Lie Algebra

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Chapter 1

Definition of Lie Algebra

1.1 Definiton and Construction

We begin with a motivation. Consider in three dimensional Euclidean space, the cross product operation of vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in the vector space V over a field F satisfy the following:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0 \tag{1.1}$$

This can be easily proven by Lagrange's formula. We say these three vectors satisfy the **Jacobi Identity** with operation $\times$. Of course, focus only on cross product operation will be too constrained and will be limiting our generalization of the theory. Therefore, we generalize such identity to arbitrary bilinear operation on vector spaces, which is called the **bracket**.

$$[\cdot, \cdot] : V \times V \to V \tag{1.2}$$

and the Jacobi identity is now read as;

$$[\mathbf{a}, [\mathbf{b}, \mathbf{c}]] + [\mathbf{c}, [\mathbf{a}, \mathbf{b}]] + [\mathbf{b}, [\mathbf{c}, \mathbf{a}]] = 0, \forall\ \mathbf{a}, \mathbf{b}, \mathbf{c} \in V. \tag{1.3}$$

Moreover, we want impose

$$[\mathbf{x}, \mathbf{x}] = 0, \forall\ \mathbf{x} \in V \tag{1.4}$$

just like the cross product operation. We are ready to define Lie algebra.

Definition 1.1.1. A Lie algebra $(V, [\cdot, \cdot])$ is a vector space V (over a certain field F) equipped with the bracket operation $[\cdot, \cdot]$ that satisfies 1.2, 1.3, 1.4.

Few remarks are due, but before that, let us see an example where such algebra arises.

Example 1.1.1. Let us consider the set of 2×2 matrices over field F with trace 0, how can we construct a Lie algebra from this set? In another word, if ever possible, what kind of bilinear operation should we equip to it so that 1.3, 1.4 is satisfied? 1.4 require that $[A, A] = 0$ for all 2×2 trace-free A matrix and we also require the closure of Lie algebra. We first try matrix multiplication, since that is the first natrual thing we would like to try; however, such operation failed our purpose, consider the following

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1.5}$$

The result of these two matrices failed to have vanishing trace. Thus we are inclined to quit guessing and try to find such opeartion in a systematic way. Consider the trace of product of two arbitrary 2×2 trace free matrices A, Ψ

$$Tr\{A\Psi\} = Tr\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix}\} = a\epsilon + b\sigma + c\gamma + d\omega \tag{1.6}$$

This is not 0 in general, but it is 0 if we subtract it from itself.

$$0 = a\epsilon + b\sigma + c\gamma + d\omega - a\epsilon + b\sigma + c\gamma + d\omega = a\epsilon + b\sigma + c\gamma + d\omega - \epsilon a - \gamma c - \sigma b - \omega d =$$
$$= Tr\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix} - \begin{bmatrix} \epsilon & \gamma \\ \sigma & \omega \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix}\} = Tr\{A\Psi \ - \Psi A\} \quad (1.7)$$

and

$$Tr\{AA \ - AA\} = Tr\{0\} = 0. \quad (1.8)$$

Thus if we choose our operation to be $[A, \Psi] = A\Psi - \Psi A$, axiom 1.4 will be satisfied, we call such a operation **commutator**: $[\diamond, \star] = \diamond \star - \star \diamond$. After tedious works, we can also show commutator operation also satisfy axiom 1.3 (problem 1).

So the set of 2×2 matrices over field F with trace 0 equipped with commutator $[\cdot, \cdot]$ is a Lie algebra, and we denote it by $\mathfrak{sl}(2, F)$ and it is also called as **special linear Lie algebra of order 2**. It will be crucial important for our theory.

If you are following me carefully enough, you found that commutator will vanish every product of two matrices, not just trace free ones, since we did not used the trace free condition at all. It is also can be showed that for $n \times n$ matrices, commutator also vanishes the product. see problem 2. △

Here we remark on the definiton of Lie algebra:

Remark. Consider the bracket of any arbitrary vectors $\mathbf{x}, \mathbf{y}$ from the vector space

$$0 \underset{by\ 1.4}{=} [\mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}] = [\mathbf{x}, \mathbf{x}] + [\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] + [\mathbf{y}, \mathbf{y}] \underset{by\ 1.4}{=} [\mathbf{x}, \mathbf{y}] + [\mathbf{y}, \mathbf{x}] \quad (1.9)$$

Notice that this is different from

$$[\mathbf{x}, \mathbf{y}] = [\mathbf{y}, \mathbf{x}] \quad (1.10)$$

Identity 1.9 is from axoims and it is true for all Lie algebra while 1.10 is not ture in general. Whenever a Lie algebra satisfies 1.10, we call that Lie algebra abelian. Whenver a Lie algebra equipped with commutator and two elements from the algebra have vanishing commutator, we call those two elements **commute**.

We now are going to see a Physics example where Lie algebra appears

Example 1.1.2. In physics, quantum mechaincally we describe physical states of an object by specifying its obvervable quantities such as position, momentum, and charge and quantum mechanics represent them as vectors in Hilbert space. Moreover, we want to know how those quantites involve with time. Turns out, we can find a quantity H ,called Hamiltonian, such that it govern the time evolution of any quantity A of a physical object:

$$\frac{d}{dt}A = \partial_t A + \frac{i}{\hbar}[H, A]. \quad (1.11)$$

Where $i = \sqrt{-1}$ and $\hbar$ is some physical constant. The bracket in equation 1.11 is the commutator. When ever a physical quantity commute with Hamiltonian and doesn't explicitly depends on time, it is time independent and called **conserved** quantity. You might heard energy is conserved, often time Hamiltonian is the energy of that physical object (but not always), whenever energy is not explicitly depending on time, that is nothing but application of axiom 1.4:

$$\frac{d}{dt}H = \partial_t H + \frac{i}{\hbar}[H, H] = \underbrace{0}_{Energy\ don't\ depends\ on\ time\ explicitly} + \underbrace{0}_{by\ 1.4} = 0. \quad (1.12)$$

Thus H is constant. Equation 1.11 is called Heisenberg picture.

