Institute of Theoretical Computer Science Chair of Automata Theory

UNIFICATION MODULO BOOLEAN RINGS

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Structure

Equational Unification

Boolean Rings

Unification modulo Boolean Rings

Unification Problem

$$S := \left\{ f(a, x) \overset{?}{\approx} x \right\}$$

Unification Problem

$I := \{f(x, x) \approx x\}$

$$S := \left\{ f(a, x) \overset{?}{\approx} x \right\}$$

Equational Unification

Unification Problem

$I := \{f(x,x) \approx x\}$

$$S := \left\{ f(a,x) \overset{?}{\approx} x \right\} \qquad \Rightarrow \qquad S := \left\{ f(a,x) \overset{?}{\approx}_{I} x \right\}$$

Unification Classes

Let S be an I-unification problem . . .

$$\text{elementary:}\quad S := \left\{ f(y,x) \overset{?}{\approx}_I x \right\}$$

$$\text{with constants:}\quad S := \left\{ f(a,x) \overset{?}{\approx}_I x \right\}$$

$$\text{general:}\quad S := \left\{ f(g(a),x) \overset{?}{\approx}_I x \right\}$$

Equational Unification

MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$

$$S := \left\{ f(x, y) \overset{?}{\approx}_C f(a, b) \right\}$$

Equational Unification
MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$

$$S := \left\{ f(x, y) \stackrel{?}{\approx}_C f(a, b) \right\}$$

$$\sigma_1 := \{x \mapsto a, y \mapsto b\} \qquad \sigma_2 := \{x \mapsto b, y \mapsto a\}$$

More General

$$\sigma < \sigma'$$

iff

$$\exists \delta : \delta(\sigma) = \sigma'$$

More General

$$
\begin{array}{ccc}
\sigma < \sigma' & & \sigma \lesssim^{\mathrm{X}}_{\mathrm{E}} \sigma' \\
\text{iff} & \Rightarrow & \text{iff} \\
\exists \delta : \delta(\sigma) = \sigma' & & \exists \delta : \forall \mathrm{x} \in \mathrm{X} : \\
 & & \delta(\sigma(\mathrm{x})) \approx_{\mathrm{E}} \sigma'(\mathrm{x})
\end{array}
$$

Minimal Complete Sets

Let S be an E-unification problem. A minimal complete set of S is a set of substitutions $\mathcal{M}$ that satisfy the following properties.

- each $\sigma \in \mathcal{M}$ is an E-unifier of S
- for all E-unifiers θ of S there exists a $\sigma \in \mathcal{M}$ such that $\sigma \lesssim_{\mathrm{E}}^{\mathrm{X}} \theta$
- for all $\sigma, \sigma' \in \mathcal{M},\ \sigma \lesssim_{\mathrm{E}}^{\mathrm{X}} \sigma'$ implies $\sigma = \sigma'$.

Unification Type of $\approx_E$

unitary iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1.

Equational Unification

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Unification Type of $\approx_E$

unitary	iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1.
finitary	iff for all E-unification problems S there exists a minimal complete set with finite cardinality.
infinitary	iff for all E-unification problems S there exists a minimal complete set, and there exists an E-unification problem for which this set is infinite.
zero	iff there exists an E-unification problem that does not have a minimal complete set.

Boolean Rings

$$
B := \left\{ \begin{array}{cc}
x + y \approx y + x, & x * y \approx y * x, \\
(x + y) + z \approx x + (y + z), & (x * y) * z \approx x * (y * z), \\
x + x \approx 0, & x * x \approx x, \\
0 + x \approx x, & 0 * x \approx 0, \\
x * (y + z) \approx (x * y) + (x * z), & 1 * x \approx x
\end{array} \right\}
$$

Interpretation

$\mathcal{B}_2 := (\Delta^{\mathcal{B}_2}, \cdot^{\mathcal{B}_2})$

$$\Delta^{\mathcal{B}_2} := \{\bot, \top\}$$

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$$\Delta^{\mathcal{B}_2} := \{\bot, \top\}$$

$$\begin{aligned}
(\mathrm{x} + \mathrm{y})^{\mathcal{B}_2} &:= \left(\mathrm{x}^{\mathcal{B}_2} \wedge \neg \mathrm{y}^{\mathcal{B}_2}\right) \vee \left(\neg \mathrm{x}^{\mathcal{B}_2} \wedge \mathrm{y}^{\mathcal{B}_2}\right) \\
(\mathrm{x} * \mathrm{y})^{\mathcal{B}_2} &:= \mathrm{x}^{\mathcal{B}_2} \wedge \mathrm{y}^{\mathcal{B}_2} \\
0^{\mathcal{B}_2} &:= \bot \\
1^{\mathcal{B}_2} &:= \top
\end{aligned}$$

Example

$$
\begin{aligned}
(1+0)^{\mathcal{B}_2} &= \left(1^{\mathcal{B}_2} \wedge \neg 0^{\mathcal{B}_2}\right) \vee \left(\neg 1^{\mathcal{B}_2} \wedge 0^{\mathcal{B}_2}\right) \\
&= (\top \wedge \neg\bot) \vee (\neg\top \wedge \bot) \\
&= \top \vee \bot \\
&= \top
\end{aligned}
$$

Polynomial Form

asd

Reproductive E-unifier

$$\sigma \text{ is an mgu of } \mathrm{S}$$

$$\text{iff}$$

$$\forall \tau \in \mathcal{U}_{\mathrm{E}}(\mathrm{S}) : \exists \theta : \forall \mathrm{x} \in \mathrm{X} :$$

$$\theta(\sigma(\mathrm{x})) \approx_{\mathrm{E}} \tau(\mathrm{x})$$

Reproductive E-unifier

σ is an mgu of S		σ is a reproductive E-unifier of S
iff	$\Rightarrow$	iff
$\forall\tau \in \mathcal{U}_{\mathrm{E}}(\mathrm{S}) : \exists\theta : \forall \mathrm{x} \in \mathrm{X} :$		$\forall\tau \in \mathcal{U}_{\mathrm{E}}(\mathrm{S}) : \forall \mathrm{x} :$
$\theta(\sigma(\mathrm{x})) \approx_{\mathrm{E}} \tau(\mathrm{x})$		$\tau(\sigma(\mathrm{x})) \approx_{\mathrm{E}} \tau(\mathrm{x})$

Löwenheim's Formula

Let τ be a B-unifier of $\mathrm{t} \overset{?}{\approx}_{\mathrm{B}} 0$. The substitution σ defined by

$$\sigma(\mathrm{x}) := \begin{cases} (\mathrm{t}+1) * \mathrm{x} + \mathrm{t} * \tau(\mathrm{x}) & \text{if } \mathrm{x} \in \mathcal{V}\mathrm{ar}(\mathrm{t}) \\ \mathrm{x} & \text{if } \mathrm{x} \notin \mathcal{V}\mathrm{ar}(\mathrm{t}) \end{cases}$$

is a reproductive B-unifier of $\mathrm{t} \overset{?}{\approx}_{\mathrm{B}} 0$.