

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>System P</b>	<b>2</b>
2.1	Definitions . . . . .	2
2.2	Provability in System P is undecidable . . . . .	3

# 1 Introduction

$FV(\Gamma) = \bigcup \{FV(t) \mid (x : t) \in \Gamma\}$   
 $\lambda 2$  deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
( $\lambda$ -Introduction)	$\frac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x. e : t_1 \rightarrow t_2}$	
( $\lambda$ -Elimination)	$\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
( $\forall$ -Introduction)	$\frac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. t}$	$\alpha \notin FV(\Gamma)$
( $\forall$ -Elimination)	$\frac{\Gamma \vdash e : \forall \alpha. t}{\Gamma \vdash e t' : t[\alpha := t']}$	

## 2 System P

### 2.1 Definitions

$FV(\Gamma) = \bigcup \{FV(A) \mid A \in \Gamma\}$   
Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
( $\rightarrow$ -Introduction)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
( $\rightarrow$ -Elimination)	$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
( $\forall$ -Introduction)	$\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin FV(\Gamma)$
( $\forall$ -Elimination)	$\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$	

An Interpretation  $I$  of a P formula is a tuple  $I = (\Delta, \cdot^I)$  where  $\Delta$  is a set (called domain),  $P^I \subseteq \Delta^k$  and  $\alpha^I \in \Delta \dots$

If we interpret *false* with the logical constant false ( $\perp$ ) (denoted by  $\vdash_f$ ) we can add a new deduction rule.

$$(\exists\text{-Elimination}) \quad \frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false} \vdash_f B} \quad a \notin FV(\Gamma, A, B)$$

*Proof.* Let  $I = (\Delta, \cdot^I)$  be a model of  $\Gamma, \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false}$  with  $\text{false}^I = \perp$ .

$$\begin{aligned} I \models \Gamma, \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false} &\Rightarrow I \models \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false} \\ &\Rightarrow (\forall\alpha(A \rightarrow \text{false}))^I \rightarrow \text{false}^I \\ &\Rightarrow (\forall\alpha(A \rightarrow \text{false}))^I \rightarrow \perp \\ &\Rightarrow \neg(\forall\alpha(A \rightarrow \text{false}))^I \\ &\Rightarrow \neg(\forall a \in \Delta : (A \rightarrow \text{false})^{I[\alpha \mapsto d]}) \\ &\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \text{false}^{I[\alpha \mapsto d]}) \\ &\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\ &\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\ &\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]} \end{aligned}$$

Together with  $a \notin FV(\Gamma, A)$ , it follows that  $I[a \mapsto d]$  is a model of  $\Gamma, A[\alpha := a]$ . Which implies  $I[a \mapsto d] \models B$ . Since  $a$  is not free in  $B$  we conclude that  $I$  is also a model of  $B$ .  $\square$

## 2.2 Provability in System P is undecidable

$\Gamma_C :$

- $Q(a)$
  - $P_1(a, a_0), P(a_{i-1}, a_i)$  for  $i \in \{1, \dots, m\}$
  - $P_2(a, b_0), P(b_{i-1}, b_i)$  for  $i \in \{1, \dots, n\}$
  - $D(a) \dots$
- $+(Q, 1, Q') :$
- $\forall\alpha\beta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$   
change of state
  - $\forall\alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow P_1(\beta, \delta) \rightarrow P(\delta, \gamma))$   
increment register 1
  - $\forall\alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma))$   
prevent zero
  - $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$   
do not change register 2

$-(Q, 1, Q_1, Q_2) :$

- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$   
jump on zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow P_1(\beta, \gamma))$   
register 1 stays zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$   
change state if register 1 is greater zero
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow P_1(\beta, \delta))$   
decrement register 1
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$   
do not change register 2

**Lemma 1.**

$M$  terminates on input  $(0, 0)$     iff     $\Gamma_M \vdash \text{false}$  holds in system  $P$ .

**Claim 2.**  $\Gamma_C \cup \Gamma \vdash \text{false} \dots$

*Proof.* For the tableau proofs we will abbreviate *false* by  $f$ .

Induction Base trivial ...

Induction Step

$C \rightarrow D$

Basic idea:

$$\frac{\frac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \quad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since  $I \models \text{false}$  holds trivially if  $I$  interprets *false* with  $\top$  we only need to consider models (there are none if  $M$  terminates) of  $\Gamma_C \cup \Gamma$  that interpret *false* with  $\perp$  (so we can use our new deduction rule)

We will just drop  $\Gamma_C \cup \Gamma$  and only write new formulas on the left side We first introduce the new variables needed for  $\Gamma_D$  (let  $b, \in V_P \setminus \text{FV}(\Gamma_C \cup \Gamma)$ ):

$$\frac{\frac{\frac{S(a, b), D(b) \vdash_f f}{S(a, b) \vdash_f D(b) \rightarrow f} \quad \frac{\frac{S(a, b) \vdash_f \forall \alpha \beta S(\alpha, \beta) \rightarrow D(\beta)}{S(a, b) \vdash_f S(a, b) \rightarrow D(b)}}{S(a, b) \vdash_f D(b)} \quad \frac{S(a, b) \vdash_f f}{\vdash_f (\forall \beta (S(a, \beta) \rightarrow f) \rightarrow f)} \quad \frac{\vdash_f \forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f \forall \beta (S(a, \beta) \rightarrow f) \rightarrow f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

Now we create  $\Gamma_D$

$$\frac{\frac{Q'(b) \vdash f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\frac{\frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f Q'(b)}$$

Alternative tableau with tikz:

$$\frac{\frac{\frac{Q'(b) \vdash f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f Q'(b)} \quad \vdash_f f$$

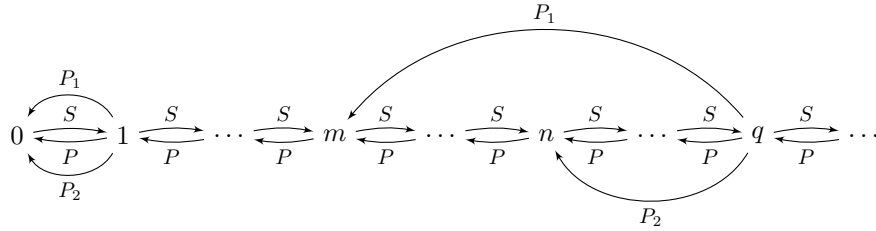
□

### Claim 3.

$$\Gamma_M \vdash \text{false holds in system } P \implies M \text{ terminates on input } (0, 0)$$

*Proof.* Assume  $M$  does not terminate then there is an infinite chain  $C_0 \Rightarrow_M C_1 \Rightarrow_M C_3 \Rightarrow_M \dots$  ( $C_i = \langle Q_i, m_i, n_i \rangle$ ) Now we construct a model of  $\Gamma_M$  which interprets *false* with  $\perp$  this contradicts  $\Gamma_M \vdash \text{false}$ .

The idea looks like this:



Formal definition:

$$I = (\mathbb{N}, \cdot^I)$$

$$\begin{aligned} P^I &= \{(i+1, i) \mid i \in \mathbb{N}\} & P_1^I &= \{(i, m_i) \mid i \in \mathbb{N}\} & P_2^I &= \{(i, n_i) \mid i \in \mathbb{N}\} \\ Q_j^I &= \{i \mid Q_j = Q_i, i \in \mathbb{N}\} & D^I &= \mathbb{N} \setminus \{0\} & E^I &= \{0\} \\ S^I &= \{(i, i+1) \mid i \in \mathbb{N}\} \end{aligned}$$

□