1 map concat

```
(.) :: (b -> c) -> (a -> b) -> a -> c
f . g = \x -> f (g x)

concat :: [[a]] -> [a]
concat []     = []
concat (x:xs) = x++(concat xs)

map :: (a -> b) -> [a] -> [b]
map f []     = []
map f (x:xs) = (f x):(map f xs)
```

Claim 1.1.

```
map f.concat = concat.map (map f)
```

Proof. structural induction on xs
`xs=[]`

```
(map f.concat) xs = (map f.concat) []
                  = map f (concat [])
                  = map f []
                  = []
                  = concat []
                  = concat (map (map f) [])
                  = (concat.map (map f)) []
                  = (concat.map (map f)) xs
```

`xs=(x:xs')` structural induction on x
  `x=[]`

```
(map f.concat) xs = (map f.concat) (x:xs')
                  = map f (concat ([]:xs'))
                  = map f ([]++(concat xs'))
                  = map f (concat xs')
                 IH
                  = (concat.map (map f)) xs'
                  = concat (map (map f) xs')
                  = []++(concat (map (map f) xs'))
                  = concat ([]:(map (map f) xs'))
                  = concat ((map f []):(map (map f) xs'))
                  = concat (map (map f) ([]:xs'))
                  = concat (map (map f) xs)
                  = (concat.map (map f)) xs
```

  `x=(y:ys)`

```
(map f.concat) xs = (map f.concat) (x:xs')
                  = map f (concat ((y:ys):xs'))
                  = map f ((y:ys)++(concat xs'))
                  = map f (y:(ys++(concat xs')))
                  = (f y):(map f (ys++(concat xs')))
```

```
= (f y):(map f (concat (ys:xs')))
IH
= (f y):((concat.map (map f)) (ys:xs'))
= (f y):(concat (map (map f) (ys:xs')))
= (f y):(concat ((map f ys):(map (map f) xs')))
= (f y):((map f ys)++(concat (map (map f) xs')))
= ((f y):(map f ys))++(concat (map (map f) xs'))
= (map f (y:ys))++(concat (map (map f) xs'))
= (map f x)++(concat (map (map f) xs'))
= concat ((map f x):(map (map f) xs'))
= concat (map (map f) (x:xs'))
= concat (map (map f) xs)
= (concat.map (map f)) xs
```

$\square$

alternative proof

Proof. we show the equivalent claim
`map f (concat xs) = concat (map (map f) xs)`
structural induction on xs
`xs=[]`

```
map f (concat xs) = map f (concat [])
                  = map f []
                  = []
                  = concat []
                  = concat (map (map f) [])
                  = concat (map (map f) xs)
```

`xs=(x:xs')` structural induction on x
  `x=[]`

```
map f (concat xs) = map f (concat ([]:xs'))
                  = map f ([]++(concat xs'))
                  = map f (concat xs')
                 IH
                  = concat (map (map f) xs')
                  = []++(concat (map (map f) xs'))
                  = concat ([]:(map (map f) xs'))
                  = concat ((map f []):(map (map f) xs'))
                  = concat (map (map f) ([]:xs'))
                  = concat (map (map f) xs)
```

  `x=(y:ys)`

```
map f (concat xs) = map f (concat ((y:ys):xs'))
                  = map f ((y:ys)++(concat xs'))
                  = map f (y:(ys++(concat xs')))
                  = (f y):(map f (ys++(concat xs')))
                  = (f y):(map f (concat (ys:xs')))
                 IH
                  = (f y):(concat (map (map f) (ys:xs')))
                  = (f y):(concat ((map f ys):(map (map f) xs')))
                  = (f y):((map f ys)++(concat (map (map f) xs')))
                  = ((f y):(map f ys))++(concat (map (map f) xs'))
                  = (map f (y:ys))++(concat (map (map f) xs'))
                  = (map f x)++(concat (map (map f) xs'))
                  = concat ((map f x):(map (map f) xs'))
                  = concat (map (map f) (x:xs'))
                  = concat (map (map f) xs)
```

