Contents

1 Introduction

$FV(\Gamma) = \bigcup \{ FV(t) \mid (x : t) \in \Gamma \}$

$\lambda 2$ deduction Rules

(Axiom)	$\Gamma, x : t \vdash x : t$
(λ-Introduction)	$\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x.e : t_1 \to t_2}$
(λ-Elimination)	$\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$
($\forall$-Introduction)	$\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda \alpha.e : \forall \alpha.t} \qquad \alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash e : \forall \alpha.t}{\Gamma \vdash e t' : t\,[\alpha := t']}$

2 Basic Definitions

We will denote the set $\{1, \ldots, n\}$ by $[n]$.

Definition 1. A <u>ranked set</u> is a tuple (Σ, rk), where Σ is a countable set and $rk : \Sigma \to \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk). The set of all elements with a certain rank k in Σ, denoted by $\Sigma^{(k)}$, is defined by $\Sigma^{(k)} := rk^{-1}(k)$. In the following we will write $\Sigma = \{P^{(0)}, Q^{(3)}\}$ to say that $\Sigma = \{P, Q\}$, $rk(P) = 0$, and $rk(Q) = 3$.

First-order logic

Definition 2. Let $\mathcal{V} = \{x_0, x_1, \ldots\}$ be a countable set (of variables), $\mathcal{F} = \{\}$ a ranked set (of function symbols), and $\mathcal{P} = \{\}$ a ranked set (of predicate symbols). Then the set of <u>terms over $(\mathcal{V}, \mathcal{F})$</u>, denoted by $\mathcal{T}_{(\mathcal{V}, \mathcal{F})}$, is the smallest set $\mathcal{T}$ satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and

- for every $k \in \mathbb{N}$ if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \ldots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \ldots, t_k) \in \mathcal{T}$.

The set of <u>first-order formulas over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$</u>, denoted by $\mathcal{L}_{(\mathcal{V}, \mathcal{F}, \mathcal{P})}$, is the smallest set $\mathcal{L}$ satisfying the following conditions:

- for every $k \in \mathbb{N}$ if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \ldots, t_k \in \mathcal{T}$ then $P(t_1, t_2, \ldots, t_k) \in \mathcal{L}$.

- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \to \psi), \neg\varphi \in \mathcal{L}$, and

- if $x \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists x \varphi, \forall x \varphi \in \mathcal{L}$.

To define the free variables of a formula we first need to define variables of a term.

Definition 3. The <u>variables of a term t</u>, denoted by $\mathrm{V}(t)$, are defined by:

$$\mathrm{V}(t) = \begin{cases} \{x\} & \text{if } t = x \\ \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \cdots \cup \mathrm{V}(t_k) & \text{if } t = f(t_1, t_2, \ldots, t_k) \end{cases}$$

Definition 4. The <u>free variables of a formula φ</u>, denoted by $\mathrm{FV}(\varphi)$, are defined as follows:

$$\mathrm{FV}(\varphi) = \begin{cases} \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \cdots \cup \mathrm{V}(t_k) & \text{if } \varphi = P(t_1, t_2, \ldots, t_k) \\ \mathrm{FV}(\varphi_1) \cup \mathrm{FV}(\varphi_2) & \text{if } \varphi = \varphi_1 \circ \varphi_2, \circ \in \{\wedge, \vee, \to\} \\ \mathrm{FV}(\psi) \setminus \{x\} & \text{if } \varphi = Qx\psi, \, Q \in \{\forall, \exists\} \end{cases}$$

Now we come to the semantics of first-order formulas.

Definition 5. An <u>interpretation I over $(\mathcal{V}, \mathcal{F}, \mathcal{P})$</u> is a triple $(\Delta, \cdot^I, \omega)$ where $\quad \Delta \quad$ is a set (which we call domain),
$\qquad \cdot^I \quad$ is a function such that
$\qquad\qquad f^I : \Delta^k \to \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$ and
$\qquad\qquad P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $f \in \mathcal{P}^{(k)}$
$\qquad \omega \quad$ is a function from $\mathcal{V}$ to Δ.

