9.1 Let W_t be a standard one-dimensional Brownian motion with $W_0 = 1$ and let r be a real number. Let T be the first time that $W_t = 0$. Let $R_t = W_t^r$.

(a) Write the stochastic differential equation for R_t (valid for $t < T$), i.e., find f, g such that

$$dR_t = f(R_t)\,dt + g(R_t)\,dW_t.$$

(b) Find a function F such that $M_{t\wedge T}$ is a martingale where

$$M_t = R_t \exp\left\{\int_0^t F(R_s)\,ds\right\}.$$

$r \in \mathbb{R}$, $T = \min\{t : B_t = 0\}$

$R_t = B_t^r$

(a) Sea $h(x) = x^r$,

$\Rightarrow h'(x) = r x^{r-1}$ $\quad h''(x) = r(r-1)x^{r-2}$

Itô:

$$dR_t = r B_t^{r-1} dB_t + \frac{r(r-1)}{2} B_t^{r-2} dt$$

Si

$$g(x) = r x^{1-\frac{1}{r}} \qquad f(x) = \frac{r(r-1)}{2} x^{1-\frac{2}{r}}.$$

(b) = T Ldp $\quad M_t = R_t \exp\left\{\int_0^t F(R_s)ds\right\} \stackrel{\cdot\cdot}{=} R_t J_t$

¿Qué función F hace de $M_{t\wedge T}$ una mgl? F suave

Fórmula del producto $\quad$ ejercicio.

$$dM_t = R_t\, dJ_t + J_t\, dR_t + \overset{\text{ejercicio}}{0}$$

$$= R_t\, dJ_t + J_t\left[g(R_t)dB_t + f(R_t)dt\right]$$

$$= \underbrace{R_t J_t F(R_t)dt}_{} + J_t g(R_t) dB_t + \underbrace{J_t f(R_t)dt}_{}$$

$$\Rightarrow F(R_t) = -\frac{J_t f(R_t)}{J_t R_t}$$

$$\boxed{F(x) = -\frac{f(x)}{x}}$$

$$dM_{t\wedge T} = -\frac{f(R_{t\wedge T})}{R_{t\wedge T}}$$

9.3 Let W_t be a standard one-dimensional Brownian motion and let $a, b > 0$. Let $T_{a,-b}$ be the first time t such that $W_t = a$ or $W_t = -b$.

✓(a) Use the martingale W_t to find $\mathbb{P}\{W_{T_{a,-b}} = a\}$.

✓(b) Use the martingale $W_t^2 - t$ to find $\mathbb{E}[T_{a,-b}]$.

✓(c) Explain why the random variables $T_{a,-a}$ and $W_{T_{a,-a}}$ are independent.

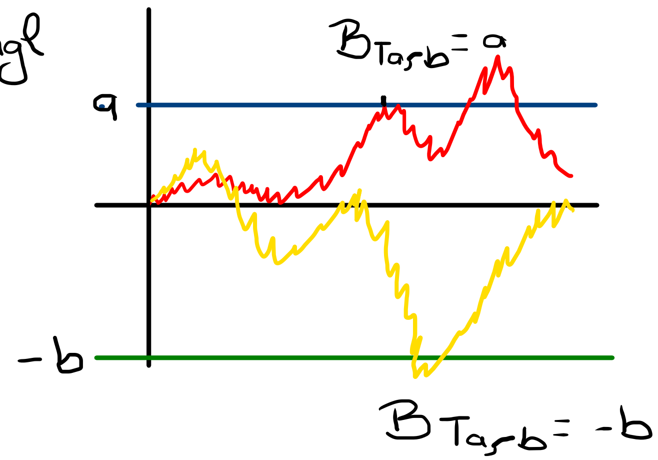
✓(d) Are the random variables $T_{a,-b}$ and $W_{T_{a,-b}}$ independent for all a, b?

✓(c) Use the martingale $e^{\lambda W_t - (\lambda/2)t}$ to compute the moment generating function for $T_{a,-a}$.

<u>Solución</u> $a,b>0$ $T_{a,-b} = \min\{t : B_t = a \text{ ó } B_t = -b\}$

- Recordemos que $B_t \text{ es } \mathcal{F}_t\text{-mgl}$
- $\mathbb{P}(T_{a,-b} < \infty) = 1$

$$\sup_{t\in\mathbb{R}} B_t = \begin{cases} \infty & B_t \\ -\infty & -B_t \end{cases}$$



