TECHNISCHE
UNIVERSITÄT
DRESDEN

SMOOTH DIGRAPHS MODULO PP-CONSTRUCTABILITY

Florian Starke

Definitions

Let $\mathfrak{A}$ be a structure.

A structure $\mathfrak{B}$ is in $\mathrm{H}(\mathfrak{A})$ if there are homomorphisms $f\colon \mathfrak{A} \to \mathfrak{B}$ and $g\colon \mathfrak{B} \to \mathfrak{A}$.

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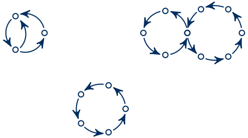

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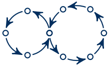

Let $\mathfrak{S}$ be the poset induced by the quasi order $\geq$ on all finite smooth digraphs.

Dividing the graphs

$$[G] \in \mathfrak{S}$$

Dividing the graphs

G can pp-construct every finite structure

G has 4-ary Siggers and its core is a disjoint union of cycles

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Poset first glance

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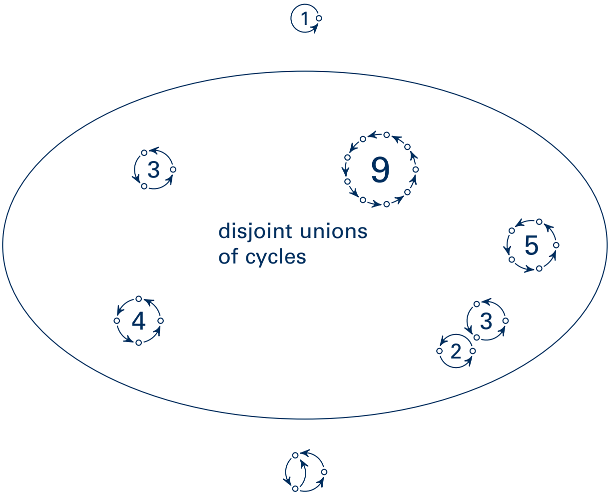

Multiples

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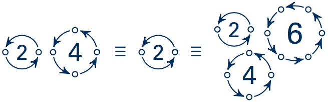

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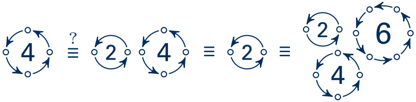

Division

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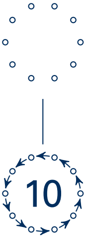
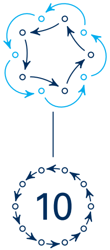


Division

$\Phi_E(x, y) = x \xrightarrow{2} y$



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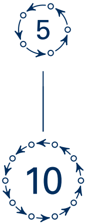

Division

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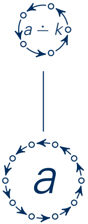
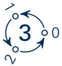


$$a \dot{-} k = \frac{a}{\gcd(a, k)}$$

Multiplication



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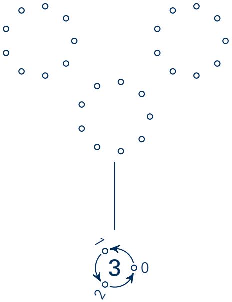

Multiplication

$$\Phi_E\begin{pmatrix} x_1, x_2, x_3, \\ y_1, y_2, y_3 \end{pmatrix} = x_1 \to y_3$$
$$\wedge\ x_2 = y_1$$
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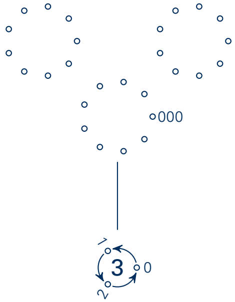

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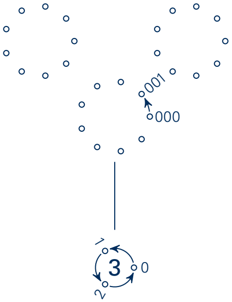

