Contents

1 Introduction

1.1 Conventions

$\lambda 2$-types: $t, t', t_1, t_2, \dots$
$\lambda 2$-terms: $M, M_1, M_2, \dots, N, N_1, N_2, \dots$
type-variables: $\alpha, a, \beta, b, \dots$
value-variables: $x, x_1, x_2, \dots$
Predicate-symbols: P, Q
P-variables:
P-formulas: A, B

2 Basic Definitions

2.1 λ-calculus $\boldsymbol{\lambda 2}$

In the following let $\mathcal{V}_T = \{\alpha, a, \beta, b, \dots\}$ be a countably infinite set (of type-variables) and $\mathcal{V}_V = \{x, x_1, x_2, \dots\}$ be a countably infinite set (of value-variables).

Definition 1. The set of all $\lambda 2$ types over $\mathcal{V}_T$, denoted by $\mathrm{T}_{\lambda 2}$, is the smallest set T satisfying the following conditions:

- $\mathcal{V}_T \subseteq \mathrm{T}$,
- if $t_1, t_2 \in \mathrm{T}$ then $(t_1 \to t_2) \in \mathrm{T}$, and
- if $t \in \mathrm{T}$ and $\alpha \in \mathcal{V}_T$ then $\forall \alpha . t \in \mathrm{T}$.

The set of all $\lambda 2$ terms over $\mathcal{V}_T$ and $\mathcal{V}_V$, denoted by $\Lambda_{\mathrm{T}_{\lambda 2}}$, is the smallest set Λ_{T} satisfying the following conditions:

- $\mathcal{V}_V \subseteq \Lambda_{\mathrm{T}}$,
- if $e_1, e_2 \in \Lambda_{\mathrm{T}}$ then $e_1 e_2 \in \Lambda_{\mathrm{T}}$,
- if $x \in \mathcal{V}_V$, $t \in \mathrm{T}_{\lambda 2}$, and $e \in \Lambda_{\mathrm{T}}$ then $\lambda x : t.e \in \Lambda_{\mathrm{T}}$,
- if $\alpha \in \mathcal{V}_T$ and $e \in \Lambda_{\mathrm{T}}$ then $\Lambda \alpha . e \in \Lambda_{\mathrm{T}}$, and
- if $e \in \Lambda_{\mathrm{T}}$ and $t \in \mathrm{T}_{\lambda 2}$ then $e\, t \in \Lambda_{\mathrm{T}}$.

If we have a type of the form $(t_1 \to (t_2 \to (\cdots \to (t_{n-1} \to t_n) \cdots)))$ we will often omit the brackets and just write $(t_1 \to t_2 \to \cdots \to t_{n-1} \to t_n)$ or $t_1 \to t_2 \to \cdots \to t_{n-1} \to t_n$ instead.

Definition 2. Let $e \in \Lambda_{\mathrm{T}_{\lambda 2}}$. The free variables of e, denoted by $\mathrm{FV}(e)$, are defined inductively as follows:

$$\mathrm{FV}(e) = \begin{cases} \{x\} & \text{if } e = x \\ \mathrm{FV}(e_1) \cup \mathrm{FV}(e_2) & \text{if } e = e_1 e_2 \\ \mathrm{FV}(e') \setminus \{x\} & \text{if } e = \lambda x : t.e' \\ \mathrm{FV}(e') & \text{if } e = \Lambda\alpha.e' \\ \mathrm{FV}(e') & \text{if } e = e'\,t \end{cases}$$

Definition 3. Let $\mathcal{V} = \{x_1, \ldots, x_n\}$ be a finite subset of $\mathcal{V}_V$ and $t_1, \ldots, t_n \in \Lambda_{\mathrm{T}_{\lambda 2}}$. A **$\boldsymbol{\lambda}$2**-basis $\Gamma = \{(x_1 : t_1), \ldots, (x_n : t_n)\}$ is a mapping from $\mathcal{V}$ to $\mathrm{T}_{\lambda 2}$. If the kind of basis is clear from the context we abbreviate **$\boldsymbol{\lambda}$2**-basis to basis.

The free variables of a basis Γ, denoted by $\mathrm{FV}(\Gamma)$, are $\bigcup\{\mathrm{FV}(t) \mid (x : t) \in \Gamma\}$.

For a basis Γ and another basis Σ, $x \in \mathcal{V}_V \setminus \mathrm{dom}(\Gamma)$, and $t \in \mathrm{T}_{\lambda 2}$ we will abbreviate $\Gamma \cup \{(x : t)\}$ to $\Gamma, x : t$ and $\Gamma \cup \Sigma$ to Γ, Σ.

Definition 4. Let e be in $\Lambda_{\mathrm{T}_{\lambda 2}}$, t in $\mathrm{T}_{\lambda 2}$, and Γ be a basis. A statement $e : t$ is derivable from Γ, denoted by $\Gamma \vdash e : t$, if $e : t$ can be produced using the following rules.

(Axiom)	$\Gamma, x : t \vdash x : t$	
(λ-Introduction)	$\dfrac{\Gamma, x : t_1 \vdash e : t_2}{\Gamma \vdash \lambda x : t_1.e : t_1 \to t_2}$	
(λ-Elimination)	$\dfrac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash e_1 e_2 : t_2}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash e : t}{\Gamma \vdash \Lambda\alpha.e : \forall\alpha.t}$	$\alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash e : \forall\alpha.t}{\Gamma \vdash e\,t' : t\,[\alpha := t']}$	

Definition 5. The inhabitation problem for $\lambda 2$, denoted by **INHAB**, is defined as follows. Given a $\lambda 2$ type t.

Is there a $\lambda 2$ term M such that $\emptyset \vdash M : t$?

But we can rephrase this problem so that it becomes more general: Given a basis Γ and a $\lambda 2$ type t.

Is there a $\lambda 2$ term M such that $\Gamma \vdash M : t$?

Obviously the second version is a special case of the first one. For the other direction consider a basis $\Gamma = \{(x_1 : t_1), \ldots, (x_n : t_n)\}$ and a $\lambda 2$ type t. Clearly, for every term M, $\Gamma \vdash M : t$ holds iff $\emptyset \vdash \lambda x_1 : t_1. \ldots \lambda x_n : t_n.M : t_1 \to \cdots \to t_n \to t$.

2.2 first-order logic

Definition 6. A ranked set is a tuple (Σ, rk), where Σ is a countable set and $rk\colon \Sigma \to \mathbb{N}$ is a function that maps every symbol from Σ to a natural number (its rank).

If the function rk is understood we will just write Σ instead of (Σ, rk). The set of all elements in Σ with a certain rank k, denoted by $\Sigma^{(k)}$, is defined as $\Sigma^{(k)} := rk^{-1}(k)$.

For the remainder of this subsection let $\mathcal{V} = \{y, y_1, y_2, \dots\}$ be a countable set (of variables), $\mathcal{F}$ a ranked set (of function symbols), and $\mathcal{P}$ a ranked set (of predicate symbols).

