Stochastic Calculus

Master in Advanced Mathematics 2017-2018

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Set 3. Gaussian distributions, Gaussian processes, Brownian motion.

Exercise 1. (1) Show that for every $f, g \in L^2_{a,T}$ the following identity holds:

$$\mathbb{E}\left[\left(\int_0^T f(u)dB_u\right)\left(\int_0^T g(u)dB_u\right)\right] = \int_0^T \mathbb{E}[f(u)g(u)]du.$$

(2) Prove that if $f : [0,T] \to \mathbb{R}$ is a deterministic continuous function, the process $(X_t)_{t\in[0,T]}$ defined by

$$X_t := \int_0^t f(u)dB_u$$

is a zero-mean Gaussian process with independent increments.

(3) Assume that the deterministic continuous function $f : [0,T] \to \mathbb{R}$ satisfies $f(t) > 0$ for all $t > 0$ and that

$$\int_0^t f^2(u)du \to T, \text{as } t \to T.$$

For every $t \in [0,T]$ define

$$\tau(t) := \inf\{s \in [0,T] : \int_0^s f^2(u)du \geq t\}$$

and

$$Y_t := \int_0^{\tau(t)} f(u)dB_u.$$

Prove that $(Y_t)_{t\in[0,T]}$ is standard Brownian motion.

Solution. (1) Using the formula $ab = \frac{1}{2}((a+b)^2 - a^2 - b^2)$ we get

$$\left(\int_0^T f(u)dB_u\right)\left(\int_0^T g(u)dB_u\right) = \frac{1}{2}\left[\left(\int_0^T (f(u)+g(u))dB_u\right)^2 - \left(\int_0^T f(u)dB_u\right)^2 - \left(\int_0^T g(u)dB_u\right)^2\right]$$

By the Ito isometry we observe the following

$$\mathbb{E}\left[\left(\int_0^T (f(u)+g(u))dB_u\right)^2\right] = \mathbb{E}\left[\int_0^T (f(u)+g(u))^2du\right] = \mathbb{E}\left[\int_0^T (f^2(u)+2f(u)g(u)+g^2(u))du\right]$$

$$\mathbb{E}\left[\left(\int_0^T f(u)dB_u\right)^2\right] = \mathbb{E}\left[\int_0^T f(u)^2du\right]$$

$$\mathbb{E}\left[\left(\int_0^T g(u)dB_u\right)^2\right] = \mathbb{E}\left[\int_0^T g(u)^2du\right]$$

Then,

$$\frac{1}{2}\mathbb{E}\left[\left(\int_0^T (f(u)+g(u))dB_u\right)^2 - \left(\int_0^T f(u)dB_u\right)^2 - \left(\int_0^T g(u)dB_u\right)^2\right] = \mathbb{E}\left[\int_0^T f(u)g(u)du\right].$$

(2) By definition,

$$X_t = \int_0^T f(u)\mathbb{1}_{[0,t]}(u)du.$$

$t_0 < t_1 < \cdots < t_n < T$, we want

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent. By definition, the random variables are Gaussian so the vector is Gaussian. Also,

$$\begin{aligned}
\operatorname{Cov}(X_{t_i} - X_{t_{i-1}}, X_{t_j} - X_{t_{j-1}}) &= \mathbb{E}\left[(X_{t_i} - X_{t_{i-1}})(X_{t_j} - X_{t_{j-1}})\right] \\
&= \mathbb{E}\left[\left(\int_0^T f(u)\mathbb{1}_{[t_{i-1},t_i]}(u)dB_u\right)\left(\int_0^T f(u)\mathbb{1}_{[t_{j-1},t_j]}(u)dB_u\right)\right] \\
&= \int_0^T \mathbb{E}[f(u)^2\mathbb{1}_{[t_{i-1},t_i]\cap[t_{j-1},t_j]}]du \\
&= \int_0^T f(u)^2\mathbb{1}_{[t_{i-1},t_i]\cap[t_{j-1},t_j]}du \\
&= 0
\end{aligned}$$

(3) We compute the covariance $\operatorname{Cov}(Y_t, Y_s)$.

