Symbolic Logic and Mechanical Theorem Proving

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Chapter 1

Introduction

1.1 ARTIFICIAL INTELLIGENCE, SYMBOLIC LOGIC, AND THEOREM PROVING

Ever since the birth of the first modern computer, computer technology has developed at a fantastic speed. Today we see computers being used not only to solve computationally difficult problems, such as carrying out a fast Fourier transform or inverting a matrix of high dimensionality, but also being asked to perform tasks that would be called intelligent if done by human beings. Some such tasks are writing programs, answering questions, and proving theorems. Artificial intelligence is a field in computer science that is concerned with performing such tasks [Ernst and Newell, 1969; Feigenbaum and Feldman, 1963; Nilsson, 1971; Slagle, 1971].

The second half of the 1960s has been phenomenal in the field of artificial intelligence for the increase of interest in mechanical theorem proving. The widespread intensive interest in mechanical theorem proving is caused not only by the growing awareness that the ability to make logical deductions is an integral part of human intelligence, but is perhaps more a result of the status of mechanical theorem-proving techniques in the late 1960s. The foundation of mechanical theorem proving was developed by Herbrand in 1930. His method was impractical to apply until the invention of the digital computer. It was not until the landmark paper by J. A. Robinson in 1965,

together with the development of the resolution principle, that major steps were taken to achieve realistic computer-implemented theorem provers. Since 1965, many refinements of the resolution principle have been made.

Parallel to the progress in improving mechanical theorem-proving techniques was the progress in applying theorem-proving techniques to various problems in artificial intelligence. These were first applied to deductive question answering and later to problem solving, program synthesis, program analysis, and many others.

There are many points of view from which we can study symbolic logic. Traditionally, it has been studied from philosophical and mathematical orientations. In this book, we are interested in the *applications* of symbolic logic to solving intellectually difficult problems. That is, we want to use symbolic logic to represent problems and to obtain their solutions.

We shall now present some very simple examples to demonstrate how symbolic logic can be used to represent problems. Since we have not yet formally discussed symbolic logic, the reader should rely on his intuition to follow the text at this moment.

Let us consider a simple example. Assume that we have the following facts:

F_1: If it is hot and humid, then it will rain.
F_2: If it is humid, then it is hot.
F_3: It is humid now.

The question is: Will it rain?

The above facts are given in English. We shall use symbols to represent them. Let P, Q, and R represent "It is hot," "It is humid," and "It will rain," respectively. We also need some logical symbols. In this case, we shall use $\wedge$ to represent "and" and $\rightarrow$ to represent "imply." Then the above three facts are represented as:

F_1: $P \wedge Q \rightarrow R$

F_2: $Q \rightarrow P$

F_3: Q.

Thus we have translated English sentences into logical formulas. The reader will see later that whenever F_1, F_2, and F_3 are true, the formula

F_4: R

is true. Therefore, we say that F_4 *logically follows from* F_1, F_2, and F_3. That is, it will rain.

Let us consider another example. We have the following facts:

F_1: Confucius is a man.
F_2: Every man is mortal.

To represent F_1 and F_2, we need a new concept, called a predicate. We may let $P(x)$ and $Q(x)$ represent "x is a man" and "x is mortal," respectively. We also use $(\forall x)$ to represent "for all x." Then the above facts are represented by

F_1: $P(\text{Confucius})$
F_2: $(\forall x)(P(x) \rightarrow Q(x))$.

Again, the reader will see later that from F_1 and F_2, we can *logically deduce*

F_3: $Q(\text{Confucius})$

which means that Confucius is mortal.

In the above two examples, we essentially had to prove that a formula *logically follows from* other formulas. We shall call a statement that a formula logically follows from formulas a *theorem*. A demonstration that a theorem is true, that is that a formula logically follows from other formulas, will be called a *proof* of the theorem. The *problem of mechanical theorem proving* is to consider mechanical methods for finding proofs of theorems.

There are many problems that can be conveniently transformed into theorem-proving problems; we shall list some of them.

1. In a *question-answering system*, facts can be represented by logical formulas. Then, to answer a question from the facts, we prove that a formula corresponding to the answer is derivable from the formulas representing the facts.

2. In the *program-analysis problem*, we can describe the execution of a program by a formula A, and the condition that the program will terminate by another formula B. Then, verifying that the program will terminate is equivalent to proving that formula B follows from formula A.

3. In the *graph-isomorphism problem*, we want to know whether a graph is isomorphic to a subgraph of another graph. This problem is not merely an interesting problem in mathematics; it is also a practical problem. For example, the structure of an organic compound can be described by a graph. Therefore, testing whether a substructure of the structure of an organic compound is the structure of another organic compound is a graph-isomorphism problem. For this problem, we can describe graphs by formulas. Thus, the problem may be stated as proving that the formula representing a graph follows from the formula representing another graph.

4. In the *state-transformation problem*, there are a collection of states and a collection of (state) operators. When one operator is applied to a state, a new state will be obtained. Starting with an initial state, one tries to find a sequence of operators that will transform the initial state into a desired state. In this case, we can describe states and the state transition rules by logical formulas. Thus, transforming the initial state into the desired state may be

regarded as verifying that the formula representing the desired state follows from the formula representing both the states and the state transition rules.

Since many problems can be formulated as theorem-proving problems, theorem proving is a very important field in artificial intelligence science. Thanks to the relentless efforts of many researchers, great progress has been made in using computers to prove theorems. In this book, both the theory and applications of mechanical theorem proving will be considered.

1.2 MATHEMATICAL BACKGROUND

In this section, we provide some basic mathematical concepts that will be used in this book. All of these concepts can be easily found in elementary mathematical textbooks [Lightstone, 1964; Pfeiffer, 1964; Stoll, 1961]. The following basic definitions of set theory are used.

A *set* is a collection of elements (members). A set that contains no elements is called the *empty set* $\varnothing$. Let A and B be two sets. $x \in A$ is used to denote that x is a member of A, or that x belongs to A.

The set A is *identical* to the set B, denoted $A = B$, if and only if both A and B have the same elements.

The set A is a *subset* of the set B, denoted $A \subseteq B$, if and only if each element of A is an element of B.

The set A is a *proper subset* of the set B, denoted $A \subset B$, if and only if $A \subseteq B$ and $A \neq B$.

The *union* of two sets A and B, denoted by $A \cup B$, is the set consisting of all elements that belong to A or to B.

The *intersection* of two sets A and B, denoted by $A \cap B$, is the set consisting of all elements that belong both to A and to B.

The *difference* of two sets A and B, denoted by $A - B$, is the set consisting of all elements that belong to A but do not belong to B.

We now define relations and functions.

An *ordered pair* of elements is denoted by (x, y), where x is called the first *coordinate* and y is called the second *coordinate*.

A *relation* is a set of ordered pairs. For example, the equality relation is a set of ordered pairs, each of which has the first coordinate equal to its second coordinate. The *domain* of a relation R is the set of all first coordinates of elements of R, and its *range* is the set of all second coordinates.

A *function* is a relation such that no two distinct elements have the same first coordinate. If f is a function and x is an element of its domain, then $f(x)$ denotes the second coordinate of the unique element of f whose first coordinate is x. $f(x)$ is called the *value* of f at x, and we say that f assigns the value $f(x)$ to x.

Throughout this book, we use many conventional symbols. For example,

$>$ means "greater than"; $\geqslant$, "greater than or equal to"; $<$, "less than"; $\leqslant$, "less than or equal to"; $=$, "equal to"; $\neq$, "not equal to"; $\triangleq$, "defined as"; and so on. The equality sign "$=$" will be used for many purposes. It may be used to mean "is defined by," "is identical to," "is equivalent to," or "is equal to." This will not cause the reader any confusion since he can always determine its exact meaning from the text.

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Chapter 2

The Propositional Logic

2.1 INTRODUCTION

Symbolic logic considers languages whose essential purpose is to symbolize reasoning encountered not only in mathematics but also in daily life. In this chapter, we shall first study the simplest symbolic logic—the propositional logic (or the propositional calculus). In the next chapter, we shall consider a more general one—the first-order logic (or the first-order predicate calculus).

In the propositional logic, we are interested in declarative sentences that can be either *true* or *false*, but not both. Any such declarative sentence is called a proposition. More formally, a *proposition* is a declarative sentence that is either true or false, but not both. Examples of propositions are: "Snow is white," "Sugar is a hydrocarbon," "Smith has a Ph.D. degree." The "true" or "false" assigned to a proposition is called the *truth value* of the proposition. Customarily, we represent "true" by $\boldsymbol{T}$ and "false" by $\boldsymbol{F}$. Furthermore, for convenience, we shall use an uppercase symbol or a string of uppercase symbols to denote a proposition. For instance, we may denote the propositions above as follows:

$P \triangleq$ Snow is white,
$Q \triangleq$ Sugar is a hydrocarbon,
$R \triangleq$ Smith has a Ph.D. degree.

The symbols, such as P, Q, and R, that are used to denote propositions are called *atomic formulus*, or *atoms*.

From propositions, we can build *compound* propositions by using *logical connectives*. Examples of compound propositions are: "Snow is white *and* the sky is clear" and "*If* John is not at home, *then* Mary is at home." The logical connectives in the above two compund propositions are "and" and "if $\cdots$ then." In the propositional logic, we shall use five logical connectives: $\sim$ (*not*), $\wedge$ (*and*), $\vee$ (*or*), $\rightarrow$ (*if* $\cdots$ *then*), and $\leftrightarrow$ (*if and only if*). These five logical connectives can be used to build compound propositions from propositions. More generally, they can be used to construct more complicated compound propositions from compound propositions by applying them repeatedly. For example, let us represent "Humidity is high" by P, "Temperature is high" by Q, and "One feels comfortable" by C. Then the sentence "If the humidity is high and the temperature is high, then one does not feel comfortable" may be represented by $((P \wedge Q) \rightarrow (\sim C))$. Therefore, we see that a compound proposition can express a rather complicated idea. In the propositional logic, an expression that represents a proposition, such as P, or a compound proposition, such as $((P \wedge Q) \rightarrow (\sim C))$, is called a well-formed formula.

Definition *Well-formed formulas*, or *formulas* for short, in the propositional logic are defined recursively as follows:

1. An atom is a formula.
2. If G is a formula, then $(\sim G)$ is a formula.
3. If G and H are formulas, then $(G \wedge H)$, $(G \vee H)$, $(G \rightarrow H)$, and $(G \leftrightarrow H)$ are formulas.
4. All formulas are generated by applying the above rules.

It is not difficult to see that expressions such as $(P \rightarrow)$ and $(P \vee)$ are not formulas. Throughout the book, when no confusion is possible, some pairs of parentheses may be dropped. For example, $P \vee Q$ and $P \rightarrow Q$ are the formulas $(P \vee Q)$ and $(P \rightarrow Q)$, respectively. We can further omit the use of parentheses by assigning decreasing ranks to the propositional connectives as follows:

$$\leftrightarrow, \quad \rightarrow, \quad \wedge, \quad \vee, \quad \sim,$$

and requiring that the connective with greater rank always reaches further. Thus $P \rightarrow Q \wedge R$ will mean $(P \rightarrow (Q \wedge R))$, and $P \rightarrow Q \wedge \sim R \vee S$ will mean $(P \rightarrow (Q \wedge ((\sim R) \vee S)))$.

Let G and H be two formulas. Then the truth values of the formulas $(\sim G)$, $(G \wedge H)$, $(G \vee H)$, $(G \rightarrow H)$, and $(G \leftrightarrow H)$ are related to the truth values of G and H in the following way:

1. $\sim G$ is true when G is false, and is false when G is true. $\sim G$ is called the *negation* of G.

2. $(G \wedge H)$ is true if G and H are both true; otherwise, $(G \wedge H)$ is false. $(G \wedge H)$ is called the *conjunction* of G and H.

3. $(G \vee H)$ is true if at least one of G and H is true; otherwise, $(G \vee H)$ is false. $(G \vee H)$ is called the *disjunction* of G and H.

4. $(G \rightarrow H)$ is false if G is true and H is false; otherwise, $(G \rightarrow H)$ is true. $(G \rightarrow H)$ is read as "*If G, then H*," or "*G implies H*."

5. $(G \leftrightarrow H)$ is true whenever G and H have the same truth values; otherwise $(G \leftrightarrow H)$ is false. $(G \leftrightarrow H)$ is read as "*G* if and only if *H*."

The above relations can be represented conveniently as in Table 2.1. Based on this table, we shall describe ways to evaluate the truth values of a formula in terms of the truth values of atoms occurring in the formula.

TABLE 2.1

G	H	$\sim G$	$(G \wedge H)$	$(G \vee H)$	$(G \rightarrow H)$	$(G \leftrightarrow H)$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

2.2 INTERPRETATIONS OF FORMULAS IN THE PROPOSITIONAL LOGIC

Suppose that P and Q are two atoms and that the truth values of P and Q are T and F, respectively. Then, according to the second row of Table 2.1 with P and Q substituted for G and H, respectively, we find that the truth values of $(\sim P)$, $(P \wedge Q)$, $(P \vee Q)$, $(P \rightarrow Q)$, and $(P \leftrightarrow Q)$ are F, F, T, F, and F, respectively. Similarly, the truth value of any formula can be evaluated in terms of truth values of atoms.

Example 2.1

Consider the formula

$$G \triangleq (P \wedge Q) \rightarrow (R \leftrightarrow (\sim S)).$$

The atoms in this formula are P, Q, R, and S. Suppose the truth values of P, Q, R, and S are T, F, T, and T, respectively. Then $(P \wedge Q)$ is F since Q is false; $(\sim S)$ is F since S is T; $(R \leftrightarrow (\sim S))$ is F since R is T and $(\sim S)$ is F; and $(P \wedge Q) \rightarrow (R \leftrightarrow (\sim S))$ is T since $(P \wedge Q)$ is F and $(R \leftrightarrow (\sim S))$ is F. Therefore, the formula G is T if P, Q, R, and S are assigned T, F, T, and T,

respectively. The assignment of the truth values $\{T, F, T, T\}$ to $\{P, Q, R, S\}$, respectively, will be called an *interpretation* of the formula G. Since each one of P, Q, R, and S can be assigned either T or F, there are $2^4 = 16$ interpretations of the formula G. In Table 2.2, we give the truth values of the formula G under all these 16 interpretations.

TABLE 2.2

TRUTH TABLE OF $(P \wedge Q) \rightarrow (R \leftrightarrow (\sim S))$

P	Q	R	S	$\sim S$	$(P \wedge Q)$	$(R \leftrightarrow (\sim S))$	$(P \wedge Q) \rightarrow (R \leftrightarrow (\sim S))$
T	T	T	T	F	T	F	F
T	T	T	F	T	T	T	T
T	T	F	T	F	T	T	T
T	T	F	F	T	T	F	F
T	F	T	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	T	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	F	F	F	T
F	T	T	F	T	F	T	T
F	T	F	T	F	F	T	T
F	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T
F	F	T	F	T	F	T	T
F	F	F	T	F	F	T	T
F	F	F	F	T	F	F	T

A table, such as Table 2.2, that displays the truth values of a formula G for all possible assignments of truth values to atoms occurring in G is called a *truth table* of G.

We now give a formal definition of an interpretation of a propositional formula.

Definition Given a propositional formula G, let $A_1, A_2, \ldots, A_n$ be the atoms occurring in the formula G. Then an *interpretation* of G is an assignment of truth values to $A_1, \ldots, A_n$ in which every A_i is assigned either T or F, but not both.

Definition A formula G is said to be *true under* (*or in*) *an interpretation* if and only if G is evaluated to T in the interpretation; otherwise, G is said to be *false under the interpretation.*

If there are n distinct atoms in a formula, then there will be 2^n distinct interpretations for the formula. Sometimes, if $A_1, \ldots, A_n$ are all atoms

occurring in a formula, it may be more convenient to represent an interpretation by a set $\{m_1, \ldots, m_n\}$, where m_i is either A_i or $\sim A_i$. For example, the set $\{P, \sim Q, \sim R, S\}$ represents an interpretation in which P, Q, R, and S are, respectively, assigned T, F, F, and T. That is, if an atom A is in a set that represents an interpretation, then A is assigned T; while if the negation of the atom A is in the set, then A is assigned F. This convention will be kept throughout the book.

2.3 VALIDITY AND INCONSISTENCY IN THE PROPOSITIONAL LOGIC

In this section, we shall consider formulas that are true under all their possible interpretations and formulas that are false under all their possible interpretations.

Example 2.2

Let us consider the formula

$$G \triangleq ((P \to Q) \wedge P) \to Q.$$

The atoms in this formula are P and Q. Hence, the formula G has $2^2 = 4$ interpretations. The truth values of G under all its four interpretations are given in Table 2.3. We see that the formula G is true under all its interpretations. This formula will be called a *valid formula* (or a *tautology*).

TABLE 2.3

TRUTH TABLE OF $((P \to Q) \wedge P) \to Q$

P	Q	$(P \to Q)$	$(P \to Q) \wedge P$	$((P \to Q) \wedge P) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Example 2.3

Consider the formula

$$G \triangleq (P \to Q) \wedge (P \wedge \sim Q).$$

The truth table of G is given in Table 2.4. We see that G is false under all its interpretations. This formula will be called an *inconsistent formula* (or a *contradiction*).

TABLE 2.4

TRUTH TABLE OF $(P \to Q) \wedge (P \wedge \sim Q)$

P	Q	$\sim Q$	$(P \to Q)$	$(P \wedge \sim Q)$	$(P \to Q) \wedge (P \wedge \sim Q)$
T	T	F	T	F	F
T	F	T	F	T	F
F	T	F	T	F	F
F	F	T	T	F	F

We now give formal definitions for validity and inconsistency.

Definition A formula is said to be *valid* if and only if it is true under all its interpretations. A formula is said to be *invalid* if and only if it is not valid.

Definition A formula is said to be *inconsistent* (or *unsatisfiable*) if and only if it is false under all its interpretations. A formula is said to be *consistent* (or *satisfiable*) if and only if it is not inconsistent.

By the above definitions, the following observations are obvious:

1. A formula is valid if and only if its negation is inconsistent.
2. A formula is inconsistent if and only if its negation is valid.
3. A formula is invalid if and only if there is at least one interpretation under which the formula is false.
4. A formula is consistent if and only if there is at least one interpretation under which the formula is true.
5. If a formula is valid, then it is consistent, but not vice versa.
6. If a formula is inconsistent, then it is invalid, but not vice versa.

Example 2.4

By using the truth table techniques, the reader should be able to establish the following:

a. $(P \wedge \sim P)$ is inconsistent; therefore also invalid.
b. $(P \vee \sim P)$ is valid; therefore also consistent.
c. $(P \to \sim P)$ is invalid, yet it is consistent.

If a formula F is true under an interpretation I, then we say that I *satisfies* F, or F *is satisfied* by I. On the other hand, if a formula F is false under an interpretation I, then we say that I *falsifies* F or F *is falsified* by I. For example, the formula $(P \wedge (\sim Q))$ is satisfied by the interpretation $\{P, \sim Q\}$, but is falsified by the interpretation $\{P, Q\}$. When an interpretation I satisfies a formula F, I is also called a *model* of F.

It will be shown later that the proof of the validity or inconsistency of a

formula is a very important problem. In the propositional logic, since the number of interpretations of a formula is finite, one can always decide whether or not a formula in the propositional logic is valid (inconsistent) by exhaustively examining all of its possible interpretations.

2.4 NORMAL FORMS IN THE PROPOSITIONAL LOGIC

As will be clear later, it is often necessary to transform a formula from one form to another, especially to a "normal form." This is accomplished by replacing a formula in the given formula by a formula "equivalent" to it and repeating this process until the desired form is obtained. By "equivalent," we mean the following:

Definition Two formulas F and G are said to be *equivalent* (or F is *equivalent* to G), denoted $F = G$, if and only if the truth values of F and G are the same under every interpretation of F and G.

Example 2.5

We can verify that $(P \to Q)$ is equivalent to $(\sim P \vee Q)$ by examining the truth table, Table 2.5.

TABLE 2.5
TRUTH TABLE OF $(P \to Q)$ AND $(\sim P \vee Q)$

P	Q	$(P \to Q)$	$(\sim P \vee Q)$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

We need an adequate supply of equivalent formulas in order to carry out the transformation of formulas. In the propositional logic, let ■ denote the formula that is always true and □ the formula that is always false. Then we have some useful pairs of equivalent formulas given in Table 2.6, where F, G, and H are all formulas. For simplicity, we shall call each of them a "law."

The laws in Table 2.6 can be verified by using truth tables. Laws (2.3a), (2.3b) are often called *commutative* laws; (2.4a), (2.4b) *associative* laws; (2.5a), (2.5b), *distributive* laws; and (2.10a), (2.10b), *De Morgan's* laws.

Because of the associative laws, the parentheses in $(F \vee G) \vee H$ or $F \vee (G \vee H)$ can be dropped. That is, we can write $F \vee G \vee H$ for $(F \vee G) \vee H$ and $F \vee (G \vee H)$. More generally, we can write $F_1 \vee F_2 \vee \cdots \vee F_n$ without

TABLE 2.6

(2.1)	$F \leftrightarrow G = (F \rightarrow G) \wedge (G \rightarrow F)$	
(2.2)	$F \rightarrow G = \sim F \vee G$	
(2.3)	(a) $F \vee G = G \vee F$;	(b) $F \wedge G = G \wedge F$
(2.4)	(a) $(F \vee G) \vee H = F \vee (G \vee H)$;	(b) $(F \wedge G) \wedge H = F \wedge (G \wedge H)$
(2.5)	(a) $F \vee (G \wedge H) = (F \vee G) \wedge (F \vee H)$;	(b) $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$
(2.6)	(a) $F \vee \square = F$;	(b) $F \wedge \blacksquare = F$
(2.7)	(a) $F \vee \blacksquare = \blacksquare$;	(b) $F \wedge \square = \square$
(2.8)	(a) $F \vee \sim F = \blacksquare$;	(b) $F \wedge \sim F = \square$
(2.9)	$\sim(\sim F) = F$	
(2.10)	(a) $\sim(F \vee G) = \sim F \wedge \sim G$;	(b) $\sim(F \wedge G) = \sim F \vee \sim G$

any ambiguity, where $F_1, F_2, \ldots, F_n$ are formulas. $F_1 \vee F_2 \vee \cdots \vee F_n$ is true whenever at least one of the F_i, $1 \leqslant i \leqslant n$, is true; otherwise, it is false. $F_1 \vee F_2 \vee \cdots \vee F_n$ is called the *disjunction* of $F_1, \ldots, F_n$. Similarly, we can write $F_1 \wedge F_2 \wedge \cdots \wedge F_n$, which is true if all $F_1, F_2, \ldots, F_n$ are true, and which is false otherwise. $F_1 \wedge F_2 \wedge \cdots \wedge F_n$ is called the *conjunction* of $F_1, \ldots, F_n$.

Note that the order in which the F_i appear in a disjunction or a conjunction is immaterial.

For example,

$$F_1 \vee F_2 \vee F_3 = F_1 \vee F_3 \vee F_2 = F_2 \vee F_1 \vee F_3 = F_3 \vee F_2 \vee F_1$$
$$= F_2 \vee F_3 \vee F_1 = F_3 \vee F_1 \vee F_2.$$

We now define normal forms as follows.

Definition A *literal* is an atom or the negation of an atom.

Definition A formula F is said to be in a *conjunctive normal form* if and only if F has the form $F \triangleq F_1 \wedge \cdots \wedge F_n, n \geqslant 1$, where each of $F_1, \ldots, F_n$ is a disjunction of literals.

Example 2.6

Let P, Q, and R be atoms. Then $F \triangleq (P \vee \sim Q \vee R) \wedge (\sim P \vee Q)$ is a formula in a conjunctive normal form. For this formula, $F_1 = (P \vee \sim Q \vee R)$ and $F_2 = (\sim P \vee Q)$. Clearly, F_1 is a disjunction of the literals P, $\sim Q$, and R, and F_2 is a disjunction of the literals $\sim P$ and Q.

Definition A formula F is said to be in a *disjunctive normal form* if and only if F has the form of $F \triangleq F_1 \vee \cdots \vee F_n$, $n \geqslant 1$, where each of $F_1, \ldots, F_n$ is a conjunction of literals.

Example 2.7

Let P, Q, and R be atoms. Then $F \triangleq (\sim P \wedge Q) \vee (P \wedge \sim Q \wedge \sim R)$ is a formula in a disjunctive normal form. For this formula, $F_1 = (\sim P \wedge Q)$

and $F_2 = (P \wedge \sim Q \wedge \sim R)$. Clearly, F_1 is a conjunction of the literals $\sim P$ and Q, and F_2 is a conjunction of the literals P, $\sim Q$, and $\sim R$.

Any formula can be transformed into a normal form. This is accomplished easily by using the laws given in Table 2.6. The following is an *outline of a transformation procedure:*

Step 1 Use the laws

(2.1) $F \leftrightarrow G = (F \rightarrow G) \wedge (G \rightarrow F)$

(2.2) $F \rightarrow G = \sim F \vee G$

to eliminate the logical connectives $\leftrightarrow$ and $\rightarrow$.

Step 2 Repeatedly use the law

(2.9) $\sim(\sim F) = F$

and De Morgan's laws

(2.10a) $\sim(F \vee G) = \sim F \wedge \sim G$

(2.10b) $\sim(F \wedge G) = \sim F \vee \sim G$

to bring the negation signs immediately before atoms.

Step 3 Repeatedly use the distributive laws

(2.5a) $F \vee (G \wedge H) = (F \vee G) \wedge (F \vee H)$

(2.5b) $F \wedge (G \vee H) = (F \wedge G) \vee (F \wedge H)$

and the other laws to obtain a normal form.

Example 2.8

Obtain a disjunctive normal form for the formula $(P \vee \sim Q) \rightarrow R$.
Since

$$\begin{aligned}(P \vee \sim Q) \rightarrow R &= \sim(P \vee \sim Q) \vee R && \text{by (2.2)}\\ &= (\sim P \wedge \sim(\sim Q)) \vee R && \text{by (2.10a)}\\ &= (\sim P \wedge Q) \vee R && \text{by (2.9),}\end{aligned}$$

a disjunctive normal form of $(P \vee \sim Q) \rightarrow R$ is $(\sim P \wedge Q) \vee R$.

Example 2.9

Obtain a conjunctive normal form for the formula $(P \wedge (Q \rightarrow R)) \rightarrow S$.

$$
\begin{aligned}
&(P \wedge (Q \to R)) \to S \\
&\quad = (P \wedge (\sim Q \vee R)) \to S && \text{by (2.2)} \\
&\quad = \sim(P \wedge (\sim Q \vee R)) \vee S && \text{by (2.2)} \\
&\quad = (\sim P \vee \sim(\sim Q \vee R)) \vee S && \text{by (2.10b)} \\
&\quad = (\sim P \vee (\sim(\sim Q) \wedge \sim R)) \vee S && \text{by (2.10a)} \\
&\quad = (\sim P \vee (Q \wedge \sim R)) \vee S && \text{by (2.9)} \\
&\quad = ((\sim P \vee Q) \wedge (\sim P \vee \sim R)) \vee S && \text{by (2.5a)} \\
&\quad = S \vee ((\sim P \vee Q) \wedge (\sim P \vee \sim R)) && \text{by (2.3a)} \\
&\quad = (S \vee (\sim P \vee Q)) \wedge (S \vee (\sim P \vee \sim R)) && \text{by (2.5a)} \\
&\quad = (S \vee \sim P \vee Q) \wedge (S \vee \sim P \vee \sim R) && \text{by (2.4a).}
\end{aligned}
$$

A conjunctive normal form of $(P \wedge (Q \to R)) \to S$ is $(S \vee \sim P \vee Q) \wedge (S \vee \sim P \vee \sim R)$.

2.5 LOGICAL CONSEQUENCES

In mathematics as well as in daily life, we often have to decide whether one statement follows from some other statements. This leads to the concept of "logical consequence." Before giving a formal definition of logical consequence, let us first consider the following example.

Example 2.10

Suppose the stock prices go down if the prime interest rate goes up. Suppose also that most people are unhappy when stock prices go down. Assume that the prime interest rate does go up. Show that you can conclude that most people are unhappy.

To show the above conclusion, let us denote the statements as follows:

$P \triangleq$ Prime interest rate goes up,

$S \triangleq$ Stock prices go down,

$U \triangleq$ Most people are unhappy.

There are four statements in this example. They are

(1) If the prime interest rate goes up, stock prices go down.
(2) If stock prices go down, most people are unhappy.

(3) The prime interest rate goes up.
(4) Most people are unhappy.

These statements are first symbolized as

(1′) $P \to S$

(2′) $S \to U$

(3′) P

(4′) U.

We now show that (4′) is true whenever (1′) $\wedge$ (2′) $\wedge$ (3′) is true.

Let us first transform $((P \to S) \wedge (S \to U) \wedge P)$ (representing (1′) $\wedge$ (2′) $\wedge$ (3′)) into a normal form:

$$
\begin{aligned}
((P \to S) \wedge (S \to U) \wedge P) &= ((\sim P \vee S) \wedge (\sim S \vee U) \wedge P) && \text{by (2.2)}\\
&= (P \wedge (\sim P \vee S) \wedge (\sim S \vee U)) && \text{by (2.3b)}\\
&= (((P \wedge \sim P) \vee (P \wedge S)) \wedge (\sim S \vee U)) && \text{by (2.5b)}\\
&= ((\square \vee (P \wedge S)) \wedge (\sim S \vee U)) && \text{by (2.8b)}\\
&= (P \wedge S) \wedge (\sim S \vee U) && \text{by (2.6a)}\\
&= (P \wedge S \wedge \sim S) \vee (P \wedge S \wedge U) && \text{by (2.5b)}\\
&= (P \wedge \square) \vee (P \wedge S \wedge U) && \text{by (2.8b)}\\
&= \square \vee (P \wedge S \wedge U) && \text{by (2.7b)}\\
&= P \wedge S \wedge U && \text{by (2.6a)}.
\end{aligned}
$$

Therefore, if $((P \to S) \wedge (S \to U) \wedge P)$ is true, then $(P \wedge S \wedge U)$ is true. Since $(P \wedge S \wedge U)$ is true only if P, S, and U are all true, we conclude that U is true. Because U is true whenever $(P \to S)$, $(S \to U)$, and P are true, in logic, U is called a *logical consequence* of $(P \to S)$, $(S \to U)$, and P. More formally, we define a logical consequence as follows.

Definition Given formulas $F_1, F_2, \ldots, F_n$ and a formula G, G is said to be a *logical consequence* of $F_1, \ldots, F_n$ (or G *logically follows from* $F_1, \ldots, F_n$) if and only if for any interpretation I in which $F_1 \wedge F_2 \wedge \cdots \wedge F_n$ is true, G is also true. $F_1, F_2, \ldots, F_n$ are called *axioms* (or *postulates, premises*) of G.

Theorem 2.1 Given formulas $F_1, \ldots, F_n$ and a formula G, G is a logical consequence of $F_1, \ldots, F_n$ if and only if the formula $((F_1 \wedge \cdots \wedge F_n) \to G)$ is valid.

Proof ($\Rightarrow$) Suppose G is a logical consequence of $F_1, \ldots, F_n$. Let I be an arbitary interpretation. If $F_1, \ldots, F_n$ are true in I, then, by the definition of

logical consequence, G is true in I. Hence, $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is true in I. On the other hand, if $F_1, \ldots, F_n$ are false in I, then $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is true in I. Thus, we have shown that $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is true under any interpretation. That is, $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is a valid formula.

($\Leftarrow$) Suppose $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is a valid formula. For any interpretation I, if $(F_1 \wedge \cdots \wedge F_n)$ is true in I, G must be true in I. Therefore, G is a logical consequence of $F_1, \ldots, F_n$. Q.E.D.

Theorem 2.2 Given formulas $F_1, \ldots, F_n$ and a formula G, G is a logical consequence of $F_1, \ldots, F_n$ if and only if the formula $(F_1 \wedge \cdots \wedge F_n \wedge \sim G)$ is inconsistent.

Proof By Theorem 2.1, G is a logical consequence of $F_1, \ldots, F_n$ if and only if the formula $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is valid. Hence, G is a logical consequence of $F_1, \ldots, F_n$ if and only if the negation of $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is inconsistent. Since

$$\begin{aligned} \sim((F_1 \wedge \cdots \wedge F_n) \rightarrow G) &= \sim(\sim(F_1 \wedge \cdots \wedge F_n) \vee G) \\ &= (\sim(\sim(F_1 \wedge \cdots \wedge F_n)) \wedge \sim G) \\ &= (F_1 \wedge \cdots \wedge F_n) \wedge \sim G \\ &= F_1 \wedge \cdots \wedge F_n \wedge \sim G, \end{aligned}$$

we conclude that Theorem 2.2 is true.

Theorems 2.1 and 2.2 are very important. They show that proving that a particular formula is a logical consequence of a finite set of formulas is equivalent to proving that a certain related formula is valid or inconsistent.

If G is a logical consequence of $F_1, \ldots, F_n$, the formula $((F_1 \wedge \cdots \wedge F_n) \rightarrow G)$ is called a *theorem*, and G is also called the *conclusion of the theorem*. In mathematics as well as in other fields, many problems can be formulated as problems of proving theorems. This will be elaborated in the next section. Before going to the next section, let us now consider one simple example to show how we can utilize Theorems 2.1 and 2.2.

Example 2.11

Consider the formulas

$$F_1 \triangleq (P \rightarrow Q), \qquad F_2 \triangleq \sim Q, \qquad G \triangleq \sim P.$$

Show that G is a logical consequence of F_1 and F_2.

Method 1 We can use the truth-table technique to show that G is true in every model of $(P \rightarrow Q) \wedge \sim Q$. From Table 2.7, one notes that there is only one model for $(P \rightarrow Q) \wedge \sim Q$, namely, $\{\sim P, \sim Q\}$. $\sim P$ is certainly true in

TABLE 2.7

TRUTH TABLE OF $(P \to Q) \wedge \sim Q$ AND $\sim P$

P	Q	$P \to Q$	$\sim Q$	$(P \to Q) \wedge \sim Q$	$\sim P$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	T	F	F	T
F	F	T	T	T	T

this model. Thus, by the definintion of logical consequence, we conclude that $\sim P$ is a logical consequence of $(P \to Q)$ and $\sim Q$.

Method 2 We can use Theorem 2.1. This can be done simply by extending the truth table in Table 2.7 or by evaluating the formula $((P \to Q) \wedge \sim Q) \to \sim P$.

Table 2.8 shows that $((P \to Q) \wedge \sim Q) \to \sim P$ is true in every interpretation. Therefore $((P \to Q) \wedge \sim Q) \to \sim P$ is valid and, according to Theorem 2.1, $\sim P$ is a logical consequence of $(P \to Q)$ and $\sim Q$.

TABLE 2.8

TRUTH TABLE OF $((P \to Q) \wedge \sim Q) \to \sim P$

P	Q	$P \to Q$	$\sim Q$	$(P \to Q) \wedge \sim Q$	$\sim P$	$((P \to Q) \wedge \sim Q) \to \sim P$
T	T	T	F	F	F	T
T	F	F	T	F	F	T
F	T	T	F	F	T	T
F	F	T	T	T	T	T

We can also prove the validity of $((P \to Q \wedge \sim Q) \to \sim P$ by transforming it into a conjunctive normal form.

$$
\begin{aligned}
((P \to Q) \wedge \sim Q) \to \sim P &= \sim((P \to Q) \wedge \sim Q) \vee \sim P && \text{by (2.2)} \\
&= \sim((\sim P \vee Q) \wedge \sim Q) \vee \sim P && \text{by (2.2)} \\
&= \sim((\sim P \wedge \sim Q) \vee (Q \wedge \sim Q)) \vee \sim P && \text{by (2.5b)} \\
&= \sim((\sim P \wedge \sim Q) \vee \square) \vee \sim P && \text{by (2.8b)} \\
&= \sim((\sim P \wedge \sim Q)) \vee \sim P && \text{by (2.6a)} \\
&= (P \vee Q) \vee \sim P && \text{by (2.10b)} \\
&= (Q \vee P) \vee \sim P && \text{by (2.3a)}
\end{aligned}
$$

$$= Q \vee (P \vee \sim P) \qquad \text{by (2.4a)}$$
$$= Q \vee \blacksquare \qquad \text{by (2.8a)}$$
$$= \blacksquare \qquad \text{by (2.7a).}$$

Thus, $((P \to Q) \wedge \sim Q) \to \sim P$ is valid.

Method 3 We can use Theorem 2.2. In this case, we prove that

$$((P \to Q) \wedge \sim Q) \wedge (\sim(\sim P)) = (P \to Q) \wedge \sim Q \wedge P$$

is inconsistent.

Again, as in Method 2, we can use the truth table technique to show that $(P \to Q) \wedge \sim Q \wedge P$ is false in every interpretation. From Table 2.9, we conclude that $(P \to Q) \wedge \sim Q \wedge P$ is inconsistent and, according to Theorem 2.2, $\sim P$ is a logical consequence of $(P \to Q)$ and $\sim Q$.

TABLE 2.9

TRUTH TABLE OF $(P \to Q) \wedge \sim Q \wedge P$

P	Q	$P \to Q$	$\sim Q$	$(P \to Q) \wedge \sim Q \wedge P$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

We can also prove the inconsistency of $(P \to Q) \wedge \sim Q \wedge P$ by transforming it into a disjunctive normal form:

$$(P \to Q) \wedge \sim Q \wedge P = (\sim P \vee Q) \wedge \sim Q \wedge P \qquad \text{by (2.2)}$$
$$= (\sim P \wedge \sim Q \wedge P) \vee (Q \wedge \sim Q \wedge P) \qquad \text{by (2.5b)}$$
$$= \square \vee \square \qquad \text{by (2.8b)}$$
$$= \square \qquad \text{by (2.6a).}$$

Thus, $(P \to Q) \wedge \sim Q \wedge P$ is inconsistent.

2.6 APPLICATIONS OF THE PROPOSITIONAL LOGIC

Having discussed the various concepts in the previous sections, we now consider applications of the propositional logic. They are best illustrated by examples.

Example 2.12

Given that if the congress refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns, will the strike not be over if the congress refuses to act and the strike just starts?

We first transform statements into symbols:

P: The congress refuses to act,
Q: The strike is over,
R: The president of the firm resigns,
S: The strike lasts more than one year.

Then the facts given in the example can be represented by formulas:

F_1: $(P \rightarrow (\sim Q \vee (R \wedge S))) \triangleq$ If the congress refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns,
F_2: $P \triangleq$ The congress refuses to act,
F_3: $\sim S \triangleq$ The strike just starts.

From the facts F_1, F_2, and F_3, can we conclude that the strike will not be over? That is, can we show that $\sim Q$ is a logical consequence of F_1, F_2, and F_3? By Theorem 2.1, this is equivalent to showing that

TABLE 2.10

TRUTH TABLE OF $(F_1 \wedge F_2 \wedge F_3) \rightarrow \sim Q$, WHERE $F_1 \triangleq (P \rightarrow (\sim Q \vee (R \wedge S)))$, $F_2 \triangleq P$, $F_3 \triangleq \sim S$.

P	Q	R	S	F_1	F_2	F_3	$\sim Q$	$(F_1 \wedge F_2 \wedge F_3) \rightarrow \sim Q$
T	T	T	T	T	T	F	F	T
T	T	T	F	F	T	T	F	T
T	T	F	T	F	T	F	F	T
T	T	F	F	F	T	T	F	T
T	F	T	T	T	T	F	T	T
T	F	T	F	T	T	T	T	T
T	F	F	T	T	T	F	T	T
T	F	F	F	T	T	T	T	T
F	T	T	T	T	F	F	F	T
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	F	F	T
F	T	F	F	T	F	T	F	T
F	F	T	T	T	F	F	T	T
F	F	T	F	T	F	T	T	T
F	F	F	T	T	F	F	T	T
F	F	F	F	T	F	T	T	T

$((P \to (\sim Q \vee (R \wedge S))) \wedge P \wedge \sim S) \to \sim Q$ is a valid formula. The truth values of the above formula under all the interpretations are shown in Table 2.10.

From Table 2.10, there is no interpretation under which the formula is false. Hence the formula $((P \to (\sim Q \vee (R \wedge S))) \wedge P \wedge \sim S) \to \sim Q$ is a valid formula. Therefore, $\sim Q$ is a logical consequence of F_1, F_2, and F_3. That is, we can conclude $\sim Q$ from F_1, F_2, and F_3. Hence, the answer is "The strike will not be over."

Example 2.13 (Chemical Synthesis Problem)

Suppose we can perform the following chemical reactions:

$$\mathrm{MgO} + \mathrm{H_2} \to \mathrm{Mg} + \mathrm{H_2O}$$

$$\mathrm{C} + \mathrm{O_2} \to \mathrm{CO_2}$$

$$\mathrm{CO_2} + \mathrm{H_2O} \to \mathrm{H_2CO_3}.$$

Suppose we have some quantities of MgO, H_2, O_2, and C. Show that we can make H_2CO_3.

For this problem, we may consider MgO, H_2, O_2, and C as atomic formulas. Then the above chemical reactions can be represented by the following formulas:

A_1: $(\mathrm{MgO} \wedge \mathrm{H_2}) \to (\mathrm{Mg} \wedge \mathrm{H_2O})$

A_2: $(\mathrm{C} \wedge \mathrm{O_2}) \to \mathrm{CO_2}$

A_3: $(\mathrm{CO_2} \wedge \mathrm{H_2O}) \to \mathrm{H_2CO_3}$.

Since we have MgO, H_2, O_2, and C, these facts can be represented by the following formulas:

A_4: MgO

A_5: H_2

A_6: O_2

A_7: C.

Now, the problem can be regarded as proving that H_2CO_3 is a logical consequence of $A_1, \ldots, A_7$, which, by Theorem 2.2, is true if $(A_1 \wedge \cdots \wedge A_7) \wedge \sim \mathrm{H_2CO_3}$ is inconsistent. We prove this by transforming the formula

$$(A_1 \wedge \cdots \wedge A_7 \wedge \sim \mathrm{H_2CO_3})$$

into a disjunctive normal form.

$$
\begin{aligned}
(A_1 \wedge \cdots \wedge A_7 \wedge \sim H_2CO_3) &= ((MgO \wedge H_2) \rightarrow (Mg \wedge H_2O)) \wedge ((C \wedge O_2) \\
&\quad \rightarrow CO_2) \wedge ((CO_2 \wedge H_2O) \rightarrow H_2CO_3) \\
&\quad \wedge MgO \wedge H_2 \wedge O_2 \wedge C \wedge \sim H_2CO_3 \\
&= (\sim MgO \vee \sim H_2 \vee Mg) \wedge (\sim MgO \vee \sim H_2 \\
&\quad \vee H_2O) \wedge (\sim C \vee \sim O_2 \vee CO_2) \\
&\quad \wedge (\sim CO_2 \vee \sim H_2O \vee H_2CO_3) \\
&\quad \wedge MgO \wedge H_2 \wedge O_2 \wedge C \wedge \sim H_2CO_3 \\
&= (\sim MgO \vee \sim H_2 \vee Mg) \wedge (\sim MgO \vee \sim H_2 \\
&\quad \vee H_2O) \wedge MgO \wedge H_2 \wedge (\sim C \vee \sim O_2 \\
&\quad \vee CO_2) \wedge C \wedge O_2 \\
&\quad \wedge (\sim CO_2 \vee \sim H_2O \vee H_2CO_3) \wedge \sim H_2CO_3 \\
&= Mg \wedge H_2O \wedge MgO \wedge H_2 \wedge CO_2 \wedge C \wedge O_2 \\
&\quad \wedge (\sim CO_2 \vee \sim H_2O) \wedge \sim H_2CO_3 \\
&= (\sim CO_2 \vee \sim H_2O) \wedge H_2O \wedge CO_2 \wedge Mg \\
&\quad \wedge MgO \wedge H_2 \wedge C \wedge O_2 \wedge \sim H_2CO_3 \\
&= \square \wedge Mg \wedge MgO \wedge H_2 \wedge C \wedge O_2 \\
&\quad \wedge \sim H_2CO_3 \\
&= \square.
\end{aligned}
$$

Since $\square$ is always false, the formula $(A_1 \wedge \cdots \wedge A_7 \wedge \sim H_2CO_3)$ is inconsistent. Therefore H_2CO_3 is a logical consequence of $A_1, \ldots, A_7$. That is, we can make H_2CO_3 from MgO, H_2, O_2, and C. The above procedure for proving an inconsistent formula by transforming it into $\square$ is sometimes called the *multiplication method*, because the transformation process is very similar to multiplying out an arithmetic expression. Example 2.13 is only a simple chemical synthesis problem. In reality, there are hundreds of chemical reactions. In order to use computers efficiently, we need an efficient computer program for proving theorems. This will be discussed in detail in Chapters 4–9.

In the above examples, we have shown that the propositional logic can be applied to many problems. The approach is first to symbolize problems by formulas and then to prove that the formulas are valid or inconsistent. We have used the truth-table method and the multiplication method to prove

whether a formula is valid (inconsistent). In Chapters 4–9, more efficient methods will be given for proving valid and inconsistent formulas.

