Stochastic Calculus

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Problem 1. Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = [0,1]$, $\mathcal{F}$ the Borel σ-field $\mathcal{B}([0,1])$ and $\mathbb{P}$ the Lebesgue measure. Describe the one-dimensional and two-dimensional distributions of the stochastic process defined by $X_t(\omega) = t\omega$, $t \in [0,1]$.

Solution:

For one-dimensional distribution we have to compute the law of X_t, for all $t \in [0,1]$ fix. For this, we compute the distribution function of X_t.
We have 3 cases:

→ For $t=0$, we have that the law of X_t is given by the Dirac mass since $\mathbb{P}(X_t(\omega) \le x) = 1$ if $x \geq 0$ and $\mathbb{P}(X_t(\omega) \le x) = 0$ if $x < 0$.

→ For $t > 0$ and $x < t$, then

$$\begin{aligned}
\mathbb{P}(\{\omega \in \Omega : X_t(\omega) \le x\}) &= \mathbb{P}(\{\omega \in \Omega : t\omega \le x\}) \\
&= \mathbb{P}(\{\omega \in \Omega : \omega \le \tfrac{x}{t}\}) \\
&= \mathbb{P}(\{\omega \in \Omega : 0 \le \omega \le \tfrac{x}{t}\}) \\
&= \mathbb{P}([0, x/t]) \\
&= \frac{x}{t}.
\end{aligned}$$

→ For $t > 0$ and $x \geq t$, we have

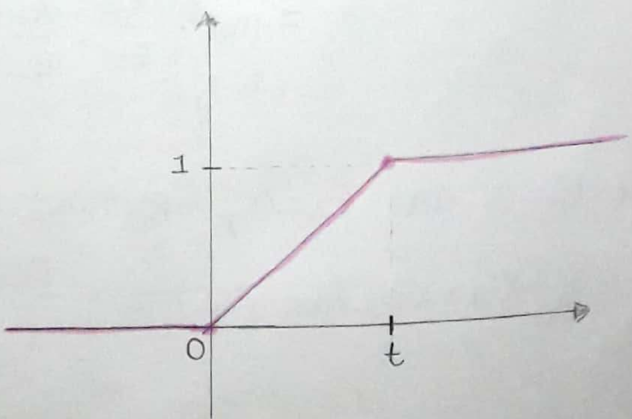
$$\mathbb{P}(\{\omega \in \Omega : X_t(\omega) \le x\}) = \mathbb{P}(\{\omega \in \Omega : t\omega \le x\})$$

but $\{\omega \in \Omega : t\omega \le x\} = [0,1]$,

then $\mathbb{P}(X_t \le x) = 1$

Therefore,

$$F_X(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{t}, & \text{if } x < t \\ 1, & x \geq t \end{cases}$$



We can conclude that X_t has a uniform distribution on $[0,t]$.

Now, for two dimensional distributions we have to compute the law of (X_{t_1}, X_{t_2}), for all $t_1, t_2 \in [0,1]$ fix. Then, the jointly distribution function of (X_{t_1}, X_{t_2}) is given in the following cases for $\omega \in \Omega$:

→ For $x_1, x_2 < 0$, then $\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = 0$

→ For $x_1 \geq t_1$ and $x_2 \geq t_2$, we have

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = 1$$

since $\{t_1\omega \leq x_1\} \cap \{t_2\omega \leq x_2\} = [0,1]$.

→ For $x_1 < t_1$ and $x_2 \geq t_2$,

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \mathbb{P}(t_1\omega \leq x_1, t_2\omega \leq x_2)$$

$$= \mathbb{P}\left(\omega \leq \frac{x_1}{t_1}, t_2\omega \leq x_2\right)$$

$\left\{\frac{x_1}{t_1}\right\} < 1$

$$= \mathbb{P}\left(\omega \leq \frac{x_1}{t_1}\right) = \frac{x_1}{t_1}$$

By the same way for $x_2 < t_2$ and $x_1 \geq t_1$, we have.