Definition 7. The set of terms over $\mathcal{V}$ and $\mathcal{F}$, denoted by $\mathcal{T}_{(\mathcal{V},\mathcal{F})}$, is the smallest set $\mathcal{T}$ satisfying the following conditions:

- $\mathcal{V} \subseteq \mathcal{T}$, and
- for every $k \in \mathbb{N}$, if $f \in \mathcal{F}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}$ then $f(t_1, t_2, \dots, t_k) \in \mathcal{T}$.

The set of first-order formulas over $\mathcal{V}$, $\mathcal{F}$, and $\mathcal{P}$, denoted by $\mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$, is the smallest set $\mathcal{L}$ satisfying the following conditions:

- for every $k \in \mathbb{N}$, if $P \in \mathcal{P}^{(k)}$ and $t_1, t_2, \dots, t_k \in \mathcal{T}_{(\mathcal{V},\mathcal{F})}$ then $P(t_1, t_2, \dots, t_k) \in \mathcal{L}$.
- If $\varphi, \psi \in \mathcal{L}$ then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\neg\varphi \in \mathcal{L}$, and
- if $y \in \mathcal{V}$ and $\varphi \in \mathcal{L}$ then $\exists y.\varphi$, $\forall y.\varphi \in \mathcal{L}$.

We introduce an additional binary operation $\rightarrow$ on formulas, where for some φ, $\psi \in \mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$ the formula $(\varphi \rightarrow \psi)$ is defined as $(\neg\varphi \vee \psi)$, if we have a formula of the form $(\varphi_1 \rightarrow (\varphi_2 \rightarrow (\cdots \rightarrow (\varphi_{n-1} \rightarrow \varphi_n)\cdots)))$ we will often omit the brackets and just write $(\varphi_1 \rightarrow \varphi_2 \rightarrow \cdots \rightarrow \varphi_{n-1} \rightarrow \varphi_n)$ or $\varphi_1 \rightarrow \varphi_2 \rightarrow \cdots \rightarrow \varphi_{n-1} \rightarrow \varphi_n$ instead.

For nullary relation symbols P we will abbreviate $P()$ to P. If a formula φ is of the form $Qy.(\psi)$ (where $Q \in \{\exists, \forall\}$, $y \in \mathcal{V}$, and $(\psi) \in \mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$) we often drop the dot and write $Qy(\psi)$ instead. If a formula φ has multiple variables bound by the same quantifier (i.e. $\varphi = Qy_1.Qy_2.\dots Qy_n.\psi$ for $Q \in \{\exists, \forall\}$, some $n \in \mathbb{N}$, $y_1, y_2, \dots, y_n \in \mathcal{V}$, and $\psi \in \mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$) we abbreviate φ to $Qy_1y_2\dots y_n.\psi$ or to $Q\vec{y}.\psi$ where $\vec{y} = (y_1, y_2, \dots, y_n)^\top$.

Definition 8. The variables of a term $t \in \mathcal{T}_{(\mathcal{V},\mathcal{F})}$, denoted by $\mathrm{V}(t)$, are defined by:

$$\mathrm{V}(t) = \begin{cases} \{y\} & \text{if } t = y \\ \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \cdots \cup \mathrm{V}(t_k) & \text{if } t = f(t_1, t_2, \dots, t_k) \end{cases}$$

The free variables of a formula $\varphi \in \mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$, denoted by $\mathrm{FV}(\varphi)$, are defined as follows:

$$\mathrm{FV}(\varphi) = \begin{cases} \mathrm{V}(t_1) \cup \mathrm{V}(t_2) \cup \cdots \cup \mathrm{V}(t_k) & \text{if } \varphi = P(t_1, t_2, \dots, t_k) \\ \mathrm{FV}(\psi) & \text{if } \varphi = \neg\psi \\ \mathrm{FV}(\varphi_1) \cup \mathrm{FV}(\varphi_2) & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \text{ or } \varphi = (\varphi_1 \vee \varphi_2) \\ \mathrm{FV}(\psi) \setminus \{y\} & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \end{cases}$$

Definition 9. Let y be in $\mathcal{V}$ and $t, t' \in \mathcal{T}_{(\mathcal{V},\mathcal{F})}$. The <u>substitution of y by t' in t</u>, denoted by $t\,[y := t']$, is defined as follows:

$$t\left[y := t'\right] = \begin{cases} t' & \text{if } t = y \\ z & \text{if } t = z \text{ and } z \neq y \\ f(t_1\,[y := t']\,, \dots, t_k\,[y := t']) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Now we can lift this definition to formulas, let φ be in $\mathcal{L}_{(\mathcal{V},\mathcal{F},\mathcal{P})}$. The <u>substitution of y by t' in φ</u>, denoted by $\varphi\,[y := t']$, is defined as follows:

$$\varphi\left[y := t'\right] = \begin{cases} P(t_1\,[y := t']\,, \dots, t_k\,[y := t']) & \text{if } \varphi = P(t_1, \dots, t_k) \\ \neg(\psi\,[y := t']) & \text{if } \varphi = \neg\psi \\ \varphi_1\,[y := t'] \circ \varphi_2\,[y := t'] & \text{if } \varphi = (\varphi_1 \circ \varphi_2)\ ,\ \circ \in \{\wedge, \vee\} \\ \varphi & \text{if } \varphi = \forall y.\psi \text{ or } \varphi = \exists y.\psi \\ Qz.(\psi\,[y := t']) & \text{if } \varphi = Qz.\psi,\ Q \in \{\forall, \exists\} \text{ and } z \neq y \end{cases}$$

Now we come to the semantics of first-order formulas.

Definition 10. An <u>interpretation I over $\mathcal{V}$, $\mathcal{F}$, and $\mathcal{P}$</u> is a triple $(\Delta, \cdot^I, \omega)$,

where Δ is a nonempty set (which we call domain),

$\cdot^I$ is a function such that

$f^I \colon \Delta^k \to \Delta$ is a function for every $k \in \mathbb{N}$, $f \in \mathcal{F}^{(k)}$ and

$P^I \subseteq \Delta^k$ is a relation for every $k \in \mathbb{N}$, $P \in \mathcal{P}^{(k)}$

ω is a function from $\mathcal{V}$ to Δ.

Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation, $y \in \mathcal{V}$, and $d \in \Delta$ the interpretation $I\,[y \mapsto d]$ is defined as $(\Delta, \cdot^I, \omega\,[y \mapsto d])$ where

$$(\omega\,[y \mapsto d])(z) = \begin{cases} d & \text{if } z = y \\ \omega(y) & \text{otherwise.} \end{cases}$$

Definition 11. Let $I = (\Delta, \cdot^I, \omega)$ be an interpretation and t a term. The <u>interpretation of t under I</u>, denoted by t^I, is defined as follows:

$$t^I = \begin{cases} \omega(y) & \text{if } t = y \\ f^I(t_1^I, \dots, t_k^I) & \text{if } t = f(t_1, \dots, t_k) \end{cases}$$

Let φ be a formula. The <u>interpretation of φ under I</u>, denoted by φ^I, is defined recursively as follows:

$$\varphi^I = \begin{cases} \top & \text{if } \varphi = P(t_1, \dots, t_k) \text{ and } (t_1^I, \dots, t_k^I) \in P^I \\ \bot & \text{if } \varphi = P(t_1, \dots, t_k) \text{ and } (t_1^I, \dots, t_k^I) \notin P^I \\ \text{not } \psi^I & \text{if } \varphi = \neg\psi \\ \varphi_1^I \text{ and } \varphi_2^I & \text{if } \varphi = (\varphi_1 \wedge \varphi_2) \\ \varphi_1^I \text{ or } \varphi_2^I & \text{if } \varphi = (\varphi_1 \vee \varphi_2) \\ \text{exists } d \in \Delta\ \psi^{I[y \mapsto d]} & \text{if } \varphi = \exists y.\psi \\ \text{forall } d \in \Delta\ \psi^{I[y \mapsto d]} & \text{if } \varphi = \forall y.\psi \end{cases}$$

The interpretation I is a model of φ, denoted by $I \models \varphi$, if $\varphi^I = \top$.

When we define an interpretation I and we have a nullary predicate symbol P we write $P^I = \top$ instead of $P^I = \{()\}$ and $P^I = \bot$ for $P^I = \emptyset$ (this works because $P()^I = \top$ iff $() \in P^I$).

Definition 12. Let Γ be a finite set of first-order formulas.

We say that an interpretation I is a model of Γ, denoted by $I \models \Gamma$, if $I \models \psi$ for every ψ in Γ.

The formula φ is a semantic consequence of Γ, denoted by $\Gamma \vdash \varphi$, if every model of Γ is also a model of φ.

The free variables of Γ, denoted by $\mathrm{FV}(\Gamma)$, are $\bigcup\{\mathrm{FV}(\varphi) \mid \varphi \in \Gamma\}$.

2.3 two-counter automaton

We will use a version of two-counter automaton which only has two types of transitions. First it can increment a register and second it can decrement a register or jump if the register is already zero. Formally:

Definition 13. A deterministic two-counter automaton is a 4-tuple $M = (\mathcal{Q}, Q_0, Q_f, R)$,

where $\mathcal{Q}$ is a finite set (of states),

Q_0 is in $\mathcal{Q}$ (the initial state),

Q_f is in $\mathcal{Q}$ (the final state), and

R is a function from $\mathcal{Q} \setminus \{Q_f\}$ to $\mathcal{R}_{\mathcal{Q}}$,

where
$$\mathcal{R}_{\mathcal{Q}} = \{+(i, Q') \mid i \in \{1,2\}, Q' \in \mathcal{Q}\} \cup \{-(i, Q_1, Q_2) \mid i \in \{1,2\}, Q_1, Q_2 \in \mathcal{Q}\}$$

A configuration C of our automaton is a triple $\langle Q, m, n\rangle$, where $Q \in \mathcal{Q}$ and $m, n \in \mathbb{N}$. Let r be in $R(\mathcal{Q} \setminus \{Q_f\})$, then $\Rightarrow_M^r$ is a binary relation on the configurations of M such that two configurations $\langle Q, m, n\rangle$, $\langle \widehat{Q}, \widehat{m}, \widehat{n}\rangle$ of M are in the in the relation if all of the following conditions hold:

- $Q \neq Q_f$, $r = R(Q)$,
- if $r = +(1, Q')$ for some $Q' \in \mathcal{Q}$ then $\widehat{Q} = Q'$, $\widehat{m} = m + 1$, and $\widehat{n} = n$,
- if $r = +(2, Q')$ for some $Q' \in \mathcal{Q}$ then $\widehat{Q} = Q'$, $\widehat{m} = m$, and $\widehat{n} = n + 1$,
- if $r = -(1, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then
  if $m = 0$ then $\widehat{Q} = Q_2$, $\widehat{m} = 0$, and $\widehat{n} = n$,
  if $m \geq 1$ then $\widehat{Q} = Q_1$, $\widehat{m} = m - 1$, and $\widehat{n} = n$,
- if $r = -(2, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then
  if $n = 0$ then $\widehat{Q} = Q_2$, $\widehat{m} = m$, and $\widehat{n} = 0$,
  if $n \geq 1$ then $\widehat{Q} = Q_1$, $\widehat{m} = m$, and $\widehat{n} = n - 1$.

The transition relation of M, denoted by $\Rightarrow_M$, is defined as $\bigcup_{r \in R(Q \setminus \{Q_f\})} \Rightarrow^r_M$. We denote the transitive reflexive closure of $\Rightarrow_M$ by $\Rightarrow^*_M$

Let m, n be in $\mathbb{N}$, we say that M terminates on input (m, n) if there exist $\widehat{m}, \widehat{n} \in \mathbb{N}$ such that $\langle Q_0, m, n\rangle \Rightarrow^*_M \langle Q_f, \widehat{m}, \widehat{n}\rangle$ (It follows that there exists an $i \in \mathbb{N}$ and configurations $D_1, \ldots, D_i$ of M such that $\langle Q_0, m, n\rangle = D_1 \Rightarrow_M \cdots \Rightarrow_M D_i = \langle Q_f, \widehat{m}, \widehat{n}\rangle$, we call this chain a computation with length i).

Definition 14. The halting problem for two-counter automaton, denoted by **HALT**, is defined as follows. Given a two-counter automaton M.

Does M terminate on input $(0, 0)$?

It is well known that **HALT** is undecidable.

3 System P

3.1 Definitions

In the following let $\mathcal{V}_P = \{\alpha, a, \beta, b, \ldots\}$ be a countably infinite subset of $\mathcal{V}_T$ (of variables). Let $\mathcal{P}_P = \{P, Q, \ldots\}$ be a set (of predicate symbols) and $\mathcal{P}$ a ranked set such that $\mathcal{P}^{(0)} = \{\mathbf{false}\}$, $\mathcal{P}^{(2)} = \mathcal{P}_P$, and $\mathcal{P}^{(k)} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0, 2\}$. A first-order logic formula φ over $\mathcal{V}_P$, $\emptyset$, and $\mathcal{P}$ is an

atomic formula if $\varphi = \mathbf{false}$ or $\varphi = P(a, b)$ for some $P \in \mathcal{P}_P$ and $a, b \in \mathcal{V}_P$.

universal formula if $\varphi = \forall\vec{\alpha}(A_1 \to A_2 \to \cdots \to A_n)$ for some $n \in \mathbb{N}$ and where A_i is an atomic formula for $i \in \{1, \ldots, n\}$, $A_i \neq \mathbf{false}$ for $i \in \{1, \ldots, n-1\}$ and for each $\alpha \in \mathrm{FV}(A_n)$ there exists an $i \in \{1, \ldots, n-1\}$ such that $\alpha \in \mathrm{FV}(A_i)$.

existential formula if there is an $n \in \mathbb{N}^+$, atomic formulas $A_i \neq \mathbf{false}$ for $i \in \{1, \ldots, n\}$ such that $\varphi = \forall\vec{\alpha}(A_1 \to A_2 \to \cdots \to A_{n-1} \to \forall\beta(A_n \to \mathbf{false}) \to \mathbf{false})$.

