1 WFI or not?

```
take 0 xs     = []
take n []     = []
take n (x:xs) = [x]++(take (n-1) xs)

drop 0 xs     = xs
drop n []     = []
drop n (x:xs) = drop (n-1) xs
```

Claim 1.1. The following equation holds.

```
(take n xs)++(drop n xs) = xs
```

Proof. by induction over n.

BC: n=0:

```
  (take n xs)++(drop n xs)
= (take 0 xs)++(drop 0 xs)
= []++xs
= xs
```

IH: `(take n' xs)++(drop n' xs) = xs`

holds for an arbitrary but fixed n' and all xs.

IS: n=n'+1

Case 1: xs=[]

```
  (take n xs)++(drop n xs)
= (take (n'+1) [])++(drop (n'+1) [])
= []++[]
= []
= xs
```

Case 2: xs=(x:xs')

```
  (take n xs)++(drop n xs)
= (take (n'+1) (x:xs'))++(drop (n'+1) (x:xs'))
= [x]++(take n' xs')++(drop n' xs')
= [x]++xs'
= xs
```

□

```
zip [] ys          = []
zip xs []          = []
zip (x:xs) (y:ys)  = [(x,y)]++(zip xs ys)

length []      = 0
length (x:xs)  = 1+(length xs)

min x y
   |x>y    = y
   |true   = x
```

Claim 1.2. The following equation holds.

```
length (zip xs ys) = min (length xs) (length ys)
```

Proof. by structural induction over xs.

BC: xs=[]

```
  length (zip xs ys)
= length (zip [] ys)
= length []
= 0
= min 0 (length ys) --since (length ys) >= 0
= min (length xs) (length ys)
```

IH: `length (zip xs' ys) = min (length xs') (length ys)`

holds for an arbitrary but fixed xs' and all ys.

IS: xs=(x:xs')

Case 1: ys=[]

```
  length (zip xs ys)
= length (zip (x:xs') [])
= length []
= min (length (x:xs')) 0 --since (length (x:xs')) > 0
= min (length xs) (length ys)
```

Case 2: ys=(y:ys')

```
  length (zip xs ys)
= length (zip (x:xs') (y:ys'))
= length ([(x,y)]++(zip xs' ys'))
= 1+length (zip xs' ys')
= 1+(min (length xs') (length ys'))
= min (1+(length xs')) (1+(length ys'))
= min (length (x:xs')) (length (y:ys'))
= min (length xs) (length ys)
```

□

But now just for the fun of it let us proof Claim 1.2 by well founded induction.

$$(xs, ys) \prec ((x : xs), (y : ys))$$

We denote the transitive closure of $\prec$ by $<$. Obviously $<$ is a well-founded order.

Proof. by well-founded induction with the predicate

```
P((xs,ys))=(length (zip xs ys) = min (length xs) (length ys))
```

xs=[]: (Note that ([],ys) has no successors w.r.t. <.)

```
  length (zip xs ys)
= length (zip [] ys)
= length []
= 0
= min 0 (length ys) --since (length ys) >= 0
= min (length xs) (length ys)
```

ys=[]: This case can be treated analogously to xs=[].

xs=(x:xs'),ys=(y:ys)

```
  length (zip xs ys)
= length (zip (x:xs') (y:ys'))
= length ([(x,y)]++(zip xs' ys'))
= 1+(length (zip xs' ys'))
= 1+(min (length xs') (length ys')) {-since (xs',ys')<(xs,ys)
                                          P((xs',ys')) holds-}
= min (1+(length xs')) (1+(length ys'))
= min (length (x:xs')) (length (y:ys'))
= min (length xs) (length ys)
```

□

1.1 Lexicographic Order

```
a 0 m = m+1
a n 0 = a (n-1) 1
a n m = a (n-1) (a n (m-1))
```

```
alist []     ys     = [1]++ys
alist (x:xs) []     = alist xs [x]
alist (x:xs) (y:ys) = alist xs (alist (x:xs) ys)
```

Claim 1.3. The following equation holds.

```
length (alist xs ys) = a (length xs) (length ys)
```

$$\begin{aligned}(xs, ys) &\prec_l (\quad xs\ , (y : ys))\\ (xs, zs) &\prec_l ((x : xs), \quad ys\ )\end{aligned}$$

We denote the transitive closure of $\prec_l$ by $<_l$. Obviously $<_l$ is a well-founded order.

