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# 1 Introduction

$FV(\Gamma) = \bigcup \{FV(t) \mid (x : t) \in \Gamma\}$   
 $\lambda 2$  deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$	
( $\lambda$ -Introduction)	$\frac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x. e : t_1 \rightarrow t_2}$	
( $\lambda$ -Elimination)	$\frac{\Gamma \vdash e_1 : t_1 \rightarrow t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
( $\forall$ -Introduction)	$\frac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha. e : \forall \alpha. t}$	$\alpha \notin FV(\Gamma)$
( $\forall$ -Elimination)	$\frac{\Gamma \vdash e : \forall \alpha. t}{\Gamma \vdash e t' : t [\alpha := t']}$	

## 2 Basic Definitions

We will denote the set  $\{1, \dots, n\}$  by  $[n]$ .

**Definition 1.** A ranked set is a tuple  $(\Sigma, rk)$ , where  $\Sigma$  is a countable set and  $rk : \Sigma \rightarrow \mathbb{N}$  is a function that maps every symbol from  $\Sigma$  to a natural number (its rank).

If the function  $rk$  is understood we will just write  $\Sigma$  instead of  $(\Sigma, rk)$ . The set of all elements with a certain rank  $k$  in  $\Sigma$ , denoted by  $\Sigma^{(k)}$ , is defined by  $\Sigma^{(k)} := rk^{-1}(k)$ . In the following we will write  $\Sigma = \{P^{(0)}, Q^{(3)}\}$  to say that  $\Sigma = \{P, Q\}$ ,  $rk(P) = 0$ , and  $rk(Q) = 3$ .

First-order logic

**Definition 2.** Let  $\mathcal{V} = \{x_0, x_1, \dots\}$  be a countable set (of variables),  $\mathcal{F} = \{\}$  a ranked set (of function symbols), and  $\mathcal{P} = \{\}$  a ranked set (of predicate symbols). Then the set of terms over  $(\mathcal{V}, \mathcal{F})$ , denoted by  $\mathcal{T}_{(\mathcal{V}, \mathcal{F})}$ , is the smallest set  $\mathcal{T}$  satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$ , and
- for every  $k \in \mathbb{N}$  if  $f \in \mathcal{F}^{(k)}$  and  $t_1, t_2, \dots, t_k \in \mathcal{T}$  then  $f(t_1, t_2, \dots, t_k) \in \mathcal{T}$ .

The set of first-order formulas over  $(\mathcal{V}, \mathcal{F}, \mathcal{P})$ , denoted by  $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$ , is the smallest set  $\mathcal{L}$  satisfying the following conditions:

- for every  $k \in \mathbb{N}$  if  $P \in \mathcal{P}^{(k)}$  and  $t_1, t_2, \dots, t_k \in \mathcal{T}$  then  $P(t_1, t_2, \dots, t_k) \in \mathcal{L}$ .
- If  $\varphi, \psi \in \mathcal{L}$  then  $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi), \neg \varphi \in \mathcal{L}$ , and
- if  $x \in \mathcal{V}$  and  $\varphi \in \mathcal{L}$  then  $\exists x \varphi, \forall x \varphi \in \mathcal{L}$ .

### 3 System P

#### 3.1 Definitions

Let  $\mathcal{V}_P = \{\alpha, \beta, \dots\}$  be a countably infinite set (of variables) and  $\mathcal{P}_P = \{false^{(0)}, P^{(2)}, Q^{(2)}, \dots\}$  a ranked set (of predicate symbols) such that  $\mathcal{P}_P^{(0)} = \{false\}$  and  $\mathcal{P}_P^{(k)} = \emptyset$  for all  $k \in \mathbb{N} \setminus \{0, 2\}$ . A first-order logic formula  $\varphi$  over  $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$  is an

**atomic formula** if  $\varphi = false$  or  $\varphi = P(\alpha, \beta)$  for some  $P \in \mathcal{P}_P$  and  $\alpha, \beta \in \mathcal{V}_P$ .

**universal formula** if  $\varphi = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_n)$  where  $A_i$  is an atomic formula for  $i \in [n]$ ,  $A_i \neq false$  for  $i \in [n-1]$  and for each  $\alpha \in FV(\varphi) \cap FV(A_n)$  there exists an  $i \in [n-1]$  such that  $\alpha \in FV(A_i)$ .

**existential formula** if there exists  $n \geq 0$ , atomic formulas  $A_i \neq false$  for  $i \in [n]$  such that  $\varphi = \forall \vec{\alpha} (A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_{n-1} \rightarrow \forall \beta (A_n \rightarrow false) \rightarrow false)$ .