□

2 foldl map

```
foldl :: (a -> b -> a) -> a -> [b] -> a
foldl f a []     = a
foldl f a (x:xs) = foldl f (f a x) xs
```

Claim 2.1.

```
foldl f a.map g = foldl h a
          h b x = f b (g x)
```

Proof. we show the equivalent claim

```
foldl f a (map g xs) = foldl h a xs
                h b x = f b (g x)
```

by structural induction on xs

`xs=[]`

```
foldl f a (map g xs) = foldl f a (map g [])
                     = foldl f a []
                     = a
                     = foldl h a []
                     = foldl h a xs
```

`xs=(x:xs')`

```
foldl f a (map g xs) = foldl f a (map g (x:xs'))
                     = foldl f a ([g x]++(map g xs'))
                     = foldl f (f a (g x)) (map g xs')
                     IH
                     = foldl h (f a (g x)) xs'
                     = foldl h (h a x) xs'
                     = foldl h a (x:xs')
                     = foldl h a xs
```

□

3 bag

```
data Bag a = ListBag [(a,Integer)] deriving (Eq,Show)

add :: Eq a => (a,Integer) -> (Bag a) -> (Bag a)
add (ele,count) (ListBag bagx)
    |ele 'elem' map fst bagx = ListBag (map (inc (ele,count)) bagx)
    |otherwise               = ListBag (bagx++[(ele,count)])

inc :: Eq a => (a,Integer) -> (a,Integer) -> (a,Integer)
inc (e,c) (x,z)
    |e==x    = (e,c+z)
    |othwise = (x,z)

unite :: Eq a => (Bag a) -> (Bag a) -> (Bag a)
unite (ListBag [])     bagy = bagy
unite (ListBag (x:xs)) bagy = unite (ListBag xs) (add x bagy)

bag :: Eq a => [a] -> (Bag a)
bag []     = Listbag []
bag (x:xs) = add (x,1) (bag xs)
```

Claim 3.1.

```
bag (xs++ys) = unite (bag xs) (bag ys)
```

used but unproven statements:

```
S1: map f (xs++ys) = (map f xs)++(map f ys)
S2: map f (map g xs) = map (f.g) xs
```

Lemma 3.2.

```
(inc (e,z)).(inc (e,c)) = inc (e,z+c)
```

Proof. of Lemma 3.2 Case 1 : `A1: f!=e`

```
(inc (e,z)).(inc (e,c)) (f,y) = inc (e,z) ((inc (e,c)) (f,y))
                             =A1 inc (e,z) (f,y)
                             =A1 (f,y)
                             =A1 inc (e,z+c) (f,y)
```

Case 2 : `A1: f==e`

```
(inc (e,z)).(inc (e,c)) (f,z) =A1 inc (e,z) ((inc (e,c)) (e,y))
                              = inc (e,z) (e,y+c)
                              = (e,y+c+z)
                              = inc (e,z+c) (f,z)
```

∎

Lemma 3.3.

(inc (f,z)).(inc (e,c)) = (inc (e,c)).(inc (f,z))

Proof. of Lemma 3.3 Case 1 : A1: g!=e

(inc (f,z)).(inc (e,c)) (g,y) = inc (f,z) ((inc (e,c)) (g,y))
$\overset{\texttt{A1}}{=}$ inc (f,z) (g,y)

Case 1.1 : A2: g!=f

$\overset{\texttt{A1}}{=}$ inc (f,z) (g,y)
$\overset{\texttt{A2}}{=}$ (g,y)
$\overset{\texttt{A1}}{=}$ inc (e,c) (g,y)
$\overset{\texttt{A2}}{=}$ inc (e,c) ((inc (f,z)) (g,y))
= (inc (e,c)).(inc (f,z)) (g,y)