Before people knew quantum mechanics, physical observable quantities were described by some functions of position q, momentum p, and time t, $f(p, q, t)$ on spacetime in Classical Mechanics and its time evolution is described again by (to be distinguished from equation 1.11, we denote bracket here as $\{\cdot, \cdot\}$)

$$\frac{d}{dt}f = \partial_t f + \{H, f\}. \tag{1.13}$$

where $\{\diamond, \star\} \equiv \sum_i (\frac{\partial \diamond}{\partial q_i}\frac{\partial \star}{\partial p_i} - \frac{\partial \star}{\partial q_i}\frac{\partial \diamond}{\partial p_i})$ and it is called Poisson bracket.

Thus we can see Lie aglebra arises from nature and it is very powerful when comes to physical applications. $\triangle$

Of course, we wish to investigate structure of Lie Algebra and possibly categorize it, we are ready to define structure preserving map :

Definition 1.1.2. We say two Lie Algebra $(V, [\cdot,\cdot]), (V', [\cdot,\cdot]')$ are isomorphic if there is a vector space isomorphism ϕ such that

$$\phi : V \to V' \tag{1.14}$$

and satisfies

$$\phi[\mathbf{x}, \mathbf{y}] = [\phi(\mathbf{x}), \phi(\mathbf{y})]',\ \forall \mathbf{x}, \mathbf{y} \in V. \tag{1.15}$$

We call such ϕ isomorphism (of Lie Algebras).

Remark. This definition is as natural as it can be, the bracket grant the vector space extra structure, and if two vector space is isomorphic and their bracket operation behave in the same way, they have exact same structure (Lie algebraically). This is very similar to other definition of structure preserving maps such as group isomorphism.

Naturally, a Lie subalgebra is defined as the following

Definition 1.1.3. We say a Lie algebra $(V', [\cdot,\cdot]')$ is a Lie subalgebra of a Lie algebra $(V, [\cdot,\cdot])$ if $V' \lhd V$ (V' is a subspace of V) and $[\cdot,\cdot]' = [\cdot,\cdot]|'_V$.

We will primarily focus on finite dimensional Lie algebras unless otherwise stated.

1.1.1 Problems

You can use any results and definitions introduced previously.

1.Show commutator operation satisfy axiom 1.3.

2.Show the result of commutator of two $n \times n$ matrices has vanishing trace. (Is it possible to show this by induction on the dimension?)

3.Verify the usual $\mathbb{R}^3$ equipped with cross product actually consistutes a Lie algebra. Choose the canonical basis for $\mathbb{R}^3$ and write down every possible result of cross product operation between them. The set of such result is called structure constant of the Lie algebra.

4. Argue why abelian Lie algebra is boring.

1.2 Algebra From Endomorphsim

Recall that whenever we need to investigate the transformation between the elements within vector space V (for example, we might want to know how the basis is changed), we are inclined to deal with endomorphisms. Also recall the general linear group $GL(V)$ is the set of all invertible endomorphisms of V with map composition operation and the natural question to ask is that can we construct a Lie algebra with endomorphisms? If so, how?

Let $End\ V$ be the set of all endomorphisms of vector space V over a field F

$$End\ V = \{\phi | \phi : V \to V\} \tag{1.16}$$

Reader can easily check that $End\ V$ form a vector space. All we have to do is try to find a operation that satisfy 1.2, 1.3, and 1.4, if possible.

But we already investigated the $\mathfrak{sl}(2, F)$ algebra, whose set is the set of endomorphisms (with trace 0), and in there we showed together that the commutator worked. Now we are going to take a leap of faith and try the commutator for $End\ V$.

Remark. There is a dark secret in Mathematics that people want to be as elegant as they can and hide all the trials and errors. To me this is nothing but being hypocritical and showing trials and all the fail attempts are wonderful for learning and teaching Mathematics, as you can already seen in the very first example of this book. To me, the result need to be elegant, but the path leads to it should be muddy.

If we equip *End V* with commutator $[\diamond, \star] = \diamond\star - \star\diamond$, let us try if the commutator actually satisfies all three axioms of Lie algebra. The first thing to check is that the commutator is bilinear, this is trivial (nonetheless important). Moreover, we already saw the commutator satisfies 1.3 for all vector space (problem 1 of 1.1). I will show here the 1.4 is also satisfied by the commutator although it is also trivial:

$$[\mathbf{x}, \mathbf{x}] = \mathbf{xx} - \mathbf{xx} = 0, \forall\ \mathbf{x} \in V. \tag{1.17}$$

Thus, we see that not only the commutator can be used to make *End V* as a Lie algebra, it can be used as the algebraic operation that make *any* vector space into a Lie algebra. Of course this is remarkable and no wonder commutator appears in nature so often, especially in physical science. From now on we denote the general bracket as $[\![\cdot,\cdot]\!]$ and commutator as $[\cdot,\cdot]$.

We now focus on the Lie algebra $(End\, V, [\cdot,\cdot])$, it is traditionally denote such lie algebra as $\mathfrak{gl}(V)$ or $\mathfrak{gl}(n, F)$ for n dimensional vector space V over the field F and called **general linear algebra**. The subalgebra of $\mathfrak{gl}(V)$ is called **linear Lie algebra**.

Futhermore, we can show $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$. (Problem 1.) For now we are going deeper into $\mathfrak{sl}(2, F)$ to dig out all the structure (in sense of Lie algebra) of it.

Recall that $\mathfrak{sl}(2, F)$ is the Lie algebra of all 2×2 trace free matrices. Since 2×2 matrix has $2\times 2 = 4$ entries and with trace free condition, we have three degree of freedom to fix an element in $\mathfrak{sl}(2, F)$, we can naturally construct the following basis:

$$\{e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\} \tag{1.18}$$

Such that for any element $\mathbf{x}$ in $\mathfrak{sl}(2, F)$, it can be written as F-linear combination of the elements in the base set:

$$\mathbf{x} = \gamma e + \sigma f + \iota h, \gamma, \sigma, \iota \in F. \tag{1.19}$$

All the possible result of commutator between basis elements, are

$$[e, f] = h, [e, h] = -2e, [f, h] = 2f. \tag{1.20}$$

As you can see, we can recover all the elements in the base set (with constant coefficient) and we are going to investigate such phenomena in the next section, but before that, let us practice more on finding the structure of a specific Lie algebra by calculating explicitly all the possible commutator results.