- TMO M_t $\mathcal{F}_t$-mgl cont

$T < \infty$ c.s. $\mathbb{E}[M_T] = \mathbb{E}[M_0]$

Queremos calcular $\mathbb{P}(B_{T_{a,-b}} = a)$

$$\mathbb{E}[B_{T_{a,-b}}] = 0$$
$$\parallel$$
$$\mathbb{E}[B_{T_{a,-b}} \mathbb{1}\{B_{T_{a,-b}} = a\}] + \mathbb{E}[B_{T_{a,-b}} \mathbb{1}\{B_{T_{a,-b}} = -b\}]$$

$$= a\,\mathbb{P}(B_{T_{a,-b}} = a) + (-b)\mathbb{P}(B_{T_{a,-b}} = -b) = 0 = a\mathbb{P}(B_{T_{a,-b}} = a) + (-b)(1 - \mathbb{P}(B_{T_{a,-b}} = a)) = 0$$

$$= (a+b)\mathbb{P}(B_{T_{a,-b}} = a) - b = 0$$

$$\mathbb{P}(B_{T_{a,-b}} = a) = \frac{b}{b+a}$$

Nota. $T_{a,-a} = T$ $\mathbb{P}(B_T = a) = \frac{1}{2}$

(b) $\mathbb{E}[T_{a,-b}]$

$M_t = B_t^2 - t$ $\mathcal{F}_t$-mgl

$$\mathbb{E}[M_{T_{a,-b}}] = 0 \quad \text{TMO}$$
$$\parallel$$
$$\mathbb{E}[B^2_{T_{a,-b}}] = \mathbb{E}[T_{a,-b}]$$
$$\parallel$$
$$-ab = \mathbb{E}[T_{a,-b}]$$

$T = \min\{t \mid |B_t| = 1\}$ $\mathbb{E}[T] = 1$

(c) Ej 7

(d) No lo son.

(e) $M_t = e^{\lambda B_t - \frac{\lambda t}{2}}$ mgl

TMO $\mathbb{E}[M_T] = 1$ $B_T = 1$ $B_T = -1$

$$\parallel$$

$$\mathbb{E}[e^{\lambda B_T - \frac{\lambda T}{2}}] = \mathbb{E}[e^{\lambda - \frac{\lambda T}{2}} + e^{-\lambda - \frac{\lambda T}{2}}]$$

$$= \mathbb{E}[e^{-\frac{\lambda T}{2}}](e^{\lambda} + e^{-\lambda}) = 1$$

$$\mathbb{E}[e^{-\frac{\lambda T}{2}}] = 2\cosh(\lambda)^{-1}$$

Ejercicio 3. Sea

$$X_t = X_0 + \int_0^t H_s dB_s + \int_0^t V_s ds, \qquad t \geq 0$$

un proceso de Itô.

(1) Sea $Y_t := \exp\left(-X_t + \frac{1}{2}\langle X\rangle_t\right)$, $t \geq 0$. Demuestra que existen reales $\alpha \in \mathbb{R}$ y $\beta \in \mathbb{R}$ tales que

$$Y_t = Y_0 + \alpha \int_0^t Y_s dX_s + \beta \int_0^t Y_s d\langle X\rangle_s, \qquad t \geq 0.$$

(2) Sea $(Z_t)_{t\geq 0}$ un proceso de Itô tal que

$$Z_t = Z_0 + \int_0^t Z_s dX_s, \qquad t \geq 0.$$

Determina $\langle Y, Z\rangle$.

Ejercicio 4. Supongamos que existe un real $\gamma \in \mathbb{R}$ tal que

$$X_t := (B_t + \gamma t)e^{-B_t - \frac{1}{2}t}, \qquad t \geq 0,$$

es una martingala. ¿Cuál es el valor de γ?

Solución ej 3.

Nota. X_t es proceso de Itô.

Sea $f(t,x) = \exp\{-x + \frac{1}{2}g(t)\}$

$$\dot{f}(t,x) = \frac{1}{2}g'(t)f(t,x)$$

$$f'(t,x) = -f(t,x)$$

$$f''(t,x) = f(t,x).$$

Aplicamos Itô:

$$Y_t = Y_0 + \frac{1}{2}\int_0^t Y_s d\langle X\rangle_s - \int_0^t Y_s dX_s + \frac{1}{2}\int_0^t Y_s d\langle X\rangle_s$$

$$\Rightarrow Y_t = Y_0 - \int_0^t Y_s dX_s + \int_0^t Y_s d\langle X\rangle_s \Rightarrow \alpha = -1 \text{ y } \beta = 1$$

$$\langle Y, Z\rangle_t = \int_0^t -Y_s Z_s d\langle X\rangle_s$$

Nota.

$$dY_t = -Y_t dX_t + Y_t d\langle X\rangle_t$$

$$dY_t = -Y_t H_t dB_t - Y_t V_t dt + Y_t d\langle X\rangle_t$$

$$dZ_t = Z_t dX_t = Z_t H_t dB_t + Z_t V_t dt$$

Obs. Esto prueba que la mgl exp de un proceso de Itô

$$Z_t = \exp\{X_t - \frac{1}{2}\langle X\rangle_t\}$$

es <u>única</u>. (Ejercicio).

