Unification modulo Boolean Rings

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Structure

Equational Unification

Boolean Rings

Unification modulo Boolean Rings

Equational Unification

Unification Problem

$$S := \left\{ f(a, x) \stackrel{?}{=} x \right\}$$

Equational Unification

Unification Problem

$$I := \{f(x,x) \approx x\}$$

$$S := \left\{ f(a,x) \stackrel{?}{=} x \right\}$$

Equational Unification

Unification Problem

$$I := \{f(x, x) \approx x\}$$

$$S := \left\{ f(a, x) \stackrel{?}{=} x \right\} \qquad \Rightarrow \qquad S := \left\{ f(a, x) \stackrel{?}{\approx}_I x \right\}$$

Let S be an I-unification problem ...

elementary: $S := \left\{ f(y, x) \overset{?}{\approx}_I x \right\}$

with **constants**: $S := \left\{ f(a, x) \overset{?}{\approx}_I x \right\}$

general: $S := \left\{ f(g(a), x) \overset{?}{\approx}_I x \right\}$

Equational Unification

MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$

$$S := \left\{ f(x, y) \overset{?}{\approx}_C f(a, b) \right\}$$

Equational Unification

MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$

$$S := \left\{ f(x, y) \overset{?}{\approx}_C f(a, b) \right\}$$

$$\sigma_1 := \{x \mapsto a, y \mapsto b\} \qquad \sigma_2 := \{x \mapsto b, y \mapsto a\}$$

Equational Unification

More General

$$\sigma \leq \sigma'$$

iff

$$\exists \delta : \delta\sigma = \sigma'$$

Equational Unification

More General

$$\sigma \leq \sigma' \quad \textbf{iff} \quad \exists \delta : \delta\sigma = \sigma'$$

$$\Rightarrow$$

$$\sigma \lesssim_E^X \sigma' \quad \textbf{iff} \quad \exists \delta : \forall x \in X : \delta(\sigma(x)) \approx_E \sigma'(x)$$

Let S be an E-unification problem. A **minimal complete** set for S is a set of substitutions $\mathcal{M}$ that satisfy the following properties:

- each $\sigma \in \mathcal{M}$ is an E-unifier of S
- for all $\theta \in \mathcal{U}_E(S)$ there exists a $\sigma \in \mathcal{M}$ such that $\sigma \lesssim_E^X \theta$
- for all $\sigma, \sigma' \in \mathcal{M},\ \sigma \lesssim_E^X \sigma'$ implies $\sigma = \sigma'$.

Equational Unification

Unification Type of $\approx_E$

unitary iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1.

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zero iff there exists an E-unification problem that does not have a minimal complete set.

Boolean Rings

$$B := \left\{ \begin{array}{cc} x + y \approx y + x, & x * y \approx y * x, \\ (x + y) + z \approx x + (y + z), & (x * y) * z \approx x * (y * z), \\ x + x \approx 0, & x * x \approx x, \\ 0 + x \approx x, & 0 * x \approx 0, \\ x * (y + z) \approx (x * y) + (x * z), & 1 * x \approx x \end{array} \right\}$$

Boolean Rings

Interpretation

$$\mathcal{B}_2 := (\Delta^{\mathcal{B}_2}, \cdot^{\mathcal{B}_2})$$

$$\Delta^{\mathcal{B}_2} := \{\bot, \top\}$$

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$$\begin{aligned}
(x+y)^{\mathcal{B}_2} &:= \left(x^{\mathcal{B}_2} \wedge \neg y^{\mathcal{B}_2}\right) \vee \left(\neg x^{\mathcal{B}_2} \wedge y^{\mathcal{B}_2}\right) \\
(x*y)^{\mathcal{B}_2} &:= x^{\mathcal{B}_2} \wedge y^{\mathcal{B}_2} \\
0^{\mathcal{B}_2} &:= \bot \\
1^{\mathcal{B}_2} &:= \top
\end{aligned}$$

Unification modulo Boolean Rings

In the following we will only consider elementary B-unification.

We will see:

- how to find a B-unifier.
- how to turn this B-unifier into an mgu.
- that elementary B-unification is unitary.