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \frac{x_2}{t_2}.$$

→ For $x_1 < t_1$ and $x_2 < t_2$

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \mathbb{P}(t_1\omega \leq x_1, t_2\omega \leq x_2)$$

$$= \mathbb{P}\left(\omega \leq \frac{x_1}{t_1}, \omega \leq \frac{x_2}{t_2}\right)$$

$$= \mathbb{P}\left(\omega \leq \min\left\{\frac{x_1}{t_1}, \frac{x_2}{t_2}\right\}\right)$$

$$= \min\left\{\frac{x_1}{t_1}, \frac{x_2}{t_2}\right\}$$

→ For $t_1 = 0$ and $t_2 = 0$, we have that $X_{t_1} = 0$ and $X_{t_2} = 0$ a.s.

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \begin{cases} 0 & , x_1, x_2 < 0 \\ 1 & , x_1, x_2 \geq 0 \end{cases}$$

→ For $t_1 = 0$ or $t_2 = 0$. We have the following:

* If $t_2 = 0$ and $t_1 > 0$, then

↳ for $x_1 \geq t_1$

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \mathbb{P}(t_1\omega \leq x_1, X_{t_2} = 0)$$
$$= \mathbb{P}(t_1\omega \leq x_1, t_2\omega = 0)$$
$$= 1$$

↳ for $x_1 < t_1$

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \mathbb{P}(\omega \leq \frac{x_1}{t_1}, t_2\omega = 0)$$
$$= \mathbb{P}(\omega \leq \frac{x_1}{t_1})$$
$$= \frac{x_1}{t_1}$$

* If $t_1 = 0$ and $t_2 > 0$

↳ for $x_2 \geq t_2$, we have that

$$\mathbb{P}(X_{t_1} = 0, t_2\omega \leq x_2) = 1$$

↳ for $x_2 < t_2$, we have

$$\mathbb{P}(X_{t_1} \leq x_1, X_{t_2} \leq x_2) = \mathbb{P}(X_{t_1} = 0, \omega \leq \frac{x_2}{t_2})$$
$$= \mathbb{P}(\omega \leq \frac{x_2}{t_2})$$
$$= \frac{x_2}{t_2}.$$

Therefore, we can conclude that

$$F_{X_{t_1}, X_{t_2}}(x_1, x_2) = \begin{cases} 0 & , \text{if } x_1, x_2 < 0 \\ \frac{x_1}{t_1} & , \text{if } x_1 < t_1 \text{ and } t_2 = 0 \\ \frac{x_2}{t_2} & , \text{if } x_2 < t_2 \text{ and } t_1 = 0 \\ \min\left\{\frac{x_1}{t_1}, \frac{x_2}{t_2}\right\} & , \text{if } x_1 < t_1 \text{ and } x_2 < t_2 \\ 1 & , \text{if } x_1 \geq t_1 \text{ and } x_2 \geq t_2 \end{cases}$$

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Problem 2. Let X_1, X_2 be two independent random variables with law $\mathcal{N}(0,1)$. Let $\{Y_t, t\geq 0\}$ be the stochastic process defined by

$$Y_t = (X_1 + X_2)\, t, \quad t \geq 0.$$

(1) Describe the finite-dimensional distributions of the process.

(2) Let A be the set of non-negative sample paths. Argue that $A \in \mathcal{F}$ and compute $\mathbb{P}(A)$.

Solution:

(1) For the finite-dimensional distributions we have to compute the law of $(Y_{t_1}, \dots, Y_{t_n})$ for all $t_1, \dots, t_n \geq 0$.

For this, we compute the joint distribution function of $(Y_{t_1}, \dots, Y_{t_n})$.