Solución del ej 4

Sea $f(t,x) = (x + \gamma t)e^{-x - \frac{1}{2}t}$

$$\dot{f}(t,x) = \left(\gamma - \frac{x + \gamma t}{2}\right)e^{-x-\frac{1}{2}t}$$

$$f'(t,x) = (1 - x - \gamma t)e^{-x-\frac{1}{2}t}$$

$$f''(t,x) = (-2 + x + \gamma t)e^{-x-\frac{1}{2}t}$$

Por Itô:

$$X_t = \int_0^t (1 - B_s - \gamma s)e^{-B_s - \frac{1}{2}s} dB_s + (\gamma - 1)\int_0^t e^{-B_s - \frac{1}{2}s} ds$$

Para que X_t sea mgl, basta que

$$\gamma = 1$$

Ejercicio. Verifique que, $\forall\, t \geq 0$

$$\mathbb{E}\left[\int_0^t (1 - B_s - s)^2 e^{-2B_s - s} ds\right] < \infty.$$

The Black-Scholes formula is a way to calculate the current value of an option that is based on the price of a stock following a stochastic differential equation. Suppose S_t denotes the price of a stock, and S_t satisfies

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t.$$

By (9.6), the solution of this is

$$S_t = S_0 \exp\left\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\right\}.$$

Assume also that one can buy or sell a bond with guaranteed interest rate r. If we let Y_t be the amount of money invested in bonds, then if we do not buy or sell any bonds the amount grows according to the equation

$$dY_t = r\, Y_t\, dt.$$

A *European call option (with strike price K at time T)* is an opportunity to buy one share of the stock at time T for price K. If $S_T \leq K$ such an option is useless, but if $S_T > K$, then it has a value of $S_T - K$, which is the profit obtained by buying the stock and then selling it immediately. We can write the value as $(S_T - K)_+$ where $x_+ = \max\{x, 0\}$. The *Black-Scholes formula* determines the value of this option at a time $t \leq T$ under the assumption that there are no *arbitrage* opportunities. Let V_t denote this value. Clearly $V_T = (S_T - K)_+$, and V_t should be measurable with respect to $\mathcal{F}_t$, the information at time t. It is reasonable to assume that $V_t = V(t, S_t)$; we will determine this function. Note that $V(T, x) = (x - K)_+$.

We can think of the option as an asset with value V_t at time $t \leq T$. Suppose we sell such an option at time $t < T$ and invest the money in a portfolio consisting of a combination of the stock and the bond, say X_t shares of the stock and Y_t invested in the bond. We assume we have a buying and selling strategy between bonds and stocks based on the stock price at a certain time. Here Y_t is determined by the X_t and the relationship that stocks are bought only with money obtained from selling bonds and vice versa.

The value of the total portfolio (one option sold plus the total of assets in bonds and stocks) at time s is

$$U_t = -V(t, S_t) + O_t,$$

where

$$O_t = X_t\, S_t + Y_t. \tag{9.16}$$

For ease, let us assume that $U_0 = 0$, i.e., at time $t = 0$ we sold one option and invested that money in some combination of bond and stock.

Suppose we monitor this investment up to time T (switching between shares of the stock and the bond based on the price of the stock) using a strategy that *guarantees* that $U_T \geq 0$. If it is also true that with positive probability $U_T > 0$, then we have found a way to gain money (with positive probability) without any risk. This is called an *arbitrage*. Similarly, if there is a strategy to guarantee $U_T \leq 0$ with a chance that $U_T < 0$, then there are arbitrage possibilities by buying an option. The main assumption in the Black-Scholes formula is: there are no arbitrage opportunities with "self-financing" strategies.

The self-financing assumption is that the change in the total value of the bond/stock portfolio is given by

$$dO_t = X_t\, dS_t + r\, Y_t\, dt. \tag{9.17}$$

In other words, the change in the value is the number of shares of stock times the change in stock price plus the number of units of the bond times the change in bond price. Assuming (9.17), we can use Itô's formula to write

$$\begin{aligned} dU_t &= -dV(t, S_t) + dO_t \\ &= -\dot{V}(t, S_t)\, dt - V'(t, S_t)\, dS_t - \frac{1}{2} V''(t, S_t)\, d\langle S\rangle_t \\ &\qquad + X_t\, dS_t + r\, Y_t\, dt \end{aligned}$$

Now, to remove the randomness from the value of the portfolio we choose $X_t = V'(t, S_t)$. This makes the coefficient of dW_t zero and

$$dU_t = \left[-\dot{V}(t, S_t) - \frac{1}{2} V''(t, S_t)\, \sigma^2\, S_t^2 + r\, Y_t\right]\, dt. \tag{9.18}$$

The assumption of no arbitrage tells us that this must equal zero.

