



UNIFICATION MODULO BOOLEAN RINGS

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Structure

Equational Unification

Boolean Rings

Unification modulo Boolean Rings

Equational Unification

Unification Problem

$$S := \left\{ f(a, x) \stackrel{?}{=} x \right\}$$

Equational Unification

Unification Problem

$$I := \{f(x, x) \approx x\}$$

$$S := \left\{ f(a, x) \stackrel{?}{=} x \right\}$$

Equational Unification Unification Problem

$$I := \{f(x, x) \approx x\}$$

$$S := \left\{ f(a, x) \stackrel{?}{=} x \right\} \quad \Rightarrow \quad S := \left\{ f(a, x) \stackrel{?}{\approx}_I x \right\}$$

Equational Unification

Unification Classes

Let S be an I-unification problem ...

$$\text{elementary: } S := \left\{ f(y, x) \stackrel{?}{\approx}_I x \right\}$$

$$\text{with constants: } S := \left\{ f(a, x) \stackrel{?}{\approx}_I x \right\}$$

$$\text{general: } S := \left\{ f(g(a), x) \stackrel{?}{\approx}_I x \right\}$$

Equational Unification

MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$

$$S := \left\{ f(x, y) \stackrel{?}{\approx}_C f(a, b) \right\}$$

Equational Unification

MGU?

$$C := \{f(x, y) \approx f(y, x)\}$$

$$S := \left\{ f(x, y) \stackrel{?}{\approx}_C f(a, b) \right\}$$

$$\sigma_1 := \{x \mapsto a, y \mapsto b\}$$

$$\sigma_2 := \{x \mapsto b, y \mapsto a\}$$

Equational Unification

More General

$$\begin{aligned} \sigma &\leq \sigma' \\ \text{iff} \\ \exists \delta : \delta(\sigma) &= \sigma' \end{aligned}$$

Equational Unification

More General

$$\begin{array}{ccc} \sigma \leq \sigma' & & \sigma \lesssim_E^X \sigma' \\ \text{iff} & \Rightarrow & \text{iff} \\ \exists \delta : \delta(\sigma) = \sigma' & & \exists \delta : \forall x \in X : \\ & & \delta(\sigma(x)) \approx_E \sigma'(x) \end{array}$$

Equational Unification

Minimal Complete Sets

Let S be an E-unification problem. A minimal complete set for S is a set of substitutions \mathcal{M} that satisfy the following properties:

- each $\sigma \in \mathcal{M}$ is an E-unifier of S
- for all $\theta \in \mathcal{U}_E(S)$ there exists a $\sigma \in \mathcal{M}$ such that $\sigma \lesssim_E^X \theta$
- for all $\sigma, \sigma' \in \mathcal{M}$, $\sigma \lesssim_E^X \sigma'$ implies $\sigma = \sigma'$.

Equational Unification

Unification Type of \approx_E

unitary iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1 .

Equational Unification

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- infinitary iff for all E-unification problems S there exists a minimal complete set, and there exists an E-unification problem for which this set is infinite.

Equational Unification

Unification Type of \approx_E

- unitary iff for all E-unification problems S there exists a minimal complete set of cardinality ≤ 1 .
- finitary iff for all E-unification problems S there exists a minimal complete set with finite cardinality.
- infinitary iff for all E-unification problems S there exists a minimal complete set, and there exists an E-unification problem for which this set is infinite.
- zero iff there exists an E-unification problem that does not have a minimal complete set.

Boolean Rings

$$B := \left\{ \begin{array}{ll} x + y \approx y + x, & x * y \approx y * x, \\ (x + y) + z \approx x + (y + z), & (x * y) * z \approx x * (y * z), \\ x + x \approx 0, & x * x \approx x, \\ 0 + x \approx x, & 0 * x \approx 0, \\ x * (y + z) \approx (x * y) + (x * z), & 1 * x \approx x \end{array} \right\}$$

Boolean Rings

Interpretation

$$\mathcal{B}_2 := (\Delta^{\mathcal{B}_2}, \cdot^{\mathcal{B}_2})$$

$$\Delta^{\mathcal{B}_2} := \{\perp, \top\}$$

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$$(x + y)^{\mathcal{B}_2} := \left(x^{\mathcal{B}_2} \wedge \neg y^{\mathcal{B}_2} \right) \vee \left(\neg x^{\mathcal{B}_2} \wedge y^{\mathcal{B}_2} \right)$$

$$(x * y)^{\mathcal{B}_2} := x^{\mathcal{B}_2} \wedge y^{\mathcal{B}_2}$$

$$0^{\mathcal{B}_2} := \perp$$

$$1^{\mathcal{B}_2} := \top$$

Boolean Rings

Example

$$\begin{aligned}(1 + 0)^{\mathcal{B}_2} &= \left(1^{\mathcal{B}_2} \wedge \neg 0^{\mathcal{B}_2}\right) \vee \left(\neg 1^{\mathcal{B}_2} \wedge 0^{\mathcal{B}_2}\right) \\&= (\top \wedge \neg \perp) \vee (\neg \top \wedge \perp) \\&= \top \vee \perp \\&= \top\end{aligned}$$

Unification modulo Boolean Rings

In the following we will only consider elementary B-unification.

We will see:

- how to find a B-unifier.
- how to turn this B-unifier into an mgu.
- that elementary B-unification is unitary.