Example 1.2.1. In Physics, people often look at $\mathfrak{su}(2)$, which is closely related to $\mathfrak{sl}(2,\mathbb{C})$ and can be represented by constructing the following basis:

$$\{u_1 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, u_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\} \tag{1.21}$$

and all the possible commutator between them are (you are strongly encouraged to verify this)

$$[u_1, u_2] = 2u_3, [u_2, u_3] = 2u_1, [u_3, u_1] = 2u_2. \tag{1.22}$$

Moreover there is a Lie algebra $\mathfrak{so}(3, 1)$, called the Lie algebra of Lorentz transformation, govern the transformation of physical states in Quantum Mechanics. it has base set:

$$\{K_1, K_2, K_3, J_1, J_2, J_3\} \tag{1.23}$$

(The Ks are called boost transformation and will "entangle" spacetime (this is why you heard time is not constant in theory of relativity, the time is "coupled" with space when boosted) and Js are regular three dimensional rotation generators.) All the resulting commutation realtion between them are

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, [K_i, K_j] = -i\varepsilon_{ijk}J_k, [J_i, K_j] = i\varepsilon_{ijk}K_k \tag{1.24}$$

where ε is the Levi-Civita symbol. If you are not familiar with tensor theory, this is nothing but a compact way to write the following:

$$[K_1, K_2] = -iJ_3, [K_2, K_3] = -iJ_1, [K_3, K_1] = -iJ_2, \tag{1.25}$$

$$[J_1, K_1] = [J_2, K_2] = [J_3, K_3] = 0, \tag{1.26}$$

$$[J_1, K_2] = -iK_3, [J_2, K_3] = -iK_1, [J_3, K_1] = -iK_2, [K_1, J_2] = iK_3, [K_2, J_3] = iK_1, [K_3, J_1] = iK_2 \tag{1.27}$$

$$[J_1, J_2] = -iJ_3, [J_2, J_3] = -iJ_1, [J_3, J_1] = -iJ_2. \tag{1.28}$$

If we define new elements:

$$U_1 = K_1 + iJ_1, U_2 = K_2 + iJ_2, U_3 = K_3 + iJ_3 \tag{1.29}$$

Then we can see that

$$[U_1, U_2] = iU_3, [U_2, U_3] = iU_1, [U_3, U_1] = iU_2. \tag{1.30}$$

Similarly, if we define new elements:

$$T_1 = K_1 - iJ_1, T_2 = K_2 - iJ_2, T_3 = K_3 - iJ_3 \tag{1.31}$$

We can conclude that

$$[T_1, T_2] = iT_3, [T_2, T_3] = iT_1, [T_3, T_1] = iT_2. \tag{1.32}$$

We are doing nothing but changing the basis of $\mathfrak{so}(3,1)$ from Ks, Js into Us, Ts.

Notice the dimension situation here, $\mathfrak{su}(2)$ has dimension 3, where $\mathfrak{so}(3,1)$ has 6. Now consider the subalgebra of Us along, can we construct a isomorphism between it and $\mathfrak{su}(2)$? what about the subalgebra of Ts and $\mathfrak{su}(2)$? sure we can (problem 2). That means $\mathfrak{so}(3,1)$ is isomorphic to $\mathfrak{su}(2) \bigotimes \mathfrak{su}(2)$, we denoted as

$$\mathfrak{so}(3,1) \cong \mathfrak{su}(2) \bigotimes \mathfrak{su}(2) \tag{1.33}$$

And we say that $\mathfrak{su}(2)$ is a **double cover** of $\mathfrak{so}(3,1)$.

Moreover, we can realize that $\mathfrak{su}(2) + i\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$ (Problem 3). △

1.2.1 Problem

You can use any results and definition introduced previously

1. Show $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

2. Construct the isomorphism between the subalgebra spanned by Us and $\mathfrak{su}(2)$ in example 1.2.1. Do the same for subalgebra spanned by Ts and $\mathfrak{su}(2)$. Show that the span of Us and Ts are actually subalgebras.

3. Argue why $\mathfrak{su}(2) + i\mathfrak{su}(2) = \mathfrak{sl}(2, \mathbb{C})$.

Chapter 2

Structure of Lie algebra

2.1 Simple and Semi-Simple Algebras

Technically, we can list every possible result of bracket within certain Lie algebra and recover the full information about that Lie algebra (since we are only dealing with finite dimensional algebra). But we all agree that this will be tedious and lack of elegance.

If we want to discover the general structure of a Lie algebra, it is no surprise that maps that serve the similar purpose as homomorphism in group theory will come in handy; in another word, we wish to put elements in to "blocks" such that those blocks serve the fundamental acting elements for constructing the structure of that Lie algebra. Thus we define the homomorphism between Lie algebra $\mathfrak{L}, \mathfrak{L}'$;

Definition 2.1.1. We call a map $\phi : \mathfrak{L} \to \mathfrak{L}'$ a homomorphism iff $\phi[\![\mathbf{x}, \mathbf{y}]\!] = [\![\phi(\mathbf{x}), \phi(\mathbf{y})]\!]',\ \forall \mathbf{x}, \mathbf{y} \in \mathfrak{L}$.

Remark. Obviously and as always, isomorphism (of Lie algebra) is just a homomorphism that is also one-to-one.

As in group theory, the pre-image of identity element in the target set (the kernel of the homomorphism) can tell us a lot of structural properties in the domain; we make the following definition

Definition 2.1.2. Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism between Lie algebra $\mathfrak{L}, \mathfrak{L}'$, we call the kernel of ϕ the **Ideal** of $\mathfrak{L}$.

Now we are ready for our first lemma;

Lemma 2.1.1. *An ideal $\mathfrak{I}$ of a Lie algebra $\mathfrak{L}$ is a subalgebra; moreover, $\mathfrak{I}$ is an ideal of $\mathfrak{L}$ iff $[\![\mathbf{x}, \mathfrak{I}]\!] \subset \mathfrak{I}, \forall\ \mathbf{x} \in \mathfrak{L}$.*

(whenever a slot of the bracket fill with a set of elements, it means the bracket is performed for every elements in that set.)