Using the product rule (9.9) on (9.16), we see that

$$dO_t = X_t\, dS_t + dY_t + S_t\, dX_t + d\langle X, S\rangle_t.$$

Hence, the self-financing condition (9.17) can be written as

$$dY_t = r\, Y_t\, dt - S_t\, dX_t - d\langle X, S\rangle_t.$$

Since $X_t = V'(t, S_t)$, Itô's formula gives

$$dX_t = [\dot{V}'(t, S_t) + V''(t, S_t)\, \mu\, S_t + \frac{1}{2} V'''(t, S_t)\, \sigma^2\, S_t^2]\, dt$$

$$+ V''(t, S_t)\, \sigma\, S_t\, dW_t.$$

Hence Y_t must satisfy

$$dY_t = r\,Y_t\,dt - \Big[\dot V'(t,S_t)\,S_t + V''(t,S_t)\,(\mu+\sigma^2)\,S_t^2$$

$$+\frac{1}{2}V'''(t,S_t)\,\sigma^2\,S_t^3\Big]\;dt - V''(t,S_t)\,\sigma\,S_t^2\,dW_t. \tag{9.19}$$

Let

$$Y_t = V(t,S_t) - S_t\,X_t = V(t,S_t) - S_t\,V'(t,S_t),$$

and assume that $V(t,x)$ has been chosen so the quantity in (9.18) vanishes, i.e.,

$$\dot V(t,x) + \frac{1}{2}\,x^2\,\sigma^2\,V''(t,x) + r\,x\,V'(t,x) - r\,V(t,x) = 0. \tag{9.20}$$

Then an Itô's formula calculation shows that (9.19) holds.

One can get lost in the calculation, so it is worth understanding why it works. If there are no arbitrage opportunities and the option is priced properly, then any strategy that produces no randomness must also produce no gain or loss. Hence the current value of the portfolio, O_t, must also be the price of the option at that time, i.e., $V(t,S_t) = O_t$. Since we know that we must have $V'(t,S_t)$ shares of the stock to hedge the option, the assets in bonds must be

$$Y_t = O_t - X_t\,S_t = V(t,S_t) - V'(t,S_t)\,S_t.$$

Plugging into (9.18) we get the *Black-Scholes equation* (9.20).

Note that the Black-Scholes equation has r and σ^2 as parameters but μ does not appear! The value of the option depends only on the bond rate and the variance parameter (sometimes called the *volatility*) σ^2. We need to find the solution of this equation with boundary condition $V(T,x) = (x-K)_+$. The dependence on r can be removed by a simple change of variables: if V satisfies (9.20) with $r=0$,

$$\dot V(t,x) + \frac{1}{2}\,x^2\,\sigma^2\,V''(t,x) = 0, \tag{9.21}$$

and $\tilde V(t,x) = e^{r(t-T)}\,V(t,e^{r(T-t)}x)$, then $\tilde V(t,x)$ satisfies (9.20) and $\tilde V(T,x) = V(T,x)$. This can be checked by differentiation (Exercise 9.7); however, there is a simple reason why this is true. If money grows at rate r, then x dollars at time T is the equivalent of $e^{r(t-T)}x$ dollars at time t. Hence, it suffices to solve the equation when $r=0$.

A probabilistic form for the solution of (9.21) is given by the Feynman-Kac formula (9.15); in fact, this form can be used for options with different payoffs $V(T,x) = g(x)$. Assume $r=0$. Remembering that $V(t,S_t) = O_t$, we get

$$dV(t,S_t) = V'(t,S_t)\;dS_t.$$

If $V(t,S_t)$ were a martingale, we would know that

$$\mathbb{E}\,[V(t,S_t)] = \mathbb{E}\,[V(T,S_T)] = \mathbb{E}\,[g(S_T)].$$

Recall that S_t satisfies

$$dS_t = \mu\,S_t\;dt + \sigma\,S_t\;dW_t.$$

This is a martingale only if $\mu=0$. However, we have seen that the value of the option does not depend on the value of μ, so we can set $\mu=0$. If $\mu=0$ the solution to the stochastic differential equation is

$$S_t = \exp\left\{\sigma\,W_t - \frac{\sigma^2}{2}t\right\}.$$

Then we have

$$\begin{aligned}
V(T-t,x) &= \mathbb{E}\,[g(S_t)\mid S_t = x] \\
&= \mathbb{E}\left[g\left(\exp\{\sigma\,W_t - \frac{\sigma^2\,t}{2}\}\right)\mid W_t = \frac{\log x}{\sigma}\right] \\
&= \mathbb{E}\left[g\left(x\,e^{-\sigma^2 t/2}\,e^{\sigma\,\sqrt{t}\,N}\right)\right],
\end{aligned}$$

where N is a standard unit normal.

Suppose $g(y) = (y-K)_+$. Then,

$$V(T-t,e^{\sigma^2 t/2}x) = \mathbb{E}\,[(x\,e^{\sigma\sqrt{t}N} - K)_+].$$

A straightforward, although tedious, calculation (see Exercise 9.4) shows that the right-hand side is

$$x\,e^{\sigma^2 t/2}\,\Phi\left(\frac{\log(x/K)+\sigma^2 t}{\sigma\,\sqrt{t}}\right) - K\,\Phi\left(\frac{\log(x/K)}{\sigma\,\sqrt{t}}\right),$$

where Φ denote the standard normal distribution function. Hence $V(T-t,x)$ is given by

$$x\,\Phi\left(\frac{\log(x/K)+(1/2)\,\sigma^2 t}{\sigma\,\sqrt{t}}\right) - K\,\Phi\left(\frac{\log(x/K)-(1/2)\,\sigma^2 t}{\sigma\,\sqrt{t}}\right),$$

This is the solution for $r=0$, and we can easily convert it to the solution for general r.