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EXERCISES

Section 2.1

1. Symbolize the following statements by formulas.

   (a) A relation is an equivalence relation if and only if it is reflexive, symmetric, and transitive.
   (b) If the humidity is so high, it will rain either this afternoon or this evening.
   (c) Cancer will not be cured unless its cause is determined and a new drug for cancer is found.
   (d) It requires courage and skills to climb that mountain.
   (e) If he is a man who campaigns so hard, he probably will be elected.

2. Let

   $P \triangleq$ He needs a doctor, $Q \triangleq$ He needs a lawyer,

   $R \triangleq$ He has an accident, $S \triangleq$ He is sick,

   $U \triangleq$ He is injured.

   State the following formulas in English.

   (a) $(S \to P) \wedge (R \to Q)$ (b) $P \to (S \vee U)$

   (c) $(P \wedge Q) \to R$ (d) $(P \wedge Q) \leftrightarrow (S \wedge U)$

   (e) $\sim(S \vee U) \to \sim P$.

Section 2.2

3. Complete the following truth table (Table 2.11) of the formula

$$(\sim P \vee Q) \wedge (\sim(P \wedge \sim Q)).$$

TABLE 2.11

P	Q	$\sim P$	$\sim Q$	$\sim P \vee Q$	$P \wedge \sim Q$	$\sim(P \wedge \sim Q)$	$(\sim P \vee Q) \wedge (\sim(P \wedge \sim Q))$
T	T						
T	F						
F	T						
F	F						

Section 2.3

4. For each of the following formulas, determine whether it is valid, invalid, inconsistent, consistent, or some combination of these.

   (a) $\sim(\sim P) \rightarrow P$ (b) $P \rightarrow (P \wedge Q)$

   (c) $\sim(P \vee Q) \vee \sim Q$ (d) $(P \vee Q) \rightarrow P$

   (e) $(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$ (f) $(P \rightarrow Q) \rightarrow (Q \rightarrow P)$

   (g) $P \vee (P \rightarrow Q)$ (h) $(P \wedge (Q \rightarrow P)) \rightarrow P$

   (i) $P \vee (Q \rightarrow \sim P)$ (j) $(P \vee \sim Q) \wedge (\sim P \vee Q)$

   (k) $\sim P \wedge (\sim(P \rightarrow Q))$ (l) $P \rightarrow \sim P$

   (m) $\sim P \rightarrow P$.

5. Consider the following statement:

   If the congress refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns, and if either the congress enacts new laws or the strike is not over then the strike lasts more than one year.

   Is the above statement contradictory? Explain.

Section 2.4

6. Transform the following into disjunctive normal forms

   (a) $(\sim P \wedge Q) \rightarrow R$ (b) $P \rightarrow ((Q \wedge R) \rightarrow S)$

   (c) $\sim(P \vee \sim Q) \wedge (S \rightarrow T)$ (d) $(P \rightarrow Q) \rightarrow R$

   (e) $\sim(P \wedge Q) \wedge (P \vee Q)$.

7. Transform the following into conjunctive normal forms

   (a) $P \vee (\sim P \wedge Q \wedge R)$ (b) $\sim(P \rightarrow Q) \vee (P \vee Q)$

   (c) $\sim(P \rightarrow Q)$ (d) $(P \rightarrow Q) \rightarrow R$

   (e) $(\sim P \wedge Q) \vee (P \wedge \sim Q)$.

8. Is it possible to have a formula that is in conjunctive normal form as well as disjunctive normal form? If yes, give an example.
9. Verify each of the following pairs of equivalent formulas by transforming formulas on both sides of the sign $=$ into the same normal form.

   (a) $P \wedge P = P$, and $P \vee P = P$

   (b) $(P \to Q) \wedge (P \to R) = (P \to (Q \wedge R))$

   (c) $(P \to Q) \to (P \wedge Q) = (\sim P \to Q) \wedge (Q \to P)$

   (d) $P \wedge Q \wedge (\sim P \vee \sim Q) = \sim P \wedge \sim Q \wedge (P \vee Q)$

   (e) $P \vee (P \to (P \wedge Q)) = \sim P \vee \sim Q \vee (P \wedge Q)$.

Sections 2.5 and 2.6

10. Prove that $(\sim Q \to \sim P)$ is a logical consequence of $(P \to Q)$.
11. If the congress refuses to enact new laws, then the strike will not be over unless it lasts more than one year and the president of the firm resigns. Suppose the congress refuses to act, the strike is over, and the president of the firm does not resign. Has the strike lasted more than one year?
12. Consider the following statements:

    $F_1 \triangleq$ Tom cannot be a good student unless he is smart and his father supports him.

    $F_2 \triangleq$ Tom is a good student only if his father supports him.

    Show that F_2 is a logical consequence of F_1.
13. Show that for the following statements, F_2 is a logical consequence of F_1.

    $F_1 \triangleq$ If the president does not have the appropriate authority or if he does not want to take the responsibility, then neither order will be restored nor will the riots stop spreading unless the rioters become tired of rioting and the local authorities begin to take conciliatory actions.

    $F_2 \triangleq$ If the president does not want to take the responsibility and the rioters are not tired of rioting, then riots will spread.
14. Show that Q is a logical consequence of $(P \to Q)$ and P. This is related to the so-called *modus ponens* rule.

Chapter 3

The First-Order Logic

3.1 INTRODUCTION

In the propositional logic, the most basic elements are atoms. Through atoms we build up formulas. We then use formulas to express various complex ideas. As discussed in Chapter 2, in this simple logic an atom represents a declarative sentence that can be either true or false, but not both. An atom is treated as a single unit. Its structure and composition are suppressed. However, there are many ideas that cannot be treated in this simple way. For example, consider the following deduction of statements:

Every man is mortal.
Since Confucius is a man, he is mortal.

The above reasoning is intuitively correct. However, if we denote

P: Every man is mortal,
Q: Confucius is a man,
R: Confucius is mortal,

then R is not a logical consequence of P and Q within the framework of the propositional logic. This is because the structures of P, Q, and R are not used in the propositional logic. In this chapter, we shall introduce the first-order logic, which has three more logical notions (called *terms*, *predicates*,

and *quantifiers*) than does the propositional logic. It will be made clear later that much of everyday and mathematical language can be symbolized by the first-order logic.

Just as in the propositional logic, we first have to define atoms in the first-order logic. Before giving the formal definition of an atom, we first consider some examples.

Suppose we want to represent "x is greater than 3." We first define a predicate $GREATER(x,y)$ to mean "x is greater than y." (Note that a predicate is a relation.) Then the sentence "x is greater than 3" is represented by $GREATER(x,3)$.

Similarly, we can represent "x loves y" by the predicate $LOVE(x,y)$. Then "John loves Mary" can be represented by $LOVE(\text{John}, \text{Mary})$.

We can also use function symbols in the first-order logic. For example, we can use $plus(x,y)$ to denote "$x + y$" and $father(x)$ to mean the father of x. The sentences "$x + 1$ is greater than x" and "John's father loves John" can be symbolized as $GREATER(plus(x,1),x)$ and $LOVE(father(\text{John}), \text{John})$, respectively.

In the above examples, $GREATER(x,3)$, $LOVE(\text{John}, \text{Mary})$, $GREATER(plus(x,1),x)$, and $LOVE(father(\text{John}), \text{John})$ are all atoms in the first-order logic, where *GREATER* and *LOVE* are *predicate symbols*, x is a *variable*, 3, John, and Mary are *individual symbols or constants*, and *father* and *plus* are *function symbols*.

In general, we are allowed to use the following four types of symbols to construct an atom:

i. Individual symbols or constants: These are usually names of objects, such as John, Mary, and 3.

ii. Variable symbols: These are customarily *lowercase* unsubscripted or subscripted letters, $x, y, z, \ldots$.

iii. Function symbols: These are customarily *lowercase* letters $f, g, h, \ldots$ or expressive strings of *lowercase* letters such as *father* and *plus*.

iv. Predicate symbols: These are customarily *uppercase* letters $P, Q, R, \ldots$ or expressive strings of *uppercase* letters such as *GREATER* and *LOVE*.

Any function or predicate symbol takes a specified number of arguments. If a function symbol f takes n arguments, f is called an *n-place function symbol*. It is noted that an individual symbol or a constant may be considered a function symbol that takes no argument. Similarly, if a predicate symbol P takes n arguments, P is called an *n-place predicate symbol*. For example, *father* is a one-place function symbol, and *GREATER* and *LOVE* are two-place predicate symbols.

A function is a mapping that maps a list of constants to a constant. For example, *father* is a function that maps a person named John to a person

who is John's father. Therefore, *father*(John) represents a person, even though his name is unknown. We call *father*(John) a *term* in the first-order logic. More formally, we have the following definition.

Definition *Terms* are defined recursively as follows:

i. A constant is a term.
ii. A variable is a term.
iii. If f is an n-place function symbol, and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.
iv. All terms are generated by applying the above rules.

Example 3.1

Since x and 1 are both terms and *plus* is a two-place function symbol, $plus(x, 1)$ is a term according to the above definition.

Furthermore, the reader can see that $plus(plus(x,1),x)$ and *father*(*father* (John)) are also terms; the former denotes $(x+1)+x$ and the latter denotes the grandfather of John.

A predicate is a mapping that maps a list of constants to T or F. For example, $GREATER$ is a predicate. $GREATER(5,3)$ is T, but $GREATER(1,3)$ is F. Having defined terms, we can now formally define an atom in the first-order logic.

Definition If P is an n-place predicate symbol, and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is an *atom*. No other expressions can be atoms.

Once atoms are defined, we can use the same five logical connectives given in Chapter 2 to build up formulas. Furthermore, since we have introduced variables, we use two special symbols $\forall$ and $\exists$ to characterize variables. The symbols $\forall$ and $\exists$ are called, respectively, the *universal* and *existential quantifiers*. If x is a variable, then $(\forall x)$ is read as "*for all x*," "*for each x*," or "*for every x*," while $(\exists x)$ is read as "*there exists an x*," "*for some x*," or "*for at least one x*." Let us consider a few examples to see how the quantifiers can be used.

Example 3.2

Symbolize the following statements:

(a) Every rational number is a real number.
(b) There exists a number that is a prime.
(c) For every number x, there exists a number y such that $x < y$.

Denote "x is a prime number" by $P(x)$, "x is a rational number" by $Q(x)$, "x is a real number" by $R(x)$, and "x is less than y" by $LESS(x,y)$. Then

the above statements can be denoted, respectively, as

(a′) $(\forall x)(Q(x) \rightarrow R(x))$

(b′) $(\exists x)P(x)$

(c′) $(\forall x)(\exists y)LESS(x, y)$.

Each of the expressions (a′), (b′), and (c′) is called a formula. Before we give a formal definition of a formula, we have to distinguish between *bound* variables and *free* variables. To do this, we first define the *scope* of a quantifier occurring in a formula as the formula to which the quantifier applies. For example, the scope of both the universal and existential quantifiers in the formula $(\forall x)(\exists y)LESS(x, y)$ is $LESS(x, y)$. The scope of the universal quantifier in the formula $(\forall x)(Q(x) \rightarrow R(x))$ is $(Q(x) \rightarrow R(x))$.

Definition An occurrence of a variable in a formula is *bound* if and only if the occurrence is within the scope of a quantifier employing the variable, or is the occurrence in that quantifier. An occurrence of a variable in a formula is *free* if and only if this occurrence of the variable is not bound.

Definition A variable is *free* in a formula if at least one occurrence of it is free in the formula. A variable is *bound* in a formula if at least one occurrence of it is bound.

In the formula $(\forall x)P(x, y)$, since both the occurrences of x are bound, the variable x is bound. However, the variable y is free since the only occurrence of y is free. We note that a variable can be both free and bound in a formula. For example, y is both free and bound in the formula $(\forall x)P(x, y) \wedge (\forall y)Q(y)$.

We now can formally define a formula by using atoms, logical connectives, and quantifiers.

Definition *Well-formed formulas*, or *formulas* for short, in the first-order logic are defined recursively as follows:

i. An atom is a formula. (Note that "atom" is an abbreviation for an atomic formula.)

ii. If F and G are formulas, then $\sim(F)$, $(F \vee G)$, $(F \wedge G)$, $(F \rightarrow G)$, and $(F \leftrightarrow G)$ are formulas.

iii. If F is a formula and x is a free variable in F, then $(\forall x)F$ and $(\exists x)F$ are formulas.

iv. Formulas are generated only by a finite number of applications of (i), (ii), and (iii).

Throughout this book, parentheses may be omitted by the same conventions established in Chapter 2. We extend these conventions by agreeing that quantifiers have the least ranks. For example, $(\exists x)A \vee B$ stands for $(((\exists x)A) \vee (B))$.

Example 3.3

Translate the statement "Every man is mortal. Confucius is a man. Therefore, Confucius is mortal." into a formula.

Denote "x is a man" by $MAN(x)$, and "x is mortal" by $MORTAL(x)$. Then "every man is mortal" can be represented by

$$(\forall x)(MAN(x) \rightarrow MORTAL(x)),$$

"Confucius is a man" by

$$MAN(\text{Confucius}),$$

and "Confucius is mortal" by

$$MORTAL(\text{Confucius}).$$

The whole statement can now be represented by

$$(\forall x)(MAN(x) \rightarrow MORTAL(x)) \wedge MAN(\text{Confucius}) \rightarrow MORTAL(\text{Confucius}).$$

Example 3.4

The basic axioms of natural numbers are the following:

A_1: For every number, there is one and only one immediate successor.
A_2: There is no number for which 0 is the immediate successor.
A_3: For every number other than 0, there is one and only one immediate predecessor.

Let $f(x)$ and $g(x)$ represent the immediate successor and predecessor of x, respectively. Let "x is equal to y" be denoted by $E(x,y)$. Then the axioms can be represented by the following formulas:

A_1': $(\forall x)(\exists y)(E(y,f(x)) \wedge (\forall z)(E(z,f(x)) \rightarrow E(y,z)))$

A_2': $\sim((\exists x)E(0,f(x)))$

A_3': $(\forall x)(\sim E(x,0) \rightarrow ((\exists y)(E(y,g(x)) \wedge (\forall z)(E(z,g(x)) \rightarrow E(y,z)))))$.

3.2 INTERPRETATIONS OF FORMULAS IN THE FIRST-ORDER LOGIC

In the propositional logic, an interpretation is an assignment of truth values to atoms. In the first-order logic, since there are variables involved, we have to do more than that. To define an interpretation for a formula in the

first-order logic, we have to specify two things, namely, the domain and an assignment to constants, function symbols, and predicate symbols occurring in the formula. The following is the formal definition of an interpretation of a formula in the first-order logic.

Definition An *interpretation* of a formula F in the first-order logic consists of a nonempty domain D, and an assignment of "values" to each constant, function symbol, and predicate symbol occurring in F as follows:

1. To each constant, we assign an element in D.
2. To each n-place function symbol, we assign a mapping from D^n to D. (Note that $D^n = \{(x_1, \ldots, x_n) \mid x_1 \in D, \ldots, x_n \in D\}$).
3. To each n-place predicate symbol, we assign a mapping from D^n to $\{\boldsymbol{T}, \boldsymbol{F}\}$.

Sometimes, to emphasize the domain D, we speak of an interpretation of the formula *over* D. When we evaluate the truth value of a formula in an interpretation over the domain D, $(\forall x)$ will be interpreted as "for all elements x in D," and $(\exists x)$ as "there is an element in D."

For every interpretation of a formula over a domain D, the formula can be evaluated to $\boldsymbol{T}$ or $\boldsymbol{F}$ according to the following rules:

1. If the truth values of formulas G and H are evaluated, then the truth values of the formulas $\sim G$, $(G \wedge H)$, $(G \vee H)$, $(G \rightarrow H)$, and $(G \leftrightarrow H)$ are evaluated by using Table 2.1 in Chapter 2.
2. $(\forall x)G$ is evaluated to $\boldsymbol{T}$ if the truth value of G is evaluated to $\boldsymbol{T}$ for every d in D; otherwise, it is evaluated to $\boldsymbol{F}$.
3. $(\exists x)G$ is evaluated to $\boldsymbol{T}$ if the truth value of G is $\boldsymbol{T}$ for at least one d in D; otherwise, it is evaluated to $\boldsymbol{F}$.

We note that any formula containing free variables cannot be evaluated. In the rest of the book, we shall assume either that formulas do not contain free variables, or that free variables are treated as constants.

Example 3.5

Let us consider the formulas

$$(\forall x)\, P(x) \qquad \text{and} \qquad (\exists x) \sim P(x).$$

Let an interpretation be as follows:

Domain: $D = \{1, 2\}$.

Assignment for P:

$P(1)$	$P(2)$
$\boldsymbol{T}$	$\boldsymbol{F}$

It should be easy for the reader to confirm that $(\forall x)P(x)$ is F in this interpretation because $P(x)$ is not T for both $x = 1$ and $x = 2$. On the other hand, since $\sim P(2)$ is T in this interpretation, $(\exists x) \sim P(x)$ is T in this interpretation.

Example 3.6

Consider the formula

$$(\forall x)(\exists y)P(x,y).$$

Let us define an interpretation as follows:

$D = \{1,2\}$

$P(1,1)$	$P(1,2)$	$P(2,1)$	$P(2,2)$
T	F	F	T

If $x = 1$, we can see that there is a y, namely 1, such that $P(1,y)$ is T.
If $x = 2$, there is also a y, namely 2, such that $P(2,y)$ is T.
Therefore, in the above interpretation, for every x in D, there is a y such that $P(x,y)$ is T; that is $(\forall x)(\exists y)P(x,y)$ is T in this interpretation.

Example 3.7

Consider the formula

G: $(\forall x)(P(x) \rightarrow Q(f(x),a))$.

There are one constant a, one one-place function symbol f, one one-place predicate symbol P, and one two-place predicate symbol Q in G. The following is an interpretation I of G.

Domain: $D = \{1,2\}$.

Assignment for a:

a
1

Assignment for f:

$f(1)$	$f(2)$
2	1

Assignment for P and Q:

$P(1)$	$P(2)$	$Q(1,1)$	$Q(1,2)$	$Q(2,1)$	$Q(2,2)$
F	T	T	T	F	T

If $x = 1$, then

$$\begin{aligned} P(x) \to Q(f(x), a) &= P(1) \to Q(f(1), a) \\ &= P(1) \to Q(2, 1) \\ &= \boldsymbol{F} \to \boldsymbol{F} = \boldsymbol{T}. \end{aligned}$$

If $x = 2$, then

$$\begin{aligned} P(x) \to Q(f(x), a) &= P(2) \to Q(f(2), a) \\ &= P(2) \to Q(1, 1) \\ &= \boldsymbol{T} \to \boldsymbol{T} = \boldsymbol{T}. \end{aligned}$$

Since $P(x) \to Q(f(x), a)$ is true for all elements x in the domain D, the formula $(\forall x)(P(x) \to Q(f(x), a))$ is true under the interpretation I.

Example 3.8

Evaluate the truth values of the following formulas under the interpretation given in Example 3.7.

(a) $(\exists x)(P(f(x)) \wedge Q(x, f(a)))$,

(b) $(\exists x)(P(x) \wedge Q(x, a))$,

(c) $(\forall x)(\exists y)(P(x) \wedge Q(x, y))$.

For (a): If $x = 1$,

$$\begin{aligned} P(f(x)) \wedge Q(x, f(a)) &= P(f(1)) \wedge Q(1, f(a)) \\ &= P(2) \wedge Q(1, f(1)) \\ &= P(2) \wedge Q(1, 2) \\ &= \boldsymbol{T} \wedge \boldsymbol{T} = \boldsymbol{T}. \end{aligned}$$

If $x = 2$,

$$\begin{aligned} P(f(x)) \wedge Q(x, f(a)) &= P(f(2)) \wedge Q(2, f(1)) \\ &= P(1) \wedge Q(2, 1) \\ &= \boldsymbol{F} \wedge \boldsymbol{F} = \boldsymbol{F}. \end{aligned}$$

Since there is an element in the domain D, that is, $x = 1$, such that $P(f(x)) \wedge Q(x, f(a))$ is true, the truth value of the formula $(\exists x)(P(f(x)) \wedge Q(x, f(a)))$ is true under the interpretation I.

For (b): If $x = 1$,

$$P(x) \wedge Q(x,a) = P(1) \wedge Q(1,1)$$
$$= \boldsymbol{F} \wedge \boldsymbol{T} = \boldsymbol{F}.$$

If $x = 2$,

$$P(x) \wedge Q(x,a) = P(2) \wedge Q(2,1)$$
$$= \boldsymbol{T} \wedge \boldsymbol{F} = \boldsymbol{F}.$$

Since there is no element in the domain D such that $P(x) \wedge Q(x,a)$ is true, the formula $(\exists x)(P(x) \wedge Q(x,a))$ is evaluated to be false under the interpretation I.

For (c): If $x = 1$, then $P(x) = P(1) = \boldsymbol{F}$. Therefore $P(x) \wedge Q(x,y) = \boldsymbol{F}$ for $y = 1$ and $y = 2$. Since there exists an x, that is, $x = 1$, such that $(\exists y)(P(x) \wedge Q(x,y))$ is false, the formula $(\forall x)(\exists y)(P(x) \wedge Q(x,y))$ is false under the interpretation I, that is, the formula is falsified by I.

Once interpretations are defined, all the concepts, such as validity, inconsistency, and logical consequence, defined in Chapter 2 can be defined analogously for formulas of the first-order logic.

Definition A formula G is *consistent* (*satisfiable*) if and only if there exists an interpretation I such that G is evaluated to $\boldsymbol{T}$ in I. If a formula G is $\boldsymbol{T}$ in an interpretation I, we say that I is a *model* of G and I *satisfies* G.

Definition A formula G is *inconsistent* (*unsatisfiable*) if and only if there exists no interpretation that satisfies G.

Definition A formula G is *valid* if and only if every interpretation of G satisfies G.

Definition A formula G is a *logical consequence* of formulas $F_1, F_2, \ldots, F_n$ if and only if for every interpretation I, if $F_1 \wedge \cdots \wedge F_n$ is true in I, G is also true in I.

The relations between validity (inconsistency) and logical consequence as stated in Theorems 2.1 and 2.2 are also true for the first-order logic. In fact, the first-order logic can be considered as an extension of the propositional logic. When a formula in the first-order logic contains no variables and quantifiers, it can be treated just as a formula in the propositional logic.

Example 3.9

It is left as an exercise for the reader to prove the following:

(1) $(\forall x)P(x) \wedge (\exists y) \sim P(y)$ is inconsistent.

(2) $(\forall x)P(x) \rightarrow (\exists y)P(y)$ is valid.

(3) $P(a) \rightarrow \sim ((\exists x)P(x))$ is consistent.

(4) $(\forall x)P(x) \vee ((\exists y) \sim P(y))$ is valid.

Example 3.10

Consider formulas

F_1: $(\forall x)(P(x) \rightarrow Q(x))$

F_2: $P(a)$.

We will now prove that formula $Q(a)$ is a logical consequence of F_1 and F_2.

Consider any interpretation I that satisfies $(\forall x)(P(x) \rightarrow Q(x)) \wedge P(a)$. Certainly in this interpretation, $P(a)$ is $\boldsymbol{T}$. Assume $Q(a)$ is not $\boldsymbol{T}$ in this interpretation; then $\sim P(a) \vee Q(a)$, that is, $P(a) \rightarrow Q(a)$ is $\boldsymbol{F}$ in I. This means that $(\forall x)(P(x) \rightarrow Q(x))$ is $\boldsymbol{F}$ in I, which is impossible. Therefore, $Q(a)$ must be $\boldsymbol{T}$ in every interpretation that satisfies

$$(\forall x)(P(x) \rightarrow Q(x)) \wedge P(a).$$

This means that $Q(a)$ is a logical consequence of F_1 and F_2.

In the first-order logic, since there are an infinite number of domains, in general, there are an infinite number of interpretations of a formula. Therefore, unlike in the propositional logic, it is not possible to verify a valid or an inconsistent formula by evaluating the formula under all the possible interpretations. In the subsequent chapters, we shall give procedures for verifying inconsistent formulas in the first-order logic.

3.3 PRENEX NORMAL FORMS IN THE FIRST-ORDER LOGIC

In the propositional logic, we have introduced two normal forms—the conjunctive normal form and the disjunctive normal form. In the first-order logic, there is also a normal form called the "prenex normal form." The reason for considering a prenex normal form of a formula is to simplify proof procedures, which will be discussed in the sequel.

Definition A formula F in the first-order logic is said to be in a *prenex normal form* if and only if the formula F is in the form of

$$(Q_1 x_1) \cdots (Q_n x_n)(M)$$

where every $(Q_i x_i)$, $i = 1, \ldots, n$, is either $(\forall x_i)$ or $(\exists x_i)$, and M is a formula containing no quantifiers. $(Q_1 x_1) \cdots (Q_n x_n)$ is called the *prefix* and M is called the *matrix* of the formula F.

Here are some formulas in prenex normal form:

$$(\forall x)(\forall y)(P(x,y) \wedge Q(y)), \qquad (\forall x)(\forall y)(\sim P(x,y) \rightarrow Q(y)),$$
$$(\forall x)(\forall y)(\exists z)(Q(x,y) \rightarrow R(z)).$$

Given a formula, we now consider a method of transforming it into a prenex normal form. This is to be accomplished by first considering some basic pairs of equivalent formulas in the first-order logic. We remember that two formulas F and G are *equivalent*, denoted by $F = G$, if and only if the truth values of F and G are the same under every interpretation. The basic pairs of equivalent formulas given in Table 2.6 in Chapter 2 are still true for the first-order logic. In addition, there are other pairs of equivalent formulas containing quantifiers. We now consider these additional pairs of equivalent formulas. Let F be a formula containing a free variable x. To emphasize that the free variable x is in F, we represent F by $F[x]$. Let G be a formula that does not contain variable x. Then we have the following pairs of equivalent formulas, where Q is either $\forall$ or $\exists$. For simplicity, we shall call each of the following pairs of equivalent formulas a "law."

(3.1a) $(Qx)F[x] \vee G = (Qx)(F[x] \vee G)$.

(3.1b) $(Qx)F[x] \wedge G = (Qx)(F[x] \wedge G)$.

(3.2a) $\sim((\forall x)F[x]) = (\exists x)(\sim F[x])$.

(3.2b) $\sim((\exists x)F[x]) = (\forall x)(\sim F[x])$.

Laws (3.1a) and (3.1b) are obviously true, since G does not contain x and therefore can be brought into the scope of the quantifier Q. Laws (3.2a) and (3.2b) are not difficult to prove. Let I be any arbitrary interpretation over a domain D. If $\sim((\forall x)F[x])$ is true in I, then $(\forall x)F[x]$ is false in I. This means there is an element e in D such that $F[e]$ is false, that is, $\sim F[e]$ is true in I. Therefore, $(\exists x)(\sim F[x])$ is true in I. On the other hand, if $\sim((\forall x)F[x])$ is false in I, then $(\forall x)F[x]$ is true in I. This means that $F[x]$ is true for every element x in D, that is, $\sim F[x]$ is false for every element x in D. Therefore, $(\exists x)(\sim F[x])$ is false in I. Since $\sim((\forall x)F[x])$ and $(\exists x)(\sim F[x])$ always assume the same truth value for every arbitrary interpretation, by definition, $\sim((\forall x)F[x]) = (\exists x)(\sim F[x])$. Hence, law (3.2a) is proved. Similarly, we can prove law (3.2b).

Suppose $F[x]$ and $H[x]$ are two formulas containing x. There are two other laws:

(3.3a) $(\forall x)F[x] \wedge (\forall x)H[x] = (\forall x)(F[x] \wedge H[x])$.

(3.3b) $(\exists x)F[x] \vee (\exists x)H[x] = (\exists x)(F[x] \vee H[x])$.

That is, the universal quantifier $\forall$ and the existential quantifier $\exists$ can distribute over $\wedge$ and $\vee$, respectively.

The proofs of (3.3a) and (3.3b) are not difficult. We leave the proofs to the reader. However, the universal quantifier $\forall$ and the existential quantifier $\exists$ *cannot* distribute over $\vee$ and $\wedge$, respectively. That is,

$$(\forall x)F[x] \vee (\forall x)H[x] \neq (\forall x)(F[x] \vee H[x])$$

and

$$(\exists x)F[x] \wedge (\exists x)H[x] \neq (\exists x)(F[x] \wedge H[x]).$$

For cases like these, we have to do something special. Since every bound variable in a formula can be considered as a dummy variable, every bound variable x can be renamed z, and the formula $(\forall x)H[x]$ becomes $(\forall z)H[z]$; that is, $(\forall x)H[x] = (\forall z)H[z]$. Suppose we choose the variable z that does not appear in $F[x]$. Then,

$$\begin{aligned}(\forall x)F[x] \vee (\forall x)H[x] &= (\forall x)F[x] \vee (\forall z)H[z] \\ &\quad \text{(by renaming all } x\text{'s occurring in } (\forall x)\,H[x] \text{ as } z) \\ &= (\forall x)(\forall z)(F[x] \vee H[z]) \\ &\quad \text{(by 3.1a).}\end{aligned}$$

Similarly, we can have

$$\begin{aligned}(\exists x)F[x] \wedge (\exists x)H[x] &= (\exists x)F[x] \wedge (\exists z)H[z] \\ &\quad \text{(by renaming all } x\text{'s occurring in } (\exists x)\,H[x] \text{ as } z) \\ &= (\exists x)(\exists z)(F[x] \wedge H[z]) \\ &\quad \text{(by 3.1b).}\end{aligned}$$

Therefore, for these two cases, we still can bring all the quantifiers to the left of the formula. In general, we have

(3.4a) $(Q_1 x)F[x] \vee (Q_2 x)H[x] = (Q_1 x)(Q_2 z)(F[x] \vee H[z])$

(3.4b) $(Q_3 x)F[x] \wedge (Q_4 x)H[x] = (Q_3 x)(Q_4 z)(F[x] \wedge H[z])$

where Q_1, Q_2, Q_3, and Q_4 are either $\forall$ or $\exists$, and z does not appear in $F[x]$. Of course, if $Q_1 = Q_2 = \exists$ and $Q_3 = Q_4 = \forall$, then we do not have to rename x's in $(Q_2 x)H[x]$ or $(Q_4 x)H[x]$. We can use (3.3) directly.

Using laws (2.1)–(2.10) and laws (3.1)–(3.4), we can always transform a given formula into prenex normal form. The following is an outline of the transforming procedure.

Transforming Formulas into Prenex Normal Form

Step 1 Use the laws

$$F \leftrightarrow G = (F \rightarrow G) \wedge (G \rightarrow F) \tag{2.1}$$

$$F \rightarrow G = \sim F \vee G \tag{2.2}$$

to eliminate the logical connectives $\leftrightarrow$ and $\rightarrow$.

Step 2 Repeatedly use the law

$$\sim(\sim F) = F \tag{2.9}$$

De Morgan's laws

$$\sim(F \vee G) = \sim F \wedge \sim G \tag{2.10a}$$

$$\sim(F \wedge G) = \sim F \vee \sim G \tag{2.10b}$$

and the laws

$$\sim((\forall x) F[x]) = (\exists x)(\sim F[x]) \tag{3.2a}$$

$$\sim((\exists x) F[x]) = (\forall x)(\sim F[x]) \tag{3.2b}$$

to bring the negation signs immediately before atoms.

Step 3 Rename bound variables if necessary.

Step 4 Use the laws

$$(Qx)F[x] \vee G = (Qx)(F[x] \vee G) \tag{3.1a}$$

$$(Qx)F[x] \wedge G = (Qx)(F[x] \wedge G) \tag{3.1b}$$

$$(\forall x)F[x] \wedge (\forall x)H[x] = (\forall x)(F[x] \wedge H[x]) \tag{3.3a}$$

$$(\exists x)F[x] \vee (\exists x)H[x] = (\exists x)(F[x] \vee H[x]) \tag{3.3b}$$

$$(Q_1 x)F[x] \vee (Q_2 x)H[x] = (Q_1 x)(Q_2 z)(F[x] \vee H[z]) \tag{3.4a}$$

$$(Q_3 x)F[x] \wedge (Q_4 x)H[x] = (Q_3 x)(Q_4 z)(F[x] \wedge H[z]) \tag{3.4b}$$

to move the quantifiers to the left of the entire formula to obtain a prenex normal form.

Example 3.11

Transform the formula $(\forall x)P(x) \rightarrow (\exists x)Q(x)$ into prenex normal form.

$$\begin{aligned}(\forall x)P(x) \rightarrow (\exists x)Q(x) &= \sim((\forall x)P(x)) \vee (\exists x)Q(x) && \text{by (2.2)}\\ &= (\exists x)(\sim P(x)) \vee (\exists x)Q(x) && \text{by (3.2a)}\\ &= (\exists x)(\sim P(x) \vee Q(x)) && \text{by (3.3b).}\end{aligned}$$

Therefore, a prenex normal form of $(\forall x)P(x) \rightarrow (\exists x)Q(x)$ is $(\exists x)(\sim P(x) \vee Q(x))$.

Example 3.12

Obtain a prenex normal form for the formula

$$(\forall x)(\forall y)((\exists z)(P(x,z) \wedge P(y,z)) \rightarrow (\exists u)Q(x,y,u)).$$

$$
\begin{aligned}
&(\forall x)(\forall y)((\exists z)(P(x,z) \wedge P(y,z)) \rightarrow (\exists u)Q(x,y,u)) \\
&\quad = (\forall x)(\forall y)\big(\sim((\exists z)(P(x,z) \wedge P(y,z))) && \text{by (2.2)} \\
&\qquad \vee (\exists u)Q(x,y,u)\big) \\
&\quad = (\forall x)(\forall y)((\forall z)(\sim P(x,z) \vee \sim P(y,z)) && \text{by (3.2b) and (2.10b)} \\
&\qquad \vee (\exists u)Q(x,y,u)) \\
&\quad = (\forall x)(\forall y)(\forall z)(\exists u)(\sim P(x,z) && \text{using (3.1a), move the} \\
&\qquad \vee \sim P(y,z) \vee Q(x,y,u)) && \text{quantifiers to the left.}
\end{aligned}
$$

Therefore, we obtain the last formula as a prenex normal form of the first formula.

3.4 APPLICATIONS OF THE FIRST-ORDER LOGIC

In this section we shall give a few examples to illustrate some applications of the first-order logic to problem solving. As in the propositional logic, the usual approach is first to symbolize problems by formulas and then to prove that the formulas are valid or inconsistent.

Example 3.13

Consider Example 3.3. There are two axioms:

A_1: $(\forall x)(MAN(x) \rightarrow MORTAL(x))$.
A_2: $MAN(\text{Confucius})$.

From A_1 and A_2, show that Confucius is mortal. That is, show that $MORTAL(\text{Confucius})$ is a logical consequence of A_1 and A_2.

We have

$A_1 \wedge A_2$: $(\forall x)(MAN(x) \rightarrow MORTAL(x)) \wedge MAN(\text{Confucius})$.

If $A_1 \wedge A_2$ is true in an interpretation I, then both A_1 and A_2 are true in I. Since $(MAN(x) \rightarrow MORTAL(x))$ is true for all x, when x is replaced by "Confucius," $(MAN(\text{Confucius}) \rightarrow MORTAL(\text{Confucius}))$ is true in I. That is, $\sim MAN(\text{Confucius}) \vee MORTAL(\text{Confucius})$ is true in I. However, $\sim MAN(\text{Confucius})$ is false in I since $MAN(\text{Confucius})$ is true in I. Hence, $MORTAL(\text{Confucius})$ must be true in I. We have therefore shown that $MORTAL(\text{Confucius})$ is true in I whenever $(A_1 \wedge A_2)$ is true in I. By definition, $MORTAL(\text{Confucius})$ is a logical consequence of A_1 and A_2.

Example 3.14

No used-car dealer buys a used car for his family. Some people who buy used cars for their families are absolutely dishonest. Conclude that some absolutely dishonest people are not used-car dealers.

We let $U(x)$, $B(x)$, and $D(x)$ denote "x is a used-car dealer," "x buys

a used car for his family," and "x is absolutely dishonest," respectively. Then we have

A_1: $(\forall x)(U(x) \rightarrow \sim B(x))$

A_2: $(\exists x)(B(x) \wedge D(x))$.

We have to show that

A_3: $(\exists x)(D(x) \wedge \sim U(x))$

is a logical consequence of A_1 and A_2.

Assume that A_1 and A_2 are true in an interpretation I over a domain D. Since A_2 is true in I, there is an x in D, say a, such that $B(a) \wedge D(a)$ is true in I. Therefore $B(a)$ is true in I, that is, $\sim B(a)$ is false in I. A_1 can be written in the following form:

A_1: $(\forall x)(\sim U(x) \vee \sim B(x))$.

Since A_1 is true in I and $\sim B(a)$ is false in I, $\sim U(a)$ must be true in I. However, since $B(a) \wedge D(a)$ is true in I, $D(a)$ is true in I. Therefore, $D(a) \wedge \sim U(a)$ is true in I. Thus, A_3, that is, $(\exists x)(D(x) \wedge \sim U(x))$, is true in I. Hence, A_3 is a logical consequence of A_1 and A_2.

Example 3.15

Some patients like all doctors. No patient likes any quack. Therefore, no doctor is a quack. Denote

$P(x)$: x is a patient,

$D(x)$: x is a doctor,

$Q(x)$: x is a quack,

$L(x, y)$: x likes y.

Then the facts and the conclusion may be symbolized as follows:

F_1: $(\exists x)(P(x) \wedge (\forall y)(D(y) \rightarrow L(x, y)))$

F_2: $(\forall x)(P(x) \rightarrow (\forall y)(Q(y) \rightarrow \sim L(x, y)))$

G: $(\forall x)(D(x) \rightarrow \sim Q(x))$.

We now show that G is a logical consequence of F_1 and F_2. Let I be an arbitrary interpretation over a domain D. Suppose F_1 and F_2 are true in I. Since F_1, that is, $(\exists x)(P(x) \wedge (\forall y)(D(y) \rightarrow L(x, y)))$ is true in I, there is some element, say e, in D such that $(P(e) \wedge (\forall y)(D(y) \rightarrow L(e, y)))$ is true in I. That is, both $P(e)$ and $(\forall y)(D(y) \rightarrow L(e, y))$ are true in I. On the other hand, since $(P(x) \rightarrow (\forall y)(Q(y) \rightarrow \sim L(x, y)))$ is true in I for all elements of x in D,

certainly $(P(e) \rightarrow (\forall y)(Q(y) \rightarrow \sim L(e,y)))$ is true in I. Since $P(e)$ is true in I, $(\forall y)(Q(y) \rightarrow \sim L(e,y))$ must be true in I. Hence we know that for every element y in D, both $(D(y) \rightarrow L(e,y))$ and $(Q(y) \rightarrow \sim L(e,y))$ are true in I. If $D(y)$ is false in I, then $(D(y) \rightarrow \sim Q(y))$ is true in I. If $D(y)$ is true in I, then $L(e,y)$ must be true in I, since $(D(y) \rightarrow L(e,y))$ is true in I. Therefore, $Q(y)$ must be false in I, since $(Q(y) \rightarrow \sim L(e,y))$ is true in I. Consequently, $(D(y) \rightarrow \sim Q(y))$ is true in I. Therefore, $(D(y) \rightarrow \sim Q(y))$ is true for every y in D: i.e., $(\forall y)(D(y) \rightarrow \sim Q(y))$ is true in I. Hence, we have shown that if F_1 and F_2 are true in I, $(\forall y)(D(y) \rightarrow \sim Q(y))$ is true in I. This shows that G is a logical consequence of F_1 and F_2.

In the above examples, we have shown that the conclusions follow from the given facts. A demonstration that a conclusion follows from axioms is called a *proof*. A procedure for finding proofs is called a *proof procedure*. In Chapters 4–9, we shall give proof procedures that *mechanically* use inference rules to find a proof. These proof procedures are very efficient. The reader will find out that the above three examples are extremely easy to prove if these proof procedures are used.

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EXERCISES

Section 3.1

1. Let $P(x)$ and $Q(x)$ represent "x is a rational number" and "x is a real number," respectively. Symbolize the following sentences:

   1.1 Every rational number is a real number.
   1.2 Some real numbers are rational numbers.
   1.3 Not every real number is a rational number.

2. Let $C(x)$ mean "x is a used-car dealer," and $H(x)$ mean "x is honest." Translate each of the following into English:

   2.1 $(\exists x)C(x)$

2.2 $(\exists x)H(x)$

2.3 $(\forall x)(C(x) \rightarrow \sim H(x))$

2.4 $(\exists x)(C(x) \wedge H(x))$

2.5 $(\exists x)(H(x) \rightarrow C(x))$.

3. Let $P(x)$, $L(x)$, $R(x,y,z)$, and $E(x,y)$ represent "x is a point," "x is a line," "z passes through x and y," and "$x = y$," respectively. Translate the following:

   For every two points, there is one and only one line passing through both points.

4. An Abelian group is a set A with a binary operator $+$ that has certain properties. Let $P(x,y,z)$ and $E(x,y)$ represent $x+y=z$ and $x=y$, respectively. Express the following axioms for Abelian groups symbolically.

   (a) For every x and y in A, there exists a z in A such that $x+y=z$ (closure).
   (b) If $x+y=z$ and $x+y=w$, then $z=w$ (uniqueness).
   (c) $(x+y)+z = x+(y+z)$ (associativity).
   (d) $x+y=y+x$ (symmetry).
   (e) For every x and y in A, there exists a z such that $x+z=y$ (right solution).

Section 3.2

5. For the following interpretation ($D = \{a,b\}$),

$P(a,a)$	$P(a,b)$	$P(b,a)$	$P(b,b)$
T	F	F	T

   determine the truth value of the following formulas:

   (a) $(\forall x)(\exists y)P(x,y)$
   (b) $(\forall x)(\forall y)P(x,y)$
   (c) $(\exists x)(\forall y)P(x,y)$
   (d) $(\exists y) \sim P(a,y)$
   (e) $(\forall x)(\forall y)(P(x,y) \rightarrow P(y,x))$
   (f) $(\forall x)P(x,x)$.

6. Consider the following formula:

   A: $(\exists x)P(x) \rightarrow (\forall x)P(x)$.

   a. Prove that this formula is always true if the domain D contains only one element.
   b. Let $D = \{a,b\}$. Find an interpretation over D in which A is evaluated to F.

7. Consider the following interpretation:

Domain: $D = \{1,2\}$.

Assignment of constants a and b:

a	b
1	2

Assignment for function f:

$f(1)$	$f(2)$
2	1

Assignment for predicate P:

$P(1,1)$	$P(1,2)$	$P(2,1)$	$P(2,2)$
T	T	F	F

Evaluate the truth value of the following formulas in the above interpretation:

(1) $P(a, f(a)) \wedge P(b, f(b))$

(2) $(\forall x)(\exists y)\, P(y, x)$

(3) $(\forall x)(\forall y)(P(x, y) \rightarrow P(f(x), f(y)))$.

8. Let F_1 and F_2 be as follows:

F_1: $(\forall x)(P(x) \rightarrow Q(x))$

F_2: $\sim Q(a)$.

Prove that $\sim P(a)$ is a logical consequence of F_1 and F_2.

Section 3.3

9. Transform the following formulas into prenex normal forms:

(1) $(\forall x)(P(x) \rightarrow (\exists y)\, Q(x, y))$

(2) $(\exists x)(\sim ((\exists y)\, P(x, y)) \rightarrow ((\exists z)\, Q(z) \rightarrow R(x)))$

(3) $(\forall x)(\forall y)((\exists z)\, P(x, y, z) \wedge ((\exists u)\, Q(x, u) \rightarrow (\exists v)\, Q(y, v)))$.

Section 3.4

10. Consider the following statements:

F_1: Every student is honest.
F_2: John is not honest.

From the above statements, prove that John is not a student.

11. Consider the following premises:

    (1) Every athelete is strong.
    (2) Everyone who is both strong and intelligent will succeed in his career.
    (3) Peter is an athelete.
    (4) Peter is intelligent.

    Try to conclude that Peter will succeed in his career.

12. Assume that St. Francis is loved by everyone who loves someone. Also assume that no one loves nobody. Deduce that St. Francis is loved by everyone.

Chapter 4

Herbrand's Theorem

4.1 INTRODUCTION

In the previous chapters, we have discussed how to solve problems by proving theorems. In this and the following chapters, we shall consider proof procedures. Actually, finding a general decision procedure to verify the validity (inconsistency) of a formula was considered long ago. It was first tried by Leibniz (1646–1716) and further revived by Peano around the turn of the century and by Hilbert's school in the 1920s. It was not until Church [1936] and Turing [1936] that this was proved impossible. Church and Turing independently showed that there is no general decision procedure to check the validity of formulas of the first-order logic. However, there are proof procedures which can verify that a formula is valid if indeed it is valid. For invalid formulas, these procedures in general will never terminate. In view of the result of Church and Turing, this is the best we can expect to get from a proof procedure.

A very important approach to mechanical theorem proving was given by Herbrand in 1930. By definition, a valid formula is a formula that is true under all interpretations. Herbrand developed an algorithm to find an interpretation that can falsify a given formula. However, if the given formula is indeed valid, no such interpretation can exist and his algorithm will halt after a finite number of trials. Herbrand's method is the basis for most modern automatic proof procedures.