→ Suppose that $t_i > 0$, $\forall i \in \{1, \dots, n\}$ fix.

$$\begin{aligned}
\mathbb{P}(Y_{t_1} \leq y_1, \dots, Y_{t_n} \leq y_n) &= \mathbb{P}\left((X_1+X_2)t_1 \leq y_1, \dots, (X_1+X_2)t_n \leq y_n\right) \\
&= \mathbb{P}\left(X_1+X_2 \leq \frac{y_1}{t_1}, \dots, X_1+X_2 \leq \frac{y_n}{t_n}\right) \\
&= \mathbb{P}\left(X_1+X_2 \leq \min\left\{\frac{y_1}{t_1}, \dots, \frac{y_n}{t_n}\right\}\right) \\
&= F_{X_1+X_2}\left(\min\left\{\frac{y_i}{t_i}\right\}\right).
\end{aligned}$$

Recall that if $Z = X_1 + X_2$, then the law of Z is given by a gaussian random variable with mean 0 and variance 2 since

$$\begin{aligned}
f_Z(z) &= f_{X_1} * f_{X_2}(z) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-(z-x_2)^2/2} e^{-x_2^2/2}\, dx_2 \\
&= \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-(x_2 - z/2)^2}\, dx_2 \\
&= \frac{1}{2\pi} e^{-z^2/4} \sqrt{\pi}\left[\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(x_2 - z/2)^2}\, dx_2\right]
\end{aligned}$$

and $\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(x_2 - z/2)^2}\, dx_2$ is the integral of gaussian density function with $\mu=0$ and $\sigma=\sqrt{2}$. So, we have

$$f_Z(z)=\frac{1}{\sqrt{4\pi}}e^{-z^2/4}.$$

→ For $t_i=0$ for all $i\in\{1,\dots,n\}$, we have that the law of $(Y_{t_1},\dots,Y_{t_n})$ is given by the Dirac mass.

→ If not all of t_i are 0, under a rearrangement:
we can assume that the first j terms $t_1,\dots,t_j$ are not 0 and the other n-j terms $t_{j+1},\dots,t_n$ are equal to 0, then
For $j\in\{1,\dots,n\}$, we look at the law of $(Y_{t_1},\dots,Y_{t_j},Y_{t_{j+1}}=0,\dots,Y_{t_n})$
for this, we compute $\mathbb{P}(Y_{t_1}\le y_1,\dots,Y_{t_j}\le y_j, Y_{t_{j+1}}=0,\dots,Y_{t_n}=0)$
but

$$\begin{aligned}\mathbb{P}(Y_{t_1}\le y_1,\dots,Y_{t_j}\le y_j,Y_{t_{j+1}}=0,\dots,Y_{t_n}=0)&=\mathbb{P}(Y_{t_1}\le y_1,\dots,Y_{t_j}\le y_j)\\&=\mathbb{P}((X_1+X_2)t_1\le y_1,\dots,(X_1+X_2)t_j\le y_j)\\&=\mathbb{P}\left(X_1+X_2\le \min_{i<j}\left\{\frac{y_i}{t_i}\right\}\right).\end{aligned}$$

and this has a law gaussiau with mean 0 and variance 2.

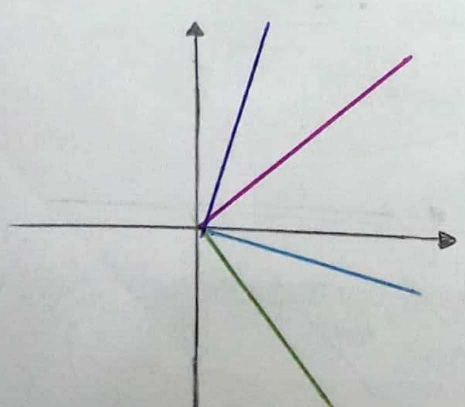
(2) For this, let be $A=\{\omega\in\Omega: Y_t>0\}$.

The set A is measurable, since $A=\{\omega\in\Omega: Y_t>0\}=Y_t^{-1}(0,\infty)\in\mathcal{B}$ and Y_t are random variables. Then,

$$\mathbb{P}(A)=\int_0^\infty f_{X_1+X_2}\left(\min\left\{\frac{y_i}{t_i}\right\}\right)dx$$

but $f_{X_1+X_2}\left(\min\left\{\frac{y_i}{t_i}\right\}\right)$ has a gaussian law with mean zero.

and by symetry we have that $\mathbb{P}(A)=\mathbb{P}(Y_t>0)=\frac{1}{2}$.



The sample paths are lines with slope 0.