$$\begin{aligned}
\operatorname{Cov}(Y_t, Y_s) &= \mathbb{E}\left[\left(\int_0^T f(u)\mathbb{1}_{\{u\le\tau(t)\}}dB_u\right)\left(\int_0^T f(u)\mathbb{1}_{\{u\le\tau(s)\}}dB_u\right)\right] \\
&= \int_0^T \mathbb{E}[f(u)\mathbb{1}_{\{u\le\tau(t)\}}f(u)\mathbb{1}_{\{u\le\tau(s)\}}]du \\
&= \int_0^T \mathbb{E}[f^2(u)\mathbb{1}_{\{u\le\tau(t)\}}\mathbb{1}_{\{u\le\tau(s)\}}]du \\
&= \int_0^T \mathbb{E}[f^2(u)\mathbb{1}_{\{u\le\tau(t)\}\cap\{u\le\tau(s)\}}]du \\
&= \int_0^T \mathbb{E}[f^2(u)\mathbb{1}_{\{u\le\min\{\tau(t),\tau(s)\}\}}]du \\
&= \int_0^{\min\{\tau(t),\tau(s)\}} f^2(u)du \\
? &= \min\{s, t\}.
\end{aligned}$$

□

Exercise 2. Let $X = \int_a^b f(t)[\sin(B_t) + \cos(B_t)]dB_t$ with $f \in L^2[a, b]$.

(1) Argue that the stochastic integral is well-defined.
(2) Compute the variance of X.

Solution. (1) $\int_a^b f^2(t)[\sin(B_t) + \cos(B_t)]^2dt < \infty$

(2) We have that

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}\left[\left(\int_a^b f(t)[\sin(B_t)+\cos(B_t)]dB_t\right)^2\right] \\
&= \mathbb{E}\left[\int_a^b f^2(t)[\sin(B_t)+\cos(B_t)]^2 dt\right] \\
&= \mathbb{E}\left[\int_a^b f^2(t)[1+2\cos(B_t)\sin(B_t)]dt\right] \\
&= \mathbb{E}\left[\int_a^b f^2(t)dt + 2\int_a^b f^2(t)\cos(B_t)\sin(B_t)dt\right] \\
&= \mathbb{E}\left[\int_a^b f^2(t)dt\right] + 2\mathbb{E}\left[\int_a^b f^2(t)\cos(B_t)\sin(B_t)dt\right] \\
&= \int_a^b f^2(t)dt + 2\int_a^b f^2(t)\mathbb{E}\left[\cos(B_t)\sin(B_t)\right]dt.
\end{aligned}$$

Now, we use the following formulas for $cos(x)$ and $\sin(x)$:

$$\cos(x) = \frac{1}{2}\left(e^{ix}+e^{-ix}\right) \qquad \text{and } \sin(x) = \frac{1}{2i}\left(e^{ix}-e^{-ix}\right).$$

Then,

$$\cos(x)\sin(x) = \frac{1}{4i}\left(e^{2ix}-e^{-2ix}\right).$$

Therefore,

$$\mathbb{E}\left[\cos(B_t)\sin(B_t)\right] = \frac{1}{4i}\left(\mathbb{E}\left[e^{2iB_t}\right]-\mathbb{E}\left[e^{2iB_t}\right]\right) = 0.$$

We conclude that

$$\mathrm{Var}(X) = \int_a^b f^2(t)dt.$$

□

Exercise 3. (1) Set $X_t = \int_0^t \sqrt{s}\sin(B_s)dB_s, t \geq 0$. Is the stochastic integral well-defined? Compute de covariance between X_t and X_u, $0 \leq u \leq t$.

(2) Let $X_t = \int_0^t \mathrm{sgn}(B_s)dB_s$. Show that for $0 \leq s < t$, $X_t - X_s$ has mean zero and variance $t-s$.

Solution. (1) $\int_0^t \mathbb{E}[s\sin^2(B_s)]ds = \frac{1}{2}\int_0^t s\left(e^{-2s}-1\right)ds$ Compute

$$\begin{aligned}
\mathbb{E}[X_tX_u] &= \mathbb{E}\left[\left(\int_0^t \sqrt{s}\sin(B_s)dB_s\right)\left(\int_0^u \sqrt{s}\sin(B_s)dB_s\right)\right] \\
&= \mathbb{E}\left[\left(\int_0^t \sqrt{s}\sin(B_s)dB_s\right)\left(\int_0^t \sqrt{s}\sin(B_s)\mathbb{1}_{s\leq u}dB_s\right)\right] \\
&= \int_0^t s\mathbb{1}_{s\leq u}\mathbb{E}[\sin^2(B_s)]ds \\
&= \frac{1}{2}\int_0^u s\left(1-e^{-2s}\right)ds \\
&= \frac{1}{4}u^2 + \frac{1}{8}\left(1-(2u+1)e^{-2u}\right)
\end{aligned}$$

$$\sin^2(x) = -\frac{1}{4}\left(e^{ix}-e^{-ix}\right)^2 = -\frac{1}{4}\left(e^{2ix}-2+e^{-2ix}\right) = \frac{1}{2}-\frac{1}{4}\left(e^{2ix}+e^{-2ix}\right)$$