The set of formulas of System $\mathbf{P}$ (= set of $\mathbf{P}$-formulas) over $\mathcal{V}_P$ and $\mathcal{P}_P$ is the set of all first-order formulas in $\mathcal{L}_{(\mathcal{V}_P, \emptyset, \mathcal{P})}$ that are either an atomic, universal or existential formula.

Definition 15. A finite set of $\mathbf{P}$-formulas Γ is called $\mathbf{P}$-basis, or basis if it is clear from the context whether a $\mathbf{P}$-basis or a $\boldsymbol{\lambda}\mathbf{2}$-basis is meant.

For a $\mathbf{P}$-basis Γ, another $\mathbf{P}$-basis Σ, and a $\mathbf{P}$-formula A we will abbreviate $\Gamma \cup \{A\}$ to Γ, A and $\Gamma \cup \Sigma$ to Γ, Σ (c.f. $\boldsymbol{\lambda}\mathbf{2}$-basis).

Definition 16. Let A be a $\mathbf{P}$-formula, and Γ be a basis. The formula A is a semantic consequence of Γ, denoted by $\Gamma \vdash A$, if A can be produced using the following deduction rules.

(Axiom)	$\Gamma, A \vdash A$	
($\rightarrow$ -Introduction)	$\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \rightarrow B}$	
($\rightarrow$ -Elimination)	$\dfrac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$	
($\forall$-Introduction)	$\dfrac{\Gamma \vdash B}{\Gamma \vdash \forall \alpha B}$	$\alpha \notin \mathrm{FV}(\Gamma)$
($\forall$-Elimination)	$\dfrac{\Gamma \vdash \forall \alpha B}{\Gamma \vdash B\,[\alpha := b]}$	$b \in \mathcal{V}_P$

We define a more general consequence relation in which we demand that **false** is interpreted with $\perp$. In this relation existential formulas will behave like the name suggests. Formally:

Definition 17. Let Γ be a basis. The **P**-formula A is a <u>semantic consequence with falsity</u> of Γ, denoted by $\Gamma \vdash_{\mathrm{f}} A$, if for every interpretation I

$$I \models \Gamma \text{ and } \mathbf{false}^I = \perp \text{ implies } I \models A.$$

This allows us to add the following deduction rule.

($\exists$-Introduction)	$\dfrac{\Gamma, A\,[\alpha := a] \vdash_{\mathrm{f}} B}{\Gamma, A' := \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \vdash_{\mathrm{f}} B}$	$a \notin FV(\Gamma, A', B)$

Proof. Let $I = (\Delta, \cdot^I, \omega)$ be a model of $\Gamma, \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false}$ with $\mathbf{false}^I = \perp$ and $a \in \mathcal{V}_P$ a variable such that $a \notin FV(\Gamma, A', B)$.

$$\begin{aligned}
I \models \Gamma, \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} &\Rightarrow I \models \forall \alpha (A \rightarrow \mathbf{false}) \rightarrow \mathbf{false} \\
&\Rightarrow (\forall \alpha (A \rightarrow \mathbf{false}))^I \rightarrow \mathbf{false}^I \\
&\Rightarrow (\forall \alpha (A \rightarrow \mathbf{false}))^I \rightarrow \perp \\
&\Rightarrow \neg (\forall \alpha (A \rightarrow \mathbf{false}))^I \\
&\Rightarrow \neg (\forall d \in \Delta \colon (A \rightarrow \mathbf{false})^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta \colon \neg (A^{I[\alpha \mapsto d]} \rightarrow \mathbf{false}^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta \colon \neg (A^{I[\alpha \mapsto d]} \rightarrow \perp) \\
&\Rightarrow \exists d \in \Delta \colon \neg (\neg A^{I[\alpha \mapsto d]}) \\
&\Rightarrow \exists d \in \Delta \colon A^{I[\alpha \mapsto d]}
\end{aligned}$$

Together with $a \notin FV(\Gamma, A')$, it follows that $I\,[a \mapsto d]$ is a model of $\Gamma, A\,[\alpha := a]$. Which implies $I\,[a \mapsto d] \models B$. Since a is not free in B we conclude that I is also a model of B. $\square$

Definition 18. The problem to decide whether a given set of **P**-formulas is consistent, denoted by **CONS**, is defined as follows. Given a set of **P**-formulas Γ.

$$\text{Does } \Gamma \vdash \textbf{false} \text{ not hold?}$$

3.2 CONS is undecidable

We will show that **HALT** $\leq$ **CONS** then the undecidability of **CONS** directly follows from the undecidability of **HALT**. For a given two-counter automaton M we will effectively construct a **P**-basis Γ_M such that

$$M \text{ terminates on input } (0,0) \qquad \text{iff} \qquad \Gamma_M \vdash \textbf{false} \text{ holds in System } \mathbf{P}.$$

Let $M = (\mathcal{Q}, Q_0, Q_f, R)$ be a two-counter automaton, w.l.o.g. $S, P, R_1, R_2, E, D \notin \mathcal{Q}$. In the following we will consider **P**-formulas over $\mathcal{V}_P$ and $\mathcal{P}_P$, where $\mathcal{P}_P = \mathcal{Q} \uplus \{S, P, R_1, R_2, E, D\}$. We will abbreviate $P(a, a)$ to $P(a)$, note that this way we can use binary predicate symbols as unary ones.

The intended informal meaning for these new relation symbols is the following:

- The meaning of $Q(a)$ is "a represents a configuration and Q is the state of this configuration".
- For $i \in \{1, 2\}$, $R_i(a, m)$ denotes that "the value of register i in the configuration represented by a is represented by m" (we call m anchor of a for register i).
- With $S(a, b)$ we state that "b is a successor of a".
- The meaning of $P(a, b)$ is "b is a predecessor of a".
- And $E(a)$ marks "a as the end of chain".
- Finally $D(a)$ states that "a is not the end of a chain".