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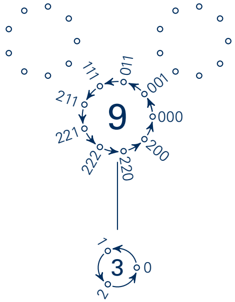

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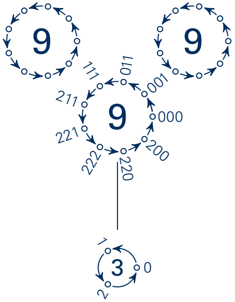

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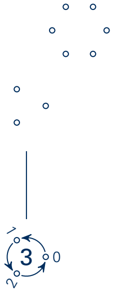

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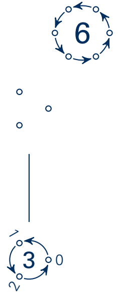

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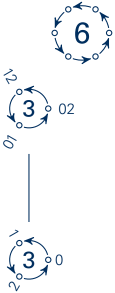

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$$
\begin{aligned}
\Phi_E\begin{pmatrix} x_1,\ldots,x_k, \\ y_1,\ldots,y_k \end{pmatrix} &= x_1 \to y_k \\
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&\qquad\vdots \\
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\end{aligned}
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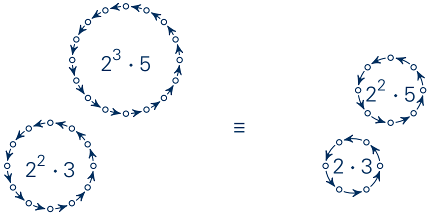
$$a \ltimes k = \prod_{\alpha_i \neq 0} p_i^{\alpha_i + \kappa_i}$$
$$3 \ltimes 3 = 9$$
$$3 \ltimes 2 = 3$$
$$3 \ltimes 6 = 9$$
$$2 \ltimes 12 = 8$$

Normal form



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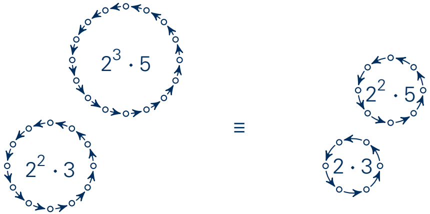

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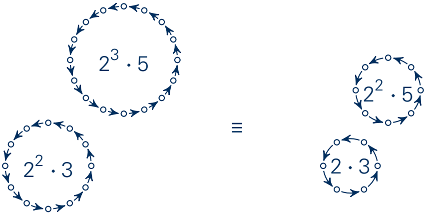

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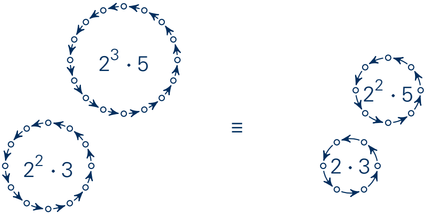
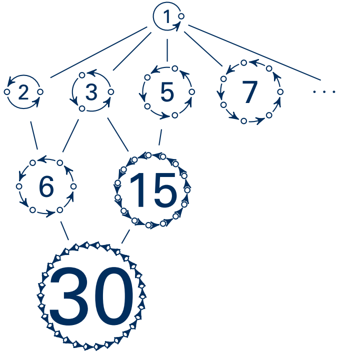


G is in **normal form** if

- for all $a, a' \in G$ we have $a \mid a'$ implies $a = a'$ and
- if for an $a \in G$ we have $p \mid a$, then there is an $a' \in G$ with $p \mid a'$ but $p^2 \nmid a'$.

