Lie Algebra

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Chapter 1

Definition of Lie Algebra

1.1 Definiton and Construction

We begin with a motivation. Consider in three dimensional Euclidean space, the cross product operation of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the vector space V over a field F satisfy the following:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0 \tag{1.1}$$

This can be easily proven by Lagrange's formula. We say these three vectors satisfy the **Jacobi Identity** with operation $\times$. Of course, focus only on cross product operation will be too constrained and will be limiting our generalization of the theory. Therefore, we generalize such identity to arbitrary bilinear operation on vector spaces, which is called the bracket.

$$[\cdot, \cdot] : V \times V \to V \tag{1.2}$$

and the Jacobi identity is now read as;

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{c}, [\mathbf{a}, \mathbf{b}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] = 0, \forall\ \mathbf{a}, \mathbf{b}, \mathbf{c} \in V. \tag{1.3}$$

Moreover, we impose

$$[\mathbf{x}, \mathbf{x}] = 0, \forall\ \mathbf{x} \in V \tag{1.4}$$

just like the cross product operation. We are ready to define Lie algebra.

Definition 1.1.1. A Lie algebra $(V, [\cdot, \cdot])$ is a vector space V (over a certain field F) equipped with the bracket operation $[\cdot, \cdot]$ that satisfies 1.2, 1.3, 1.4.

Few remarks are due, but before that, let us see an example where such algebra arises.

Example 1.1.1. Let us consider the set of 2×2 matrices over field F with trace 0, how can we construct a Lie algebra from this set? in another word, if ever possible, what kind of bilinear operation should we equip to it so that 1.3, 1.4 is satisfied? 1.4 require that $[A, A] = 0$ for all 2×2 trace-free A matrix. We first try matrix multiplication, since that is the first natrual thing we would like to try; however, such operation failed our purpose, consider the following

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.5}$$

The result of these two matrices failed to have vanishing trace. Thus we are inclined to quit guessing and try to find such opeartion in a systematic way. Consider the trace of product of two arbitrary 2×2 trace free matrices A, Ψ

$$Tr\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix}\} = a\epsilon + b\sigma + c\gamma + d\omega \tag{1.6}$$

This is not 0 in general, but it is 0 if we subtract it from itself.

$$0 = a\epsilon + b\sigma + c\gamma + d\omega - a\epsilon + b\sigma + c\gamma + d\omega = a\epsilon + b\sigma + c\gamma + d\omega - \epsilon a - \gamma c - \sigma b - \omega d = \\ = Tr\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix} - \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\} \quad (1.7)$$

Thus if we choose our operation to be $[A, \Psi] = A\Psi - \Psi A$, axiom 1.4 will be satisfied, we call such a operation **commutator**: $[\diamond, \star] = \diamond\star - \star\diamond$. After tedious works, we can also show commutator operation also satisfy axiom 1.3 (problem 1).

So the set of 2×2 matrices over field F with trace 0 equipped with commutator $[\cdot, \cdot]$ is a Lie algebra, and we denote it by $\mathfrak{sl}(2, F)$ and it is also called as **special linear Lie algebra of order 2**. It will be crucial important for our theory.

If you following me carefully enough, you found that commutator will vanish every product of two matrices, not just trace free ones, since we did not used the trace free condition at all. It is also can be showed that for $n \times n$ matrices, commutator also vanishes the product. see problem 2.

Here we remark on the definiton of Lie algebra:

Remark. Consider the bracket of any arbitrary vectors $\mathbf{x}, \mathbf{y}$ from the vector space

$$0 \underset{by\ 1.4}{=} [\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}] = [\mathbf{x}, \mathbf{x}] + [\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] + [\mathbf{y}, \mathbf{y}] \underset{by\ 1.4}{=} [\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] \quad (1.8)$$

Notice that this is different from

$$[\mathbf{x}, \mathbf{y}] = [\mathbf{y}, \mathbf{x}] \quad (1.9)$$

Identity 1.8 is from axoims and it is true for all Lie algebra while 1.9 is not ture in general. Whenever a Lie algebra satisfies 1.9, we call that Lie algebra abelian. Whenver a Lie algebra equipped with commutator and two elements from the algebra have vanishing commutator, we call those two elements **commute**.