Gilmore [1960] was one of the first persons to implement Herbrand's procedure on a computer. Since a formula is valid if and only if its negation is inconsistent, his program was designed to detect the inconsistency of the negation of the given formula. During the execution of his program, propositional formulas are generated that are periodically tested for inconsistency. If the negation of the given formula is inconsistent, his program will eventually detect this fact. Gilmore's program managed to prove a few simple formulas, but encountered decisive difficulties with most other formulas of the first order logic. Careful studies of his program revealed that his method of testing the inconsistency of a propositional formula is inefficient. Gilmore's method was improved by Davis and Putnam [1960] a few months after his result was published. However, their improvement was still not enough. Many valid formulas of the first-order logic still could not be proved by computers in a reasonable amount of time.

A major breakthrough was made by Robinson [1965a], who introduced the so-called resolution principle. Resolution proof procedure is much more efficient than any earlier procedure. Since the introduction of the resolution principle, several refinements have been suggested in attempts to further increase its efficiency. Some of these refinements are semantic resolution [Slagle, 1967; Meltzer, 1966; Robinson, 1965b; Kowalski and Hayes, 1969], lock resolution [Boyer, 1971], linear resolution [Loveland, 1970a, b; Luckham, 1970; Anderson and Bledsoe, 1970; Yates *et al.*, 1970; Reiter, 1971; Kowalski and Kuehner, 1970], unit resolution [Wos *et al.*, 1964; Chang, 1970a], and set-of-support strategy [Wos *et al.*, 1965]. In this chapter, we shall first prove Herbrand's theorem. The resolution principle and some refinements of the resolution principle will be discussed in subsequent chapters.

4.2 SKOLEM STANDARD FORMS

Herbrand's and the resolution proof procedures that are to be discussed later in this book are actually refutation procedures. That is, instead of proving a formula valid, they prove that the negation of the formula is inconsistent. This is just a matter of convenience. There is no loss of generality in using refutation procedures. Furthermore, these refutation procedures are applied to a "standard form" of a formula. This standard form was introduced by Davis and Putnam [1960], and will be used throughout this book.

Essentially what Davis and Putnam did was to exploit the following ideas:

1. A formula of the first-order logic can be transformed into prenex normal form where the matrix contains no quantifiers and the prefix is a sequence of quantifiers.

2. The matrix, since it does not contain quantifiers, can be transformed into a conjunctive normal form.

3. Without affecting the inconsistency property, the existential quantifiers in the prefix can be eliminated by using Skolem functions.

In Chapter 3, we have already discussed how to transform a formula into a prenex normal form. By the techniques given in Chapter 2, we also know how to transform the matrix into a conjunctive normal form. We now discuss how to eliminate the existential quantifiers.

Let a formula F be already in a prenex normal form $(Q_1x_1)\cdots(Q_nx_n)M$, where M is in a *conjunctive* normal form. Suppose Q_r is an existential quantifier in the prefix $(Q_1x_1)\cdots(Q_nx_n)$, $1 \leqslant r \leqslant n$. If no universal quantifier appears before Q_r, we choose a new constant c different from other constants occurring in M, replace all x_r appearing in M by c, and delete (Q_rx_r) from the prefix. If $Q_{s_1}, \ldots, Q_{s_m}$ are all the universal quantifiers appearing before Q_r, $1 \leqslant s_1 < s_2 \cdots < s_m < r$, we choose a new m-place function symbol f different from other function symbols, replace all x_r in M by $f(x_{s_1}, x_{s_2}, \ldots, x_{s_m})$, and delete (Q_rx_r) from the prefix. After the above process is applied to all the existential quantifiers in the prefix, the last formula we obtain is a *Skolem standard form* (*standard form* for short) of the formula F. The constants and functions used to replace the existential variables are called *Skolem* functions.

Example 4.1

Obtain a standard form of the formula

$$(\exists x)(\forall y)(\forall z)(\exists u)(\forall v)(\exists w)\, P(x, y, z, u, v, w).$$

In the above formula, $(\exists x)$ is preceded by no universal quantifiers, $(\exists u)$ is preceded by $(\forall y)$ and $(\forall z)$, and $(\exists w)$ by $(\forall y)$, $(\forall z)$ and $(\forall v)$. Therefore, we replace the existential variable x by a constant a, u by a two-place function $f(y, z)$, and w by a three-place function $g(y, z, v)$. Thus, we obtain the following standard form of the formula:

$$(\forall y)(\forall z)(\forall v)\, P(a, y, z, f(y, z), v, g(y, z, v)).$$

Example 4.2

Obtain a standard form of the formula

$$(\forall x)(\exists y)(\exists z)((\sim P(x, y) \wedge Q(x, z)) \vee R(x, y, z)).$$

First, the matrix is transformed into a conjunctive normal form:

$$(\forall x)(\exists y)(\exists z)((\sim P(x, y) \vee R(x, y, z)) \wedge (Q(x, z) \vee R(x, y, z))).$$

Then, since $(\exists y)$ and $(\exists z)$ are both preceded by $(\forall x)$, the existential variables y and z are replaced, respectively, by one-place functions $f(x)$ and $g(x)$.

Thus, we obtain the following standard form of the formula:

$$(\forall x)\big((\sim P(x, f(x)) \vee R(x, f(x), g(x))) \wedge (Q(x, g(x)) \vee R(x, f(x), g(x)))\big).$$

Definition A *clause* is a finite disjunction of zero or more literals.

When it is convenient, we shall regard a set of literals as synonymous with a clause. For example, $P \vee Q \vee \sim R = \{P, Q, \sim R\}$. A clause consisting of r literals is called an *r-literal* clause. A one-literal clause is called a *unit* clause. When a clause contains no literal, we call it the *empty* clause. Since the empty clause has no literal that can be satisfied by an interpretation, the empty clause is always false. We customarily denote the empty clause by $\square$.

The disjunctions $\sim P(x,f(x)) \vee R(x,f(x),g(x))$ and $Q(x,g(x)) \vee R(x,f(x),g(x))$ in the standard form in Example 4.2 are clauses. *A set S of clauses is regarded as a conjunction of all clauses in S, where every variable in S is considered governed by a universal quantifier.* By this convention, a standard form can be simply represented by a set of clauses. For example, the standard form of Example 4.2 can be represented by the set

$$\{\sim P(x,f(x)) \vee R(x,f(x),g(x)), Q(x,g(x)) \vee R(x,f(x),g(x))\}.$$

As we said in the beginning of this section, we can eliminate existential quantifiers without affecting the inconsistency property. This is shown in the following theorem.

Theorem 4.1 Let S be a set of clauses that represents a standard form of a formula F. Then F is inconsistent if and only if S is inconsistent.

Proof Without loss of generality, we may assume that F is in a prenex normal form, that is, $F = (Q_1x_1)\cdots(Q_nx_n)M[x_1, \ldots, x_n]$. (We use $M[x_1, \ldots, x_n]$ to indicate that the matrix M contains variables $x_1, \ldots, x_n$.) Let Q_r be the first existential quantifier. Let $F_1 = (\forall x_1)\cdots(\forall x_{r-1})(Q_{r+1}x_{r+1})\cdots(Q_nx_n)M[x_1, \ldots, x_{r-1}, f(x_1, \ldots, x_{r-1}), x_{r+1}, \ldots, x_n]$, where f is a Skolem function corresponding to x_r, $1 \leqslant r \leqslant n$. We want to show that F is inconsistent if and only if F_1 is inconsistent. Suppose F is inconsistent. If F_1 is consistent, then there is an interpretation I such that F_1 is true in I. That is, for all $x_1, \ldots, x_{r-1}$, there exists at least one element, which is $f(x_1, \ldots, x_{r-1})$, such that $(Q_{r+1}x_{r+1})\cdots(Q_nx_n)M[x_1, \ldots, x_{r-1}, f(x_1, \ldots, x_{r-1}), x_{r+1}, \ldots, x_n]$ is true in I. Thus, F is true in I, which contradicts the assumption that F is inconsistent. Therefore F_1 must be inconsistent. On the other hand, suppose that F_1 is inconsistent. If F is consistent, then there is an interpretation I over a domain D such that F is true in I. That is, for all $x_1, \ldots, x_{r-1}$, there exists an element x_r such that $(Q_{r+1}x_{r+1})\cdots(Q_nx_n)M[x_1, \ldots, x_{r-1}, x_r, x_{r+1}, \ldots, x_n]$ is true in I. Extend the interpretation I to include a function f that maps $(x_1, \ldots, x_{r-1})$ to x_r for all $x_1, \ldots, x_{r-1}$ in D, that is, $f(x_1, \ldots, x_{r-1}) = x_r$. Let

this extension of I be denoted by I'. Then, clearly, for all $x_1, \ldots, x_{r-1}$, $(Q_{r+1}x_{r+1}) \cdots (Q_n x_n) M[x_1, \ldots, x_{r-1}, f(x_1, \ldots, x_{r-1}), x_{r+1}, \ldots, x_n]$ is true in I'. That is, F_1 is true in I', which contradicts the assumption that F_1 is inconsistent. Therefore F must be inconsistent. Assume there are m existential quantifiers in F. Let $F_0 = F$. Let F_k be obtained from F_{k-1} by replacing the first existential quantifier in F_{k-1} by a Skolem function, $k = 1, \ldots, m$. Clearly, $S = F_m$. Using the same arguments given above, we can show that F_{k-1} is inconsistent if and only if F_k is inconsistent for $k = 1, \ldots, m$. Therefore, we conclude that F is inconsistent if and only if S is inconsistent. Q.E.D.

Let S be a standard form of a formula F. If F is inconsistent, then by Theorem 4.1, $F = S$. If F is not inconsistent, we note that, in general, F is not equivalent to S. For example, let $F \triangleq (\exists x) P(x)$ and $S \triangleq P(a)$. Clearly, S is a standard form of F. However, let I be the interpretation defined below:

Domain: $D = \{1, 2\}$.

Assignment for a:

a
1

Assignment for P:

$P(1)$	$P(2)$
F	T

Then, clearly, F is true in I, but S is false in I. Therefore $F \neq S$.

It is noted that a formula may have more than one standard form. For the sake of simplicity, when we transform a formula F into a standard form S, we should replace existential quantifiers by Skolem functions that are as simple as possible. That is, we should use Skolem functions with the least number of arguments. This means that we should move existential quantifiers to the left as far as possible. Furthermore, if we have $F = F_1 \wedge \cdots \wedge F_n$, we can separately obtain a set S_i of clauses, where each S_i represents a standard form of F_i, $i = 1, \ldots, n$. Then, let $S = S_1 \cup \cdots \cup S_n$. By arguments similar to those given in the proof of Theorem 4.1, it is not difficult to see that F is inconsistent if and only if S is inconsistent.

Example 4.3

In this example, we shall show how to express the following theorem in a standard form:

If $x \cdot x = e$ for all x in group G, where $\cdot$ is a binary operator and e is the identity in G, then G is commutative.

We shall first symbolize the above theorem together with some basic axioms in group theory and then represent the negation of the above theorem by a set of clauses.

We know that group G satisfies the following four axioms:

A_1: $x, y \in G$ implies that $x \cdot y \in G$ (closure property);

A_2: $x, y, z \in G$ implies that $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity property);

A_3: $x \cdot e = e \cdot x = x$ for all $x \in G$ (identity property);

A_4: for every $x \in G$ there exists an element $x^{-1} \in G$ such that $x \cdot x^{-1} = x^{-1} \cdot x = e$ (inverse property).

Let $P(x, y, z)$ stand for $x \cdot y = z$ and $i(x)$ for x^{-1}. Then the above axioms can be represented by

A_1': $(\forall x)(\forall y)(\exists z)\, P(x, y, z)$

A_2': $(\forall x)(\forall y)(\forall z)(\forall u)(\forall v)(\forall w)\,(P(x, y, u) \wedge P(y, z, v) \wedge P(u, z, w) \rightarrow P(x, v, w))$
$\wedge\, (\forall x)(\forall y)(\forall z)(\forall u)(\forall v)(\forall w)\,(P(x, y, u) \wedge P(y, z, v) \wedge P(x, v, w) \rightarrow P(u, z, w))$

A_3': $(\forall x)\, P(x, e, x) \wedge (\forall x)\, P(e, x, x)$

A_4': $(\forall x)\, P(x, i(x), e) \wedge (\forall x)\, P(i(x), x, e)$.

The conclusion of the theorem is

B: If $x \cdot x = e$ for all $x \in G$, then G is commutative, i.e., $u \cdot v = v \cdot u$ for all $u, v \in G$.

B can be represented by

B': $(\forall x)\, P(x, x, e) \rightarrow ((\forall u)(\forall v)(\forall w)\,(P(u, v, w) \rightarrow P(v, u, w)))$.

Now, the entire theorem is represented by the formula $F = A_1' \wedge \cdots \wedge A_4' \rightarrow B'$. Thus, $\sim F = A_1' \wedge A_2' \wedge A_3' \wedge A_4' \wedge \sim B'$. To obtain a set S of clauses for $\sim F$, we first obtain a set S_i of clauses for each axiom A_i', $i = 1, 2, 3, 4$ as follows.

S_1: $\{P(x, y, f(x, y))\}$

S_2: $\{\sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(u, z, w) \vee P(x, v, w),$
$\sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(x, v, w) \vee P(u, z, w)\}$

S_3: $\{P(x, e, x), P(e, x, x)\}$.

S_4': $\{P(x, i(x), e), P(i(x), x, e)\}$.

Since

$$\begin{aligned}
\sim B' &= \sim\big((\forall x)\,P(x,x,e) \to ((\forall u)(\forall v)(\forall w)(P(u,v,w) \to P(v,u,w)))\big)\\
&= \sim\big(\sim(\forall x)\,P(x,x,e) \vee ((\forall u)(\forall v)(\forall w)(\sim P(u,v,w) \vee P(v,u,w)))\big)\\
&= (\forall x)\,P(x,x,e) \wedge \sim((\forall u)(\forall v)(\forall w)(\sim P(u,v,w) \vee P(v,u,w)))\\
&= (\forall x)\,P(x,x,e) \wedge (\exists u)(\exists v)(\exists w)(P(u,v,w) \wedge \sim P(v,u,w)),
\end{aligned}$$

a set of clauses for $\sim B'$ is given below.

$$\begin{aligned}
T:\quad &\{P(x,x,e),\\
&\ P(a,b,c),\\
&\ \sim P(b,a,c)\}.
\end{aligned}$$

Thus, the set $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup T$ is the set consisting of the following clauses:

(1) $P(x,y,f(x,y))$

(2) $\sim P(x,y,u) \vee \sim P(y,z,v) \vee \sim P(u,z,w) \vee P(x,v,w)$

(3) $\sim P(x,y,u) \vee \sim P(y,z,v) \vee \sim P(x,v,w) \vee P(u,z,w)$

(4) $P(x,e,x)$

(5) $P(e,x,x)$

(6) $P(x,i(x),e)$

(7) $P(i(x),x,e)$

(8) $P(x,x,e)$

(9) $P(a,b,c)$

(10) $\sim P(b,a,c)$.

In Example 4.3, we have shown how to obtain a set S of clauses for the formula $\sim F$. By Theorems 2.2 and 4.1, we know that F is valid if and only if S is inconsistent. As we said at the beginning of this section, we shall use refutation procedures to prove theorems. Thus, from here on, we shall assume that the input to a refutation procedure is always a set of clauses, such as the set S obtained in the above example. Furthermore, we shall use "unsatisfiable" ("satisfiable"), instead of "inconsistent" ("consistent"), for sets of clauses.

4.3 THE HERBRAND UNIVERSE OF A SET OF CLAUSES

By definition, a set S of clauses is unsatisfiable if and only if it is false under all interpretations over all domains. Since it is inconvenient and impossible

to consider all interpretations over all domains, it would be nice if we could fix on one special domain H such that S is unsatisfiable if and only if S is false under all the interpretations over this domain. Fortunately, there does exist such a domain, which is called the *Herbrand universe* of S, defined as follows.

Definition Let H_0 be the set of constants appearing in S. If no constant appears in S, then H_0 is to consist of a single constant, say $H_0 = \{a\}$. For $i = 0, 1, 2, \ldots$, let H_{i+1} be the union of H_i and the set of all terms of the form $f^n(t_1, \ldots, t_n)$ for all n-place functions f^n occurring in S, where t_j, $j = 1, \ldots, n$, are members of the set H_i. Then each H_i is called the *i-level constant set* of S, and H_∞, or $\lim_{i\to\infty} H_i$, is called the *Herbrand universe* of S.

Example 4.4

Let $S = \{P(a), \sim P(x) \vee P(f(x))\}$. Then

$$
\begin{aligned}
H_0 &= \{a\} \\
H_1 &= \{a, f(a)\} \\
H_2 &= \{a, f(a), f(f(a))\} \\
&\vdots \\
H_\infty &= \{a, f(a), f(f(a)), f(f(f(a))), \ldots\}.
\end{aligned}
$$

Example 4.5

Let $S = \{P(x) \vee Q(x), R(z), T(y) \vee \sim W(y)\}$. Since there is no constant in S, we let $H_0 = \{a\}$. There is no function symbol in S, hence $H = H_0 = H_1 = \cdots = \{a\}$.

Example 4.6

Let $S = \{P(f(x), a, g(y), b)\}$. Then

$$
\begin{aligned}
H_0 &= \{a, b\} \\
H_1 &= \{a, b, f(a), f(b), g(a), g(b)\} \\
H_2 &= \{a, b, f(a), f(b), g(a), g(b), f(f(a)), f(f(b)), f(g(a)), f(g(b)), g(f(a)), \\
&\quad g(f(b)), g(g(a)), g(g(b))\} \\
&\vdots
\end{aligned}
$$

In the sequel, by expression we mean a term, a set of terms, an atom, a set of atoms, a literal, a clause, or a set of clauses. When no variable appears in an expression, we sometimes call the expression a *ground expression* to emphasize this fact. Thus we may use a ground term, a ground atom, a ground literal, and a ground clause to mean that no variable occurs

in the respective expressions. Furthermore, a *subexpression* of an expression E is an expression that occurs in E.

Definition Let S be a set of clauses. The set of ground atoms of the form $P^n(t_1, \ldots, t_n)$ for all n-place predicates P^n occurring in S, where $t_1, \ldots, t_n$ are elements of the Herbrand universe of S, is called the *atom set*, or the *Herbrand base* of S.

Definition A *ground instance* of a clause C of a set S of clauses is a clause obtained by replacing variables in C by members of the Herbrand universe of S.

Example 4.7

Let $S = \{P(x), Q(f(y)) \vee R(y)\}$. $C = P(x)$ is a clause in S and $H = \{a, f(a), f(f(a)), \ldots\}$ is the Herbrand universe of S. Then $P(a)$ and $P(f(f(a)))$ are both ground instances of C.

We now consider interpretations over the Herbrand universe. Let S be a set of clauses. As discussed in Chapter 3, an interpretation over the Herbrand universe of S is an assignment of constants, function symbols, and predicate symbols occurring in S. In the following, we shall define a special interpretation over the Herbrand universe of S, called the H-interpretation of S.

Definition Let S be a set of clauses; H, the Herbrand universe of S; and I, an interpretation of S over H. I is said to be an *H-interpretation* of S if it satisfies the following conditions:

1. I maps all constants in S to themselves.
2. Let f be an n-place function symbol and $h_1, \ldots, h_n$ be elements of H. In I, f is assigned a function that maps $(h_1, \ldots, h_n)$ (an element in H^n) to $f(h_1, \ldots, h_n)$ (an element in H).

There is no restriction on the assignment to each n-place predicate symbol in S. Let $A = \{A_1, A_2, \ldots, A_n, \ldots\}$ be the atom set of S. An H-interpretation I can be conveniently represented by a set

$$I = \{m_1, m_2, \ldots, m_n, \ldots\}$$

in which m_j is either A_j or $\sim A_j$ for $j = 1, 2, \ldots$. The meaning of this set is that if m_j is A_j, then A_j is assigned "true"; otherwise, A_j is assigned "false."

Example 4.8

Consider the set $S = \{P(x) \vee Q(x), R(f(y))\}$. The Herbrand universe H of S is $H = \{a, f(a), f(f(a)), \ldots\}$. There are three predicate symbols: P, Q, and R. Hence the atom set of S is

$$A = \{P(a), Q(a), R(a), P(f(a)), Q(f(a)), R(f(a)), \ldots\}.$$

Some H-interpretations for S are as follows:

$$I_1 = \{P(a), Q(a), R(a), P(f(a)), Q(f(a)), R(f(a)), \ldots\}$$

$$I_2 = \{\sim P(a), \sim Q(a), \sim R(a), \sim P(f(a)), \sim Q(f(a)), \sim R(f(a)), \ldots\}$$

$$I_3 = \{P(a), Q(a), \sim R(a), P(f(a)), Q(f(a)), \sim R(f(a)), \ldots\}.$$

An interpretation of a set S of clauses does not necessarily have to be defined over the Herbrand universe of S. Thus an interpretation may not be an H-interpretation. For example, let $S = \{P(x), Q(y, f(y, a))\}$. If the domain is $D = \{1, 2\}$, then the following is an interpretation of S.

$D = \{1, 2\}$.

a	$f(1,1)$	$f(1,2)$	$f(2,1)$	$f(2,2)$
2	1	2	2	1

$P(1)$	$P(2)$	$Q(1,1)$	$Q(1,2)$	$Q(2,1)$	$Q(2,2)$
T	F	F	T	F	T

For an interpretation such as the one defined above, we can define an H-interpretation I^* corresponding to I. We use the above example to illustrate this point. First, we find the atom set of S,

$$A = \{P(a), Q(a, a), P(f(a, a)), Q(a, f(a, a)), Q(f(a, a), a), Q(f(a, a), f(a, a)), \ldots\}.$$

Next, we evaluate each member of A by using the above table.

$$P(a) = P(2) = F$$

$$Q(a, a) = Q(2, 2) = T$$

$$P(f(a, a)) = P(f(2, 2)) = P(1) = T$$

$$Q(a, f(a, a)) = Q(2, f(2, 2)) = Q(2, 1) = F$$

$$Q(f(a, a), a) = Q(f(2, 2), 2) = Q(1, 2) = T$$

$$Q(f(a, a), f(a, a)) = Q(f(2, 2), f(2, 2)) = Q(1, 1) = F$$

$$\vdots$$

Therefore, the H-interpretation I^* corresponding to I is

$$I^* = \{\sim P(a), Q(a, a), P(f(a, a)), \sim Q(a, f(a, a)), Q(f(a, a), a), \sim Q(f(a, a), f(a, a)), \ldots\}.$$

In case there is no constant in S, the element a that is used to initiate the Herbrand universe of S can be mapped into any element of the domain

D. In this case, if there is more than one element in D, then there is more than one H-interpretation corresponding to I. For example, let $S = \{P(x), Q(y, f(y,z))\}$ and let an interpretation I for S be as follows:

$$D = \{1, 2\}$$

$f(1,1)$	$f(1,2)$	$f(2,1)$	$f(2,2)$
1	2	2	1

$P(1)$	$P(2)$	$Q(1,1)$	$Q(1,2)$	$Q(2,1)$	$Q(2,2)$
T	F	F	T	F	T

Then the two H-interpretations corresponding to I are

$$I^* = \{\sim P(a), Q(a,a), P(f(a,a)), \sim Q(a, f(a,a)), Q(f(a,a), a), \sim Q(f(a,a), f(a,a)), \ldots\} \quad \text{if} \quad a = 2,$$

$$I^* = \{P(a), \sim Q(a,a), P(f(a,a)), \sim Q(a, f(a,a)), \sim Q(f(a,a), a), \sim Q(f(a,a), f(a,a)), \ldots\} \quad \text{if} \quad a = 1.$$

We can formalize the concepts mentioned above as follows:

Definition Given an interpretation I over a domain D, an *H-interpretation I^* corresponding to I* is an H-interpretation that satisfies the following condition:

Let $h_1, \ldots, h_n$ be elements of H (the Herbrand universe of S). Let every h_i be mapped to some d_i in D. If $P(d_1, \ldots, d_n)$ is assigned $T(F)$ by I, then $P(h_1, \ldots, h_n)$ is also assigned $T(F)$ in I^*.

In fact, it is not hard to prove the following lemma. The proof is left as an exercise.

Lemma 4.1 If an interpretation I over some domain D satisfies a set S of clauses, then any one of the H-interpretations I^* corresponding to I also satisfies S.

Theorem 4.2 A set S of clauses is unsatisfiable if and only if S is false under all the H-interpretations of S.

Proof ($\Rightarrow$) The first half of the above theorem is obvious since, by definition, S is unsatisfiable if and only if S is false under all the interpretations over *any* domain.

($\Leftarrow$) To prove the second half of the above theorem, assume that S is false under all the H-interpretations of S. Suppose S is not unsatisfiable. Then there is an interpretation I over some domain D such that S is true

under I. Let I^* be an H-interpretation corresponding to I. According to Lemma 4.1, S is true under I^*. This contradicts the assumption that S is false under all the H-interpretations of S. Therefore, S must be unsatisfiable.

Q.E.D.

Thus we have obtained the objective stated at the beginning of this section. That is, we need consider only interpretations over the Herbrand universe, or more strongly, H-interpretations, for checking whether or not a set of clauses is unsatisfiable. Because of the above theorem, from here on, whenever we mention an interpretation, we mean an H-interpretation.

Let $\varnothing$ denote the empty set. Each of the following observations is obvious. We shall leave their proofs to the reader.

1. A ground instance C' of a clause C is satisfied by an interpretation I if and only if there is a ground literal L' in C' such that L' is also in I, that is, $C' \cap I \neq \varnothing$.
2. A clause C is satisfied by an interpretation I if and only if every ground instance of C is satisfied by I.
3. A clause C is falsified by an interpretation I if and only if there is at least one ground instance C' of C such that C' is not satisfied by I.
4. A set S of clauses is unsatisfiable if and only if for every interpretation I, there is at least one ground instance C' of some clause C in S such that C' is not satisfied by I.

Example 4.9

1. Consider the clause $C = \sim P(x) \vee Q(f(x))$. Let I_1, I_2, and I_3 be defined as follows:

$$I_1 = \{\sim P(a), \sim Q(a), \sim P(f(a)), \sim Q(f(a)), \sim P(f(f(a))), \sim Q(f(f(a))), \ldots\}$$
$$I_2 = \{\ P(a),\ \ Q(a),\ \ P(f(a)),\ \ Q(f(a)),\ \ P(f(f(a))),\ \ Q(f(f(a))), \ldots\}$$
$$I_3 = \{\ P(a), \sim Q(a),\ \ P(f(a)), \sim Q(f(a)),\ \ P(f(f(a))), \sim Q(f(f(a))), \ldots\}.$$

The reader should be able to see that C is satisfied by I_1 and I_2, but falsified by I_3.

2. Consider $S = \{P(x), \sim P(a)\}$. There are only two H-interpretations:

$$I_1 = \{P(a)\} \qquad \text{and} \qquad I_2 = \{\sim P(a)\}.$$

S is falsified by both H-interpretations, and therefore is unsatisfiable.

4.4 SEMANTIC TREES

Having introduced the Herbrand universe, we now consider semantic trees [Robinson, 1968a; Kowalski and Hayes, 1969]. It will be seen in the

sequel that finding a proof for a set of clauses is equivalent to generating a semantic tree.

Definition If A is an atom, then the two literals A and $\sim A$ are said to be each other's *complement*, and the set $\{A, \sim A\}$ is called a *complementary pair*.

We note that a clause is a tautology if it contains a complementary pair. In the sequel, when we use "tautology," we shall specifically mean a clause that is a tautology.

Definition Given a set S of clauses, let A be the atom set of S. A *semantic tree* for S is a (downward) tree T, where each link is attached with a finite set of atoms or negations of atoms from A in such a way that:

i. For each node N, there are only finitely many immediate links $L_1, \ldots, L_n$ from N. Let Q_i be the conjunction of all the literals in the set attached to L_i, $i = 1, \ldots, n$. Then $Q_1 \vee Q_2 \vee \cdots \vee Q_n$ is a valid propositional formula.

ii. For each node N, let $I(N)$ be the union of all the sets attached to the links of the branch of T down to and including N. Then $I(N)$ does not contain any complementary pair.

Definition Let $A = \{A_1, A_2, \ldots, A_k, \ldots\}$ be the atom set of a set S of clauses. A semantic tree for S is said to be *complete* if and only if for every tip node N of the semantic tree, that is, a node that has no links sprouting from it, $I(N)$ contains either A_i or $\sim A_i$ for $i = 1, 2, \ldots$.

Example 4.10

Let $A = \{P, Q, R\}$ be the atom set of a set S of clauses. Then each one of the two trees in Fig. 4.1 is a complete semantic tree for S. (See p. 58.)

Example 4.11

Consider $S = \{P(x), P(a)\}$. The atom set of S is $\{P(a)\}$. A complete semantic tree for S is shown in Fig. 4.2. (See p. 58.)

Example 4.12

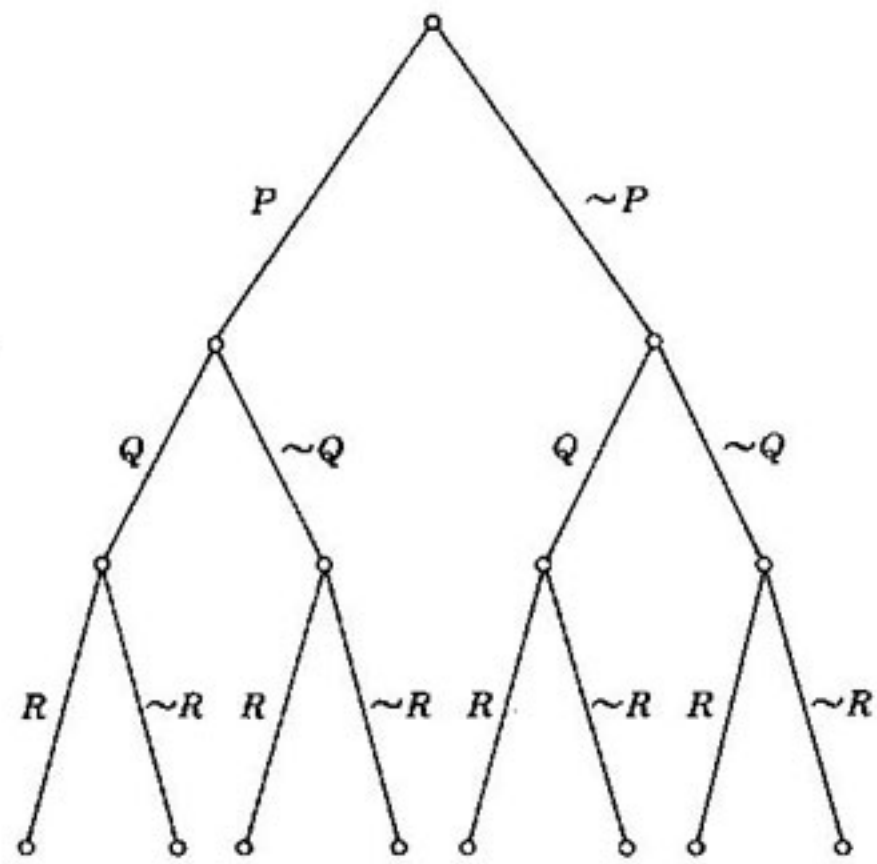
Consider $S = \{P(x), Q(f(x))\}$. The atom set of S is

$$\{P(a), Q(a), P(f(a)), Q(f(a)), P(f(f(a))), Q(f(f(a))), \ldots\}.$$

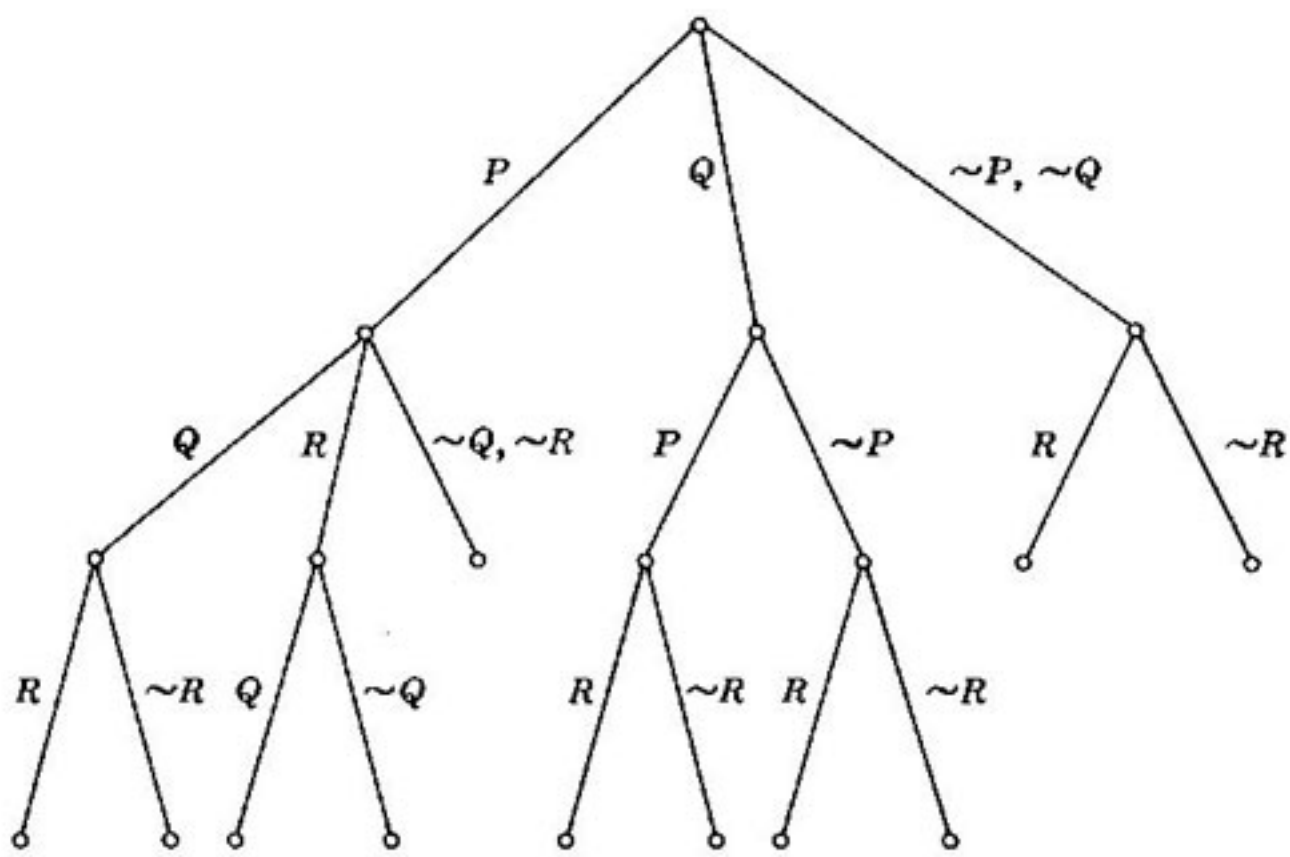
Fig. 4.3 shows a semantic tree for S.

It is noted that for each node N in a semantic tree for S, $I(N)$ is a subset of some interpretation for S. For this reason, $I(N)$ will be called a *partial interpretation* for S.

When the atom set A of a set S of clauses is infinite, any complete semantic tree for S will be infinite. As is easily seen, a complete semantic



(a)



(b)

Figure 4.1

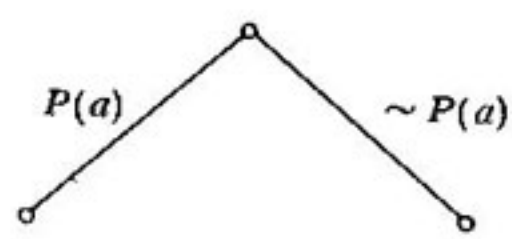


Figure 4.2

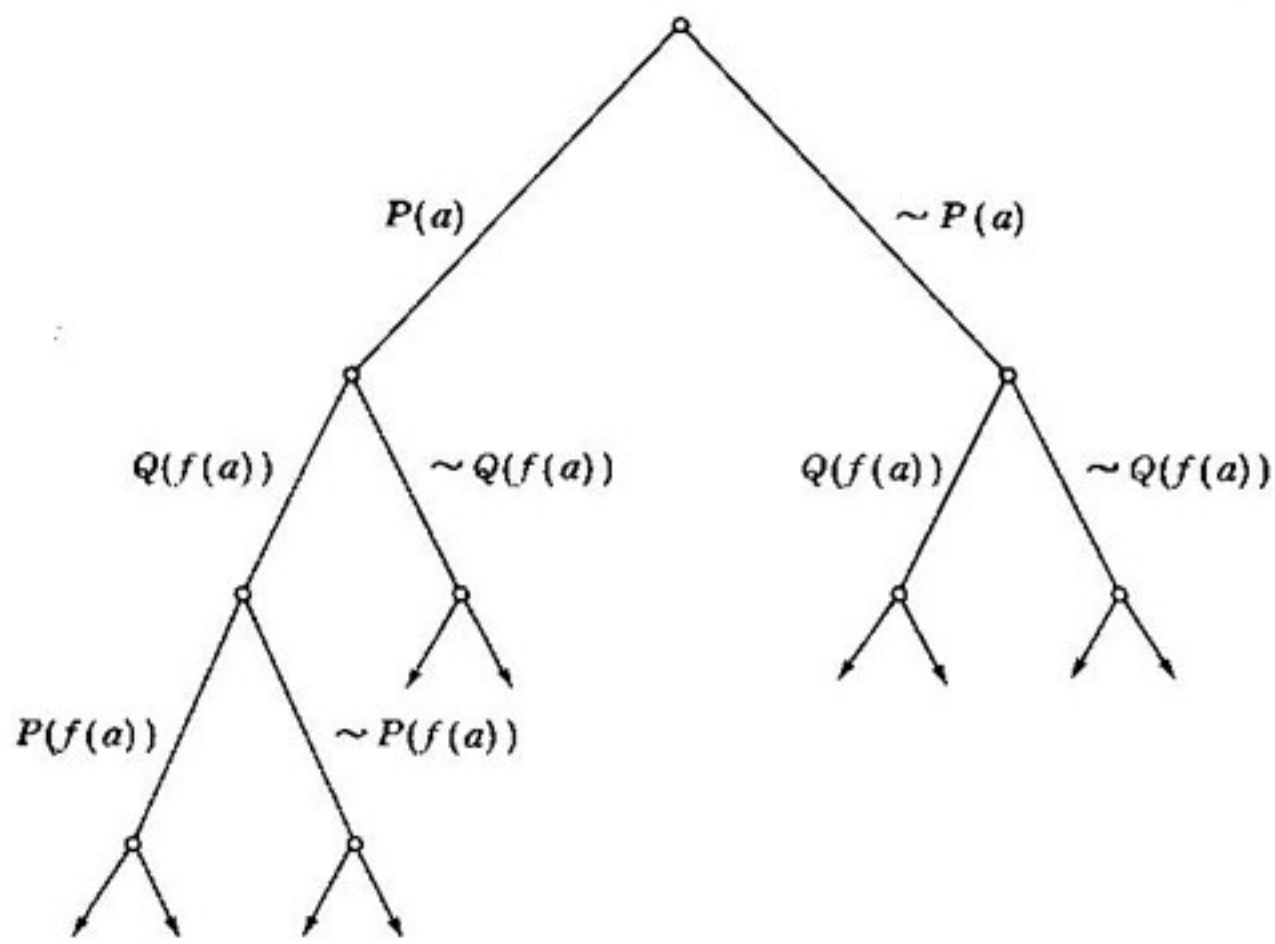


Figure 4.3

tree for S corresponds to an exhaustive survey of all possible interpretations for S. If S is unsatisfiable, then S fails to be true in each of these interpretations. Thus, we may stop expanding nodes from a node N if $I(N)$ falsifies S. This motivates the following definitions.

Definition A node N is a *failure node* if $I(N)$ falsifies some ground instance of a clause in S, but $I(N')$ does not falsify any ground instance of a clause in S for every ancestor node N' of N.

Definition A semantic tree T is said to be *closed* if and only if every branch of T terminates at a failure node.

Definition A node N of a closed semantic tree is called an *inference node* if all the immediate descendant nodes of N are failure nodes.

Example 4.13

Let $S = \{P, Q \vee R, \sim P \vee \sim Q, \sim P \vee \sim R\}$. The atom set of S is $A = \{P, Q, R\}$. Figure 4.4a is a complete semantic tree for S, while Fig. 4.4b is a closed semantic tree for S.

Example 4.14

Consider $S = \{P(x), \sim P(x) \vee Q(f(x)), \sim Q(f(a))\}$. The atom set of S is

$$A = \{P(a), Q(a), P(f(a)), Q(f(a)), \ldots\}.$$

Figure 4.5 shows a closed semantic tree for S.

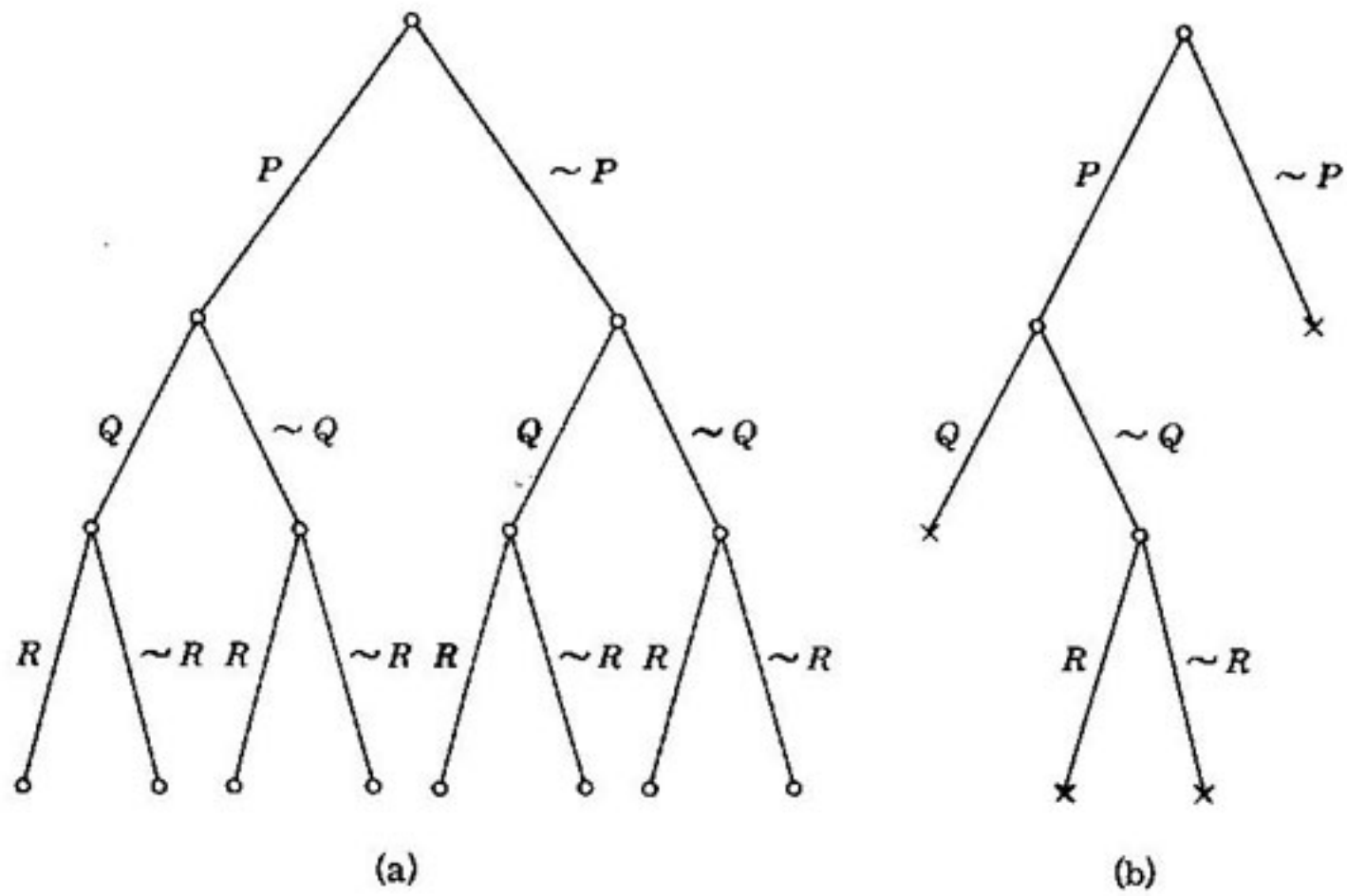


Figure 4.4

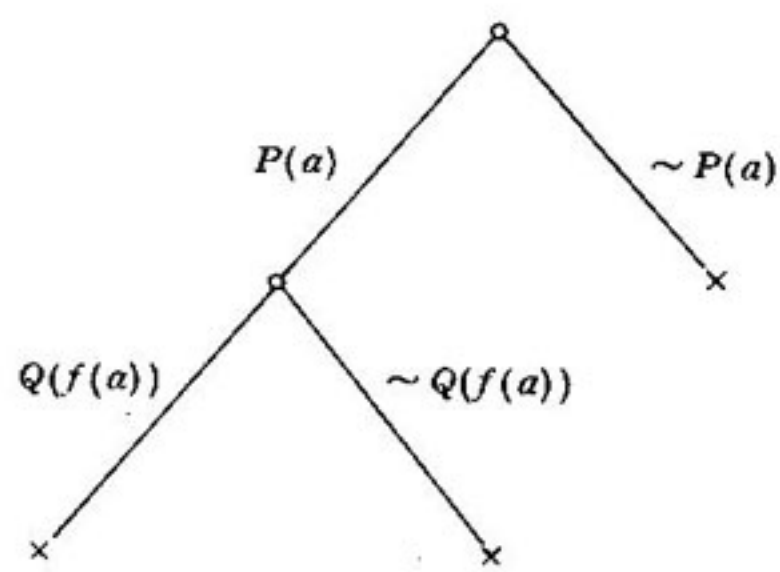


Figure 4.5

4.5 HERBRAND'S THEOREM

Herbrand's theorem is a very important theorem in symbolic logic; it is a base for most modern proof procedures in mechanical theorem proving. Herbrand's theorem is closely related to Theorem 4.2 given in Section 4.3. That is, to test whether a set S of clauses is unsatisfiable, we need consider only interpretations over the Herbrand universe of S. If S is false under all interpretations over the Herbrand universe of S, then we can conclude that S is unsatisfiable. Since there are usually many, possibly an infinite number, of these interpretations, we should organize them in some systematic way.

