Lie Algebra

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Chapter 1

Definition of Lie Algebra

1.1 Definiton and Construction

We begin with a motivation. Consider in three dimensional Euclidean space, the cross product operation of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the vector space V over a field F satisfy the following:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0 \tag{1.1}$$

This can be easily proven by Lagrange's formula. We say these three vectors satisfy the **Jacobi Identity** with operation $\times$. Of course, focus only on cross product operation will be too constrained and will be limiting our generalization of the theory. Therefore, we generalize such identity to arbitrary bilinear operation on vector spaces, which is called the **bracket**.

$$[\cdot, \cdot] : V \times V \to V \tag{1.2}$$

and the Jacobi identity is now read as;

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{c}, [\mathbf{a}, \mathbf{b}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] = 0, \forall\ \mathbf{a}, \mathbf{b}, \mathbf{c} \in V. \tag{1.3}$$

Moreover, we want impose

$$[\mathbf{x}, \mathbf{x}] = 0, \forall\ \mathbf{x} \in V \tag{1.4}$$

just like the cross product operation. We are ready to define Lie algebra.

Definition 1.1.1. A Lie algebra $(V, [\cdot, \cdot])$ is a vector space V (over a certain field F) equipped with the bracket operation $[\cdot, \cdot]$ that satisfies 1.2, 1.3, 1.4.

Few remarks are due, but before that, let us see an example where such algebra arises.

Example 1.1.1. Let us consider the set of 2×2 matrices over field F with trace 0, how can we construct a Lie algebra from this set? In another word, if ever possible, what kind of bilinear operation should we equip to it so that 1.3, 1.4 is satisfied? 1.4 require that $[A, A] = 0$ for all 2×2 trace-free A matrix and we also require the closure of Lie algebra. We first try matrix multiplication, since that is the first natrual thing we would like to try; however, such operation failed our purpose, consider the following

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.5}$$

The result of these two matrices failed to have vanishing trace. Thus we are inclined to quit guessing and try to find such opeartion in a systematic way. Consider the trace of product of two arbitrary 2×2 trace free matrices A, Ψ

$$Tr\{A\Psi\} = Tr\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix}\} = a\epsilon + b\sigma + c\gamma + d\omega \tag{1.6}$$

This is not 0 in general, but it is 0 if we subtract it from itself.

$$0 = a\epsilon + b\sigma + c\gamma + d\omega - a\epsilon + b\sigma + c\gamma + d\omega = a\epsilon + b\sigma + c\gamma + d\omega - \epsilon a - \gamma c - \sigma b - \omega d =$$
$$= Tr\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix} - \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\} = Tr\{A\Psi \ - \Psi A\} \quad (1.7)$$

and

$$Tr\{AA \ - AA\} = Tr\{0\} = 0. \quad (1.8)$$

Thus if we choose our operation to be $[A, \Psi] = A\Psi - \Psi A$, axiom 1.4 will be satisfied, we call such a operation **commutator**: $[\diamond, \star] = \diamond\star - \star\diamond$. After tedious works, we can also show commutator operation also satisfy axiom 1.3 (problem 1).

So the set of 2×2 matrices over field F with trace 0 equipped with commutator $[\cdot, \cdot]$ is a Lie algebra, and we denote it by $\mathfrak{sl}(2, F)$ and it is also called as **special linear Lie algebra of order 2**. It will be crucial important for our theory.

If you are following me carefully enough, you found that commutator will vanish every product of two matrices, not just trace free ones, since we did not used the trace free condition at all. It is also can be showed that for $n \times n$ matrices, commutator also vanishes the product. see problem 2. △

Here we remark on the definiton of Lie algebra:

Remark. Consider the bracket of any arbitrary vectors $\mathbf{x}, \mathbf{y}$ from the vector space

$$0 \underset{by\ 1.4}{=} [\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}] = [\mathbf{x}, \mathbf{x}] + [\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] + [\mathbf{y}, \mathbf{y}] \underset{by\ 1.4}{=} [\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] \quad (1.9)$$

Notice that this is different from

$$[\mathbf{x}, \mathbf{y}] = [\mathbf{y}, \mathbf{x}] \quad (1.10)$$

Identity 1.9 is from axoims and it is true for all Lie algebra while 1.10 is not ture in general. Whenever a Lie algebra satisfies 1.10, we call that Lie algebra abelian. Whenver a Lie algebra equipped with commutator and two elements from the algebra have vanishing commutator, we call those two elements **commute**.