Then,

$$\mathbb{E}[\sin^2(B_s)] = \frac{1}{2}-\frac{1}{4}\left(\mathbb{E}[e^{2iB_s}]+\mathbb{E}[e^{-2iB_s}]\right) = \frac{1}{2}-\frac{1}{4}\left(2e^{-2s}\right) = \frac{1}{2}\left(1-e^{-2s}\right).$$

(2) $0 \leq s < t$. Then,

$$X_t - X_s = \int_s^t \operatorname{sgn}(B_s) dB_s$$

$\square$

Exercise 4. Using the definition of the Ito integral, prove that:

(1) $\int_0^t s dB_s = tB_t - \int_0^t B_s ds$.
(2) $\int_0^t B_s^2 dB_s = \frac{1}{3} B_t^3 - \int_0^t B_s ds$.

Solution. (1)

$$\begin{aligned}
\sum_i s_{t_{i-1}}(B_{t_i} - B_{t_{i-1}}) &= \sum_i (s_{t_{i-1}} B_{t_i} - s_{t_{i-1}} B_{t_{i-1}}) \\
&= \sum_i (s_{t_{i-1}} B_{t_i} - s_{t_i} B_{t_i} + s_{t_i} B_{t_i} - s_{t_{i-1}} B_{t_{i-1}}) \\
&= \sum_i \left(s_{t_i} B_{t_i} - s_{t_{i-1}} B_{t_{i-1}}\right) - \sum_i (s_{t_i} - s_{t_{i-1}}) \\
&= tB_t - \sum_i B_{t_i}(s_{t_i} - s_{t_{i-1}}).
\end{aligned}$$

(2) We use the formula $a^2(b-a) = \frac{1}{3}(b^3 - a^3 - (b-a)^3 - 3a(b-a)^2)$.

$$\begin{aligned}
\sum_i B_{t_{i-1}}^2 \left(B_{t_i} - B_{t_{i-1}}\right) &= \frac{1}{3} \sum_i \left[(B_{t_i}^3 - B_{t_{i-1}}^3) - (B_{t_i} - B_{t_{i-1}})^3 - 3B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})^2\right] \\
&= \frac{1}{3} B_t^3 - \frac{1}{3} \sum_i (B_{t_i} - B_{t_{i-1}})^3 - \sum_i B_{t_{i-1}}(B_{t_i} - B_{t_{i-1}})^2.
\end{aligned}$$

Notice that,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_i (B_{t_i} - B_{t_{i-1}})^3\right)^2\right] &= \mathbb{E}\left[\sum_i (B_{t_i} - B_{t_{i-1}})^6\right] + 2\mathbb{E}\left[\sum_{i,j} (B_{t_i} - B_{t_{i-1}})^3 (B_{t_j} - B_{t_{j-1}})^3\right] \\
&= \sum_i \mathbb{E}\left[(B_{t_i} - B_{t_{i-1}})^6\right] \\
&= 15 \sum_i (t_i - t_{i-1})^3 \\
&\leq 15 |\Pi^n|^2 t \to 0
\end{aligned}$$

Let us denote $X_i := B_{t_{i-1}}\left((B_{t_i} - B_{t_{i-1}})^2 - (t_i - t_{i-1})\right)$. We have that

$$\mathbb{E}\left[\left(\sum_i X_i\right)^2\right] = \sum_i \mathbb{E}[X_i^2].$$

Indeed, for $j < i$

$$\begin{aligned}
\mathbb{E}[X_i X_j] &= \mathbb{E}[\mathbb{E}[X_i X_j \mid \mathcal{F}_{t_{j-1}}]] \\
&= \mathbb{E}[X_i B_{t_{j-1}} \mathbb{E}[(B_{t_j} - B_{t_{j-1}})^2 - (t_j - t_{j-1}) \mid \mathcal{F}_{t_{j-1}}]] \\
&= \mathbb{E}[X_i B_{t_{j-1}} \left(\mathbb{E}[(B_{t_j} - B_{t_{j-1}})^2] - (t_j - t_{j-1})\right)] \\
&= 0.
\end{aligned}$$