This can be done by using a semantic tree. We shall give two versions of Herbrand's theorem. The one stated most often in the literature is the second version; however, the first version is useful in this book.

Theorem 4.3 (Herbrand's Theorem, Version I) A set S of clauses is unsatisfiable if and only if corresponding to every complete semantic tree of S, there is a finite closed semantic tree.

Proof ($\Rightarrow$) Suppose S is unsatisfiable. Let T be a complete semantic tree for S. For each branch B of T, let I_B be the set of all literals attached to all links of the branch B. Then I_B is an interpretation for S. Since S is unsatisfiable, I_B must falsify a ground instance C' of a clause C in S. However, since C' is finite, there must exist a failure node N_B (which is a finite number of links away from the root node) on the branch B. Since every branch of T has a failure node, there is a closed semantic tree T' for S. Furthermore, since only a finite number of links are connected to each node of T', T' must be finite (that is, the number of nodes in T' is finite), for otherwise, by Konig's lemma [Knuth, 1968], we could find an infinite branch containing no failure node. Thus we complete the proof of the first half of the theorem.

($\Leftarrow$) Conversely, if corresponding to every complete semantic tree T for S there is a finite closed semantic tree, then every branch of T contains a failure node. This means that every interpretation falsifies S. Hence S is unsatisfiable. This completes the proof of the second half of the theorem.

Theorem 4.4 (Herbrand's Theorem, Version II). A set S of clauses is unsatisfiable if and only if there is a finite unsatisfiable set S' of ground instances of clauses of S.

Proof ($\Rightarrow$) Suppose S is unsatisfiable. Let T be a complete semantic tree for S. Then, by Herbrand's theorem (version I), there is a finite closed semantic tree T' corresponding to T. Let S' be the set of all the ground instances of clauses that are falsified at all the failure nodes of T'. S' is finite since there are a finite number of failure nodes in T'. Since S' is false in every interpretation of S', S' is unsatisfiable.

($\Leftarrow$) Suppose there is a finite unsatisfiable set S' of ground instances of clauses in S. Since every interpretation I of S contains an interpretation I' of S', if I' falsifies S', then I must also falsify S'. However, S' is falsified by every interpretation I'. Consequently, S' is falsified by every interpretation I of S. Therefore, S is falsified by every interpretation of S. Hence, S is unsatisfiable. Q.E.D.

Example 4.15

Let $S = \{P(x), \sim P(f(a))\}$. This set S is unsatisfiable. Hence, by Herbrand's theorem, there is a finite unsatisfiable set S' of ground instances of clauses in S. We have found that one of these sets is $S' = \{P(f(a)), \sim P(f(a))\}$.

Example 4.16

Let $S = \{\sim P(x) \vee Q(f(x), x), P(g(b)), \sim Q(y, z)\}$. This set S is unsatisfiable. One of the unsatisfiable sets of ground instances of clauses in S is

$$S' = \{\sim P(g(b)) \vee Q(f(g(b)), g(b)), P(g(b)), \sim Q(f(g(b)), g(b))\}.$$

Example 4.17

Let the set S consist of the following clauses:

$$\begin{aligned} S = \{ & \sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(x, v, w) \vee P(u, z, w), \\ & \sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(u, z, w) \vee P(x, v, w), \\ & P(g(x, y), x, y), P(x, h(x, y), y), P(x, y, f(x, y)), \\ & \sim P(k(x), x, k(x))\}. \end{aligned}$$

This set S is also unsatisfiable. However, it is not very easy to find by hand a finite unsatisfiable set S' of ground instances of clauses in S. One way to find such a set S' is to generate a closed semantic tree T' for S. Then the set S' of all the ground instances falsified at all the failure nodes of T' is such a desired set. The following is a desired set S'. The reader may want to check that each ground clause in S' is a ground instance of some clause in S, and that S' is unsatisfiable.

$$\begin{aligned} S' = \{ & P(a, h(a, a), a), \\ & \sim P(k(h(a, a)), h(a, a), k(h(a, a))), \\ & P(g(a, k(h(a, a))), a, k(h(a, a))), \\ & \sim P(g(a, k(h(a, a))), a, k(h(a, a))) \vee \sim P(a, h(a, a), a) \\ & \vee \sim P(g(a, k(h(a, a))), a, k(h(a, a))) \vee P(k(h(a, a)), h(a, a), k(h(a, a)))\}. \end{aligned}$$

4.6 IMPLEMENTATION OF HERBRAND'S THEOREM

The second version of Herbrand's theorem suggests a refutation procedure. That is, given an unsatisfiable set S of clauses to prove, if there is a mechanical procedure that can successively generate sets $S_1', \ldots, S_n', \ldots$ of ground instances of clauses in S and can successively test $S_1', S_2', \ldots$ for unsatisfiability, then, as guaranteed by Herbrand's theorem, this procedure can detect a finite N such that S_N' is unsatisfiable.

Gilmore was one of the first men to implement the above idea [Gilmore, 1960]. In 1960, he wrote a computer program that successively generated sets $S_0', S_1', \ldots$, where S_i' is the set of all the ground instances obtained by replacing the variables in S by the constants in the i-level constant set H_i of S. Since each S_i' is a conjunction of ground clauses, one can use any method

available in the propositional logic to check its unsatisfiability. Gilmore used the multiplication method. That is, as each S_i' is produced, S_i' is multiplied out into a disjunctive normal form. Any conjunction in the disjunctive normal form containing a complementary pair is removed. Should some S_i' be empty, then S_i' is unsatisfiable and a proof is found.

Example 4.18

Consider

$$S = \{P(x), \sim P(a)\}.$$

$$H_0 = \{a\}$$

$$S_0' = P(a) \wedge \sim P(a) = \square.$$

Thus S is proved to be unsatisfiable.

Example 4.19

Consider

$$S = \{P(a), \sim P(x) \vee Q(f(x)), \sim Q(f(a))\}.$$

$$H_0 = \{a\}.$$

$$\begin{aligned} S_0' &= P(a) \wedge (\sim P(a) \vee Q(f(a))) \wedge \sim Q(f(a)) \\ &= (P(a) \wedge \sim P(a) \wedge \sim Q(f(a))) \vee (P(a) \wedge Q(f(a)) \wedge \sim Q(f(a))) \\ &= \square \vee \square = \square. \end{aligned}$$

Thus S is proved to be unsatisfiable.

The multiplication method used by Gilmore is inefficient. As is easily seen, even for a small set of ten two-literal ground clauses, there are 2^{10} conjunctions. To overcome this inefficiency, Davis and Putnam [1960] introduced a more efficient method for testing the unsatisfiability of a set of ground clauses. We shall now describe their method with some modification.

The Method of Davis and Putnam

Let S be a set of ground clauses. Essentially, the method consists of the following four rules.

I. *Tautology Rule* Delete all the ground clauses from S that are tautologies. The remaining set S' is unsatisfiable if and only if S is.

II. *One-Literal Rule* If there is a unit ground clause L in S, obtain S' from S by deleting those ground clauses in S containing L. If S' is empty,

S is satisfiable. Otherwise, obtain a set S'' from S' by deleting $\sim L$ from S'. S'' is unsatisfiable if and only if S is. Note that if $\sim L$ is a ground unit clause, then the clause becomes $\square$ when $\sim L$ is deleted from the clause.

III. *Pure-Literal Rule* A literal L in a ground clause of S is said to be *pure* in S if and only if the literal $\sim L$ does not appear in any ground clause in S. If a literal L is pure in S, delete all the ground clauses containing L. The remaining set S' is unsatisfiable if and only if S is.

IV. *Splitting Rule* If the set S can be put into the form

$$(A_1 \vee L) \wedge \cdots \wedge (A_m \vee L) \wedge (B_1 \vee \sim L) \wedge \cdots \wedge (B_n \vee \sim L) \wedge R,$$

where A_i, B_i, and R are free of L and $\sim L$, then obtain the sets $S_1 = A_1 \wedge \cdots \wedge A_m \wedge R$ and $S_2 = B_1 \wedge \cdots \wedge B_n \wedge R$. S is unsatisfiable if and only if $(S_1 \vee S_2)$ is unsatisfiable, that is, both S_1 and S_2 are unsatisfiable.

We can now show that the above rules are sound. That is, if the original set S is unsatisfiable, then the remaining set after one of the rules is applied is still unsatisfiable, and vice versa.

For Rule I Since a tautology is satisfied by every interpretation, S' is unsatisfiable if and only if S is.

For Rule II If S' is empty, then all the ground clauses in S contain L. Hence any interpretation containing L can satisfy S. Therefore S is satisfiable. We still have to show that S'' is unsatisfiable if and only if S is unsatisfiable. Suppose S'' is unsatisfiable. If S is satisfiable, then there is a model M of S containing L. For S'', M must satisfy all the clauses which do not contain L. Furthermore, since M falsifies $\sim L$, M must satisfy all the clauses that originally contain $\sim L$. Therefore, M must satisfy S''. This contradicts the assumption that S'' is unsatisfiable. Hence, S must be unsatisfiable. Conversely, suppose S is unsatisfiable. If S'' is satisfiable, then there is a model M'' of S''. Thus any interpretation of S containing M'' and L must be a model of S. This contradicts the assumption that S has no model. Hence S'' must be unsatisfiable. Therefore, S'' is unsatisfiable if and only if S is.

For Rule III Suppose S' is unsatisfiable. Then S must be unsatisfiable since S' is a subset of S. Conversely, suppose S is unsatisfiable. If S' is satisfiable, then there is a model M of S'. Since neither L nor $\sim L$ is in S', neither L nor $\sim L$ is in M. Thus any interpretation of S that contains M and L is a model of S. This contradicts the assumption that S has no model. Hence S' must be unsatisfiable. Therefore, S' is unsatisfiable if and only if S is unsatisfiable.

For Rule IV Suppose S is unsatisfiable. If $(S_1 \vee S_2)$ is satisfiable, then either S_1 or S_2 has a model. If $S_1(S_2)$ has a model M, then any interpretation

of S containing $\sim L(L)$ is a model of S. This contradicts the assumption that S has no model. Hence $(S_1 \vee S_2)$ is unsatisfiable. Conversely, suppose $(S_1 \vee S_2)$ is unsatisfiable. If S is satisfiable, S must have a model M. If M contains $\sim L(L)$, M can satisfy $S_1(S_2)$. This contradicts the assumption that $(S_1 \vee S_2)$ is unsatisfiable. Hence S must be unsatisfiable. Therefore, S is unsatisfiable if and only if $(S_1 \vee S_2)$ is.

The above rules are all very important. We shall see in the subsequent chapters that these rules have many extensions. We now give some examples to show how these rules can be used.

Example 4.20

Show that $S = (P \vee Q \vee \sim R) \wedge (P \vee \sim Q) \wedge \sim P \wedge R \wedge U$ is unsatisfiable.

(1)	$(P \vee Q \vee \sim R) \wedge (P \vee \sim Q) \wedge \sim P \wedge R \wedge U$	
(2)	$(Q \vee \sim R) \wedge (\sim Q) \wedge R \wedge U$	Rule II on $\sim P$
(3)	$\sim R \wedge R \wedge U$	Rule II on $\sim Q$
(4)	$\square \wedge U$	Rule II on $\sim R$.

Since the last formula contains the empty clause $\square$, S is unsatisfiable.

Example 4.21

Show that $S = (P \vee Q) \wedge \sim Q \wedge (\sim P \vee Q \vee \sim R)$ is satisfiable.

(1)	$(P \vee Q) \wedge \sim Q \wedge (\sim P \vee Q \vee \sim R)$	
(2)	$P \wedge (\sim P \vee \sim R)$	Rule II on $\sim Q$
(3)	$\sim R$	Rule II on P
(4)	$\blacksquare$	Rule II on $\sim R$.

The last set is an empty set. Hence S is satisfiable.

Example 4.22

Show that $S = (P \vee \sim Q) \wedge (\sim P \vee Q) \wedge (Q \vee \sim R) \wedge (\sim Q \vee \sim R)$ is satisfiable.

(1)	$(P \vee \sim Q) \wedge (\sim P \vee Q) \wedge (Q \vee \sim R) \wedge (\sim Q \vee \sim R)$	
(2)	$(\sim Q \wedge (Q \vee \sim R) \wedge (\sim Q \vee \sim R))$	
	$\vee (Q \wedge (Q \vee \sim R) \wedge (\sim Q \vee \sim R))$	Rule IV on P
(3)	$\sim R \vee \sim R$	Rule II on $\sim Q$ and Q
(4)	$\blacksquare \vee \blacksquare$	Rule II on $\sim R$.

Since both of the split sets are satisfiable, S is satisfiable.

Example 4.23

Show that $S = (P \vee Q) \wedge (P \vee \sim Q) \wedge (R \vee Q) \wedge (R \vee \sim Q)$ is satisfiable.

(1) $(P \vee Q) \wedge (P \vee \sim Q) \wedge (R \vee Q) \wedge (R \vee \sim Q)$

(2) $(R \vee Q) \wedge (R \vee \sim Q)$ Rule III on P

(3) ■ Rule III on R.

Thus, S is satisfiable.

The above method for testing the unsatisfiability (inconsistency) is more efficient than the multiplication method. This method can be applied to any formula in the propositional logic. That is, first transform the given propositional formulas into a conjunctive normal form, and then apply the above four rules on the conjunctive normal form. The reader is encouraged to apply this method to Example 2.13 of Chapter 2.

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EXERCISES

Section 4.2

1. Find a standard form for each of the following formulas:

   (a) $\sim((\forall x)\, P(x) \rightarrow (\exists y)(\forall z)\, Q(y, z))$

   (b) $(\forall x)\big((\sim E(x, 0) \rightarrow ((\exists y)\,(E(y, g(x)) \wedge (\forall z)\,(E(z, g(x)) \rightarrow E(y, z))))\big)$

   (c) $\sim((\forall x)\, P(x) \rightarrow (\exists y)\, P(y))$.

2. Suppose $(\exists x)(\forall y)\, M[x, y]$ is a prenex normal form of a formula F, where $M[x, y]$ is the matrix that contains only variables x and y. Let f be a function symbol not occurring in $M[x, y]$. Prove that F is valid if and only if $(\exists x)\, M[x, f(x)]$ is valid.
3. Let S_1 and S_2 be standard forms of formulas F_1 and F_2, respectively. If $S_1 = S_2$, is it true that $F_1 = F_2$? Explain.
4. Consider the following statements:

   F_1: Everyone who saves money earns interst.
   F_2: If there is no interest, then nobody saves money.

   Let $S(x, y)$, $M(x)$, $I(x)$, and $E(x, y)$ represent "x saves y," "x is money," "x is interest," and "x earns y," respectively.

   (1) Symbolize F_1 and F_2.
   (2) Find the standard forms of F_1 and $\sim F_2$.

Section 4.3

5. Let $S = \{P(f(x), a, g(f(x), b))\}$.
   (1) Find H_0 and H_1.
   (2) Find all the ground instances of S over H_0.
   (3) Find all the ground instances of S over H_1.
6. Prove Lemma 4.1 of this chapter.
7. Let F denote a formula. Let S be a standard form of $\sim F$. Find the necessary and sufficient condition for F such that the Herbrand universe of S is finite.
8. Consider the following clause C and interpretation I:

$$C: \quad P(x) \vee Q(x, f(x))$$

$$\begin{aligned} I: \quad & \{\sim P(a), \sim P(f(a)), \sim P(f(f(a))), \ldots, \\ & \sim Q(a, a), Q(a, f(a)), \sim Q(a, f(f(a))), \ldots, \\ & \sim Q(f(a), a), Q(f(a), f(a)), \sim Q(f(a), f(f(a))), \ldots\}. \end{aligned}$$

   Does I satisfy C?
9. Consider the following set S of clauses:

$$S = \begin{cases} P(x) \\ Q(f(y)). \end{cases}$$

   Let an interpretation I be defined as below:

$$\begin{aligned} I = \{ & P(a), P(f(a)), P(f(f(a))), \ldots, \\ & Q(a), \sim Q(f(a)), Q(f(f(a))), \ldots\}. \end{aligned}$$

   Does I satisfy S?
10. Consider $S = \{P(x), \sim P(f(y))\}$.
    1. Give H_0, H_1, H_2, and H_3 of S.
    2. Is it possible to find an interpretation that satisfies S? If yes, give one. If no, why?

Section 4.4

11. Let $S = \{P, \sim P \vee Q, \sim Q\}$. Give a closed semantic tree of S.
12. Consider $S = \{P(x), \sim P(x) \vee Q(x, a), \sim Q(y, a)\}$.
    (a) Give the atom set of S.
    (b) Give a complete semantic tree of S.
    (c) Give a closed semantic tree of S.

Section 4.5

13. Consider $S = \{P(x, a, g(x, b)), \sim P(f(y), z, g(f(a), b))\}$. Find an unsatisfiable set S' of ground instances of clauses in S.
14. Let $S = \{P(x), Q(x, f(x)) \vee \sim P(x), \sim Q(g(y), z)\}$. Find an unsatisfiable set S' of ground instances of clauses in S.

Section 4.6

15. Use the rules suggested by Davis and Putnam to prove that the following formulas are unsatisfiable.

    (1) $(\sim P \vee Q) \wedge \sim Q \wedge P$

    (2) $(P \vee Q) \wedge (R \vee Q) \wedge \sim R \wedge \sim Q$

    (3) $(P \vee Q) \wedge (\sim P \vee Q) \wedge (\sim R \vee \sim Q) \wedge (R \vee \sim Q)$.

16. Use the rules suggested by Davis and Putnam to prove that the following formulas are satisfiable.

    (1) $P \wedge Q \wedge R$

    (2) $(P \vee Q) \wedge (\sim P \vee Q) \wedge R$

    (3) $(P \vee Q) \wedge \sim Q$.

Chapter 5

The Resolution Principle

5.1 INTRODUCTION

In the previous chapters, we have considered Herbrand's theorem. Based upon this theorem, we have given a refutation procedure. However, Herbrand's procedure has one major drawback: It requires the generation of sets S_1', S_2',... of ground instances of clauses. For most cases, this sequence grows exponentially. To see this, let us give one simple example. Consider

$$S = \{P(x, g(x), y, h(x, y), z, k(x, y, z)), \sim P(u, v, e(v), w, f(v, w), x)\}.$$

Since

$$\begin{aligned} H_0 &= \{a\}, \\ H_1 &= \{a, g(a), h(a, a), k(a, a, a), e(a), f(a, a)\} \\ &\vdots \end{aligned},$$

the number of elements in S_0', S_1', ... is 2, 1512, ..., respectively. It is interesting to note that the earliest set that is unsatisfiable is S_5'. However, H_5' has of the order of 10^{64} elements. Consequently, S_5' has of the order of 10^{256} elements. With current computer technology, it is impossible even to store S_5' in a computer, not to mention test its unsatisfiability.

In order to avoid the generation of sets of ground instances as required

in Herbrand's procedure, we shall introduce in this chapter the resolution principle due to Robinson [1965]. It can be applied directly to any set S of clauses (not necessarily ground clauses) to test the unsatisfiability of S.

The essential idea of the resolution principle is to check whether S contains the empty clause $\square$. If S contains $\square$, then S is unsatisfiable. If S does not contain $\square$, the next thing to check is whether $\square$ can be derived from S. It will be clear later that, by Herbrand's theorem (version I), checking for the presence of $\square$ is equivalent to counting the number of nodes of a closed semantic tree for S. By Theorem 4.3, S is unsatisfiable if and only if there is a finite closed semantic tree T for S. Clearly S contains $\square$ if and only if T consists of only one node—the root node. If S does not contain $\square$, T must contain more than one node. However, if we can reduce the number of nodes in T to one, eventually $\square$ can be forced to appear. This is what the resolution principle does. Indeed, we can view the resolution principle as an inference rule that can be used to generate new clauses from S. If we put these new clauses into S, some nodes of the original T can be forced to become failure nodes. Thus the number of nodes in T can be reduced and the empty clause $\square$ will be eventually derived.

In this chapter, we shall first consider the resolution principle for the propositional logic. Then we shall extend it to the first-order logic.

5.2 THE RESOLUTION PRINCIPLE FOR THE PROPOSITIONAL LOGIC

The resolution principle is essentially an extension of the *one-literal rule* of Davis and Putnam, given in Section 4.6 of Chapter 4. Consider the following clauses:

C_1: P

C_2: $\sim P \vee Q$.

Using the one-literal rule, from C_1 and C_2 we can obtain a clause

C_3: Q.

What the one-literal rule has required us to do is first to examine whether there is a complementary pair of a literal (for example, P) in C_1 and a literal (for example, $\sim P$) in C_2, then to delete this pair from C_1 and C_2 to obtain clause C_3, which is Q.

Extending the above rule and applying it to any pair of clauses (not necessarily unit clauses), we have the following rule, which is called the *resolution principle: For any two clauses C_1 and C_2, if there is a literal L_1 in C_1 that is complementary to a literal L_2 in C_2, then delete L_1 and L_2 from C_1 and C_2, respectively, and construct the disjunction of the remaining clauses. The constructed clause is a resolvent of C_1 and C_2.*

Example 5.1

Consider the following clauses:

C_1: $P \vee R$

C_2: $\sim P \vee Q$.

Clause C_1 has the literal P, which is complementary to $\sim P$ in C_2. Therefore, by deleting P and $\sim P$ from C_1 and C_2, respectively, and constructing the disjunction of the remaining clauses R and Q, we obtain a resolvent $R \vee Q$.

Example 5.2

Consider clauses

C_1: $\sim P \vee Q \vee R$

C_2: $\sim Q \vee S$.

The resolvent of C_1 and C_2 is $\sim P \vee R \vee S$.

Example 5.3

Consider clauses

C_1: $\sim P \vee Q$

C_2: $\sim P \vee R$.

Since there is no literal in C_1 that is complementary to any literal in C_2, there is no resolvent of C_1 and C_2.

An important property of a resolvent is that any resolvent of two clauses C_1 and C_2 is a logical consequence of C_1 and C_2. This is stated in the following theorem.

Theorem 5.1 *Given two clauses C_1 and C_2, a resolvent C of C_1 and C_2 is a logical consequence of C_1 and C_2.*

Proof Let C_1, C_2, and C be denoted as follows: $C_1 = L \vee C_1'$, $C_2 = \sim L \vee C_2'$, and $C = C_1' \vee C_2'$, where C_1' and C_2' are disjunctions of literals. Suppose C_1 and C_2 are true in an interpretation I. We want to prove that the resolvent C of C_1 and C_2 is also true in I. To prove this, note that either L or $\sim L$ is false in I. Assume L is false in I. Then C_1 must not be a unit clause, for otherwise, C_1 would be false in I. Therefore, C_1' must be true in I. Thus, the resolvent C, that is, $C_1' \vee C_2'$, is true in I. Similarly, we can show that if $\sim L$ is false in I, then C_2' must be true in I. Therefore, $C_1' \vee C_2'$ must be true in I. Q.E.D.

If we have two unit clauses, then the resolvent of them, if there is one, is the empty clause $\square$. More importantly, if a set S of clauses is unsatisfiable, we can use the resolution principle to generate $\square$ from S. This result will be presented as a theorem in Section 5.6 of this chapter. Meanwhile, we define the notion of deduction.

Definition Given a set S of clauses, a (resolution) *deduction* of C from S is a finite sequence $C_1, C_2, \ldots, C_k$ of clauses such that each C_i either is a clause in S or a resolvent of clauses preceding C_i, and $C_k = C$. A deduction of $\square$ from S is called a *refutation*, or a *proof* of S.

We say that a clause C can be *deduced* or *derived* from S if there is a deduction of C from S. We shall now present several examples to show how the resolution principle can be used to prove the unsatisfiability of a set of clauses.

Example 5.4

Consider the set

$$\left.\begin{array}{ll} (1) & \sim P \vee Q \\ (2) & \sim Q \\ (3) & P \end{array}\right\} S.$$

From (1) and (2), we can obtain a resolvent

(4) $\sim P$.

From (4) and (3), we can obtain a resolvent

(5) $\square$.

Since $\square$ is derived from S by resolution, according to Theorem 5.1 $\square$ is a logical consequence of S. However, $\square$ can only be a logical consequence of an unsatisfiable set of clauses. Hence, S is unsatisfiable.

Example 5.5

For the set

$$\left.\begin{array}{ll} (1) & P \vee Q \\ (2) & \sim P \vee Q \\ (3) & P \vee \sim Q \\ (4) & \sim P \vee \sim Q \end{array}\right\} S$$

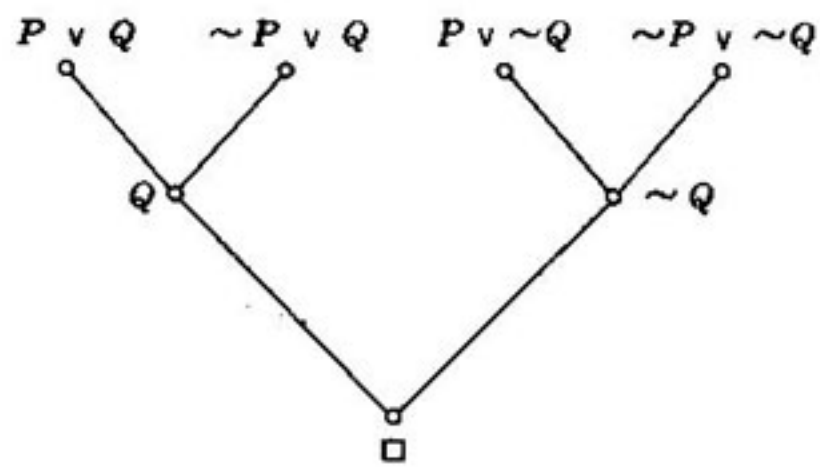

Figure 5.1

we generate the following resolvents

(5) Q from (1) and (2)

(6) $\sim Q$ from (3) and (4)

(7) $\square$ from (5) and (6).

Since $\square$ is derived, S is unsatisfiable. The above deduction can be represented by the tree in Fig. 5.1, called a *deduction tree*.

The resolution principle is a very powerful inference rule. In the sequel, we shall define this principle in the context of the first-order logic. We shall also show that the resolution principle is *complete* in proving the unsatisfiability of a set of clauses. That is, given a set S of clauses, S is unsatisfiable if and only if there is a resolution deduction of the empty clause $\square$ from S. In Section 5.7, we shall apply the resolution principle to several examples so that the reader can appreciate the usefulness of this inference rule.

5.3 SUBSTITUTION AND UNIFICATION

In the last section, we considered the resolution principle for the propositional logic. We shall extend it to the first-order logic in later sections. In Section 5.2, we noted that the most important part of applying the resolution principle is finding a literal in a clause that is complementary to a literal in another clause. For clauses containing no variables, this is very simple. However, for clauses containing variables, it is more complicated. For example, consider the clauses:

C_1: $P(x) \vee Q(x)$

C_2: $\sim P(f(x)) \vee R(x)$.

There is no literal in C_1 that is complementary to any literal in C_2. However, if we substitute $f(a)$ for x in C_1 and a for x in C_2, we obtain

C_1': $P(f(a)) \vee Q(f(a))$

C_2': $\sim P(f(a)) \vee R(a)$.

We know that C_1' and C_2' are ground instances of C_1 and C_2, respectively, and $P(f(a))$ and $\sim P(f(a))$ are complementary to each other. Therefore, from C_1' and C_2', we can obtain a resolvent

C_3': $Q(f(a)) \vee R(a)$.

More generally, if we substitute $f(x)$ for x in C_1, we obtain

C_1^*: $P(f(x)) \vee Q(f(x))$.

Again, C_1^* is an *instance* of C_1. This time the literal $P(f(x))$ in C_1^* is complementary to the literal $\sim P(f(x))$ in C_2. Therefore, we can obtain a resolvent from C_1^* and C_2,

C_3: $Q(f(x)) \vee R(x)$.

C_3' is an instance of clause C_3. By substituting appropriate terms for the variables in C_1 and C_2 as above, we can generate new clauses from C_1 and C_2. Furthermore, clause C_3 is the *most general clause* in the sense that all the other clauses which can be generated by the above process are instances of C_3. C_3 will also be called a resolvent of C_1 and C_2. In the sequel, we shall consider how to generate resolvents from clauses (possibly containing variables). Since obtaining resolvents from clauses frequently requires substitutions for variables, we give the following definitions.

Definition A *substitution* is a finite set of the form $\{t_1/v_1, \ldots, t_n/v_n\}$, where every v_i is a variable, every t_i is a term different from v_i, and no two elements in the set have the same variable after the stroke symbol. When $t_1, \ldots, t_n$ are ground terms, the substitution is called a *ground substitution*. The substitution that consists of no elements is called the *empty substitution* and is denoted by ε. We shall use Greek letters to represent substitutions.

Example 5.6

The following two sets are substitutions:

$$\{f(z)/x, y/z\}, \qquad \{a/x, g(y)/y, f(g(b))/z\}.$$

Definition Let $\theta = \{t_1/v_1, \ldots, t_n/v_n\}$ be a substitution and E be an expression. Then $E\theta$ is an expression obtained from E by replacing simultaneously each occurrence of the variable v_i, $1 \leqslant i \leqslant n$, in E by the term t_i. $E\theta$ is called an *instance* of E. (We note that the definition of an instance is compatible with that of a ground instance of a clause, defined in Chapter 4.)

Example 5.7

Let $\theta = \{a/x, f(b)/y, c/z\}$ and $E = P(x,y,z)$. Then $E\theta = P(a,f(b),c)$.

Definition Let $\theta = \{t_1/x_1, \ldots, t_n/x_n\}$ and $\lambda = \{u_1/y_1, \ldots, u_m/y_m\}$ be two substitutions. Then the *composition* of θ and λ is the substitution, denoted by $\theta \circ \lambda$, that is obtained from the set

$$\{t_1\lambda/x_1, \ldots, t_n\lambda/x_n, u_1/y_1, \ldots, u_m/y_m\}$$

by deleting any element $t_j\lambda/x_j$ for which $t_j\lambda = x_j$, and any element u_i/y_i such that y_i is among $\{x_1, x_2, \ldots, x_n\}$.

Example 5.8

Let

$$\theta = \{t_1/x_1, t_2/x_2\} = \{f(y)/x, z/y\}$$

$$\lambda = \{u_1/y_1, u_2/y_2, u_3/y_3\} = \{a/x, b/y, y/z\}.$$

Then

$$\{t_1\lambda/x_1, t_2\lambda/x_2, u_1/y_1, u_2/y_2, u_3/y_3\} = \{f(b)/x, y/y, a/x, b/y, y/z\}.$$

However, since $t_2\lambda = x_2, t_2\lambda/x_2$, i.e., y/y, should be deleted from the set. In addition, since y_1 and y_2 are among $\{x_1, x_2, x_3\}$, u_1/y_1 and u_2/y_2, that is, a/x and b/y, should be deleted. Thus we obtain

$$\theta \circ \lambda = \{f(b)/x, y/z\}.$$

It should be noted that the composition of substitutions is associative, and that the empty substitution ε is both a left and a right identity. That is, $(\theta \circ \lambda) \circ \mu = \theta \circ (\lambda \circ \mu)$ and $\varepsilon \circ \theta = \theta \circ \varepsilon$ for all θ, λ, and μ.

In the resolution proof procedure, in order to identify a complementary pair of literals, very often we have to unify (match) two or more expressions. That is, we have to find a substitution that can make several expressions identical. Therefore, we now consider the unification of expressions.

Definition A substitution θ is called a *unifier* for a set $\{E_1, \ldots, E_k\}$ if and only if $E_1\theta = E_2\theta = \cdots = E_k\theta$. The set $\{E_1, \ldots, E_k\}$ is said to be *unifiable* if there is a unifier for it.

Definition A unifier σ for a set $\{E_1, \ldots, E_k\}$ of expressions is a *most general unifier* if and only if for each unifier θ for the set there is a substitution λ such that $\theta = \sigma \circ \lambda$.

Example 5.9

The set $\{P(a,y), P(x,f(b))\}$ is unifiable since the substitution $\theta = \{a/x, f(b)/y\}$ is a *unifier* for the set.

5.4 UNIFICATION ALGORITHM

In this section, we shall give a unification algorithm for finding a most general unifier for a finite unifiable set of nonempty expressions. When the set is not unifiable, the algorithm will also detect this fact.

Consider $P(a)$ and $P(x)$. These two expressions are not identical: The disagreement is that a occurs in $P(a)$ but x in $P(x)$. In order to unify $P(a)$ and $P(x)$, we first have to find the disagreement, and then try to eliminate it. For $P(a)$ and $P(x)$, the disagreement is $\{a,x\}$. Since x is a variable, x can be replaced by a, and thus the disagreement can be eliminated. This is the idea upon which the unification algorithm is based.

Definition The disagreement set of a nonempty set W of expressions is obtained by locating the first symbol (counting from the left) at which not all the expressions in W have exactly the same symbol, and then extracting from each expression in W the subexpression that begins with the symbol occupying that position. The set of these respective subexpressions is the *disagreement set* of W.

Example 5.10

If W is $\{P(x,\underline{f(y,z)}), P(x,\underline{a}), P(x,\underline{g(h(k(x)))})\}$, then the first symbol position at which not all atoms in W are exactly the same is the fifth, since they all have the first four symbols $P(x,$ in common. Thus, the disagreement set consists of the respective subexpressions (underlined terms) that begin in symbol position number five, and it is in fact the set $\{f(y,z),a,g(h(k(x)))\}$.

Unification Algorithm

Step 1 Set $k=0$, $W_k = W$, and $\sigma_k = \varepsilon$.

Step 2 If W_k is a singleton, stop; σ_k is a most general unifier for W. Otherwise, find the disagreement set D_k of W_k.

Step 3 If there exist elements v_k and t_k in D_k such that v_k is a variable that does not occur in t_k, go to Step 4. Otherwise, stop; W is not unifiable.

Step 4 Let $\sigma_{k+1} = \sigma_k\{t_k/v_k\}$ and $W_{k+1} = W_k\{t_k/v_k\}$. (Note that $W_{k+1} = W\sigma_{k+1}$.)

Step 5 Set $k=k+1$ and go to Step 2.

Example 5.11

Find a most general unifier for $W = \{P(a,x,f(g(y))), P(z,f(z),f(u))\}$.

1. $\sigma_0 = \varepsilon$ and $W_0 = W$. Since W_0 is not a singleton, σ_0 is not a most general unifier for W.

2. The disagreement set $D_0 = \{a,z\}$. In D_0, there is a variable $v_0 = z$ that does not occur in $t_0 = a$.

3. Let

$$\begin{aligned}\sigma_1 &= \sigma_0 \circ \{t_0/v_0\} = \varepsilon \circ \{a/z\} = \{a/z\},\\ W_1 &= W_0\{t_0/v_0\}\\ &= \{P(a, x, f(g(y))), P(z, f(z), f(u))\}\{a/z\}\\ &= \{P(a, x, f(g(y))), P(a, f(a), f(u))\}.\end{aligned}$$

4. W_1 is not a singleton, hence we find the disagreement set D_1 of W_1:

$$D_1 = \{x, f(a)\}.$$

5. From D_1, we find that $v_1 = x$ and $t_1 = f(a)$.
6. Let

$$\begin{aligned}\sigma_2 &= \sigma_1 \circ \{t_1/v_1\} = \{a/z\} \circ \{f(a)/x\} = \{a/z, f(a)/x\},\\ W_2 &= W_1\{t_1/v_1\}\\ &= \{P(a, x, f(g(y))), P(a, f(a), f(u))\}\{f(a)/x\}\\ &= \{P(a, f(a), f(g(y))), P(a, f(a), f(u))\}.\end{aligned}$$

7. W_2 is not a singleton, hence we find the disagreement set D_2 of W_2: $D_2 = \{g(y), u\}$. From D_2, we find that $v_2 = u$ and $t_2 = g(y)$.
8. Let

$$\begin{aligned}\sigma_3 &= \sigma_2 \circ \{t_2/v_2\} = \{a/z, f(a)/x\} \circ \{g(y)/u\} = \{a/z, f(a)/x, g(y)/u\},\\ W_3 &= W_2\{t_2/v_2\}\\ &= \{P(a, f(a), f(g(y))), P(a, f(a), f(u))\}\{g(y)/u\}\\ &= \{P(a, f(a), f(g(y))), P(a, f(a), f(g(y)))\}\\ &= \{P(a, f(a), f(g(y)))\}.\end{aligned}$$

9. Since W_3 is a singleton, $\sigma_3 = \{a/z, f(a)/x, g(y)/u\}$ is a most general unifier for W.

Example 5.12

Determine whether or not the set $W = \{Q(f(a), g(x)), Q(y, y)\}$ is unifiable.

1. Let $\sigma_0 = \varepsilon$ and $W_0 = W$.
2. W_0 is not a singleton, hence we find the disagreement set D_0 of W_0: $D_0 = \{f(a), y\}$. From D_0, we know that $v_0 = y$ and $t_0 = f(a)$.
3. Let

$$\begin{aligned}\sigma_1 &= \sigma_0 \circ \{t_0/v_0\} = \varepsilon \circ \{f(a)/y\} = \{f(a)/y\},\\ W_1 &= W_0\{t_0/v_0\} = \{Q(f(a), g(x)), Q(y, y)\}\{f(a)/y\}\\ &= \{Q(f(a), g(x)), Q(f(a), f(a))\}.\end{aligned}$$

4. W_1 is not a singleton, hence we find the disagreement set D_1 of W_1: $D_1 = \{g(x), f(a)\}$.

5. However, no element of D_1 is a variable. Hence the unification algorithm is terminated and we conclude that W is not unifiable.

It is noted that the above unification algorithm will always terminate for any finite nonempty set of expressions, for otherwise there would be generated an infinite sequence $W\sigma_0, W\sigma_1, W\sigma_2, \ldots$ of finite nonempty sets of expressions with the property that each successive set contains one less variable than its predecessor (namely, $W\sigma_k$ contains v_k but $W\sigma_{k+1}$ does not). This is impossible since W contains only finitely many distinct variables.

We indicated before that if W is unifiable, the unification algorithm will always find a most general unifier for W. This is proved in the following theorem.

Theorem 5.2 (Unification Theorem) If W is a finite nonempty unifiable set of expressions, then the unification algorithm will always terminate at Step 2, and the last σ_k is a most general unifier for W.

Proof Since W is unifiable, we let θ be any unifier for W. For $k = 0, 1, \ldots$, we show that there is a substitution λ_k such that $\theta = \sigma_k \circ \lambda_k$ by induction on k. For $k = 0$, we let $\lambda_0 = \theta$. Then $\theta = \sigma_0 \circ \lambda_0$, since $\sigma_0 = \varepsilon$. Suppose $\theta = \sigma_k \circ \lambda_k$ holds for $0 \leqslant k \leqslant n$. If $W\sigma_n$ is a singleton, then the unification algorithm terminates at Step 2. Since $\theta = \sigma_n \circ \lambda_n$, σ_n is a most general unifier for W. If $W\sigma_n$ is not a singleton, then the unification algorithm will find the disagreement set D_n of $W\sigma_n$. Because $\theta = \sigma_n \circ \lambda_n$ is a unifier of W, λ_n must unify D_n. However, since D_n is the disagreement set, there must be a variable in D_n. Let t_n be any other element different from v_n. Then, since λ_n unifies D_n, $v_n\lambda_n = t_n\lambda_n$. Now if v_n occurs in t_n, then $v_n\lambda_n$ occurs in $t_n\lambda_n$. However, this is impossible since v_n and t_n are distinct, and $v_n\lambda_n = t_n\lambda_n$. Therefore, v_n does not occur in t_n. Hence the unification algorithm will not terminate at Step 3, but will go to Step 4 to set $\sigma_{n+1} = \sigma_n \circ \{t_n/v_n\}$. Let $\lambda_{n+1} = \lambda_n - \{t_n\lambda_n/v_n\}$. Then, since v_n does not occur in t_n, $t_n\lambda_{n+1} = t_n(\lambda_n - \{t_n\lambda_n/v_n\}) = t_n\lambda_n$. Thus, we have

$$\begin{aligned}
\{t_n/v_n\} \circ \lambda_{n+1} &= \{t_n\lambda_{n+1}/v_n\} \cup \lambda_{n+1} \\
&= \{t_n\lambda_n/v_n\} \cup \lambda_{n+1} \\
&= \{t_n\lambda_n/v_n\} \cup (\lambda_n - \{t_n\lambda_n/v_n\}) \\
&= \lambda_n.
\end{aligned}$$

That is, $\lambda_n = \{t_n/v_n\} \circ \lambda_{n+1}$. Therefore,

$$\theta = \sigma_n \circ \lambda_n = \sigma_n \circ \{t_n/v_n\} \circ \lambda_{n+1} = \sigma_{n+1} \circ \lambda_{n+1}.$$

Hence, for all $k \geqslant 0$, there is a substitution λ_k such that $\theta = \sigma_k \circ \lambda_k$. Since

the unification algorithm must terminate, and since it will not terminate at Step 3, it must terminate at Step 2. Furthermore, since $\theta = \sigma_k \circ \lambda_k$ for all k, the last σ_k is a most general unifier for W. Q.E.D.

5.5 THE RESOLUTION PRINCIPLE FOR THE FIRST-ORDER LOGIC

Having introduced the unification algorithm in the last section, we can now consider the resolution principle for the first-order logic.

Definition If two or more literals (with the same sign) of a clause C have a most general unifier σ, then $C\sigma$ is called a *factor* of C. If $C\sigma$ is a unit clause, it is called a *unit factor* of C.

Example 5.13

Let $C = \underline{P(x)} \vee \underline{P(f(y))} \vee \sim Q(x)$. Then the first and the second literals (underlined) have a most general unifier $\sigma = \{f(y)/x\}$. Hence, $C\sigma = P(f(y)) \vee \sim Q(f(y))$ is a factor of C.

Definition Let C_1 and C_2 be two clauses (called *parent clauses*) with no variables in common. Let L_1 and L_2 be two literals in C_1 and C_2, respectively. If L_1 and $\sim L_2$ have a most general unifier σ, then the clause

$$(C_1\sigma - L_1\sigma) \cup (C_2\sigma - L_2\sigma)$$

is called a *binary resolvent* of C_1 and C_2. The literals L_1 and L_2 are called the *literals resolved upon.*

Example 5.14

Let $C_1 = P(x) \vee Q(x)$ and $C_2 = \sim P(a) \vee R(x)$. Since x appears in both C_1 and C_2, we rename the variable in C_2 and let $C_2 = \sim P(a) \vee R(y)$. Choose $L_1 = P(x)$ and $L_2 = \sim P(a)$. Since $\sim L_2 = P(a)$, L_1 and $\sim L_2$ have the most general unifier $\sigma = \{a/x\}$. Therefore,

$$\begin{aligned}
&(C_1\sigma - L_1\sigma) \cup (C_2\sigma - L_2\sigma) \\
&\quad = (\{P(a), Q(a)\} - \{P(a)\}) \cup (\{\sim P(a), R(y)\} - \{\sim P(a)\}) \\
&\quad = \{Q(a)\} \cup \{R(y)\} = \{Q(a), R(y)\} = Q(a) \vee R(y).
\end{aligned}$$

Thus $Q(a) \vee R(y)$ is a binary resolvent of C_1 and C_2. $P(x)$ and $\sim P(a)$ are the literals resolved upon.

Definition A *resolvent* of (parent) clauses C_1 and C_2 is one of the following binary resolvents:

1. a binary resolvent of C_1 and C_2,
2. a binary resolvent of C_1 and a factor of C_2,
3. a binary resolvent of a factor of C_1 and C_2,
4. a binary resolvent of a factor of C_1 and a factor of C_2.

Example 5.15

Let $C_1 = P(x) \vee P(f(y)) \vee R(g(y))$ and $C_2 = \sim P(f(g(a))) \vee Q(b)$. A factor of C_1 is $C_1' = P(f(y)) \vee R(g(y))$. A binary resolvent of C_1' and C_2 is $R(g(g(a))) \vee Q(b)$. Therefore, $R(g(g(a))) \vee Q(b)$ is a resolvent of C_1 and C_2.