Black-Scholes Formula. *Suppose $V(t,x)$ is the solution to (9.19) satisfying $V(T,x) = (x-K)_+$. Then $V(T-t,x)$ equals*

$$x\,\Phi\left(\frac{\log(x/K) + (r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right) - K\,e^{-rt}\,\Phi\left(\frac{\log(x/K) + (r-\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}\right),$$

where Φ is the standard normal distribution function.

Let us generalize and assume that S_t satisfies

$$dS_t = \mu(t,S_t)\,S_t\,dt + \sigma(t,S_t)\,S_t\,dW_t,$$

where $\mu(t,x), \sigma(t,x)$ are given functions. We cannot give an explicit solution to this stochastic differential equation. However, we can still give an expression for the value of a European call option. We assume that we have a self-financing portfolio with value $O_t = X_t\,S_t + Y_t$ that "hedges" the option. If $V(t,x)$ denotes the value of the option, then we choose $X_t = -V'(t,S_t)$ in order to remove the randomness. Assuming no arbitrage, the value of the portfolio using the hedging strategy is exactly the same as the value of the option at that time. Therefore $Y_t = O_t - X_t\,S_t = V(t,S_t) - V'(t,S_t)\,S_t$. Hence, we again obtain the Black-Scholes equation (9.20) where σ^2 is replaced with $\sigma^2(t,x)$. We need to find the solution to

$$\dot{V}(t,x) + \frac{1}{2}\,\sigma^2(t,x)\,V''(t,x) + r\,x\,V'(t,x) - r\,V(t,x) = 0,$$

with $V(T,x) = g(x)$. Note again that $\mu(t,x)$ does not appear in the equation. In most cases, there is no closed form for this solution. However, the Feynman-Kac formula (9.15) gives the value in terms of an expectation that can be estimated by simulation.

Ejercicio 7. Sea B un movimiento browniano que comienza en cero y $\gamma \in \mathbb{R}$. Sea

$$T = \inf\{t \geq 0 : |B_t + \gamma t| = 1\}.$$

(1) Pruebe que si $\gamma = 0$ entonces T y B_T son independiente.

(2) Al utilizar el teorema de Girsanov muestre la independencia entre T y B_T cuando $\gamma \neq 0$.

Ejercicio 8 (Transformación de deriva). Considere la ecuación diferencial estocástica

$$dX_t = dB_t + b(X_t)dt \qquad X_0 = x$$

donde b es continua y acotada. Suponga qe bajo $\mathbb{P}$, X es un movimiento browniano que comienza en x. Utilice el teorema de Girsanov para encontrar una medida de probabilidad $\widetilde{\mathbb{P}}$ tal que si definimos a

$$B_t = X_t - \int_0^t b(X_s)ds$$

entonces $(B_t)_{t\leq 1}$ es un $\widetilde{\mathbb{P}}$-movimiento browniano

Solución 7. $\gamma \in \mathbb{R}$ $T = \inf\{t \geq 0 : |B_t + \gamma t| = 1\}$

(Girsanov)

(1) $\gamma = 0$ T y B_T son indeps.

Obs. $-B$ es mb y $(-B)_T = -B_T$ $T = \inf\{t \geq 0 : |-B_t| = 1\}$

$$2\mathbb{P}(B_T = 1) = \mathbb{P}(B_T = 1) + \mathbb{P}((-B)_T = 1)$$

$$= \mathbb{P}(B_T = 1) + \mathbb{P}(B_T = -1)$$

$$= 1 \qquad \mathbb{P}(B_T = 1) = \frac{1}{2}$$

$$\mathbb{P}(B_T \in \{-1, 1\}, T \in A) = \frac{1}{2}\mathbb{P}(T \in A) = \mathbb{P}(B_T = 1)\mathbb{P}(T \in A)$$

$$\mathbb{P}(B_T = -1, T \in A) = \mathbb{P}(B_T = 1, T \in A)$$

(2) $\gamma \neq 0$ Por Girsanov, $\exists$ $\mathbb{Q}$ m. de prob $\ni$ $\mathbb{P} \ll \mathbb{Q}$

$A \in \mathcal{F}_t$

$$\underline{\mathbb{Q}(A) = e^{-\gamma B_t - \gamma^2 \frac{t}{2}} \mathbb{P}(A)}$$

$$X_t^\gamma = B_t + \gamma t \Rightarrow X^\gamma \text{ es } \mathbb{Q}\text{-mb}$$

Sabemos que $T < \infty$ c.s. $\Rightarrow T < \infty$ $\mathbb{Q}$-c.s.