Case 1.2 : A2: g==f

$\overset{\texttt{A1}}{=}$ inc (f,z) (g,y)
$\overset{\texttt{A2}}{=}$ inc (f,z) (f,y)
= (f,y+z)
$\overset{\texttt{A1,A2}}{=}$ inc (e,c) (f,y+z)
= inc (e,c) ((inc (f,z)) (f,y))
$\overset{\texttt{A1}}{=}$ (inc (e,c)).(inc (f,z)) (g,y)

Case 2 : A1: f==e

(inc (f,z)).(inc (e,c)) (g,y) $\overset{\texttt{A1}}{=}$ inc (f,z) ((inc (e,c)) (e,y))
= inc (f,z) (e,c+y)

Case 2.1 : A2: e!=f

= inc (f,z) (e,c+y)
$\overset{\texttt{A2}}{=}$ (e,c+y)
= inc (e,c) (e,y)
$\overset{\texttt{A2}}{=}$ inc (e,c) ((inc (f,z)) (e,y))
$\overset{\texttt{A1}}{=}$ (inc (e,c)).(inc (f,z)) (g,y)

Case 2.2 : A2: e==f

= inc (f,z) (e,c+y)
$\overset{\texttt{A2}}{=}$ inc (f,z) (f,c+y)
= (f,z+c+y)
= inc (f,c) (f,y+z)
= inc (f,c) ((inc (f,z)) (f,y))
$\overset{\texttt{A1,A2}}{=}$ (inc (e,c)).(inc (f,z)) (g,y)

□

Lemma 3.4. *if* A1: not e 'elem' map fst xs *inc does not do anything*

xs = map (inc (e,c)) xs

Proof. of Lemma 3.4 by structural induction on xs
`A1: not e 'elem' map fst xs`
`xs=[]`

```
xs = []
   = map (inc (e,c)) []
   = map (inc (e,c)) xs
```

`xs=((f,z):xs')`

```
xs = (f,z):xs'
   IH
   = (f,z):(map (inc (e,c)) xs')
   A1
   = (inc (e,c) (f,z)):(map (inc (e,c)) xs')
   = map (inc (e,c)) ((f,z):xs')
   = map (inc (e,c)) xs
```

□

Lemma 3.5. *adding same element*

```
add (e,z) (add (e,c) (ListBag xs)) = add (e,c+z) (ListBag xs)
```

Proof. of Lemma 3.5
Case 1 : `A1: not e 'elem' map fst xs`

```
add (e,z) (add (e,c) (ListBag xs))
                A1
                = add (e,z) (ListBag (xs++[(e,c)]))
                = ListBag (map (inc (e,z)) (xs++[(e,c)]))
                S1
                = ListBag ((map (inc (e,z)) xs)++(map (inc (e,z)) [(e,c)]))
            3.4,A1
                = ListBag (xs++(map (inc (e,z)) [(e,c)]))
                = ListBag (xs++((inc (e,z) (e,c)):(map (inc (e,z)) [])))
                = ListBag (xs++((inc (e,z) (e,c)):[]))
                = ListBag (xs++((e,z+c):[]))
                = ListBag (xs++[(e,z+c)])
                A1
                = add (e,c+z) (ListBag xs)
```

Case 2 : `A1: e 'elem' map fst xs`

```
add (e,z) (add (e,c) (ListBag xs))
                A1
                = add (e,z) (ListBag (map (inc (e,c)) xs))
                A1
                = ListBag (map (inc (e,z)) (map (inc (e,c)) xs))
                S2
                = ListBag (map (inc (e,z)).(inc (e,c)) xs)
               3.2
                = ListBag (map (inc (e,c+z)) xs)
                A1
                = add (e,c+z) (ListBag xs)
```

□

Lemma 3.6. *adding different elements*
`A1: e 'elem' map fst xs`
`A2: f!=e`

`add (f,z) (add (e,c) (ListBag xs)) = add (e,c) (add (f,z) (ListBag xs))`

Proof. of Lemma 3.6
`A1: e 'elem' map fst xs`
`A2: f!=e`
Case 1 : `A3: not f 'elem' map fst xs`

