Stochastic Calculus

Master in Advanced Mathematics 2017-2018

Claudia Cristina Reyes Montes de Oca

Set 2. Gaussian distributions, Gaussian processes, Brownian motion.

Exercise 1. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. Prove that for every $0 < s < t$ and $n \in \mathbb{N}$,

$$\mathbb{E}\left[(B_t - B_s)^{2n}\right] = \frac{(2n)!}{2^n n!}(t-s)^n.$$

Solution. Since $\{B_t, t \geq 0\}$ is a standard Brownian motion, we know that $X := B_t - B_s$ has a Gaussian law with mean 0 and variance $t - s$. Then, by definition of characteristic function for X, expanding around 0 in power series, we have

$$\begin{aligned}\varphi_X(u) &= \exp\left(-\frac{1}{2}u^2(t-s)\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}u^2(t-s)\right)^k \\ &= \sum_{k=0}^{\infty} \frac{i^{2k}u^{2k}(t-s)^k}{2^k k!}.\end{aligned}$$

Remark. Note that $\mathbb{E}[X^{2n}]$ is finite. In fact, if $X \sim \mathcal{N}(0,\sigma^2)$ then $\mathbb{E}[|X|^p]$ is finite for $p \geq 1$. Indeed, we have that there is a constant C_p[1] such that

$$\mathbb{E}[|X|^p] \leq C_p \mathbb{E}[\cosh(X)] = \frac{C_p}{2}\left(\mathbb{E}\left[e^X + e^{-X}\right]\right) = C_p e^{\frac{1}{2}\sigma^2} < +\infty$$

because $\mathbb{E}[e^{tX}] = \exp\left(\sigma^2 t^2/2\right)$ for all $t \in \mathbb{R}$.

Alternatively, we can argue as follows: since the function $x \mapsto |x|^{p+2}e^{-x^2/2}$ converge to 0 at infinity, then $|x|^p e^{-x^2/2}$ is of orden $\frac{1}{x^2}$ in $-\infty$ and $+\infty$. From this, it follows that the integral $\int_{\mathbb{R}} |x|^p e^{-x^2/2}dx$ converges.

Then, by a property of characteristic function, the characteristic function is of class $\mathcal{C}^{2n}$ and

$$i^{2n}\mathbb{E}\left[X^{2n}\right] = \varphi_X^{(2n)}(0).$$

Furthermore, differentiating with respect to u:

$$\begin{aligned}\varphi_X^{(1)}(u) &= \frac{i^2 2u(t-s)^1}{2^1 1!} + \frac{i^4 4u^3(t-s)^2}{2^2 2!} + \frac{i^6 6u^5(t-s)^3}{2^3 3!} + \frac{i^8 8u^7(t-s)^4}{2^4 4!} + \dots \\ \varphi_X^{(2)}(u) &= \frac{i^2(2\cdot 1)(t-s)^1}{2^1 1!} + \frac{i^4(4\cdot 3)u^2(t-s)^2}{2^2 2!} + \frac{i^6(6\cdot 5)u^4(t-s)^3}{2^3 3!} + \frac{i^8(8\cdot 7)u^6(t-s)^4}{2^4 4!} + \dots \\ \varphi_X^{(3)}(u) &= \frac{i^4(4\cdot 3\cdot 2)u(t-s)^2}{2^2 2!} + \frac{i^6(6\cdot 5\cdot 4)u^3(t-s)^3}{2^3 3!} + \frac{i^8(8\cdot 7\cdot 6)u^5(t-s)^4}{2^4 4!} + \dots \\ \varphi_X^{(4)}(u) &= \frac{i^4(4\cdot 3\cdot 2\cdot 1)(t-s)^2}{2^2 2!} + \frac{i^6(6\cdot 5\cdot 4\cdot 3)u^2(t-s)^3}{2^3 3!} + \frac{i^8(8\cdot 7\cdot 6\cdot 5)u^4(t-s)^4}{2^4 4!} + \dots \\ &\vdots \\ \varphi_X^{(2n)}(u) &= \frac{i^{2n}2n!(t-s)^n}{2^n n!} + u\cdot S(u),\end{aligned}$$

[1] This follows from the estimate: $|x|^p \leq C_p\cosh(x)$. Since $|x|$ and $\cosh(x)$ are even, we can suppose that $x \geq 0$. if $x = 0$ the inequality is trivial, so we can assume that $x > 0$. Let $f(x) = \frac{x^p}{\cosh(x)}$. Then $f'(x) = 0$ if and only if $x = 0$ or $x^{p-1}(p\cosh(x) - x\sinh(x)) = 0$. So, $p\cosh(x) - x\sinh(x) = 0$ if and only if $\frac{e^x}{2}(p-x) + \frac{e^{-x}}{2}(x-p) = 0$ if and only if $x = \pm p$. Then, the function f has its maximum at $f(-p) = f(p) = C_p$.

where $S(u)$ is a series that depends on u. Evaluating on $u = 0$, we obtain that

$$i^{2n}\mathbb{E}\left[X^{2n}\right] = \frac{i^{2n}2n!(t-s)^n}{2^n n!}.$$

Therefore,

$$\mathbb{E}\left[(B_t - B_s)^{2n}\right] = \frac{2n!}{2^n n!}(t-s)^n.$$

□

Exercise 2. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of Gaussian random variables. Prove that if there is a random variable X such that

$$\lim_{n\to\infty} \mathbb{E}\left[|X - X_n|^2\right] = 0,$$

then X is also Gaussian.

Solution. By hypothesis X_n converges to X in $\mathrm{L}^2(\Omega)$ therefore converges in law to X, and by the weak version of Levy's theorem: let $X, X_1, X_2, \ldots$ be a random variables on $\mathbb{R}^d$, then $X_n \to X$ in law if and only if $\varphi_{X_n}(t) \to \varphi_X(t)$, for all $t \in \mathbb{R}^d$.

Let $\mu_n = E(X_n)$ and $\sigma_n^2 = \mathrm{Var}(X_n)$ for all $n \in \mathbb{N}$, and since that X_n has a Gaussian law, we have that the characteristic function of X_n is:

$$\varphi_{X_n}(t) = \exp\left(it\mu_n - \frac{\sigma_n^2 t^2}{2}\right), \quad t \in \mathbb{R}.$$

By hypothesis we have that $\mathbb{E}[|X - X_n|^2] \to 0$ and by Minkowski's inequality we have that

$$0 \le \left|\mathbb{E}[|X|^2]^{1/2} - \mathbb{E}[|X_n|^2]^{1/2}\right| \le \mathbb{E}[|X - X_n|^2]^{1/2}$$

and therefore $\lim_{n\to\infty} \mathbb{E}[|X_n|^2] = \mathbb{E}[|X|^2]$. Also we note that convergence in $\mathrm{L}^2(\Omega)$ implies convergence in $\mathrm{L}^1(\Omega)$, since by Holder's inequality and as we have a finite measure space, we have that

$$0 \le \mathbb{E}\left[|X - X_n|\right] \le \left(\mathbb{E}\left[|X - X_n|^2\right]\right)^{1/2},$$

and therefore we have

$$\lim_{n\to\infty} \mathbb{E}[|X - X_n|] = 0,$$

i.e., $X_n \to X$ in L^1. Then, by triangle's inequality

$$0 \le \left|\mathbb{E}[X] - \mathbb{E}[X_n]\right| = \left|\mathbb{E}[X - X_n]\right| \le \mathbb{E}[|X - X_n|],$$

and taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathbb{E}\left[X_n\right] = \mathbb{E}\left[X\right].$$

By convergence in $\mathrm{L}^2(\Omega)$ we have that

$$\lim_{n\to\infty} \mathbb{E}\left[X_n^2\right] = \mathbb{E}[X^2],$$

and by convergence in $\mathrm{L}^1(\Omega)$,

$$\lim_{n\to\infty} \left(\mathbb{E}\left[X_n\right]\right)^2 = \left(\mathbb{E}\left[X\right]\right)^2.$$

By definition of variance we have

$$\begin{aligned}
\lim_{n\to\infty} \mathrm{Var}\left(X_n\right) &= \lim_{n\to\infty} \left(\mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2\right) \\
&= \mathbb{E}[X^2] - \left(\mathbb{E}\left[X\right]\right)^2 \\
&= \mathrm{Var}(X).
\end{aligned}$$

Therefore, we have

$$\lim_{n\to\infty} \mu_n = \mu \quad \text{and} \quad \lim_{n\to\infty} \sigma_n^2 = \sigma^2.$$

By continuity, we can deduce that

$$\lim_{n\to\infty} \varphi_{X_n}(t) = \varphi_X(t) = \exp\left(it\mu - \frac{\sigma^2 t^2}{2}\right),$$

then we can say that X is Gaussian because φ_X is the characteristic function of a Gaussian random variable.