Now,

$$\begin{aligned}
\mathbb{E}[X_i^2] &= \mathbb{E}[B_{t_{i-1}}^2\left((B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\right)^2] \\
&= t_{i-1}\mathbb{E}[\left((B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\right)^2] \\
&= t_{i-1}\left(\mathbb{E}\left[(B_{t_i}-B_{t_{i-1}})^4\right]-2\mathbb{E}[(B_{t_i}-B_{t_{i-1}})^2](t_i-t_{i-1})+(t_i-t_{i-1})^2\right) \\
&= t_{i-1}\left(3(t_i-t_{i-1})^2-2(t_i-t_{i-1})^2+(t_i-t_{i-1})^2\right) \\
&= 2t_{i-1}(t_i-t_{i-1})^2.
\end{aligned}$$

We estimate,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_i B_{t_{i-1}}(B_{t_i}-B_{t_{i-1}})^2-\sum_i B_{t_{i-1}}(t_i-t_{i-1})\right)^2\right] &= \mathbb{E}\left[\left(\sum_i B_{t_{i-1}}\left((B_{t_i}-B_{t_{i-1}})^2-(t_i-t_{i-1})\right)\right)^2\right] \\
&= \mathbb{E}\left[\left(\sum_i X_i\right)^2\right] \\
&= \sum_i \mathbb{E}[X_i^2] \\
&= 2\sum_i t_{i-1}(t_i-t_{i-1})^2 \\
&\leq 2|\Pi^n|t^2 \to 0
\end{aligned}$$

□

Exercise 5. Let $u \in L^2_{a,T}$. Prove that the process

$$\left(\int_0^t u_s dBs\right)^2 - \int_0^t u_s^2 ds, \qquad t \in [0,T],$$

is a martingale.

Solution. □

Exercise 6. Prove that the following stochastic processes are martingales with respect to the natural filtration of the Brownian motion.

(1) $M_t = B_t^3 - 3tB_t, t \geq 0$;
(2) $X_t = \exp\left(\frac{1}{2}t\right)\cos(B_t), t \geq 0$;
(3) $X_t = (B_t + t)\exp\left(-B_t - \frac{1}{2}t\right)$.

Solution. (1) We use the items (1) and (2) in Problem 4 and we write

$$\begin{aligned}
M_t &= 3\left(\int_0^t B_s^2 dB_s + \int_0^t B_s ds - \int_0^t s dB_s - \int_0^t B_s ds\right) \\
&= \int_0^t 3(B_s^2 - s) dB_s.
\end{aligned}$$

(2) We consider the function

$$\begin{aligned}
f : [0,T]\times\mathbb{R} &\to \mathbb{R} \\
(t,x) &\mapsto \exp\left(\frac{1}{2}t\right)\cos(x).
\end{aligned}$$

$$\partial_t f(t,x) = \frac{1}{2}\exp\left(\frac{1}{2}t\right)\cos(x).$$

$$\partial_x f(t,x) = -\exp\left(\frac{1}{2}t\right)\sin(x).$$
$$\partial_{xx}^2 f(t,x) = -\exp\left(\frac{1}{2}t\right)\cos(x).$$

Applying the Ito formula we obtain

$$X_t = f(0,B_0) + \int_0^t \frac{1}{2}\exp\left(\frac{1}{2}s\right)\cos(B_s)ds - \int_0^t \exp\left(\frac{1}{2}s\right)\sin(B_s)dB_s - \frac{1}{2}\int_0^t \exp\left(\frac{1}{2}s\right)\cos(B_s)ds$$
$$= 1 - \int_0^t \exp\left(\frac{1}{2}s\right)\sin(B_s)dB_s.$$

(3) We consider the function

$$f : [0,T]\times\mathbb{R} \to \mathbb{R}$$
$$(t,x) \mapsto (x+t)\exp\left(-x-\frac{1}{2}t\right).$$

$$\partial_t f(t,x) = -\frac{1}{2}(x+t)\exp\left(-x-\frac{1}{2}t\right) + \exp\left(-x-\frac{1}{2}t\right) = \exp\left(-x-\frac{1}{2}t\right)\left(1-\frac{1}{2}(x+t)\right)$$
$$\partial_x f(t,x) = -(x+t)\exp\left(-x-\frac{1}{2}t\right) + \exp\left(-x-\frac{1}{2}t\right) = \exp\left(-x-\frac{1}{2}t\right)(1-(x+t))$$
$$\partial_{xx}^2 f(t,x) = -(1-(x+t))\exp\left(-x-\frac{1}{2}t\right) - \exp\left(-x-\frac{1}{2}t\right) = -\exp\left(-x-\frac{1}{2}t\right)(2-x-t).$$

