

Contents

1	Introduction	2
2	Basic Definitions	2
3	System P	3
3.1	Definitions	3
3.2	Provability in System P is undecidable	4

1 Introduction

$FV(\Gamma) = \bigcup \{FV(t) \mid (x : t) \in \Gamma\}$
 $\lambda 2$ deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ -Introduction)	$\frac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x. e : t_1 \rightarrow t_2}$	
(λ -Elimination)	$\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
(\forall -Introduction)	$\frac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. t}$	$\alpha \notin FV(\Gamma)$
(\forall -Elimination)	$\frac{\Gamma \vdash e : \forall \alpha. t}{\Gamma \vdash e t' : t [\alpha := t']}$	

2 Basic Definitions

We will denote the set $\{1, \dots, n\}$ by $[n]$.

Definition 1. A ranked set is a tuple (Σ, rk) , where Σ is a countable set and $rk : \Sigma \rightarrow \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk) . The set of all elements with a certain rank k in Σ , denoted by $\Sigma^{(k)}$, is defined by $\Sigma^{(k)} := rk^{-1}(k)$. In the following we will write $\Sigma = \{P^{(0)}, Q^{(3)}\}$ to say that $\Sigma = \{P, Q\}$, $rk(P) = 0$, and $rk(Q) = 3$.

First-order logic

Definition 2. Let $\mathcal{V} = \{x_0, x_1, \dots\}$ be a countable set (of variables), $\mathcal{F} = \{\}$ a ranked set (of function symbols), and $\mathcal{P} = \{\}$ a ranked set (of predicate symbols). Then the set of terms over $(\mathcal{V}, \mathcal{F})$, denoted by $\mathcal{T}_{(\mathcal{V}, \mathcal{F})}$, is the smallest set \mathcal{T} satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and
- for every $k \in \mathbb{N}$ if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \dots, t_k) \in \mathcal{T}$.

The set of first-order formulas over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$, denoted by $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$, is the smallest set \mathcal{L} satisfying the following conditions:

- for every $k \in \mathbb{N}$ if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}$ then $P(t_1, t_2, \dots, t_k) \in \mathcal{L}$.

- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), \neg\varphi \in \mathcal{L}$, and
- if $x \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists x\varphi, \forall x\varphi \in \mathcal{L}$.

Definition 3. The free variables of a formula φ , denoted by $\text{FV}(\varphi)$, are defined as follows:

$$\text{FV}(\varphi) = \begin{cases} V(t_1) \cup V(t_2) \cup \dots \cup V(t_k) & \text{if } \varphi = P(t_1, t_2, \dots, t_k) \\ \text{FV}(\varphi_1) \cup \text{FV}(\varphi_2) & \text{if } \varphi = \varphi_1 \circ \varphi_2, \circ \in \{\wedge, \vee, \rightarrow\} \\ \text{FV}(\psi) \setminus \{x\} & \text{if } \varphi = Qx\psi, Q \in \{\forall, \exists\} \end{cases}$$

Definition 4. An interpretation I over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$ is a triple $(\Delta, \cdot^I, \omega)$ where

- Δ is a set (which we call domain),
- \cdot^I is a function such that
 - $f^I : \Delta^k \rightarrow \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$ and
 - $P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $P \in \mathcal{P}^{(k)}$
- ω is a function from \mathcal{V} to Δ .

3 System P

3.1 Definitions

Let $\mathcal{V}_P = \{\alpha, \beta, \dots\}$ be a countably infinite set (of variables) and $\mathcal{P}_P = \{false^{(0)}, P^{(2)}, Q^{(2)}, \dots\}$ a ranked set (of predicate symbols) such that $\mathcal{P}_P^{(0)} = \{false\}$, $\mathcal{P}_P^{(2)} = \{P, Q, \dots\}$ is a countable infinite set, and $\mathcal{P}_P^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ is an

atomic formula if $\varphi = false$ or $\varphi = P(\alpha, \beta)$ for some $P \in \mathcal{P}_P$ and $\alpha, \beta \in \mathcal{V}_P$.

universal formula if $\varphi = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n)$ where A_i is an atomic formula for $i \in [n]$, $A_i \neq false$ for $i \in [n-1]$ and for each $\alpha \in \text{FV}(\varphi) \cap \text{FV}(A_n)$ there exists an $i \in [n-1]$ such that $\alpha \in \text{FV}(A_i)$.

existential formula if there exists $n \geq 0$, atomic formulas $A_i \neq false$ for $i \in [n]$ such that $\varphi = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n-1} \rightarrow \forall \beta (A_n \rightarrow false) \rightarrow false)$.