For a configuration $C = \langle Q, m, n \rangle$ of M we define a set of **P**-formulas Γ_C. It contains the following formulas:

- $Q(a)$
- $R_1(a, a_0), P(a_{i-1}, a_i)$ for $i \in \{1, \ldots, m\}$
- $R_2(a, b_0), P(b_{i-1}, b_i)$ for $i \in \{1, \ldots, n\}$
- $D(a_i), D(b_j)$ for $i \in \{0, \ldots, m-1\}$ and $j \in \{0, \ldots, n-1\}$
- $E(a_m), E(b_n)$

Next we need sets of **P**-formulas for all possible transitions. For every $Q \in \mathcal{Q} \setminus \{Q_f\}$ and $r \in \mathcal{R}_\mathcal{Q}$ we define $\Gamma_{Q,r}$. If $r = +(1, Q')$ for some $Q' \in \mathcal{Q}$ then $\Gamma_{Q,+(1,Q')}$ contains the following formulas:

- $\forall\alpha\beta(Q(\alpha) \to S(\alpha, \beta) \to Q'(\beta))$
  change of state
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to R_1(\beta, \delta) \to P(\delta, \gamma))$
  increment register 1
- $\forall\alpha\beta\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\beta, \delta) \to D(\delta))$
  prevent zero in register 1
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change the value register 2

If $r = -(1, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ then $\Gamma_{Q,-(1,Q_1,Q_2)}$ contains the following formulas:

- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to Q_2(\beta))$
  jump to Q_2 if register 1 is zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to E(\gamma) \to R_1(\beta, \gamma))$
  if register 1 is zero it stays zero
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to Q_1(\beta))$
  change state to Q_1 if register 1 is greater zero
- $\forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha, \beta) \to R_1(\alpha, \gamma) \to D(\gamma) \to P(\gamma, \delta) \to R_1(\beta, \delta))$
  decrement register 1 if possible
- $\forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha, \beta) \to R_2(\alpha, \gamma) \to R_2(\beta, \gamma))$
  do not change register 2 in both cases

For $r = +(2, Q')$ for some $Q' \in \mathcal{Q}$ or $r = -(2, Q_1, Q_2)$ for some $Q_1, Q_2 \in \mathcal{Q}$ the sets $\Gamma_{Q,r}$ are defined analogously.

We also need a set Γ_1 to ensure that our representation works correctly. The following formula are in Γ_1:

- $\forall\alpha(\forall\beta(R_1(\alpha, \beta) \to \mathbf{false}) \to \mathbf{false})$
  every element represents a configuration so it has a value for register 1
- $\forall\alpha(\forall\beta(R_2(\alpha, \beta) \to \mathbf{false}) \to \mathbf{false})$
  every element represents a configuration so it has a value for register 2
- $\forall\alpha(\forall\beta(S(\alpha, \beta) \to \mathbf{false}) \to \mathbf{false})$
  every element has a successor

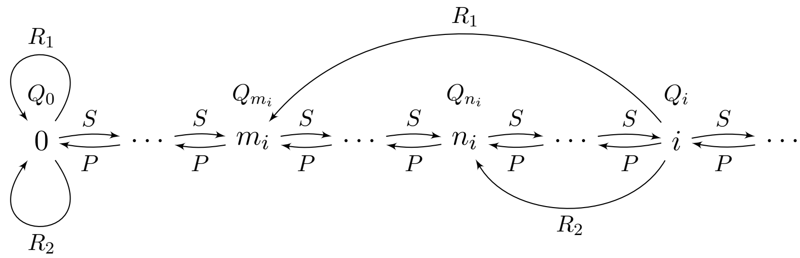
We define $\Gamma_{\overline{M}}$ as $\bigcup_{Q \in \mathcal{Q} \setminus \{Q_f\}} \Gamma_{Q,R(Q)} \cup \{\forall\alpha(Q_f(\alpha) \to \textbf{false})\} \cup \Gamma_1$. We have added the formula $\forall\alpha(Q_f(\alpha) \to \textbf{false})$ to be able to deduce **false** if our automaton terminates. Finally we can define Γ_M as $\Gamma_{C_0} \cup \Gamma_{\overline{M}}$, where $C_0 = \langle Q_0, 0, 0\rangle$ is the initial configuration.

Claim 19.

$$\Gamma_M \vdash \textbf{false} \text{ holds in system P} \qquad \Longrightarrow \qquad M \text{ terminates on input } (0,0)$$

Proof. Assume M does not terminate then there is an infinite chain $C_0 \Rightarrow_M C_1 \Rightarrow_M C_2 \Rightarrow_M \cdots$ ($C_i = \langle Q_i, m_i, n_i\rangle$ for $i \in \mathbb{N}$). Now we construct a model of Γ_M which interprets **false** with $\bot$ this contradicts $\Gamma_M \vdash \textbf{false}$.

To illustrate the idea we will use a graphical notation for an interpretation I. By $d_1 \xrightarrow{\text{R}} d_2$ we say that $(d_1, d_2) \in R^I$. And we use $\overset{\text{P}}{d}$ to say that $(d,d) \in P^I$ for predicate symbols that are used as unary predicate symbols. As domain for our interpretation we will use the natural numbers. Every number will have two tasks: firstly it will represent itself as a possible value for register 1 or 2 and secondly every number i will also represent the i^{th} configuration of our infinite computation. Now the idea for our model of Γ_M looks like this:



We have $0 \in E^I$ and all other numbers are in D^I.
Here is the more formal definition of our model $I = (\mathbb{N}, \cdot^I, \omega)$.

$$P^I = \{(i+1, i) \mid i \in \mathbb{N}\} \qquad R_1^I = \{(i, m_i) \mid i \in \mathbb{N}\} \qquad R_2^I = \{(i, n_i) \mid i \in \mathbb{N}\}$$
$$S^I = \{(i, i+1) \mid i \in \mathbb{N}\} \qquad D^I = \{(i,i) \mid i \in \mathbb{N}^+\} \qquad E^I = \{(0,0)\}$$
$$Q^I = \{(i,i) \mid i \in \mathbb{N}, Q = Q_i\} \text{ for every } Q \in \mathcal{Q} \qquad \textbf{false}^I = \bot$$

$$a^I = 0 \qquad a_0^I = 0 \qquad b_0^I = 0$$

Since there are no free variables in Γ_M we can just set $\omega(x) = 0$ for every $x \in \mathcal{V}_P$. It is easy to see that I is indeed a model of Γ_M. □

We proof the other direction by induction on the length of the computation. But to be able to use the induction hypothesis we need a slightly more general statement (this is why we defined $\Gamma_{\overline{M}}$ and not just Γ_M right away).

Claim 20. Let $C = \langle Q, m, n\rangle$ be a configuration of M. If a final configuration (i.e. a configuration $\langle Q_f, \widehat{m}, \widehat{n}\rangle$ for some $\widehat{m}, \widehat{n} \in \mathbb{N}$) is reachable from C then $\Gamma_C \cup \Gamma_{\overline{M}} \vdash \mathbf{false}$.

Proof. By induction on the length i of the computation.