We now are going to see a Physics example where Lie algebra appears

Example 1.1.2. In physics, quantum mechaincally we describe physical states of an object by specifying its obvervable quantities such as position, momentum, and charge and quantum mechanics represent them as vectors in Hilbert space. Moreover, we want to know how those quantites involve with time. Turns out, we can find a quantity H ,called Hamiltonian, such that it govern the time evolution of any quantity A of a physical object:

$$\frac{d}{dt}A = \partial_t A + \frac{i}{\hbar}[H, A]. \quad (1.11)$$

Where $i = \sqrt{-1}$ and $\hbar$ is some physical constant. The bracket in equation 1.11 is the commutator. When ever a physical quantity commute with Hamiltonian and doesn't explicitly depends on time, it is time independent and called **conserved** quantity. You might heard energy is conserved, often time Hamiltonian is the energy of that physical object (but not always), whenever energy is not explicitly depending on time, that is nothing but application of axiom 1.4:

$$\frac{d}{dt}H = \partial_t H + \frac{i}{\hbar}[H, H] = \underbrace{0}_{Energy\ don't\ depends\ on\ time\ explicitly} + \underbrace{0}_{by\ 1.4} = 0. \quad (1.12)$$

Thus H is constant. Equation 1.11 is called Heisenberg picture.

Before people knew quantum mechanics, physical observable quantities were described by some functions of position q, momentum p, and time t, $f(p, q, t)$ on spacetime in Classical Mechanics and its time evolution is described again by (to be distinguished from equation 1.11, we denote bracket here as $\{\cdot, \cdot\}$)

$$\frac{d}{dt}f = \partial_t f + \{H, f\}. \tag{1.13}$$

where $\{\diamond, \star\} \equiv \sum_i (\frac{\partial\diamond}{\partial q_i}\frac{\partial\star}{\partial p_i} - \frac{\partial\star}{\partial q_i}\frac{\partial\diamond}{\partial p_i})$ and it is called Poisson bracket.

Thus we can see Lie aglebra arises from nature and it is very powerful when comes to physical applications. $\triangle$

Of course, we wish to investigate structure of Lie Algebra and possibly categorize it, we are ready to define structure preserving map :

Definition 1.1.2. We say two Lie Algebra $(V, [\cdot,\cdot]), (V', [\cdot,\cdot]')$ are isomorphic if there is a vector space isomorphism ϕ such that

$$\phi : V \to V' \tag{1.14}$$

and satisfies

$$\phi[\mathbf{x}, \mathbf{y}] = [\phi(\mathbf{x}), \phi(\mathbf{y})]', \ \forall \mathbf{x}, \mathbf{y} \in V. \tag{1.15}$$

We call such ϕ isomorphism (of Lie Algebras).

Remark. This definition is as natural as it can be, the bracket grant the vector space extra structure, and if two vector space is isomorphic and their bracket operation behave in the same way, they have exact same structure (Lie algebraically). This is very similar to other definition of structure preserving maps such as group isomorphism.

Naturally, a Lie subalgebra is defined as the following

Definition 1.1.3. We say a Lie algebra $(V', [\cdot,\cdot]')$ is a Lie subalgebra of a Lie algebra $(V, [\cdot,\cdot])$ if $V' \triangleleft V$ (V' is a subspace of V) and $[\cdot,\cdot]' = [\cdot,\cdot]|'_V$.

We will primarily focus on finite dimensional Lie algebras unless otherwise stated.

Problems

You can use any results and definitions introduced previously.

1.Show commutator operation satisfy axiom 1.3.

2.Show the result of commutator of two $n \times n$ matrices has vanishing trace. (Is it possible to show this by induction on the dimension?)

3.Verify the usual $\mathbb{R}^3$ equipped with cross product actually consistutes a Lie algebra. Choose the canonical basis for $\mathbb{R}^3$ and write down every possible result of cross product operation between them. The set of such result is called structure constant of the Lie algebra.