`add (f,z) (add (e,c) (ListBag xs))`
$\overset{\text{A1}}{=}$ `add (f,z) (ListBag (map (inc (e,c)) xs))`
$\overset{\text{A3}}{=}$ `ListBag ((map (inc (e,c)) xs)++[(f,z)])`
$\overset{\text{A2}}{=}$ `ListBag ((map (inc (e,c)) xs)++[inc (e,c) (f,z)])`
$=$ `ListBag ((map (inc (e,c)) xs)++(map (inc (e,c)) [(f,z)]))`
$\overset{\text{S1}}{=}$ `ListBag (map (inc (e,c)) (xs++[(f,z)]))`
$\overset{\text{A1,A2}}{=}$ `add (e,c) (ListBag (xs++[(f,z)]))`
$\overset{\text{A3}}{=}$ `add (e,c) (add (f,z) (ListBag xs))`

Case 2 : `A3: f 'elem' map fst xs`

`add (f,z) (add (e,c) (ListBag xs))`
$\overset{\text{A1}}{=}$ `add (f,z) (ListBag (map (inc (e,c)) xs))`
$\overset{\text{A3}}{=}$ `ListBag (map (inc (f,z)) (map (inc (e,c)) xs))`
$\overset{\text{S2}}{=}$ `ListBag (map (inc (f,z)).(inc (e,c)) xs)`
$\overset{3.3}{=}$ `ListBag (map (inc (e,c)).(inc (f,z)) xs)`
$\overset{\text{S2}}{=}$ `ListBag (map (inc (e,c)) (map (inc (f,z)) xs))`
$\overset{\text{A1}}{=}$ `add (e,c) (ListBag (map (inc (f,z)) xs))`
$\overset{\text{A3}}{=}$ `add (e,c) (add (f,z) (ListBag xs))`

□

Lemma 3.7. *if* `A1: not e 'elem' map fst xs`

```
unite (ListBag ((e,c):(xs++[(e,z)]++ys))) bagy
                = unite (ListBag ((e,c+z):(xs++ys))) bagy
```

Proof. of Lemma 3.7 by structural induction on xs
`A1: not e 'elem' map fst xs`
`xs=[]`

`unite (ListBag ((e,c):(xs++[(e,z)]++ys))) bagy`
$=$ `unite (ListBag ((e,c):([]++[(e,z)]++ys))) bagy`
$=$ `unite (ListBag ((e,c):([(e,z)]++ys))) bagy`
$=$ `unite (ListBag [(e,z)]++ys) (add (e,c) bagy)`
$=$ `unite (ListBag ys) (add (e,z) (add (e,c) bagy))`
$\overset{3.5}{=}$ `unite (ListBag ys) (add (e,c+z) bagy)`
$=$ `unite (ListBag ((e,c+z):ys)) bagy`
$=$ `unite (ListBag ((e,c+z):(xs++ys))) bagy`

`xs=xs'++[x]`

```
unite (ListBag ((e,c):(xs++[(e,z)]++ys))) bagy
  = unite (ListBag (xs++[(e,z)]++ys)) (add (e,c) bagy)
  = unite (ListBag (xs'++[x]++[(e,z)]++ys)) (add (e,c) bagy)
  = unite (ListBag ys) (add (e,z) (add x (add ... (add (e,c) bagy)...)))
3.6= unite (ListBag ys) (add x (add (e,z) (add ... (add (e,c) bagy)...)))
  = unite (ListBag (xs'++[(e,z)]++[x]++ys)) (add (e,c) bagy)
  = unite (ListBag ((e,c):(xs'++[(e,z)]++[x]++ys))) bagy
 IH= unite (ListBag ((e,c+z):(xs'++[x]++ys))) bagy
  = unite (ListBag ((e,c+z):(xs++ys))) bagy
```