Applying the Ito formula we obtain

$$X_t = \int_0^t \exp\left(-B_s-\frac{1}{2}s\right)\left(1-\frac{1}{2}(B_s+s)\right)ds + \int_0^t \exp\left(-B_s-\frac{1}{2}s\right)(1-(B_s+s))\,dB_s$$
$$-\frac{1}{2}\int_0^t \exp\left(-B_s-\frac{1}{2}s\right)(2-B_s-s)ds$$
$$= \int_0^t \exp\left(-B_s-\frac{1}{2}s\right)(1-(B_s+s))\,dB_s.$$

□

Exercise 7. Write the Ito differential of the following processes.

(1) $X_t = 2+t+\exp(B_t)$;
(2) $X_t = B_t^2$;
(3) $X_t = (t+\lambda, B_t)$,where $\lambda > 0$ is a constant;
(4) $X_t = B_1^2(t)+B_2^2(t)$, where $\{(B_1(t),B_2(t)), t\geq 0\}$ is a 2-dimensional Brownian motion;
(5) $X_t = (B_1(t)+B_2(t)+B_3(t), B_2(t)^2 - B_1(t)B_3(t))$, where $\{(B_1(t),B_2(t),B_3(t)), t\geq 0\}$ is a 3-dimensional Brownian motion.
(6) $X_t = \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(t)\right)$, where $\{(B_1(t),...,B_n(t)), t\geq 0\}$ is an n-dimensional Brownian motion, and $\alpha_0,\dots,\alpha_n\in\mathbb{R}$ are some constants.

Solution. (1) We consider the function

$$f : [0,T]\times\mathbb{R} \to \mathbb{R}$$
$$(t,x) \mapsto 2+t+\exp(x).$$

$$\partial_t f(t,x) = 1.$$
$$\partial_x f(t,x) = \exp(x).$$
$$\partial_{xx}^2 f(t,x) = \exp(x).$$

Ito:

$$X_t = 3 + \int_0^t ds + \int_0^t \exp(B_s) dB_s + \frac{1}{2}\int_0^t \exp(B_s) ds$$

Then,

$$dX_t = \left(1 + \frac{1}{2}\exp(B_t)\right) dt + \exp(B_t) dB_t.$$

(2) We consider the function

$$f : [0,T] \times \mathbb{R} \to \mathbb{R}$$
$$(t,x) \mapsto x^2.$$

$\partial_t f(t,x) = 0.$
$\partial_x f(t,x) = 2x.$
$\partial^2_{xx} f(t,x) = 2.$
Ito:

$$X_t = \int_0^t 2B_s dB_s + \frac{1}{2}\int_0^t 2ds$$

Then,

$$dX_t = 2B_t dB_t + dt.$$

(3) In this case, we have

$$dX_t = \begin{pmatrix} dt \\ dB_t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \begin{pmatrix} 0 \\ 1 \end{pmatrix} dB_t.$$

(4) We consider the function

$$f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$$
$$(t,x,y) \mapsto x^2 + y^2.$$

$\partial_t f(t,x,y) = 0.$
$\partial_x f(t,x,y) = 2x.$
$\partial_y f(t,x,y) = 2y.$
$\partial_{xy} f(t,x,y) = 0 = \partial_{yx} f(t,x,y).$
$\partial^2_{xx} f(t,x,y) = 2 = \partial^2_{yy} f(t,x,y).$
Ito:

$$\begin{aligned} X_t &= \int_0^t 2B_1(s)dB_1(s) + \int_0^t 2B_2(s)dB_2(s) + \frac{1}{2}\int_0^t 2dB_1(s)dB_1(s) + \frac{1}{2}\int_0^t 2dB_2(s)dB_2(s) \\ &= \int_0^t 2B_1(s)dB_1(s) + \int_0^t 2B_2(s)dB_2(s) + \int_0^t ds + \int_0^t ds \end{aligned}$$

Then,

$$dX_t = 2B_1(t)dB_1(t) + 2B_2(t)dB_2(t) + 2dt.$$

(5) We consider the following function

$$f_1 : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$$
$$(t, x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$$

and

$$\begin{aligned} f_2 : [0,T] \times \mathbb{R}^3 &\to \mathbb{R} \\ (t,x,y,z) &\mapsto y^2 - xz. \end{aligned}$$