4. Argue why abelian Lie algebra is boring.

1.2 Algebra From Endomorphsim

Recall that whenever we need to investigate the transformation between the elements within vector space V (for example, we might want to know how the basis is changed), we are inclined to deal with endomorphisms. Also recall the general linear group $GL(V)$ is the set of all invertible endomorphisms of V with map composition operation and the natural question to ask is that can we construct a Lie algebra with endomorphisms? If so, how?

Let $End\ V$ be the set of all endomorphisms of vector space V over a field F

$$End\ V = \{\phi | \phi : V \to V\} \tag{1.16}$$

Reader can easily check that $End\ V$ form a vector space. All we have to do is try to find a operation that satisfy 1.2, 1.3, and 1.4, if possible.

But we already investigated the $\mathfrak{sl}(2, F)$ algebra, whose set is the set of endomorphisms (with trace 0), and in there we showed together that the commutator worked. Now we are going to take a leap of faith and try the commutator for $End\ V$.

Remark. There is a dark secret in Mathematics that people want to be as elegant as they can and hide all the trials and errors. To me this is nothing but being hypocritical and showing trials and all the fail attempts are wonderful for learning and teaching Mathematics, as you can already seen in the very first example of this book. To me, the result need to be elegant, but the path leads to it should be muddy.

If we equip *End V* with commutator $[\diamond, \star] = \diamond\star - \star\diamond$, let us try if the commutator actually satisfies all three axioms of Lie algebra. The first thing to check is that the commutator is bilinear, this is trivial (nonetheless important). Moreover, we already saw the commutator satisfies 1.3 for all vector space (problem 1 of 1.1). I will show here the 1.4 is also satisfied by the commutator although it is also trivial:

$$[\mathbf{x}, \mathbf{x}] = \mathbf{xx} - \mathbf{xx} = 0, \forall\ \mathbf{x} \in V. \tag{1.17}$$

Thus, we see that not only the commutator can be used to make *End V* as a Lie algebra, it can be used as the algebraic operation that make *any* vector space into a Lie algebra. Of course this is remarkable and no wonder commutator appears in nature so often, especially in physical science. From now on we denote the general bracket as $[\![\cdot,\cdot]\!]$ and commutator as $[\cdot,\cdot]$.

We now focus on the Lie algebra $(End\, V, [\cdot,\cdot])$, it is traditionally denote such lie algebra as $\mathfrak{gl}(V)$ or $\mathfrak{gl}(n, F)$ for n dimensional vector space V over the field F and called **general linear algebra**. The subalgebra of $\mathfrak{gl}(V)$ is called **linear Lie algebra**.

Futhermore, we can show $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. (Problem 1.) For now we are going deeper into $\mathfrak{sl}(2, F)$ to dig out all the structure (in sense of Lie algebra) of it.

Recall that $\mathfrak{sl}(2, F)$ is the Lie algebra of all 2×2 trace free matrices. Since 2×2 matrix has $2 \times 2 = 4$ entries and with trace free condition, we have three degree of freedom to fix an element in $\mathfrak{sl}(2, F)$, we can naturally construct the following basis:

$$\{e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\} \tag{1.18}$$

Such that for any element $\mathbf{x}$ in $\mathfrak{sl}(2, F)$, it can be written as F-linear combination of the elements in the base set:

$$\mathbf{x} = \gamma e + \sigma f + \iota h, \gamma, \sigma, \iota \in F. \tag{1.19}$$

All the possible result of commutator between basis elements, are

$$[e, f] = h, [e, h] = -2e, [f, h] = 2f. \tag{1.20}$$

As you can see, we can recover all the elements in the base set (with constant coefficient) and we are going to investigate such phenomena in the next section, but before that, let us practice more on finding the structure of a specific Lie algebra by calculating explicitly all the possible commutator results.