Sea f, g cont y acotadas

$$\mathbb{P}(f(X_T^\gamma) g(T)) \overset{\mathbb{P}\ d}{=} \mathbb{P}(f(X_T^\gamma)) \mathbb{P}(g(T))$$

$$\| \qquad\qquad\qquad \|$$

$$\mathbb{Q}\left(e^{\gamma X_T^\gamma} f(X_T^\gamma) e^{-\gamma^2 \frac{T}{2}} g(T)\right) \overset{(1)}{\underset{(*)}{=}} \mathbb{Q}\left(e^{\gamma X_T^\gamma} f(X_T^\gamma)\right) \mathbb{Q}\left(e^{-\gamma^2 \frac{T}{2}} g(T)\right)$$

$$\mathbb{P}(f(X_T^\gamma)) = \frac{\mathbb{Q}\left(e^{\gamma X_T^\gamma} f(X_T^\gamma)\right)}{\mathbb{Q}\left(e^{\gamma X_T^\gamma}\right)} \qquad \mathbb{P}(g(T)) = \frac{\mathbb{Q}\left(e^{-\gamma^2 \frac{T}{2}} g(T)\right)}{\mathbb{Q}\left(e^{-\gamma^2 \frac{T}{2}}\right)}$$

$$\mathbb{Q}\left(e^{\gamma X_T^\gamma}\right) \mathbb{Q}\left(e^{-\gamma^2 \frac{T}{2}}\right) \underset{(*)}{=} 1$$

Solución del ej 8 Mgl Dóleans $(B_t)_{t\geq 0}$ mbs.

Girsanov: $\mathcal{E}(\tilde{X})_t = \exp\left\{\tilde{X}_t - \frac{1}{2}\langle \tilde{X}\rangle_t\right\} > 0$ mgl donde $\{\tilde{X}_t\}$ mdbl y adaptado a $\sigma(B_s, s\leq t)$

positiva (hipótesis)

$\exists\ \mathbb{Q}$ m. prob sobre $(\Omega, \mathcal{F})$ s. $\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(\tilde{X})_t$ $\mathbb{Q}|_{\mathcal{F}_t} = \mathcal{E}(\tilde{X})_t \cdot \mathbb{P}|_{\mathcal{F}_t}$

$dX_t = dB_t + b(X_t)dt \qquad X_0 = x$

Hip. Bajo $\mathbb{P}$, X es mb que comienza en x.

Mostrar que $\exists\ \tilde{\mathbb{P}}$ s. $dB_t = dX_t - b(X_t)dt$, $t\leq 1$ sea $\tilde{\mathbb{P}}$-mb.

Por el teo de Girsanov, exhibir a $\mathcal{E}(\tilde{X}_t)$

Sea

$$\mathcal{E}_t = \exp\left\{\int_0^t b(X_s)dX_s - \frac{1}{2}\int_0^t b^2(X_s)ds\right\}$$

$$\Rightarrow \quad \mathbb{Q}|_{\mathcal{F}_t} = \mathcal{E}_t \cdot \mathbb{P}|_{\mathcal{F}_t}$$

9.5 Let $X_1, X_2, \ldots$ be independent $N(0,1)$ random variables and let f be a bounded continuous function. Let $Z_0 = 0$ and for $n > 0$,

$$Z_n = Z_{n-1} + f(Z_{n-1}) + X_n.$$

We will do the Girsanov transformation for Z_n to make Z_n a martingale (with respect to $\mathcal{F}_n$, where $\mathcal{F}_n$ is the information in $X_1, \ldots, X_n$).

(a) If a is a real number, compute $\mathbb{E}\left[X_1 e^{aX_1}\right]$. (One can do it directly, or one can differentiate the moment generating function $\mathbb{E}\left[e^{aX_1}\right]$ with respect to a.)

(b) Let $M_0 = 1$ and for $n > 0$,

$$M_n = \exp\left\{ -\sum_{j=1}^{n} f(Z_{j-1})\, X_j - \sum_{j=1}^{n} \frac{f(Z_{j-1})^2}{2} \right\}.$$

Show that M_n is a martingale with respect to $\mathcal{F}_n$.

(c) Show that $M_n Z_n$ is a martingale with respect to $\mathcal{F}_n$.

(d) Show that Z_n is a $\tilde{\mathbb{P}}$-martingale where $d\tilde{\mathbb{P}} = M_n\, d\mathbb{P}$.

$\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ $\quad X_j \sim N(0,1)$ indeps

$Z_0 = 0 \qquad Z_n = Z_{n-1} + f(Z_{n-1}) + X_n$

(a) $M_{X_1}(a) = \mathbb{E}\left[e^{aX_1}\right] = e^{a^2/2}$

$M'_{X_1}(a) = \mathbb{E}\left[X_1 e^{aX_1}\right] = a e^{a^2/2}$

(b) M_n es $\mathcal{F}_n$-mgl.