Proof. Let $\phi : \mathfrak{L} \to \mathfrak{L}'$ be a homomorphism between Lie algebra $\mathfrak{L}, \mathfrak{L}'$, let $\mathfrak{I}$ be the ideal corresponds to ϕ and 0 be the zero in $\mathfrak{L}'$. $\forall \mathbf{x}, \mathbf{y} \in \mathfrak{L}$, $\phi[\![\mathbf{x}, \mathbf{y}]\!] = [\![\phi(\mathbf{x}), \phi(\mathbf{y})]\!] = [\![0, 0]\!] = 0$. Thus we conclude $[\![\mathbf{x}, \mathbf{y}]\!] \in \mathfrak{I}$, and $\mathfrak{I}$ is an algebra, therefore a subalgebra of $\mathfrak{L}$. Now let $\mathbf{x} \in \mathfrak{L}, \mathbf{i} \in \mathfrak{I}$ be arbitrary, and consider $\phi([\![\mathbf{x}, \mathbf{i}]\!]) = [\![\phi(\mathbf{x}), \phi(\mathbf{i})]\!] = [\![\phi(\mathbf{x}), 0]\!] = 0$. Thus we see $[\![\mathbf{x}, \mathbf{i}]\!] \in \mathfrak{I}$. The reverse direction is trivial. ∎

Remark. We see that Ideal serve as role like black hole, the Ideal set suck every element in the Ideal under the bracket operation.

For every Lie algebra, there are two ideals for sure: 0 and the entire Lie algebra. For any other ideals, we call those Ideals **proper ideals**. For $\mathfrak{sl}(2, F)$ (and $\mathfrak{sl}(n, F)$), there is no proper Ideals. (see the end of section 1.2) However for an abelian Lie algebra, every subset is an ideal.

Remark. If you know Topology theory, this is strikingly similar to the trivial topology and discrete topology.

Ideals behave well under addition (of vectors), namely the following lemma

Lemma 2.1.2. *If $\mathfrak{I}, \mathfrak{I}'$ are two ideals of Lie algebra $\mathfrak{L}$, $\mathfrak{I} + I' = \{\mathfrak{I} + \mathbf{i}' | \mathbf{i} \in \mathfrak{I}, \mathbf{i}' \in \mathfrak{I}'\}$ is also an ideal of $\mathfrak{L}$.*

Proof. Let $\mathfrak{I}, \mathfrak{I}'$ are two ideals of Lie algebra $\mathfrak{L}$, consider $[\![\mathbf{i}+\mathbf{i}', \mathbf{x}]\!], \forall \mathbf{i} \in \mathfrak{I}, \mathbf{i}' \in \mathfrak{I}', \mathbf{x} \in \mathfrak{L}$. $[\![\mathbf{i}+\mathbf{i}', \mathbf{x}]\!] \underset{by\ bilinearity}{=} [\![\mathbf{i}, \mathbf{x}]\!] + [\![\mathbf{i}', \mathbf{x}]\!] \in \mathfrak{I} + \mathfrak{I}'$. ■

You are asked in problem 3 to prove If $\mathfrak{I}, \mathfrak{I}'$ are two ideals of Lie algebra $\mathfrak{L}$, then $[\![\mathfrak{I}, \mathfrak{I}']\!]$ is also an ideal.

We don't have to stop here; we can also consider a more "powerful" black hole that not just sucks every element in but annihilate every element in the algebra into 0. We call such set the **center** $Z(\mathfrak{L}) = \{\mathbf{x} \in \mathfrak{L} | [\![\mathbf{x}, \mathbf{y}]\!] = 0, \forall \mathbf{y} \in \mathfrak{L}\}$. Naturally, the center of an abelian Lie algebra is the entire algebra itself. Also notice that every center is an ideal, but the reverse is not true in general.

If a non-abelian Lie algebra has no proper ideals, then we call it **simple**. A simple Lie algebra $\mathfrak{L}$ satisfies $[\![\mathfrak{L}, \mathfrak{L}]\!] = \mathfrak{L}$. If $[\![\mathfrak{L}, \mathfrak{L}]\!] \neq \mathfrak{L}$, we realize that $[\![\mathfrak{L}, \mathfrak{L}]\!] \lhd \mathfrak{L}$, and naturally, we want to continue this process until we "stablize" the result, we consider the following series:

$$\mathfrak{L}^{(1)} \equiv [\![\mathfrak{L}, \mathfrak{L}]\!] \lhd \mathfrak{L}, \mathfrak{L}^{(2)} \equiv [\![\mathfrak{L}^{(1)}, \mathfrak{L}^{(1)}]\!] \lhd \mathfrak{L}^{(1)}, \mathfrak{L}^{(3)} \equiv [\![\mathfrak{L}^{(2)}, \mathfrak{L}^{(2)}]\!] \lhd \mathfrak{L}^{(2)}, \cdots, \mathfrak{L}^{(i)} \equiv [\![\mathfrak{L}^{(i-1)}, \mathfrak{L}^{(i-1)}]\!] \lhd \mathfrak{L}^{(i)}. \tag{2.1}$$

You are asked in problem 1 why this series has to terminate in a sense that there exist a minimium m such that $\mathfrak{L}^{(n)} = \mathfrak{L}^{(m)}, \forall n > m, n \in \mathbb{N}$. Let m be such minimum number and if $\mathfrak{L}^{(m)} = 0$, we call $\mathfrak{L}$ **solvable**, otherwise not solvable. For example, simple Lie algebra is not solvable. $\mathfrak{sl}(n, F)$ is not solvable. You are asked in problem 2 to try to construct a Lie algebra such that this series has length 3.

Obivously, $[\![\mathfrak{L}^{(m)}, \mathfrak{L}^{(m)}]\!] = \mathfrak{L}^{(m)}$, so $\mathfrak{L}^{(m)}$ is simple; the natural qeustion now is that how about we exclude $\mathfrak{L}^{(m)}$ from $\mathfrak{L}$? is the rest of the elements (with 0) form a simple Lie algebra? Well let $\mathfrak{L}_1 = (\mathfrak{L} \setminus \mathfrak{L}^{(m)}) \bigcup \{0\}$, and consider the following series

$$\mathfrak{L}_1^{(1)} \equiv [\![\mathfrak{L}_1, \mathfrak{L}_1]\!] \lhd \mathfrak{L}_1, \mathfrak{L}_1^{(2)} \equiv [\![\mathfrak{L}_1^{(1)}, \mathfrak{L}_1^{(1)}]\!] \lhd \mathfrak{L}_1^{(1)}, \mathfrak{L}_1^{(3)} \equiv [\![\mathfrak{L}_1^{(2)}, \mathfrak{L}_1^{(2)}]\!] \lhd \mathfrak{L}_1^{(2)}, \cdots, \mathfrak{L}_1^{(j)} \equiv [\![\mathfrak{L}_1^{(j-1)}, \mathfrak{L}_1^{(j-1)}]\!] \lhd \mathfrak{L}_1^{(j)}. \tag{2.2}$$