Poset second glance



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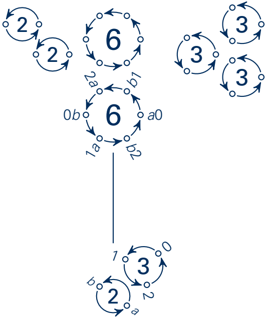

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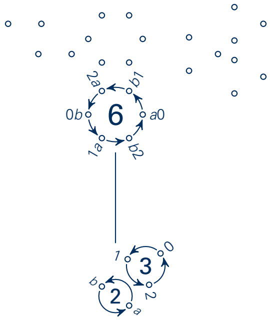

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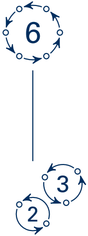

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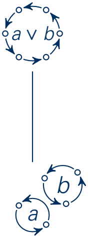
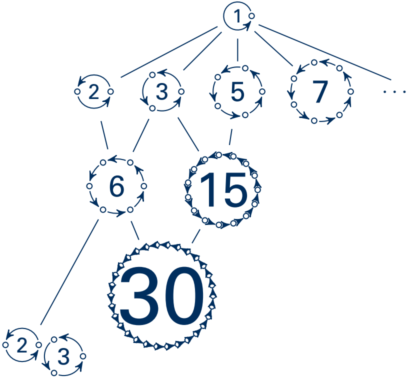
$$\land\ x_1 \xrightarrow{a} x_1$$
$$\land\ x_2 \xrightarrow{b} x_2$$



$a \nmid b, b \nmid a$
$a \vee b = \mathrm{lcm}(a, b)$

Poset final glance



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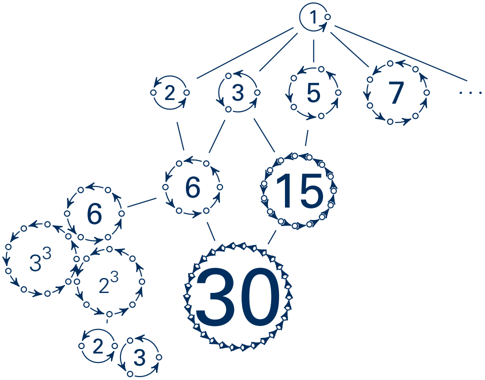

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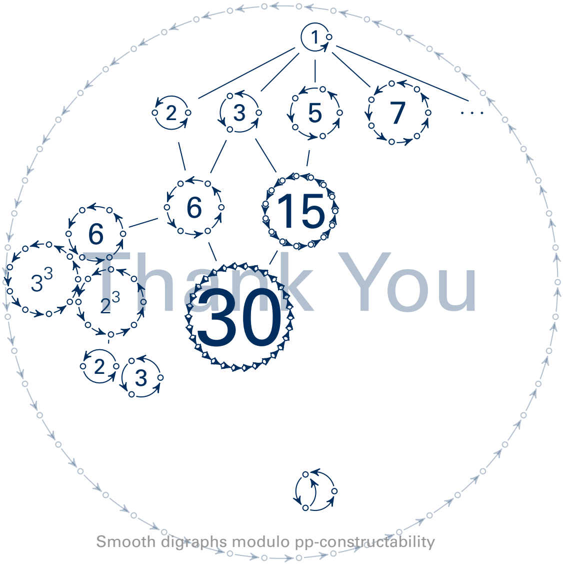

Summary

$$a \vee b = \mathrm{lcm}(a, b) \qquad a \dotminus k = \frac{a}{\gcd(a, k)} \qquad a \ltimes k = \prod_{\alpha_i \neq 0} p_i^{\alpha_i + \kappa_i}$$

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$$G \dot{-} (k_1, \ldots, k_d) = \{(a_1 \dot{-} k1) \vee \cdots \vee (a_d \dot{-} k_d) \mid a_1, \ldots, a_d \in G\}$$
$$G \ltimes (k_1, \ldots, k_d) = \{(a_1 \ltimes k1) \vee \cdots \vee (a_d \ltimes k_d) \mid a_1, \ldots, a_d \in G\}$$

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$$\begin{tikzcd} 2 \cdot 3, 5 \cdot 7 \dot{-} (2 \cdot 5, 3 \cdot 7) \arrow[d, dash] \\ 2 \cdot 3, 5 \cdot 7 \end{tikzcd}$$

