Stochastic Calculus

Master in Advanced Mathematics 2017-2018

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Set 4. Comprehensive problems on SDEs.

Exercise 1. Consider the stochastic differential equation

$$X_t = X_0 + \int_0^t [A(s)X_s + a(s)]ds + \int_0^t \sigma(s)dB_s \tag{1}$$

where $A(t)$ is a $m \times m$ matrix, $a(t)$ is a m-dimensional vector and $\sigma(t)$ is a $d \times m$ matrix of continuous functions.

(a) Using Ito formula prove that if $\Phi(t)$ is a solution to

$$\begin{cases} \dot{\Phi}(t) = A(t)\Phi(t) \\ \Phi(0) = Id \end{cases}$$

then

$$X_t = \Phi(t)\left[X_0 + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dB_s\right]$$

is a solution to the linear equation (1).

(b) Let $d = m = 1$. Consider the particular case

$$X_t = a + B_t + \int_0^t \frac{b - X_s}{T - s}ds, \ 0 \leq t \leq T.$$

Give an explicit expression for the solution of this equation. Prove that $\{X_t, 0 \leq t < T\}$ is a Gaussian process and write the mean and the covariance function.

(c) Consider again $d = 1 = m$. Let $a(t) = 0, \sigma(t) = \sigma > 0, -\beta \leq A(t) \leq -\alpha < 0$,for any $t \in [0, \infty)$, and $X_0 = x$. Prove the following estimate

$$\mathbb{E}[X_t^2] \leq (x^2 + \sigma^2 C_{\beta,t}) \exp(-2\alpha t),$$

with $C_{\beta,t}$ a positive constant depending on β and t.

Solution. (a) We write

$$X_t = \Phi(t)Z_t, \qquad Z_t = X_0 + \int_0^t \Phi^{-1}(s)a(s)ds + \int_0^t \Phi^{-1}(s)\sigma(s)dB_s.$$

Using Ito formula we obtain

$$\begin{aligned}
X_t &= X_0 + \int_0^t \Phi(s)dZ_s + \int_0^t Z_s d\Phi(s) \\
&= X_0 + \int_0^t \Phi(s)\left(\Phi^{-1}(s)a(s)ds + \Phi^{-1}(s)\sigma(s)dB_s\right) + \int_0^t Z_s A(s)\Phi(s)ds \\
&= X_0 + \int_0^t a(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t X_s\Phi^{-1}(s)A(s)\Phi(s)ds \\
&= X_0 + \int_0^t a(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t A(s)X_s ds \\
&= X_0 + \int_0^t [A(s)X_s + a(s)]ds + \int_0^t \sigma(s)dB_s.
\end{aligned}$$

(b) Consider

$$X_t = a + B_t + \int_0^t \frac{b - X_s}{T-s} ds, \; 0 \le t < T.$$

By inspection,

$$A(t) = -\frac{1}{T-t}, \qquad a(t) = \frac{b}{T-t}, \qquad \sigma(t) = 1 \qquad \text{and} \qquad X_0 = a$$

Furthermore, we have that $\Phi(t)$ is a solution of

$$\begin{cases} \dot{\Phi}(t) = \frac{1}{t-T}\Phi(t) \\ \Phi(0) = 1 \end{cases}$$

So,

$$\Phi(t) = \exp\left(\int_0^t \frac{-1}{T-s} ds\right) = \exp\left(\ln(T-t) - \ln T\right) = \frac{T-t}{T}.$$

The explicit expression for the solution of this equation is

$$\begin{aligned} X_t &= \frac{T-t}{T}\left[a + \int_0^t \frac{T}{T-s}\frac{b}{T-s} ds + \int_0^t \frac{T}{T-s} dB_s\right] \\ &= \frac{T-t}{T}a + b\int_0^t \frac{T-t}{(T-s)^2} ds + (T-t)\int_0^t \frac{1}{T-s} dB_s \\ &= \frac{T-t}{T}a + \frac{t}{T}b + (T-t)\int_0^t \frac{1}{T-s} dB_s. \end{aligned}$$