□

Exercise 3. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion, and let $\alpha > 0$ be some constant. Show that the processes defined by

$$\{Y_t = tB(1/t), t > 0, Y_0 = 0\} \quad \text{and} \quad \left\{Z_t = \sqrt{\alpha}B(t/\alpha), t \geq 0, \right\},$$

are also standard Brownian motions.

Solution. In order to show that a stochastic process is a Brownian motion, we need to prove that is Gaussian, that have expectation zero and that their covariance function is $t \wedge s$.

Remark. We recall first that the Brownian motion is a Gaussian process, i.e, for every choice $t_1, \ldots, t_n \in \mathbb{R}_+$ every linear combination of the components of the random vector $(B_{t_1}, \ldots, B_{t_n})$ is a Gaussian random variable. This follows because there is a linear transformation $A \in Mat_{n\times n}(\mathbb{R})$ [2] such that

$$A(B_{t_1}, \ldots, B_{t_n}) = (B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$$

and, by definition, the random vector $(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ is Gaussian.

(1) For the process $\{Y_t = tB(1/t), t > 0, Y_0 = 0\}$, we have:

- Is a Gaussian process, since for every choice $t_1, \ldots, t_n \in \mathbb{R}_+$, we have the linear combination

$$\begin{aligned}\sum_{i=1}^n a_i Y_{t_i} &= \sum_{i=1}^n a_i t_i B(1/t_i) \\ &= \sum_{i=1}^n b_i B(s_i),\end{aligned}$$

  where $b_i = a_i t_i$ and $s_i = 1/t_i > 0$ for $i = 1\ldots, n$. In this form, we obtain that $(Y_{t_1}, \ldots, Y_{t_n})$ is a Gaussian vector.

- Now, we compute the expectation,

$$\mathbb{E}[Y_t] = \mathbb{E}[tB(1/t)] = t\mathbb{E}[B(1/t)] = 0.$$

- We compute the covariance,

$$\begin{aligned}\operatorname{Cov}(Y_t, Y_s) &= \mathbb{E}[Y_t Y_s] \\ &= \mathbb{E}[tB(1/t)sB(1/s)] \\ &= ts\mathbb{E}[B(1/t)B(1/s)] \\ &= ts\min\{1/t, 1/s\} \\ &= \min\{t, s\}.\end{aligned}$$

Therefore $\{Y_t = tB(1/t), t > 0, Y_0 = 0\}$ is a standard Brownian motion.

[2] The matrix A is the change of coordinates matrix (with respect to canonical basis) of the linear transformation $h : \mathbb{R}^n \to \mathbb{R}^n, (x_1, \ldots, x_n) \mapsto (x_1, x_2 - x_1, \ldots, x_n - x_{n-1})$.

(2) For the process $\{Z_t = \sqrt{\alpha}B(t/\alpha), t \geq 0,\}$, we have:

- Is a Gaussian process since, for every choice $t_1, \ldots, t_n \in \mathbb{R}_+$ and $\alpha > 0$, we have the linear combination

$$\begin{aligned}\sum_{i=1}^{n} a_i Z_{t_i} &= \sum_{i=1}^{n} a_i \sqrt{\alpha} B(t_i/\alpha) \\ &= \sum_{i=1}^{n} b_i B(s_i),\end{aligned}$$

where $b_i = a_i\sqrt{\alpha}$ and $s_i = t_i/\alpha > 0$ for $i = 1 \ldots, n$. In this form, we obtain that $(Z_{t_1}, \ldots, Z_{t_n})$ is a Gaussian vector.

- Now, we compute the expectation,

$$\mathbb{E}\left[Z_t\right] = \mathbb{E}\left[\sqrt{\alpha}B(t/\alpha)\right] = \sqrt{\alpha}\mathbb{E}\left[B(t/\alpha)\right] = 0.$$

- We compute the covariance,

$$\begin{aligned}\mathrm{Cov}\left(Z_t, Z_s\right) &= \mathbb{E}\left[Z_t Z_s\right] \\ &= \mathbb{E}\left[\sqrt{\alpha}B(t/\alpha)\sqrt{\alpha}B(t/\alpha)\right] \\ &= \alpha\mathbb{E}\left[B(t/\alpha)B(s/\alpha)\right] \\ &= \alpha \min\{t/\alpha, s/\alpha\} \\ &= \alpha/\alpha \min\{t, s\} \\ &= \min\{t, s\}\end{aligned}$$

Therefore $\{Z_t = \sqrt{\alpha}B(t/\alpha), t \geq 0,\}$ is a standard Brownian motion.

□

Remark. The following definition of independent increments of the standard Brownian motion, $\{B_t, t \geq 0\}$, is given in the lectures notes: for any $0 \leq s < t$, the random variable $B_t - B_s$ is independent of the σ-field generated by B_r, $0 \leq r \leq s$, $\sigma(B_r, 0 \leq r \leq s)$. This is equivalent to say that, for any choice $t_0 = 0 < t_1 < \cdots < t_n$, the variables $B_{t_i} - B_{t_{i-1}}$, $i \leq n$ are independent. We refer to Le Gall "Brownian Motion, Martingales and Stochastic Calculus" Prop. 2.3.

In the following, we take both definitions as the same.

Exercise 4. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. Prove that the process defined by $\{Y_t = B_{2t} - B_t, t \geq 0\}$ is not a Brownian motion.

Solution. For the process $\{Y_t = B_{2t} - B_t, t \geq 0\}$ we have,

$$Y_0 = B_0 - B_0 = 0 \text{ a.s.}$$

Suppose that $\{Y_t, t \geq 0\}$ has independent increments, then for $0 < t < s < 2t$, we have

$$\mathbb{E}\left[Y_t(Y_{2t} - Y_s)\right] = \mathbb{E}\left[Y_t\right]\mathbb{E}\left[Y_{2t} - Y_s\right] = 0.$$

since Y_t has expectation zero, in effect, $\mathbb{E}[Y_t] = \mathbb{E}[B_{2t} - B_t] = \mathbb{E}[B_{2t}] - \mathbb{E}[B_t] = 0$. Now, by definition of covariance function, we have

$$\begin{aligned}
\mathbb{E}\left[Y_t\left(Y_{2t} - Y_s\right)\right] &= \mathbb{E}\left[Y_t Y_{2t} - Y_t Y_s\right] \\
&= \mathbb{E}\left[Y_t Y_{2t}\right] - \mathbb{E}\left[Y_t^2\right] \\
&= \mathbb{E}\left[(B_{2t} - B_t)(B_{4t} - B_{2t})\right] - \mathbb{E}\left[(B_{2t} - B_t)(B_{2s} - B_s)\right] \\
&= \mathbb{E}[B_{2t}B_{4t}] - \mathbb{E}[B_{2t}B_{2t}] - \mathbb{E}[B_t B_{4t}] + \mathbb{E}[B_t B_{2t}] - \mathbb{E}[B_{2t}B_{2s}] + \mathbb{E}[B_{2t}B_s] + \mathbb{E}[B_t B_{2s}] - \mathbb{E}[B_t B_s] \\
&= \min\{2t, 4t\} - 2t - \min\{t, 4t\} + \min\{t, 2t\} - \min\{2t, 2s\} + \min\{2t, s\} + \min\{t, 2s\} - \min\{t, s\} \\
&= -2t + s
\end{aligned}$$

From this, we have that $s = 2t$, this in not true because we are assuming that $s < 2t$. Therefore $\{Y_t, t \geq 0\}$ does not have independent increments and we can conclude that $\{Y_t, t \geq 0\}$ is not a Brownian motion. $\square$

Exercise 5. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. Define $X = \{X_t = e^{-\alpha \frac{t}{2}} B_{e^{\alpha t}}, t \geq 0\}$.