Example 1.2.1. In Physics, people often look at $\mathfrak{su}(2)$, which is closely related to $\mathfrak{sl}(2, \mathbb{C})$ and can be represented by constructing the following basis:

$$\{u_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, u_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\} \tag{1.21}$$

and all the possible commutator between them are (you are strongly encouraged to verify this)

$$[u_1, u_2] = 2u_3, [u_2, u_3] = 2u_1, [u_3, u_1] = 2u_2. \tag{1.22}$$

Moreover there is a Lie algebra $\mathfrak{so}(3, 1)$, called the Lie algebra of Lorentz transformation, govern the transformation of physical states in Quantum Mechanics. it has base set:

$$\{K_1, K_2, K_3, J_1, J_2, J_3\} \tag{1.23}$$

(The Ks are called boost transformation and will "entangle" spacetime (this is why you heard time is not constant in theory of relativity, the time is "coupled" with space when boosted) and Js are regular three dimensional rotation generators.) All the resulting commutation realtion between them are

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, [K_i, K_j] = -i\varepsilon_{ijk}J_k, [J_i, K_j] = i\varepsilon_{ijk}K_k \tag{1.24}$$

where ε is the Levi-Civita symbol. If you are not familiar with tensor theory, this is nothing but a compact way to write the following:

$$[K_1, K_2] = -iJ_3, [K_2, K_3] = -iJ_1, [K_3, K_1] = -iJ_2, \tag{1.25}$$

$$[J_1, K_1] = [J_2, K_2] = [J_3, K_3] = 0, \tag{1.26}$$

$$[J_1, K_2] = -iK_3, [J_2, K_3] = -iK_1, [J_3, K_1] = -iK_2, [K_1, J_2] = iK_3, [K_2, J_3] = iK_1, [K_3, J_1] = iK_2 \tag{1.27}$$

$$[J_1, J_2] = -iJ_3, [J_2, J_3] = -iJ_1, [J_3, J_1] = -iJ_2. \tag{1.28}$$

If we define new elements:

$$U_1 = K_1 + iJ_1, U_2 = K_2 + iJ_2, U_3 = K_3 + iJ_3 \tag{1.29}$$

Then we can see that

$$[U_1, U_2] = iU_3, [U_2, U_3] = iU_1, [U_3, U_1] = iU_2. \tag{1.30}$$

Similarly, if we define new elements:

$$T_1 = K_1 - iJ_1, T_2 = K_2 - iJ_2, T_3 = K_3 - iJ_3 \tag{1.31}$$

We can conclude that

$$[T_1, T_2] = iT_3, [T_2, T_3] = iT_1, [T_3, T_1] = iT_2. \tag{1.32}$$

We are doing nothing but changing the basis of $\mathfrak{so}(3, 1)$ from Ks, Js into Us, Ts.

Notice the dimension situation here, $\mathfrak{su}(2)$ has dimension 3, where $\mathfrak{so}(3, 1)$ has 6. Now consider the subalgebra of Us along, can we construct a isomorphism between it and $\mathfrak{su}(2)$? what about the subalgebra of Ts and $\mathfrak{su}(2)$? sure we can (problem 2). That means $\mathfrak{so}(3, 1)$ is isomorphic to $\mathfrak{su}(2) \bigotimes \mathfrak{su}(2)$, we denoted as

$$\mathfrak{so}(3, 1) \cong \mathfrak{su}(2) \bigotimes \mathfrak{su}(2) \tag{1.33}$$

And we say that $\mathfrak{su}(2)$ is a **double cover** of $\mathfrak{so}(3, 1)$.

Moreover, we can realize that $\mathfrak{su}(2) + i\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$ (Problem 3). △

Problem

You can use any results and definition introduced previously

1. Show $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.
2. Construct the isomorphism between the subalgebra spanned by Us and $\mathfrak{su}(2)$ in example 1.2.1. Do the same for subalgebra spanned by Ts and $\mathfrak{su}(2)$. Show that the span of Us and Ts are actually subalgebras.
3. Argue why $\mathfrak{su}(2) + i\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$.

Chapter 2

Structure of Lie algebra

2.1 Structure

Technically, we can list every possible result of bracket within certain Lie algebra and recover the full information about that Lie algebra (since we are only dealing with finite dimensional algebra). But we all agree that this will be tedious and lack of elegance.

If we want to discover the general structure of a Lie algebra, it is no surprise that maps that serve the similar purpose as homomorphism in group theory will come in handy; in another word, we wish to put elements in to "blocks" such that those blocks serve the fundamental acting elements for constructing the structure of that Lie algebra. Thus we define the homomorphism between Lie algebra $\mathfrak{L}, \mathfrak{L}'$;

Definition 2.1.1. We call a map $\phi : \mathfrak{L} \to \mathfrak{L}'$ a homomorphism $\phi[\![\mathbf{x}, \mathbf{y}]\!] = [\![\phi(\mathbf{x}), \phi(\mathbf{y})]\!]', \ \forall \mathbf{x}, \mathbf{y} \in \mathfrak{L}$.