Just like before, there exist a minimum number m_1 such that $[\![\mathfrak{L}_1^{(m_1)}, \mathfrak{L}_1^{(m_1)}]\!] = \mathfrak{L}_1^{(m_1)}$. If $\mathfrak{L}_1^{(m_1)} \neq 0$, we let $\mathfrak{L}_2 = (\mathfrak{L}_1 \setminus \mathfrak{L}_1^{(m_1)}) \bigcup \{0\}$ and consider the following series

$$\mathfrak{L}_2^{(1)} \equiv [\![\mathfrak{L}_2, \mathfrak{L}_2]\!] \lhd \mathfrak{L}_2, \mathfrak{L}_2^{(2)} \equiv [\![\mathfrak{L}_2^{(1)}, \mathfrak{L}_2^{(1)}]\!] \lhd \mathfrak{L}_2^{(1)}, \mathfrak{L}_2^{(3)} \equiv [\![\mathfrak{L}_2^{(2)}, \mathfrak{L}_2^{(2)}]\!] \lhd \mathfrak{L}_2^{(2)}, \cdots, \mathfrak{L}_2^{(k)} \equiv [\![\mathfrak{L}_2^{(k-1)}, \mathfrak{L}_2^{(k-1)}]\!] \lhd \mathfrak{L}_2^{(k)}. \tag{2.3}$$

And if $\mathfrak{L}^{(m_2)}$ is not 0, we continue to construct $\mathfrak{L}_3$ by the similar procedure and check $\mathfrak{L}^{(m_3)}$. Since we are concerned with finite dimensional Lie algebra, this procedure will terminate and we will get a chain of simple algebras

$$\mathfrak{L}_1, \mathfrak{L}_2, \mathfrak{L}_3, \cdots, \mathfrak{L}_t, t \in \mathbb{N}. \tag{2.4}$$

and they satisfy the following

$$\mathfrak{L}_i \bigcap \mathfrak{L}_j = \{0\} \tag{2.5}$$

and

$$\mathfrak{L} = \mathfrak{L}_1 \bigoplus \mathfrak{L}_2 \bigoplus \mathfrak{L}_3 \bigoplus \cdots \bigoplus \mathfrak{L}_t \tag{2.6}$$

Now we are ready for the important definition

Definition 2.1.3. if the series 2.4 exist, namely we have $\mathfrak{L}_i \neq 0, \forall 0 < i < t$, we call $\mathfrak{L}$ **semi-simple**.

We realize every semi-simple Lie algebra can be decomposed into direct sum of simple Lie algebras, such that the only pair-wise common element is the zero element. This is remarkable structure and we will explore much more later. Turns out semi-simple Lie algebras are fully categorized and the theory related to it is stunningly beautiful, however, for those Lie algebras that are not semi-simple, categorization is much harder.

For our convenience, we call all the elements been "sucked in" by a set the normalizer of that set, formally, we define **normalizer** $N_{\mathfrak{L}}(H) = \{\mathbf{x} \in \mathfrak{L} | [\![\mathbf{x}, H]\!] \subset H\}$. Obviously, for every ideal, its normalizer is the entire algebra. If $H = N_{\mathfrak{L}}(H)$, we call H **self-normalizing**. Furthermore, the **centrializer** of a set H is the lements been annihilated by U: $C_{\mathfrak{L}}(H) = \{\mathbf{x} \in \mathfrak{L} | [\![\mathbf{x}, H]\!] \ = 0\}$.

Before we leaving this section, I have to mention that some people introduce semi-simple Lie algebra as the algebra has maximal solvable ideal, which is also called the **Radical**, as zero. We simply treat this as a lemma and prove it here. Before doing that, we need to introduce a lemma to prove it:

Lemma 2.1.3. *If $\mathfrak{I}, \mathfrak{I}'$ are two ideals of a Lie algebra $\mathfrak{L}$, then the intersection between them is 0 or either of the ideals.*

You are asked to prove this in problem 5.

Lemma 2.1.4. *$\mathfrak{L}$ is a semi-simple Lie algebra iff its maximal solvable ideal is zero.*

Proof. Let $\mathfrak{R}$ be the Radical of a Lie algebra $\mathfrak{L}$. ($\Rightarrow$) Let $\mathfrak{L}$ be a semi-simple Lie algebra, thus we can get a chain of nonzero simple Lie algebras 2.4. If $\mathfrak{R}$ is a proper subset of $\mathfrak{L}_i$ for any $\mathfrak{L}_i$ in the series 2.4, then it will contradict the requirement of no proper subalgebra of a simple Lie algebra. If $\mathfrak{R}$ is any $\mathfrak{L}_i$ in the series 2.4, then it will contradict to the fact that simple Lie algebra is not solvable. If $\mathfrak{R}$ intersects more than one simple algebra, then by lemma 2.1.3, it is zero. ($\Leftarrow$), we assume $\mathfrak{R}$ is zero, then none of the $\mathfrak{L}_i$ in 2.4 is zero, thus $\mathfrak{L}$ is semi-simple. ■

2.1.1 Problems

You can use all the results and definitions introduced previously.

1. argue why 2.3 has to terminate.
2. can you construct a Lie algebra such that the series 2.3 has length 3?

3.prove If I, I' are two ideals of Lie algebra $\mathfrak{L}$, then $[I, I']$ is also an ideal.

4.give an example of finite dimensional Lie algebra that is not semi-simple.

5. Prove the "two black holes" lemma 2.1.3.

2.2 Nilpotency

As we already argued, the $\mathfrak{L}^{(1)} \equiv [\![\mathfrak{L}, \mathfrak{L}]\!] \lhd \mathfrak{L}$ is an ideal of the entire algebra $\mathfrak{L}$. It is natrual instinct of us to see where does this ideal takes every elements in the algebra to; or following our black hole analogy, we want to know where does everything landed after being sucked in to black hole. In formal language, this means we want to investigate the following objects:

$$\mathfrak{L}^{[i]} \equiv [\![\mathfrak{L}^{[i-1]}, \mathfrak{L}]\!] \tag{2.7}$$

Where

$$\mathfrak{L}^{(1)} = \mathfrak{L}^{[1]} \equiv [\![\mathfrak{L}, \mathfrak{L}]\!] \tag{2.8}$$

For examples :

$$\mathfrak{L}^{[2]} \equiv [\![\mathfrak{L}^{[1]}, \mathfrak{L}]\!], \mathfrak{L}^{[3]} \equiv [\![\mathfrak{L}^{[2]}, \mathfrak{L}]\!], \cdots \tag{2.9}$$

We get a sense that $\mathfrak{L}^{[i]}$ is "larger" (in the sense of Lie algebra containment) than $\mathfrak{L}^{(i)}$, and we are right about our intuitation this time:

Lemma 2.2.1. *Let $\mathfrak{L}$ be a Lie algebra, then $\mathfrak{L}^{[i]} \lhd \mathfrak{L}^{(i)}$.*

Try to prove this before reading on.