Induction Base: $i = 0$

Since a final configuration is reachable in 0 steps C must be this final configuration. So $C = \langle Q_f, m, n\rangle$ for some $m, n \in \mathbb{N}$. Hence, $Q_f(a)$ is in Γ_C for some $a \in \mathcal{V}_P$ and $\forall\alpha(Q_f(\alpha) \to \mathbf{false})$ is in $\Gamma_{\overline{M}}$, we can easily deduce false.

$$\dfrac{\dfrac{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \forall\alpha(Q_f(\alpha) \to \mathbf{false})}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a) \to \mathbf{false}} \qquad \Gamma_C \cup \Gamma_{\overline{M}} \vdash Q_f(a)}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash \mathbf{false}}$$

Induction Step: $i = i' + 1$

Since $I \models \mathbf{false}$ holds trivially if I interprets $\mathbf{false}$ with $\top$ we only need to consider models of $\Gamma_C \cup \Gamma_{\overline{M}}$ that interpret $\mathbf{false}$ with $\bot$ (note that there are no such models if M terminates which is exactly what we want to proof). As result of this observation we can use the $\exists$-Introduction rule.

From the fact that a final configuration is reachable from C in i steps we can deduce that there exists a configuration $D = \langle \widehat{Q}, \widehat{m}, \widehat{n}\rangle$ such that $C \Rightarrow^r_M D$ for some $r \in \mathcal{R}_{\mathcal{Q}}$ and a final configuration is reachable from D in i' steps. We also know that $C = \langle Q, m, n\rangle$ for some $Q \in \mathcal{Q} \setminus \{Q_f\}$ and some $m, n \in \mathbb{N}$. The set Γ_C contains the formulas:

$R_1(a, a_0)$, $P(a_{i-1}, a_i)$ and $D(a_{i-1})$ for $i \in \{1, \dots, m\}$,

$R_2(a, b_0)$, $P(b_{i-1}, b_i)$ and $D(b_{i-1})$ for $i \in \{1, \dots, n\}$,

$Q(a)$, $E(a_m)$ and $E(b_n)$.

And Γ_D contains the formulas:

$R_1(\widehat{a}, \widehat{a}_0)$, $P(\widehat{a}_{i-1}, \widehat{a}_i)$ and $D(\widehat{a}_{i-1})$ for $i \in \{1, \dots, \widehat{m}\}$,

$R_2(\widehat{a}, \widehat{b}_0)$, $P(\widehat{b}_{i-1}, \widehat{b}_i)$ and $D(\widehat{b}_{i-1})$ for $i \in \{1, \dots, \widehat{n}\}$,

$\widehat{Q}(\widehat{a})$, $E(\widehat{a}_{\widehat{m}})$ and $E(\widehat{b}_{\widehat{n}})$.

The basic idea is to deduce Γ_D from $\Gamma_C \cup \Gamma_{\overline{M}}$ and then apply the induction hypothesis to $\Gamma_D \cup \Gamma_{\overline{M}}$.

$$\dfrac{\dfrac{\text{Induction Hypothesis}}{\Gamma_C \cup \Gamma_{\overline{M}} \cup \Gamma_D \vdash_{\mathrm{f}} \mathbf{false}} \qquad \Gamma_C \cup \Gamma_{\overline{M}} \vdash_{\mathrm{f}} \Gamma_D}{\Gamma_C \cup \Gamma_{\overline{M}} \vdash_{\mathrm{f}} \mathbf{false}}$$

We achieve this by looking at the four possible cases for the type of the rule r. We will only consider the cases $r = +(1, Q')$ and $r = -(1, Q_1, Q_2)$, because the two remaining cases $r = +(2, Q')$ and $r = -(2, Q_1, Q_2)$ follow by exchanging the roles of register 1 and register 2 in the first two cases.

First we need a new free variable representing the configuration D. Also the value in register 2 does not change, because in both cases we are only concerned with register 1. For the succeeding tableau proofs we will abbreviate **false** by $\mathbf{f}$ and we will drop $\Gamma_C \cup \Gamma_{\overline{M}}$ and only write new formulas on the left side of $\vdash_{\mathrm{f}}$.

We first introduce a new variable representing the new configuration D (let $b \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C)$, note that $\mathrm{FV}(\Gamma_{\overline{M}}) = \emptyset$).

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{S(a,b) \vdash_{\mathrm{f}} \mathbf{f}}}{\forall\beta(S(a,\beta) \to \mathbf{f}) \to \mathbf{f} \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} (\forall\beta(S(a,\beta) \to \mathbf{f}) \to \mathbf{f}) \to \mathbf{f}} \qquad \dfrac{\vdash_{\mathrm{f}} \forall\alpha(\forall\beta(S(\alpha,\beta) \to \mathbf{f}) \to \mathbf{f})}{\vdash_{\mathrm{f}} \forall\beta(S(a,\beta) \to \mathbf{f}) \to \mathbf{f}}}{\vdash_{\mathrm{f}} \mathbf{f}}$$

Since register 2 should not change we need $R_2(b, b_0)$. Again we will just drop $S(a, b)$ on the left side for comprehensibility.

$$\dfrac{\dfrac{\dfrac{\vdots}{R_2(b,b_0) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} R_2(b,b_0) \to \mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_2(\alpha,\gamma) \to R_2(\beta,\gamma))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_2(a,b_0) \to R_2(b,b_0)} \quad \vdash_{\mathrm{f}} R_2(a,b_0)}{\vdash_{\mathrm{f}} R_2(b,b_0)}}{\vdash_{\mathrm{f}} \mathbf{f}}$$

For the case that $\boldsymbol{r = +(1, Q')}$, we have that $\widehat{Q} = Q'$, $\widehat{m} = m + 1$, and $\widehat{n} = n$. So we need to increment register 1 and ensure that the state of b is Q'.

$$\dfrac{\dfrac{\dfrac{\vdots}{Q'(b) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} Q'(b) \to \mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta(Q(\alpha) \to S(\alpha,\beta) \to Q'(\beta))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to Q'(b)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to Q'(b)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} Q'(b)}}{\vdash_{\mathrm{f}} \mathbf{f}}$$

To increment register 1 we need a new free variable as anchor for register 1 (let $d \in \mathcal{V}_P \setminus \mathrm{FV}(\Gamma_C)$ and $d \neq b$).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{R_1(b,d) \vdash_{\mathrm{f}} \mathbf{f}}}{\forall\beta(R_1(b,\beta) \to \mathbf{f}) \to \mathbf{f} \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} (\forall\beta(R_1(b,\beta) \to \mathbf{f}) \to \mathbf{f}) \to \mathbf{f}} \qquad \dfrac{\vdash_{\mathrm{f}} \forall\alpha(\forall\beta(R_1(\alpha,\beta) \to \mathbf{f}) \to \mathbf{f})}{\vdash_{\mathrm{f}} \forall\beta(R_1(b,\beta) \to \mathbf{f}) \to \mathbf{f}}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

Now we need to connect d with a_0 (the anchor of a for register 1).

$$
\dfrac{\dfrac{\dfrac{\vdots}{P(d,a_0) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} P(d,a_0) \to \mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to R_1(\beta,\delta) \to P(\delta,\gamma))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to P(d,a_0)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to R_1(a,a_0) \to R_1(b,d) \to P(d,a_0)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_1(a,a_0) \to R_1(b,d) \to P(d,a_0)} \quad \vdash_{\mathrm{f}} R_1(a,a_0)}{\vdash_{\mathrm{f}} R_1(b,d) \to P(d,a_0)} \quad \vdash_{\mathrm{f}} R_1(b,d)}{\vdash_{\mathrm{f}} P(d,a_0)}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