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$G\, f\, (t_1, \dots, t_d)$, **where** $f \in \{\dot{-}, \ltimes\}^n$ **and** $t_i \in \{1, 2, \dots\}^n$

$$\begin{array}{ccc}
2 \cdot 3, 5 \cdot 7, 2 \cdot 7, 3 \cdot 5 & & 2^3, 2 \cdot 3, 3^3 \\
\| & & \| \\
2 \cdot 3, 5 \cdot 7 \mathbin{\dot{-}} (2 \cdot 5, 3 \cdot 7) & & 2, 3 \ltimes (4, 9) \\
| & & | \\
2 \cdot 3, 5 \cdot 7 & & 2, 3
\end{array}$$

Summary

$$a \vee b = \operatorname{lcm}(a, b) \qquad a \mathbin{\dot{-}} k = \frac{a}{\gcd(a, k)} \qquad a \ltimes k = \prod_{\alpha_i \neq 0} p_i^{\alpha_i + \kappa_i}$$

$$G \mathbin{\dot{-}} (k_1, \dots, k_d) = \{(a_1 \mathbin{\dot{-}} k1) \vee \cdots \vee (a_d \mathbin{\dot{-}} k_d) \mid a_1, \dots, a_d \in G\}$$
$$G \ltimes (k_1, \dots, k_d) = \{(a_1 \ltimes k1) \vee \cdots \vee (a_d \ltimes k_d) \mid a_1, \dots, a_d \in G\}$$

$G\, f\, (t_1, \dots, t_d)$, where $f \in \{\dot{-}, \ltimes\}^n$ and $t_i \in \{1, 2, \dots\}^n$

$$(G_1\, f_1\, T_1) \vee \cdots \vee (G_n\, f_n\, T_n)$$
$$G$$

$$\begin{array}{c} 2 \cdot 3, 5 \cdot 7, 2 \cdot 7, 3 \cdot 5 \\ \parallel \\ 2 \cdot 3, 5 \cdot 7 \mathbin{\dot{-}} (2 \cdot 5, 3 \cdot 7) \end{array}$$
$$2 \cdot 3, 5 \cdot 7$$

$$\begin{array}{c} 2^3, 2 \cdot 3, 3^3 \\ \parallel \\ 2, 3 \ltimes (4, 9) \end{array}$$
$$2, 3$$

For all $a \in G \setminus G_i$ we have $a \nmid \operatorname{lcm}(G_i)$.

Summary

$$a \vee b = \mathrm{lcm}(a, b) \qquad a \mathbin{\dot-} k = \frac{a}{\gcd(a, k)} \qquad a \ltimes k = \prod_{\alpha_i \neq 0} p_i^{\alpha_i + \kappa_i}$$

$$G \mathbin{\dot-} (k_1, \dots, k_d) = \{(a_1 \mathbin{\dot-} k1) \vee \cdots \vee (a_d \mathbin{\dot-} k_d) \mid a_1, \dots, a_d \in G\}$$
$$G \ltimes (k_1, \dots, k_d) = \{(a_1 \ltimes k1) \vee \cdots \vee (a_d \ltimes k_d) \mid a_1, \dots, a_d \in G\}$$

$G\, f\, (t_1, \dots, t_d)$, where $f \in \{\dot-, \ltimes\}^n$ and $t_i \in \{1, 2, \dots\}^n$

Thank You

$$(G_1\, f_1\, T_1) \vee \cdots \vee (G_n\, f_n\, T_n)$$
$$G$$

$$2 \cdot 3, 5 \cdot 7, 2 \cdot 7, 3 \cdot 5 = 2 \cdot 3, 5 \cdot 7 \mathbin{\dot-} (2 \cdot 5, 3 \cdot 7)$$
$$2 \cdot 3, 5 \cdot 7$$

$$2^3, 2 \cdot 3, 3^3 = 2, 3 \ltimes (4, 9)$$
$$2, 3$$

For all $a \in G \setminus G_i$ we have $a \nmid \mathrm{lcm}(G_i)$.