For f_1, the derivatives
$\partial_t f(t, x_1, x_2, x_3) = 0.$
$\partial_{x_i} f(t, x_1, x_2, x_3) = 1$ for $i = 1, \dots, 3$.
$\partial^2_{x_k x_l} f(t, x_1, x_2, x_3) = 0$ for all $k, l = 1, \dots, 3$.
Ito:

$$f_1(t, B_1(t), B_2(t), B_3(t)) = \int_0^t dB_1(s) + \int_0^t dB_2(s) + \int_0^t dB_3(s).$$

Then,

$$df_1(t, B_1(t), B_2(t), B_3(t)) = dB_1(t) + dB_2(t) + dB_3(t).$$

Now, consider the following function

$$\begin{aligned} f_1 : [0,T] \times \mathbb{R}^3 &\to \mathbb{R} \\ (t, x_1, x_2, x_3) &\mapsto x_2^2 - x_1 x_3 \end{aligned}$$

For f_2, the derivatives
$\partial_t f(t, x_1, x_2, x_3) = 0.$
$\partial_{x_1} f(t, x_1, x_2, x_3) = -x_3.$
$\partial_{x_2} f(t, x_1, x_2, x_3) = 2x_2.$
$\partial_{x_3} f(t, x_1, x_2, x_3) = -x_1.$
$\partial^2_{x_k x_k} f(t, x_1, x_2, x_3) = 0$ for $k \neq 2$ and $\partial^2_{x_2 x_2} f(t, x_1, x_2, x_3) = 2.$
$\partial^2_{x_1 x_3} f(t, x_1, x_2, x_3) = \partial^2_{x_1 x_3} f(t, x_1, x_2, x_3) = -1$
$\partial_{x_1 x_2} f(t, x_1, x_2, x_3) = \partial_{x_2 x_3} f(t, x_1, x_2, x_3) = \partial_{x_3 x_2} f(t, x_1, x_2, x_3) = \partial_{x_2 x_3} f(t, x_1, x_2, x_3) = 0$
Ito:

$$\begin{aligned} f_2(t, B_1(t), B_2(t), B_3(t)) &= \int_0^t -B_3(s) dB_1(s) + \int_0^t 2B_2(s) dB_2(s) + \int_0^t -B_1(s) dB_3(s) \\ &\quad + \frac{1}{2}\int_0^t 2dB_2(s)dB_2(s) + \frac{1}{2}\int_0^t -dB_1(s)dB_3(s) + \frac{1}{2}\int_0^t -dB_3(s)dB_1(s) \\ &= \int_0^t -B_3(s) dB_1(s) + \int_0^t 2B_2(s) dB_2(s) + \int_0^t -B_1(s) dB_3(s) \\ &\quad + \int_0^t ds. \end{aligned}$$

Then,

$$df_2(t, B_1(t), B_2(t), B_3(t)) = dt - B_3(t) dB_1(t) + 2B_2(t) dB_2(t) - B_1(t) dB_3(t).$$

Finally,

$$\begin{aligned} dX_t &= (df_1(t, B_1(t), B_2(t), B_3(t)), df_2(t, B_1(t), B_2(t), B_3(t))) \\ &= (dB_1(t) + dB_2(t) + dB_3(t), dt - B_3(t) dB_1(t) + 2B_2(t) dB_2(t) - B_1(t) dB_3(t)) \end{aligned}$$

or $dX_t = \begin{pmatrix} 0 \\ 1 \end{pmatrix} dt + \begin{pmatrix} 1 & 1 & 1 \\ -B_3(t) & 2B_2(t) & -B_1(t) \end{pmatrix} \begin{pmatrix} dB_1(t) \\ dB_2(t) \\ dB_3(t) \end{pmatrix}$

(6) Now, consider the following function. For $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$ some constants.

$$\begin{aligned} f : [0,T] \times \mathbb{R}^n &\to \mathbb{R} \\ (t, x_1, \ldots, x_n) &\mapsto \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j x_j\right) \end{aligned}$$

$$\partial_t f(t, x_1, \ldots, x_n) = \alpha_0 \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j x_j\right).$$

$$\partial_{x_j} f(t, x_1, \ldots, x_n) = \alpha_j \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j x_j\right).$$

$$\partial^2_{x_j x_j} f(t, x_1, \ldots, x_n) = {\alpha_j}^2 \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j x_j\right).$$

$$\partial_{x_j x_l} f(t, x_1, \ldots, x_n) = \partial_{x_l x_j} f(t, x_1, \ldots, x_n) = \alpha_j \alpha_l \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j x_j\right).$$