At last we have to make sure that we do not get an artificial zero. We achieve this by deducing $D(d)$.

$$
\dfrac{\dfrac{\dfrac{\vdots}{D(d) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} D(d) \to \mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\beta,\delta) \to D(\delta))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to R_1(b,d) \to D(d)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_1(b,d) \to D(d)} \quad \vdash_{\mathrm{f}} R_1(b,d)}{\vdash_{\mathrm{f}} D(d)}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

Now we already have deduced Γ_D, to see why define $\widehat{a} := b$, $\widehat{b}_i := b_i$ for $i \in \{0,\dots,n\}$, $\widehat{a}_0 := d$, and $\widehat{a}_{i+1} := a_i$ for $i \in \{0,\dots,m\}$. Hence we can deduce **false** by induction hypothesis.

The other case, that $\boldsymbol{r = -(Q, 1, Q_1, Q_2)}$, has to be split into two cases again. If $\boldsymbol{m = 0}$ then $\widehat{Q} = Q_2$, $\widehat{m} = 0$, and $\widehat{n} = n$. We only need to ensure that the successor state is Q_2 and that register 1 is still zero.

$$
\dfrac{\dfrac{\dfrac{\vdots}{Q_2(b) \vdash_{\mathrm{f}} \mathbf{f}}}{\vdash_{\mathrm{f}} Q_2(b) \to \mathbf{f}} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to Q_2(\beta))}{\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_{\mathrm{f}} Q(a)}{\vdash_{\mathrm{f}} S(a,b) \to R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_{\mathrm{f}} S(a,b)}{\vdash_{\mathrm{f}} R_1(a,a_0) \to E(a_0) \to Q_2(b)} \quad \vdash_{\mathrm{f}} R_1(a,a_0)}{\vdash_{\mathrm{f}} E(a_0) \to Q_2(b)} \quad \vdash_{\mathrm{f}} E(a_0)}{\vdash_{\mathrm{f}} Q_2(b)}}{\vdash_{\mathrm{f}} \mathbf{f}}
$$

Register 1 stays zero.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$R_1(b,a_0) \vdash_{\mathrm{f}} \mathbf{f}$}
\UnaryInfC{$\vdash_{\mathrm{f}} R_1(b,a_0) \to \mathbf{f}$}
\AxiomC{$\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to E(\gamma) \to R_1(\beta,\gamma))$}
\UnaryInfC{$\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_1(a,a_0) \to E(a_0) \to R_1(b,a_0)$}
\AxiomC{$\vdash_{\mathrm{f}} Q(a)$}
\BinaryInfC{$\vdash_{\mathrm{f}} S(a,b) \to R_1(a,a_0) \to E(a_0) \to R_1(b,a_0)$}
\AxiomC{$\vdash_{\mathrm{f}} S(a,b)$}
\BinaryInfC{$\vdash_{\mathrm{f}} R_1(a,a_0) \to E(a_0) \to R_1(b,a_0)$}
\AxiomC{$\vdash_{\mathrm{f}} R_1(a,a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} E(a_0) \to R_1(b,a_0)$}
\AxiomC{$\vdash_{\mathrm{f}} E(a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} R_1(b,a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} \mathbf{f}$}
\end{prooftree}
$$

If we define $\widehat{a} := b$, $\widehat{b}_i := b_i$ for $i \in \{0,\dots,n\}$, and $\widehat{a}_0 := a_0$ then it is clear that we have deduced all formulas required for Γ_D. So we can use the induction hypothesis to deduce **false**.

In the last case $\boldsymbol{m > 0}$, so $\widehat{Q} = Q_1$, $\widehat{m} = m-1$, and $\widehat{n} = n$. First we ensure that b is in state Q_1.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$Q_1(b) \vdash_{\mathrm{f}} \mathbf{f}$}
\UnaryInfC{$\vdash_{\mathrm{f}} Q_1(b) \to \mathbf{f}$}
\AxiomC{$\vdash_{\mathrm{f}} \forall\alpha\beta\gamma(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to D(\gamma) \to Q_1(\beta))$}
\UnaryInfC{$\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_1(a,a_0) \to D(a_0) \to Q_1(b)$}
\AxiomC{$\vdash_{\mathrm{f}} Q(a)$}
\BinaryInfC{$\vdash_{\mathrm{f}} S(a,b) \to R_1(a,a_0) \to D(a_0) \to Q_1(b)$}
\AxiomC{$\vdash_{\mathrm{f}} S(a,b)$}
\BinaryInfC{$\vdash_{\mathrm{f}} R_1(a,a_0) \to D(a_0) \to Q_1(b)$}
\AxiomC{$\vdash_{\mathrm{f}} R_1(a,a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} D(a_0) \to Q_1(b)$}
\AxiomC{$\vdash_{\mathrm{f}} D(a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} Q_1(b)$}
\BinaryInfC{$\vdash_{\mathrm{f}} \mathbf{f}$}
\end{prooftree}
$$

Now we decrement register 1 by taking a_1 (the predecessor of a_0) as anchor of b for register 1.

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\UnaryInfC{$R_1(b,a_1) \vdash_{\mathrm{f}} \mathbf{f}$}
\UnaryInfC{$\vdash_{\mathrm{f}} R_1(b,a_1) \to \mathbf{f}$}
\AxiomC{$\vdash_{\mathrm{f}} \forall\alpha\beta\gamma\delta(Q(\alpha) \to S(\alpha,\beta) \to R_1(\alpha,\gamma) \to D(\gamma) \to P(\gamma,\delta) \to R_1(\beta,\delta))$}
\UnaryInfC{$\vdash_{\mathrm{f}} Q(a) \to S(a,b) \to R_1(a,a_0) \to D(a_0) \to P(a_0,a_1) \to R_1(b,a_1)$}
\AxiomC{$\vdash_{\mathrm{f}} Q(a)$}
\BinaryInfC{$\vdash_{\mathrm{f}} S(a,b) \to R_1(a,a_0) \to D(a_0) \to P(a_0,a_1) \to R_1(b,a_1)$}
\AxiomC{$\vdash_{\mathrm{f}} S(a,b)$}
\BinaryInfC{$\vdash_{\mathrm{f}} R_1(a,a_0) \to D(a_0) \to P(a_0,a_1) \to R_1(b,a_1)$}
\AxiomC{$\vdash_{\mathrm{f}} R_1(a,a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} D(a_0) \to P(a_0,a_1) \to R_1(b,a_1)$}
\AxiomC{$\vdash_{\mathrm{f}} D(a_0)$}
\BinaryInfC{$\vdash_{\mathrm{f}} P(a_0,a_1) \to R_1(b,a_1)$}
\AxiomC{$\vdash_{\mathrm{f}} P(a_0,a_1)$}
\BinaryInfC{$\vdash_{\mathrm{f}} R_1(b,a_1)$}
\BinaryInfC{$\vdash_{\mathrm{f}} \mathbf{f}$}
\end{prooftree}
$$