Definition 6. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and t a term the interpretation of t under I, denoted by t^I, is defined as follows:

$$t^I = \begin{cases} \omega(x) & \text{if } t = x \\ f^I(t_1^I, \ldots, t_k^I) & \text{if } t = f(t_1, \ldots, t_k) \end{cases}$$

Definition 7. An interpretation I is a <u>model of a formula φ</u>, denoted by $I \models \varphi$, if

3   System P

3.1   Definitions

Let $\mathcal{V}_P = \{\alpha, \beta, \ldots\}$ be a countably infinite set (of variables) and $\mathcal{P}_P = \{false^{(0)}, P^{(2)}, Q^{(2)}, \ldots\}$ a ranked set (of predicate symbols) such that $\mathcal{P}_P^{(0)} = \{false\}$, $\mathcal{P}_P^{(2)} = \{P, Q, \ldots\}$ is a countable infinite set, and $\mathcal{P}_P^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ is an

atomic formula if $\varphi = false$ or $\varphi = P(\alpha, \beta)$ for some $P \in \mathcal{P}_P$ and $\alpha, \beta \in \mathcal{V}_P$.

universal formula if $\varphi = \forall\overrightarrow{\alpha}(A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n)$ where A_i is an atomic formula for $i \in [n]$, $A_i \neq$ *false* for $i \in [n-1]$ and for each $\alpha \in \mathrm{FV}(\varphi) \cap \mathrm{FV}(A_n)$ there exists an $i \in [n-1]$ such that $\alpha \in \mathrm{FV}(A_i)$.

existential formula if there exits $n \geq 0$, atomic formulas $A_i \neq$ *false* for $i \in [n]$ such that $\varphi = \forall\overrightarrow{\alpha}(A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_{n-1} \rightarrow \forall\beta(A_n \rightarrow \textit{false}) \rightarrow \textit{false})$.

The set of formulas of System **P** over $(\mathcal{V}_P, \mathcal{P}_P)$ is the set of all first-order formulas over $(\mathcal{V}_P, \emptyset, \mathcal{P}_P)$ that are either an atomic, universal or existential formula.
$\mathrm{FV}(\Gamma) = \bigcup\{\mathrm{FV}(A) \mid A \in \Gamma\}$
Deduction Rules

(Axiom)	$\Gamma, A \vdash A$
($\rightarrow$-Introduction)	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$
($\rightarrow$-Elimination)	$\dfrac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$
($\forall$-Introduction)	$\dfrac{\Gamma \vdash B}{\Gamma \vdash \forall\alpha B}$ $\qquad \alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash \forall\alpha B}{\Gamma \vdash B\,[\alpha := b]}$

An Interpretation I of a P formula is a tuple $I = (\Delta, \cdot^I)$ where Δ is a set (called domain), $P^I \subseteq \Delta^k$ and $\alpha^I \in \Delta$....
If we interpret *false* with the logical constant false ($\perp$) (denoted by $\vdash_f$) we can add a new deduction rule.

($\exists$-Introduction)	$\dfrac{\Gamma, A\,[\alpha := a] \vdash_f B}{\Gamma, \forall\alpha(A \rightarrow \textit{false}) \rightarrow \textit{false} \vdash_f B}$	$a \notin FV(\Gamma, A, B)$

Proof. Let $I = (\Delta, \cdot^I)$ be a model of $\Gamma, \forall \alpha(A \to false) \to false$ with $false^I = \bot$.

$$
\begin{aligned}
I \models \Gamma, \forall \alpha(A \to false) \to false &\Rightarrow I \models \forall \alpha(A \to false) \to false \\
&\Rightarrow (\forall \alpha(A \to false))^I \to false^I \\
&\Rightarrow (\forall \alpha(A \to false))^I \to \bot \\
&\Rightarrow \neg(\forall \alpha(A \to false))^I \\
&\Rightarrow \neg(\forall d \in \Delta : (A \to false)^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \to false^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : \neg(A^{I[\alpha \mapsto d]} \to \bot) \\
&\Rightarrow \exists d \in \Delta : \neg(\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta : A^{I[\alpha \mapsto d]}
\end{aligned}
$$

Together with $a \notin FV(\Gamma, A)$, it follows that $I[a \mapsto d]$ is a model of $\Gamma, A[\alpha := a]$. Which implies $I[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. $\qquad \square$

3.2 Provability in System P is undecidable

$\Gamma_C :$

- $Q(a)$

- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$

- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$

- $D(a), D(a_i), D(b_j)$ for $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$

- $E(a_m), E(b_n)$

$+(Q, 1, Q') :$

- $\forall \alpha \beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state

- $\forall \alpha \beta \gamma \delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to R_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1

- $\forall \alpha \beta \delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\beta, \delta) \to D(\delta))$
  prevent zero

- $\forall \alpha \beta \gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2

$-(Q, 1, Q_1, Q_2) :$

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- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to Q_2(\beta))$
  jump on zero

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to R_1(\beta,\gamma))$
  register 1 stays zero

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to D(\gamma) \to Q_1(\beta))$
  change state if register 1 is greater zero

- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to D(\gamma) \to P(\gamma,\delta) \to R_1(\beta,\delta))$
  decrement register 1

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_2(\alpha,\gamma) \to R_2(\beta,\gamma))$
 do not change register 2

Lemma 8.