The set of formulas of System **P** over  $(\mathcal{V}_P, \mathcal{P}_P)$  is the set of all first-order formulas over  $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$  that are either an atomic, universal or existential formula.

$FV(\Gamma) = \bigcup \{FV(A) \mid A \in \Gamma\}$

Deduction Rules

(Axiom)	$\Gamma, A \vdash A$	
( $\rightarrow$ -Introduction)	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
( $\rightarrow$ -Elimination)	$\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
( $\forall$ -Introduction)	$\frac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin FV(\Gamma)$
( $\forall$ -Elimination)	$\frac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B[\alpha := b]}$	

An Interpretation  $I$  of a P formula is a tuple  $I = (\Delta, \cdot^I)$  where  $\Delta$  is a set (called domain),  $P^I \subseteq \Delta^k$  and  $\alpha^I \in \Delta \dots$

If we interpret *false* with the logical constant false ( $\perp$ ) (denoted by  $\vdash_f$ ) we can add a new deduction rule.

( $\exists$ -Introduction)	$\frac{\Gamma, A[\alpha := a] \vdash_f B}{\Gamma, \forall \alpha (A \rightarrow false) \rightarrow false \vdash_f B}$	$a \notin FV(\Gamma, A, B)$
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*Proof.* Let  $I = (\Delta, \cdot^I)$  be a model of  $\Gamma, \forall\alpha(A \rightarrow false) \rightarrow false$  with  $false^I = \perp$ .

$$\begin{aligned}
I \models \Gamma, \forall\alpha(A \rightarrow false) \rightarrow false &\Rightarrow I \models \forall\alpha(A \rightarrow false) \rightarrow false \\
&\Rightarrow (\forall\alpha(A \rightarrow false))^I \rightarrow false^I \\
&\Rightarrow (\forall\alpha(A \rightarrow false))^I \rightarrow \perp \\
&\Rightarrow \neg(\forall\alpha(A \rightarrow false))^I \\
&\Rightarrow \neg(\forall d \in \Delta : (A \rightarrow false)^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow false^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}$$

Together with  $a \notin FV(\Gamma, A)$ , it follows that  $I[a \mapsto d]$  is a model of  $\Gamma, A[\alpha := a]$ . Which implies  $I[a \mapsto d] \models B$ . Since  $a$  is not free in  $B$  we conclude that  $I$  is also a model of  $B$ .  $\square$

### 3.2 Provability in System P is undecidable

$\Gamma_C :$

- $Q(a)$
  - $R_1(a, a_0), P(a_{i-1}, a_i)$  for  $i \in \{1, \dots, m\}$
  - $R_2(a, b_0), P(b_{i-1}, b_i)$  for  $i \in \{1, \dots, n\}$
  - $D(a), D(a_i), D(b_j)$  for  $i \in \{1, \dots, m\}$  and  $j \in \{1, \dots, n\}$
  - $E(a_m), E(b_n)$
- $+(Q, 1, Q') :$
- $\forall\alpha\beta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))$   
change of state
  - $\forall\alpha\beta\gamma\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))$   
increment register 1
  - $\forall\alpha\beta\delta(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\beta, \delta) \rightarrow D(\delta))$   
prevent zero
  - $\forall\alpha\beta\gamma(Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$   
do not change register 2
- $-(Q, 1, Q_1, Q_2) :$

- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))$   
jump on zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow R_1(\beta, \gamma))$   
register 1 stays zero
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow Q_1(\beta))$   
change state if register 1 is greater zero
- $\forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow D(\gamma) \rightarrow P(\gamma, \delta) \rightarrow R_1(\beta, \delta))$   
decrement register 1
- $\forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))$   
do not change register 2

**Lemma 3.**

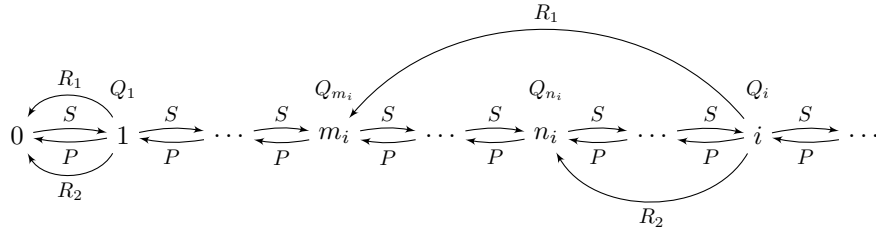
$M$  terminates on input  $(0, 0)$       iff       $\Gamma_M \vdash \text{false}$  holds in system  $P$ .