Unification modulo Boolean Rings

Solution in $\mathcal{B}_2 \Rightarrow \text{B-unifier}$

$$S := x + y + z \stackrel{?}{\approx}_{\mathcal{B}} z + 1$$

Unification modulo Boolean Rings

Solution in $\mathcal{B}_2 \Rightarrow$ B-unifier

$$S := x + y + z \stackrel{?}{\approx}_B z + 1$$

$$\varphi(w) := \begin{cases} \perp & \text{if } w = x \\ \top & \text{if } w \neq x \end{cases}$$

Unification modulo Boolean Rings

Solution in $\mathcal{B}_2 \Rightarrow$ B-unifier

$$S := x + y + z \stackrel{?}{\approx}_B z + 1$$

$$\varphi(w) := \begin{cases} \perp & \text{if } w = x \\ \top & \text{if } w \neq x \end{cases} \quad \Rightarrow \quad \sigma' := \{x \mapsto 0, y \mapsto 1, z \mapsto 1\}$$

Unification modulo Boolean Rings Transformation

$$S = \left\{ s_1 \overset{?}{\approx}_B t_1, \dots, s_n \overset{?}{\approx}_B t_n \right\}$$

Unification modulo Boolean Rings Transformation

$$\begin{aligned} S &= \left\{ s_1 \overset{?}{\approx}_B t_1, \dots, s_n \overset{?}{\approx}_B t_n \right\} \\ \Rightarrow \quad S &= \left\{ s_1 + t_1 \overset{?}{\approx}_B 0, \dots, s_n + t_n \overset{?}{\approx}_B 0 \right\} \end{aligned}$$

Unification modulo Boolean Rings Transformation

$$S = \left\{ s_1 \overset{?}{\approx}_B t_1, \dots, s_n \overset{?}{\approx}_B t_n \right\}$$

$$\Rightarrow S = \left\{ s_1 + t_1 \overset{?}{\approx}_B 0, \dots, s_n + t_n \overset{?}{\approx}_B 0 \right\}$$

$$\Rightarrow S = \left\{ (s_1 + t_1 + 1) * \dots * (s_n + t_n + 1) \overset{?}{\approx}_B 1 \right\}$$

Unification modulo Boolean Rings

Transformation

$$S = \left\{ s_1 \overset{?}{\approx}_B t_1, \dots, s_n \overset{?}{\approx}_B t_n \right\}$$

$$\Rightarrow S = \left\{ s_1 + t_1 \overset{?}{\approx}_B 0, \dots, s_n + t_n \overset{?}{\approx}_B 0 \right\}$$

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$$\Rightarrow S = \left\{ (s_1 + t_1 + 1) * \dots * (s_n + t_n + 1) + 1 \overset{?}{\approx}_B 0 \right\}$$

Unification modulo Boolean Rings

Reproductive E-unifier

σ is a most general E-unifier of S

iff

$$\forall \tau \in \mathcal{U}_E(S) : \exists \theta : \forall x \in X :$$

$$\theta(\sigma(x)) \approx_E \tau(x)$$

Unification modulo Boolean Rings

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\Rightarrow

σ is a reproductive E-unifier of S

iff

$$\forall \tau \in \mathcal{U}_E(S) : \forall x :$$

$$\tau(\sigma(x)) \approx_E \tau(x)$$

Unification modulo Boolean Rings

Löwenheim's Formula

Let τ be a B-unifier of $t \stackrel{?}{\approx}_B 0$. The substitution σ defined by

$$\sigma(x) := \begin{cases} (t + 1) * x + t * \tau(x) & \text{if } x \in \mathcal{V}\text{ar}(t) \\ x & \text{if } x \notin \mathcal{V}\text{ar}(t) \end{cases}$$

is a reproductive B-unifier of $t \stackrel{?}{\approx}_B 0$.

Unification modulo Boolean Rings

Löwenheim's Formula

Why is σ reproductive?

Let τ' be an arbitrary B-unifier of S.

$$\begin{aligned}\tau'(\sigma(x)) &= \tau'((t + 1) * x + t * \tau(x)) \\ &= (\tau'(t) + 1) * \tau'(x) + \tau'(t) * \tau'(\tau(x)) \\ &\approx_B (0 + 1) * \tau'(x) + 0 * \tau'(\tau(x)) \\ &\approx_B \tau'(x)\end{aligned}$$

Unification modulo Boolean Rings

Löwenheim's Formula Example

B-unification problem: $xy \stackrel{?}{\approx}_B 0$

$x = 0, y = 0 :$

$x = 0, y = 1 :$

$x = 1, y = 0 :$

Unification modulo Boolean Rings

Löwenheim's Formula Example

B-unification problem: $xy \stackrel{?}{\approx}_B 0$

$$x = 0, y = 0 : \quad \sigma_1(x) = (xy + 1) * x + xy * 0 \approx_B xy + x$$

$$\sigma_1(y) = (xy + 1) * y + xy * 0 \approx_B xy + y$$

$$x = 0, y = 1 :$$

$$x = 1, y = 0 :$$

Unification modulo Boolean Rings

Löwenheim's Formula Example

B-unification problem: $xy \stackrel{?}{\approx}_B 0$

$$x = 0, y = 0 : \quad \sigma_1(x) = (xy + 1) * x + xy * 0 \approx_B xy + x$$

$$\sigma_1(y) = (xy + 1) * y + xy * 0 \approx_B xy + y$$

$$x = 0, y = 1 : \quad \sigma_2(x) = (xy + 1) * x + xy * 0 \approx_B xy + x$$

$$\sigma_2(y) = (xy + 1) * y + xy * 1 \approx_B y$$

$$x = 1, y = 0 : \quad \text{similar to } x = 0, y = 1.$$

Unification modulo Boolean Rings

Summary

Equational unification needs minimal complete sets of unifiers.

Elementary B-unification is unitary.

Finding an mgu is NP-complete.