Ito,

$$\begin{aligned} X_t = 1 &+ \int_0^t \alpha_0 \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) ds + \sum_{j=1}^n \int_0^t \alpha_j \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) dB_j(s) \\ &+ \frac{1}{2} \sum_{j=1}^n \int_0^t \alpha_j^2 \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) dB_j(s) dB_j(s) \\ &+ \frac{1}{2} \sum_{j \neq l} \int_0^t \alpha_j \alpha_l \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) dB_j(s) dB_l(s) \\ = 1 &+ \int_0^t \alpha_0 \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) ds + \sum_{j=1}^n \int_0^t \alpha_j \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) dB_j(s) \\ &+ \frac{1}{2} \sum_{j=1}^n \int_0^t \alpha_j^2 \exp\left(\alpha_0 t + \sum_{j=1}^n \alpha_j B_j(s)\right) ds \\ = 1 &+ \int_0^t \alpha_0 X_s ds + \sum_{j=1}^n \int_0^t \alpha_j X_s dB_j(s) + \frac{1}{2} \sum_{j=1}^n \int_0^t \alpha_j^2 X_s ds \\ = 1 &+ \int_0^t \left(\alpha_0 + \frac{1}{2} \sum_{j=1}^n \alpha_j^2\right) X_s ds + \int_0^t X_s \sum_{j=1}^n \alpha_j dB_j. \end{aligned}$$

Therefore,

$$dX_t = \left(\alpha_0 + \frac{1}{2} \sum_{j=1}^n \alpha_j^2\right) X_t dt + X_t \left(\sum_{j=1}^n \alpha_j dB_j\right).$$

□

Exercise 8. Consider two positive Ito processes:

$$dX_t^{(i)} = X_t^{(i)} \left(\alpha_t^{(i)} dt + \beta_t^{(i)} dB_t\right), \qquad i = 1, 2$$

Find the Ito differential of the log-ratio process

$$Y_t := \ln\left(\frac{X_t^{(1)}}{X_t^{(2)}}\right), \qquad t \geq 0.$$

Solution. Consider the function $f(x,y) = \ln\left(\frac{x}{y}\right)$. Then,

$$\begin{aligned} \partial_x f(x,y) &= \frac{1}{x} \qquad \partial_{xx} f(x,y) = -\frac{1}{x^2}; \\ \partial_y f(x,y) &= -\frac{1}{y} \qquad \partial_{yy} f(x,y) = \frac{1}{y^2}; \\ \partial_{xy} f(x,y) &= 0 \qquad \partial_{yx} f(x,y) = 0. \end{aligned}$$

$$\begin{aligned} dX_t^{(i)} dX_t^{(i)} &= (X_t^{(i)})^2 \left(\alpha_t^{(i)} dt + \beta_t^{(i)} dB_t\right)^2 \\ &= (X_t^{(i)})^2 \left((\alpha_t^{(i)})^2 dt dt + 2\alpha_t^{(i)} \beta_t^{(i)} dt dB_t + (\beta_t^{(i)})^2 dB_t dB_t\right) \\ &= (X_t^{(i)})^2 (\beta_t^{(i)})^2 dt. \end{aligned}$$

Ito:

$$\begin{aligned} Y_t &= c + \int_0^t \frac{1}{X_s^{(1)}} dX_s^{(1)} + \int_0^t -\frac{1}{X_s^{(2)}} dX_s^{(2)} + \frac{1}{2}\left(\int_0^t -\frac{1}{(X_s^{(1)})^2} dX_s^{(1)} dX_s^{(1)} + \int_0^t \frac{1}{(X_s^{(2)})^2} dX_s^{(2)} dX_s^{(2)}\right) \\ &= c + \int_0^t \left(\alpha_s^{(1)} ds + \beta_t^{(1)} dB_s\right) + \int_0^t -\left(\alpha_s^{(2)} ds + \beta_t^{(2)} dB_s\right) + \frac{1}{2}\left(\int_0^t -(\beta_s^{(1)})^2 ds + \int_0^t (\beta_s^{(2)})^2 ds\right) \\ &= c + \int_0^t \left(\alpha_s^{(1)} - \alpha_s^{(2)} - \frac{1}{2}\beta_s^{(1)} + \frac{1}{2}\beta_s^{(2)}\right) ds + \int_0^t \left(\beta_s^{(1)} - \beta_s^{(2)}\right) dB_s \end{aligned}$$

Thus, we get

$$dY_t = bt + a dB_t$$

where

$$b = \alpha_s^{(1)} - \alpha_s^{(2)} - \frac{1}{2}\beta_s^{(1)} + \frac{1}{2}\beta_s^{(2)} \qquad \text{and} \qquad a = \beta_s^{(1)} - \beta_s^{(2)}.$$