$at+b$ wher b is the slope b

Problem 3. Let X be a Gaussian standar random variable, and let $\alpha \in (1,\infty)$. Consider the stochastic process defined by

$$Y_t = X + \alpha t \quad , \quad t \geq 0$$

Describe its sample paths. Fix a countable set $D \subset [0,\infty)$. What is the probability that $Y_t = 0$ at least one $t \in D$?

Solution:

For the sample paths we have

$$Y_t : \Omega \longrightarrow \mathbb{R}$$
$$\omega \longmapsto Y_t(\omega) = X(\omega) + \alpha t$$

Then the sample paths are lines with slope $\alpha > 1$ plus a random variable.

Now, let $D \subset [0,\infty)$ a countable set. We consider

$$A_t = \{\omega \in \Omega : Y_t(\omega) = 0\} = Y_t^{-1}(0) \text{ is measurable}$$

and let

$$A = \bigcup_{t \in D} A_t \text{ a countable union of measurable sets}$$

~~and~~

then A is measurable.

Then by sub-additivity of probability measure we have

$$\mathbb{P}(A) = \mathbb{P}\left(\bigcup_{t \in D} A_t\right) \leq \sum_{t \in D} \mathbb{P}(A_t)$$

and

$$\begin{aligned}\mathbb{P}(A_t) &= \mathbb{P}(Y_t = 0) \\ &= \mathbb{P}(X + \alpha t = 0) \\ &= \mathbb{P}(X = -\alpha t) \\ &= 0\end{aligned}$$

Therefore, $\mathbb{P}(A) = 0$.

Problem 4. Let A and U be two independent random variables such that $\mathbb{E}(A)=0$, $\mathbb{E}(A^2)<\infty$, and U is uniformly distributed on $[0,2\pi]$. Consider the stochastic process

$$X_t = A\cos(U+\lambda t)\ ,\ t\geq 0 \text{ and } \lambda\in\mathbb{R}.$$

(1) Give a graphic representation of the sample paths.

(2) What is the value of $\mathbb{E}(X_t)$? Why?

(3) Show that the covariance between two different variables X_t, X_s is

$$\mathbb{E}(X_t X_s)=\frac{1}{2}\mathbb{E}(A^2)\cos\lambda(t-s).$$

Solution:

In the following we will use the next fact:

If $X_1,\dots,X_n$ are independent random variables, then

$$\mathbb{E}(g_1(X_1)\cdots g_n(X_n))=\mathbb{E}(g_1(X_1))\cdots\mathbb{E}(g_n(X_n)),$$

where g_i are measurable functions such that $\mathbb{E}(|g_i(X_i)|)<\infty$.

(2) We have that,

$$\begin{aligned}\mathbb{E}(X_t)&=\mathbb{E}(A\cos(U+\lambda t))\\&=\mathbb{E}(A)\,\mathbb{E}(\cos(U+\lambda t))\end{aligned}$$

since A is independento of U by hypothesis, and we take $g_1:\mathbb{R}\to\mathbb{R}$, $g_1(x)=x$ with $x\in\mathbb{R}$. This function is measurable because $g_1^{-1}((-\infty,\infty))=\mathbb{R}\in\mathcal{B}(\mathbb{R})$, and $\mathbb{E}(|g_1(A)|)=\mathbb{E}(|A|)=0<\infty$. Now, let $g_2:\mathbb{R}\to\mathbb{R}$ defined by $g_2(x)=\cos(x+\lambda t)$ with $x\in\mathbb{R}$, $t\geq 0$ and $\lambda\in\mathbb{R}$, is measurable since $g_2^{-1}((-\infty,\infty))=\mathbb{R}\in\mathcal{B}(\mathbb{R})$, and

$$\mathbb{E}(|g_2(U)|)=\mathbb{E}(|\cos(U+\lambda t)|)\leq\mathbb{E}(1)=1<\infty.$$

Therefore,

$$\mathbb{E}(X_t)=0$$

since $\mathbb{E}(A)=0$.