Again it is obvious that we have deduced Γ_D ($\widehat{a} := b$, $\widehat{b}_i := b_i$ for $i \in \{0,\dots,n\}$, and $\widehat{a}_{i-1} := a_i$ for $i \in \{1,\dots,m\}$). Hence, by induction hypothesis, we can deduce **false**. $\square$

Lemma 21.

$$M \textit{ terminates on input } (0,0) \qquad \textit{iff} \qquad \Gamma_M \vdash \textit{\textbf{false}} \textit{ holds in system } P.$$

Proof. The $\Leftarrow$ directions is proven in Claim 19. And the $\Rightarrow$ direction is a direct consequence of Claim 20 with $C = \langle Q_0, 0, 0\rangle$. $\square$

Theorem 22. ***CONS*** *is undecidable.*

Proof. Since by Lemma 21 for a given two-counter automaton M we can effectively construct a set of **P**-formulas Γ_M such that M terminates on input $(0,0)$ iff Γ_M is not consistent. It follows that **HALT** $\leq$ **CONS**. Since **HALT** is undecidable we have shown that **CONS** is undecidable too. $\square$

4 INHAB is undecidable

Now we can show that the inhabitation problem in $\lambda 2$ is undecidable by reducing **CONS** to **INHAB**. Given a **P**-basis Γ we construct a $\boldsymbol{\lambda 2}$-basis $\overline{\Gamma}$ such that

$$\Gamma \vdash \mathbf{false} \qquad \text{iff} \qquad \overline{\Gamma} \vdash \mathbf{false}$$

where **false**

Definition 23. For a **P**-basis Γ and a $\lambda 2$ type t we define a set $\mathcal{U}(t)$, it contains the ????

$$(x_t : t \to \eta_2)$$

and for every P in $Predicateysymbols\ of\ \Gamma$, $i \in \{1,2\}$ the ???

$$(x_{t,p_i} : (t \to p_i) \to \eta_1).$$

Definition 24. We define a function ...

For a **P**-formula A, if A is an atomic formula then

$$\overline{A} = \begin{cases} \mathbf{false} & \text{if } A = \mathbf{false} \\ (\alpha \to p_1) \to (\beta \to p_2) \to p & \text{if } A = P(\alpha, \beta) \end{cases}$$

if A is an universal formula, it follows that there is an $n \in \mathbb{N}$ and atomic formulas $A_1, A_2, \ldots, A_n$ such that $A = \forall \vec{\alpha}(A_1 \to A_2 \to \cdots \to A_n)$, then

$$\overline{A} = \forall \vec{\alpha}(\mathcal{U}(\vec{\alpha}) \to \overline{A_1} \to \overline{A_2} \to \cdots \to \overline{A_n})$$

if A is an existential formula, it follows that for some $n \in \mathbb{N}^+$ and some atomic formulas $A_1, \ldots, A_n$ it holds that $A = \forall \vec{\alpha}(A_1 \to \cdots \to A_{n-1} \to \forall \beta((A_n) \to \mathbf{false}) \to \mathbf{false})$, then

$$\overline{A} = \forall \vec{\alpha}(\mathcal{U}(\vec{\alpha}) \to \overline{A_1} \to \cdots \to \forall \beta(\mathcal{U}(\beta) \to \overline{A_n} \to \mathbf{false}) \to \mathbf{false})$$

For a **P**-basis Γ we define $\overline{\Gamma}$ as $\{\overline{A} \mid A \in \Gamma\} \cup \{\mathcal{U}(a) \mid a \in \mathrm{FV}(\Gamma)\}$.

Claim 25. Let α, β be type-variables, x a value-variable, and Γ a basis such that β is not a target in Γ. Furthermore assume that $\Gamma, (x : \alpha \to \beta) \vdash e : \beta$ for some term e. It follows that there exists a Γ' such that $\Gamma \subseteq \Gamma'$, $\Gamma, (x : \alpha \to \beta) \vdash t$ for each $t \in \{t \mid (x : t) \in \Gamma'\}$, and $\Gamma', (x : \alpha \to \beta) \vdash e' : \alpha$ for some term e'.

Proof. By structural induction on the term e.

$\underline{e = y}$ for some $y \in \mathcal{V}_V$.
asd

$\underline{e = e_1 e_2}$ for some $e_1, e_2 \in \Lambda_{\mathrm{T}_{\lambda 2}}$.

$$\frac{\Gamma, (x : \beta \to \alpha) \vdash e_1 e_2 : t}{\exists t' : \quad \Gamma, (x : \beta \to \alpha) \vdash e_1 : t' \to t \qquad \Gamma, (x : \beta \to \alpha) \vdash e_2 : t'}$$

$\underline{e = \lambda y : t_1.e'}$ for some $e' \in \Lambda_{\mathrm{T}_{\lambda 2}}$, some $y \in \mathcal{V}_V \setminus \mathrm{dom}(\Gamma)$, $y \neq x$, and $t_1 \in \mathrm{T}_{\lambda 2}$.
It follows that $t = t_1 \to t_2$ for some $t_2 \in \mathrm{T}_{\lambda 2}$.

$$\frac{\Gamma, (x : \beta \to \alpha) \vdash \lambda y : t_1.e' : t_1 \to t_2}{\Gamma, (x : \beta \to \alpha), (y : t_1) \vdash e' : t_2}$$

$\underline{e = \Lambda\gamma.e'}$ for some $e' \in \Lambda_{\mathrm{T}_{\lambda 2}}$, $\gamma \in \mathcal{V}_T \setminus \mathrm{FV}(\Gamma)$,$\gamma \neq \beta$, and $\gamma \neq \alpha$.
It follows that $t = \forall\gamma.t'$ for some $t' \in \mathrm{T}_{\lambda 2}$.

$$\frac{\Gamma, (x : \beta \to \alpha) \vdash \Lambda\gamma.e' : \forall\gamma.t'}{\Gamma, (x : \beta \to \alpha) \vdash e' : t'}$$

$\underline{e = e'\,t'}$ for some $e' \in \Lambda_{\mathrm{T}_{\lambda 2}}$ and some $t' \in \mathrm{T}_{\lambda 2}$.

$$\frac{\Gamma, (x : \beta \to \alpha) \vdash e'\,t' : t}{\exists\gamma \in \mathcal{V}_T, \widehat{t} \in \mathrm{T}_{\lambda 2} : \Gamma, (x : \beta \to \alpha) \vdash e' : \forall\gamma.\widehat{t} \wedge \widehat{t}[\gamma := t'] = t}$$

□

References

[1] H.P. Barendregt, 1993. Lambda Calculi with Types, Handbook of Logic in Computer Science, Volume II, 34-68.