M *terminates on input* $(0,0)$ $\quad$ *iff* $\quad$ $\Gamma_M \vdash$ false *holds in system P.*

Claim 9.

$\Gamma_M \vdash$ false *holds in system P* $\quad\implies\quad$ M *terminates on input* $(0,0)$

Proof. Assume M does not terminate then there is an infinite chain $C_1 \Rightarrow_M C_2 \Rightarrow_M C_3 \Rightarrow_M \ldots$ ($C_i = \langle Q_i, m_i, n_i \rangle$) Now we construct a model of Γ_M which interprets *false* with $\bot$ this contradicts $\Gamma_M \vdash$ *false.*

To illustrate the idea we will use a graphical notation for an interpretation I. By $d_1 \xrightarrow{R} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{P}{\underset{d}{}}$ to say that $d \in P^I$ for unary predicate symbols. Now the idea for our model of Γ_M looks like this:
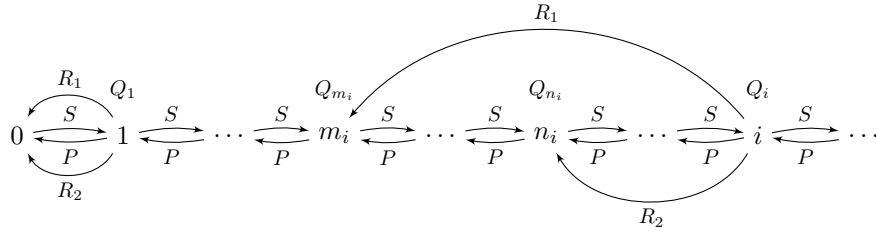
We have $0 \in E^I$ and all other numbers are in D^I.
Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I)$.

$$P^I = \{(i+1, i) \mid i \in \mathbb{N}\} \qquad R_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} \quad R_2^I = \{(i, n_i) \mid i \in \mathbb{N}\}$$
$$Q^I = \{i \in \mathbb{N} \setminus \{0\} \mid Q = Q_i\} \quad D^I = \mathbb{N} \setminus \{0\} \qquad E^I = \{0\}$$
$$S^I = \{(i, i+1) \mid i \in \mathbb{N}\}$$

$$a^I = 1 \qquad\qquad a_0^I = 0 \qquad\qquad b_0^I = 0$$

$\square$

Claim 10. *If a final state is reachable from C then $\Gamma_C \cup \Gamma \vdash$ false.*

Proof. By induction on the length of the computation. For the tableau proofs we will abbreviate *false* by f.
Induction Base trivial ...
Induction Step
$C \Rightarrow^r_M D$
We need to make a case distinction on the rule r.
Case $r = +(Q, 1, Q')$
Basic idea:

$$
\cfrac{
\cfrac{IH}{\Gamma_C \cup \Gamma \cup \Gamma_D \vdash f} \quad \overline{\Gamma_C \cup \Gamma \vdash \Gamma_D}
}{\Gamma_C \cup \Gamma \vdash f}
$$

Since $I \models$ *false* holds trivially if I interprets *false* with $\top$ we only need to consider models (note that there are none if M terminates which is exactly what we want to proof) of $\Gamma_C \cup \Gamma$ that interpret *false* with $\bot$ (so we can use our new deduction rule).
We will just drop $\Gamma_C \cup \Gamma$ and only write new formulas on the left side.
We first introduce the new variables needed for Γ_D (let $b, d \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C \cup \Gamma)$). Intuitively b will represent the successor state and d will be the anchor for register one.

$$
\cfrac{
\cfrac{
\cfrac{
\cfrac{\vdots}{S(a,b), D(b) \vdash_f f}
}{S(a,b) \vdash_f D(b) \to f}
\quad
\cfrac{
S(a,b) \vdash_f \forall\alpha\beta(S(\alpha,\beta) \to D(\beta))
\cfrac{}{S(a,b) \vdash_f S(a,b) \to D(b) \quad S(a,b) \vdash_f S(a,b)}
}{S(a,b) \vdash_f D(b)}
}{S(a,b) \vdash_f f}
}{
\cfrac{
\cfrac{\forall\beta(S(a,\beta) \to f) \to f \vdash_f f}{\vdash_f (\forall\beta(S(a,\beta) \to f) \to f) \to f}
\quad
\cfrac{\vdash_f \forall\alpha(\forall\beta(S(\alpha,\beta) \to f) \to f)}{\vdash_f \forall\beta(S(a,\beta) \to f) \to f}
}{\vdash_f f}
}
$$

The formula $R_1(b, d)$ can be acquired in a similar way. Again we will just drop $S(a, b)$ and $D(b)$ on the left side for comprehensibility.