Proof. Prove this by induction. we know $\mathfrak{L}^{[1]} \lhd \mathfrak{L}^{(1)}$, and we know that if $\mathfrak{L}^{[i]} \lhd \mathfrak{L}^{(i)}$, then $\mathfrak{L}^{[i+1]} \equiv [\![\mathfrak{L}^{[i]}, \mathfrak{L}]\!] \lhd [\![\mathfrak{L}^{(i)}, \mathfrak{L}^{(i)}]\!] \equiv \mathfrak{L}^{(i+1)}$, since $\mathfrak{L} \lhd \mathfrak{L}^{(i)}$. ■

Similar as before, we are ready to define the **nilpotent**, that is similar to solvable.

Definition 2.2.1. A Lie algebra $\mathfrak{L}$ is called **nilpotent** iff there exist a $m \in \mathbb{N}$ such that $\mathfrak{L}^{[m]} = 0$.

Let us see an example of nilpotent algebra

Example 2.2.1. Let $(\mathfrak{T}^{++}(n, \mathbb{R}), [\cdot, \cdot])$ be the algebra of $n \times n$ strictly upper triangle real matrices with commutator. Then $\mathfrak{T}^{++}(n, \mathbb{R})$ is nilpotent. You are asked to show why in problem 1. △

Furthermore, we can realize the following lemma

Lemma 2.2.2. *Every nilpotent Lie algebra is solvable.*

You should try to prove this before reading me proving it:

Proof. Let $\mathfrak{L}$ be a nilpotent Lie algebra, and we realize that from lemma 2.2 that $\mathfrak{L}^{[i]} \lhd \mathfrak{L}^{(i)}$, since $\mathfrak{L}$ is nilpotent there exist a $m \in \mathbb{N}$ such that $\mathfrak{L}^{[m]} = 0$, we conclude $\mathfrak{L}^{(m)} = 0$, and $\mathfrak{L}$ is solvable. ■

Remark. If we think carefully, if a Lie algebra $\mathfrak{L}$ is not nilpotent, then there are elements in the $\mathfrak{L}$ that are being the "potential troubled kids" and keep us from getting 0 from the brakcet. Now since THE ART OF WAR teachs us deal with the easy first, we look at the "good kids" first. But we already know who are the good kids, they are the center of $\mathfrak{L}$, $Z(\mathfrak{L})$, which bracket every elements in the algebra into 0; they are behaving in the best way to make a Lie algebra nilpotent. Since they are behaving equally well, we put them into an equivalent class and turn our focus on the potential troubled kids. But realize what has happened here, we are reducing the problem down to potential troubled kids; as long as they can make the entire $\mathfrak{L}$ nilpotent, the good kids are going to collaborate no matter what. Thus we conjure the following

Lemma 2.2.3. *Let $\mathfrak{L}$ be a Lie algebra, and if the quotient algebra $\mathfrak{L}/Z(\mathfrak{L})$ (the quotient is in the sense of linear algebra) is nilpotent, then $\mathfrak{L}$ is nilpotent.*

Proof. You are asked to prove this in problem 3. ■

2.2.1 Problems

You can use all the results and definitions introduced previously

1. Show $\mathfrak{T}^{++}(n)$ is nilpotent. (hint, if you know tensor theory, use it.)
2. Show the converse of lemma 2.2.2 is false.
3. Prove theorem 2.2.3.

Chapter 3

Representation Theory

3.1 Definition of Representation and Adjoint Representation

So far we have dealt largely the abstract side of Lie algebra, but there is also a great advantage of pushing the theory down to earth. For example we looked at example 1.2.1 which I secretly used a representation for $\mathfrak{su}(2)$ and $\mathfrak{so}(3,1)$: I used matrices to represents the basis of $\mathfrak{su}(2)$ and K, J to represent $\mathfrak{so}(3,1)$. However we also saw there that we can transform our K, J representation into U, T representation; representation is not unique.

But we have to discuss what can be a representation of a Lie algebra $\mathfrak{L}$. Recall that if we know result of every possible bracket, we will recover all the information of the Lie algebra. This is suggesting we think the "action" of each element in the Lie algebra to any other arbitrary element. Moreover, we know that $\mathfrak{L}$ is closed under the bracket, and all these arguments encourage us to think elements in the $\mathfrak{L}$ as endormorphism that preserve the structure of the $\mathfrak{L}$; how can we make such representation preserve the structure? We use homomorphism (recall 1.1.2). We are ready for the definiton of a representation:

Definition 3.1.1. A **representation** of a Lie algebra $\mathfrak{L}$ is a homomorphism ϕ such that

$$\phi : \mathfrak{L} \to \mathfrak{gl}(V) \tag{3.1}$$

where $\mathfrak{gl}(V)$ is the endormorphism Lie algebra of V (over a field F) such that the elements are linear endormorphism and the operation is the commutator.

Remark. If you are careful, you might realize that the validity of definition 3.1.1 need to be checked; in another word, are we sure that we can find a representation for an arbitrary Lie algebra? Especially we are choosing the the endormorphism that are linear and the operation to be commutator: both conditions are seemingly very strong. Of course, if we can find a representation to any Lie algebra, it will be remarkable since we can get our hands on any finite dimensional Lie algebra by using representation.

We just need to be a little bit sneaky (or cleaver if you prefer). Let me emphasize again that we can recover all the information about an arbitrary Lie algebra $\mathfrak{L}$ by just knowing every possible result of bracket $[\![\cdot,\cdot]\!]$. What if we make a representation that does exactly that? namely we maps every element $\mathbf{x}$ in the $\mathfrak{L}$ to the endormorphism $[\![\mathbf{x},\cdot]\!]$ (this is indeed an endormorphism, since you can "plug in" any element in the algebra and get another element) and as you can check, this defines a new Lie algebra $\mathfrak{L}'$ with elements $[\![\mathbf{x},\cdot]\!]$ and we can define the bracket operation in $\mathfrak{L}'$ as $< \cdot,\cdot >$.