The resolution principle, or resolution for short, is an inference rule that generates resolvents from a set of clauses. This rule was introduced in 1965 by Robinson. It is more efficient than the earlier proof procedures such as the direct implementation of Herbrand's theorem used by Gilmore and Davis and Putnam. Furthermore, resolution is *complete*. That is, it will always generate the empty clause $\square$ from an unsatisfiable set of clauses. Before we prove this statement, we first give an example in plane geometry.

Example 5.16

Show that alternate interior angles formed by a diagonal of a trapezoid are equal.

To prove this theorem, we first axiomatize it. Let $T(x,y,u,v)$ mean that $xyuv$ is a trapezoid with upper-left vertex x, upper-right vertex y, lower-right vertex u, and lower-left vertex v; let $P(x,y,u,v)$ mean that the line segment xy is parallel to the line segment uv; and let $E(x,y,z,u,v,w)$ mean that the angle xyz is equal to the angle uvw. Then we have the following axioms:

A_1:	$(\forall x)(\forall y)(\forall u)(\forall v)[T(x,y,u,v) \rightarrow P(x,y,u,v)]$	definition of a trapezoid
A_2:	$(\forall x)(\forall y)(\forall u)(\forall v)[P(x,y,u,v) \rightarrow E(x,y,v,u,v,y)]$	alternate interior angles of parallel lines are equal
A_3:	$T(a,b,c,d)$	given in Fig. 5.2.

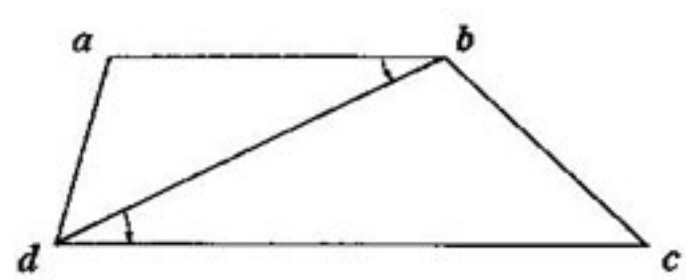

Figure 5.2

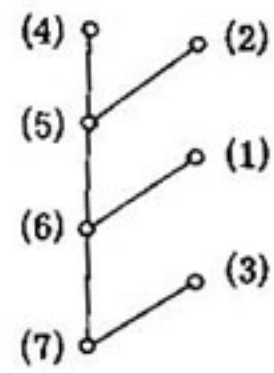

Figure 5.3

From these axioms, we should be able to conclude that $E(a,b,d,c,d,b)$ is true, that is,

$$A_1 \wedge A_2 \wedge A_3 \rightarrow E(a,b,d,c,d,b)$$

is a valid formula. Since we want to prove this by refutation, we negate the conclusion and prove that

$$A_1 \wedge A_2 \wedge A_3 \wedge \sim E(a,b,d,c,d,b)$$

is unsatisfiable. To do this, we transform it into a standard form as follows:

$$S = \{\sim T(x,y,u,v) \vee P(x,y,u,v), \sim P(x,y,u,v) \vee E(x,y,v,u,v,y),$$
$$T(a,b,c,d), \sim E(a,b,d,c,d,b)\}.$$

The above standard form S is a set of four clauses. We now show that the set S is unsatisfiable by resolution:

(1)	$\sim T(x,y,u,v) \vee P(x,y,u,v)$	a clause in S
(2)	$\sim P(x,y,u,v) \vee E(x,y,v,u,v,y)$	a clause in S
(3)	$T(a,b,c,d)$	a clause in S
(4)	$\sim E(a,b,d,c,d,b)$	a clause in S
(5)	$\sim P(a,b,c,d)$	a resolvent of (2) and (4)
(6)	$\sim T(a,b,c,d)$	a resolvent of (1) and (5)
(7)	$\square$	a resolvent of (3) and (6).

Since the last clause is the empty clause that is derived from S, we conclude that S is unsatisfiable. The above proof steps can be easily represented by a tree shown in Fig. 5.3.

The tree in Fig. 5.3 is called a deduction tree. That is, a *deduction tree* from a set S of clauses is an (upward) tree, to each initial node of which is attached a clause in S, and to each noninitial node of which is attached a resolvent of clauses attached to its immediate predecessor nodes. We call

the deduction tree a *deduction tree of R* if R is the clause attached to the root node of the deduction tree. Since a deduction tree is merely the tree representation of a deduction, in the sequel we shall use deduction and deduction tree interchangeably.

5.6 COMPLETENESS OF THE RESOLUTION PRINCIPLE

In Chapter 4, we presented the concept of semantic trees to prove Herbrand's theorem. In this section, we shall use the semantic tree to prove the completeness of the resolution principle. In fact, there is a close relationship between the semantic tree and resolution, as is demonstrated by the following example.

Example 5.17

Consider the following set S of clauses:

(1) P

(2) $\sim P \vee Q$

(3) $\sim P \vee \sim Q$.

The atom set of S is $\{P,Q\}$. Let T be a complete semantic tree as shown in Fig. 5.4a. T has a closed semantic tree T' shown in Fig. 5.4b. Node (2) in Fig. 5.4b is an inference node. Its two successors, nodes (4) and (5), are failure nodes. The clauses falsified at nodes (4) and (5) are $\sim P \vee \sim Q$ and $\sim P \vee Q$, respectively. As can be seen, these two clauses must have a complementary pair of literals, and therefore can be resolved. Resolving

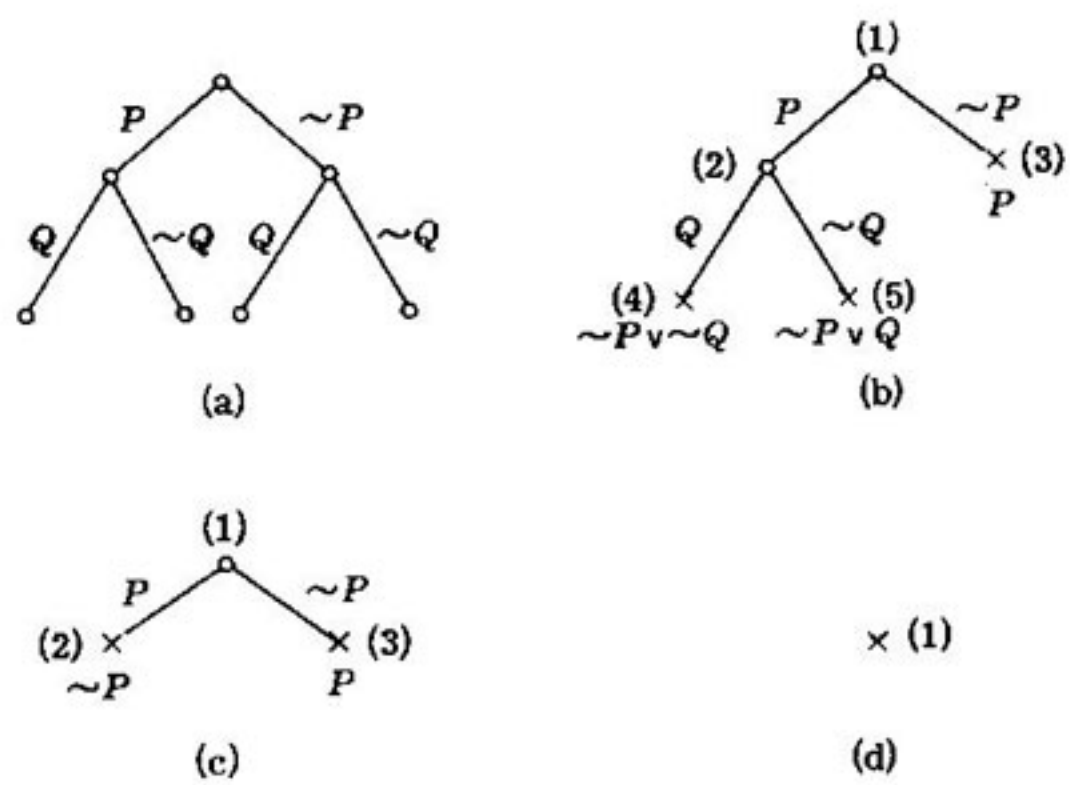

Figure 5.4 (a) T. (b) T'. $(\sim P \vee Q)$ and $(\sim P \vee \sim Q)$ can be resolved to give $\sim P$. (c) T''. P and $\sim P$ can be resolved to give $\square$. (d) T'''.

$\sim P \vee \sim Q$ and $\sim P \vee Q$, we obtain $\sim P$. Note that $\sim P$ is falsified by the partial interpretation corresponding to node (2). If we put $\sim P$ into S, then we can have a closed semantic tree T'' for $S \cup \{\sim P\}$, as shown in Fig. 5.4c. In Fig. 5.4c, node (1) is an inference node. This time we can obtain $\square$ by resolving P and $\sim P$. Putting $\square$ into $S \cup \{\sim P\}$, we obtain the closed semantic tree T''' for $S \cup \{\sim P\} \cup \{\square\}$ as shown in Fig. 5.4d. The above "collapsing" of the semantic tree actually corresponds to a resolution proof as follows:

(4) $\sim P$ a resolvent of (2) and (3)

(5) $\square$ a resolvent of (4) and (1).

In the following, we shall use the concept developed above to prove the completeness of the resolution principle. That is, after constructing a closed semantic tree for an unsatisfiable set of clauses, we can gradually force the tree to collapse and at the same time obtain a resolution proof. Before proving the completeness theorem, we prove a lemma called the *lifting lemma*.

Lemma 5.1 (Lifting Lemma) If C_1' and C_2' are instances of C_1 and C_2, respectively, and if C' is a resolvent of C_1' and C_2', then there is a resolvent C of C_1 and C_2 such that C' is an instance of C.

Proof We rename, if necessary, the variables in C_1 and C_2 so that variables in C_1 are all different from those in C_2. Let L_1' and L_2' be the literals resolved upon, and let

$$C' = (C_1'\gamma - L_1'\gamma) \cup (C_2'\gamma - L_2'\gamma),$$

where γ is a most general unifier of L_1' and $\sim L_2'$. Since C_1' and C_2' are instances of C_1 and C_2, respectively, there is a substitution θ such that $C_1' = C_1\theta$ and $C_2' = C_2\theta$. Let $L_i^1, \ldots, L_i^{r_i}$ be the literals in C_i corresponding to L_i' (i.e., $L_i^1\theta = \cdots = L_i^{r_i}\theta = L_i'$), $i = 1,2$. If $r_i > 1$, obtain a most general unifier λ_i for $\{L_i^1, \ldots, L_i^{r_i}\}$ and let $L_i = L_i^1\lambda_i$, $i = 1,2$. (Note that $L_i^1\lambda_i, \ldots, L_i^{r_i}\lambda_i$ are the same, since λ_i is a most general unifier.) Then L_i is a literal in the factor $C_i\lambda_i$ of C_i. If $r_i = 1$, let $\lambda_i = \varepsilon$ and $L_i = L_i^1\lambda_i$. Let $\lambda = \lambda_1 \cup \lambda_2$. Thus, clearly L_i' is an instance of L_i. Since L_i' and $\sim L_2'$ are unifiable, L_1 and $\sim L_2$ are unifiable. Let σ be a most general unifier of L_1 and $\sim L_2$. Let

$$\begin{aligned} C &= ((C_1\lambda)\sigma - L_1\sigma) \cup ((C_2\lambda)\sigma - L_2\sigma) \\ &= ((C_1\lambda)\sigma - (\{L_1^1, \ldots, L_1^{r_1}\}\lambda)\sigma) \cup ((C_2\lambda)\sigma - (\{L_2^1, \ldots, L_2^{r_2}\}\lambda)\sigma) \\ &= (C_1(\lambda \circ \sigma) - \{L_1^1, \ldots, L_1^{r_1}\}(\lambda \circ \sigma)) \cup (C_2(\lambda \circ \sigma) - \{L_2^1, \ldots, L_2^{r_2}\}(\lambda \circ \sigma)). \end{aligned}$$

C is a resolvent of C_1 and C_2. Clearly, C' is an instance of C since

$$\begin{aligned} C' &= (C_1'\gamma - L_1'\gamma) \cup (C_2'\gamma - L_2'\gamma) \\ &= ((C_1\theta)\gamma - (\{L_1^1, \ldots, L_1^{r_1}\}\theta)\gamma) \cup ((C_2\theta)\gamma - (\{L_2^1, \ldots, L_2^{r_2}\}\theta)\gamma) \\ &= (C_1(\theta \circ \gamma) - \{L_1^1, \ldots, L_1^{r_1}\}(\theta \circ \gamma)) \cup (C_2(\theta \circ \gamma) - \{L_2^1, \ldots, L_2^{r_2}\}(\theta \circ \gamma)) \end{aligned}$$

and $\lambda \circ \sigma$ is more general than $\theta \circ \gamma$. Thus we complete the proof of this lemma.

Theorem 5.3 (Completeness of the Resolution Principle) A set S of clauses is unsatisfiable if and only if there is a deduction of the empty clause $\square$ from S.

Proof ($\Rightarrow$) Suppose S is unsatisfiable. Let $A = \{A_1, A_2, A_3, \ldots\}$ be the atom set of S. Let T be a complete semantic tree as shown in Fig. 5.5. By Herbrand's theorem (version I), T has a finite closed semantic tree T'. If T' consists of only one (root) node, then $\square$ must be in S, for no other clause can be falsified at the root of a semantic tree. In this case, the theorem is obviously true. Assume T' consists of more than one node. Then T' has at least one inference node, for, if it did not, then every node would have at least one nonfailure descendent. We could then find an infinite branch through T', violating the fact that T' is a finite closed semantic tree. Let N be an inference node in T'. Let N_1 and N_2 be the failure nodes immediately below N. Let

$$\begin{aligned} I(N) &= \{m_1, m_2, \ldots, m_n\}, \\ I(N_1) &= \{m_1, m_2, \ldots, m_n, m_{n+1}\}, \\ I(N_2) &= \{m_1, m_2, \ldots, m_n, \sim m_{n+1}\}. \end{aligned}$$

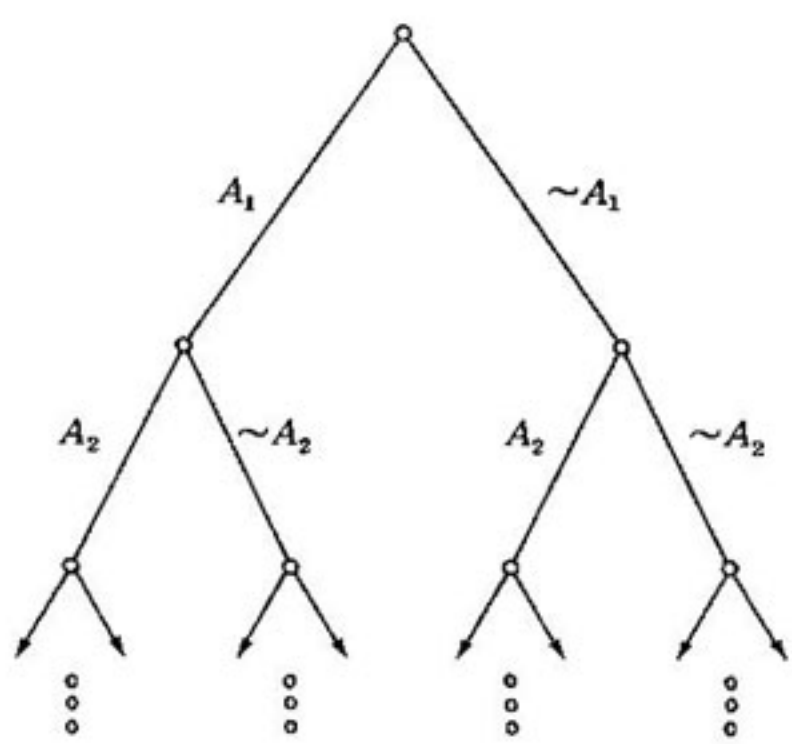

Figure 5.5

Since N_1 and N_2 are failure nodes but N is not a failure node, there must exist two ground instances C_1' and C_2' of clauses C_1 and C_2 such that C_1' and C_2' are false in $I(N_1)$ and $I(N_2)$, respectively, but both C_1' and C_2' are not falsified by $I(N)$. Therefore, C_1' must contain $\sim m_{n+1}$ and C_2' must contain m_{n+1}. Let $L_1' = \sim m_{n+1}$ and $L_2' = m_{n+1}$. By resolving upon the literals L_1' and L_2', we can obtain a resolvent C' of C_1' and C_2', namely,

$$C' = (C_1' - L_1') \cup (C_2' - L_2').$$

C' must be false in $I(N)$ since both $(C_1' - L_1')$ and $(C_2' - L_2')$ are false in $I(N)$. By Lemma 5.1, there is a resolvent C of C_1 and C_2 such that C' is a ground instance of C. Let T'' be the closed semantic tree for $(S \cup \{C\})$, obtained from T' by deleting any node or link that is below the first node where the resolvent C' is falsified. Clearly, the number of nodes in T'' is fewer than that in T'. Applying the above process on T'' again, we can obtain another resolvent of clauses in $(S \cup \{C\})$. Putting this resolvent into $(S \cup \{C\})$, we can get another smaller closed semantic tree. This process is repeated again and again until the closed semantic tree that consists of only the root node is generated. This is possible only when $\square$ is derived. Therefore, there is a deduction of $\square$ from S.

($\Leftarrow$) Conversely, suppose there is a deduction of $\square$ from S. Let R_1, $R_2, \ldots, R_k$ be the resolvents in the deduction. Assume S is satisfiable. Then there is a model M of S. However, if a model satisfies clauses C_1 and C_2, it must also satisfy any resolvent of C_1 and C_2 (Theorem 5.1). Therefore, M satisfies $R_1, R_2, \ldots, R_k$. However, this is impossible because one of these resolvents is $\square$. Hence, S must be unsatisfiable. Q.E.D.

5.7 EXAMPLES USING THE RESOLUTION PRINCIPLE

In the previous chapters, we have used many methods to prove the inconsistency of formulas. In this section we shall use examples to demonstrate how resolution can be efficiently used to prove theorems.

Example 5.18

Let us reconsider Example 2.10. We have four statements:

(1′) $P \rightarrow S$

(2′) $S \rightarrow U$

(3′) P

(4′) U.

To show that (4′) follows from (1′), (2′), and (3′), we first transform all these statements into standard forms. Thus we have

(1) $\sim P \vee S$

(2) $\sim S \vee U$

(3) P

(4) U.

We prove that U is a logical consequence of (1), (2), and (3) by refutation. We negate (4) and obtain the following proof:

(1)	$\sim P \vee S$	
(2)	$\sim S \vee U$	
(3)	P	
(4)	$\sim U$	negation of conclusion
(5)	S	a resolvent of (3) and (1)
(6)	U	a resolvent of (5) and (2)
(7)	$\square$	a resolvent of (6) and (4).

Example 5.19

Reconsider Example 2.12 in Chapter 2. We have three formulas F_1, F_2, and F_3. We want to prove that $\sim Q$ is a logical consequence of $F_1 \wedge F_2 \wedge F_3$. We negate $\sim Q$ and transform $F_1 \wedge F_2 \wedge F_3 \wedge Q$ into a standard form. This gives rise the following set of clauses:

(1)	$\sim P \vee \sim Q \vee R$	from F_1
(2)	$\sim P \vee \sim Q \vee S$	
(3)	P	from F_2
(4)	$\sim S$	from F_3
(5)	Q	negation of conclusion.

Using resolution, we obtain the following proof:

(6)	$\sim Q \vee S$	a resolvent of (3) and (2)
(7)	S	a resolvent of (6) and (5)
(8)	$\square$	a resolvent of (7) and (4).

Example 5.20

Consider the following set of formulas:

F_1: $(\forall x)(C(x) \rightarrow (W(x) \wedge R(x)))$

F_2: $(\exists x)(C(x) \wedge O(x))$

G: $(\exists x)(O(x) \wedge R(x))$.

Our problem is to show that G is a logical consequence of F_1 and F_2. We transform F_1, F_2, and $\sim G$ into standard forms and obtain the following five clauses:

(1)	$\sim C(x) \vee W(x)$	from F_1
(2)	$\sim C(x) \vee R(x)$	
(3)	$C(a)$	from F_2
(4)	$O(a)$	
(5)	$\sim O(x) \vee \sim R(x)$	from $\sim G$.

The above set of the clauses is unsatisfiable. This can be proved by resolution as follows:

(6)	$R(a)$	a resolvent of (3) and (2)
(7)	$\sim R(a)$	a resolvent of (5) and (4)
(8)	$\square$	a resolvent of (7) and (6).

Therefore, G is a logical consequence of F_1 and F_2.

Example 5.21

Reconsider Example 3.15. We have

F_1: $(\exists x)(P(x) \wedge (\forall y)(D(y) \rightarrow L(x,y)))$

F_2: $(\forall x)(P(x) \rightarrow (\forall y)(Q(y) \rightarrow \sim L(x,y)))$

G: $(\forall x)(D(x) \rightarrow \sim Q(x))$.

As usual, F_1, F_2, and $\sim G$ are transformed into the following clauses:

(1)	$P(a)$	from F_1
(2)	$\sim D(y) \vee L(a, y)$	
(3)	$\sim P(x) \vee \sim Q(y) \vee \sim L(x, y)$	from F_2
(4)	$D(b)$	from $\sim G$.
(5)	$Q(b)$	

Using resolution, we obtain the following proof:

(6)	$L(a,b)$	a resolvent of (4) and (2)
(7)	$\sim Q(y) \vee \sim L(a,y)$	a resolvent of (3) and (1)
(8)	$\sim L(a,b)$	a resolvent of (5) and (7)
(9)	$\square$	a resolvent of (6) and (8).

The resolution principle is quite natural. Let us try to translate the above proof into English:

(a) From F_1, we can assume that there is a patient a who likes every doctor (clauses (1) and (2)).
(b) Let us assume that the conclusion is wrong. That is, assume that b is both a doctor and a quack (clauses (4) and (5)).
(c) Since the patient a likes every doctor, a likes b (clause (6)).
(d) Since a is a patient, a does not like any quack (clause (7)).
(e) However, b is a quack. Therefore, a does not like b ((clause (8)).
(f) This is impossible because of (c). Thus, we complete the proof.

Example 5.22

Premises: The custom officials searched everyone who entered this country who was not a VIP. Some of the drug pushers entered this country and they were only searched by drug pushers. No drug pusher was a VIP.

Conclusion: Some of the officials were drug pushers.

Let $E(x)$ mean "x entered this country," $V(x)$ mean "x was a VIP," $S(x,y)$ mean "y searched x," $C(x)$ mean "x was a custom official," and $P(x)$ mean "x was a drug pusher."

The premises are represented by the following formulas:

$$(\forall x)(E(x) \wedge \sim V(x) \rightarrow (\exists y)(S(x,y) \wedge C(y)))$$

$$(\exists x)(P(x) \wedge E(x) \wedge (\forall y)(S(x,y) \rightarrow P(y)))$$

$$(\forall x)(P(x) \rightarrow \sim V(x))$$

and the conclusion is

$$(\exists x)(P(x) \wedge C(x)).$$

Transforming premises into clauses, we obtain

(1) $\sim E(x) \vee V(x) \vee S(x, f(x))$

(2) $\sim E(x) \vee V(x) \vee C(f(x))$

(3) $P(a)$

(4) $E(a)$

(5) $\sim S(a,y) \vee P(y)$

(6) $\sim P(x) \vee \sim V(x)$.

The negation of the conclusion is

(7) $\sim P(x) \vee \sim C(x)$.

The resolution proof is as follows:

(8)	$\sim V(a)$	from (3) and (6)
(9)	$V(a) \vee C(f(a))$	from (2) and (4)
(10)	$C(f(a))$	from (8) and (9)
(11)	$V(a) \vee S(a, f(a))$	from (1) and (4)
(12)	$S(a, f(a))$	from (8) and (11)
(13)	$P(f(a))$	from (12) and (5)
(14)	$\sim C(f(a))$	from (13) and (7)
(15)	$\square$	from (10) and (14).

Thus, we have established the conclusion.

Example 5.23

The premise is that everyone who saves money earns interest. The conclusion is that if there is no interest, then nobody saves money. Let $S(x, y)$, $M(x)$, $I(x)$, and $E(x, y)$ denote "x saves y," "x is money," "x is interest," and "x earns y," respectively. Then the premise is symbolized as

$$(\forall x)((\exists y)(S(x, y) \wedge M(y)) \rightarrow (\exists y)(I(y) \wedge E(x, y)))$$

and the conclusion is

$$\sim(\exists x) I(x) \rightarrow (\forall x)(\forall y)(S(x, y) \rightarrow \sim M(y)).$$

Translating the premise into clauses, we have

(1) $\sim S(x, y) \vee \sim M(y) \vee I(f(x))$

(2) $\sim S(x, y) \vee \sim M(y) \vee E(x, f(x))$.

The negation of the conclusion is

(3) $\sim I(z)$

(4) $S(a, b)$

(5) $M(b)$.

The proof given in the following is indeed very simple:

(6)	$\sim S(x, y) \vee \sim M(y)$	from (3) and (1)
(7)	$\sim M(b)$	from (6) and (4)
(8)	$\square$	from (7) and (5).

Thus the conclusion is established.

Example 5.24

Premise: Students are citizens.

Conclusion: Students' votes are citizens' votes.

Let $S(x)$, $C(x)$, and $V(x,y)$ denote "x is a student," "x is a citizen," and "x is a vote of y," respectively. Then the premise and the conclusion are symbolized as follows:

$(\forall y)(S(y) \rightarrow C(y))$ (premise)

$(\forall x)((\exists y)(S(y) \wedge V(x, y)) \rightarrow (\exists z)(C(z) \wedge V(x, z)))$ (conclusion).

A standard form of the premise is

(1) $\sim S(y) \vee C(y)$.

Since

$$\begin{aligned}
&\sim((\forall x)((\exists y)(S(y) \wedge V(x,y)) \rightarrow (\exists z)(C(z) \wedge V(x,z))))\\
&\quad = \sim((\forall x)((\forall y)(\sim S(y) \vee \sim V(x,y)) \vee (\exists z)(C(z) \wedge V(x,z))))\\
&\quad = \sim((\forall x)(\forall y)(\exists z)(\sim S(y) \vee \sim V(x,y) \vee (C(z) \wedge V(x,z))))\\
&\quad = (\exists x)(\exists y)(\forall z)(S(y) \wedge V(x,y) \wedge (\sim C(z) \vee \sim V(x,z)))
\end{aligned}$$

we have three clauses for the negation of the conclusion,

(2) $S(b)$

(3) $V(a,b)$

(4) $\sim C(z) \vee \sim V(a,z)$.

The proof proceeds as follows:

(5) $C(b)$ from (1) and (2)

(6) $\sim V(a,b)$ from (5) and (4)

(7) $\square$ from (6) and (3).

Again, although the above proof is quite mechanical, it is actually very natural. Indeed, we can translate the above proof into English as follows: Let us assume that b is a student, a is b's vote, and a is not a vote of any citizen. Since b is a student, b must be a citizen. Furthermore, a must not be a vote of b because b is a citizen. This is impossible. Thus we complete the proof.

5.8 DELETION STRATEGY

In the previous sections, we have shown that resolution is complete. Resolution is also more efficient than the earlier methods such as Herbrand's procedure used by Gilmore. However, unlimited applications of resolution may cause many irrelevant and redundant clauses to be generated. To see this, let us consider a simple example. Suppose we want to show, by resolution, that the set $S = \{P \vee Q, \sim P \vee Q, P \vee \sim Q, \sim P \vee \sim Q\}$ is unsatisfiable. A straightforward way of carrying out resolution on the set S is to compute all resolvents of pairs of clauses in S, add these resolvents to the set S, compute all further resolvents, and repeat this process until the empty clause $\square$ is found. That is, we generate the sequences, $S^0, S^1, S^2, \ldots$, where

$$S^0 = S,$$

$$S^n = \{\text{resolvents of } C_1 \text{ and } C_2 \mid C_1 \in (S^0 \cup \cdots \cup S^{n-1}) \text{ and } C_2 \in S^{n-1}\},$$
$$n = 1, 2, 3, \ldots.$$

This procedure is called the *level-saturation (resolution) method.* To program this method on computers, we may first list clauses of $(S^0 \cup \cdots \cup S^{n-1})$ in order, and then compute resolvents by comparing every clause $C_1 \in (S^0 \cup \cdots \cup S^{n-1})$ with a clause $C_2 \in S^{n-1}$ that is listed after C_1. When a resolvent is computed, it is appended to the end of the list so far generated. This is what we do in generating the following list of clauses:

S^0:	(1)	$P \vee Q$	S
	(2)	$\sim P \vee Q$	
	(3)	$P \vee \sim Q$	
	(4)	$\sim P \vee \sim Q$	
S^1:	(5)	Q	from (1) and (2)
	(6)	P	from (1) and (3)
	(7)	$Q \vee \sim Q$	from (1) and (4)
	(8)	$P \vee \sim P$	from (1) and (4)
	(9)	$Q \vee \sim Q$	from (2) and (3)
	(10)	$P \vee \sim P$	from (2) and (3)
	(11)	$\sim P$	from (2) and (4)
	(12)	$\sim Q$	from (3) and (4)

S^2:	(13)	$P \vee Q$	from (1) and (7)
	(14)	$P \vee Q$	from (1) and (8)
	(15)	$P \vee Q$	from (1) and (9)
	(16)	$P \vee Q$	from (1) and (10)
	(17)	Q	from (1) and (11)
	(18)	P	from (1) and (12)
	(19)	Q	from (2) and (6)
	(20)	$\sim P \vee Q$	from (2) and (7)
	(21)	$\sim P \vee Q$	from (2) and (8)
	(22)	$\sim P \vee Q$	from (2) and (9)
	(23)	$\sim P \vee Q$	from (2) and (10)
	(24)	$\sim P$	from (2) and (12)
	(25)	P	from (3) and (5)
	(26)	$P \vee \sim Q$	from (3) and (7)
	(27)	$P \vee \sim Q$	from (3) and (8)
	(28)	$P \vee \sim Q$	from (3) and (9)
	(29)	$P \vee \sim Q$	from (3) and (10)
	(30)	$\sim Q$	from (3) and (11)
	(31)	$\sim P$	from (4) and (5)
	(32)	$\sim Q$	from (4) and (6)
	(33)	$\sim P \vee \sim Q$	from (4) and (7)
	(34)	$\sim P \vee \sim Q$	from (4) and (8)
	(35)	$\sim P \vee \sim Q$	from (4) and (9)
	(36)	$\sim P \vee \sim Q$	from (4) and (10)
	(37)	Q	from (5) and (7)
	(38)	Q	from (5) and (9)
	(39)	$\square$	from (5) and (12).

Many irrelevant and redundant clauses have been generated. For example, (7), (8), (9), and (10) are tautologies. Since a tautology is true in any interpretation, if we delete a tautology from an unsatisfiable set of clauses, the remaining set must still be unsatisfiable. Therefore, a tautology is an irrelevant clause and should not be generated. If one is generated, except in a very few cases, it should be deleted. Otherwise, it may interact with other clauses and produce other unwanted redundant clauses. For example, (13)–(16), (20)–(23), (26)–(29), (33)–(36), and (37)–(38) are all such clauses. Furthermore, the clauses P, Q, $\sim P$, and $\sim Q$ are repeatedly generated. Actually, to get a proof for S, we need only generate clauses (5), (12), and (39). To solve this redundancy problem, we shall consider in this section a deletion strategy. Before we describe it, we consider the following definition.

Definition A clause C *subsumes* a clause D if and only if there is a substitution σ such that $C\sigma \subseteq D$. D is called a *subsumed* clause.

Example 5.25

Let $C = P(x)$ and $D = P(a) \vee Q(a)$. If $\sigma = \{a/x\}$, then $C\sigma = P(a)$. Since $C\sigma \subseteq D$, C subsumes D.

It is noted that if D is identical to C, or D is an instance of C, then D is subsumed by C. *The deletion strategy is the deletion of any tautology and any subsumed clause whenever possible.* The deletion strategy is a generalization of Davis and Putnam's tautology rule given in Section 4.6. The completeness of the deletion strategy depends upon how tautologies and subsumed clauses are deleted. (See [Kowalski, 1970b].) The deletion strategy will be complete if it is used with the level-saturation method in the following way: First, list clauses of $(S^0 \cup \cdots \cup S^{n-1})$ in order; then compute resolvents by comparing every clause $C_1 \in (S^0 \cup \cdots \cup S^{n-1})$ with a clause $C_2 \in S^{n-1}$ that is listed after C_1. When a resolvent is computed, it is appended to the end of the list so far generated if it is neither a tautology nor subsumed by any clause in the list. Otherwise, it is deleted. The proof of the completeness of this deletion strategy is left as an exercise to the reader. We apply this deletion strategy to the above example and obtain the list as follows:

$$S^0: \left.\begin{array}{ll} (1) & P \vee Q \\ (2) & \sim P \vee Q \\ (3) & P \vee \sim Q \\ (4) & \sim P \vee \sim Q \end{array}\right\} S$$

S^1:	(5)	Q	from (1) and (2)
	(6)	P	from (1) and (3)
	(7)	$\sim P$	from (2) and (4)
	(8)	$\sim Q$	from (3) and (4)
S^2:	(9)	$\square$	from (5) and (8).

We note that this list is much shorter than the list generated before. Therefore, the deletion strategy can improve the efficiency of the resolution principle.

In order to use the deletion strategy, we must be able to decide whether a clause is a tautology and whether a clause subsumes another clause. It is very easy to check whether a clause is a tautology by testing whether a complementary pair is present in the clause. However, the subsumption test is not so simple. We now describe an algorithm for this test.

Let C and D be clauses. Let $\theta = \{a_1/x_1, \ldots, a_n/x_n\}$, where $x_1, \ldots, x_n$ are variables occurring in D, and $a_1, \ldots, a_n$ are new distinct constants not occurring in C or D. Suppose $D = L_1 \vee L_2 \vee \cdots \vee L_m$. Then $D\theta = L_1\theta \vee L_2\theta \vee \cdots \vee L_m\theta$. Note that $D\theta$ is a ground clause. $\sim D\theta = \sim L_1\theta \wedge \cdots \wedge \sim L_m\theta$. The following algorithm tests whether or not C subsumes D.

Subsumption Algorithm

Step 1 Let $W = \{\sim L_1\theta, \ldots, \sim L_m\theta\}$.

Step 2 Set $k = 0$ and $U^0 = \{C\}$.

Step 3 If U^k contains $\square$, terminate; C subsumes D. Otherwise, let $U^{k+1} = \{\text{resolvents of } C_1 \text{ and } C_2 \mid C_1 \in U^k \text{ and } C_2 \in W\}$.

Step 4 If U^{k+1} is empty, terminate; C does not subsume D. Otherwise, set $k = k+1$ and go to Step 3.

In the above algorithm, we note that each clause in U^{k+1} is smaller by one literal than the clause in U^k from which it was obtained. Therefore, the sequence $U^0, U^1, \ldots,$ must eventually contain a set that contains $\square$ or is empty. The subsumption algorithm is correct. That is, C subsumes D if and only if the subsumption algorithm terminates in Step 3. This is proved as follows: If C subsumes D, there is a substitution σ such that $C\sigma \subseteq D$. Hence $C(\sigma \circ \theta) \subseteq D\theta$. Thus literals in $C(\sigma \circ \theta)$ can be resolved by using unit ground clauses in W. However $C(\sigma \circ \theta)$ is an instance of C. Therefore, literals in C can be resolved away by using unit clauses in W. This means that we will eventually find a U^k that contains $\square$. Therefore, the algorithm will terminate at Step 3. Conversely, if the algorithm terminates at Step 3, then we obtain

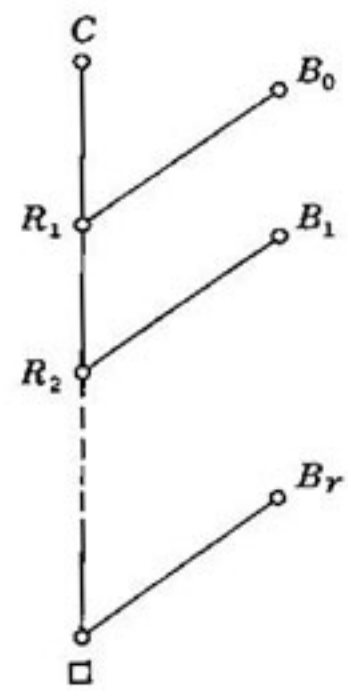

Figure 5.6

a refutation shown in Fig. 5.6, where $B_0, \ldots, B_r$ are unit clauses from W. R_1 is a resolvent of C and B_0, and R_i is a resolvent of R_{i-1} and B_{i-1}, for $i = 2, \ldots, r$.

Let σ_0 be the most general unifier obtained when resolving C and B_0, and let σ_i be the most general unifier obtained when resolving R_i and B_i, $i = 1, \ldots, r$. Then $C(\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_r) = \{\sim B_0, \sim B_1, \ldots, \sim B_r\} \subseteq D\theta$. Let $\lambda = \sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_r$. Then $C\lambda \subseteq D\theta$. Let σ be the substitution obtained from λ by the replacement, in each component of λ, of a_i by x_i, for $i = 1, \ldots, n$. Then $C\sigma \subseteq D$. Therefore, C subsumes D. Q.E.D.

Example 5.26

Let $C = \sim P(x) \vee Q(f(x), a)$ and $D = \sim P(h(y)) \vee Q(f(h(y)), a) \vee \sim P(z)$. Decide whether C subsumes D.

1. y and z are variables in D. Let $\theta = \{b/y, c/z\}$. Note that b and c are not in C or D. Then $D\theta = \sim P(h(b)) \vee Q(f(h(b)), a) \vee \sim P(c)$. Therefore, $\sim D\theta = P(h(b)) \wedge \sim Q(f(h(b)), a) \wedge P(c)$. Therefore

$$W = \{P(h(b)), \sim Q(f(h(b)), a), P(c)\}$$
$$U^0 = \{\sim P(x) \vee Q(f(x), a)\}.$$

2. Since U^0 does not contain □, we obtain

$$U^1 = \{Q(f(h(b)), a), \sim P(h(b)), Q(f(c), a)\}.$$

3. Since U^1 is not empty and does not contain □, we obtain $U^2 = \{\square\}$.
4. Since U^2 contains □, we terminate the algorithm and conclude that C subsumes D.

Example 5.27

Let $C = P(x, x)$ and $D = P(f(x), y) \vee P(y, f(x))$. Determine whether or not C subsumes D.

1. x and y are variables in D. Choose new constants a and b different from any constant in C or D. Let $\theta = \{a/x, b/y\}$. Then,

$$D\theta = P(f(a),b) \vee P(b,f(a)).$$

$$\sim D\theta = \sim P(f(a),b) \wedge \sim P(b,f(a)).$$

Thus,

$$W = \{\sim P(f(a),b), \sim P(b,f(a))\}$$

$$U^0 = \{P(x,x)\}.$$

2. Since U^0 does not contain $\square$, we obtain

$$U^1 = \varnothing.$$

3. Since U^1 is empty, we terminate the algorithm and conclude that C does not subsume D.

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EXERCISES

Section 5.2

1. Prove the following set of clauses is unsatisfiable by resolution:

$$P \vee Q \vee R, \quad \sim P \vee R, \quad \sim Q, \quad \sim R.$$

2. For the set $S = \{P \vee Q, \sim Q \vee R, \sim P \vee Q, \sim R\}$ derive an empty clause from S by resolution.

Section 5.3

3. Let $\theta = \{a/x, b/y, g(x,y)/z\}$ be a substitution and $E = P(h(x),z)$. Find $E\theta$.
4. Let $\theta_1 = \{a/x, f(z)/y, y/z\}$ and $\theta_2 = \{b/x, z/y, g(x)/z\}$. Find the composition of θ_1 and θ_2.
5. Prove the following theorem: Let θ, λ, and μ all be substitutions. Then $(\theta \circ \lambda) \circ \mu = \theta \circ (\lambda \circ \mu)$ (associativity).

Section 5.4

6. Determine whether each of the following sets is unifiable. If yes, obtain a most general unifier.

   (1) $W = \{Q(a), Q(b)\}$

   (2) $W = \{Q(a,x), Q(a,a)\}$

   (3) $W = \{Q(a,x,f(x)), Q(a,y,y)\}$

   (4) $W = \{Q(x,y,z), Q(u,h(v,v),u)\}$

   (5) $W = \{P(x_1, g(x_1), x_2, h(x_1,x_2), x_3, k(x_1,x_2,x_3)),$
   $P(y_1, y_2, e(y_2), y_3, f(y_2,y_3), y_4)\}$.

Section 5.5

7. Determine whether the following clauses have factors. If yes, give the factors.

   (1) $P(x) \vee Q(y) \vee P(f(x))$

   (2) $P(x) \vee P(a) \vee Q(f(x)) \vee Q(f(a))$

   (3) $P(x,y) \vee P(a,f(a))$

   (4) $P(a) \vee P(b) \vee P(x)$

   (5) $P(x) \vee P(f(y)) \vee Q(x,y)$.

8. Find all the possible resolvents (if any) of the following pairs of clauses:

   (1) $C = \sim P(x) \vee Q(x,b), D = P(a) \vee Q(a,b)$

   (2) $C = \sim P(x) \vee Q(x,x), D = \sim Q(a,f(a))$

   (3) $C = \sim P(x,y,u) \vee \sim P(y,z,v) \vee \sim P(x,v,w) \vee P(u,z,w),$
   $D = P(g(x,y),x,y)$

   (4) $C = \sim P(v,z,v) \vee P(w,z,w), D = P(w,h(x,x),w)$.

Section 5.7

9. Reconsider Example 2.13 in Section 2.6. This time show that H_2CO_3 can be made by resolution.
10. In Exercise 14 of Chapter 2, you were asked to show that Q is a logical consequence of P and $(P \rightarrow Q)$. Can you prove this again by using the resolution principle?
11. Reconsider Exercise 12 of Chapter 3. Use the resolution principle this time.

12. Consider Example 4.16. Prove that the set of clauses in this example is unsatisfiable by the resolution principle.
13. Use the resolution principle to prove that the set of clauses in Exercise 14 of Chapter 4 is unsatisfiable.
14. Prove that $(\sim Q \rightarrow \sim P)$ is a logical consequence of $(P \rightarrow Q)$ by resolution.

Section 5.8

15. Let $C = P(x,y) \vee Q(z)$ and $D = Q(a) \vee P(b,b) \vee R(u)$. Determine whether C subsumes D.
16. Let $C = P(x,y) \vee R(y,x)$ and $D = P(a,y) \vee R(z,b)$. Prove that C does not subsume D.
17. Let $C = \sim P(x) \vee P(f(x))$ and $D = \sim P(x) \vee P(f(f(x)))$. Show that C implies D, but C does not subsume D.

Chapter 6

Semantic Resolution and Lock Resolution

6.1 INTRODUCTION

In the last chapter, we considered the resolution principle as an inference rule that can be used to generate new clauses from old ones. We also have seen that unlimited applications of resolution may generate many irrelevant and redundant clauses besides useful ones. Although the deletion strategy can be used to delete some of these irrelevant and redundant clauses after they are generated, time has already been wasted by generating them. Furthermore, if useless clauses are generated, many resources such as computer time and memory are required to determine whether they are indeed irrelevant and redundant. Therefore, in order to have efficient theorem-proving procedures, we must prevent large numbers of useless clauses from being generated. This leads to the discussion of refinements of resolution.

There are many important refinements of resolution. Each refinement has its own merit. In this book, we shall consider three important refinements, namely, semantic resolution, lock resolution, and linear resolution. Semantic resolution was proposed by Slagle [1967]. It unifies Robinson's hyper-resolution [1965b], Meltzer's renamable resolution [1966], and the set-of-support strategy of Wos, Robinson, and Carson [1965]. Lock resolution was proposed by Boyer [1971]. It is a very efficient rule. Linear resolution was independently proposed by Loveland [1970a] and Luckham [1970]. It was

later strengthened by Anderson and Bledsoe [1970], Reiter [1971], Loveland [1972], and Kowalski and Kuehner [1971]. Semantic resolution, lock resolution, and linear resolution are complete. That is, each can be used alone to derive the empty clause $\square$ from an unsatisfiable set of clauses.

We shall discuss semantic resolution and lock resolution in this chapter, and linear resolution in Chapter 7. In Chapter 7, we shall also consider a subcase of linear resolution, namely, input resolution. It was shown by Chang [1970a] that input resolution is equivalent to another kind of resolution, namely, unit resolution. In Chapter 8, special treatments for the equality relation will be given. Finally, in Chapter 9, some other types of proof procedures will be introduced, including variable-constrained resolution [Chang, 1972].

6.2 AN INFORMAL INTRODUCTION TO SEMANTIC RESOLUTION

Consider the following example.

Example 6.1

Let $S = \{\sim P \vee \sim Q \vee R, P \vee R, Q \vee R, \sim R\}$. To prove the unsatisfiability of the set S, if the ordinary resolution principle is used, the following clauses will be generated.

(1)	$\sim P \vee \sim Q \vee R$	S
(2)	$P \vee R$	
(3)	$Q \vee R$	
(4)	$\sim R$	
(5)	$\sim Q \vee R$	from (1) and (2)
(6)	$\sim P \vee R$	from (1) and (3)
(7)	$\sim P \vee \sim Q$	from (1) and (4)
(8)	P	from (2) and (4)
(9)	Q	from (3) and (4)
(10)	$\sim Q \vee R$	from (1) and (8)
(11)	$\sim P \vee R$	from (1) and (9)
(12)	R	from (2) and (6)
(13)	$\sim Q$	from (4) and (5)
(14)	$\sim P$	from (4) and (6)
(15)	$\square$	from (4) and (12).