$n=2$

$$M_2 = e^{-(f(0)X_1 + f(Z_1)X_2)}\, e^{-\frac{1}{2}(f(0)^2 + f(Z_1)^2)}$$

$$\mathbb{E}[M_2 \mid \mathcal{F}_1] = e^{-\frac{1}{2}f(0)^2}\, e^{-f(0)X_1}\, \mathbb{E}\left[e^{f(Z_1)X_2 - \frac{1}{2}f(Z_1)^2} \mid \mathcal{F}_1\right] = M_1$$

$$Z_n = Z_{n-1} + f(Z_{n-1}) + X_n = Z_{n-2} + f(Z_{n-2}) + X_{n-1} + f(Z_{n-1}) + X_n$$

Por inducción (Note que $Z_n = \sum_{j=1}^{n-1} f(Z_j) + X_{j+1}$)

(Ejercicio).

(c) $M_n Z_n = \underbrace{M_n \sum_{j=1}^{n-1} f(Z_j)}_{\text{mgl}} + M_n Y_n \qquad Y_n = \sum_{j=1}^{n} X_j$

$\mathbb{E}[M_{n+1} Y_{n+1} \mid \mathcal{F}_n] = M_n Y_n \quad$ inducción + (b)

(d) Ejercicio: Aplica Girsanov.

Solución ej 2.

Sea B mb

$$H_n(x,t) = \frac{d}{d\lambda^n}\Big|_{\lambda=0} e^{\lambda x - \lambda^2 t/2}$$

Polinomios de Hermite

(a) $(H_n(B_t,t), t\geq 0)$ son mgls c.r.a $\mathcal{F}_t = \sigma(B_s, s\leq t)$

(b) Mgls $n=1,2$. ✓

$$H_1(x,t) = (x-\lambda t)e^{\lambda x - \lambda^2 t/2}\Big|_{\lambda=0} = x$$

$\underline{H_1(B_t,t) = B_t \text{ mgl}}$

$$H_2(x,t) = (x-\lambda t)^2 e^{\lambda x - \lambda^2 t/2} + e^{\lambda x - \lambda^2 t/2}(-t)\Big|_{\lambda=0}$$

$$= x^2 - t$$

$\underline{H_2(B_t,t) = B_t^2 - t \text{ mgl}}$

$$H_3(x,t) = (x-\lambda t)^3 e^{\lambda x - \lambda^2 t/2} + e^{\lambda x - \lambda^2 t/2}\left[2(x-\lambda t)(-t)\right] - t(x-\lambda t)e^{\lambda x - \lambda^2 t/2}\Big|_{\lambda=0}$$

$$= x^3 - 3xt$$

$\underline{H_3(B_t,t) = B_t^3 - 3tB_t \text{ mgl}}$

(a) $\mathcal{F}_t = \sigma(B_s, s\leq t)$ canónica.

Obs $\underline{e^{\lambda B_t - \lambda^2 t/2} \text{ mgl}}$ $n=0$

Podemos probar que

$$H_n^\lambda(B_t,t) = \frac{d^n}{d\lambda^n} e^{\lambda B_t - \lambda^2 t/2} \quad \text{es mgl}$$

$n=0$ ✓ Sup n pd $n+1$

$$H_n^\lambda(B_s,s) = \mathbb{E}\left[H_n^\lambda(B_t,t)\,|\,\mathcal{F}_s\right] \quad s\leq t$$

Sea $\varepsilon > 0$

$$\frac{H_n^{\lambda+\varepsilon}(B_s,s) - H_n^\lambda(B_s,s)}{\varepsilon} = \mathbb{E}\left[\frac{H_n^{\lambda+\varepsilon}(B_t,t) - H_n^\lambda(B_t,t)}{\varepsilon}\,\Big|\,\mathcal{F}_s\right]$$

TCD $\quad \downarrow \varepsilon\to 0 \qquad\qquad \downarrow \varepsilon\to 0$

$$H_{n+1}^\lambda(B_s,s) = \mathbb{E}\left[H_{n+1}^\lambda(B_t,t)\,|\,\mathcal{F}_s\right] \checkmark$$

Ejercicio 8 (Opciones europeas y americanas). Sea $dS_t = S_t(rdt + \sigma_t dB_t)$ el precio de un activo en donde σ es un proceso adaptado y acotado. Sea $C = \mathbb{E}[e^{-rT}(S_T - K)^+]$ el precio de un *call* europeo y $C^{Am} = \sup_\tau \mathbb{E}[e^{-r\tau}(S_\tau - K)^+]$ el precio de un *call* americano, en donde el supremo corre sobre todos los tiempos de paro con valores en $[0,T]$. Denótese $P = \mathbb{E}[e^{-rT}(Ke^{rT} - S_T)^+]$ y $P^{Am} = \sup_\tau \mathbb{E}[e^{-r\tau}(Ke^{r\tau} - S_\tau)^+]$ el precio de *puts* con *strikes* actualizados. Demuestra que, para todo $t < u \le T$,

$$\mathbb{E}[e^{-ru}g(S_u) \mid \mathcal{F}_t] \le \mathbb{E}[e^{-rT}g(S_T) \mid \mathcal{F}_t],$$

en donde g es una función convexa de clase C^2 tal que $g(0) = 0$. Deduce que $C = C^{Am}$ y $P = P^{Am}$.