**Claim 4.**

$\Gamma_M \vdash \text{false}$  holds in system  $P$        $\implies$        $M$  terminates on input  $(0, 0)$

*Proof.* Assume  $M$  does not terminate then there is an infinite chain  $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \dots$  ( $C_i = \langle Q_i, m_i, n_i \rangle$ ) Now we construct a model of  $\Gamma_M$  which interprets *false* with  $\perp$  this contradicts  $\Gamma_M \vdash \text{false}$ .

To illustrate the idea we will use a graphical notation for an interpretation  $I$ . By  $d_1 \xrightarrow{R} d_2$  we say that  $(d_1, d_2) \in R^I$ . And we use  $\overset{P}{d}$  to say that  $d \in P^I$  for unary predicate symbols. Now the idea for our model of  $\Gamma_M$  looks like this:



We have  $0 \in E^I$  and all other numbers are in  $D^I$ .

Here is the more formal definition of our model  $I = (\mathbb{N}, \cdot^I)$ .

$$\begin{aligned}
 P^I &= \{(i+1, i) \mid i \in \mathbb{N}\} & R_1^I &= \{(i, m_i) \mid i \in \mathbb{N}\} & R_2^I &= \{(i, n_i) \mid i \in \mathbb{N}\} \\
 Q^I &= \{i \in \mathbb{N} \setminus \{0\} \mid Q = Q_i\} & D^I &= \mathbb{N} \setminus \{0\} & E^I &= \{0\} \\
 S^I &= \{(i, i+1) \mid i \in \mathbb{N}\}
 \end{aligned}$$

$$a^I = 1$$

$$a_0^I = 0$$

$$b_0^I = 0$$

□

**Claim 5.** *If a final state is reachable from  $C$  then  $\Gamma_C \cup \Gamma \vdash \text{false}$ .*

*Proof.* By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by  $f$ .

Induction Base trivial ...

Induction Step

$C \Rightarrow_M^r D$

We need to make a case distinction on the rule  $r$ .

Case  $r = +(Q, 1, Q')$

Basic idea:

$$\frac{\frac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \quad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}}{\Gamma_C \cup \Gamma \vdash f}$$

Since  $I \models \text{false}$  holds trivially if  $I$  interprets *false* with  $\top$  we only need to consider models (note that there are none if  $M$  terminates which is exactly what we want to prove) of  $\Gamma_C \cup \Gamma$  that interpret *false* with  $\perp$  (so we can use our new deduction rule).

We will just drop  $\Gamma_C \cup \Gamma$  and only write new formulas on the left side.

We first introduce the new variables needed for  $\Gamma_D$  (let  $b, d \in \mathcal{V}_P \setminus \text{FV}(\Gamma_C \cup \Gamma)$ ).

Intuitively  $b$  will represent the successor state and  $d$  will be the anchor for register one.

$$\frac{\frac{\vdots}{S(a, b), D(b) \vdash_f f} \quad \frac{S(a, b) \vdash_f \forall \alpha \beta (S(\alpha, \beta) \rightarrow D(\beta))}{S(a, b) \vdash_f S(a, b) \rightarrow D(b)} \quad S(a, b) \vdash_f S(a, b)}{S(a, b) \vdash_f D(b) \rightarrow f} \quad \frac{S(a, b) \vdash_f D(b)}{S(a, b) \vdash_f f}$$

$$\frac{\frac{\forall \beta (S(a, \beta) \rightarrow f) \rightarrow f \vdash_f f}{\vdash_f (\forall \beta (S(a, \beta) \rightarrow f) \rightarrow f) \rightarrow f} \quad \frac{\vdash_f \forall \alpha (\forall \beta (S(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f \forall \beta (S(a, \beta) \rightarrow f) \rightarrow f}}{\vdash_f f}$$

The formula  $R_1(b, d)$  can be acquired in a similar way. Again we will just drop  $S(a, b)$  and  $D(b)$  on the left side for comprehensibility.

$$\frac{\frac{\vdots}{R_1(b, d) \vdash_f f} \quad \frac{\vdash_f \forall \alpha (D(\alpha) \rightarrow \forall \beta (R_1(\alpha, \beta) \rightarrow f) \rightarrow f)}{\vdash_f D(b) \rightarrow \forall \beta (R_1(b, \beta) \rightarrow f) \rightarrow f} \quad \vdash_f D(b)}{\vdash_f (\forall \beta (R_1(b, \beta) \rightarrow f) \rightarrow f) \rightarrow f} \quad \frac{\vdash_f \forall \beta (R_1(b, \beta) \rightarrow f) \rightarrow f}{\vdash_f f}$$