Among all these clauses generated, only (6) and (12) are actually used in the proof. All other clauses are irrelevant and redundant. Is it possible to have some mechanisms that can reduce the number of useless clauses generated? Indeed, there are such mechanisms and we shall now discuss them in detail.

Let us imagine that we can, in some way, divide a set of clauses into two group—S_1 and S_2. Let us further imagine that we do not allow clauses within the same group to be resolved with each other. This will obviously cut down the number of clauses generated.

For example, let us divide the clauses in Example 6.1 into S_1 (clause (2) and clause (3)) and S_2 (clause (1) and clause (4)). Since clause (1) and clause (4) are both in S_2, (1) and (4) will not be resolved. Thus we have succeeded in blocking the resolution of (1) and (4). As we shall see, this blocking will not affect our ability to deduce the empty clause $\square$.

The question is: How do we divide the clauses into two groups? In semantic resolution, we use an interpretation to divide clauses. This is why this kind of resolution is called semantic resolution. Note that in Example 6.1, (2) and (3) are all falsified by the interpretation $I = \{\sim P, \sim Q, \sim R\}$, while (1) and (4) are satisfied by I. Therefore, this interpretation can be used to divide S into S_1 and S_2, where S_1 contains all the clauses falsified by I and S_2 contains those satisfied by I. The reader should remember that we are dealing with an unsatisfiable set of clauses. No interpretation can satisfy or falsify all the clauses. Therefore, every interpretation divides S into two nonempty sets of clauses. We shall see in the next section that any interpretation can be used.

So far we have succeeded in blocking the resolution of (1) and (4). Can we impose some further restrictions on resolution so that more resolutions will be blocked? Note that in Example 6.1, (2) and (3) can still be resolved with (4). We shall show that these resolutions can also be blocked.

To block these resolutions, we can use an ingenious scheme, namely, the ordering of predicate symbols. Suppose we order all the predicate symbols in S as follows:

$$P > Q > R.$$

Suppose we further require that, when we resolve two clauses (one from S_1 and one from S_2), the resolved literal in the clause from S_1 contain the largest predicate symbol in that clause. With this restriction, we cannot resolve (2) with (4), nor (3) with (4), because R is *not* the largest literal in (2) and (3).

The foregoing ideas—the use of an interpretation to divide clauses into two groups and of an ordering to reduce the number of possible resolutions —are important concepts in semantic resolution. We shall elaborate on these ideas later. Meanwhile, we shall introduce another important concept, the clash.

Example 6.2

Consider the clauses in Example 6.1,

(1) $\sim P \vee \sim Q \vee R$

(2) $P \vee R$

(3) $Q \vee R$

(4) $\sim R$.

Let $I = \{\sim P, \sim Q, \sim R\}$ and let the ordering of predicate symbols be $P > Q > R$. Using the two restrictions discussed above, we can only generate two clauses,

(5) $\sim Q \vee R$ from (1) and (2)

(6) $\sim P \vee R$ from (1) and (3).

Both of these clauses are satisfied by I; therefore we put them into S_2 and resolve them with the clauses in S_1. We find that (5) can be resolved with (2), and (6) with (3). Both result in the same clause R:

(7) R from (3) and (5)

(8) R from (2) and (6).

The clause R is false in I, hence we put it into S_1 and resolve it with the clauses in S_2. We find that $\sim R$ is in S_2. Therefore, resolving R with $\sim R$, we obtain the empty clause $\square$.

Figure 6.1 shows the two ways in which R was derived. Both these deductions of R use clauses (1), (2), and (3); they differ only in that they use these clauses in different orders. Since only one of them is needed in the proof, it is quite wasteful to generate both of them. In order to avoid this redundancy, we introduce the clash. The idea of a clash is to generate R directly from (1), (2), and (3) without going through the "intermediate" clause (5) or (6). The set $\{(1),(2),(3)\}$ will be called a clash. A detailed description of a clash will be given in the next section. It will be also shown later that we can combine the concepts of interpretation and ordering of predicate symbols with that of the clash to restrict many resolutions.

Figure 6.1

6.3 FORMAL DEFINITIONS AND EXAMPLES OF SEMANTIC RESOLUTION

Definition Let I be an interpretation. Let P be an ordering of predicate symbols. A finite set of clauses $\{E_1, \ldots, E_q, N\}$, $q \geqslant 1$, is called a *semantic clash* with respect to P and I (or *PI-clash* for short), if and only if $E_1, \ldots, E_q$ (called *electrons*) and N (called the *nucleus*) satisfy the following conditions:

1. $E_1, \ldots, E_q$ are false in I.
2. Let $R_1 = N$. For each $i = 1, \ldots, q$, there is a resolvent R_{i+1} of R_i and E_i.
3. The literal in E_i, which is resolved upon, contains the largest predicate symbol in E_i, $i = 1, \ldots, q$.
4. R_{q+1} is false in I.

R_{q+1} is called a *PI-resolvent* (of the *PI*-clash $\{E_1, \ldots, E_q, N\}$).

Example 6.3

Let

$$E_1 = A_1 \vee A_3, \qquad E_2 = A_2 \vee A_3, \qquad N = \sim A_1 \vee \sim A_2 \vee A_3.$$

Let $I = \{\sim A_1, \sim A_2, \sim A_3\}$ and P be an ordering in which $A_1 > A_2 > A_3$. Then $\{E_1, E_2, N\}$ is a *PI*-clash. The *PI*-resolvent of this clash is A_3. Note that A_3 is false in I.

Example 6.4

In Example 6.3, neither $\{E_1, N\}$ nor $\{E_2, N\}$ is a *PI*-clash. For instance, we can see that the resolvent of E_1 and N is $\sim A_2 \vee A_3$, which is true in I. Thus $\{E_1, N\}$ is not a *PI*-clash. We can also see that, if we change P into an ordering in which $A_3 > A_2 > A_1$, then $\{E_1, E_2, N\}$ is not a *PI*-clash since the third condition for a *PI*-clash is not satisfied.

Example 6.5

Let

$$\begin{aligned} E_1 &= \sim Q(z) \vee \sim Q(a), \\ E_2 &= R(b) \vee S(c), \\ N &= Q(x) \vee Q(a) \vee \sim R(y) \vee \sim R(b) \vee S(c). \end{aligned}$$

Let $I = \{Q(a), Q(b), Q(c), \sim R(a), \sim R(b), \sim R(c), \sim S(a), \sim S(b), \sim S(c)\}$. Let P be an ordering of predicate symbols in which $Q > R > S$. Let $R_1 = N$. There is a resolvent of R_1 and E_1, namely, $\sim R(y) \vee \sim R(b) \vee S(c)$. Call this resolvent

R_2. Then there is a resolvent of R_2 and E_2, namely, $S(c)$. Denote $S(c)$ by R_3. R_3 is false in I. Since $\{E_1, E_2, N\}$ satisfies all the four conditions, it is a PI-clash. The PI-resolvent of this clash is $S(c)$.

In a PI-clash $\{E_1, E_2, \ldots, E_q, N\}$, E_1 is considered as the first electron, E_2 the second, ..., and E_q the last. In fact, the order of these electrons is immaterial. We can relabel any one of these electrons as the first, any of the remaining electrons as the second, and so on. No matter what order of the electrons we use, we can always get the same PI-resolvent of this clash. Therefore, we can avoid generating too many deductions of the same resolvent, as discussed in Example 6.2. This is one reason for using the concept of a clash.

Definition Let I be an interpretation for a set S of clauses, and P be an ordering of predicate symbols appearing in S. A deduction from S is called a *PI-deduction* if and only if each clause in the deduction is either a clause in S, or a PI-resolvent.

Example 6.6

Let $S = \{A_1 \vee A_2, A_1 \vee \sim A_2, \sim A_1 \vee A_2, \sim A_1 \vee \sim A_2\}$. Let $I = \{A_1, \sim A_2\}$ and P be an ordering of the predicate symbols in which $A_1 < A_2$. The deduction shown in Fig. 6.2 is a PI-deduction of $\square$ from S. In Fig. 6.2 there are three PI-resolvents, namely, $\sim A_1$, A_2, and $\square$.

An interesting thing about semantic resolution is that we can use any ordering and any interpretation. For instance, if in Example 6.6 we let $I = \{\sim A_1, \sim A_2\}$ and $A_1 > A_2$, then we obtain a PI-deduction of $\square$ from S shown in Fig. 6.3. In fact, PI-resolution is complete; that is, from any unsatisfiable set of clauses, we can always derive the empty clause $\square$ by using some PI-resolution. We shall devote the next section to proving this.

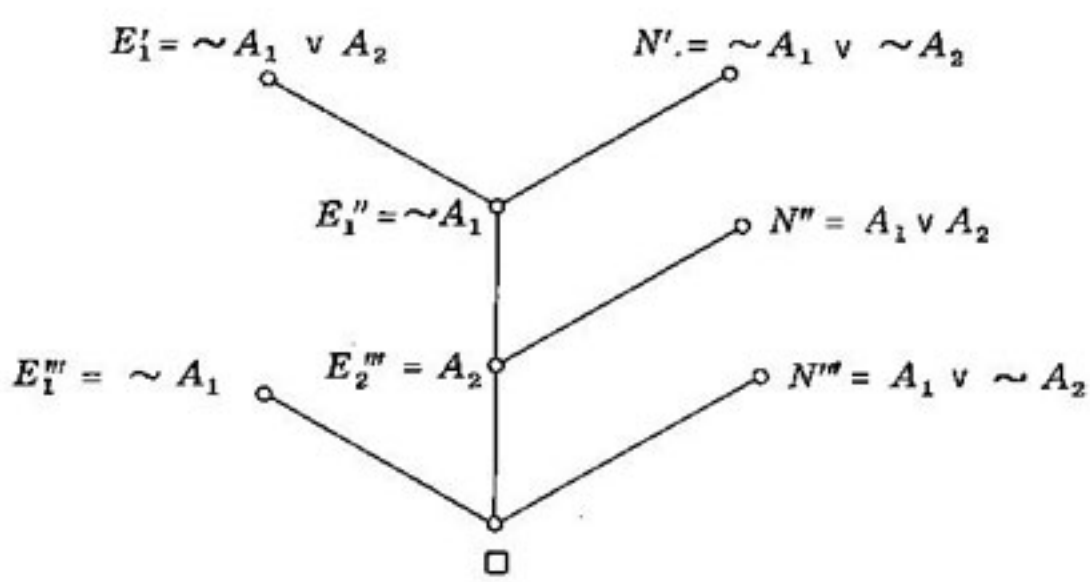

Figure 6.2

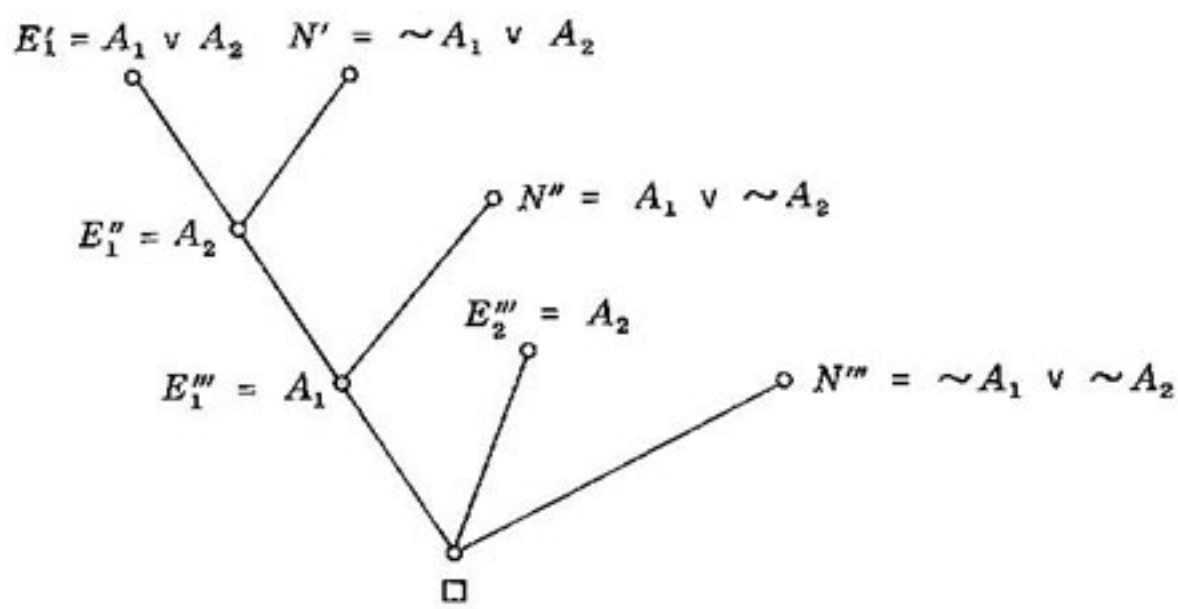

Figure 6.3

6.4 COMPLETENESS OF SEMANTIC RESOLUTION

Lemma 6.1 If P is an ordering of predicate symbols in a finite unsatisfiable set S of ground clauses, and if I is an interpretation of S, then there is a PI-deduction of $\square$ from S.

Proof This lemma can be proved by induction. Let A be the atom set of S. If A consists of a single element, say Q, then among the elements of S there are the clauses Q and $\sim Q$. Clearly, the resolvent of Q and $\sim Q$ is $\square$. Since one of Q and $\sim Q$ must be false in I, and $\square$ is false in I, $\square$ is a PI-resolvent. Therefore, Lemma 6.1 holds for this case.

Assume Lemma 6.1 holds when A consists of i elements, $1 \leqslant i \leqslant n$. To complete the induction we consider A such that A consists of exactly $n+1$ elements. There are two possible cases:

Case 1 S contains a unit clause L that is false in I. (L is a literal.) Let S' be that set obtained from S by deleting those clauses containing the literal L and by deleting $\sim L$ from the remaining clauses. Clearly, S' is unsatisfiable. (See the one-literal rule in Section 4.6.) Since S' contains n or fewer than n atoms, by the induction hypothesis there is a PI-deduction D' of $\square$ from S'. From the deduction D', we can obtain a PI-deduction of $\square$ from S. This is done as follows: First, for each PI-clash $\{E_1', \ldots, E_q', N'\}$, where $E_1', \ldots, E_q', N'$ are clauses attached to initial nodes of D', if N' is obtained from a clause N in S by deleting $\sim L$ from N, replace the clash $\{E_1', \ldots, E_q', N'\}$ by the PI-clash $\{E_1', \ldots, E_q', L, N\}$ (where $E_1', \ldots, E_q', L$ are electrons and N is the nucleus). Second, if E_i' is obtained from a clause E_i in S by deleting $\sim L$ from E_i, attach the PI-clah $\{L, E_i\}$ above the node of E_i'. After we perform the above process, it is clear that we have obtained a PI-deduction of $\square$ from S.

Case 2 S does not contain a unit clause that is false in I. In this case, choose an element B in the atom set A of S such that B contains the

smallest predicate symbol. Either B or $\sim B$ must be false in I. Let L be that element in $\{B, \sim B\}$ that is false in I. Let S' be that set obtained from S by deleting those clauses containing the literal $\sim L$ and by deleting L from the remaining clauses. S' is unsatisfiable. Since S' contains n or fewer than n atoms, by the induction hypothesis there is a PI-deduction D' of $\square$ from S'. Let D_1 be the deduction obtained from D' by putting the literal L back to those clauses from which it was deleted. D_1 is still a PI-deduction since L contains the smallest predicate symbol and L is false in I. D_1 is either a PI-deduction of $\square$, or L. If it is the former, we are done. If it is the latter, consider the set $(S \cup \{L\})$. Since $(S \cup \{L\})$ contains a unit clause L that is false in I, by the proof of Case 1 given above there is a PI-deduction D_2 of $\square$ from $(S \cup \{L\})$. Combining D_1 and D_2, we can obtain a PI-deduction of $\square$ from S. This completes the proof of Lemma 6.1. Q.E.D.

Theorem 6.1 (Completeness of *PI*-resolution) If P is an ordering of predicate symbols in a finite and unsatisfiable set S of clauses, and if I is an interpretation of S, then there is a PI-deduction of $\square$ from S.

Proof Since S is unsatisfiable, by Herbrand's theorem (version II) there is a finite unsatisfiable set S' of ground instances of clauses in S. By Lemma 6.1, there is a PI-deduction D' of $\square$ from S'. From the PI-deduction D', we now show that we can produce a PI-deduction of $\square$ from S. To see this, we merely attach to each node of D' a clause over or above the ground clause already there as follows: To each initial node, attach a clause in S of which the ground clause already there is an instance. Then, for each noninitial node, if clauses have been attached in this way to each of its immediate predecessor nodes and constitute a PI-clash, attach to it the PI-resolvent of which the ground clause already there is an instance. (This is possible because of the lifting lemma, Lemma 5.1.) In this fashion, a clause is attached to each node of which the ground clause already at the node is an instance. The clause attached to the terminal node must be $\square$, since the clause already there is $\square$. It is easy to see that the deduction tree, together with the attached clauses, is a PI-deduction of $\square$ from S. This completes the proof.

It is noted that, without losing the completeness property, the deletion strategy given in Section 5.8 can be used together with PI-resolution.

6.5 HYPERRESOLUTION AND THE SET-OF-SUPPORT STRATEGY: SPECIAL CASES OF SEMANTIC RESOLUTION

In this section, we shall introduce two special kinds of interpretations to be used in semantic resolution. One interpretation leads to *hyperresolution* and the other leads to the *set-of-support strategy*.

A Hyperresolution

Let us consider an interpretation I in which every literal is the negation of an atom. If this interpretation is used, every electron and every PI-resolvent must contain only atoms. Similarly, if every literal in I is an atom, then every electron and every PI-resolvent must contain only negations of atoms. Hyperresolution is based on these considerations.

Definition A clause is called *positive* if it does not contain any negation sign. A clause is called *negative* if every literal of it contains the negation sign. A clause is called *mixed* if it is neither positive nor negative.

Definition A *positive hyperresolution* is a special case of PI-resolution in which every literal in the interpretation I contains the negation sign. It is called positive hyperresolution because all the electrons and PI-resolvents in this case are positive.

Definition A *negative hyperresolution* is a special case of PI-resolution in which the interpretation I does not contain any negation sign. It is called negative hyperresolution because all the electrons and PI-resolvents in this case are negative.

From Theorem 6.1, we can see that both positive and negative hyperresolutions are complete.

Example 6.7

For the following unsatisfiable set of clauses,

$$\begin{aligned} S = \{&Q(a) \vee R(x), \\ &\sim Q(x) \vee R(x), \\ &\sim R(a) \vee \sim S(a), \\ &S(x)\} \end{aligned}$$

let P be an ordering defined as $R < Q < S$. Then we obtain positive and negative hyperdeductions shown in Fig. 6.4a and Fig. 6.4b, respectively.

It is often the case that the axioms of a theorem are represented by some positive and mixed clauses, and the negation of the conclusion is represented by a negative clause. In this case, the positive hyperresolution roughly corresponds to "thinking forward," while the negative hyperresolution corresponds to "thinking backward." This can be best explained by the following example.

Example 6.8

Let the axioms be Q, R, and $\sim Q \vee \sim R \vee S$. Let the negation of the conclusion be $\sim S$. Then we have the following four clauses:

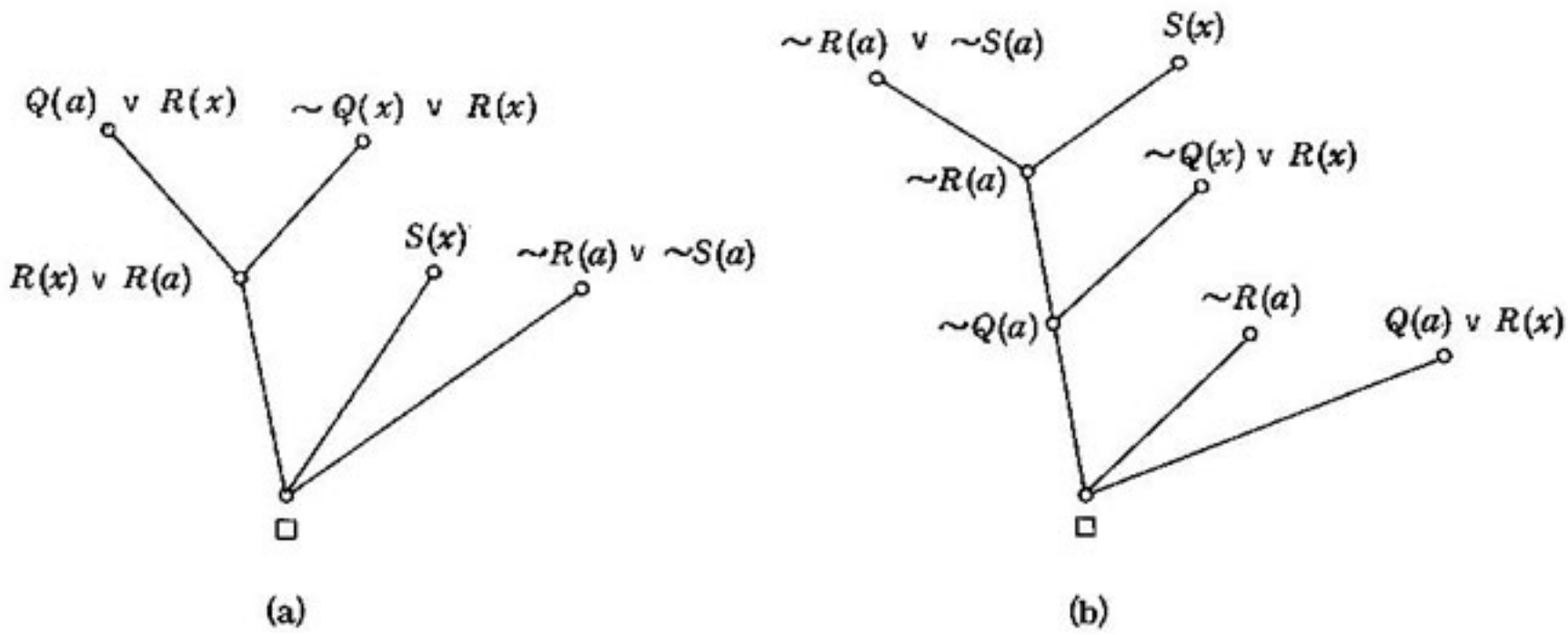

Figure 6.4

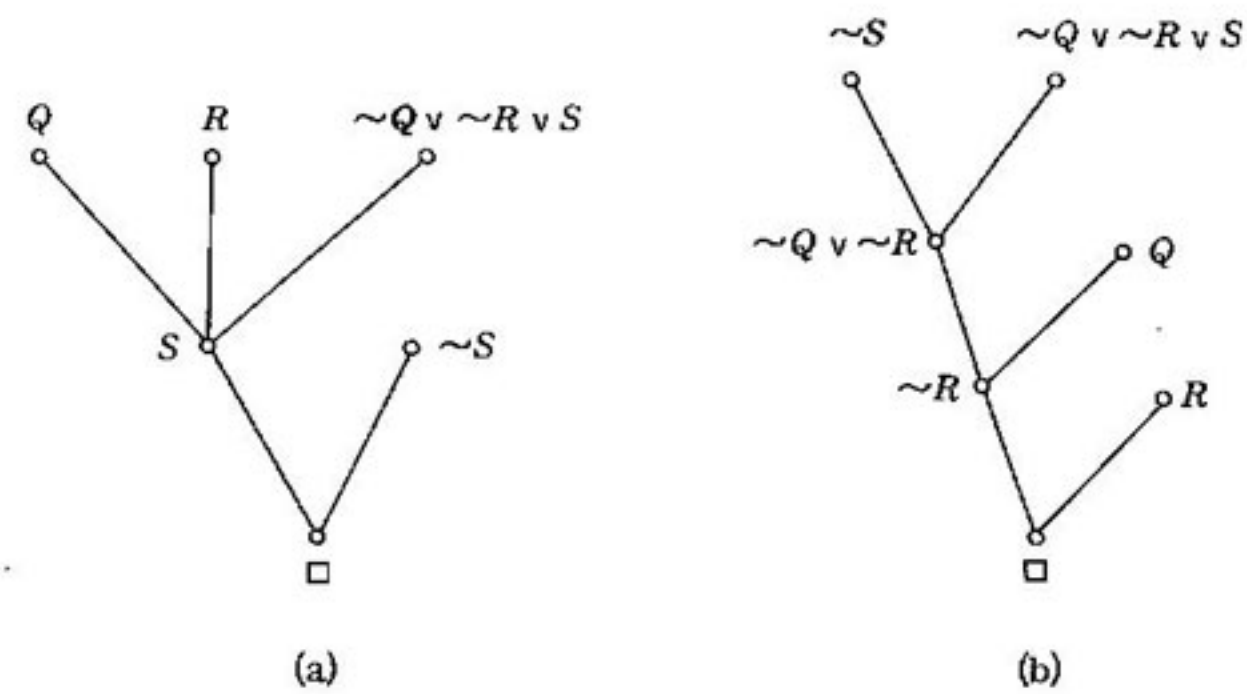

Figure 6.5

(1) Q

(2) R

(3) $\sim Q \vee \sim R \vee S$

(4) $\sim S$.

Let the ordering of predicate symbols be $Q > R > S$. Then Fig. 6.5a is a positive hyperdeduction of $\square$. The reader can see that the deduction starts with axioms and derives S, which is contradictory to the negation of the conclusion $\sim S$. In Fig. 6.5b, the deduction is a negative hyperdeduction of $\square$. This time the deduction starts from the negation of the conclusion. Assuming the conclusion is false, we derive a contradiction from this assumption.

B The Set-of-Support Strategy

The set-of-support strategy was proposed by Wos, Robinson, and Carson [1965]. As discussed in the previous chapters, a theorem consists of a set

of axioms $A_1, A_2, \ldots, A_n$ and a conclusion B. To prove the theorem, we are essentially proving that $A_1 \wedge A_2 \wedge \cdots \wedge A_n \wedge \sim B$ is unsatisfiable. Since $A_1 \wedge A_2 \wedge \cdots \wedge A_n$ is usually satisfiable, it might be wise to avoid resolving clauses in $A_1 \wedge A_2 \wedge \cdots \wedge A_n$. This is what the set-of-support strategy tries to accomplish.

Definition A subset T of a set S of clauses is called a *set of support* of S if $S - T$ is satisfiable. A *set-of-support resolution* is a resolution of two clauses that are not both from $S - T$. A *set-of-support deduction* is a deduction in which every resolution is a set-of-support resolution.

We shall now prove that the set-of-support resolution is complete. This can be easily done by using Theorem 6.1.

Theorem 6.2 (Completeness of the Set-of-Support Strategy) If S is a finite unsatisfiable set of clauses and T is a subset of S such that $S - T$ is satisfiable, then there is a set-of-support deduction of $\square$ from S with T as a set of support.

Proof Since $S - T$ is satisfiable, there is an interpretation I that satisfies $S - T$. Choose any ordering P of predicate symbols in S. By Theorem 6.1, there is a PI-deduction D of $\square$ from S. From D, consider each PI-clash $\{E_1, E_2, \ldots, E_q, N\}$. The PI-resolvent of this clash is obtained by first resolving E_1 and N, then resolving E_2 with this resolvent, and so on. Every resolvent involves an electron E_i. However electron is false in I. Therefore, for every corresponding resolution, the two clauses cannot both come from $S - T$. Thus, the deduction D can be transformed into a set-of-support deduction of $\square$ from S. Q.E.D.

Example 6.9

Let S be a set consisting of the following clauses:

(1) $P(g(x_1, y_1), x_1, y_1)$

(2) $\sim P(x_2, h(x_2, y_2), y_2)$

(3) $\sim P(x_3, y_3, u_3) \vee P(y_3, z_3, v_3) \vee \sim P(x_3, v_3, w_3) \vee P(u_3, z_3, w_3)$

(4) $\sim P(k(x_4), x_4, k(x_4))$.

Let T be a set consisting of clause (4). Then the following deduction is a set-of-support deduction with T as the set of support. We see that no resolution is performed among (1), (2), and (3).

(5) $\sim P(x_3, y_3, k(z_3)) \vee P(y_3, z_3, v_3) \vee \sim P(x_3, v_3, k(z_3))$

a resolvent of (4) and (3)

(6) $\sim P(x_3, y_3, k(h(y_3, v_3))) \vee \sim P(x_3, v_3, k(h(y_3, v_3)))$

a resolvent of (5) and (2)

(7) $\square$ a resolvent of (6) and (1).

6.6 SEMANTIC RESOLUTION USING ORDERED CLAUSES

In this section, we shall consider some orderings other than orderings of predicate symbols. For the propositional logic, orderings of predicate symbols used in conjunction with semantic resolution are quite effective. For any electron E, we have no difficulty in singling out a literal in E to be resolved upon. For example, consider $E_1 = A_1 \vee A_3$, $E_2 = A_2 \vee A_3$, and $N = \sim A_1 \vee \sim A_2 \vee A_3$ in Example 6.3. Using $I = \{\sim A_1, \sim A_2, \sim A_3\}$ and the ordering P in which $A_1 > A_2 > A_3$, we can easily decide that the literals A_1 and A_2 in E_1 and E_2, respectively, have to be resolved upon. However, for the first-order logic where variables are involved, in an electron E there may be more than one literal that contains the same largest predicate symbol in E. Any one of these literals is a candidate to be resolved upon. This is best illustrated by the following example.

Example 6.10

Consider

$$E = Q(a) \vee Q(b) \vee Q(c) \vee Q(d)$$

$$N = \sim Q(x).$$

Let I be an interpretation in which every literal is negative, and P be any ordering of predicate symbols. Thus, clearly, $\{E, N\}$ is a PI-clash. Since all the four literals in E contain the same predicate symbol Q, every one of them can be resolved with N. Therefore, from the PI-clash $\{E, N\}$, we can obtain four PI-resolvents, namely, $Q(b) \vee Q(c) \vee Q(d)$, $Q(a) \vee Q(c) \vee Q(d)$, $Q(a) \vee Q(b) \vee Q(d)$, and $Q(a) \vee Q(b) \vee Q(c)$.

From the above example we have seen that, using orderings of predicate symbols, we may not be able to single out *uniquely* a literal in an electron. Thus we may be forced to generate *more than one* semantic resolvent from a semantic clash. In order to remedy this situation, we consider *ordered clauses* [Reiter, 1971; and Slagle and Norton 1971].

The idea of ordered clauses is to consider a clause as a *sequence* of literals rather than a set of literals. In so doing, we set the order of all literals in a clause. We agree that a literal L_2 is greater than a literal L_1 in a clause just in case L_2 follows L_1 in the sequence specified by the

clause. Thus, the last literal in an ordered clause will always be considered the largest literal in the clause. For example, if we consider $A_3 \vee A_1 \vee A_2$ as an ordered clause, then A_2 is the largest literal. We now formally give the following definitions.

Definition An *ordered clause* is a sequence of distinct literals.

As a clause, an ordered clause is also interpreted as a disjunction of all the literals in the ordered clause. The only difference is that the order of literals in a clause is immaterial, while the order of literals in an ordered clause is deliberately specified. An ordered clause will also be written as a disjunction of literals. This will not be confused since we shall always use the word "ordered" for ordered clauses.

Definition A literal L_2 is said to be *greater than* a literal L_1 in an ordered clause (or L_1 is *smaller than* L_2) if and only if L_2 follows L_1 in the sequence specified by the ordered clause.

We shall consider semantic resolution using ordered clauses. However, we first define ordered resolution for ordered clauses.

Definition If two or more literals (with the same sign) of an ordered clause C have a most general unifier σ, then the ordered clause obtained from the sequence $C\sigma$ by deleting any literal that is identical to a smaller literal in the sequence is called an *ordered factor* of C.

Example 6.11

Consider the ordered clause $C = \underline{P(x)} \vee Q(x) \vee \underline{P(a)}$. The first and the last literals (underlined) of C have a most general unifier $\sigma = \{a/x\}$. Hence, $C\sigma = P(a) \vee Q(a) \vee P(a)$. In this sequence $C\sigma$, there are two identical literals. Since the second $P(a)$ is identical to a smaller literal, namely, the first $P(a)$, we delete the second $P(a)$ from $C\sigma$ and obtain the sequence $P(a) \vee Q(a)$, which is an ordered factor of C. We note that $Q(a) \vee P(a)$ is *not* an ordered factor of C since it is obtained from $C\sigma$ by deleting the first $P(a)$.

Definition Let C_1 and C_2 be ordered clauses with no variables in common. Let L_1 and L_2 be two literals in C_1 and C_2, respectively. If L_1 and $\sim L_2$ have a most general unifier σ, and if C is the ordered clause obtained by concatenating the sequences $C_1\sigma$ and $C_2\sigma$, removing $L_1\sigma$ and $L_2\sigma$, and deleting any literal that is identical to a smaller literal in the remaining sequence, then C is called an *ordered binary resolvent* of C_1 against C_2. The literals L_1 and L_2 are called the *literals resolved upon.*

The above definition is the same as the definition of a binary resolvent given in Section 5.5 of Chapter 5, except that the order of literals in an

ordered binary resolvent is deliberately specified. Note that we use the word "against" to emphasize that an ordered binary resolvent of C_1 against C_2 is not the same as one of C_2 against C_1.

Example 6.12

Consider the ordered clauses

$$C_1 = P(x) \vee Q(x) \vee R(x) \qquad \text{and} \qquad C_2 = \sim P(a) \vee Q(a).$$

Choose $L_1 = P(x)$ and $L_2 = \sim P(a)$. Since $\sim L_2 = P(a)$, L_1 and $\sim L_2$ have a most general unifier $\sigma = \{a/x\}$. First, concatenating the sequences $C_1\sigma$ and $C_2\sigma$, we obtain the sequence, $P(a) \vee Q(a) \vee R(a) \vee \sim P(a) \vee Q(a)$. Second, removing $L_1\sigma$ and $L_2\sigma$ from the above sequence, we obtain the sequence $Q(a) \vee R(a) \vee Q(a)$. Finally, since in $Q(a) \vee R(a) \vee Q(a)$ the second $Q(a)$ is identical to a smaller literal, namely, the first $Q(a)$, we delete the second $Q(a)$ and obtain $Q(a) \vee R(a)$. The sequence $Q(a) \vee R(a)$ is an ordered binary resolvent of C_1 against C_2. $P(x)$ and $\sim P(a)$ are the literals resolved upon.

Definition An *ordered resolvent* of an ordered clause C_1 against an ordered clause C_2 is one of the following ordered binary resolvents:

1. an ordered binary resolvent of C_1 against C_2;
2. an ordered binary resolvent of C_1 against an ordered factor of C_2;
3. an ordered binary resolvent of an ordered factor of C_1 against C_2;
4. an ordered binary resolvent of an ordered factor of C_1 against an ordered factor of C_2.

Example 6.13

Consider the ordered clauses

$$C_1 = P(x) \vee Q(x) \vee R(x) \vee P(a),$$
$$C_2 = \sim P(a) \vee Q(a).$$

An ordered factor of C_1 is $C_1' = P(a) \vee Q(a) \vee R(a)$. An ordered binary resolvent of C_1' against C_2 is $Q(a) \vee R(a)$. Therefore, $Q(a) \vee R(a)$ is an ordered resolvent of C_1 against C_2.

Ordered resolution is an inference rule that generates ordered resolvents from a set of ordered clauses. It is not difficult to see that ordered resolution is complete. That is, ordered resolution will always generate the empty clause $\square$ from an unsatisfiable set of ordered clauses. Proof of this is left to the reader as an exercise.

We have defined ordered resolution. Now, we shall consider semantic

resolution for ordered clauses. We shall still use the concepts of interpretation and clash. However, we shall use the concept of ordered clauses rather than that of ordering of predicate symbols. As discussed in the beginning of this section, this has an advantage because now we can always uniquely single out a literal in an electron to resolve upon.

Definition Let I be an interpretation. A finite sequence of ordered clauses, $(E_1, E_2, \ldots, E_q, N), q \geqslant 1$, is called an *ordered semantic clash* with respect to I (or *OI-clash* for short) if and only if $E_1, E_2, \ldots, E_q$ (called *ordered electrons*) and N (called the *ordered nucleus*) satisfy the following conditions:

1. $E_1, E_2, \ldots, E_q$ are false in I.
2. Let $R_q = N$. For each $i = q, q-1, \ldots, 1$, there is an ordered resolvent R_{i-1} of E_i against R_i.
3. The literal in E_i that is resolved upon is the "last literal" in $E_i, i = 1, \ldots, q$; the literal in R_i that is resolved upon is the "largest literal" that has an instance true in I.
4. R_0 is false in I.

R_0 is called an *OI-resolvent* of the *OI*-clash $(E_1, E_2, \ldots, E_q, N)$.

Except for condition (3), the above definition is similar to that of the *PI*-resolvent given in Section 6.3. We now give an example to illustrate how an *OI*-clash can be found.

Example 6.14

Consider the ordered clauses

(1) $A_1 \vee A_2$

(2) $A_1 \vee A_3$

(3) $\sim A_3 \vee \sim A_2 \vee A_1$.

Let $I = \{\sim A_1, \sim A_2, \sim A_3\}$. Since (1) and (2) are false in I, they can be used as ordered electrons. Since (3) has a literal that is true in I, (3) can be used as an ordered nucleus. Therefore, let $N = \sim A_3 \vee \sim A_2 \vee A_1$. In N, there are two literals, namely, $\sim A_3$ and $\sim A_2$, that are true in I. Therefore, we need two ordered electrons. Let $R_2 = N$, i.e., $R_2 = \sim A_3 \vee \sim A_2 \vee A_1$. In R_2, the largest literal that has an instance (itself in this case) true in I is $\sim A_2$. Since $\sim A_2$ and the last literal of (1) can be resolved upon, we have $E_2 = A_1 \vee A_2$. The ordered resolvent R_1 of E_2 against R_2 is given as $R_1 = A_1 \vee \sim A_3$. Now, in R_1, the largest literal that has an instance (itself in this case) true in I is $\sim A_3$. Since $\sim A_3$ and the last literal of (2) can be resolved upon, we have $E_1 = A_1 \vee A_3$. The ordered resolvent R_0 of E_1 against R_1 is given as $R_0 = A_1$. Since R_0 is false in I, we conclude that (E_1, E_2, N), or $((2), (1), (3))$, is an *OI*-clash, and A_1 is the *OI*-resolvent of this clash.

From the above example, we can see that the order of the ordered electrons is determined by the order of literals in the ordered nucleus. Therefore, for the above example, ((1), (2), (3)) would not be an *OI*-clash.

Example 6.15

Consider the clauses given in Example 6.10,

(1) $Q(a) \vee Q(b) \vee Q(c) \vee Q(d)$

(2) $\sim Q(x)$.

Suppose that we treat (1) and (2) as ordered clauses. Let I be an interpretation in which every literal is negative. Since (1) is false in I, (1) can be used as an electron. Resolving the last literal of (1) against $\sim Q(x)$, we obtain an ordered resolvent $Q(a) \vee Q(b) \vee Q(c)$, which is false in I. Therefore, ((1),(2)) is an *OI*-clash with (1) and (2) as the ordered electron and the ordered nucleus, respectively. There is only one *OI*-resolvent of this clash, as opposed to four *PI*-resolvents using an ordering of predicate symbols.

Definition Let I be an interpretation for a set S of ordered clauses. A deduction from S is called an *OI*-deduction if and only if each ordered clause in the deduction is either an ordered clause in S or an *OI*-resolvent.

Example 6.16

Consider the following ordered clauses:

(1) $Q(a) \vee R(x)$

(2) $\sim Q(x) \vee R(x)$

(3) $\sim R(x) \vee \sim S(a)$

(4) $S(x)$.

Let I be an interpretation in which every literal is negative. Then the deduction shown in Fig. 6.6 is an *OI*-deduction of $\square$ from S.

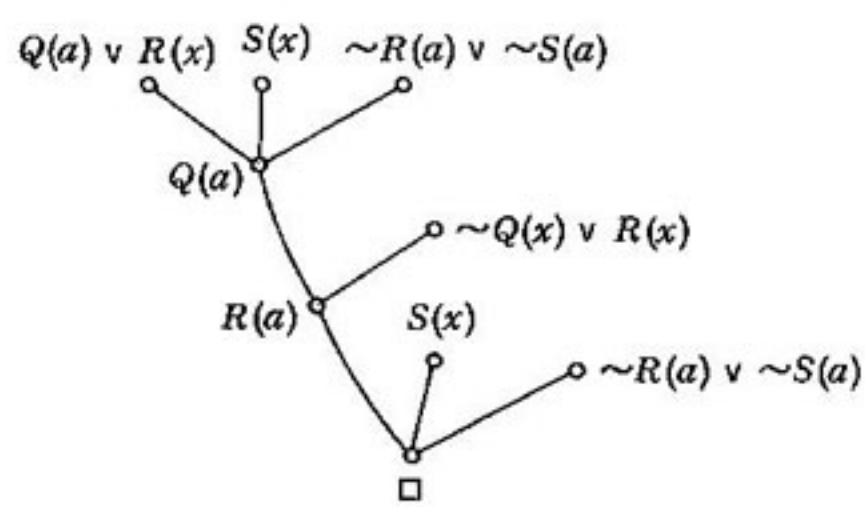

Figure 6.6

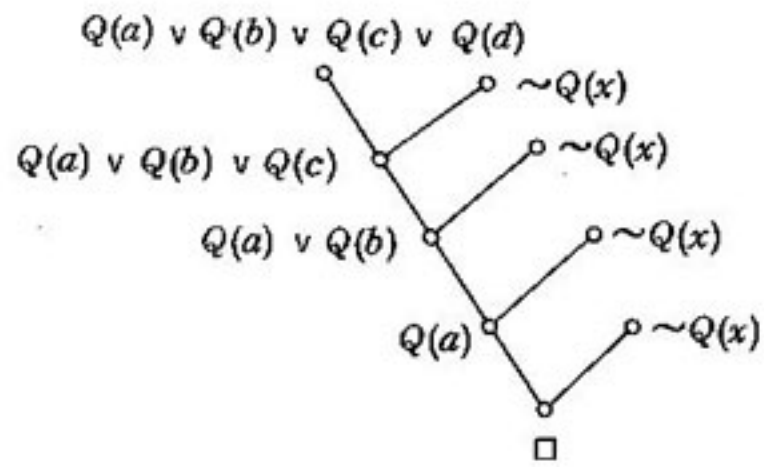

Figure 6.7

Example 6.17

Consider the set S consisting of the ordered clauses $Q(a) \vee Q(b) \vee Q(c) \vee Q(d)$ and $\sim Q(x)$, as given in Example 6.10. S is unsatisfiable. Let I be an interpretation in which every literal is negative. Then the deduction shown in Fig. 6.7 is an OI-deduction of $\square$ from S. Only four ordered OI-resolvents have been generated. However, if PI-resolution were used, 40 PI-resolvents would be generated by the level-saturation method before $\square$ could be found.

Slagle and Norton [1971] experimented with OI-resolution; their results indicated that it is quite efficient. Many theorems can be proved by OI-resolution; however, unfortunately, OI-resolution is *not* complete. The following counterexample is given by Anderson [1971].

Example 6.18

Consider the following set S of ordered clauses:

$$\left.\begin{array}{ll} (1) & P \vee Q \\ (2) & Q \vee R \\ (3) & R \vee W \\ (4) & \sim R \vee \sim P \\ (5) & \sim W \vee \sim Q \\ (6) & \sim Q \vee \sim R \end{array}\right\} S.$$

Let I be an interpretation in which every literal is negative. Thus, clauses (1)–(3) can be used as ordered electrons, and clauses (4)–(6) can be used as an ordered nucleus. From clauses (1)–(6), we can obtain the following OI-resolvents:

(7) $R \vee P$ from OI-clash ((3), (1), (5))

(8) $P \vee Q$ from OI-clash ((1), (2), (6)).

From clauses (1)–(8), we can obtain the following OI-resolvent:

(9) $Q \vee R$ from OI-clash ((2), (7), (4)).

We note that clauses (8) and (9) are in S. Therefore, from clauses (1)–(9) we can not produce any new OI-resolvents. That is, $\square$ cannot be produced by OI-resolution. However, S is unsatisfiable. Hence, OI-resolution is not complete.

6.7 IMPLEMENTATION OF SEMANTIC RESOLUTION

In this section, we shall consider how PI-resolution is implemented. Although OI-resolution is not complete, we can use the concept of ordered clauses to implement PI-resolution. We shall say that a *positive ordered clause* is an ordered clause that does not contain any negation sign, and a *negative ordered clause* is an ordered clause of which every literal contains the negation sign. A *nonpositive* (*nonnegative*) *ordered clause* is an ordered clause that is not positive (negative). In order to relieve us from specifying an interpretation every time PI-resolution is used, we shall consider only positive hyperresolution. (Negative hyperresolution can be treated in a similar manner.) Thus we need consider only positive ordered clauses as candidates for electrons and nonpositive ones for the nucleus. *We agree that for any nonpositive ordered clauses, negative literals are put after positive literals. Let S be a set of ordered clauses. Let P be an ordering of predicate symbols in S.* The following algorithm can compute positive hyperresolvents.