Remark. Obviously and as always, isomorphism (of Lie algebra) is just a homomorphism that is also one-to-one.

As in group theory, the pre-image of identity element in the target set (the kernel of the homomorphism) can tell us a lot of structural properties in the domain; we make the following definition

Definition 2.1.2. Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism between Lie algebra $\mathfrak{L}, \mathfrak{L}'$, we call the kernel of ϕ the **Ideal** of $\mathfrak{L}$.

Now we are ready for our first lemma;

Lemma 2.1.1. *An ideal I of a Lie algebra $\mathfrak{L}$ is a subalgebra; moreover, I is an ideal of $\mathfrak{L}$ iff $[\![\mathbf{x}, I]\!] \subset I, \forall\, \mathbf{x} \in \mathfrak{L}$.*

(whenever a slot of the bracket fill with a set of elements, it means the bracket is performed for every elements in that set.)

Proof. Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism between Lie algebra $\mathfrak{L}, \mathfrak{L}'$, let I be the ideal corresponds to ϕ and 0 be the zero in $\mathfrak{L}'$. $\forall \mathbf{x}, \mathbf{y} \in \mathfrak{L}$, $\phi[\![\mathbf{x}, \mathbf{y}]\!] = [\![\phi(\mathbf{x}), \phi(\mathbf{y})]\!] = [\![0, 0]\!] = 0$. Thus we conclude $[\![\mathbf{x}, \mathbf{y}]\!] \in I$, and I is an algebra, therefore a subalgebra of $\mathfrak{L}$. Now let $\mathbf{x} \in \mathfrak{L}, \mathbf{i} \in I$ be arbitrary, and consider $\phi([\![\mathbf{x}, \mathbf{i}]\!]) = [\![\phi(\mathbf{x}), \phi(\mathbf{i})]\!] = [\![\phi(\mathbf{x}), 0]\!] = 0$. Thus we see $[\![\mathbf{x}, \mathbf{i}]\!] \in I$. The reverse direction is trivial. ■

Remark. We see that Ideal serve as role like black hole, the Ideal set suck every element in the Ideal under the bracket operation.

For every Lie algebra, there are two ideals for sure: 0 and the entire Lie algebra. For any other ideals, we call those Ideals **proper ideals**. For $\mathfrak{sl}(2, F)$ (and $\mathfrak{sl}(n, F)$), there is no proper Ideals. (see the end of section 1.2) However for an abelian Lie algebra, every subset is an ideal.

Remark. If you know Topology theory, this is strikingly similar to the trivial topology and discrete topology.

Ideals behave well under addition (of vectors), namely the following lemma

Lemma 2.1.2. *If I, I' are two ideals of Lie algebra $\mathfrak{L}$, $I + I' = \{\mathbf{i} + \mathbf{i}' | \mathbf{i} \in I, \mathbf{i}' \in I'\}$ is also an ideal of $\mathfrak{L}$.*

Proof. Let I, I' are two ideals of Lie algebra $\mathfrak{L}$, consider $[\![\mathbf{i}+\mathbf{i}', \mathbf{x}]\!], \forall \mathbf{i} \in I, \mathbf{i}' \in I', \mathbf{x} \in \mathfrak{L}$. $[\![\mathbf{i}+\mathbf{i}', \mathbf{x}]\!] \underset{by\ bilinearity}{=} [\![\mathbf{i}, \mathbf{x}]\!] + [\![\mathbf{i}', \mathbf{x}]\!] \in I + I'$. ■

You are asked in problem 3 to prove If I, I' are two ideals of Lie algebra $\mathfrak{L}$, then $[\![I, I']\!]$ is also an ideal.

We don't have to stop here; we can also consider a more "powerful" black hole that not just sucks every element in but annihilate every element in the algebra into 0. We call such set the **center** $Z(\mathfrak{L}) = \{\mathbf{x} \in \mathfrak{L} | [\![\mathbf{x}, \mathbf{y}]\!] = 0, \forall \mathbf{y} \in \mathfrak{L}\}$. Naturally, the center of an abelian Lie algebra is the entire algebra itself. Also notice that every center is an ideal, but the reverse is not true in general.

