Determinant Identity In Terms Of Trace Log Exponential

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Consider a general square matrix A. We show that

$$det(exp(A)) = exp(tr(A)) \tag{1}$$

and for compelx invertible matrix B, we have

$$det(B) = exp(tr(\ln(B))) \tag{2}$$

Proof. By definition

$$exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} \tag{3}$$

Consider matrix O such that take A in to A's cannonical Jordan form J:

$$O^{-1}AO = J \tag{4}$$

so equation 1 starts with

$$det(exp(OJO^{-1})) \tag{5}$$

Consider the following

$$exp(OJO^{-1}) = exp(\sum_{n=0}^{\infty} \frac{(OJO^{-1})^n}{n!}) \tag{6}$$

For any $n \in \mathbb{N}$, we have (easily verified.)

$$(OJO^{-1})^n = OJ^nO^{-1} \tag{7}$$

so equation 5 becomes

$$det(exp(OJO^{-1})) = det(Oexp(J)O^{-1}) = det(exp(J)) \tag{8}$$

For Jordan canonical form J, this is $\prod_i exp(J_i^i)$ and we know

$$\prod_i exp(J_i^i) = exp(\prod_i J_i^i) = exp(tr(J)) \tag{9}$$

But we know that J and A are similar; they have the same trace, so we have $exp(tr(J)) = exp(tr(A))$

$$det(exp(A)) = exp(tr(A)) \tag{10}$$

So for identity 2, we have to ensure B can be written as exponential of some matrix. This is true iff B is invertible and complex by using holomorphic functional ■