Weyl Spinor

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We will derive the Weyl equations from Klein-Gordon equation.

Let $S = \mathbb{R}^3 \times T$ be the spacetime, where $\mathbb{R}^3$ is the space and T is the time. Further let $\phi(\mathbf{x}) : S \to \mathbb{C}$ be class C^2. We have K-G equation as

$$(-\eta^{\alpha\beta}\partial_\alpha\partial_\beta + \mu^2)\phi = 0, \mu \in \mathbb{C}. \tag{1}$$

with the metric

$$\eta^{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

So ϕ is in the kernel of operator $-\eta^{\alpha\beta}\partial_\alpha\partial_\beta + \mu^2$.

we can write the map $Q \equiv -\eta^{\alpha\beta}\partial_\alpha\partial_\beta$ as map compostion of the map $W \equiv i\partial_0 + \gamma^a\partial_a$ and the map $Y \equiv i\partial_0 + \xi_b\partial^b$, $a, b \in \{1, 2, 3\}$, $\gamma, \xi \in \mathbb{C}$.

We can realize the following lemma

Lemma 1. *$Q = W \circ Y$ iff $\gamma = -\xi$ and $(\gamma^a)^2 = -1$ $\forall a \in \{1, 2, 3\}$.*

(the proof of this is just simple algebra over complex field.)

With that in mind, we can express operator

$$\partial_a\partial^a = \gamma^a\xi_b\partial_a\partial^b \tag{3}$$

With intention of obtaining a Clifford algebra generator, we apply identity transformation to obtain $\gamma^a\xi_b\partial_a\partial^b = \frac{1}{2}(\gamma^a\xi_b + \gamma_b\xi^a)\partial_a\partial^b$, which means

$$\partial_a\partial^a = \frac{1}{2}(\gamma^a\xi_b + \gamma_b\xi^a)\partial_a\partial^b \tag{4}$$

We use lemma 1 to obtain

$$\partial_a\partial^a = \frac{1}{2}(\gamma^a\xi_b + \gamma_b\xi^a)\partial_a\partial^b = \partial_a\partial^a = \frac{1}{2}(\gamma^a(-\gamma_b) + \gamma_b(-\gamma^a))\partial_a\partial^b = -\frac{1}{2}(\gamma^a\gamma_b + \gamma_b\gamma^a)\partial_a\partial^b \tag{5}$$

We eliminating annoying negative sign by letting

$$\gamma_a \equiv i\sigma_a \tag{6}$$

and obtain

$$\partial_a\partial^a = \frac{1}{2}(\sigma^a\sigma_b + \sigma_b\sigma^a)\partial_a\partial^b \tag{7}$$

and we obtain the following algebra relation

$$[\sigma^a, \sigma_b]_+ = 2\delta^a_b \tag{8}$$

And here is where we hit a wall. if $\sigma \in \mathbb{C}$, then above equation can not be satisfied. So we see $\gamma, \xi \notin \mathbb{C}$, and we need to reconsider what space does γ, ξ belong.

The equation 8 is very obvious solveable if σs are 2×2 matrices. So we let $\gamma, \xi \in \mathbb{M}(2, \mathbb{C})$.

It is easy to see σ matrices are just three Pauli matrices. And we have K-G equation as

$$[(i\partial_0 + i\sigma^a \partial_a)(i\partial_0 - i\sigma^b \partial_b) - \mu^2]\phi \equiv [K \circ P - \mu^2]\phi = 0 \tag{9}$$

where the map $K = i\partial_0 + i\sigma^a \partial_a$ and the map $P = i\partial_0 - i\sigma^b \partial_b$.

We denote $P(\phi) \equiv \mu\chi$ and obtain $K(\chi) = \mu\phi$ and notice the K-G equation is satisfied.

The equation $P(\phi) \equiv \mu\chi$ gives us

$$i\partial_0 \phi - i\sigma^b \partial_b \phi - \mu\chi = 0 \tag{10}$$

and $K(\chi) = \mu\phi$ gives us

$$i\partial_0 \chi + i\sigma^b \partial_b \chi - \mu\phi = 0 \tag{11}$$

Coupled Equations 10 and 11 are called Weyl equations, and solutions ϕ and χ are called the left Weyl spinor and the right Weyl spinor resepctively.

We observe that if μ(mass term) vanishes, Weyl equations decouple. So the resulting equation describe massless particles. We denote operator (which is a map)$i\partial_0 \equiv E$ (the energy operator) and $-i\partial_a p_a$ (momentum operator). For massless particle, we have (with relation $E^2 - p_a p^a - \mu^2 = 0$)

$$(\sigma^a p_a + \mathbb{1})\phi = 0 \tag{12}$$

and

$$(\sigma^a p_a - \mathbb{1})\chi = 0 \tag{13}$$

Recall that $\sigma^a p_a$ is the *helicity*. So for massless particle, the state ϕ has helicity -1 and χ has helicity of $+1$.

From Weyl equation, we see that ϕ, χ are two component spinors:

$$\phi_a = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \chi^{\dot{a}} = \begin{pmatrix} \chi^{\dot{1}} \\ \chi^{\dot{2}} \end{pmatrix}, \phi_1, \phi_2, \chi^{\dot{1}}, \chi^{\dot{2}} \in \mathbb{C}. \tag{14}$$

Convention is to use undotted index to represent left spinor and dotted index to represent right spinor.

The transformation of Weyl spinors are particular interesting and important. From equation 14, we see that if we denote the transformation of Weyl spinors as N^a_b (for left spinor) and $N^{\dot{b}}_{\dot{a}}$, those transformations need to be unitary(for quantum mechanical purpose) and the entries are complex numbers. Well, this means they are elements of $SL(2, \mathbb{C})$. More interestingly is its Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. (Notice we can apply Cartan decomposition to $\mathfrak{sl}(2, \mathbb{C})$.) Complexification of $\mathfrak{so}(3, 1)$ is isomorphic to the direct sum of two $\mathfrak{sl}(2, \mathbb{C})$, which in turns isomorphic to complexification of $\mathfrak{sl}(2, \mathbb{C})$.