We observe that the function $f : [0,T) \to \mathbb{R}$ given by $f(t) = \frac{1}{T-t}$ is a deterministic continuous function. By previous homework, the process $\{X_t, 0 \le t < T\}$ is Gaussian. The mean is

$$\mathbb{E}[X_t] = \frac{T-t}{T}a + \frac{t}{T}b,$$

and the covariance is

$$\begin{aligned} \operatorname{Cov}(X_t, X_r) &= \mathbb{E}[(X_t - \mathbb{E}[X_t])(X_r - \mathbb{E}[X_r])] \\ &= (T-t)(T-r)\mathbb{E}\left[\int_0^t \frac{1}{T-s} dB_s \int_0^r \frac{1}{T-s} dB_s\right] \\ &= (T-t)(T-r)\int_0^m \frac{1}{(T-s)^2} ds \\ &= \frac{(T-t)(T-r)m}{T(T-m)} \\ &= m - \frac{rt}{T} \end{aligned}$$

where $m := \min\{t, r\}$.

(c) We consider $d = 1 = m$. Under the conditions we have that (1) is now

$$X_t = x + \int_0^t A(s)X_s ds + \sigma B_t.$$

Therefore, by the point (a),

$$X_t = \Phi(t)\left[x + \sigma\int_0^t \Phi^{-1}(s) dB_s\right],$$

where

$$\Phi(t) = \exp\left(\int_0^t A(s) ds\right).$$

Now,

$$X_t^2 = \Phi(t)^2\left[x^2 + 2x\int_0^t \Phi^{-1}(s) dB_s + \sigma^2\left(\int_0^t \Phi^{-1}(s) dB_s\right)^2\right]$$

By Ito isometry,

$$\mathbb{E}[X_t^2] = \Phi(t)^2 \left[x^2 + \sigma^2 \int_0^t (\Phi^{-1}(s))^2 ds\right].$$

Now, observe the following,

$$\Phi(t)^2 = \exp\left(2\int_0^t A(s)ds\right) \leq \exp\left(\int_0^t -2\alpha ds\right) = \exp\left(-2\alpha t\right),$$

and

$$(\Phi^{-1}(t))^2 = \exp\left(-2\int_0^t A(s)ds\right) \leq \exp\left(2\beta t\right).$$

We have then,

$$\int_0^t (\Phi^{-1}(s))^2 ds \leq \int_0^t \exp(2\beta s)ds = \frac{1}{2\beta}\left(\exp(2\beta t) - 1\right) = C_{\beta,t}.$$

Thus,

$$\mathbb{E}[X_t^2] \leq \exp(-2\alpha t)\left(x^2 + \sigma^2 C_{\beta,t}\right).$$

□

Exercise 2. Let X^ε be the solution to the one-dimensional stochastic differential equation

$$\begin{aligned} dX_t^\varepsilon &= b(X_t^\varepsilon)dt + \varepsilon\sigma(X_t^\varepsilon)dBt, \\ X_0^\varepsilon &= x, \end{aligned}$$

where b and σ are Lipschitz functions (the Lipschitz constant will be denoted by L), and σ is bounded. Consider the ordinary differential equation

$$\begin{cases} \gamma_t' = b(\gamma_t), \\ \gamma_0 = x \end{cases}$$

We want to prove that X_t^ε converges to γ_t , when $\varepsilon \to 0$, in probability, and give the speed of convergence, by completing the next steps:

(a) Let $\eta_t^\varepsilon = X_t^\varepsilon - \gamma_t$. Prove that

$$\begin{cases} d\eta_t^\varepsilon = (b(\eta_t^\varepsilon + \gamma_t) - b(\gamma_t))dt + \varepsilon\sigma(\eta_t^\varepsilon + \gamma_t)dB_t, \\ \eta_0^\varepsilon = 0. \end{cases}$$

(b) Prove that $T > 0$, $c > 0$

$$\left\{\sup_{0\leq t\leq T}\left|\varepsilon\int_0^t \sigma(\eta_s^\varepsilon + \gamma_s)dB_s\right| < c\right\} \subset \left\{\sup_{0\leq t\leq T}|\eta_t^\varepsilon| < c\exp(LT)\right\}.$$

(c) As a consequance of (b), prove that

$$\mathbb{P}\left(\sup_{0\leq s\leq T}|X_s^\varepsilon - \gamma_s| > \alpha\right) \leq 2\exp\left(-\frac{\alpha^2 e^{-2LT}}{2\varepsilon^2 T|\sigma|_\infty^2}\right).$$

(d) Deduce that X_t^ε converges to γ_t as $\varepsilon \to 0$, and say in which type of convergence.