Solución.

$dS_t = S_t(rdt + \sigma_t dB_t)$ Feynmac-Kac

σ_t adapta a $\mathcal{F}_t$ y acotado. Volatilidad.

Call europeo $C = \mathbb{E}[e^{-rT}(S_T - K)^+]$ Put $P = \mathbb{E}[e^{-rT}(Ke^{rT} - S_T)^+]$

Call americano $C^{Am} = \sup_{\tau\ \text{tdp}} \mathbb{E}[e^{-r\tau}(S_\tau - K)^+]$ $P^{Am} = \sup_{\tau\ \text{tdp}} \mathbb{E}[e^{-r\tau}(Ke^{r\tau} - S_\tau)^+]$

p.d. $t < u \le T$ $\mathbb{E}[e^{-ru}g(S_u) \mid \mathcal{F}_t] \le \mathbb{E}[e^{-rT}g(S_T) \mid \mathcal{F}_t]$

$g \in C^2$ $g(0) = 0$ convexa

Prerrequisito a) g convexa Jensen $\mathbb{E}[g(X) \mid \mathcal{G}] \ge g(\mathbb{E}[X \mid \mathcal{G}])$

b) τ tdp acotado $\tau \le T$ y Z submgl $\Rightarrow \mathbb{E}[Z_T] \ge \mathbb{E}[Z_\tau]$
TMO

c) $g \in C^2$ $g(0) = 0$ convexa
$\forall x\ \forall \alpha \ge 1$ $g(x) \le \frac{1}{\alpha}g(\alpha x)$ Inmediato

Sea $g(x) = (x - K)^+$. Es convexa Nota/Ejercicio $(e^{-rt}S_t - K)^+$ es submgl.

$$C_t^{Am} \ge C_t := \mathbb{E}[e^{-rT}(S_T - K)^+ \mid \mathcal{F}_t] = \mathbb{E}[e^{-rT}g(S_T) \mid \mathcal{F}_t]$$

Ahora bien, $t \le u < T$

$$e^{-ru}g(S_u) \underset{(c)}{\le} e^{-rT}g(S_u e^{r(T-u)}) = e^{-rT}g(e^{rT}\mathbb{E}[S_T e^{-rT} \mid \mathcal{F}_u])$$

$$= e^{-rT}g(\mathbb{E}[S_T \mid \mathcal{F}_u]) \underset{\text{prop de torre + Jensen}}{\le} \mathbb{E}[e^{-rT}g(S_T) \mid \mathcal{F}_t]$$

$C_t \le C_t^{Am}$

$e^{-ru}g(S_u) \le \mathbb{E}[e^{-rT}g(S_T) \mid \mathcal{F}_t] = C_t$ Aplicando esp cond se tiene la igualdad.

$P \le P^{Am}$ y $(K - S_t e^{-rt})^+$ submgl ⊠

Ejercicio 5. Sea $(B_t)_{t\geq 0}$ un movimiento browniano que comienza en 0 con su filtración canónica. Sea $f:\mathbb{R}\to\mathbb{R}$ una función continua. Demuestra que el proceso $M_t = f(B_t)$ es una martingala con respecto a la filtración canónica si y sólo si la función f es afín.

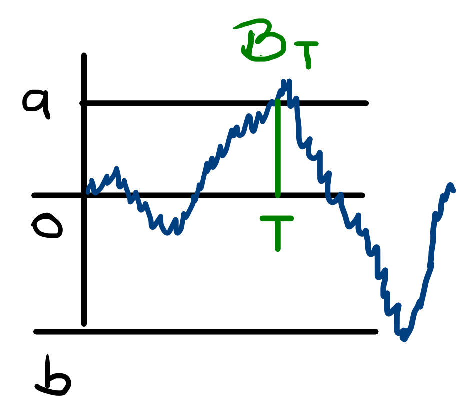
Sugerencias:

Sea $M_t = f(B_t)$ con f continua

$\Rightarrow$) Sup M_t es $\sigma(B_s, s<t) = \mathcal{F}_s$ - mgl

1.- Sean $a<0<b$ y considere $T = \inf\{t\geq 0 : B_t \notin (a,b)\}$

Prueba que T es tdp



2.- Usa TMO para mostrar que

$$\mathbb{E}[f(B_T)] = f(0)$$

(Es necesario verificar detalladamente que se cumplen las hipótesis de TMO)

3.- Calcula $\mathbb{E}[f(B_T)]$ y verifica que

$$\frac{b}{b-a} f(a) + \frac{-a}{b-a} f(b) = f(0) \quad \text{(Ver Ej 1 Lawler 9.3)}$$

4.- Concluye que f es afín

$\Leftarrow$) Ejercicio fácil.