Generation of Positive Hyperresolvents

Step 0 Let M and N be the sets of all positive and nonpositive ordered clauses in S, respectively.

Step 1 Set $j = 1$.

Step 2 Let $A_0 = \varnothing$ and $B_0 = N$.

Step 3 Set $i = 0$.

Step 4 If A_i contains $\square$, terminate; a contradiction is found. Otherwise, go to the next step.

Step 5 If B_i is empty, go to Step 8. Otherwise, go to the next step.

Step 6 Compute the set

$W_{i+1} = \{$ordered resolvents of C_1 against C_2, where C_1 is an ordered clause or an ordered factor of an ordered clause

in M, C_2 is an ordered clause in B_i, the resolved literal of C_1 contains the "largest" predicate symbol in C_1, and the resolved literal of C_2 is the "last" literal of C_2}.

Let A_{i+1} and B_{i+1} be the sets of all the positive and nonpositive ordered clauses in W_{i+1}, respectively.

Step 7 Set $i = i+1$ and go to Step 4.

Step 8 Set $T = A_0 \cup \cdots \cup A_i$ and $M = T \cup M$.

Step 9 Set $j = j+1$.

Step 10 Compute the set

$R = \{$ordered resolvents of C_1 against C_2, where C_1 is an ordered clause or an ordered factor of an ordered clause in T, C_2 is an ordered clause in N, and the resolved literal of C_1 contains the "largest" predicate symbol in C_1}.

(Note that in the above definition of R, the resolved literal of C_2 can be any literal of C_2, not necessarily the last literal.) Let A_0 and B_0 be the sets of all the positive and nonpositive ordered clauses in R, respectively.

Step 11 Go to Step 3.

In the above algorithm, for each cycle, that is for each j, B_i will eventually be empty since the maximum number of negative literals in any ordered clause of B_i decreases by one as i increases by one. All the clauses in each A_i are positive hyperresolvents. It is not hard to see that if S is unsatisfiable, then $\square$ can be generated by the above algorithm. The deletion strategy can be also incorporated into the above algorithm without losing the completeness property. That is, for T and M obtained in Step 8 of the algorithm, any clause in T or M subsumed by other clauses in T or M can be deleted. (Note that no tautologies are in T or M since all clauses in T or M are positive.) However, in the following example, we do not use the deletion strategy.

Example 6.19

Let S be a set of ordered clauses given as $S = \{P_6 \vee P_4, P_5 \vee P_4, P_4 \vee P_1, \sim P_1 \vee \sim P_2, P_3 \vee \sim P_6, \sim P_4, P_2 \vee \sim P_5 \vee \sim P_3\}$. For simplicity, we denote $P_1, P_2, \ldots, P_6$ by $1, 2, \ldots, 6$, respectively. Therefore, S is written as $S = \{6 \vee 4, 5 \vee 4, 4 \vee 1, \sim 1 \vee \sim 2, 3 \vee \sim 6, \sim 4, 2 \vee \sim 5 \vee \sim 3\}$. Let P be an ordering of predicate symbols in which $1 < 2 < 3 < 4 < 5 < 6$. From S, we obtain

$$M = \{6 \vee 4, 5 \vee 4, 4 \vee 1\}$$

$$N = \{\sim 1 \vee \sim 2, 3 \vee \sim 6, \sim 4, 2 \vee \sim 5 \vee \sim 3\}.$$

Now, we apply the above algorithm.

1. $j = 1$.

   a. $A_0 = \varnothing$

   $B_0 = N = \{\sim 1 \vee \sim 2, 3 \vee \sim 6, \sim 4, 2 \vee \sim 5 \vee \sim 3\}$.

   b. Since neither A_0 contains $\square$ nor B_0 is empty, we apply Step 6 and obtain

   $W_1 = \{4 \vee 3, 1\}$

   $A_1 = \{4 \vee 3, 1\}$

   $B_1 = \varnothing$.

   c. Since B_1 is empty, we go to Step 8 to set

   $T = A_0 \cup A_1 = \{4 \vee 3, 1\}$

   $M = T \cup M = \{4 \vee 3, 1, 6 \vee 4, 5 \vee 4, 4 \vee 1\}$.

2. $j = 2$.

   a. Computing R, we obtain $R = \{3, \sim 2\}$. From R, we obtain $A_0 = \{3\}$, $B_0 = \{\sim 2\}$.

   b. Since neither A_0 contains $\square$ nor B_0 is empty, we apply Step 6 and obtain $W_1 = \varnothing$, $A_1 = \varnothing$, $B_1 = \varnothing$.

   c. Since B_1 is empty, we let

   $T = A_0 \cup A_1 = \{3\}$

   $M = T \cup M = \{3, 4 \vee 3, 1, 6 \vee 4, 5 \vee 4, 4 \vee 1\}$.

3. $j = 3$.

   a. Applying Step 10, we obtain $R = \{2 \vee \sim 5\}$, $A_0 = \varnothing$, $B_0 = \{2 \vee \sim 5\}$.

   b. Since neither A_0 contains $\square$ nor B_0 is empty, we apply Step 6 and obtain $W_1 = \{4 \vee 2\}$, $A_1 = \{4 \vee 2\}$, $B_1 = \varnothing$.

   c. Since B_1 is empty, we let

   $T = A_0 \cup A_1 = \{4 \vee 2\}$

   $M = T \cup M = \{4 \vee 2, 3, 4 \vee 3, 1, 6 \vee 4, 5 \vee 4, 4 \vee 1\}$.

4. $j = 4$.

   a. Applying Step 10, we obtain $R = \{2\}$, $A_0 = \{2\}$, $B_0 = \varnothing$.

   b. Since B_0 is empty, we let

   $T = A_0 = \{2\}$

   $M = T \cup M = \{2, 4 \vee 2, 3, 4 \vee 3, 1, 6 \vee 4, 5 \vee 4, 4 \vee 1\}$.

5. $j = 5$.

a. Applying Step 10, we obtain $R = \{\sim 1\}$, $A_0 = \varnothing$, $B_0 = \{\sim 1\}$.

b. Since neither A_0 contains $\square$ nor B_0 is empty, we apply Step 6 and obtain $W_1 = \{\square\}$, $A_1 = \{\square\}$, $B_1 = \varnothing$.

c. Since A_1 contains $\square$, we terminate the algorithm; a contradiction is found.

We have generated six positive hyperresolvents, namely, $\square$, 2, $4 \vee 2$, 3, $4 \vee 3$ and 1. To get the hyperdeduction of $\square$, we first trace the ancestry of $\square$ as shown in Fig. 6.8, which is then easily transformed into the hyperdeduction of $\square$ shown in Fig. 6.9. We note that all hyperresolvents generated by the algorithm are used in the proof. In this example, neither irrelevant nor redundant resolvents have been generated by the algorithm.

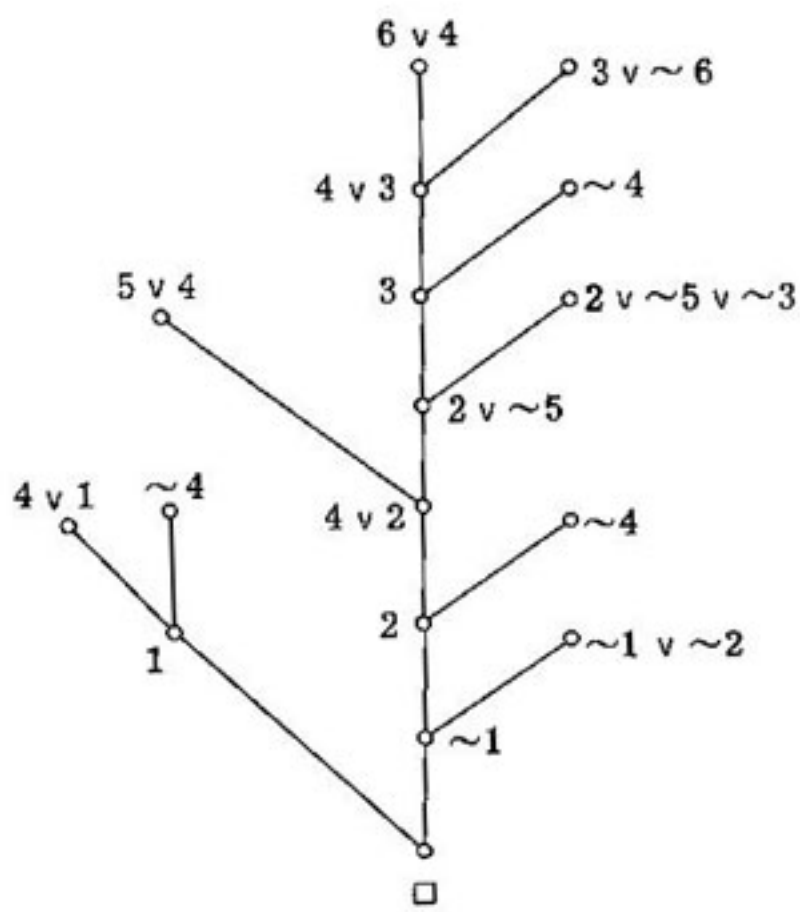

Figure 6.8

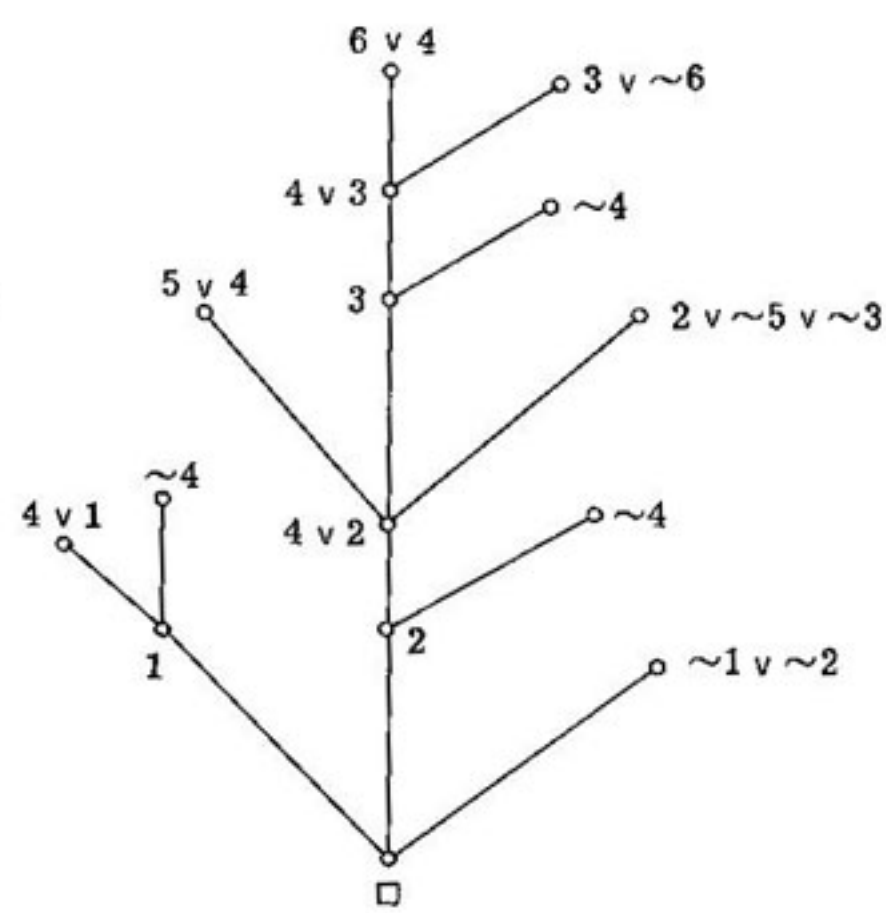

Figure 6.9

6.8 LOCK RESOLUTION

Lock resolution was introduced by Boyer [1971]. It is a refinement of resolution which uses a concept similar to that of ordered clauses. Given a set S of clauses, essentially, the idea of lock resolution is to use indices to order literals of clauses in S. That is, it involves arbitrarily indexing each occurrence of a literal in S with an integer; different occurrences of the same literal in S may be indexed differently. Resolution is then permitted only on literals of lowest index in each clause. The literals in resolvents

inherit their indices from their parent clauses. If a literal in a resolvent has more than one possible inherited index, the lowest index is assigned to this literal. Before we give a formal definition of lock resolution, we first consider the following example.

Example 6.20

Consider the following two clauses:

(1) ${}_1P \vee {}_2Q$

(2) ${}_3{\sim}P \vee {}_4Q$.

The integer beneath a literal is the index associated with that literal. The literals P and Q of clause (1) have indices 1 and 2, respectively. We shall use ${}_1P$ and ${}_2Q$ to refer to the literals P and Q of clause (1), respectively. Similarly, we shall use ${}_3{\sim}P$ and ${}_4Q$ to refer to the literals ${\sim}P$ and Q of clause (2), respectively. Since the index of ${}_1P$ is lower than that of ${}_2Q$, ${}_1P$ is permitted to be resolved upon. Similarly, since the index of ${}_3{\sim}P$ is lower than that of ${}_4Q$, ${}_3{\sim}P$ is permitted to be resolved upon. Thus, resolving clauses (1) and (2) upon ${}_1P$ and ${}_3{\sim}P$, we obtain the following clause:

(3) ${}_2Q \vee {}_4Q$.

Now ${}_2Q$ and ${}_4Q$ mean the same literal Q. Since 2 is lower than 4, Q is indexed with 2. Therefore, we obtain

(4) ${}_2Q$.

Clause (4) is called a lock resolvent of clauses (1) and (2). Note that if the literals of clause (2) are indexed differently as

(2*) ${}_4{\sim}P \vee {}_3Q$,

then the literal in clause (2*) that is permitted to be resolved upon is ${}_3Q$. However, ${}_1P$ and ${}_3Q$ cannot be resolved. Therefore, there is no lock resolvent of clauses (1) and (2*).

Definition Let C be a clause that has every one of its literals indexed with integers. If two or more literals (with the same sign) of C have a most general unifier σ, then the clause obtained from $C\sigma$ by deleting any literal that is identical to a literal of lower index is called a *lock factor* of C.

Example 6.21

Let $C = {}_2P(x) \vee {}_8Q(a) \vee {}_{11}P(a) \vee {}_5Q(x)$. ${}_2P(x)$ and ${}_{11}P(a)$ have a most general unifier $\sigma = \{a/x\}$. Thus $C\sigma = {}_2P(a) \vee {}_8Q(a) \vee {}_{11}P(a) \vee {}_5Q(a)$. In $C\sigma$, except for its index, ${}_2P(a)$ is identical to ${}_{11}P(a)$. Since ${}_{11}P(a)$ has higher index than does ${}_2P(a)$, ${}_{11}P(a)$ will be deleted. Similarly, since ${}_8Q(a)$ has higher

index than does ${}_5Q(a)$, ${}_8Q(a)$ will be deleted. Therefore, deleting ${}_{11}P(a)$ and ${}_8Q(a)$, we obtain ${}_2P(a) \vee {}_5Q(a)$, which is a lock factor of C.

In a clause, if there is more than one of the same literal, we always keep only the literal with the lowest index and delete the other identical literals. This operation is called *merging low* for identical literals. For instance, in the above example, we have merged low for ${}_2P(a)$ and ${}_{11}P(a)$, and also for ${}_8Q(a)$ and ${}_5Q(a)$.

Definition Let C_1 and C_2 be two clauses with no variable in common and with every literal in them indexed. Let L_1 and L_2 be two literals of "lowest" index in C_1 and C_2, respectively. If L_1 and $\sim L_2$ have a most general unifier σ, and if C is the clause obtained from $(C_1\sigma \vee C_2\sigma)$ by removing $L_1\sigma$ and $L_2\sigma$ and by merging low for any identical literals in the remaining clause, then C is called a *binary lock resolvent* of C_1 and C_2. The literals L_1 and L_2 are called the *literals resolved upon.*

Example 6.22

Consider the clauses: $C_1 = {}_1P(x) \vee {}_2Q(x) \vee {}_3R(x)$ and $C_2 = {}_4\sim P(a) \vee {}_5Q(a)$. Since ${}_1P(x)$ and ${}_4\sim P(a)$ have the lowest indices in C_1 and C_2, respectively, we choose $L_1 = P(x)$ and $L_2 = \sim P(a)$. L_1 and $\sim L_2$ have a most general unifier $\sigma = \{a/x\}$. Therefore, $(C_1\sigma \vee C_2\sigma) = {}_1P(a) \vee {}_2Q(a) \vee {}_3R(a) \vee {}_4\sim P(a) \vee {}_5Q(a)$. Removing $L_1\sigma$ and $L_2\sigma$, that is, ${}_1P(a)$ and ${}_4\sim P(a)$, from $(C_1\sigma \vee C_2\sigma)$, we obtain ${}_2Q(a) \vee {}_3R(a) \vee {}_5Q(a)$. Now, ${}_2Q(a)$ and ${}_5Q(a)$ are identical literals with different indices. Merging low for ${}_2Q(a)$ and ${}_5Q(a)$ in ${}_2Q(a) \vee {}_3R(a) \vee {}_5Q(a)$, we obtain ${}_2Q(a) \vee {}_3R(a)$, which is a binary lock resolvent of C_1 and C_2. ${}_1P(x)$ and ${}_4\sim P(a)$ are the literals resolved upon.

Definition Let C_1 and C_2 be two clauses with every literal in them indexed. A *lock resolvent* of C_1 and C_2 is one of the following binary lock resolvents:

1. a binary lock resolvent of C_1 and C_2;
2. a binary lock resolvent of C_1 and a lock factor of C_2;
3. a binary lock resolvent of a lock factor of C_1 and C_2;
4. a binary lock resolvent of a lock factor of C_1 and a lock factor of C_2.

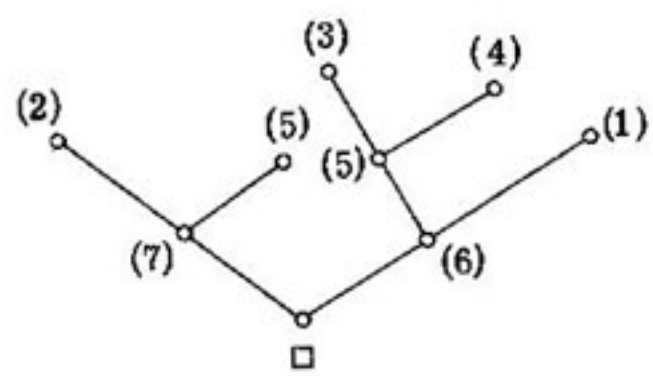

Figure 6.10

Definition Let S be a set of clauses, where every literal in S is indexed with an integer. A deduction from S is called a *lock deduction* if and only if every clause in the deduction is either a clause in S or a lock resolvent.

Example 6.23

Consider the following set S of clauses:

$$\left.\begin{array}{ll} (1) & {}_1P \vee {}_2Q \\ (2) & {}_3P \vee {}_4{\sim}Q \\ (3) & {}_6{\sim}P \vee {}_5Q \\ (4) & {}_8{\sim}P \vee {}_7{\sim}Q \end{array}\right\} S.$$

From clauses (1)–(4), there is only one lock resolvent:

(5) ${}_6{\sim}P$ from (3) and (4).

From clauses (1)–(5), there are only two lock resolvents:

(6) ${}_2Q$ from (1) and (5)

(7) ${}_4{\sim}Q$ from (2) and (5).

Finally, resolving clauses (6) and (7), we obtain

(8) $\square$.

Thus we obtain a lock deduction of $\square$ shown in Fig. 6.10. We note that altogether only three lock resolvents were generated. However, if ordinary (unrefined) resolution were used, 37 resolvents would be generated by the level-saturation method before $\square$ could be generated. (See Section 5.8.)

Example 6.24

Consider the following set of clauses:

$$\left.\begin{array}{ll} (1) & {}_5P(y,a) \vee {}_1P(f(y),y) \\ (2) & {}_6P(y,a) \vee {}_2P(y,f(y)) \\ (3) & {}_8{\sim}P(x,y) \vee {}_3P(f(y),y) \\ (4) & {}_9{\sim}P(x,y) \vee {}_4P(y,f(y)) \\ (5) & {}_{10}{\sim}P(x,y) \vee {}_7{\sim}P(y,a) \end{array}\right\} S.$$

From S, we can obtain the following lock deduction of $\square$:

(6) ${}_5P(a,a) \vee {}_{10}{\sim}P(x,f(a))$ a lock resolvent of (1) and (5)

(7) ${}_8{\sim}P(x,a) \vee {}_{10}{\sim}P(y,f(a))$ a lock resolvent of (3) and (5)

(8)	${}_{10}\sim P(x, f(a)) \vee {}_{10}\sim P(y, f(a))$	a lock resolvent of (6) and (7)
(9)	${}_{6}P(a, a)$	a lock resolvent of (8) and (2)
(10)	${}_{9}\sim P(x, a)$	a lock resolvent of (8) and (4)
(11)	$\square$	a lock resolvent of (9) and (10).

6.9 COMPLETENESS OF LOCK RESOLUTION

In this section, we shall prove the completeness of lock resolution. As usual, this is done by first proving the completeness of lock resolution for the ground case, which is then "lifted" to the general case. For a clause C that has every one of its literals indexed, any instance of C is obtained by first making substitutions for variables and then merging low for identical literals. That is, if σ is a substitution, first we obtain $C\sigma$ and then merge low for identical literals in $C\sigma$ to obtain an instance of C. For example, let $C = {}_1P(x) \vee {}_2P(y)$. If $\sigma = \{a/x, a/y\}$, then an instance of C is ${}_1P(a)$. By this convention, the following lifting lemma is obvious. We leave its proof to the reader. (Use Lemma 5.1.)

Lemma 6.2 Let C_1 and C_2 be two clauses with every literal of them indexed. If C_1' and C_2' are instances of C_1 and C_2, respectively, and if C' is a lock resolvent of C_1' and C_2', then there is a lock resolvent C of C_1 and C_2 such that C' is an instance of C.

We now prove the completeness of lock resolution for the ground case.

Lemma 6.3 Let S be a set of "ground" clauses, where every literal in S is indexed with an integer. If S is unsatisfiable, then there is a lock deduction of the empty clause $\square$ from S.

Proof Let $k(S)$ be defined to be the total number of appearances of literals in S minus the number of clauses in S. $k(S)$ is called the *excess literal parameter* by Anderson and Bledsoe [1970]. We prove Lemma 6.3 by induction on $k(S)$. If $\square$ is in S, Lemma 6.3 is obvious. Assume $\square$ is not in S. If $k(S) = 0$, then S consists of only unit clauses. Since S is unsatisfiable, there is a literal L such that ${}_iL$ and ${}_j\sim L$ are in S, where i and j are the indices of ${}_iL$ and ${}_j\sim L$, respectively. Clearly, $\square$ is a lock resolvent of ${}_iL$ and ${}_j\sim L$. Thus, Lemma 6.3 holds for $k(S) = 0$. Assume Lemma 6.3 holds when $k(S) < n$. To complete the induction, we consider the case where $k(S) = n$ and $n > 0$. Since $k(S) > 0$, there is at least one nonunit clause in S. Let r be the highest index among all the nonunit clauses of S. Let $C = C' \vee {}_rL$ be a nonunit clause that has a literal ${}_rL$ indexed with r, where C' is a nonempty clause. Let $S_1 = (S - \{C\}) \cup \{C'\}$ and $S_2 = (S - \{C\}) \cup \{{}_rL\}$. Clearly, S_1 and

S_2 are unsatisfiable. However, $k(S_1) < n$ and $k(S_2) < n$. Hence, by the induction hypothesis, there are lock deductions D_1' and D_2' of $\square$ from S_1 and S_2, respectively. Let D_1 be the deduction obtained from D_1' by putting ${}_rL$ back with C'. Clearly, D_1 is also a lock deduction since r is the highest index in a clause attached to any noninitial node of D_1. D_1 is a lock deduction of $\square$ or ${}_rL$ from S. If it is the former, we are done, If it is the latter, combining D_1 and D_2', we obtain a lock deduction of $\square$ from S. This completes the proof of Lemma 6.3.

Theorem 6.3 (Completeness of Lock Resolution) Let S be a set of clauses, where every literal in S is indexed with an integer. If S is unsatisfiable, then there is a lock deduction of the empty clause $\square$ from S.

Proof Since S is unsatisfiable, by Herbrand's theorem (vertion II) there is a finite unsatisfiable set S' of ground instances of clauses in S. By Lemma 6.3, there is a lock deduction D' of $\square$ from S'. Using Lemma 6.2 and a process similar to that given in the proof of Theorem 6.1, we can transform D' to produce a lock deduction of $\square$ from S. This completes the proof.

Although lock resolution itself is complete, it is not compatible with most other resolution strategies. (See [Boyer, 1971].) For example, the combination of lock resolution and the deletion strategy is not complete. Lock resolution is not compatible with the set-of-support strategy either. (See Exercises 20 and 21 of this chapter.) Lock resolution is a very restrictive refinement of resolution. Boyer has given many examples to show that lock resolution is indeed a very efficient inference rule.

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EXERCISES

Section 6.3

1. Determine whether the following sets of clauses, interpretations, and orderings of atoms constitute *PI*-clashes. If yes, find the *PI*-resolvents. If no, give reasons.

   (a) $E_1 = P, E_2 = Q, E_3 = R, N = \sim P \vee \sim Q \vee \sim R \vee \sim W$

   $I = \{\sim P, \sim Q, \sim R, W\}$

   $P < Q < R < W.$

   (b) $E_1 = A_1 \vee A_3, E_2 = A_2 \vee A_3, N = \sim A_1 \vee \sim A_2 \vee A_3$

   $I = \{\sim A_1, \sim A_2, \sim A_3\}$

   $A_1 > A_2 > A_3.$

   (c) $E_1 = A_1 \vee A_3, E_2 = A_2 \vee A_3, N = \sim A_1 \vee \sim A_2 \vee A_3$

   $I = \{\sim A_1, \sim A_2, \sim A_3\}$

   $A_3 > A_1 > A_2.$

   (d) $E_1 = P(a), E_2 = Q(a,b) \vee S(b,c), N = \sim P(x) \vee \sim Q(x,y) \vee R(x)$

   $I = \{\sim P(a), P(b), \sim Q(a,b), \sim S(b,c), \sim R(a)\}$

   $P > Q > R > S.$

2. Prove the following theorem: For the ground case, the nucleus N of a *PI*-clash is true in I, and for the general (nonground) case, there exists at least one instance N' of N such that N' is true in I.

3. Let $S = \{P, Q \vee \sim P, R \vee \sim P, \sim P \vee \sim Q \vee \sim R\}$. Prove that S is unsatisfiable by semantic resolution. The following cases are to be considered:

   (a) $I = \{\sim P, \sim Q, \sim R\}; P > Q > R$

   (b) $I = \{P, Q, R\}; Q > P > R$

   (c) $I = \{\sim P, \sim Q, R\}; R > Q > P$.

Section 6.4

4. Prove Theorem 6.1 by using the semantic tree technique, that is, the technique given in the proof of Theorem 5.3. (Consult [Kowalski and Hayes, 1969].)
5. Prove that the deletion strategy is compatable with PI-resolution, that is, that the combination of the deletion strategy and PI-resolution is complete.

Section 6.5

6. Let $S = \{P \vee Q, \sim P \vee Q, P \vee \sim Q, \sim P \vee \sim Q\}$. Let $P > Q$. Prove that S is unsatisfiable by (a) positive hyperresolution, and (b) negative hyperresolution.
7. Consider Example 5.16. In that example, we have the following set S of clauses:

$$S = \{\sim T(x,y,u,v) \vee P(x,y,u,v), \sim P(x,y,u,v) \vee E(x,y,v,u,v,y),$$
$$T(a,b,c,d), \sim E(a,b,d,c,d,b)\}.$$

   Prove that S is unsatisfiable by (a) positive hyperresolution, and (b) negative hyperresolution. Translate every step of each proof into English. Which deduction corresponds to thinking forward (thinking backward)?
8. Let

$$\left.\begin{array}{ll}
(1) & M(a, s(c), s(b)) \\
(2) & P(a) \\
(3) & M(x, x, s(x)) \\
(4) & \sim M(x,y,z) \vee M(y,x,z) \\
(5) & \sim M(x,y,z) \vee D(x,z) \\
(6) & \sim P(x) \vee \sim M(y,z,u) \vee \sim D(x,u) \vee D(x,y) \vee D(x,z) \\
(7) & \sim D(a,b)
\end{array}\right\} S.$$

Let the set of support consist of clause (7). Prove the unsatisfiability of S by the set-of-support strategy.

9. Reconsider Example 5.21. We have the following set of clauses:

   (1) $P(a)$

   (2) $\sim D(y) \vee L(a,y)$

   (3) $\sim P(x) \vee \sim Q(y) \vee \sim L(x,y)$

   (4) $D(b)$

   (5) $Q(b)$.

   Let the set of support consist of clause (4) and clause (5). Prove the unsatisfiability of the above set of clauses. Translate the entire proof into English.

Section 6.6

10. Consider the following set of ordered clauses:

$$S = \{P \vee Q, \sim P \vee Q, P \vee \sim Q, \sim P \vee \sim Q\}.$$

   Let $I = \{P,Q\}$. Prove that S is unsatisfiable by OI-resolution.

11. In Example 6.18, we gave a counterexample to show that OI-resolution is not complete. Can you give another counterexample?

Section 6.7

12. Let $S = \{\sim P \vee \sim Q \vee \sim R, P, Q, R\}$.

   a. Use the algorithm mentioned in Section 6.7 to prove that S is unsatisfiable.
   b. Use the same algorithm again without the concept of ordered clauses, that is, in Step 6, the resolved literal of C_2 is not restricted to the last literal of C_2.
   c. Compare the number of clauses generated in each case.

13. Use the algorithm of Section 6.7 to prove the theorems given in (a) Example 5.21, and (b) Example 5.22.

14. Show that if a set S of ordered clauses is unsatisfiable, then $\square$ can be generated by the algorithm of Section 6.7.

Section 6.8

15. Let $S = \{P, Q, R, W, \sim P \vee \sim Q \vee \sim R \vee \sim W\}$.

   a. If ordinary resolution is used, how many resolvents will be generated from S by the level-saturation method before $\square$ can be generated?

b. Index the literals of S. If lock resolution is used, how many lock resolvents will be generated from S by the level-saturation method before $\square$ can be generated?
c. Compare the number of clauses generated in each case.

16. Consider the set S of clauses given in Example 6.18. Let us index literals of S as follows:

(1) ${}_7P \vee {}_1Q$ (2) ${}_8Q \vee {}_2R$

(3) ${}_9R \vee {}_3W$ (4) ${}_{10}{\sim}R \vee {}_4{\sim}P$

(5) ${}_{11}{\sim}W \vee {}_5{\sim}Q$ (6) ${}_{12}{\sim}Q \vee {}_6{\sim}R$.

Deduce the empty clause $\square$ from S by lock resolution.

17. Consider the set of clauses (this set is taken from Example 5.22):

(1) ${}_1{\sim}E(x) \vee {}_2V(x) \vee {}_3S(x, f(x))$

(2) ${}_4{\sim}E(x) \vee {}_5V(x) \vee {}_6C(f(x))$

(3) ${}_7P(a)$

(4) ${}_8E(a)$

(5) ${}_9{\sim}S(a, y) \vee {}_{10}P(y)$

(6) ${}_{11}{\sim}P(x) \vee {}_{12}{\sim}V(x)$

(7) ${}_{13}{\sim}P(x) \vee {}_{14}{\sim}C(x)$.

Derive the empty clause $\square$ from the above set by lock resolution.

Section 6.9

18. Prove Lemma 6.2.
19. For any arbitrary set S of clauses,

(a) give a method to index every literal in S so that the set of all lock resolvents from S is equal to the set of all ordinary (unrefined) resolvents from S, that is, simulate resolution by lock resolution;
(b) simulate positive hyperresolution by lock resolution.

20. Give a counterexample to show that the combination of the deletion strategy and lock resolution is not complete.
21. Give a counterexample to show that the combination of the set-of-support strategy and lock resolution is not complete.

Chapter 7

Linear Resolution

7.1 INTRODUCTION

When proving an identity, we often start with the left-hand side of the identity, apply an inference rule to it to obtain some other expression, then apply some inference rule again to the expression we have just obtained until the left-side expression is identical to the right-side expression of the identity. The idea of linear resolution is similar to this kind of chain reasoning. It starts with a clause, resolves it against a clause to obtain a resolvent, and resolves this resolvent against some clause until the empty clause $\square$ is obtained.

The special appeal of linear deduction is its simple structure. Furthermore, linear resolution is complete and is compatible with the set-of-support strategy. In addition, as will be shown, heuristic methods can be conveniently applied with it.

Linear resolution was independently proposed by Loveland [1970a] and Luckham [1970]. It was later strengthened by Anderson and Bledsoe [1970], Yates *et al.* [1970], Reiter [1971], Loveland [1972], and Kowalski and Kuehner [1971]. In this chapter, we shall consider the version given by Kowalski and Kuehner [1971] and Loveland [1972] because it can be easily implemented on a digital computer.

7.2 LINEAR RESOLUTION

Definition Given a set S of clauses and a clause C_0 in S, a *linear deduction* of C_n from S with *top clause* C_0 is a deduction of the form shown in Fig. 7.1 where

1. for $i = 0, 1, \ldots, n-1$, C_{i+1} is a resolvent of C_i (called a *center clause*) and B_i (called a *side clause*), and
2. each B_i is either in S, or is a C_j for some j, $j < i$.

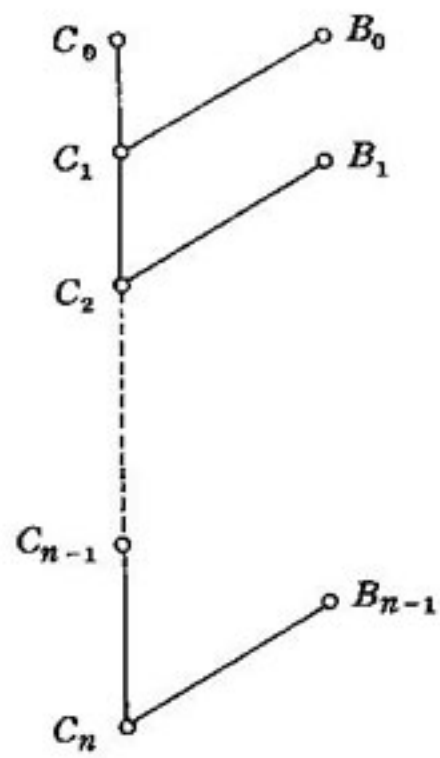

Figure 7.1

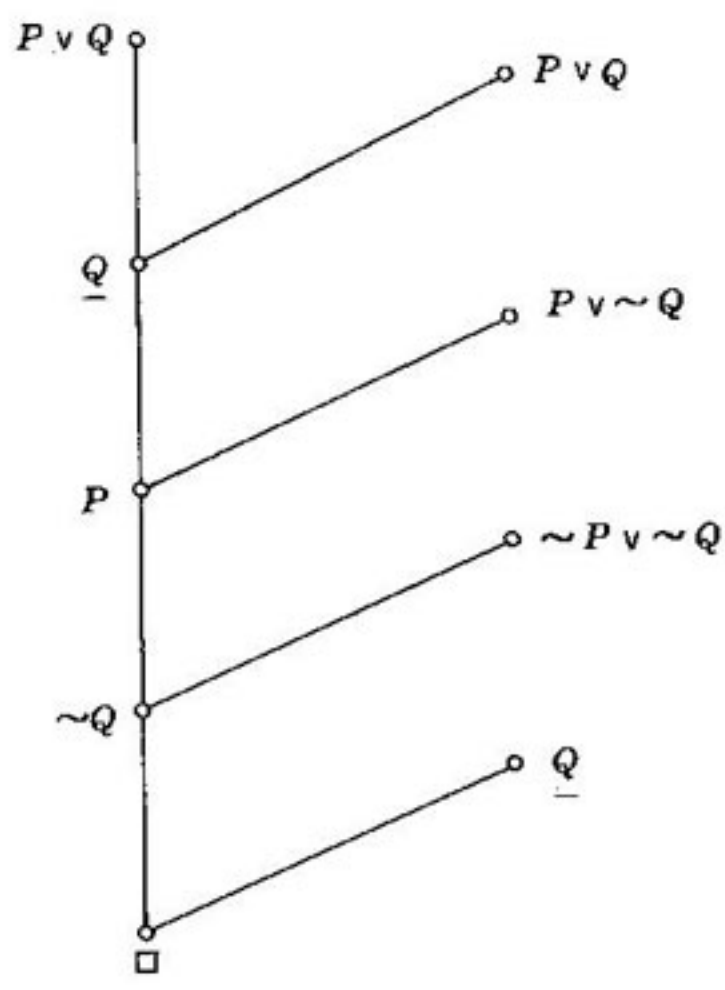

Figure 7.2

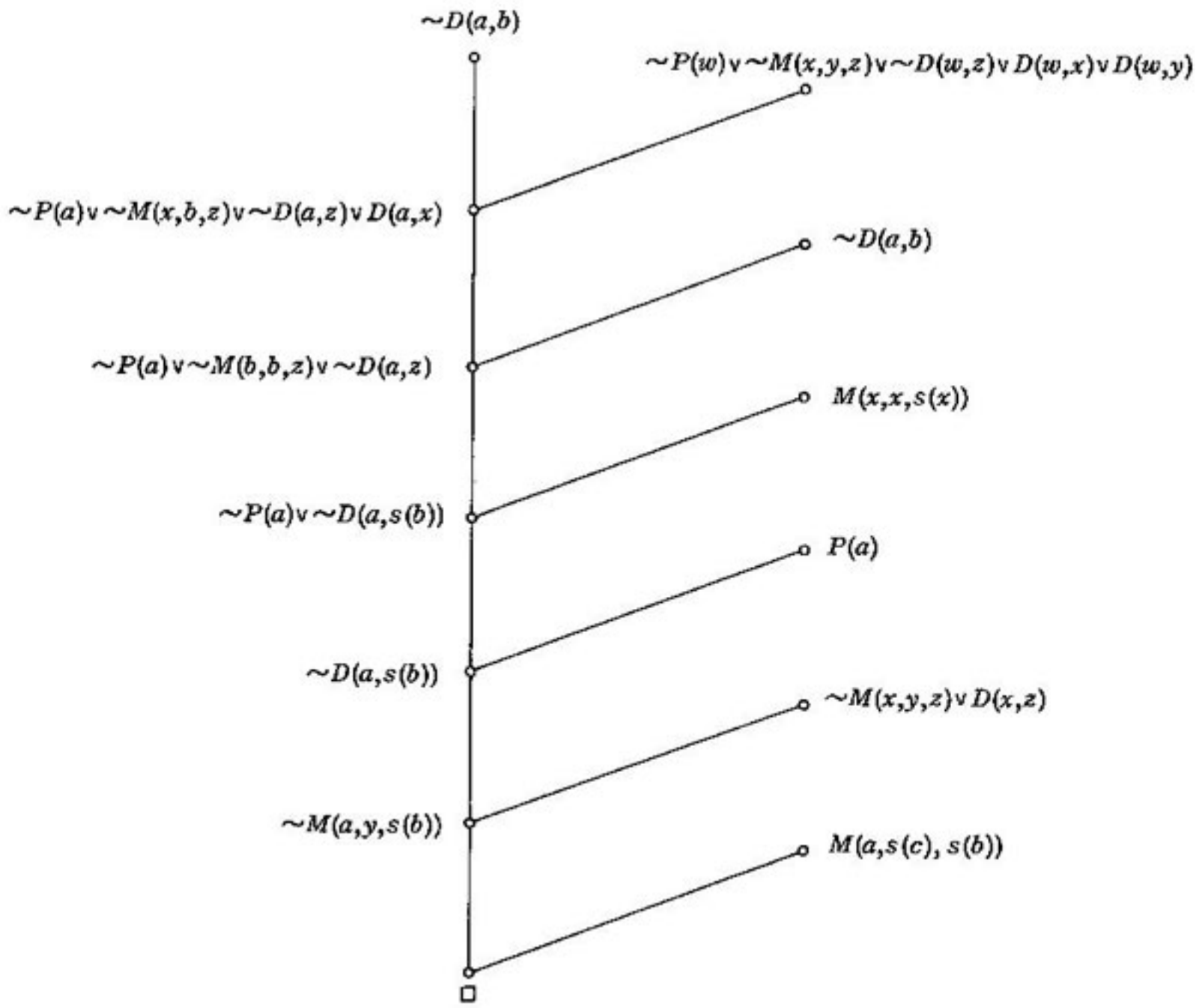

Figure 7.3

Example 7.1

Let $S = \{P \vee Q, \sim P \vee Q, P \vee \sim Q, \sim P \vee \sim Q\}$. Then Fig. 7.2 is a linear deduction of $\square$ from S with top clause $P \vee Q$. There are four side clauses. Three of them are from S, while one of them, namely, Q (underlined), is a resolvent derived before $\square$ is deduced.

Example 7.2

Consider the set of clauses $S = \{M(a, s(c), s(b)),\ P(a),\ M(x, x, s(x)),\ \sim M(x, y, z) \vee M(y, x, z),\ \sim M(x, y, z) \vee D(x, z),\ \sim P(x) \vee \sim M(y, z, u) \vee \sim D(x, u) \vee D(x, y) \vee D(x, z),\ \sim D(a, b)\}$. Figure 7.3 is a linear deduction of $\square$ from S.

The linear deduction introduced in this section is a primitive one and is later modified to the *OL-deduction* discussed in Section 7.4. The proof of the completeness of linear resolution is given in Section 7.5.

7.3 INPUT RESOLUTION AND UNIT RESOLUTION

When we consider a refinement of resolution, we would like it to be complete, that is, to guarantee that the empty clause can always be derived from an

unsatisfiable set of clauses. However, efficiency is also important in mechanical theorem proving. Sometimes, we may like to trade completeness for efficiency. That is, there may be some refinements of resolution that are efficient but incomplete. If a refinement of resolution is efficient and powerful enough to prove a large class of theorems, even though it is not complete, it may still be useful. In this section, we shall consider some incomplete but efficient refinements of resolution, namely, input resolution and unit resolution. Input resolution is a subcase of linear resolution. It is much easier to implement and, although not complete, is more efficient than linear resolution. We shall show that input resolution is equivalent to unit resolution. That is, theorems that can be proved by input resolution can also be proved by unit resolution, and vice versa.

Given a set S of clauses, since S is the original input set, we shall call each member of S an *input clause*.

Definition An *input resolution* is a resolution in which one of the two parent clauses is an input clause. An *input deduction* (to emphasize the input set S, we sometimes shall say *S-input deduction*) is a deduction in which every resolution is an input resolution. An *input refutation* is an input deduction of $\square$ from S.

An input deduction is actually a linear deduction in which every side clause is an input clause, and hence is a subcase of linear deduction.

Example 7.3

Consider the following set of clauses:

$$\sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(x, v, w) \vee P(u, z, w)$$

$$P(g(x, y), x, y)$$

$$P(x, h(x, y), y)$$

$$\sim P(k(x), x, k(x)).$$

Figure 7.4 shows an input refutation from this set.

Definition A *unit resolution* is a resolution in which a resolvent is obtained by using at least one unit parent clause, or a unit factor of a parent clause. A *unit deduction* is a deduction in which every resolution is a unit resolution. A *unit refutation* is a unit deduction of $\square$.