Proof. by well-founded induction with the predicate

```
P((xs,ys))=(length (alist xs ys) = a (length xs) (length ys))
```

xs=ys=[]: (Note that ([],[]) has no successors w.r.t. $<_l$.)

```
  length (alist xs ys)
= length (alist [] [])
= length ([1]++[])
= 1
= a 0 0
= a (length []) (length [])
```

xs=(x:xs')

Case 1: ys=[]

```
  length (alist xs ys)
= length (alist (x:xs') [])
= length (alist xs' [x])
= a (length xs') (length [x])
      {-since (xs',[x])<_l(xs,ys)
              P((xs',[])) holds-}
= a (length xs') 1
= a ((length xs')+1) 0
= a (length (x:xs')) (length [])
= a (length xs) (length ys)
```

Case 2: ys=(y:ys)

```
  length (alist xs ys)
= length (alist (x:xs') (y:ys'))
= length (alist xs' (alist (x:xs') ys'))
= a (length xs') (length (alist (x:xs') ys'))
                {-since (xs',(alist (x:xs') ys'))<_l(xs,ys)
                      P((xs',(alist (x:xs') ys'))) holds-}
= a (length xs') (a (length (x:xs')) (length ys'))
                {-since (xs,ys')<_l(xs,ys)
                       P((xs,ys') holds-}
= a (length xs') (a ((length xs')+1) (length ys'))
= a ((length xs')+1) ((length ys')+1)
= a (length (x:xs')) (length (y:ys'))
= a (length xs) (length ys)
```

∎

1.2 Multiset Order

```
r []        = []
r (x:xs)
     |x<1   = r xs
     |true = r (xs++[(x-1)]++[(x-1)])
```

Claim 1.4. The following equation holds.

```
r xs = []
```

Proof. by well-founded induction with the predicate

```
P(xs)=(r xs = [])
```

xs=[]:

```
  r xs
= r []
= []
```

xs=(x:xs')

Case 1: x<1

```
      r xs
    = r (x:xs')
    = r xs'
    = []
  {-since xs'<_m xs)
      P(xs') holds-}
```

Case 2: xs≥1

```
  r xs
= r (x:xs')
= r (xs'++[(x-1)]++[(x-1)])
= []
    {-since (xs'++[(x-1)]++[(x-1)])<_m xs
         P(xs'++[(x-1)]++[(x-1)]) holds-}
```

□

$$\begin{tikzcd}
\mathbb{N} & \mathbb{N}\times\mathbb{N} & & & \\
\vdots \arrow[d] & \vdots \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
1 \arrow[d] & (1,0) \arrow[d] \arrow[dr] \arrow[drr] \arrow[drrr] & (1,1) \arrow[l] & (1,2) \arrow[l] & \cdots \arrow[l] \\
0 & (0,0) & (0,1) \arrow[l] & (0,2) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

$\mathbb{N} \rightarrow$ finitely branching.
$\mathbb{N} \times \mathbb{N} \rightarrow$ infinitely branching.
Assume there exits a bijective mapping $B : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that

$$(a, b) <_{lex} (c, d) \textbf{ iff } B((a, b)) < B((c, d))$$

holds. Let $B((1, 0)) = k$ and consider the increasing chain $(0, 0) <_{lex} \cdots <_{lex} (0, k) <_{lex} (1, 0)$. This chain has $k + 2$ elements, the pigeon principle implies