If a non-abelian Lie algebra has no proper ideals, then we call it **simple**. A simple Lie algebra $\mathfrak{L}$ satisfies $[\![\mathfrak{L}, \mathfrak{L}]\!] = \mathfrak{L}$. If $[\![\mathfrak{L}, \mathfrak{L}]\!] \neq \mathfrak{L}$, we realize that $[\![\mathfrak{L}, \mathfrak{L}]\!] \lhd \mathfrak{L}$, and naturally, we want to continue this process until we "stablize" the result, we consider the following series:

$$\mathfrak{L}^{(1)} \equiv [\![\mathfrak{L}, \mathfrak{L}]\!] \lhd \mathfrak{L}, \mathfrak{L}^{(2)} \equiv [\![\mathfrak{L}^{(1)}, \mathfrak{L}^{(1)}]\!] \lhd \mathfrak{L}^{(1)}, \mathfrak{L}^{(3)} \equiv [\![\mathfrak{L}^{(2)}, \mathfrak{L}^{(2)}]\!] \lhd \mathfrak{L}^{(2)}, \cdots, \mathfrak{L}^{(i)} \equiv [\![\mathfrak{L}^{(i-1)}, \mathfrak{L}^{(i-1)}]\!] \lhd \mathfrak{L}^{(i)}. \tag{2.1}$$

You are asked in problem 1 why this series has to terminate in a sense that there exist a minimium m such that $\mathfrak{L}^{(n)} = \mathfrak{L}^{(m)}, \forall n > m, n \in \mathbb{N}$. Let m be such minimum number and if $\mathfrak{L}^{(m)} = 0$, we call $\mathfrak{L}$ **solvable**, otherwise not solvable. For example, simple Lie algebra is not solvable. $\mathfrak{sl}(n, F)$ is not solvable. You are asked in problem 2 to try to construct a Lie algebra such that this series has length 3.

Obivously, $[\![\mathfrak{L}^{(m)}, \mathfrak{L}^{(m)}]\!] = \mathfrak{L}^{(m)}$, so $\mathfrak{L}^{(m)}$ is simple; the natural qeustion now is that how about we exclude $\mathfrak{L}^{(m)}$ from $\mathfrak{L}$? is the rest of the elements (with 0) form a simple Lie algebra? Well let $\mathfrak{L}_1 = (\mathfrak{L} \setminus \mathfrak{L}^{(m)}) \bigcup \{0\}$, and consider the following series

$$\mathfrak{L}_1^{(1)} \equiv [\![\mathfrak{L}_1, \mathfrak{L}_1]\!] \lhd \mathfrak{L}_1, \mathfrak{L}_1^{(2)} \equiv [\![\mathfrak{L}_1^{(1)}, \mathfrak{L}_1^{(1)}]\!] \lhd \mathfrak{L}_1^{(1)}, \mathfrak{L}_1^{(3)} \equiv [\![\mathfrak{L}_1^{(2)}, \mathfrak{L}_1^{(2)}]\!] \lhd \mathfrak{L}_1^{(2)}, \cdots, \mathfrak{L}_1^{(j)} \equiv [\![\mathfrak{L}_1^{(j-1)}, \mathfrak{L}_1^{(j-1)}]\!] \lhd \mathfrak{L}_1^{(j)}. \tag{2.2}$$

Just like before, there exist a minimum number m_1 such that $[\![\mathfrak{L}_1^{(m_1)}, \mathfrak{L}_1^{(m_1)}]\!] = \mathfrak{L}_1^{(m_1)}$. If $\mathfrak{L}_1^{(m_1)} \neq 0$, we let $\mathfrak{L}_2 = (\mathfrak{L}_1 \setminus \mathfrak{L}_1^{(m_1)}) \bigcup \{0\}$ and consider the following series

$$\mathfrak{L}_2^{(1)} \equiv [\![\mathfrak{L}_2, \mathfrak{L}_2]\!] \lhd \mathfrak{L}_2, \mathfrak{L}_2^{(2)} \equiv [\![\mathfrak{L}_2^{(1)}, \mathfrak{L}_2^{(1)}]\!] \lhd \mathfrak{L}_2^{(1)}, \mathfrak{L}_2^{(3)} \equiv [\![\mathfrak{L}_2^{(2)}, \mathfrak{L}_2^{(2)}]\!] \lhd \mathfrak{L}_2^{(2)}, \cdots, \mathfrak{L}_2^{(k)} \equiv [\![\mathfrak{L}_2^{(k-1)}, \mathfrak{L}_2^{(k-1)}]\!] \lhd \mathfrak{L}_2^{(k)}. \tag{2.3}$$