The set of formulas of System **P** over $(\mathcal{V}_P, \mathcal{P}_P)$ is the set of all first-order formulas over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ that are either an atomic, universal or existential formula.

$\text{FV}(\Gamma) = \bigcup \{\text{FV}(A) \mid A \in \Gamma\}$

Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
(\rightarrow -Introduction)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
(\rightarrow -Elimination)	$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
(\forall -Introduction)	$\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin FV(\Gamma)$
(\forall -Elimination)	$\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$	

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta \dots$

If we interpret *false* with the logical constant false (\perp) (denoted by \vdash_f) we can add a new deduction rule.

(\exists -Introduction)	$\frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, \forall \alpha (A \rightarrow false) \rightarrow false \vdash_f B}$	$a \notin FV(\Gamma, A, B)$
----------------------------	---	-----------------------------

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall \alpha (A \rightarrow false) \rightarrow false$ with $false^I = \perp$.

$$\begin{aligned}
I \models \Gamma, \forall \alpha (A \rightarrow false) \rightarrow false &\Rightarrow I \models \forall \alpha (A \rightarrow false) \rightarrow false \\
&\Rightarrow (\forall \alpha (A \rightarrow false))^I \rightarrow false^I \\
&\Rightarrow (\forall \alpha (A \rightarrow false))^I \rightarrow \perp \\
&\Rightarrow \neg(\forall \alpha (A \rightarrow false))^I \\
&\Rightarrow \neg(\forall d \in \Delta : (A \rightarrow false)^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow false^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I[a \mapsto d]$ is a model of $\Gamma, A[\alpha := a]$. Which implies $I[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B . \square

3.2 Provability in System P is undecidable

$\Gamma_C :$

- $Q(a)$
 - $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \dots, m\}$
 - $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \dots, n\}$
 - $D(a), D(a_i), D(b_j)$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$
 - $E(a_m), E(b_n)$
- $+(Q, 1, Q') :$
- $\forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$
change of state
 - $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))$
increment register 1
 - $\forall \alpha \beta \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\beta, \delta) \rightarrow D(\delta))$
prevent zero
 - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$
do not change register 2
- $-(Q, 1, Q_1, Q_2) :$
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$
jump on zero
 - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow R_1(\beta, \gamma))$
register 1 stays zero
 - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$
change state if register 1 is greater zero
 - $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow R_1(\beta, \delta))$
decrement register 1
 - $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$
do not change register 2

Lemma 5.

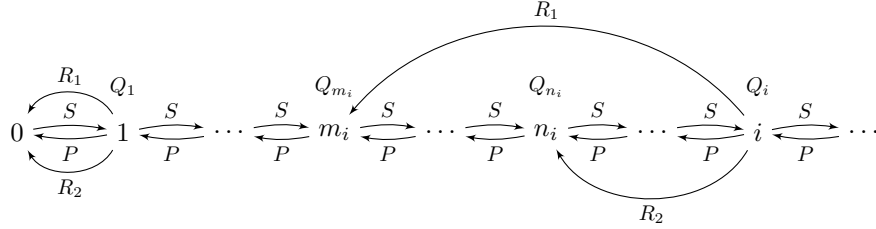
M terminates on input $(0, 0)$ iff $\Gamma_M \vdash \text{false}$ holds in system P .

Claim 6.

$\Gamma_M \vdash \text{false}$ holds in system P \implies M terminates on input $(0, 0)$

Proof. Assume M does not terminate then there is an infinite chain $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \dots$ ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of Γ_M which interprets *false* with \perp this contradicts $\Gamma_M \vdash \text{false}$.

To illustrate the idea we will use a graphical notation for an interpretation I . By $d_1 \xrightarrow{R} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{P}{d}$ to say that $d \in P^I$ for unary predicate symbols. Now the idea for our model of Γ_M looks like this:



We have $0 \in E^I$ and all other numbers are in D^I .

Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I)$.

$$\begin{aligned} P^I &= \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I &= \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I &= \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q^I &= \{i \in \mathbb{N} \setminus \{0\} \mid Q = Q_i\} & D^I &= \mathbb{N} \setminus \{0\} & E^I &= \{0\} \\ S^I &= \{(i, i+1) \mid i \in \mathbb{N}\} \end{aligned}$$

$$a^I = 1 \qquad a_0^I = 0 \qquad b_0^I = 0$$

□

Claim 7. *If a final state is reachable from C then $\Gamma_C \cup \Gamma \vdash \text{false}$.*

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by f .

Induction Base trivial ...

Induction Step

$$C \Rightarrow_M^r D$$

We need to make a case distinction on the rule r .

Case $r = +(Q, 1, Q')$

Basic idea:

$$\frac{\frac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \quad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models \text{false}$ holds trivially if I interprets *false* with \top we only need to consider models (note that there are none if M terminates which is exactly what we want to proof) of $\Gamma_C \cup \Gamma$ that interpret *false* with \perp (so we can use

our new deduction rule).

We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.

We first introduce the new variables needed for Γ_D (let $b, d \in \mathcal{V}_P \setminus \text{FV}(\Gamma_C \cup \Gamma)$).

Intuitively b will represent the successor state and d will be the anchor for register one.

$$\begin{array}{c}
\vdots \\
\frac{S(a, b), D(b) \vdash_f f}{S(a, b) \vdash_f D(b) \rightarrow f} \quad \frac{S(a, b) \vdash_f \forall \alpha \beta (S(\alpha, \beta) \rightarrow D(\beta))}{S(a, b) \vdash_f S(a, b) \rightarrow D(b)} \quad \frac{S(a, b) \vdash_f S(a, b)}{S(a, b) \vdash_f D(b)} \\
\hline
\frac{S(a, b) \vdash_f f}{\forall \beta (S(a, \beta) \rightarrow f) \rightarrow f \vdash_f f} \quad \frac{\vdash_f \forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f \forall \beta (S(a, \beta) \rightarrow f) \rightarrow f} \\
\hline
\vdash_f f
\end{array}$$

The formula $R_1(b, d)$ can be acquired in a similar way. Again we will just drop $S(a, b)$ and $D(b)$ on the left side for comprehensibility.

$$\begin{array}{c}
\vdots \\
\frac{R_1(b, d) \vdash_f f}{\forall \beta (R_1(b, \beta) \rightarrow f) \rightarrow f \vdash_f f} \quad \frac{\vdash_f \forall \alpha (D(\alpha) \rightarrow \forall \beta (R_1(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f D(b) \rightarrow \forall \beta (R_1(b, \beta) \rightarrow f) \rightarrow f} \quad \vdash_f D(b) \\
\hline
\frac{\vdash_f \forall \beta (R_1(b, \beta) \rightarrow f) \rightarrow f}{\vdash_f f}
\end{array}$$

Now we have all the new free variables we need and we continue by ensuring that these variables fulfill all the formulas in Γ_D .

$$\begin{array}{c}
\vdots \quad \frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a) \\
\frac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\vdash_f S(a, b) \rightarrow Q'(b)}{\vdash_f Q'(b)} \quad \vdash_f S(a, b) \\
\hline
\vdash_f f
\end{array}$$

Starting from $Q'(b) \vdash_f \text{false}$ we can connect d and a_0 .

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b) \quad \vdash_f Q(a)} \\
\vdots \\
\frac{\frac{P(d, a_0) \vdash_f f}{\vdash_f P(d, a_0) \rightarrow f}}{\vdash_f f}
\end{array}$$

For register one we still need $D(d)$.

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\beta, \delta) \rightarrow D(\delta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(b, d) \rightarrow D(d) \quad \vdash_f Q(a)} \\
\vdots \\
\frac{\frac{D(d) \vdash_f f}{\vdash_f D(d) \rightarrow f}}{\vdash_f f}
\end{array}$$

Since register two should not change we only need $R_2(b, b_0)$.

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0) \quad \vdash_f Q(a)} \\
\vdots \\
\frac{\frac{R_2(b, b_0) \vdash_f f}{\vdash_f R_2(b, b_0) \rightarrow f}}{\vdash_f f}
\end{array}$$

Now we have Γ_C (Since $P(a_{i-1}, a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.

Case $r = -(Q, 1, Q_1, Q_2)$ $r1 = 0$

$$\begin{array}{c}
\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b) \quad \vdash_f Q(a)} \\
\vdots \\
\frac{\frac{Q_2(b) \vdash_f f}{\vdash_f Q_2(b) \rightarrow f}}{\vdash_f f}
\end{array}$$

□