(1) Prove that X is a Gaussian process.
(2) Compute $\mathbb{E}(X_t)$ and $\mathrm{Cov}(X_s X_t)$, for any $s, t \geq 0$.
(3) Find the joint density of the vector $(X_{t_1}, \ldots, X_{t_n})$ for any $0 < t_1 < \cdots < t_n < \infty$.
(4) Show that the probability law of $(X_{t_1}, \ldots, X_{t_n})$ coincide with that of $(X_{t_1+h}, \ldots, X_{t_n+h})$, for any $h > 0$.

Solution. (1) For prove that X is a Gaussian process, we note that, for every choice $t_1, \ldots, t_n \in \mathbb{R}_+$, we have the linear combination

$$\begin{aligned}
\sum_{i=1}^{n} a_i X_{t_i} &= \sum_{i=1}^{n} a_i e^{-\alpha \frac{t_i}{2}} B(e^{\alpha t_i}) \\
&= \sum_{i=1}^{n} b_i B(s_i),
\end{aligned}$$

where $b_i = a_i e^{-\alpha \frac{t_i}{2}}$ and $s_i = e^{\alpha t_i} > 0$ for $i = 1 \ldots, n$. In this form, we obtain that $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector.

(2) Now, we compute the expectation and covariance. For $\mathbb{E}(X_t)$ with $t \geq 0$, we have

$$\mathbb{E}[X_t] = \mathbb{E}\left[e^{-\alpha \frac{t}{2}} B_{e^{\alpha t}}\right] = e^{-\alpha \frac{t}{2}} \mathbb{E}\left[B_{e^{\alpha t}}\right] = 0.$$

For $\mathrm{Cov}(X_s X_t)$, we take any $s, t \geq 0$

$$\begin{aligned}
\mathrm{Cov}(X_s X_t) &= \mathbb{E}[X_t X_s] \\
&= \mathbb{E}\left[e^{-\alpha \frac{t}{2}} B_{e^{\alpha t}} e^{-\alpha \frac{s}{2}} B_{e^{\alpha s}}\right] \\
&= e^{-\alpha \frac{t+s}{2}} \mathbb{E}\left[B_{e^{\alpha t}} B_{e^{\alpha s}}\right] \\
&= e^{-\alpha \frac{t+s}{2}} \min\{e^{\alpha t}, e^{\alpha s}\} \\
&= e^{-\frac{\alpha}{2}|t-s|}.
\end{aligned}$$

(3) We need to find the joint density of the vector $(X_{t_1}, \dots, X_{t_n})$ for any $0 < t_1 < \cdots , t_n < \infty$,

$$\begin{aligned}
F_{X_{t_1},\dots,X_{t_n}}(x_1, \dots, x_n) &= \mathbb{P}(X_{t_1} \le x_1, \dots, X_{t_n} \le x_n) \\
&= \mathbb{P}\left(e^{-\alpha\frac{t_1}{2}} B_{e^{\alpha t_1}} \le x_1, \dots, e^{-\alpha\frac{t_n}{2}} B_{e^{\alpha t_n}} \le x_n\right) \\
&= \mathbb{P}\left(B_{e^{\alpha t_1}} \le e^{\alpha\frac{t_1}{2}} x_1, \dots, B_{e^{\alpha t_n}} \le e^{\alpha\frac{t_n}{2}} x_n\right) \\
&= \int_{-\infty}^{e^{\alpha\frac{t_1}{2}} x_1} \cdots \int_{-\infty}^{e^{\alpha\frac{t_n}{2}} x_n} f_{B_{e^{\alpha t_1}},\dots,B_{e^{\alpha t_n}}}(y_1, \dots, y_n) dy_1 \cdots dy_n \\
&= F_{B_{e^{\alpha t_1}},\dots,B_{e^{\alpha t_n}}}\left(e^{\alpha\frac{t_1}{2}} x_1, \dots, e^{\alpha\frac{t_n}{2}} x_n\right).
\end{aligned}$$

Now, we compute the joint density function of the vector $(X_{t_1}, \dots, X_{t_n})$. Consider the transformation:

$$\begin{aligned}
g : \mathbb{R}^n &\to \mathbb{R}^n \\
(x_1, \dots, x_n) &\mapsto \left(e^{-\alpha\frac{t_1}{2}} x_1, \dots, e^{-\alpha\frac{t_n}{2}} x_n\right)
\end{aligned}$$

Then, for $(y_1, \dots, y_n) \in \mathbb{R}^n$,

$$g^{-1}(y_1, \dots, y_n) = \left(e^{\alpha\frac{t_1}{2}} y_1, \dots, e^{\alpha\frac{t_n}{2}} y_n\right),$$

and by the change of variable formula, we have

$$f_{X_{t_1},\dots,X_{t_n}}(x_1, \dots, x_n) = |\det(Jg^{-1})| f_{B_{e^{\alpha t_1}},\dots,B_{e^{\alpha t_n}}}\left(e^{\alpha\frac{t_1}{2}} x_1, \dots, e^{\alpha\frac{t_n}{2}} x_n\right),$$

where Jg^{-1} is the Jacobian of the transformation g^{-1}. So, in this form, we have

$$Jg^{-1} = \begin{pmatrix} e^{\alpha\frac{t_1}{2}} & 0 & \cdots & 0 \\ 0 & e^{\alpha\frac{t_2}{2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\alpha\frac{t_n}{2}} \end{pmatrix},$$

and

$$\det(Jg^{-1}) = \prod_{j=1}^{n} e^{\alpha\frac{t_j}{2}}.$$

Thus,

$$f_{X_{t_1},\dots,X_{t_n}}(x_1, \dots, x_n) = \prod_{j=1}^{n} e^{\alpha\frac{t_j}{2}} f_{B_{e^{\alpha t_1}},\dots,B_{e^{\alpha t_n}}}\left(e^{\alpha\frac{t_1}{2}} x_1, \dots, e^{\alpha\frac{t_n}{2}} x_n\right).$$

In the following, we compute the law of the random vector $(B_{s_1}, \dots, B_{s_n})$, where $0 < s_1 < \cdots < s_n < \infty$. We define the following Gaussian random variables:

$$Z_1 = B_{s_1},\ Z_2 = B_{s_2} - B_{s_1},\ \dots\ , Z_n = B_{s_n} - B_{s_{n-1}}.$$

By definition, we have that $Z_i \sim \mathcal{N}(0, s_i - s_{i-1})$ for $i = 1, \dots, n$, where we define $s_0 = 0$, are independent, and random vector $(Z_1, \dots, Z_n)$ is Gaussian; and therefore admits the density given by

$$f_{Z_1,\dots,Z_n}(z_1, \dots, z_n) = (2\pi)^{-n/2} \det(\Lambda_n)^{-1/2} \exp\left(-\frac{1}{2} z^T \Lambda_n^{-1} z\right),$$

where we write $z = (z_1, \dots, z_n)$ and

$$\Lambda_n = (\mathrm{Cov}(Z_i, Z_j))_{i,j} = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 - s_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s_n - s_{n-1} \end{pmatrix}.$$

