$$
\begin{array}{lll}
\text{(Axiom)} & \Gamma, A \vdash A & \\[1em]
(\rightarrow\text{-Introduction}) & \dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B} & \\[1em]
(\rightarrow\text{-Elimination}) & \dfrac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B} & \\[1em]
(\forall\text{-Introduction}) & \dfrac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B} & \alpha \notin \mathrm{FV}(\Gamma) \\[1em]
(\forall\text{-Elimination}) & \dfrac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B\,[\alpha := b]} &
\end{array}
$$

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta$....
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

$$
(\exists\text{-Introduction}) \quad \frac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall \alpha (A \rightarrow \mathit{false}) \rightarrow \mathit{false} \vdash_f B} \quad a \notin FV(\Gamma, A, B)
$$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall \alpha (A \rightarrow \mathit{false}) \rightarrow \mathit{false}$ with $\mathit{false}^I = \perp$.

$$
\begin{aligned}
I \models \Gamma, \forall \alpha (A \rightarrow \mathit{false}) \rightarrow \mathit{false} &\Rightarrow I \models \forall \alpha (A \rightarrow \mathit{false}) \rightarrow \mathit{false} \\
&\Rightarrow (\forall \alpha (A \rightarrow \mathit{false}))^I \rightarrow \mathit{false}^I \\
&\Rightarrow (\forall \alpha (A \rightarrow \mathit{false}))^I \rightarrow \perp \\
&\Rightarrow \neg (\forall \alpha (A \rightarrow \mathit{false}))^I \\
&\Rightarrow \neg (\forall d \in \Delta : (A \rightarrow \mathit{false})^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg (A^{I[\alpha \mapsto d]} \rightarrow \mathit{false}^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg (A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta : \neg (\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}
$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. □

3.2 Provability in System P is undecidable

Γ_C :

- $Q(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \dots, m\}$
- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \dots, n\}$
- $D(a), D(a_i), D(b_j)$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$
- $E(a_m), E(b_n)$

$+(Q, 1, Q')$:

- $\forall \alpha\beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall \alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to R_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall \alpha\beta\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\beta, \delta) \to D(\delta))$
  prevent zero
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2)$:

- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to Q_2(\beta))$
  jump on zero
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to R_1(\beta, \gamma))$
  register 1 stays zero
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to Q_1(\beta))$
  change state if register 1 is greater zero
- $\forall \alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to P(\gamma, \delta) \to R_1(\beta, \delta))$
  decrement register 1
- $\forall \alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

Lemma 5.

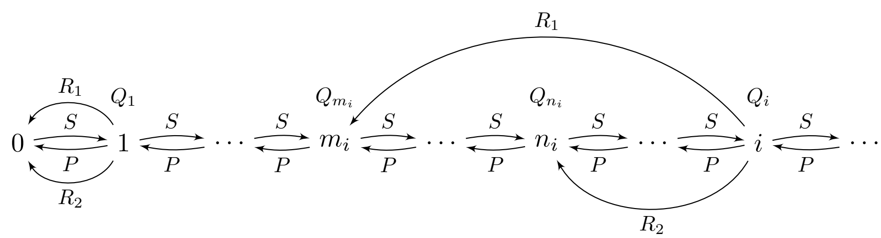
$$M \textit{ terminates on input } (0,0) \quad \textit{iff} \quad \Gamma_M \vdash \text{false} \textit{ holds in system } P.$$

Claim 6.

$$\Gamma_M \vdash \text{false} \textit{ holds in system } P \quad \Longrightarrow \quad M \textit{ terminates on input } (0,0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \ldots$. ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of Γ_M which interprets *false* with $\bot$ this contradicts $\Gamma_M \vdash$ *false*.

To illustrate the idea we will use a graphical notation for an interpretation I. By $d_1 \xrightarrow{\text{R}} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{\text{P}}{d}$ to say that $d \in P^I$ for unary predicate symbols. Now the idea for our model of Γ_M looks like this:



We have $0 \in E^I$ and all other numbers are in D^I.

Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I)$.

$$\begin{array}{lll} P^I = \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I = \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I = \{i \in \mathbb{N} \setminus \{0\} \mid Q = Q_i\} & D^I = \mathbb{N} \setminus \{0\} & E^I = \{0\} \\ S^I = \{(i, i+1) \mid i \in \mathbb{N}\} & & \end{array}$$

$$a^I = 1 \qquad\qquad a_0^I = 0 \qquad\qquad b_0^I = 0$$

□

Claim 7. *If a final state is reachable from C then $\Gamma_C \cup \Gamma \vdash$ false.*

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by f.
Induction Base trivial ...
Induction Step
$C \Rightarrow^r_M D$
We need to make a case distinction on the rule r.
<u>Case $r = +(Q, 1, Q')$</u>
Basic idea:

$$\frac{\dfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \qquad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models$ *false* holds trivially if I interprets *false* with $\top$ we only need to consider models (note that there are none if M terminates which is exactly what we want to proof) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\bot$ (so we can use

our new deduction rule).
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.
We first introduce the new variables needed for Γ_D (let $b, d \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$). Intuitively b will represent the successor state and d will be the anchor for register one.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{S(a,b), D(b) \vdash_f f}}{S(a,b) \vdash_f D(b) \to f} \quad \dfrac{\dfrac{S(a,b) \vdash_f \forall\alpha\beta(S(\alpha,\beta) \to D(\beta))}{S(a,b) \vdash_f S(a,b) \to D(b)} \quad S(a,b) \vdash_f S(a,b)}{S(a,b) \vdash_f D(b)}}{S(a,b) \vdash_f f}}{\dfrac{\forall\beta(S(a,\beta) \to f) \to f \vdash_f f}{\vdash_f (\forall\beta(S(a,\beta) \to f) \to f) \to f}} \quad \dfrac{\vdash_f \forall\alpha(\forall\beta(S(\alpha,\beta) \to f) \to f)}{\vdash_f \forall\beta(S(a,\beta) \to f) \to f}}{\vdash_f f}$$

The formula $R_1(b,d)$ can be acquired in a similar way. Again we will just drop $S(a,b)$ and $D(b)$ on the left side for comprehensibility.

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{R_1(b,d) \vdash_f f}}{\forall\beta(R_1(b,\beta) \to f) \to f \vdash_f f}}{\vdash_f (\forall\beta(R_1(b,\beta) \to f) \to f) \to f} \quad \dfrac{\dfrac{\vdash_f \forall\alpha(D(\alpha) \to \forall\beta(R_1(\alpha,\beta) \to f) \to f)}{\vdash_f D(b) \to \forall\beta(R_1(b,\beta) \to f) \to f} \quad \vdash_f D(b)}{\vdash_f \forall\beta(R_1(b,\beta) \to f) \to f}}{\vdash_f f}$$

Now we have all the new free variables we need and we continue by ensuring that these variables fulfill all the formulas in Γ_D.

$$\dfrac{\dfrac{\dfrac{\vdots}{Q'(b) \vdash_f f}}{\vdash_f Q'(b) \to f} \quad \dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f Q'(b)}}{\vdash_f f}$$

Starting from $Q'(b) \vdash_f$ *false* we can connect d and a_0.

$$
\dfrac{\dfrac{\dfrac{\vdots}{P(d,a_0) \vdash_f f}}{\vdash_f P(d,a_0) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to R_1(\beta,\delta) \to P(\delta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f S(a,b)}{\vdash_f R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f R_1(a,a_0)}{\vdash_f R_1(b,d) \to Q'(b)} \quad \vdash_f R_1(b,d)}{\vdash_f P(d,a_0)}}{\vdash_f f}
$$

For register one we still need $D(d)$.

$$
\dfrac{\dfrac{\dfrac{\vdots}{D(d) \vdash_f f}}{\vdash_f D(d) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\beta,\delta) \to D(\delta))}{\vdash_f Q(a) \to S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_f S(a,b)}{\vdash_f R_1(b,d) \to D(d)} \quad \vdash_f R_1(b,d)}{\vdash_f D(d)}}{\vdash_f f}
$$

Since register two should not change we only need $R_2(b,b_0)$.

$$
\dfrac{\dfrac{\dfrac{\vdots}{R_2(b,b_0) \vdash_f f}}{\vdash_f R_2(b,b_0) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_2(\alpha,\gamma) \to R_2(\beta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f S(a,b)}{\vdash_f R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f R_2(a,b_0)}{\vdash_f R_2(b,b_0)}}{\vdash_f f}
$$

Now we have Γ_C (Since $P(a_{i-1},a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.
Case $r = -(Q,1,Q_1,Q_2)$ $r1 = 0$

$$
\dfrac{\dfrac{\dfrac{\vdots}{Q_2(b) \vdash_f f}}{\vdash_f Q_2(b) \to f} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_f \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to Q_2(\beta))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_f Q(a)}{\vdash_f S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_f S(a,b)}{\vdash_f R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_f R_1(a,a_0)}{\vdash_f E(a_0) \to Q_2(b)} \quad \vdash_f E(a_0)}{\vdash_f Q_2(b)}}{\vdash_f f}
$$

$\square$