2024近世代数期中小测

最懒得切换输入法的一集）

1.(7p) Assume that $M \triangleleft G, N \triangleleft G$,then $G/(M \cap N)$ is isomorphic to a subgroup of $(G/M) \times (G/N)$.

2.(7p) Prove that a non-abelian group of order 6 is S_3.

3.(8p) Prove that a non-trivial group with non-trivial subgroups must be a cyclic group with a prime order.

4.(8p) Prove that for any $n \in \mathbb{Z}_{\leq 0}$, there are only finitely many mutually non-isomorphic groups of order n.

5.(10p) Assume that R is a ring. If for any element $a \in R$, $a^2 = a$ holds, then we call R a $Boole\ Ring$. Prove that a $Boole\ Ring$ must be a commutative ring, and $a + a = 0$ holds for any $a \in R$.

6.(10p) Assume that G is a finite non-abelian simple group, and p is a prime number with $p \mid |G|$. Prove that G has more than one $Sylow\ p$ -subgroup.

7.(10p) Assume that K, H are two subgroup of G, use $[K, H]$ to denote the group generated by $\{[k,h] | k \in K, h \in H\}$. Use $\langle K, H \rangle$ to denote the group generated by $K \cup H$. Try to prove that:
(1) If $H \triangleleft G$ and $H \leq K$, then $K/H \leq Z(G/H) \iff [K, G] \leq H$.
(2) $[K, H] \triangleleft \langle K, H \rangle$.

8.(10p) Assume that p is a factor of the order of G as G is a finite group, and N is a normal subgroup of G. If p and $|G/N|$ are coprime, then N contains all $Sylow\ p$ -subgroups of G.

9.(10p) Assume that N is a normal subgroup of G,and G act transitively on Ω, prove that under this action, all orbits of N acts on Ω has the same length.

10.(10p) Find all the maximal ideals and prime ideals of ring $2\mathbb{Z}$.

11.(10p) Prove that if R is a integral domain but not a field, then R contains a ideal which is not a principal ideal.