Then, we have that

$$\det(\Lambda_n) = \prod_{j=1}^{n} (s_j - s_{j-1}) \neq 0,$$

and

$$\Lambda_n^{-1} = \begin{pmatrix} \frac{1}{s_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{s_2-s_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{s_n-s_{n-1}} \end{pmatrix}.$$

We compute $z^T \Lambda_n^{-1} z$:

$$z^T \Lambda_n^{-1} z = \frac{z_1^2}{s_1} + \frac{z_2^2}{s_2 - s_1} + \cdots + \frac{z_n^2}{s_n - s_{n-1}} = \sum_{j=1}^{n} \frac{z_j^2}{s_j - s_{j-1}}.$$

Therefore

$$f_{Z_1,\dots,Z_n}(z_1, \dots, z_n) = (2\pi)^{-n/2} \left(\prod_{j=1}^{n} (s_j - s_{j-1}) \right)^{-1/2} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{z_j^2}{s_j - s_{j-1}} \right).$$

Now, consider the following transformation:

$$\begin{aligned} h : \mathbb{R}^n &\to \mathbb{R}^n \\ (z_1, \dots, z_n) &\mapsto (z_1, z_1 + z_2, \dots, z_1 + z_2 + \cdots + z_n). \end{aligned}$$

Then, for $(y_1, \dots, y_n) \in \mathbb{R}^n$,

$$h^{-1}(y_1, \dots, y_n) = (y_1, y_2 - y_1, \dots, y_n - y_{n-1})$$

and

$$Jh^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

From this, $\det(Jh^{-1}) = 1$.
Therefore, we have that $h(Z_1, \dots, Z_n) = (B_{s_1}, B_{s_2}, \dots, B_{s_n})$. Then, by writing $(y_1, \dots, y_n) = h(z_1, \dots, z_n)$ and, by the change of variable formula, we have

$$\begin{aligned} f_{B_{s_1},\dots,B_{s_n}}(y_1, \dots, y_n) &= f_{Z_1,\dots,Z_n}(h^{-1}(y_1, y_2, \dots, y_n)) \det(Jh^{-1}) \\ &= (2\pi)^{-n/2} \left(\prod_{j=1}^{n} (s_j - s_{j-1}) \right)^{-1/2} \exp\left(-\frac{1}{2} \sum_{j=1}^{n} \frac{(y_j - y_{j-1})^2}{s_j - s_{j-1}} \right) \end{aligned}$$

with $y_0 = 0$.

Now, since $0 < t_1 < \cdots < t_n < \infty$, then $0 < e^{\alpha t_1} < \cdots < e^{\alpha t_n} < \infty$. We take $\{s_j = e^{\alpha t_j}\}_{j=1}^n$ and $\{y_j = e^{\alpha \frac{t_j}{2}} x_j\}_{j=1}^n$. We obtain that $f_{B_{e^{\alpha t_1}},\dots,B_{e^{\alpha t_n}}}\left(e^{\alpha\frac{t_1}{2}}x_1,\dots,e^{\alpha\frac{t_n}{2}}x_n\right)$ is equal to

$$\frac{1}{(2\pi)^{n/2}}\left(\prod_{j=1}^n \left(e^{\alpha t_j}-e^{\alpha t_{j-1}}\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(e^{\alpha\frac{t_j}{2}}x_j - e^{\alpha\frac{t_{j-1}}{2}}x_{j-1})^2}{e^{\alpha t_j}-e^{\alpha t_{j-1}}}\right),$$

where, by convention, $t_0 = -\infty$ and $x_0 = 0$.

Thus, the joint density $f_{X_{t_1},\dots,X_{t_n}}(x_1,\dots,x_n)$ of $(X_{t_1},\dots,X_{t_n})$ is

$$(2\pi)^{-n/2}\prod_{j=1}^n e^{\alpha\frac{t_j}{2}}\left(\prod_{j=1}^n \left(e^{\alpha t_j}-e^{\alpha t_{j-1}}\right)\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(e^{\alpha\frac{t_j}{2}}x_j - e^{\alpha\frac{t_{j-1}}{2}}x_{j-1})^2}{e^{\alpha t_j}-e^{\alpha t_{j-1}}}\right)$$

which is equal to

$$(2\pi)^{-n/2}\prod_{j=1}^n \left(1-e^{-\alpha(t_j-t_{j-1})}\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(x_j - e^{-\alpha\frac{t_j-t_{j-1}}{2}}x_{j-1})^2}{1-e^{-\alpha(t_j-t_{j-1})}}\right).$$

The last part is because of the following computations:

$$\begin{aligned}\prod_{j=1}^n e^{\alpha\frac{t_j}{2}}\left(\prod_{j=1}^n \left(e^{\alpha t_j}-e^{\alpha t_{j-1}}\right)\right)^{-1/2} &= \left(\prod_{j=1}^n e^{-\alpha t_j}\left(e^{\alpha t_j}-e^{\alpha t_{j-1}}\right)\right)^{-1/2}\\ &= \left(\prod_{j=1}^n (1-e^{-\alpha(t_j-t_{j-1})})\right)^{-1/2}\end{aligned}$$

and

$$\begin{aligned}\frac{(e^{\alpha\frac{t_j}{2}}x_j - e^{\alpha\frac{t_{j-1}}{2}}x_{j-1})^2}{e^{\alpha t_j}-e^{\alpha t_{j-1}}} &= \frac{e^{-\alpha t_j}}{e^{-\alpha t_j}}\,\frac{(e^{\alpha\frac{t_j}{2}}x_j - e^{\alpha\frac{t_{j-1}}{2}}x_{j-1})^2}{e^{\alpha t_j}-e^{\alpha t_{j-1}}}\\ &= \frac{((e^{-\alpha\frac{t_j}{2}})(e^{\alpha\frac{t_j}{2}}x_j - e^{\alpha\frac{t_{j-1}}{2}}x_{j-1}))^2}{1-e^{-\alpha(t_j-t_{j-1})}}\\ &= \frac{(x_j - e^{-\alpha\frac{t_j-t_{j-1}}{2}}x_{j-1})^2}{1-e^{-\alpha(t_j-t_{j-1})}}.\end{aligned}$$

(4) For any $h > 0$, by the previous point we have that the joint density of the random vector $(X_{t_1+h},\dots,X_{t_n+h})$ is given by

$$\begin{aligned}&f_{X_{t_1+h},\dots,X_{t_n+h}}(x_1,\dots,x_n)\\ &\quad= (2\pi)^{-n/2}\prod_{j=1}^n \left(1-e^{-\alpha(t_j+h-t_{j-1}-h)}\right)^{-1/2} \exp\left(-\frac{1}{2}\sum_{j=1}^n \frac{(x_j - e^{-\alpha\frac{t_j+h-t_{j-1}-h}{2}}x_{j-1})^2}{1-e^{-\alpha(t_j+h-t_{j-1}-h)}}\right)\end{aligned}$$

which is clearly the same as $(X_{t_1},\dots,X_{t_n})$.

$\square$

Exercise 6. Let $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ be two independent standard Brownian motions. What are the properties inherited by the process $Z = \{X_t - Y_t, t \geq 0\}$? Prove the following:

(1) Z has independent increments.
(2) Z has stationary increments.
(3) Z has continuous sample paths.
(4) The sample paths of Z are γ-Holder continuous for any $\gamma \in \left(0, \frac{1}{2}\right)$.
(5) Z has finite quadratic variation.

