TECHNISCHE UNIVERSITÄT DRESDEN
DRESDEN concept
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UNIFICATION MODULO BOOLEAN RINGS

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Structure

Equational Unification

Boolean Rings

Unification Problem

$$S := \left\{ f(a, x) \overset{?}{\approx} x \right\}$$

Unification Problem

$I := \{f(x,x) \approx x\}$

$$S := \left\{ f(a,x) \stackrel{?}{\approx} x \right\}$$

Unification Problem

$\mathrm{I} := \{\mathrm{f}(\mathrm{x}, \mathrm{x}) \approx \mathrm{x}\}$

$$\mathrm{S} := \left\{\mathrm{f}(\mathrm{a}, \mathrm{x}) \stackrel{?}{\approx} \mathrm{x}\right\} \qquad \Rightarrow \qquad \mathrm{S} := \left\{\mathrm{f}(\mathrm{a}, \mathrm{x}) \stackrel{?}{\approx}_{\mathrm{I}} \mathrm{x}\right\}$$

Unification Classes

Let S be an I-unification problem ...

$$\text{elementary: } \mathrm{S} := \left\{ \mathrm{f}(\mathrm{y}, \mathrm{x}) \overset{?}{\approx}_{\mathrm{I}} \mathrm{x} \right\}$$

$$\text{with constants: } \mathrm{S} := \left\{ \mathrm{f}(\mathrm{a}, \mathrm{x}) \overset{?}{\approx}_{\mathrm{I}} \mathrm{x} \right\}$$

$$\text{general: } \mathrm{S} := \left\{ \mathrm{f}(\mathrm{g}(\mathrm{a}), \mathrm{x}) \overset{?}{\approx}_{\mathrm{I}} \mathrm{x} \right\}$$

Equational Unification
MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$
$$S := \left\{f(x, y) \overset{?}{\approx}_C f(a, b)\right\}$$

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$\sigma_1 := \{x \mapsto a, y \mapsto b\}$ $\qquad$ $\sigma_2 := \{x \mapsto b, y \mapsto a\}$

More General

$$\sigma < \sigma'$$

iff

$$\exists\delta : \delta(\sigma) = \sigma'$$

Equational Unification
More General

$$\begin{array}{ccc} \sigma < \sigma' & & \sigma \lesssim_{E}^{X} \sigma' \\ \text{iff} & \Rightarrow & \text{iff} \\ \exists \delta : \delta(\sigma) = \sigma' & & \exists \delta : \delta(\sigma(x)) \approx_{E} \sigma'(x) \text{ for all } x \in X \end{array}$$

Minimal Complete Sets

Let S be an E-unification problem. A minimal complete set of S is a set of substitutions $\mathcal{M}$ that satisfy the following properties.

- each $\sigma \in \mathcal{M}$ is an E-unifier of S
- for all E-unifiers θ of S there exists a $\sigma \in \mathcal{M}$ such that $\sigma \lesssim_{\mathrm{E}}^{\mathrm{X}} \theta$
- for all $\sigma, \sigma' \in \mathcal{M},\ \sigma \lesssim_{\mathrm{E}}^{\mathrm{X}} \sigma'$ implies $\sigma = \sigma'$.

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unitary iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1.

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Unification Type of $\approx_{\mathrm{E}}$

unitary	iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1.
finitary	iff for all E-unification problems S there exists a minimal complete set with finite cardinality.
infinitary	iff for all E-unification problems S there exists a minimal complete set, and there exists an E-unification problem for which this set is infinite.
zero	iff there exists an E-unification problem that does not have a minimal complete set.

Boolean Rings

$$B := \left\{ \begin{array}{cc} x + y \approx y + x, & x * y \approx y * x, \\ (x + y) + z \approx x + (y + z), & (x * y) * z \approx x * (y * z), \\ x + x \approx 0, & x * x \approx x, \\ 0 + x \approx x, & 0 * x \approx 0, \\ x * (y + z) \approx (x * y) + (x * z), & 1 * x \approx x \end{array} \right\}$$

Interpretation

$\mathcal{B}_2 := (\Delta^{\mathcal{B}_2}, \cdot^{\mathcal{B}_2})$

$$\Delta^{\mathcal{B}_2} := \{\bot, \top\}$$

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$$\begin{aligned}
(\mathrm{x} + \mathrm{y})^{\mathcal{B}_2} &:= \left(\mathrm{x}^{\mathcal{B}_2} \wedge \neg \mathrm{y}^{\mathcal{B}_2}\right) \vee \left(\neg \mathrm{x}^{\mathcal{B}_2} \wedge \mathrm{y}^{\mathcal{B}_2}\right) \\
(\mathrm{x} * \mathrm{y})^{\mathcal{B}_2} &:= \mathrm{x}^{\mathcal{B}_2} \wedge \mathrm{y}^{\mathcal{B}_2} \\
0^{\mathcal{B}_2} &:= \bot \\
1^{\mathcal{B}_2} &:= \top
\end{aligned}$$

Example

$$
\begin{aligned}
(1+0)^{\mathcal{B}_2} &= \left(1^{\mathcal{B}_2} \wedge \neg 0^{\mathcal{B}_2}\right) \vee \left(\neg 1^{\mathcal{B}_2} \wedge 0^{\mathcal{B}_2}\right) \\
&= (\top \wedge \neg\bot) \vee (\neg\top \wedge \bot) \\
&= \top \vee \bot \\
&= \top
\end{aligned}
$$

Polynomial Form

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