Unification modulo Boolean Rings

Solution in $\mathcal{B}_2 \Rightarrow B$-unifier

$$S := x + y + z \overset{?}{\approx}_B z + 1$$

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$$\varphi(w) := \begin{cases} \bot & \text{if } w = x \\ \top & \text{if } w \neq x \end{cases}$$

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Solution in $\mathcal{B}_2 \Rightarrow B$-unifier

$$S := x + y + z \overset{?}{\approx}_B z + 1$$

$$\varphi(w) := \begin{cases} \perp & \text{if } w = x \\ \top & \text{if } w \neq x \end{cases} \qquad \Rightarrow \qquad \sigma' := \{x \mapsto 0, y \mapsto 1, z \mapsto 1\}$$

Unification modulo Boolean Rings

Transformation

$$S = \left\{ s_1 \overset{?}{\approx}_B t_1, \ldots, s_n \overset{?}{\approx}_B t_n \right\}$$

Unification modulo Boolean Rings

Transformation

$$
\begin{aligned}
& S = \left\{ s_1 \overset{?}{\approx}_B t_1, \ldots, s_n \overset{?}{\approx}_B t_n \right\} \\
\Rightarrow \quad & S = \left\{ s_1 + t_1 \overset{?}{\approx}_B 0, \ldots, s_n + t_n \overset{?}{\approx}_B 0 \right\}
\end{aligned}
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Unification modulo Boolean Rings

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\Rightarrow \quad & S = \left\{ s_1 + t_1 \overset{?}{\approx}_B 0, \ldots, s_n + t_n \overset{?}{\approx}_B 0 \right\} \\
\Rightarrow \quad & S = \left\{ (s_1 + t_1 + 1) * \cdots * (s_n + t_n + 1) \overset{?}{\approx}_B 1 \right\}
\end{aligned}
$$

Unification modulo Boolean Rings

Transformation

$$S = \left\{ s_1 \overset{?}{\approx}_B t_1, \ldots, s_n \overset{?}{\approx}_B t_n \right\}$$

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Unification modulo Boolean Rings

Reproductive E-unifier

σ is a most general E-unifier of S

iff

$$\forall \tau \in \mathcal{U}_E(S) : \exists \theta : \forall x \in X :$$

$$\theta(\sigma(x)) \approx_E \tau(x)$$

Unification modulo Boolean Rings

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$\Rightarrow$

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iff

$\forall \tau \in \mathcal{U}_E(S) : \forall x$

$\tau(\sigma(x)) \approx_E \tau(x$

Let τ be a B-unifier of $t \overset{?}{\approx}_B 0$. The substitution σ defined by

$$\sigma(x) := \begin{cases} (t+1) * x + t * \tau(x) & \text{if } x \in \mathcal{V}ar(t) \\ x & \text{if } x \notin \mathcal{V}ar(t) \end{cases}$$

is a reproductive B-unifier of $t \overset{?}{\approx}_B 0$.

Why is σ reproductive?

Let τ' be an arbitrary B-unifier of S.

$$\begin{aligned}
\tau'(\sigma(x)) &= \tau'((t+1)*x+t*\tau(x)) \\
&= (\tau'(t)+1)*\tau'(x)+\tau'(t)*\tau'(\tau(x)) \\
&\approx_B (0+1)*\tau'(x)+0*\tau'(\tau(x)) \\
&\approx_B \tau'(x)
\end{aligned}$$

B-unification problem: $xy \overset{?}{\approx}_B 0$

$x = 0, y = 0 :$

$x = 0, y = 1 :$

$x = 1, y = 0 :$

Unification modulo Boolean Rings

Löwenheim's Formula Example

B-unification problem: $xy \overset{?}{\approx}_B 0$

$$x = 0, y = 0 : \quad \sigma_1(x) = (xy + 1) * x + xy * 0 \approx_B xy + x$$

$$\sigma_1(y) = (xy + 1) * y + xy * 0 \approx_B xy + y$$

$$x = 0, y = 1 :$$

$$x = 1, y = 0 :$$

Unification modulo Boolean Rings

Löwenheim's Formula Example

B-unification problem: $xy \overset{?}{\approx}_B 0$

$$
\begin{aligned}
x = 0, y = 0: \quad & \sigma_1(x) = (xy+1) * x + xy * 0 \approx_B xy + x \\
& \sigma_1(y) = (xy+1) * y + xy * 0 \approx_B xy + y \\
x = 0, y = 1: \quad & \sigma_2(x) = (xy+1) * x + xy * 0 \approx_B xy + x \\
& \sigma_2(y) = (xy+1) * y + xy * 1 \approx_B y \\
x = 1, y = 0: \quad & \text{similar to } x = 0, y = 1.
\end{aligned}
$$

Unification modulo Boolean Rings

Summary

Equational unification needs minimal complete sets of unifiers.

Elementary B-unification is unitary.

Finding an mgu is NP-complete.