Solution. Recall that two stochastic process X and Y are independent, if for all finite set of times $I, J \subset \mathbb{R}_+$, the random vectors X_I and Y_J are independent.

For the stochastic process Z, we have that $Z = X_0 - Y_0 = 0$ a.s., and that for every choice $t_1, \dots, t_n \in \mathbb{R}_+$

$$\begin{aligned}\sum_{i=1}^n a_i Z_{t_i} &= \sum_{i=1}^n a_i (X_{t_i} - Y_{t_i}) \\ &= \sum_{i=1}^n a_i X_{t_i} - \sum_{i=1}^n a_i Y_{t_i},\end{aligned}$$

where X_{t_i} and Y_{t_i} are Gaussian with law $\mathcal{N}(0, t_i)$ for $i = 1 \dots, n$. In this form, we obtain that $(Z_{t_1}, \dots, Z_{t_n})$ is a Gaussian vector, since that $\sum_{i=1}^n a_i X_{t_i}$ and $\sum_{i=1}^n a_i Y_{t_i}$ are independent random variables, by hypothesis.

(1) For every choice of nonnegative real numbers $0 \leq s_1 < t_1 \leq s_2 < t_2 < \cdots \leq s_n < t_n < \infty$ we know by hypothesis that

$$X_1 := X_{t_1} - X_{s_1}, \dots, X_n := X_{t_n} - X_{s_n}$$

are independent and that

$$Y_1 := Y_{t_1} - Y_{s_1}, \dots, Y_n := Y_{t_n} - Y_{s_n}$$

are also independent. We want to prove that the increments

$$Z_1 := Z_{t_1} - Z_{s_1}, \dots, Z_n := Z_{t_n} - Z_{s_n}$$

are independent. Now, since X_i and Y_i, for $i = 1, \dots, n$, are independent Gaussian random variables, the vector $(X_1, Y_1, \dots, X_n, Y_n)$ is Gaussian. We put

$$\Gamma_1 = (X_1, Y_1),\ \Gamma_2 = (X_2, Y_2),\ \dots\ , \Gamma_n = (X_n, Y_n).$$

By independence, $\mathrm{Cov}(X_i, X_j) = \mathrm{Cov}(Y_i, Y_j) = \mathrm{Cov}(X_i, Y_j) = 0$ for all $i \neq j$. Indeed,

$$\begin{aligned}\mathrm{Cov}(X_i, X_j) &= E[X_i X_j] \\ &= \mathbb{E}[(X_{t_i} - X_{s_i})(X_{t_j} - X_{s_j})] \\ &= \mathbb{E}[X_{t_i} - X_{s_i}]\mathbb{E}[X_{t_j} - X_{s_j}] = 0.\end{aligned}$$

Similarly, we have that $\mathrm{Cov}(Y_i, Y_j) = 0$.

Also,

$$\begin{aligned}\mathrm{Cov}(X_i, Y_j) &= E[X_i Y_j] \\ &= \mathbb{E}[(X_{t_i} - X_{s_i})(Y_{t_j} - Y_{s_j})] \\ &= \mathbb{E}[X_{t_i} - X_{s_i}]\mathbb{E}[Y_{t_j} - Y_{s_j}] = 0,\end{aligned}$$

by the hypothesis of independence of the Brownian motions $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$.

We can conclude that the random vectors $\Gamma_1, \dots, \Gamma_n$ are jointly independent. Finally, considering the continuous function

$$\begin{aligned} f &: \mathbb{R}^2 \to \mathbb{R} \\ &(x, y) \mapsto x - y\end{aligned}$$

then $f(\Gamma_1) = Z_1, f(\Gamma_2) = Z_2, \dots, f(\Gamma_n) = Z_n$ are independent random variables. Therefore, Z has independent increments.

(2) We have to prove that, for $0 \leq s < t$, the random variable $Z_t - Z_s \overset{d}{=} \mathcal{N}(0, 2(t-s))$. Siince $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are independent standard Brownian motions and we have that $X_t - X_s \sim \mathcal{N}(0, t-s)$ and that $Y_t - Y_s \sim \mathcal{N}(0, t-s)$, we can conclude that:

$$Z_t - Z_s = (X_t - X_s) - (Y_t - Y_s) \overset{d}{=} \mathcal{N}(0, (t-s) + (t-s)) = \mathcal{N}(0, 2(t-s)).$$

(3) For $\omega \in \Omega$ fixed, we know that

$$t \mapsto X_t(\omega) \quad \text{and} \quad t \mapsto Y_t(\omega)$$

are continuous because they are Brownian motions, then the function

$$t \mapsto Z_t(\omega) = X_t(\omega) - Y_t(\omega)$$

is continuous.

(4) Let $\gamma \in (0, \frac{1}{2})$. For $s < t$,

$$\begin{aligned} |Z_t - Z_s| &= |(X_t - X_s) + (Y_s - Y_t)| \\ &\leq |X_t - X_s| + |Y_s - Y_t| \end{aligned}$$

Then, we have

$$\sup_{t \neq s} \frac{|Z_t - Z_s|}{|t-s|^\gamma} \leq \sup_{t \neq s} \frac{|X_t - X_s|}{|t-s|^\gamma} + \sup_{t \neq s} \frac{|Y_t - Y_s|}{|t-s|^\gamma}.$$

By hypothesis, we know that $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ have Holder continuous sample paths of degree $\gamma \in (0, \frac{1}{2})$ almost surely, then there exists constants $B, C > 0$ such that

$$\sup_{t \neq s} \frac{|X_t - X_s|}{|t-s|^\gamma} \leq B < +\infty \ \text{ a.s.} \qquad \text{and} \qquad \sup_{t \neq s} \frac{|Y_t - Y_s|}{|t-s|^\gamma} \leq C < +\infty \ \text{ a.s.}$$

Then, we can conclude that

$$\sup_{t \neq s} \frac{|Z_t - Z_s|}{|t-s|^\gamma} \leq B + C < +\infty \ \text{ a.s.},$$

i.e., Z has γ-Holder continuous sample paths.

(5) We fix a time interval $[0, T]$ and we consider the sequence of partitions $(\Pi_n)_{n \in \mathbb{N}}$ given by the points $\Pi_n = \{t_0^n = 0 \leq t_1^n \leq \cdots \leq t_{r_n}^n = T\}$ such that $|\Pi_n| := \max\{t_{k+1}^n - t_k^n : 0 \leq k \leq r_n\}$ converges to 0 as $n \to \infty$.

We want to prove that the sequence $\{\sum_{k=1}^{r_n} (Z_{t_k^n} - Z_{t_{k-1}^n})^2\}_{n \geq 1}$ converges in $\mathrm{L}^2(\Omega)$ to a deterministic random variable M. That is,

$$\lim_{n \to \infty} \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (Z_{t_k^n} - Z_{t_{k-1}^n})^2 - M\right)^2\right] = 0.$$

Recall that, since $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are standard Brownian motions, then the sequences $\{\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})^2\}_{n \geq 1}$ and $\{\sum_{k=1}^{r_n} (Y_{t_k^n} - Y_{t_{k-1}^n})^2\}_{n \geq 1}$ both converges in $\mathrm{L}^2(\Omega)$ to a deterministic random variable T, i.e.,

$$\lim_{n \to \infty} \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 - T\right)^2\right] = 0 \ \text{ and } \ \lim_{n \to \infty} \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (Y_{t_k^n} - Y_{t_{k-1}^n})^2 - T\right)^2\right] = 0.$$

Then, we have

$$\begin{aligned}
\sum_{k=1}^{r_n} \left(Z_{t_k^n} - Z_{t_{k-1}^n} \right)^2 - 2T &= \sum_{k=1}^{r_n} \left((X_{t_k^n} - Y_{t_k^n}) - (X_{t_{k-1}^n} - Y_{t_{k-1}^n}) \right)^2 - 2T \\
&= \sum_{k=1}^{r_n} \left((X_{t_k^n} - X_{t_{k-1}^n}) - (Y_{t_k^n} - Y_{t_{k-1}^n}) \right)^2 - 2T \\
&= \left(\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 - T \right) + \left(\sum_{k=1}^{r_n} (Y_{t_k^n} - Y_{t_{k-1}^n})^2 - T \right) \\
&\quad - 2 \sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}).
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{k=1}^{r_n} \left(Z_{t_k^n} - Z_{t_{k-1}^n} \right)^2 - 2T \right)^2 \right] &\ll \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})^2 - T \right)^2 \right] + \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (Y_{t_k^n} - Y_{t_{k-1}^n})^2 - T \right)^2 \right] \\
&\quad + \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}) \right)^2 \right]
\end{aligned}$$

where the symbol $\ll$ means inequality up to a constant that does not depends on n (This follows from the estimate: For any real numbers $a, b \in \mathbb{R}$, $(a+b)^2 \leq 4(a^2 + b^2)$).