And if $\mathfrak{L}^{(m_2)}$ is not 0, we continue to construct $\mathfrak{L}_3$ by the similar procedure and check $\mathfrak{L}^{(m_3)}$. Since we are concerned with finite dimensional Lie algebra, this procedure will terminate and we will get a chain of simple algebras

$$\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \cdots, \mathfrak{L}_t, t \in \mathbb{N}. \tag{2.4}$$

and they satisfy the following

$$\mathfrak{L}_i \bigcap \mathfrak{L}_j = \{0\} \tag{2.5}$$

and

$$\mathfrak{L} = \mathfrak{L}_1 \bigoplus \mathfrak{L}_2 \bigoplus \mathfrak{L}_3 \bigoplus \cdots \bigoplus \mathfrak{L}_t \tag{2.6}$$

Now we are ready for the important definition

Definition 2.1.3. if the series 2.4 exist, namely we have $\mathfrak{L}_i \neq 0, \forall 0 < i < t$, we call $\mathfrak{L}$ **semi-simple**.

We realize every semi-simple Lie algebra can be decomposed into direct sum of simple Lie algebras, such that the only pair-wise common element is the zero element. This is remarkable structure and we will explore much more later. Turns out semi-simple Lie algebras are fully categorized and the theory related to it is stunningly beautiful, however, for those Lie algebras that are not semi-simple, categorization is much harder.

For our convenience, we call all the elements been "sucked in" by a set the normalizer of that set, formally, we define **normalizer** $N_{\mathfrak{L}}(H) = \{\mathbf{x} \in \mathfrak{L} | [\![\mathbf{x}, H]\!] \subset H\}$. Obviously, for every ideal, its normalizer is the entire algebra. If $H = N_{\mathfrak{L}}(H)$, we call H **self-normalizing**. Furthermore, the **centrializer** of a set H is the lements been annihilated by U: $C_{\mathfrak{L}}(H) = \{\mathbf{x} \in \mathfrak{L} | [\![\mathbf{x}, H]\!] = 0\}$.

Before we leaving this section, I have to mention that some people introduce semi-simple Lie algebra as the algebra has maximal solvable ideal as zero. We simply treat this as a lemma and prove it here.

Lemma 2.1.3. *$\mathfrak{L}$ is a semi-simple Lie algebra iff its maximal solvable ideal is zero.*

before we prove this lemma, we need a lemma to support the proof of it:

Lemma 2.1.4. *Let $\mathfrak{L}$ be a Lie algebra, then $[\![\mathfrak{L}, \mathfrak{L}]\!]$ is an ideal of $\mathfrak{L}$.*

Proof. Trivial. ■

Now we are ready to prove 2.1.3:

Proof. ($\Rightarrow$) Let $\mathfrak{L}$ be a semi-simple Lie algebra, then if its maximal solvable ideal is not zero, the in series 2.4,we have $\mathfrak{L}_i = 0,$ for some $0 < i < t$, contradicting to the definition of semi-simple Lie algebra. ($\Leftarrow$)if $\mathfrak{L}$'s maximal solvable ideal is zero, every non-zero ideal is not solvable, and none of the $\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \cdots, \mathfrak{L}_t, t \in \mathbb{N}$ is zero, thus $\mathfrak{L}$ is a semi-simple Lie algebra. ■

2.1.1 Problem

You can use all the results and definitions introduced previously.

1. argue why 2.3 has to terminate.
2. can you construct a Lie algebra such that the series 2.3 has length 3?
3.prove If I, I' are two ideals of Lie algebra $\mathfrak{L}$, then $[I, I']$ is also an ideal.
4.give an example of finite dimensional Lie algebra that is not semi-simple.