Solution. (a) We have that, for $t > 0$,

$$\begin{aligned} d\eta_t^\varepsilon &= dX_t^\varepsilon - d\gamma_t \\ &= b(X_t^\varepsilon)dt + \varepsilon\sigma(X_t^\varepsilon)dB_t - b(\gamma_t)dt \\ &= (b(X_t^\varepsilon) - b(\gamma_t))dt + \varepsilon\sigma(X_t^\varepsilon)dB_t \\ &= (b(\eta_t^\varepsilon + \gamma_t) - b(\gamma_t))dt + \varepsilon\sigma(\eta_t^\varepsilon + \gamma_t)dB_t. \end{aligned}$$

For $t = 0$,

$$\eta_0^\varepsilon = X_0^\varepsilon - \gamma_0 = x - x = 0.$$

(b) Let $T > 0$ and $c > 0$. Let

$$\omega \in \left\{ \sup_{0\le t\le T} \left| \varepsilon \int_0^t \sigma(\eta_s^\varepsilon + \gamma_s) dB_s \right| < c \right\}.$$

Then,

$$\begin{aligned}
|\eta_t^\varepsilon|(\omega) &= \left| \int_0^t (b(\eta_s^\varepsilon + \gamma_s) - b(\gamma_s)) ds + \varepsilon \int_0^t \sigma(\eta_s^\varepsilon + \gamma_s) dB_s \right| (\omega) \\
&< c + \left| \int_0^t (b(\eta_s^\varepsilon + \gamma_s) - b(\gamma_s)) ds \right| (\omega) \\
&\le c + \int_0^t |b(\eta_s^\varepsilon + \gamma_s) - b(\gamma_s)|(\omega) ds \\
&\le c + L \int_0^t |\eta_s^\epsilon|(\omega) ds
\end{aligned}$$

By Gromwall's lemma,

$$|\eta_t^\varepsilon|(\omega) < c \exp{(Lt)}.$$

Therefore,

$$\sup_{0\le t\le T} |\eta_t^\varepsilon|(\omega) < c \exp(LT).$$

Thus,

$$\omega \in \left\{ \sup_{0\le t\le T} |\eta_t^\varepsilon| < c \exp(LT) \right\}.$$

(c) Taking complement in (b), we have

$$\mathbb{P}\left(\sup_{0\le t\le T} |X_t^\varepsilon - \gamma_t| > \alpha \right) \le \mathbb{P}\left(\sup_{0\le t\le T} \left| \int_0^t \sigma(\eta_s^\varepsilon + \gamma_s) dB_s \right| > \frac{\alpha}{\exp(LT)\varepsilon} \right).$$

Now, using a version of the martingale exponential inequality with $\varphi(t) = \sigma(\eta_t^\varepsilon + \gamma_t) \in L^2_{a,T}$ and $M = \int_0^T \sigma^2(\eta_s^\varepsilon + \gamma_s) ds$, we have

$$\mathbb{P}\left(\sup_{0\le t\le T} \left| \int_0^t \sigma(\eta_s^\varepsilon + \gamma_s) dB_s \right| > \frac{\alpha}{\exp(LT)\varepsilon} \right) \le 2 \exp\left(-\frac{\alpha^2 e^{-2LT}}{2\varepsilon^2 M} \right)$$

Finally, since σ is bounded by hypothesis, $M \le T|\sigma|_\infty^2$ and

$$\mathbb{P}\left(\sup_{0\le t\le T} |X_t^\varepsilon - \gamma_t| > \alpha \right) \le 2 \exp\left(-\frac{\alpha^2 e^{-2LT}}{2\varepsilon^2 T |\sigma|_\infty^2} \right).$$

(d) We have, by the previous point, that

$$\lim_{\varepsilon\to 0} \mathbb{P}\left(\sup_{0\le t\le T} |X_s^\varepsilon - \gamma_s| > \alpha \right) = 0.$$

Since $\alpha > 0$ was arbitrary, we have the convergence in probability.

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