By hypothesis, the first two terms converges to 0 as $n \to \infty$. Now, for the third term:

$$\begin{aligned}
\mathbb{E}\left[\left(\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n}) \right)^2 \right] &= \sum_{k=1}^{r_n} \mathbb{E}\left[(X_{t_k^n} - X_{t_{k-1}^n})^2 (Y_{t_k^n} - Y_{t_{k-1}^n})^2 \right] \\
&\quad + 2 \sum_{i,j} \mathbb{E}\left[(X_{t_i^n} - X_{t_{i-1}^n})(Y_{t_j^n} - Y_{t_{j-1}^n}) \right] \\
&= \sum_{k=1}^{r_n} \mathbb{E}\left[(X_{t_k^n} - X_{t_{k-1}^n})^2 \right] \mathbb{E}\left[(Y_{t_k^n} - Y_{t_{k-1}^n})^2 \right] \\
&\quad + 2 \sum_{i,j} \mathbb{E}\left[(X_{t_i^n} - X_{t_{i-1}^n}) \right] \mathbb{E}\left[(Y_{t_j^n} - Y_{t_{j-1}^n}) \right] \\
&= \sum_{k=1}^{r_n} (t_k^n - t_{k-1}^n)^2 \\
&\leq |\Pi^n| \sum_{k=1}^{r_n} (t_k^n - t_{k-1}^n) \\
&= T|\Pi^n| \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where the second equality follows from the independence of the processes $\{X_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$, and the third equality follows from the fact that the random varibles $X_{t_i^n} - X_{t_{i-1}^n}$ and $Y_{t_j^n} - Y_{t_{j-1}^n}$ have zero mean. Therefore, the sequence $\{\sum_{k=1}^{r_n} (X_{t_k^n} - X_{t_{k-1}^n})(Y_{t_k^n} - Y_{t_{k-1}^n})\}_{n \geq 1}$ converges to 0 in $\mathrm{L}^2(\Omega)$.

Thus, we have that $\{\sum_{k=1}^{r_n} (Z_{t_k^n} - Z_{t_{k-1}^n})^2\}_{n \geq 1}$ converges to $M = 2T$ in $\mathrm{L}^2(\Omega)$ as $n \to \infty$, i.e.,

$$\lim_{n \to \infty} \mathbb{E}\left[\left(\sum_{k=1}^{r_n} (Z_{t_k^n} - Z_{t_{k-1}^n})^2 - 2T \right)^2 \right] = 0.$$

$\square$

Exercise 7. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. Let $a > 0$ and $b \in \mathbb{R}$ be two constants. Set

$$Y_t = c + bt + aB_t, \qquad t \geq 0.$$

Prove the following facts:

(1) $Y_0 = c$ a.s.
(2) $\{Y_t, t \geq 0\}$ has independent increments.
(3) The sample paths of $\{Y_t, t \geq 0\}$ are continuous a.s.
(4) What is the law of the increment $Y_t - Y_s$?
(5) Show that the density of Y_t, for any $t \geq 0$, is given by

$$p_t(x, y) = \frac{1}{\sqrt{2\pi a^2 t}} \exp\left(\frac{(y - x - bt)^2}{2a^2 t}\right), \qquad (x, y) \in \mathbb{R}^2.$$

Show that

$$p_{s+t}(x, y) = \int_{\mathbb{R}} p_s(x, z) p_t(z, y) dz.$$

(6) What are the differences between the sample paths of $\{Y_t, t \geq 0\}$ with those of brownian motion.

Solution. (1) We have that

$$\mathbb{P}(Y_0 = c + aB_0) = \mathbb{P}\left(B_0 = \frac{Y_0 - c}{a}\right).$$

Now, by definition of the Brownian motion, $P(B_0 = 0) = 1$. So,

$$1 = \mathbb{P}\left(\frac{Y_0 - c}{a} = 0\right) = \mathbb{P}(Y_0 - c = 0) = \mathbb{P}(Y_0 = c).$$

Therefore, $Y_0 = c$ almost surely.

(2) We need to prove that, for any $0 \leq s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_n < t_n < \infty$ the increments

$$Y_1 := Y_{t_1} - Y_{s_1}, Y_2 := Y_{t_2} - Y_{s_2}, \ldots, Y_n := Y_{t_n} - Y_{s_n}$$

are independent. For this, we use the characteristic function as follows.

$$\begin{aligned}
\varphi_{Y_1,\ldots,Y_n}(u_1, \ldots, u_n) &= \mathbb{E}\left[\exp\left(i \sum_j u_j Y_j\right)\right] \\
&= \mathbb{E}\left[\exp\left(i \sum_j u_j (Y_{t_j} - Y_{s_j})\right)\right] \\
&= \mathbb{E}\left[\exp\left(i \sum_j u_j (b(t_j - s_j) + a(B_{t_j} - B_{s_j}))\right)\right] \\
&= \mathbb{E}\left[\exp\left(i \sum_j u_j b(t_j - s_j)\right) \exp\left(i \sum_j u_j a(B_{t_j} - B_{s_j})\right)\right] \\
&= \mathbb{E}\left[\exp\left(i \sum_j u_j b(t_j - s_j)\right)\right] \mathbb{E}\left[\exp\left(i \sum_j u_j a(B_{t_j} - B_{s_j})\right)\right] \\
&= \prod_j \mathbb{E}\left[\exp(i u_j b(t_j - s_j))\right] \mathbb{E}\left[\exp\left(i \sum_j u_j a(B_{t_j} - B_{s_j}))\right)\right]
\end{aligned}$$

Now, by the independence of the increments of the Brownian motion, we have

$$\mathbb{E}\left[\exp\left(i\sum_j u_j a(B_{t_j} - B_{s_j})\right)\right] = \prod_j \mathbb{E}\left[\exp(iau_j(B_{t_j} - B_{s_j}))\right].$$

We then have,

$$\begin{aligned}
\varphi_{Y_1,\ldots,Y_n}(u_1,\ldots,u_n) &= \prod_j \mathbb{E}\left[\exp(iu_j b(t_j - s_j))\right] \prod_j \mathbb{E}\left[\exp(iau_j(B_{t_j} - B_{s_j}))\right] \\
&= \prod_j \mathbb{E}\left[\exp\left(iu_j\left(b(t_j - s_j) + a(B_{t_j} - B_{s_j})\right)\right)\right] \\
&= \prod_j \varphi_{Y_{t_j} - Y_{s_j}}(u_j).
\end{aligned}$$

Therefore, $\{Y_t, t \geq 0\}$ has independent increments.