Now we have all the new free variables we need and we continue by ensuring that these variables fulfill all the formulas in  $\Gamma_D$ .

$$\begin{array}{c}
\vdots \quad \frac{\vdash_f \forall \alpha \beta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow Q'(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow Q'(b)} \quad \vdash_f Q(a) \\
\frac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \rightarrow f} \quad \frac{\vdash_f S(a, b) \rightarrow Q'(b)}{\vdash_f Q'(b)} \quad \vdash_f S(a, b) \\
\hline
\vdash_f f
\end{array}$$

Starting from  $Q'(b) \vdash_f \text{false}$  we can connect  $d$  and  $a_0$ .

$$\begin{array}{c}
\vdots \quad \frac{\vdash_f \forall \alpha \beta \gamma \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow R_1(\beta, \delta) \rightarrow P(\delta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b)} \quad \vdash_f Q(a) \\
\frac{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b)}{\vdash_f R_1(a, a_0) \rightarrow R_1(b, d) \rightarrow Q'(b)} \quad \vdash_f S(a, b) \\
\frac{P(d, a_0) \vdash_f f}{\vdash_f P(d, a_0) \rightarrow f} \quad \frac{\vdash_f R_1(b, d) \rightarrow Q'(b)}{\vdash_f R_1(b, d)} \quad \vdash_f R_1(a, a_0) \\
\hline
\vdash_f f
\end{array}$$

For register one we still need  $D(d)$ .

$$\begin{array}{c}
\vdots \quad \frac{\vdash_f \forall \alpha \beta \delta (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\beta, \delta) \rightarrow D(\delta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(b, d) \rightarrow D(d)} \quad \vdash_f Q(a) \\
\frac{\vdash_f S(a, b) \rightarrow R_1(b, d) \rightarrow D(d)}{\vdash_f R_1(b, d) \rightarrow D(d)} \quad \vdash_f S(a, b) \\
\frac{D(d) \vdash_f f}{\vdash_f D(d) \rightarrow f} \quad \frac{\vdash_f R_1(b, d) \rightarrow D(d)}{\vdash_f D(d)} \quad \vdash_f R_1(b, d) \\
\hline
\vdash_f f
\end{array}$$

Since register two should not change we only need  $R_2(b, b_0)$ .

$$\begin{array}{c}
\vdots \quad \frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_2(\alpha, \gamma) \rightarrow R_2(\beta, \gamma))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0)} \quad \vdash_f Q(a) \\
\frac{\vdash_f S(a, b) \rightarrow R_2(a, b_0) \rightarrow R_2(b, b_0)}{\vdash_f R_2(a, b_0) \rightarrow R_2(b, b_0)} \quad \vdash_f S(a, b) \\
\frac{R_2(b, b_0) \vdash_f f}{\vdash_f R_2(b, b_0) \rightarrow f} \quad \frac{\vdash_f R_2(a, b_0) \rightarrow R_2(b, b_0)}{\vdash_f R_2(b, b_0)} \quad \vdash_f R_2(a, b_0) \\
\hline
\vdash_f f
\end{array}$$

Now we have  $\Gamma_C$  (Since  $P(a_{i-1}, a_i)$  is already in  $\Gamma_D$ ) and can deduce *false* by induction hypothesis.

Case  $r = -(Q, 1, Q_1, Q_2)$   $r_1 = 0$

$$\frac{\frac{\frac{\vdash_f \forall \alpha \beta \gamma (Q(\alpha) \rightarrow S(\alpha, \beta) \rightarrow R_1(\alpha, \gamma) \rightarrow E(\gamma) \rightarrow Q_2(\beta))}{\vdash_f Q(a) \rightarrow S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)} \quad \vdash_f Q(a)}{\vdash_f S(a, b) \rightarrow R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)} \quad \vdash_f S(a, b)$$

$$\vdots$$

$$\frac{\frac{\vdash_f R_1(a, a_0) \rightarrow E(a_0) \rightarrow Q_2(b)}{\vdash_f E(a_0) \rightarrow Q_2(b)} \quad \vdash_f R_1(a, a_0)}{\vdash_f E(a_0) \rightarrow Q_2(b)} \quad \vdash_f E(a_0)$$

$$\frac{Q_2(b) \vdash_f f}{\vdash_f Q_2(b) \rightarrow f} \quad \vdash_f Q_2(b)$$

$$\vdash_f f$$