□

Lemma 3.8.

```
add (e,z) (unite (ListBag xs) bagy)
                = unite (add (e,z) (ListBag xs)) bagy
```

Proof. of Lemma 3.8 by structural induction on xs
`xs=[]`

```
add (e,z) (unite (ListBag xs) bagy)
            = add (e,z) (unite (ListBag []) bagy)
            = add (e,z) bagy
            = unite (ListBag []) (add (e,z) bagy)
            = unite (ListBag [(e,z)]) bagy
            = unite (add (e,z) (ListBag [])) bagy
            = unite (add (e,z) (ListBag xs)) bagy
```

`xs=(x:xs')`

```
add (e,z) (unite (ListBag xs) bagy)
            = add (e,z) (unite (ListBag (x:xs')) bagy)
            = add (e,z) (unite (ListBag xs') (add x bagy))
          IH= unite (add (e,z) (ListBag xs')) (add x bagy)
```

Case 1 : `A1: not e 'elem' map fst xs`

```
          IH= unite (add (e,z) (ListBag xs')) (add x bagy)
          A1= unite (ListBag (xs'++[(e,z)])) (add x bagy)
            = unite (ListBag (x:(xs'++[(e,z)]))) bagy
            = unite (ListBag ((x:xs')++[(e,z)])) bagy
            = unite (ListBag (xs++[(e,z)])) bagy
          A1= unite (add (e,z) (ListBag xs)) bagy
```

Case 2.1 : `A1: e 'elem' map fst xs` and `A2: x=(f,c) f!=e`
`A1` and `A2` imply `A3: e 'elem' map fst xs'`

```
          IH= unite (add (e,z) (ListBag xs')) (add x bagy)
          A3= unite (ListBag (map (inc (e,z)) xs')) (add x bagy)
            = unite (ListBag (x:(map (inc (e,z)) xs'))) bagy
          A2= unite (ListBag ((inc (e,z) x):(map (inc (e,z)) xs'))) bagy
```

```
= unite (ListBag (map (inc (e,z)) (x:xs'))) bagy
A1
=  unite (add (e,z) (ListBag xs)) bagy
```

Case 2.2 : `A1: e 'elem' map fst xs` and `A2: x=(e,c)`
multiset, `A1` and `A2` imply `A3: not e 'elem' map fst xs'`

```
IH
=  unite (add (e,z) (ListBag xs')) (add x bagy)
A3
=  unite (ListBag (xs'++[(e,z)])) (add x bagy)
=  unite (ListBag (x:(xs'++[(e,z)]))) bagy
A2
=  unite (ListBag ((e,c):(xs'++[(e,z)]))) bagy
3.7
=  unite (ListBag ((e,c+z):xs')) bagy
=  unite (ListBag ((inc (e,z) (e,c)):xs')) bagy
3.4
=  unite (ListBag ((inc (e,z) x):(map (inc (e,z)) xs'))) bagy
=  unite (ListBag (map (inc (e,z)) (x:xs'))) bagy
=  unite (ListBag (map (inc (e,z)) xs)) bagy
A1
=  unite (add (e,z) (ListBag xs)) bagy
```

□

Proof. of Claim 3.1 by structural induction on xs
`xs=[]`

```
bag (xs++ys) = bag ([]++ys)
             = bag ys
             = unite (ListBag []) (bag ys)
             = unite (bag []) (bag ys)
             = unite (bag xs) (bag ys)
```

`xs=(x:xs')`

```
bag (xs++ys) = bag ((x:xs')++ys)
             = bag (x:(xs'++ys))
             = add (x,1) (bag (xs'++ys))
             IH
             =  add (x,1) (unite (bag xs') (bag ys))
             3.8
             =  unite (add (x,1) (bag xs')) (bag ys)
             = unite (bag (x:xs')) (bag ys)
             = unite (bag xs) (bag ys)
```

□