(3) For the continuity of $\{Y_t, t \geq 0\}$ we use that $\{B_t, t \geq 0\}$ is a standard Brownian motion, then the sample paths of B_t are continuous functions. Then, we have

$$\begin{aligned}
Y_t &: \Omega \to \mathbb{R} \\
&\omega \mapsto Y_t(\omega) = c + bt + aB_t(\omega),\ t \geq 0
\end{aligned}$$

i.e., the sample paths of $\{Y_t, t \geq 0\}$ are continuous, since it is the sum of continuous functions (and sum of continuous functions is continuous).

(4) We compute the characteristic function of $Y = Y_t - Y_s$. For $s < t$, we have

$$\begin{aligned}
\varphi_Y(u) &= \mathbb{E}\left[\exp(iu(Y_t - Y_s))\right] \\
&= \exp(iu(b(t-s)))\mathbb{E}\left[iu(a(B_t - B_s))\right].
\end{aligned}$$

Since the law of $B_t - B_s$ is Gaussian with mean 0 and variance $t - s$,

$$\begin{aligned}
\varphi_Y(u) &= \exp(iu(b(t-s)))\exp\left(-\frac{1}{2}(ua)^2(t-s)\right) \\
&= \exp\left(iu(b(t-s)) - \frac{1}{2}u^2a^2(t-s)\right)
\end{aligned}$$

It now follows that the law of $Y_t - Y_s$ is Gaussian with mean $b(t-s)$ and variance $a^2(t-s)$.

(5) We compute the characteristic function of Y_t as follows:

$$\begin{aligned}
\mathbb{E}\left[\exp(iuY_t)\right] &= \mathbb{E}[\exp(iu(c + bt + aB_t))] \\
&= \exp(iu(c + bt))\mathbb{E}[\exp(iuaB_t)] \\
&= \exp(iu(c + bt))\exp\left(-\frac{1}{2}u^2a^2t\right) \\
&= \exp\left(iu(c + bt) - \frac{1}{2}u^2a^2t\right)
\end{aligned}$$

Then, it follows that Y_t has Gaussian distribution with mean $c + bt$ and variance a^2t. With this, the density of Y_t is

$$p_t(c, y) = \frac{1}{\sqrt{2\pi a^2 t}}\exp\left(-\frac{(y - c - bt)^2}{2a^2t}\right).$$

Observe that

$$
\begin{aligned}
p_s(x,z)p_t(z,y) &= \frac{1}{\sqrt{2\pi a^2 s}}\exp\left(-\frac{(z-x-bs)^2}{2a^2 s}\right)\frac{1}{\sqrt{2\pi a^2 t}}\exp\left(-\frac{(y-z-bt)^2}{2a^2 t}\right)\\
&= \frac{1}{2\pi a^2\sqrt{st}}\exp\left(-\frac{1}{2a^2}\left[\frac{(z-x-bs)^2}{s}+\frac{(y-z-bt)^2}{t}\right]\right)\\
&= \frac{1}{2\pi a^2\sqrt{st}}\exp\left(-\frac{1}{2a^2 st}\left[stb(2(x-y)+b(s+t))+sy^2+tx^2+z^2(s+t)-2z(xt+sy)\right]\right)
\end{aligned}
$$

Now, let

$$
\alpha=\frac{s+t}{2a^2st},\qquad \beta=-\frac{2xt+2sy}{2a^2st},\qquad \gamma=\frac{sy^2+tx^2}{2a^2st}
$$

We claim that

$$
\int_{\mathbb{R}}\exp(-(\alpha z^2+\beta z+\gamma))dz=\sqrt{\frac{\pi}{\alpha}}\exp\left(\frac{\beta^2-4\alpha\gamma}{4\alpha}\right).
$$

Indeed, we have that

$$
\alpha z^2+\beta z+\gamma=\alpha\left[\left(z+\frac{\beta}{2\alpha}\right)^2-\frac{\beta^2-4\alpha\gamma}{4\alpha^2}\right]=\alpha\left(z+\frac{\beta}{2\alpha}\right)^2-\frac{\beta^2-4\alpha\gamma}{4\alpha}.
$$

So,

$$
\begin{aligned}
\int_{\mathbb{R}}\exp(-(\alpha z^2+\beta z+\gamma))dz &= \exp\left(\frac{\beta^2-4\alpha\gamma}{4\alpha}\right)\int_{\mathbb{R}}\exp\left(-\alpha\left(z+\frac{\beta}{2\alpha}\right)^2\right)dz\\
&= \exp\left(\frac{\beta^2-4\alpha\gamma}{4\alpha}\right)\int_{\mathbb{R}} e^{-\frac{\left(z+\frac{\beta}{2\alpha}\right)^2}{2\left(\sqrt{\frac{1}{2\alpha}}\right)^2}}dz
\end{aligned}
$$

Observe that the integrand correspond to the density function of a Gaussian random variable Z with mean $-\beta/2\alpha$ an variance $1/2\alpha$: $Z\sim\mathcal{N}\left(-\frac{\beta}{2\alpha},\frac{1}{2\alpha}\right)$. With this, we have

$$
\int_{\mathbb{R}} e^{-\frac{\left(z+\frac{\beta}{2\alpha}\right)^2}{2\left(\sqrt{\frac{1}{2\alpha}}\right)^2}}dz=\sqrt{\frac{2\pi}{2\alpha}}=\sqrt{\frac{\pi}{\alpha}}.
$$

we obtain that

$$
\int_{\mathbb{R}} p_s(x,z)p_t(z,y)dz=\frac{1}{2\pi a^2\sqrt{st}}\sqrt{\frac{\pi}{\alpha}}\exp\left(-\frac{1}{2a^2st}stb(2(x-y)+b(s+t))\right)\exp\left(\frac{\beta^2-4\alpha\gamma}{4\alpha}\right)
$$

$$
\begin{aligned}
\frac{\beta^2-4\alpha\gamma}{4\alpha} &= \frac{2a^2st}{4(s+t)}\left[\frac{4x^2t^2+8xtsy+4s^2y^2}{4a^4s^2t^2}-\frac{4(s+t)}{2a^2st}\frac{(sy^2+tx^2)}{2a^2st}\right]\\
&= \frac{2a^2st}{4(s+t)}\left[\frac{4x^2t^2+8xtsy+4s^2y^2-4s^2y^2-4stx^2-4tsy^2-4t^2x^2}{4a^4s^2t^2}\right]\\
&= \frac{2a^2st}{4(s+t)}\left[\frac{4x^2t^2+8xtsy+4s^2y^2-4s^2y^2-4stx^2-4tsy^2-4t^2x^2}{4a^4s^2t^2}\right]\\
&= \frac{2a^2st}{4(s+t)}\left[\frac{8xtsy-4stx^2-4tsy^2}{4a^4s^2t^2}\right]\\
&= \frac{1}{(s+t)}\left[\frac{2xtsy-stx^2-tsy^2}{2a^2st}\right]\\
&= \frac{1}{(s+t)}\left[\frac{2xy-x^2-y^2}{2a^2}\right]\\
&= \frac{1}{(s+t)}\left[\frac{-(x-y)^2}{2a^2}\right]
\end{aligned}
$$

$$-\frac{1}{2a^2}b(2(x-y)+b(s+t))+\frac{1}{(s+t)}\frac{-(x-y)^2}{2a^2} = -\frac{1}{2a^2}\left(2bx-2by+b^2s+b^2t+\frac{x^2-2xy+y^2}{(s+t)}\right)$$
$$= -\frac{1}{2a^2(s+t)}\left(-2b(y-x)(t+s)+b^2(t+s)^2+(y-x)^2\right)$$
$$= -\frac{1}{2a^2(s+t)}(y-x-b(t+s))^2.$$