(3) $\mathbb{E}(X_t X_s) = \mathbb{E}([A\cos(U+\lambda t)][A\cos(U+\lambda s)]$

$= \mathbb{E}(A^2 \cos(U+\lambda t)\cos(U+\lambda s)]$

Let $g_1 : \mathbb{R} \longrightarrow \mathbb{R}_+$, $g_1(x) = x^2$, $x \in \mathbb{R}$ is measurable since $g_1^{-1}(\mathbb{R}) = [0,\infty) \in \mathcal{B}(\mathbb{R})$ and $\mathbb{E}(|g_1(A)|) = \mathbb{E}(|A^2|) = \mathbb{E}(A^2) < \infty$ by hypothesis.

Then,

$$\mathbb{E}(X_t X_s) = \mathbb{E}(A^2)\mathbb{E}(\cos(U+\lambda t)\cos(U+\lambda s))$$
$$= \mathbb{E}(A^2)\mathbb{E}\left(\tfrac{1}{2}[\cos(U+\lambda t - U - \lambda s) + \cos(2U+\lambda t+\lambda s)]\right)$$
$$= \tfrac{1}{2}\mathbb{E}(A^2)\mathbb{E}(\cos\lambda(t-s) + \cos(2U+\lambda(t+s)))$$
$$= \tfrac{1}{2}\mathbb{E}(A^2)\mathbb{E}(\cos\lambda(t-s)) + \tfrac{1}{2}\mathbb{E}(A^2)\mathbb{E}(\cos(2U+\lambda(t+s)))$$

We compute $\mathbb{E}(\cos(2U+\lambda(t+s)))$, where $U \sim \mathcal{U}nif([0,2\pi])$. Let $r = t+s$, then we have

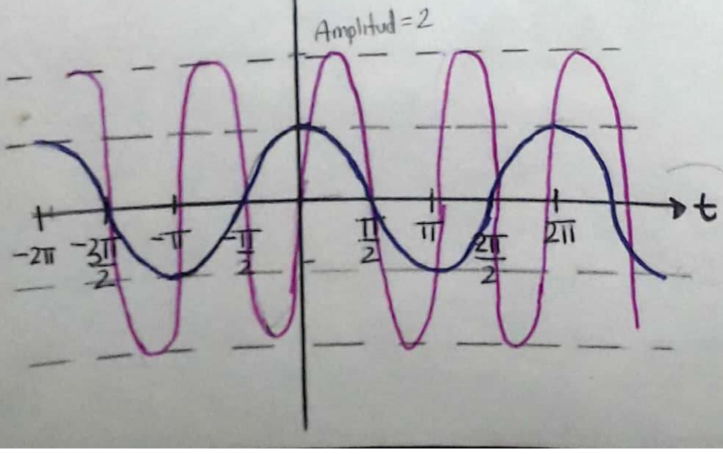
$$\mathbb{E}(\cos(U^2+\lambda r)) = \int_0^{2\pi} \cos(2U+\lambda r)\frac{1}{2\pi}du$$
$$= \frac{1}{2\pi}\frac{1}{2}\int_0^{4\pi}\cos(v+\lambda r)dv$$
$$= \frac{1}{4\pi}\left(-\sin(v+\lambda r)\Big|_0^{4\pi}\right)$$
$$= \frac{1}{4\pi}\left(-\sin(4\pi+\lambda r) + \sin(0+\lambda r)\right)$$
$$= \frac{1}{4\pi}\left(-(\sin(4\pi)\cos(\lambda r)) - \cos(4\pi)\sin(\lambda r) + \sin(\lambda r)\right)$$
$$= 0. \text{ (as } \sin(4\pi) = 0 \text{ and } \cos(4\pi) = 1)$$

$\boxed{v = 2U \quad dv = 2du}$

We use that if X is a r.v. Then the law of X, denoted by P_X is the unique measure such that
$\mathbb{E}(f(x)) = \int_\Omega f(x) P_X(dx)$
$\forall f : \Omega \to \mathbb{R}_+$ measurable

Finally, we get $\mathbb{E}(X_t X_s) = \frac{1}{2}\mathbb{E}(A^2)\cos\lambda(t-s)$.

(1) Graphic representation of the sample paths: $X_t = A\cos(U+\lambda t)$, where



$A \to$ amplitud of oscilation
$U \to$ phase
$\lambda \to$ frequency per unit time

This is one phath.