□

Exercise 9. $\{B_t, t \geq 0\}$ is a Brownian motion and $f : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is continuous, continuous differentiable in the first argument and twice continuously differentiable in the second argument ($f \in \mathcal{C}^{1,2}(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$). Moreover, we assume that

$$\sup_{(x,t)\in(0,\infty)\times\mathbb{R}} |f(t,x)| + |\partial_t f(t,x)| + |\partial_x f(t,x)| + |\partial_{xx}^2 f(t,x)| < \infty.$$

Prove that the process

$$M_t^f := f(t, B_t) - f(0,0) - \int_0^t \mathcal{L} f(s, B_s) ds, \qquad t \geq 0,$$

is a martingale with respect to the filtration generated by the Brownian motion, where

$$\mathcal{L} = \partial_t + \frac{1}{2}\partial_{xx}^2.$$

Solution. We can apply Ito

$$\begin{aligned} f(t, B_t) &= f(0,0) + \int_0^t \partial_s f(s, B_s) ds + \int_0^t \partial_x f(s, B_s) dB_s + \frac{1}{2}\int_0^t \partial_{xx}^2 f(s, B_s) ds \\ &= f(0,0) + \int_0^t \mathcal{L} f(s, B_s) ds + \int_0^t \partial_x f(s, B_s) dB_s. \end{aligned}$$

Thus,

$$M_t^f = \int_0^t \partial_x f(s, B_s) dB_s.$$

□

Exercise 10. $\{B_t, t \geq 0\}$ is a Brownian motion, $\Phi \in C_b^2(\mathbb{R})$ (twice continuously differentiable, Φ and its derivatives up to the second order are bounded), $\Phi(0) = 0$, $f, g \in L_{a,T}^2$. Prove that

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t f(s)dB_s\right)\left(\Phi\left(\int_0^t g(s)dB_s\right)\right)\right] &= \mathbb{E}\left[\int_0^t f(s)g(s)\Phi'\left(\int_0^s g(r)dB_r\right)ds\right] \\ &\quad + \frac{1}{2}\mathbb{E}\left[\int_0^t \left(\int_0^s f(r)dB_r\right)\Phi''\left(\int_0^s g(\alpha)dB_\alpha\right)g^2(s)ds\right].\end{aligned}$$

Solution. Consider the following Ito processes:

$$X_t = \int_0^t f(s)dB_s \qquad \text{and} \qquad Y_t = \int_0^t g(s)dB_s.$$

By hypothesis, we can apply Itos formula to $\Phi(Y_t)$ to obtain

$$\begin{aligned}\Phi(Y_t) &= \Phi(Y_0) + \int_0^t \Phi'(Y_s)ds + \frac{1}{2}\int_0^t \Phi''(Y_s)ds \\ &= \int_0^t \Phi'(Y_s)g(s)dB_s + \frac{1}{2}\int_0^t \Phi''(Y_s)g^2(s)ds.\end{aligned}$$

In differential form,

$$d\Phi(Y_t) = \Phi'(Y_t)g(t)dB_t + \frac{1}{2}\Phi''(Y_t)g^2(t)dt.$$

Thus, $\Phi(Y_s)$ is an Ito process with $u_t = \Phi'(Y_t)g(t)$ and $v_t = \frac{1}{2}\Phi''(Y_t)g^2 t$. By the integration by parts formula,

$$\begin{aligned}X_t\Phi(Y_t) &= \left(\int_0^t f(s)dB_s\right)\left(\Phi\left(\int_0^t g(s)dB_s\right)\right) \\ &= \int_0^t X_s d\phi(Y_s) + \int_0^t \Phi(Y_s)dX_s + \int_0^t \Phi'(Y_s)g(s)f(s)ds \\ &= \int_0^t X_s\Phi'(Y_s)g(s)dB_s + \frac{1}{2}\int_0^t X_s\Phi''(Y_s)g^2(s)ds + \int_0^t \Phi(Y_s)f(s)dB_s + \int_0^t \Phi'(Y_s)g(s)f(s)ds.\end{aligned}$$

Therefore,

$$\mathbb{E}\left[X_t\Phi(Y_t)\right] = \mathbb{E}\left[\int_0^t \Phi'(Y_s)g(s)f(s)ds\right] + \frac{1}{2}\mathbb{E}\left[\int_0^t X_s\Phi''(Y_s)g^2(s)ds\right]$$

□