Exponential of an operator function

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1 Operator expansion and as exponential

Let a function h of an operator A be $h(A)$. If from infinitesimal transformation ϵX, we can write

$$h(\theta) = \lim_{N\to\infty}(\mathbb{1} + i\frac{\theta}{N}X)^N \tag{1}$$

(where θ is the parameter of the operator function. here $\frac{\theta}{N}$ serve as ϵ in equation 1. The validity of such assumption need further rigorous mathematical investigation.)

Then we will have

$$h(\theta) = \lim_{N\to\infty}(\mathbb{1} + i\frac{\theta}{N}X)^N = e^{\theta X} \tag{2}$$

Proof. Consider

$$\ln\left(\lim_{N\to\infty}(\mathbb{1} + i\frac{\theta}{N}X)^N\right) = \lim_{N\to\infty} N\ln\left((\mathbb{1} + i\frac{\theta}{N}X)\right) = \lim_{N\to\infty}\frac{1}{N^{-1}}\ln\left((\mathbb{1} + i\frac{\theta}{N}X)\right) \tag{3}$$

Here we use L' Hospital's rule to find the limit by taking derivative of both denominator and numerator:

$$\lim_{N\to\infty}\frac{1}{N^{-1}}\ln\left((\mathbb{1} + i\frac{\theta}{N}X)\right) = \lim_{N\to\infty}\frac{-\frac{1}{1+\frac{\theta X}{N}}\frac{\theta X}{N^2}}{-\frac{1}{N^2}} = \frac{\theta X}{1+\frac{\theta X}{N}} = \theta X \tag{4}$$

So we have

$$\ln^{-1}(\ln h(\theta)) = \ln^{-1}(\theta X) = e^{\theta X} \tag{5}$$

∎

X is called the **generator**.