Don't be confused here, all we are doing is mapping an arbitrary Lie algebra $\mathfrak{L}$ with elements $\mathbf{x}$ and operation $[\![\cdot,\cdot]\!]$ in to the Lie algebra $\mathfrak{L}'$ with elements $< [\![\mathbf{x},\cdot]\!] >$ and operation $< \cdot,\cdot >$. Ultimately we want $< [\![\mathbf{x},\cdot]\!], [\![\mathbf{y},\cdot]\!] > = < [\![\mathbf{x},\mathbf{y}]\!], \cdot >$, since the homomorphism require this.

Now how can we satisfy $< [\![\mathbf{x},\cdot]\!], [\![\mathbf{y},\cdot]\!] > = < [\![\mathbf{x},\mathbf{y}]\!], \cdot >$? Remember that the commutator $[\cdot,\cdot]$ can be used as the algebraic operation that make any vector space into a Lie algebra? Our very soul is calling us to try the commutator now. So we let $< \cdot,\cdot > \equiv [\cdot,\cdot]$ and check :

$$
\begin{aligned}
< [\![\mathbf{x},\cdot]\!], [\![\mathbf{y},\cdot]\!] >= [[\![\mathbf{x},\cdot]\!], [\![\mathbf{y},\cdot]\!]] \underset{by\ definition\ of\ commutator}{=} \\
[\![\mathbf{x},\cdot]\!] \underbrace{\circ}_{composition\ of\ maps} [\![\mathbf{y},\cdot]\!] - [\![\mathbf{y},\cdot]\!] \underbrace{\circ}_{composition\ of\ maps} [\![\mathbf{x},\cdot]\!] = \\
[\![\mathbf{x}, [\![\mathbf{y},\cdot]\!]]\!] - [\![\mathbf{y}, [\![\mathbf{x},\cdot]\!]]\!] \underset{anti-symmetry\ of\ the\ bracket}{=} \\
= [\![\mathbf{x}, [\![\mathbf{y},\cdot]\!]]\!] + [\![[\![\mathbf{x},\cdot]\!], \mathbf{y}]\!] \underset{anti-symmetry\ of\ the\ bracket}{=} -[\![\mathbf{x}, [\![\cdot,\mathbf{y}]\!]]\!] - [\![\mathbf{y}, [\![\mathbf{x},\cdot]\!]]\!] \underset{Jacobi}{=} [\![\cdot, [\![\mathbf{y},\mathbf{x}]\!]]\!] \\
\underset{double\ usages\ of\ anti-symmetry\ of\ the\ bracket}{=} [\![[\![\mathbf{x},\mathbf{y}]\!],\cdot]\!]
\end{aligned} \tag{3.2}
$$

The equation 3.2 tells us that it will works if we maps the opeartion $[\![\cdot,\cdot]\!]$ into $[\cdot,\cdot]$ such that (see the second term and the last term of equation 3.2)

$$[\![[\![\mathbf{x},\mathbf{y}]\!],\cdot]\!] = [[\mathbf{x},\mathbf{y}],\cdot] = [[\mathbf{x},\cdot],[\mathbf{y},\cdot]] \tag{3.3}$$

Therefore the validity of using commutator for representation is verified. Of course, we already argued this is remarkable, since we can find a representation for an arbitrary Lie algebra by using commutator and linear endormorphism. We call this representation the **adjoint** representation and defined as the following:

Definition 3.1.2. Let $\mathfrak{L}$ be a finite dimensional Lie algebra, its adjoint representation is the homorphism ad such that

$$
\begin{aligned}
ad : \mathfrak{L} &\to \mathfrak{gl}(V) \\
\mathbf{x} &\mapsto [\mathbf{x},\cdot].
\end{aligned}
$$

By convention,

$$ad(\mathbf{x})\mathbf{y} = [\mathbf{x},\mathbf{y}] \tag{3.4}$$

Notice the kernel of ad map is the center of $\mathfrak{L}$. (you are asked to show this in problem 1), if the center of $\mathfrak{L}$ is 0, then adjoint representation becomes isomorphism, for example, the simple Lie algebras.

3.1.1 Problem

You can use ....
1. Show the kernel of ad map is the center of $\mathfrak{L}$.

3.2 Engel's Theorem

I promise you that using representation of a Lie algebra has great advantage, and now I am willing to fullfill that promise by introducing a new perspective on nilpotency.

Recall

$$\mathfrak{L}^{[i]} \equiv [\![\mathfrak{L}^{[i-1]},\mathfrak{L}]\!] \equiv [\![[\![\mathfrak{L}^{[i-2]},\mathfrak{L}]\!],\mathfrak{L}]\!] \equiv \cdots \equiv [\![[\![\cdots[\![[\![\mathfrak{L}^{[1]},\mathfrak{L}]\!],\mathfrak{L}]\!],\cdots]\!]\mathfrak{L}]\!] \tag{3.5}$$

Where

$$\mathfrak{L}^{(1)} = \mathfrak{L}^{[1]} \equiv [\![\mathfrak{L},\mathfrak{L}]\!] \tag{3.6}$$

For our convenience, we flip equation 3.5:

$$[\![\mathfrak{L},\cdots,[\![\mathfrak{L}[\![\mathfrak{L},\mathfrak{L}^{[1]}]\!]]\!]\cdots]\!] = [\![\mathfrak{L},\cdots,[\![\mathfrak{L}[\![\mathfrak{L},[\![\mathfrak{L},\mathfrak{L}]\!]]\!]]\!]\cdots]\!] \tag{3.7}$$

Of course, this will differ from 3.5 with a minus sign but that is not important for our discussion. If we use the adjoint representation of $\mathfrak{L}$ and think every element individually, this is nothing but saying

$$ad(\mathbf{x}_1)ad(\mathbf{x}_2)\cdots ad(\mathbf{x}_n)\mathbf{y}, \forall\mathbf{x}_1, \forall\mathbf{x}_2, \cdots, \forall\mathbf{x}_n, \forall\mathbf{y} \in \mathfrak{L}. \tag{3.8}$$

and if the Lie algebra is nilpotent, we have that there exist a minimum $m \in \mathbb{N}$ such that

$$ad(\mathbf{x}_1)ad(\mathbf{x}_2)\cdots ad(\mathbf{x}_m)\mathbf{y} = 0, \forall\mathbf{x}_1, \forall\mathbf{x}_2, \cdots, \forall\mathbf{x}_m, \forall\mathbf{y} \in \mathfrak{L}. \tag{3.9}$$

We also define the following for our convenience:

Definition 3.2.1. An endormorphism ϕ is nilpotent iff there exist a minimum $r \in \mathbb{N}$ such that $\phi^r = 0$.