We now are going to see a Physics example where Lie algebra appears

Example 1.1.2. In physics, quantum mechaincally we describe physical states of an object by specifying its obvervable quantities such as position, momentum, and charge and quantum mechanics represent them as vectors in Hilbert space. Moreover, we want to know how those quantites involve with time. Turns out, we can find a quantity H ,called Hamiltonian, such that it govern the time evolution of any quantity A of a physical object:

$$\frac{d}{dt}A = \partial_t A + \frac{i}{\hbar}[H, A]. \quad (1.10)$$

Where $i = \sqrt{-1}$ and $\hbar$ is some physical constant. The bracket in equation 1.11 is the commutator. When ever a physical quantity commute with Hamiltonian and doesn't explicitly depends on time, it is time independent and called **conserved** quantity. You might heard energy is conserved, often time Hamiltonian is the energy of that physical object (but not always), whenever energy is not explicitly depending on time, that is nothing but application of axiom 1.4:

$$\frac{d}{dt}H = \partial_t H + \frac{i}{\hbar}[H, H] = \underbrace{0}_{Energy\ don't\ depends\ on\ time\ explicitly} + \underbrace{0}_{by\ 1.4} = 0. \quad (1.11)$$

Thus H is constant.

Before people knew quantum mechanics, physical observable quantities were described by some functions of position q, momentum p, and time t, $f(p, q, t)$ on spacetime and its time evolution is described again by (to be distinguished from equation 1.11, we denote bracket here as $\{\cdot, \cdot\}$)

$$\frac{d}{dt}f = \partial_t f + \{H, f\}. \quad (1.12)$$

where $\{\diamond, \star\} \equiv \sum_i (\frac{\partial\diamond}{\partial q_i}\frac{\partial\star}{\partial p_i} - \frac{\partial\star}{\partial q_i}\frac{\partial\diamond}{\partial p_i})$ and it is called Poisson bracket.

Thus we can see Lie aglebra arises from nature and it is very powerful when comes to physical applications.

Of course, we wish to investigate structure of Lie Algebra and possibly categorize it, we are ready to define structure preserving map :

Definition 1.1.2. We say two Lie Algebra $(V, [\cdot, \cdot]), (V', [\cdot, \cdot]')$ are isomorphic if there is a vector space isomorphism ϕ such that

$$\phi : V \to V' \tag{1.13}$$

and satisfies

$$\phi[\mathbf{x}, \mathbf{y}] = [\phi(\mathbf{x}), \phi(\mathbf{y})]', \ \forall \mathbf{x}, \mathbf{y} \in V. \tag{1.14}$$

We call such ϕ isomorphism (of Lie Algebras).

Remark. This definition is as natural as it can be, the bracket grant the vector space extra structure, and if two vector space is isomorphic and their bracket operation behave in the same way, they have exact same structure (Lie algebraically). This is very similar to other definition of structure preserving maps such as group isomorphism.

Naturally, a Lie subalgebra is defined as the following

Definition 1.1.3. We say a Lie algebra $(V', [\cdot, \cdot]')$ is a Lie subalgebra of a Lie algebra $(V, [\cdot, \cdot])$ if $V' \lhd V$ (V' is a subspace of V) and $[\cdot, \cdot]' = [\cdot, \cdot]|'_V$.

We will primarily focus on finite dimensional Lie algebras unless otherwise stated.

Problems

You can use any results and definitions introduced previously.

1.Show commutator operation satisfy axiom 1.3.

2.Show the result of commutator of two $n \times n$ matrices has vanishing trace. (Is it possible to show this by induction on the dimension?)

3.Verify the usual $\mathbb{R}^3$ equipped with cross product actual consistutes a Lie algebra. Choose the canonnical basis for $\mathbb{R}^3$ and write down every possible result of cross product operation between them. The set of such result is called structure constant of the Lie algebra.

4. Argue why abelian Lie algebra is boring.