We obtain

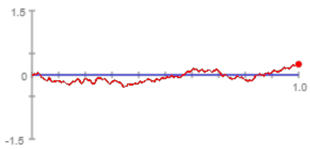
$$\int_{\mathbb{R}} p_s(x,z)p_t(z,y)dz = \frac{1}{2\pi a^2\sqrt{st}}\sqrt{\frac{\pi 2a^2 st}{(s+t)}}\exp\left(-\frac{(y-x-b(t+s))^2}{2a^2(t+s)}\right)$$
$$= \sqrt{\frac{\pi 2a^2}{4\pi^2 a^4(s+t)}}\exp\left(-\frac{(y-x-b(t+s))^2}{2a^2(t+s)}\right)$$
$$= \frac{1}{\sqrt{2\pi a^2(s+t)}}\exp\left(-\frac{(y-x-b(t+s))^2}{2a^2(t+s)}\right)$$
$$= p_{t+s}(x,y).$$

(6) For the sample paths of Brownian motion with drift we have

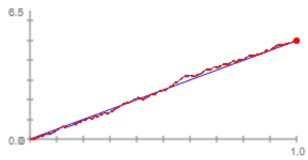
$$\begin{aligned} Y_t : \Omega &\to \mathbb{R} \\ \omega &\mapsto Y_t(\omega) = c + bt + aB_t(\omega), \end{aligned}$$

The first difference is that the Brownian motion with drift starts in c a.s., while the standard Brownian motion starts in 0 a.s. The Brownian motion with drift goes over the line bt whit slope b, where b is called the drift parameter and $a > 0$ is called the scale parameter. If $c = b = 0$ and $a = 1$ then we have the standard Brownian motion.

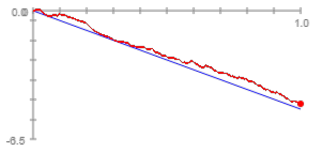
In the following figures we take $c = 0$. We see that for values of a close to zero we have that the Brownian motion with drift is closer to the slope, and when a is not close to zero, the Brownian motion moves away from the slope.



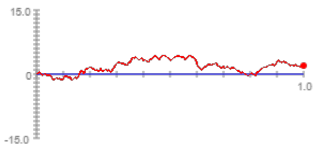
(a) $b = 0$ and $a = 0.5$



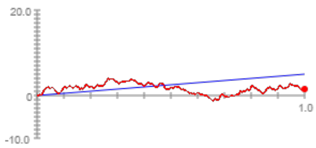
(b) $b = 5$ and $a = 0.5$



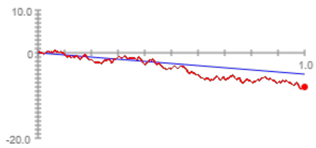
(c) $b = -5$ and $a = 0.5$



(d) $b = 0$ and $a = 5$



(e) $b = 5$ and $a = 5$



(f) $b = -5$ and $a = 5$

$\square$

Exercise 8. Let $\{B_t, t \geq 0\}$ be a standard Brownian motion. For any $n \in \mathbb{N}$ set $t_i = \frac{i}{n}$, $i = 0, \ldots, n$.

(1) Prove that for any $p \geq 1$, the sequence

$$n^{\frac{p}{2}-1}\sum_{i=0}^{n-1}|B(t_{i+1})-B(t_i)|^p, \qquad n \geq 1$$

converges in probability to a constant m_p as $n \to \infty$.

(2) Prove that the sequence

$$n^{\frac{p-1}{2}} \left(\sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|^p - n^{1-\frac{p}{2}} m_p \right), \qquad n \geq 1$$

converges in law as $n \to \infty$ to a normal random variable.

Solution. (1) Consider the continuous function $f(x) = |x|^p$, $p \geq 1$ for $x \in \mathbb{R}$ and define the following random variables

$$\begin{aligned} X_i &= |n^{\frac{1}{2}} B(t_i) - n^{\frac{1}{2}} B(t_{i-1})|^p \\ &\stackrel{d}{=} |W_i - W_{i-1}|^p = f(W_i - W_{i-1}), \qquad i \geq 1, \end{aligned}$$

where $W_i = n^{\frac{1}{2}} B(t_i)$ is a standard Brownian motion by the scaling property. Indeed, by the scaling property of the Brownian motion,

$$n^{\frac{1}{2}} B(t_i) - n^{\frac{1}{2}} B(t_{i-1}) \stackrel{d}{=} W_i - W_{i-1}.$$

Furthermore, for any measurable function $g : \mathbb{R} \to \mathbb{R}^+$, the function $g \circ f : \mathbb{R} \to \mathbb{R}^+$ is measurable. Then,

$$f(n^{\frac{1}{2}} B(t_i) - n^{\frac{1}{2}} B(t_{i-1})) \stackrel{d}{=} f(W_i - W_{i-1}),$$

since

$$\begin{aligned} \mathbb{E}\left[g(f(n^{\frac{1}{2}} B(t_i) - n^{\frac{1}{2}} B(t_{i-1}))) \right] &= \mathbb{E}\left[(g \circ f)(n^{\frac{1}{2}} B(t_i) - n^{\frac{1}{2}} B(t_{i-1})) \right] \\ &= \mathbb{E}\left[(g \circ f)(W_i - W_{i-1}) \right] \\ &= \mathbb{E}\left[g(f(W_i - W_{i-1})) \right]. \end{aligned}$$

Notice the following properties:

- The random variables $X_1, X_2, \dots$ are independents. Indeed, since the W_i, $i \geq 1$, are standard Brownian motions then the increments

$$W_1 - W_0, W_2 - W_1, \dots$$

  are independents. Then, the random variables $X_i = f(W_i - W_{i-1})$ are independent for $i \geq 1$.
- The random variables $X_1, X_2, \dots$ have the same law. Indeed, since each increment has Gaussian distribution, $W_i - W_{i-1} \sim \mathcal{N}(0, i - (i-1) = 1)$. Then, we obtain that the random variables $f(W_i - W_{i-1})$ all have the same distribution. The last part follows since, for every measurable function $g : \mathbb{R} \to \mathbb{R}_+$, the function $g \circ f : \mathbb{R} \to \mathbb{R}_+$ is still measurable and $\mathbb{E}[g \circ f(W_i - W_{i-1})] = \mathbb{E}[g \circ f(W_j - W_{j-1})]$ for all $i, j = 1, \dots, n$ because $W_i - W_{i-1}$ have all the same law.
- The expectation of X_1 is

$$\mathbb{E}[X_1] = \mathbb{E}[|W_1|^p] = m_p.$$

  By the remark of the Exercise 1, $m_p < +\infty$.

Notice the following:

$$\begin{aligned} \lim_{n \to \infty} n^{\frac{p}{2}-1} \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|^p &= \lim_{n \to \infty} n^{\frac{p}{2}-1} \sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^p \\ &= \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i. \end{aligned}$$

Thus, by the Strong Law of Large Numbers,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i = m_p$$

in probability.

(2) We use the Central Limit Theorem. We proved that $(X_i)_{i\geq 1}$ is a sequence of independent identically distributed random variables. Then, by the Central Limit Theorem, we have the following convergence in distribution:

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} X_i - nm_p}{\sqrt{n}} = \mathcal{N},$$

where $\mathcal{N} \sim \mathcal{N}\left(0, \mathrm{Var}(X_1)\right)$.

To conclude, just notice that

$$\begin{aligned}
\frac{\sum_{i=1}^{n} X_i - nm_p}{\sqrt{n}} &= n^{-\frac{1}{2}}\left(n^{\frac{p}{2}}\sum_{i=1}^{n} |B(t_i) - B(t_{i-1})|^p - nm_p\right) \\
&= n^{\frac{p-1}{2}}\sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|^p - n^{\frac{1}{2}} m_p \\
&= n^{\frac{p-1}{2}}\left(\sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)|^p - n^{1-\frac{p}{2}} m_p\right).
\end{aligned}$$

Therefore, the sequence converges in law to a Gaussian random variable as $n \to \infty$.

$\square$