Remark. Notice this definition is well defined, since endormorphism's target space is the same as the domain.

We also call adjoint map **ad-nilpotent** iff the adjoint map is nilpotent. Moreover, we call $\mathbf{x}$ is ad-nilpotent respect to $\mathbf{x}$ if there exist a minimum $n \in \mathbb{N}$ such that $ad^n(\mathbf{x})$. Of course, the nilpotency of a Lie algebra $\mathfrak{L}$ implies the ad-nilpotency of the adjoint map: just consider the equation 3.9 and chose $\mathbf{x}_1 = \mathbf{x}_2, \cdots, = \mathbf{x}_m$ and we will have

$$ad^m(\mathbf{x}) = 0, \forall\mathbf{x} \in \mathfrak{L}. \tag{3.10}$$

Therefore the adjoint map is ad-nilpotent for nilpotent Lie algebra. The interesting case is the converse: If the adjoint map is ad-nilpotent, does that implies the nilpotency of the entire Lie algebra? The answer is surprisingly yes, and it is the Engel's Theorem

Theorem 3.2.1. *(Engel) If all elements of $\mathfrak{L}$ are ad-nilpotent respect to itself, then $\mathfrak{L}$ is nilpotent.*

Remark. Before I prove it for you, you really should try to prove this first. It is a remarkable result in a sense that equation 3.9 can be deduced by just saying equation 3.10, which at first glance lacks the "crossterms" information to conclude equation 3.9.

Let us think about it for a while, recall that $ad(\mathbf{x}) = [\mathbf{x}, \cdot]$ is nothing but a map that any element $\mathbf{y}$ in the Lie algebra can be plugged into the second slot of the commutator. $ad^m(\mathbf{x})$ is also nothing but a map with one slot: $[\mathbf{x}, [\mathbf{x}, \cdots [\mathbf{x}, \underbrace{\quad\cdot\quad}_{the\ variable\ slot}], \cdots]]$ equation 3.10 is nothing but saying that for any element $\mathbf{y}$ in the Lie algebra such that

$$ad^m(\mathbf{x})\mathbf{y} = [\mathbf{x}, [\mathbf{x}, \cdots [\mathbf{x}, \mathbf{y}], \cdots]] = 0, \forall\mathbf{y} \in \mathfrak{L}. \tag{3.11}$$

Now we now that the maps $ad^m(\mathbf{x})$, which is an element in the adjoint representation (why?), annihilate every elements in the Lie algebra, but can we find an element annihilated by every nilpotent adjoint representation? if so we can find a particular $\mathbf{y}$ such that

$$ad(\mathbf{x}_1)ad(\mathbf{x}_2)\cdots ad(\mathbf{x}_m)\mathbf{y} = 0, \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m, \mathbf{y} \in \mathfrak{L}. \tag{3.12}$$

since $ad(\mathbf{x}_1)ad(\mathbf{x}_2)\cdots ad(\mathbf{x}_m)$ is an nilpotent adjoint representation (why?). The answer is yes and we will prove it now

Lemma 3.2.2. *There is a common nonzero vector $v \in \mathfrak{L}$, where $\mathfrak{L}$ is spaned by ad-nilpotent elements, such that $[\mathfrak{L}, v] = 0$.*

Proof. We prove it by strong induction on dimension. The case of dimension 1 is trivial. Let the assumption holds true for dimension m, and we denote the common vector as v, for all the subalgebra $\mathfrak{K}$, the assumption is also true. Consider $\mathfrak{L}/\mathfrak{K}$, its dimension is less than $dim(\mathfrak{L})$ thus also contain a common vector a that annihilated by every element in $\mathfrak{L}/\mathfrak{K}$, this means that $\mathfrak{K}$ is an ideal of subalgebra $\mathfrak{K} + \epsilon a, \epsilon \in F$, where F is the field. Let $\mathfrak{K}$ be maximal, then $\mathfrak{K} + \epsilon a, \epsilon \in F$ is forced to be the entire $\mathfrak{L}$. We now left to prove that $\mathbf{a}$ also annihilate a vector $\mathbf{k} \in N_{\mathfrak{K}+\epsilon\mathbf{a}}(\mathfrak{K})$ (the Normalizer of $\mathfrak{K}$). One way to show this is to realize that entire $\mathfrak{L} = \mathfrak{K} + \epsilon a$ stabilizes $N_{\mathfrak{K}+\epsilon\mathbf{a}}(\mathfrak{K})$.

$$[\mathfrak{K}, [\mathfrak{L}, N_{\mathfrak{K}+\epsilon\mathbf{a}}(\mathfrak{K})]] = 0 \tag{3.13}$$

You are asked to show why this is in problem 1. So by the assumption, there exist an element $\mathbf{k}$ such that $[\mathbf{a}, \mathbf{k}] = 0$ (you are asked to show why in problem 2) and we conclude the lemma. ■

Now we are ready to prove Engel's theorem.

Proof. (Engel). We prove this by induction. Let $\mathfrak{L}$ be the Lie algebra with all elements being ad-nilpotent. By the lemma 3.2.2, we conclude that there exist a common vector $\mathbf{x}$ such that $[\mathfrak{L}, \mathbf{x}] = 0$. We have non zero center for $\mathfrak{L}$, Now consider $\mathfrak{L}/\mathbf{Z}(\mathfrak{L})$. By assumption, it is nilpotent, therefore $\mathfrak{L}$ is nilpotent. (recall 2.2.3). ■

3.2.1 Problem

You....

1. Show equation 3.13 is true. (remember $\mathfrak{K}$ is an ideal and use Jacobi identity.)
2. Show why $[\mathbf{a}, \mathbf{k}] = 0$ in the proof of lemma 3.2.2.