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1 Introduction

$FV(\Gamma) = \bigcup \{FV(t) \mid (x : t) \in \Gamma\}$
 $\lambda 2$ deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ -Introduction)	$\frac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x. e : t_1 \rightarrow t_2}$	
(λ -Elimination)	$\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
(\forall -Introduction)	$\frac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. t}$	$\alpha \notin FV(\Gamma)$
(\forall -Elimination)	$\frac{\Gamma \vdash e : \forall \alpha. t}{\Gamma \vdash e t' : t[\alpha := t']}$	

2 System P

2.1 Definitions

$FV(\Gamma) = \bigcup \{FV(A) \mid A \in \Gamma\}$
Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
(\rightarrow -Introduction)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
(\rightarrow -Elimination)	$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
(\forall -Introduction)	$\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin FV(\Gamma)$
(\forall -Elimination)	$\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$	

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta \dots$

If we interpret *false* with the logical constant false (\perp) (denoted by \vdash_f) we can add a new deduction rule.

$$(\exists\text{-Elimination}) \quad \frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false} \vdash_f B} \quad a \notin FV(\Gamma, A, B)$$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false}$ with $\text{false}^I = \perp$.

$$\begin{aligned} I \models \Gamma, \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false} &\Rightarrow I \models \forall\alpha(A \rightarrow \text{false}) \rightarrow \text{false} \\ &\Rightarrow (\forall\alpha(A \rightarrow \text{false}))^I \rightarrow \text{false}^I \\ &\Rightarrow (\forall\alpha(A \rightarrow \text{false}))^I \rightarrow \perp \\ &\Rightarrow \neg(\forall\alpha(A \rightarrow \text{false}))^I \\ &\Rightarrow \neg(\forall a \in \Delta : (A \rightarrow \text{false})^{I[a \mapsto d]}) \\ &\Rightarrow \exists d \in \Delta : \neg(A^{I[a \mapsto d]} \rightarrow \text{false}^{I[a \mapsto d]}) \\ &\Rightarrow \exists d \in \Delta : \neg(A^{I[a \mapsto d]} \rightarrow \perp) \\ &\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[a \mapsto d]}) \\ &\Rightarrow \exists d \in \Delta : A^{I[a \mapsto d]} \end{aligned}$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I[a \mapsto d]$ is a model of $\Gamma, A[\alpha := a]$. Which implies $I[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B . \square

2.2 Provability in System P is undecidable

$\Gamma_C :$

- $Q(a)$
 - $P_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \dots, m\}$
 - $P_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \dots, n\}$
 - $D(a), D(a_i), D(b_j)$ for $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$
 - $E(a_m), E(b_n)$
- $+(Q, 1, Q') :$
- $\forall\alpha\beta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$
change of state
 - $\forall\alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow P_1(\beta, \delta) \rightarrow P(\delta, \gamma))$
increment register 1
 - $\forall\alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma))$
prevent zero

- $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$
do not change register 2
- $-(Q, 1, Q_1, Q_2) :$
- $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$
jump on zero
- $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow P_1(\beta, \gamma))$
register 1 stays zero
- $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$
change state if register 1 is greater zero
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow P_1(\beta, \delta))$
decrement register 1
- $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_2(\alpha, \gamma) \rightarrow P_2(\beta, \gamma))$
do not change register 2

Lemma 1.

M terminates on input $(0, 0)$ iff $\Gamma_M \vdash \text{false}$ holds in system P .

Claim 2. If a final state is reachable from C then $\Gamma_C \cup \Gamma \vdash \text{false}$.

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by f .

Induction Base trivial ...

Induction Step

$C \rightarrow_M^r D$

We need to make a case distinction on the rule r .

Case $r = +(Q, 1, Q')$

Basic idea:

$$\frac{\frac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \quad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since $I \models \text{false}$ holds trivially if I interprets *false* with \top we only need to consider models (note that there are none if M terminates which is exactly what we want to prove) of $\Gamma_C \cup \Gamma$ that interpret *false* with \perp (so we can use our new deduction rule).

We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.

We first introduce the new variables needed for Γ_D (let $b, d \in V_P \setminus \text{FV}(\Gamma_C \cup \Gamma)$):

$$\frac{\frac{\frac{S(a, b), D(b) \vdash_f f}{S(a, b) \vdash_f D(b) \rightarrow f} \quad \frac{\frac{S(a, b) \vdash_f \forall\alpha\beta S(\alpha, \beta) \rightarrow D(\beta)}{S(a, b) \vdash_f S(a, b) \rightarrow D(b)}}{S(a, b) \vdash_f D(b)} \quad \frac{\vdash_f \forall\alpha(\forall\beta(S(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f \forall\beta(S(a, \beta) \rightarrow f) \rightarrow f}}{\Gamma_C \cup \Gamma \vdash_f f}$$

The formula $P_1(b, d)$ can be acquired in a similar way.
Now we create Γ_D

$$\frac{\frac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\frac{\frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f Q'(b)}$$

Alternative tableau with tikz:

$$\frac{\frac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\frac{\frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f Q'(b)}$$

Starting from $Q'(b) \vdash_f \text{false}$ we can deduce:

$$\frac{\frac{\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow P_1(\alpha, \gamma) \rightarrow P_1(\beta, \delta) \rightarrow P(\delta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow P_1(a, a_0) \rightarrow P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow P_1(a, a_0) \rightarrow P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f S(a, b)}{\vdash_f P_1(a, a_0) \rightarrow P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f P_1(a, a_0)}{\vdash_f P_1(b, d) \rightarrow Q'(b)} \quad \vdash_f P_1(b, d)}{\vdash_f P(d, a_0) \rightarrow f} \quad \vdash_f P(d, a_0)$$

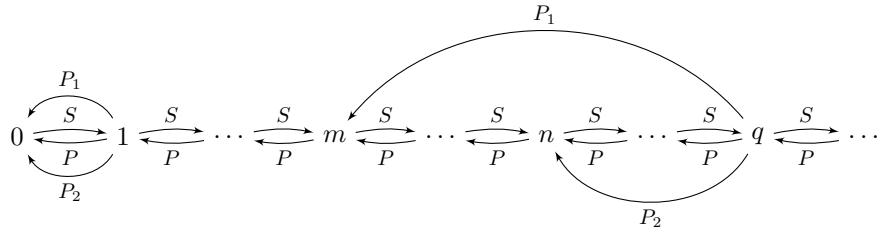
$P_2(b, b_0)$ can be deduced in the same way.
Now we have Γ_C (Since $P(a_{i-1}, a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.
Case $r = -(Q, 1, Q_1, Q_2)$ □

Claim 3.

$$\Gamma_M \vdash \text{false holds in system } P \implies M \text{ terminates on input } (0, 0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_0 \Rightarrow_M C_1 \Rightarrow_M C_3 \Rightarrow_M \dots$ ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of Γ_M which interprets *false* with \perp this contradicts $\Gamma_M \vdash \text{false}$.

The idea looks like this:



Formal definition:
 $I = (\mathbb{N}, \cdot^I)$

$$\begin{aligned}
 P^I &= \{(i+1, i) \mid i \in \mathbb{N}\} & P_1^I &= \{(i, m_i) \mid i \in \mathbb{N}\} & P_2^I &= \{(i, n_i) \mid i \in \mathbb{N}\} \\
 Q_j^I &= \{i \mid Q_j = Q_i, i \in \mathbb{N}\} & D^I &= \mathbb{N} \setminus \{0\} & E^I &= \{0\} \\
 S^I &= \{(i, i+1) \mid i \in \mathbb{N}\}
 \end{aligned}$$

□