$$
\cfrac{
\cfrac{
\cfrac{\cfrac{\vdots}{R_1(b,d) \vdash_f f}}{\forall\beta(R_1(b,\beta) \to f) \to f \vdash_f f}
}{\vdash_f (\forall\beta(R_1(b,\beta) \to f) \to f) \to f}
\quad
\cfrac{
\cfrac{
\cfrac{\vdash_f \forall\alpha(D(\alpha) \to \forall\beta(R_1(\alpha,\beta) \to f) \to f)}{\vdash_f D(b) \to \forall\beta(R_1(b,\beta) \to f) \to f}
\quad \vdash_f D(b)
}{\vdash_f \forall\beta(R_1(b,\beta) \to f) \to f}
}{}
}{\vdash_f f}
$$

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Now we have all the new free variables we need and we continue by ensuring that these variables fulfill all the formulas in Γ_D.

$$
\cfrac{
  \cfrac{\vdots}{
    \cfrac{Q'(b) \vdash_f f}{\vdash_f Q'(b) \to f}
  }
  \quad
  \cfrac{
    \cfrac{
      \cfrac{\vdash_f \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_f Q(a) \to S(a,b) \to Q'(b)} \quad \vdash_f Q(a)
    }{\vdash_f S(a,b) \to Q'(b)} \quad \vdash_f S(a,b)
  }{\vdash_f Q'(b)}
}{\vdash_f f}
$$

Starting from $Q'(b) \vdash_f$ *false* we can connect d and a_0.

$$
\cfrac{
  \cfrac{\vdots}{
    \cfrac{P(d,a_0) \vdash_f f}{\vdash_f P(d,a_0) \to f}
  }
  \quad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{\vdash_f \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to R_1(\beta,\delta) \to P(\delta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f Q(a)
        }{\vdash_f S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to Q'(b)} \quad \vdash_f S(a,b)
      }{\vdash_f R_1(a,a_0) \to R_1(b,d) \to Q'(b) \quad \vdash_f R_1(a,a_0)}
    }{\vdash_f R_1(b,d) \to Q'(b) \quad \vdash_f R_1(b,d)}
  }{\vdash_f P(d,a_0)}
}{\vdash_f f}
$$

For register one we still need $D(d)$.

$$
\cfrac{
  \cfrac{\vdots}{
    \cfrac{D(d) \vdash_f f}{\vdash_f D(d) \to f}
  }
  \quad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{\vdash_f \forall\alpha\beta\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\beta,\delta) \to D(\delta))}{\vdash_f Q(a) \to S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_f Q(a)
      }{\vdash_f S(a,b) \to R_1(b,d) \to D(d) \quad \vdash_f S(a,b)}
    }{\vdash_f R_1(b,d) \to D(d) \quad \vdash_f R_1(b,d)}
  }{\vdash_f D(d)}
}{\vdash_f f}
$$

Since register two should not change we only need $R_2(b,b_0)$.

$$
\cfrac{
  \cfrac{\vdots}{
    \cfrac{R_2(b,b_0) \vdash_f f}{\vdash_f R_2(b,b_0) \to f}
  }
  \quad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{\vdash_f \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_2(\alpha,\gamma) \to R_2(\beta,\gamma))}{\vdash_f Q(a) \to S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_f Q(a)
      }{\vdash_f S(a,b) \to R_2(a,b_0) \to R_2(b,b_0) \quad \vdash_f S(a,b)}
    }{\vdash_f R_2(a,b_0) \to R_2(b,b_0) \quad \vdash_f R_2(a,b_0)}
  }{\vdash_f R_2(b,b_0)}
}{\vdash_f f}
$$

Now we have Γ_C (Since $P(a_{i-1}, a_i)$ is already in Γ_D) and can deduce *false* by induction hypothesis.

Case $r = -(Q, 1, Q_1, Q_2)$ $r1 = 0$

$$
\cfrac{
  \cfrac{
    \vdots
  }{
    \cfrac{Q_2(b) \vdash_f f}{\vdash_f Q_2(b) \to f}
  }
  \qquad
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{\vdash_f \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to Q_2(\beta))}{\vdash_f Q(a) \to S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b) \quad \vdash_f Q(a)}
        }{\vdash_f S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b) \quad \vdash_f S(a,b)}
      }{\vdash_f R_1(a,a_0) \to E(a_0) \to Q_2(b) \quad \vdash_f R_1(a,a_0)}
    }{\vdash_f E(a_0) \to Q_2(b) \quad \vdash_f E(a_0)}
  }{\vdash_f Q_2(b)}
}{\vdash_f f}
$$

$\square$