Contents

1 Equational Unification

In the following let E be a set of identities of the form $\{e_1 \approx f_1, \dots, e_n \approx f_n\}$. Furthermore let $\mathcal{S}ig(E)$ denote the set of all function symbols occurring in E. Let Σ be a finite set of function symbols and a superset of $\mathcal{S}ig(E)$.

Definition 1.1. An E**-unification Problem** over Σ is a finite set S of the form $S = \left\{ s_1 \overset{?}{\approx}_E t_1, \dots, s_n \overset{?}{\approx}_E t_n \right\}$ with $s_1, \dots, s_n, t_1, \dots, t_n \in T(\Sigma, V)$, V being a countable set of Variables.
A substitution σ is an E**-unifier** of S iff $\sigma(s_i) \approx_E \sigma(t_i)$ for all $1 \leq i \leq n$. The set of all E-unifiers of S is denoted by $\mathcal{U}_E(S)$. S is E**-unifiable** iff $\mathcal{U}_E(S) \neq \emptyset$.

Definition 1.2. Let S be an E-unification problem over Σ.

- S is an **elementary** E-unification problem iff $\mathcal{S}ig(E) = \Sigma$.
- S is an E-unification problem **with constants** iff $\Sigma - \mathcal{S}ig(E) \subseteq \Sigma^{(0)}$ and $\mathcal{S}ig(E) \subset \Sigma$
- S is an **general** E-unification problem iff $\Sigma - \mathcal{S}ig(E)$ contains an at least unary function symbol.

One *most general unifier* does not always suffice to represent $\mathcal{U}_E(S)$. In this case we need a *minimal complete set of unifiers* but to define this set we first need an order on substitutions.

Definition 1.3. Let X be a set of variables. A substitution σ is **more general** modulo $\approx_E$ than a substitution σ' on X iff there is a substitution δ such that $\delta(\sigma(x)) \approx_E \sigma'(x)$ for all $x \in X$. We denote this by $\sigma \lesssim_E^X \sigma'$.

$\lesssim_E^X$ is a is a quasi order since it obviously is reflexive and transitive. But why do we only demand equality modulo $\approx_E$ on X and not on all Variables like we did in syntactic unification? Note that by the restriction to Variables in X more substitutions are comparable with respect to $\lesssim_E^X$ since we do not demand equality modulo $\approx_E$ on all Variables. Lets denote the Variables occurring in an E-unification problem S by $\mathcal{V}ar(S)$. It is easy to see that if $X = \mathcal{V}ar(S)$, σ' is an E-unifier of S and $\sigma \lesssim_E^X \sigma'$ then σ is also an E-unifier of S. This only shows that restriction to X does not do any damage but the reason it is useful is that there are E-unification problems S for which any *minimal complete set of E-unifiers* has to contain Variables not occurring in S. Lets consider a small example, let $\sigma := \{x \mapsto f(y)\}$ be in $\mathcal{M}$ a *minimal complete set of E-unifiers* of S with $\mathcal{V}ar(S) = \{x\}$ and $\{a \approx x\} \notin E$. Clearly $\sigma' := \{x \mapsto f(a)\}$ is also an E-unifier of S but σ and σ' are incomparable w.r.t. $\lesssim_E^{\{x,y\}}$. The substitution $\delta := \{y \mapsto a\}$ does not work here since $\delta(\sigma(y)) = a \not\approx_E y = \sigma'(y)$ which means there has to be another unifier σ'' in $\mathcal{M}$ with $\sigma'' \lesssim_E^{\{x,y\}} \sigma$. But if we restrict X to $\{x\}$ we only need that $\delta(\sigma(x)) = f(a) \approx_E f(a) = \sigma'(x)$ so $\sigma \lesssim_E^{\{x\}} \sigma'$ holds. We see that *minimal complete sets of E-unifiers* can become unnecessary large if we consider all Variables. Since we have talked about these sets a lot lets define them formally.

Definition 1.4. Let S be an E-unification problem over Σ and let $X := \mathcal{V}ar(S)$. A **complete set of E-unifiers** of S is a set of substitutions $\mathcal{C}$ that satisfies the following properties.

- each $\sigma \in \mathcal{C}$ is an E-unifier of S
- for all $\theta \in \mathcal{U}_E(S)$ there exists a $\sigma \in \mathcal{C}$ such that $\sigma \lesssim_E^X \theta$

A **minimal complete set of E-unifiers** is a complete set of E-unifiers $\mathcal{M}$ that satisfies the additional property

- for all $\sigma, \sigma' \in \mathcal{M},\ \sigma \lesssim_E^X \sigma'$ implies $\sigma = \sigma'$.

The substitution σ is a **most general E-unifier** (mgu) of S iff $\{\sigma\}$ is a minimal complete set of E-unifiers of S.