Unit resolution is essentially an extension of the one-literal rule of Davis and Putnam (see Section 4.6) to the first-order logic. This rule is especially important, for in order to deduce $\square$ from a given set of clauses, one must obtain successively shorter clauses, and unit resolution provides a

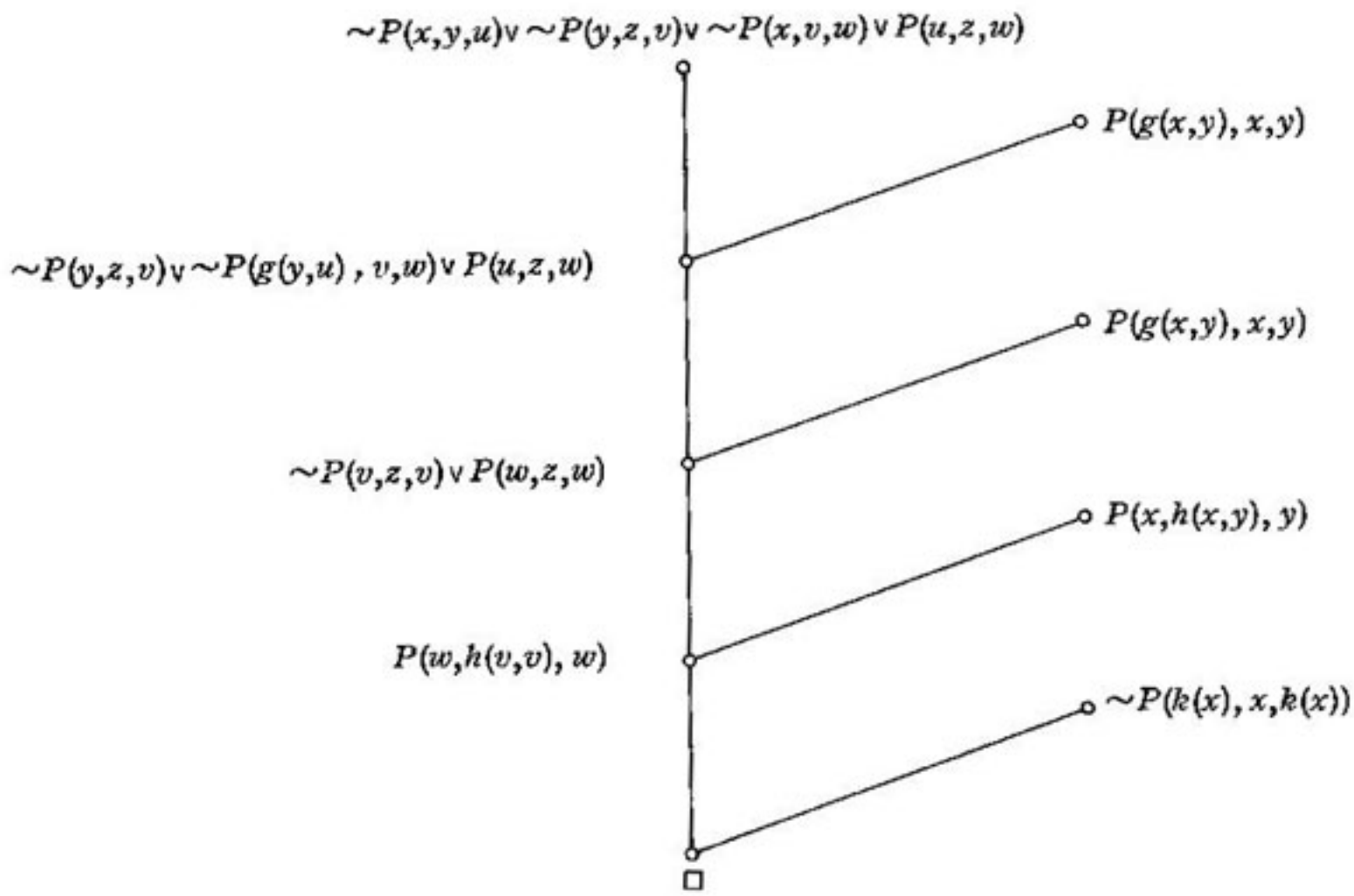

Figure 7.4

means for progressing rapidly toward shorter clauses. Unit resolution was used intensively by Wos, Carson, and Robinson [1964]. We now prove that unit resolution and input resolution are equivalent [Chang, 1970].

Lemma 7.1 There is a unit refutation from a set S of ground clauses if and only if there is an input refutation from S.

Proof Lemma 7.1 is proved by induction. Let A be the atom set of S. If A consists of a single element, say Q, then among the elements of S, there exist unit clauses Q and $\sim Q$. Clearly, the resolvent of Q and $\sim Q$ is the empty clause $\square$. This deduction is both a unit and an input refutation. Hence, Lemma 7.1 holds for this case. Assume Lemma 7.1 holds when A consists of i elements, $1 \leqslant i \leqslant n$. To complete the induction, we consider A such that A consists of exactly $n+1$ elements.

($\Rightarrow$) If there is a unit refutation from S, then S must contain at least one unit clause, say L, where L is a literal. Let S' be that set obtained from S by deleting those clauses containing the literal L and by deleting $\sim L$ from the remaining clauses. Clearly, since there is a unit refutation from S, there must be a unit refutation from S'. (Proof of this is left as an exercise for the reader.) But S' contains n or fewer than n atoms. By the induction hypothesis, there is an S'-input refutation D' from S'. Let D be the deduction obtained from D' by putting the literal $\sim L$ back into the clauses from which it was deleted. Let T be the clause attached to the root node of D. Clearly, D is an

S-input deduction of T from S. T must be either $\square$ or $\sim L$. If T is $\square$, we are done. If T is $\sim L$, we can obtain $\square$ by resolving T with L, which is an input clause. Thus, the refutation obtained from D and the resolution of T and L is an input refutation from S. Therefore we have completed the proof of the first part of Lemma 7.1.

($\Leftarrow$) Conversely, if there is an input refutation from S, then S must contain at least one unit clause, say L, where L is a literal.

Let

$$S' = \{S \cup [\text{Res}(C, L) \mid C \in S]\} - \{\text{all clauses that contain } L \text{ or } \sim L\}$$

where Res(C, L) denotes a resolvent of a clause C and the clause L. Since there is an S-input refutation from S, there must be an S'-input refutation from S'. But S' contains n or fewer than n atoms. By the induction hypothesis, there is a unit refutation from S'. However, every clause in S' is either a member of S or a resolvent obtained by applying a unit resolution on the clause L and a clause in S. Therefore, there is a unit refutation from S. This completes the proof of the second half of Lemma 7.1.

Theorem 7.1 (Equivalence of Unit and Input Resolutions) There is a unit refutation from a set S of clauses if and only if there is an input refutation from S.

Proof ($\Rightarrow$) If there is a unit refutation D_1 from S, then from D_1 we can obtain a ground unit refutation D_1' by replacing the clause C attached to each node of D_1 by an appropriate ground instance of C. Let S' be the set of ground clauses attached to the initial nodes of D_1'. By Lemma 7.1, there is an input refutation D_2' from S'. From D_2', using the lifting lemma (Lemma 5.1), we can obtain an input refutation D_2 from S.

($\Leftarrow$) The proof of the second half of Theorem 7.1 is the same as the above proof except that the words "unit" and "input" are interchanged. Q.E.D.

Given a set S of clauses, if S has an input refutation, then by Theorem 7.1 we can also derive $\square$ from S by applying unit resolution. Since unit resolution is easier to program than input resolution, we have implemented only unit resolution. A LISP program implementing unit resolution is described in Appendix A. This program has been shown to be quite efficient.

7.4 LINEAR RESOLUTION USING ORDERED CLAUSES AND THE INFORMATION OF RESOLVED LITERALS

We can strengthen linear resolution by introducing two concepts into it. One of them is the concept of ordered clauses. As discussed in Chapter 6,

incorporation of this concept into semantic resolution greatly increases its efficiency. We shall discover later that this is also true for linear resolution. Furthermore, unlike semantic resolution, incorporation of the concept of ordered clauses into linear resolution will *not* destroy its completeness. Another concept is one that uses the information of literals resolved upon. In resolution, when a resolvent is obtained, literals resolved upon are deleted. Actually, the information provided by these literals is very useful. As discovered by Loveland [1968, 1969a, b, 1972] and Kowalski and Kuehner [1971], this information can be used to improve linear resolution.

Before introducing the mechanism of using the information of resolved literals, let us first examine a rather simple case. Consider a set S of clauses defined as $S = \{P \vee Q, P \vee \sim Q, \sim P \vee Q, \sim P \vee \sim Q\}$. Fig. 7.5 shows a linear refutation from S.

Note that one of the side clauses (clause P) is not an input clause. In fact it is easy to see that there is no unit proof for this set of clauses. Thus, according to Theorem 7.1, there is no input proof either. That is, the usage of a center clause as a side clause is inevitable.

It would be nice if we could find a necessary and sufficient condition under which a side clause must be a center clause generated previously. That is, a side clause is a previously generated center clause when and only when the condition is satisfied.

We shall show that if the information of resolved literals is appropriately

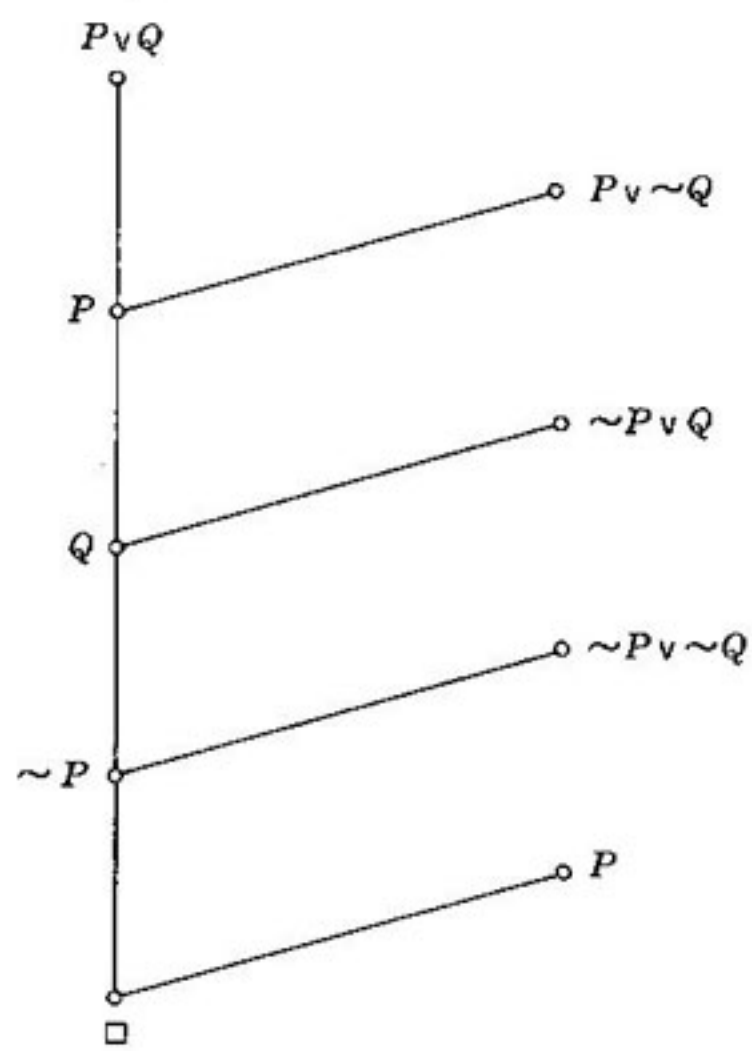

Figure 7.5

recorded and the concept of ordered clauses is used, we can clearly define this condition. It will be obvious later that using this condition will cut down the number of possible resolutions. Actually, it can also allow us to do more. If we have determined that a side clause must be a center clause previously generated, we do not even have to determine which center clause to use. We simply perform an operation on the clause and obtain a new clause. Therefore, we do not have to store previous center clauses in the computer memory. This greatly simplifies the computer implementation of linear resolution.

We now discuss the mechanism of recording the information of resolved literals. Suppose C_1 and C_2 are two ordered clauses given as

$$C_1 = P \vee Q$$
$$C_2 = \sim Q \vee R.$$

There is an ordered resolvent (namely, $P \vee R$) of C_1 against C_2, with Q and $\sim Q$ being the literals resolved upon. Since Q and $\sim Q$ are complementary to each other, we need only record one of them. Suppose we agree to record Q, the last literal of C_1. Then the ordered resolvent can be represented by $P \vee \boxed{Q} \vee R$, where the framed literal is the literal resolved upon. This is how we store the information of resolved literals. That is, literals resolved upon are indicated by framed literals in an ordered clause. The framed literals are merely for recording those literals that have been resolved upon; they do not participate in resolution.

In the above scheme, if a framed literal is not followed by any unframed literal, we shall delete this framed literal. That we can do this will be explained later.

The algorithm that employs both the concept of ordered clauses and the information of resolved literals is called OL-deduction (ordered linear deduction). Before presenting the precise algorithm, we first describe it by an example.

Example 7.4

Consider the set of ordered clauses defined as $S = \{P \vee Q, P \vee \sim Q, \sim P \vee Q, \sim P \vee \sim Q\}$. A linear refutation using the information of resolved literals and the ordered clause concept is shown in Fig. 7.6, where the center clauses are obtained as follows:

1. We start with the ordered clause $P \vee Q$ as the top clause. The last literal of this ordered clause is Q, which can be resolved with $P \vee \sim Q$ upon $\sim Q$. Therefore, resolving $P \vee Q$ against $P \vee \sim Q$ and recording the literal Q, we obtain an ordered resolvent that is represented by $P \vee \boxed{Q}$. Since $\boxed{Q}$ is not followed by any unframed literal, it is deleted. Therefore, we obtain P, as shown in Fig. 7.6.

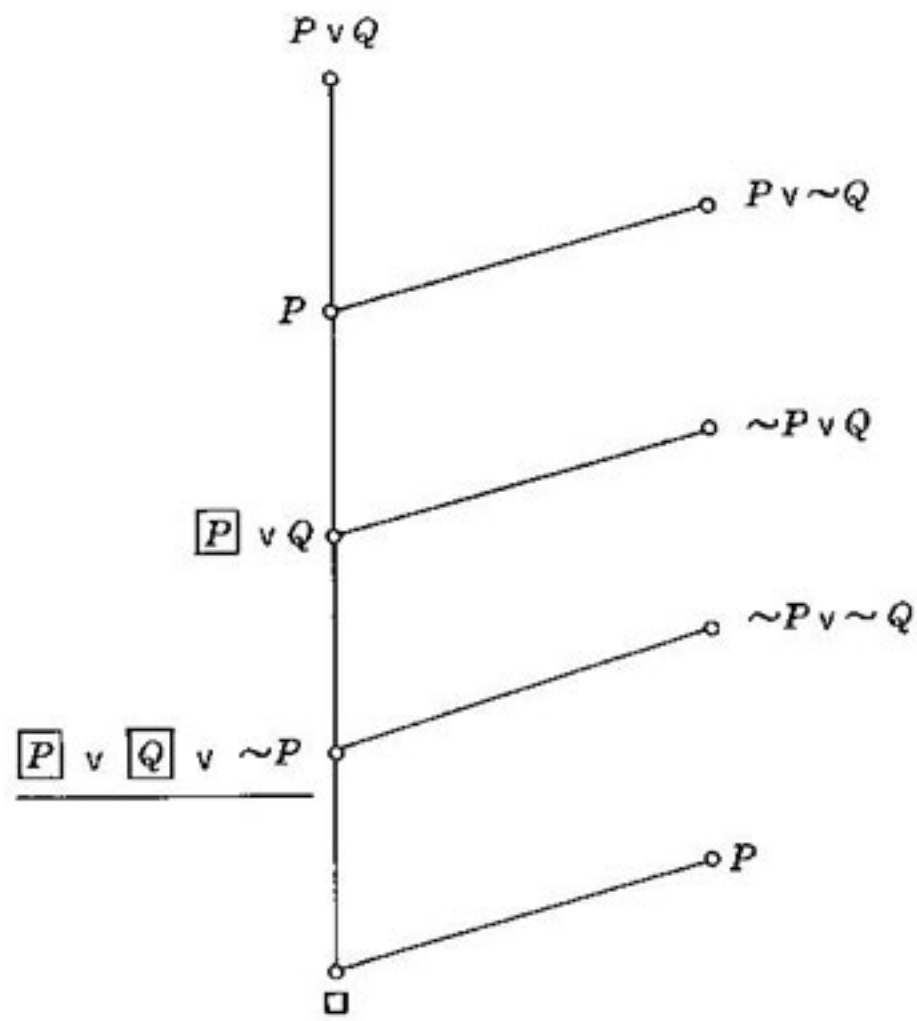

Figure 7.6

2. The last literal of the ordered clause obtained in (1) is P, which can be resolved against $\sim P \vee Q$ upon $\sim P$. Therefore, resolving P against $\sim P \vee Q$ and framing the literal P, we obtain $\boxed{P} \vee Q$.

3. The last literal of $\boxed{P} \vee Q$ is Q, which can be resolved with $\sim P \vee \sim Q$. Thus, resolving $\boxed{P} \vee Q$ against $\sim P \vee \sim Q$ and recording Q, we obtain $\boxed{P} \vee \boxed{Q} \vee \sim P$.

4. Now we have obtained the clause $\boxed{P} \vee \boxed{Q} \vee \sim P$. This clause has an interesting feature. Note that the last literal of $\boxed{P} \vee \boxed{Q} \vee \sim P$ is $\sim P$, which is complementary to one of its framed literals, namely $\boxed{P}$. This is the condition that we were trying to find before. In other words, the side clause at this step must be a center clause. In this case, the clause is P. Resolving $\boxed{P} \vee \boxed{Q} \vee \sim P$ with P upon $\sim P$, we obtain $\boxed{P} \vee \boxed{Q} \vee \boxed{\sim P}$. However, none of these framed literals is followed by unframed literals. Therefore, deleting them, we obtain $\square$.

As emphasized before, the clause $\boxed{P} \vee \boxed{Q} \vee \sim P$ is a special one because the last literal of this clause is complementary to one of its framed literals. Every time this kind of clause is generated, we shall use a center clause as the side clause. We shall call this kind of clause a reducible clause.

Definition An ordered clause C is a *reducible ordered clause* if and only if the last literal of C is unifiable with the negation of a framed literal of C.

Actually, whenever a reducible ordered clause is generated, we do not have to search the memory for a deduced center clause to resolve it with. Instead, we may simply delete the last literal from this ordered clause. For example, when $\boxed{P} \vee \boxed{Q} \vee \sim P$ is generated, we simply delete $\sim P$ from $\boxed{P} \vee \boxed{Q} \vee \sim P$ and obtain $\boxed{P} \vee \boxed{Q}$. Since $\boxed{P}$ and $\boxed{Q}$ are not followed by any unframed literal, they are deleted and thus we obtain $\square$. We shall call this kind of operation the *reduction* of a reducible ordered clause.

Definition Let C be a reducible ordered clause. Let the last literal L be unifiable with the negation of some framed literal with a most general unifier σ. The *reduced ordered clause of* C is the ordered clause obtained from $C\sigma$ by deleting $L\sigma$ and every subsequent framed literal not followed by any unframed literal.

Once the reduction of reducible clauses is incorporated into OL-deduction, we do not have to store intermediate clauses any more. This important aspect of OL-deduction makes it very suitable for computer implementation. Note that the detection of a reducible ordered clause already narrows down the choice of resolutions. The introduction of the reduction mechanism effectively reduces it to only one resolution.

OL-deduction is essentially the same as what Loveland [1968, 1969a, b, 1972] has called the model elimination method. OL-deduction is also essentially a special *SL*-deduction of Kowalski and Kuehner [1971]. We now formally define OL-deduction. In the following, an ordered clause may contain both framed and unframed literals. To take this into consideration, we repeat the definitions of an ordered factor, an ordered binary resolvent, and an ordered resolvent given in Section 6.6.

In an ordered clause, if there is more than one occurrence of the same unframed literal, we always keep only the leftmost one and delete the other identical unframed literals. This operation is called *merging left* for identical unframed literals. For instance, merging left for identical unframed literals in $P \vee Q \vee \boxed{R} \vee S \vee Q \vee P$, we obtain $P \vee Q \vee \boxed{R} \vee S$.

Definition If two or more unframed literals (with the same sign) of an ordered clause C have a most general unifier σ, the ordered clause obtained from the sequence $C\sigma$ by merging left for any identical literals in $C\sigma$ and by deleting every framed literal not followed by an unframed literal in the remaining clause is called an *ordered factor* of C.

Definition Let C_1 and C_2 be two ordered clauses with no variables in common and L_1 and L_2 be two unframed literals in C_1 and C_2, respectively. Let L_1 and $\sim L_2$ have a most general unifier σ. Let C^* be the ordered clause obtained by concatenating the sequence $C_1\sigma$ and $C_2\sigma$, framing $L_1\sigma$, removing $L_2\sigma$, and merging left for any identical unframed literals in the

remaining sequence. Let C be obtained from C^* by removing every framed literal not followed by any unframed literal in C^*. C is called an *ordered binary resolvent* of C_1 against C_2. The literals L_1 and L_2 are called the *literals resolved upon.*

Definition An *ordered resolvent* of an ordered clause C_1 against an ordered clause C_2 is any of the following ordered binary resolvents:

1. an ordered binary resolvent of C_1 against C_2;
2. an ordered binary resolvent of C_1 against an ordered factor of C_2;
3. an ordered binary resolvent of an ordered factor of C_1 against C_2;
4. an ordered binary resolvent of an ordered factor of C_1 against an ordered factor of C_2.

Definition Given a set S of ordered clauses and an ordered clause C_0 in S, an OL-deduction of C_n from S with top ordered clause C_0 is a deduction of the form shown in Fig. 7.7, which satisfies the following conditions:

1. For $i = 0, 1, 2, \ldots, n-1, C_{i+1}$ is an ordered resolvent of C_i (called a *center ordered clause*) against B_i (called a *side ordered clause*); the literal resolved upon in C_i (or an ordered factor of C_i) is the *last* literal.
2. Each B_i is either an ordered clause in S or an instance of some $C_j, j < i$. B_i is an instance of some $C_j, j < i$, if and only if C_i is a reducible ordered clause. In this case, C_{i+1} is the reduced ordered clause of C_i.
3. No tautology is in the deduction.

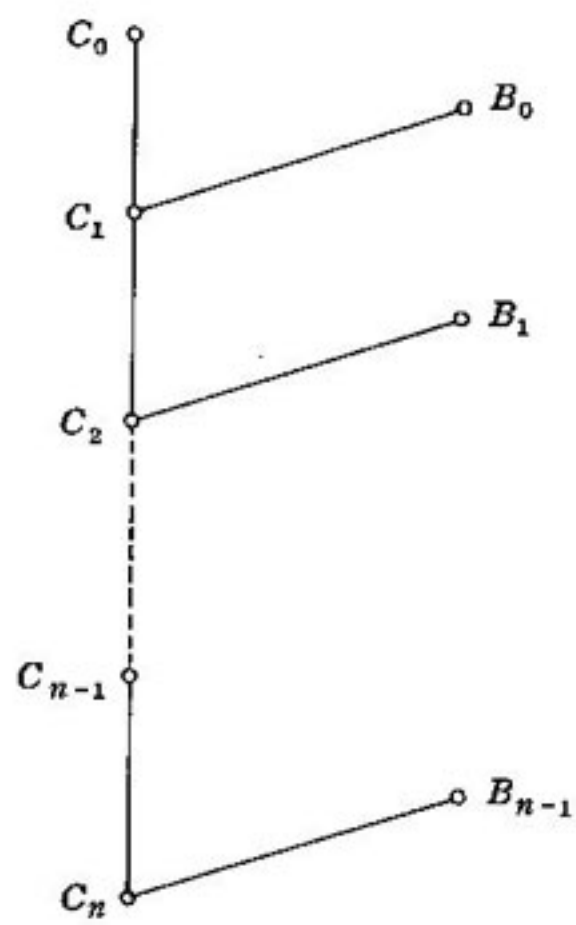

Figure 7.7

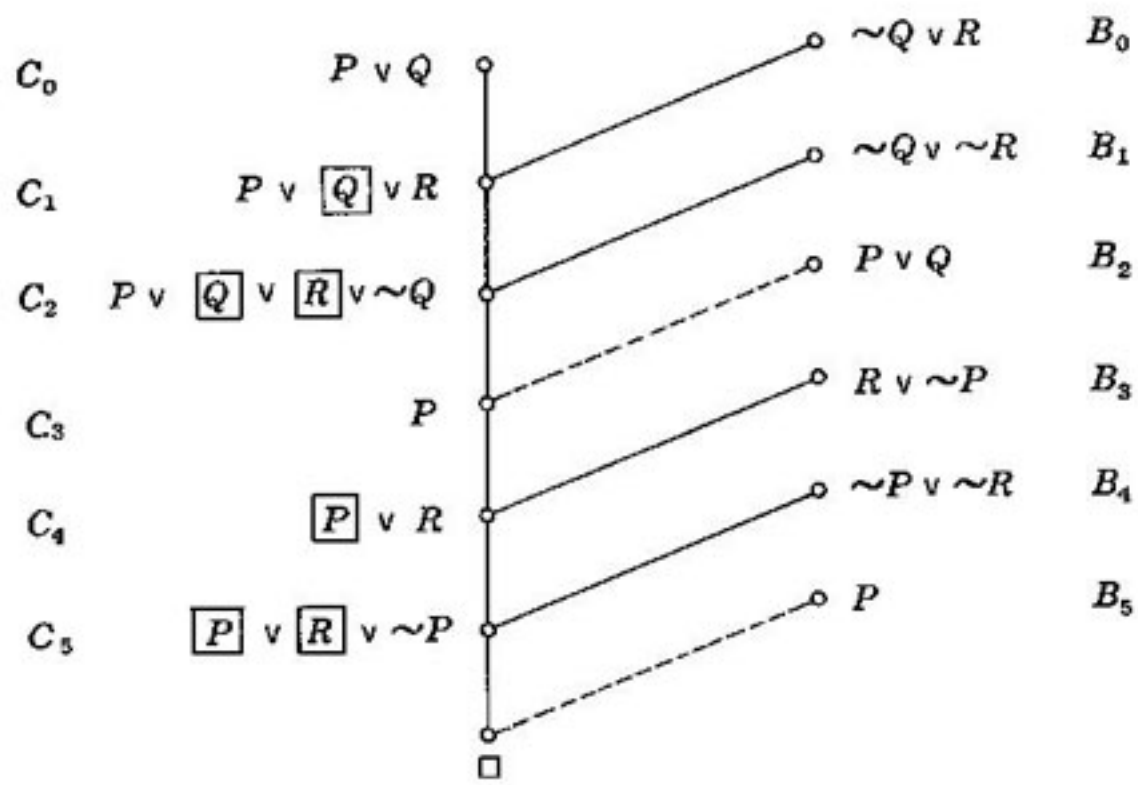

Figure 7.8

The definition of an ordered clause can be used to prove the following lemma. We leave the proof to the reader.

Lemma 7.2 In an OL-deduction, if C_i is a reducible ordered clause, then there exists a center ordered clause $C_j, j < i$, such that the reduced ordered clause C_{i+1} of C_i is an ordered resolvent of C_i against an instance of C_j.

Thus the conditions in the OL-deduction can also be stated as follows:

1. Every B_i is either in S or an instance of some $C_j, j < i$.
2. If C_i is a reducible ordered clause, then C_{i+1} is the reduced ordered clause of C_i. Otherwise, C_{i+1} is an ordered resolvent of C_i with B_i in S where the literal resolved upon in C_i is the *last* literal.
3. No tautology is in the deduction.

Definition An OL-refutation is an OL-deduction of $\square$.

Example 7.5

Consider a set of ordered clauses $S = \{P \vee Q, \sim Q \vee R, R \vee \sim P, \sim Q \vee \sim R, \sim P \vee \sim R\}$. Then Fig. 7.8 is an OL-deduction of $\square$ from S with top ordered clause $P \vee Q$. Note that C_2 and C_5 are reducible ordered clauses. Although no resolution is needed at those steps, the reader can find out what the side clauses would be by those indicated by dashed lines. For example, B_2 is $P \vee Q$, which is the top clause C_0, and B_5 is P, which is C_3 deduced in this refutation.

Example 7.6

In Example 7.2, we presented an example of a linear deduction without the concept of ordered clauses and the information of resolved literals. In Fig. 7.9, we prove that example again by using an OL-deduction.

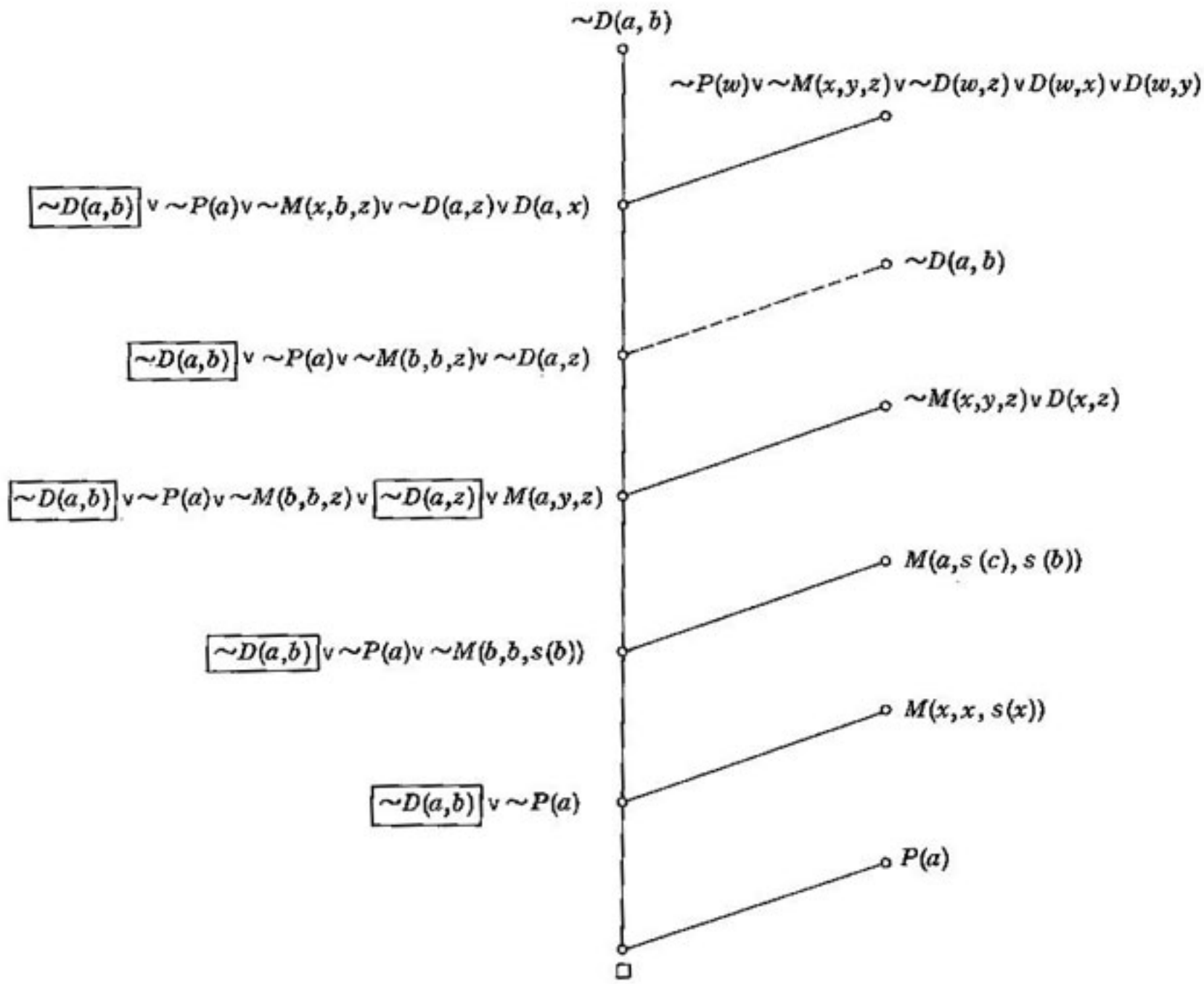

Figure 7.9

7.5 COMPLETENESS OF LINEAR RESOLUTION

In this section, we shall prove the completeness of OL-deduction. We first have to prove the following lemma.

Lemma 7.3 If C is a ground ordered clause in an unsatisfiable set S of ground ordered clauses, and if $S-\{C\}$ is satisfiable, then there exists an OL-refutation from S with top ordered clause C.

Proof We prove this lemma by induction on the number of elements in the atom set of S. Let A be the atom set of S. If A consists of a single element, say Q, then among the elements of S there are ordered clauses Q and $\sim Q$. Clearly the resolvent of Q and $\sim Q$ is $\square$. Since $S-\{C\}$ is satisfiable, one of Q and $\sim Q$ must be C. Thus Lemma 7.3 holds for this case.

Assume Lemma 7.3 holds when A consists of i elements, $1 \leqslant i \leqslant n$. To complete the induction, we consider A such that A consists of exactly $n+1$ elements.

Case 1 C is a unit clause. Let $C \triangleq L$, where L is a literal. Let S' be the set obtained from S by deleting those ordered clauses containing L and by

deleting $\sim L$ from the remaining ordered clauses. S' must be unsatisfiable. Let T' be an unsatisfiable subset of S' such that every proper subset of T' is satisfiable. (This subset can be obtained by exhaustively considering all possible subsets of S'.) T' must contain some ordered clause E' that was obtained from a clause in S by deleting the literal $\sim L$, for otherwise T' would be a subset of $S-\{C\}$. Thus $S-\{C\}$ would be unsatisfiable, violating the fact that it is satisfiable. We now have an ordered clause E' in the unsatisfiable set T' of clauses, and $T'-\{E'\}$ is satisfiable. Since T' contains n or fewer than n atoms, by the induction hypothesis there is an OL-refutation D' from T' with top ordered clause E' as shown in Fig. 7.10a. By adding the literal $\sim L$, in its proper place, back to all ordered clauses from which it was deleted except for the top most ordered clause E', and obtaining E' by resolving L against E, we obtain a deduction D of either $\square$ or $(\sim L)$ as shown in Fig. 7.10b.

In Fig. 7.10b, the symbol $(\vee \sim L)$ means the addition of the literal $\sim L$, in its proper place, back to the ordered clauses from which it was deleted. We now have to show that D is also an OL-deduction.

Note that no clause in D is a tautology since D is obtained from D', which contains neither a tautology nor the literal L. Note also that if $\sim L$ does not appear as the last literal of a center clause, then the resolution in D is exactly the same as that in D'. If $\sim L$ does appear as the last literal of a center clause, then $\sim L$ should be resolved upon. In this case, we replace that part of D shown in Fig. 7.11a by the deduction shown in Fig. 7.11b. It is obvious that $\boxed{L} \vee C_i \vee (\sim L)$ is reducible, and $\boxed{L} \vee C_i$ is obtained by the reduction operation.

After these replacements have been made whenever required in Fig. 7.10b,

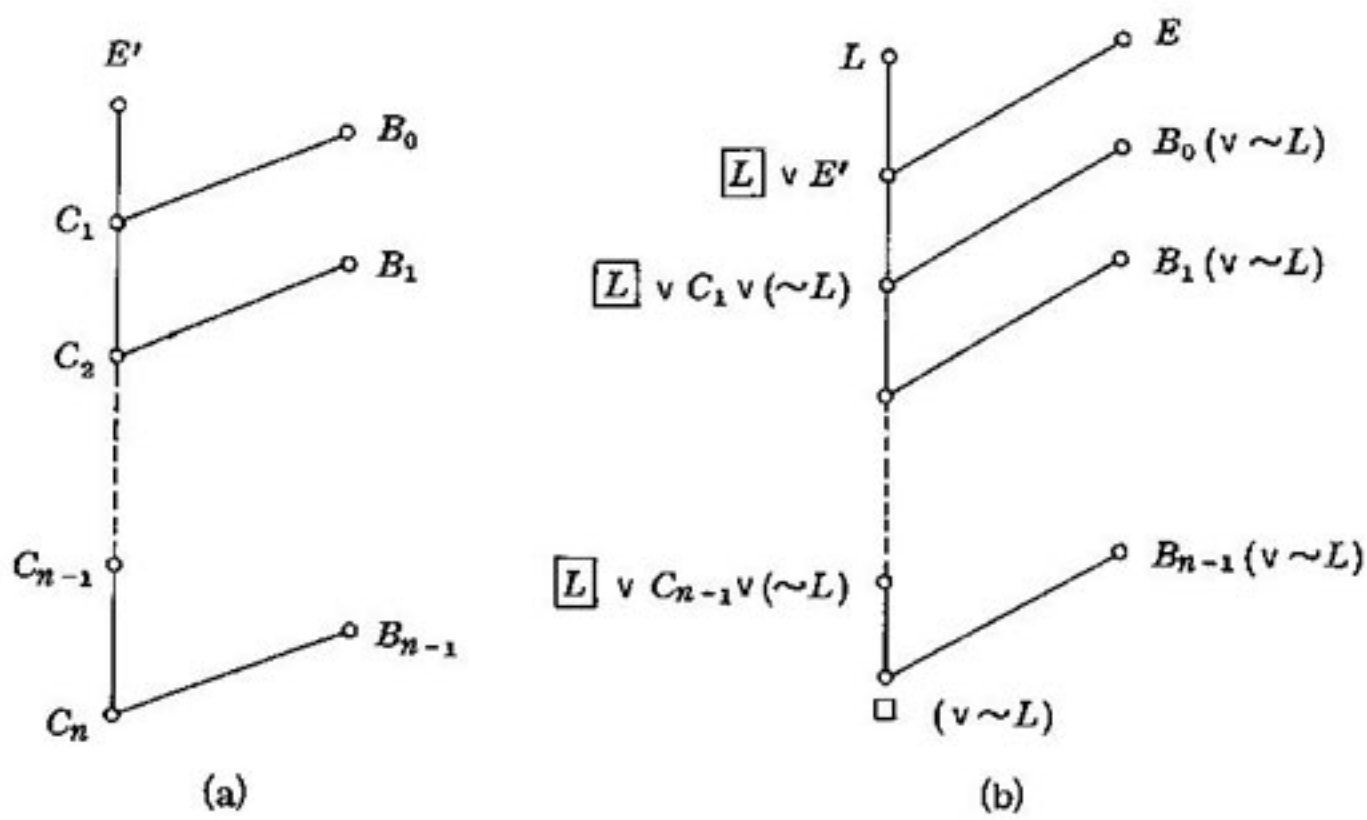

Figure 7.10

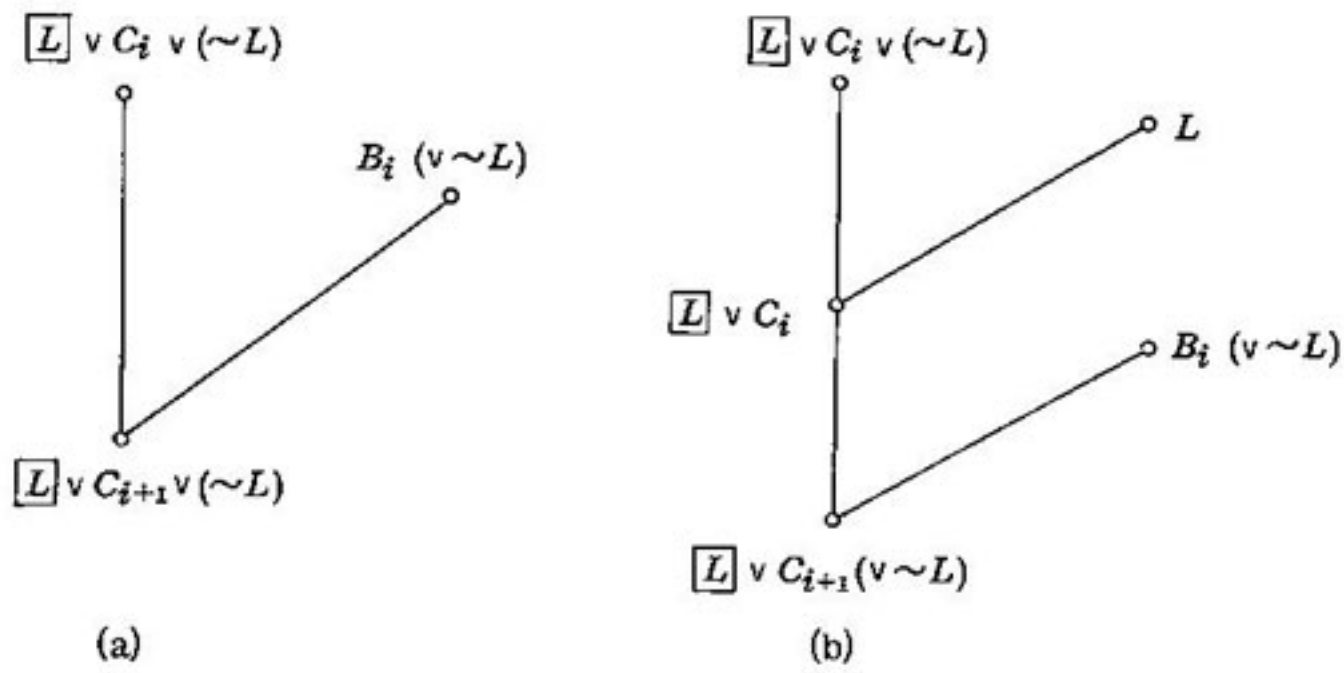

Figure 7.11

an OL-deduction D^* of $\square$ or $\boxed{L} \vee \sim L$ from S with top ordered clause C is obtained. If D^* is a deduction of $\square$, we are done. Otherwise, D^*, together with the resolution of $\boxed{L} \vee \sim L$ against L, is a desired OL-refutation from S with top ordered clause C.

Case 2 C is not a unit clause. In this case let L be the first (leftmost) literal in C, that is, $C \triangleq L \vee C'$, where C' is a nonempty ordered clause. Let S' be that set obtained from S by deleting those ordered clauses containing $\sim L$ and by deleting L from the remaining ordered clauses. S' is unsatisfiable. We shall show that $S' - \{C'\}$ is satisfiable. Let I be an interpretation that satisfies $S - \{C\}$. (Such an I exists because $S - \{C\}$ is satisfiable.) Since S is unsatisfiable, C must be false in I. Therefore, L is false in I. Consequently $S' - \{C'\}$ is true in I. Thus $S' - \{C'\}$ is satisfiable. Since S' contains n or fewer than n atoms, by the induction hypothesis there is an OL-deduction D' of $\square$ from S' with top ordered clause C'. Adding the literal L, in its proper place, to all ordered clauses from which it was deleted, we obtain an OL-deduction D_1 of L from S with top ordered clause C. Now $\{L\} \cup (S - \{C\})$ is unsatisfiable and $S - \{C\}$ is satisfiable. As proved in Case 1, there is an OL-deduction D_2 of $\square$ from $\{L\} \cup (S - \{C\})$ with top ordered clause $\{L\}$. Putting D_1 on top of D_2, we obtain an OL-refutation from S with top ordered clause C. This completes the proof.

Theorem 7.2 (Completeness of OL-deduction). If C is an ordered clause in an unsatisfiable set S of ordered clauses and if $S - \{C\}$ is satisfiable, then there is an OL-refutation from S with top ordered clause C.

Proof Since S is unsatisfiable and $S - \{C\}$ is satisfiable, by Herbrand's theorem (version II) there is a finite set S' of ground instances of ordered

clauses in S and a ground instance C' of C such that S' is unsatisfiable, C' is in S', and $S'-\{C'\}$ is satisfiable. By Lemma 7.3, there is an OL-refutation D' from S' with top ordered clause C'. Using the lifting lemma (similar to Lemma 5.1), from D' we can obtain an OL-refutation from S with top ordered clause C. This completes the proof.

It can be proved easily that set-of-support resolution is implied by OL-resolution. That is, Theorem 7.2 implies Theorem 6.2. We leave this proof as an exercise for the reader.

7.6 LINEAR DEDUCTION AND TREE SEARCHING

In the remaining sections, we shall discuss a very important problem of linear deduction, namely, how to implement linear deduction efficiently.

Let us assume that we have chosen a top clause C_0. We then can single out all the possible side clauses. After resolving C_0 with these possible side clauses, we obtain resolvents $R_1, \ldots, R_m$. Every R_i, $1 \leqslant i \leqslant m$, is a possible center clause that may lead to a proof. If some R_i is the empty clause, we are done. Otherwise, for every i, we find all possible side clauses that can be resolved with R_i and continue this process until an empty clause is generated. In order to describe the above process conveniently, a linear deduction tree such as the one shown in Fig. 7.7 will be represented by a path as shown in Fig. 7.12. That is, a side clause B_i is attached to an arc of the path, and C_i is a resolvent of C_{i-1} and B_{i-1}. In the sequel, we shall implement the OL-deduction.

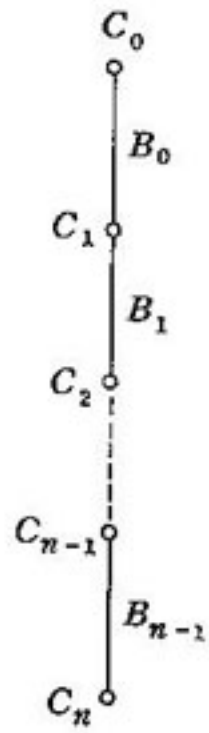

Figure 7.12

Let us first consider a simple example. (In this example, tautologies are included for illustrative purposes.) Consider the following set of clauses:

(1) $P \vee Q$

(2) $\sim P \vee Q$

(3) $P \vee \sim Q$

(4) $\sim P \vee \sim Q$.

Let us choose clause (1) to be the top clause C_0. The ordered clauses that can be the side clauses of C_0 are (3) and (4). Resolving C_0 with clauses (3) and (4), we obtain the tree shown in Fig. 7.13. We have generated two

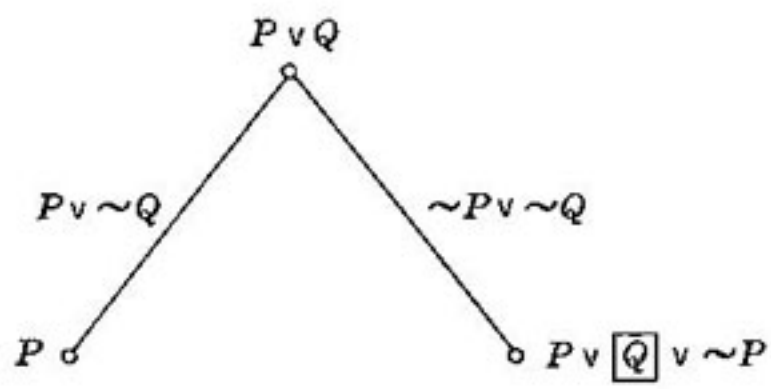

Figure 7.13

center clauses. For each such clause, we can single out all possible side clauses to be resolved and further develop the tree in Fig. 7.13 to that in Fig. 7.14.

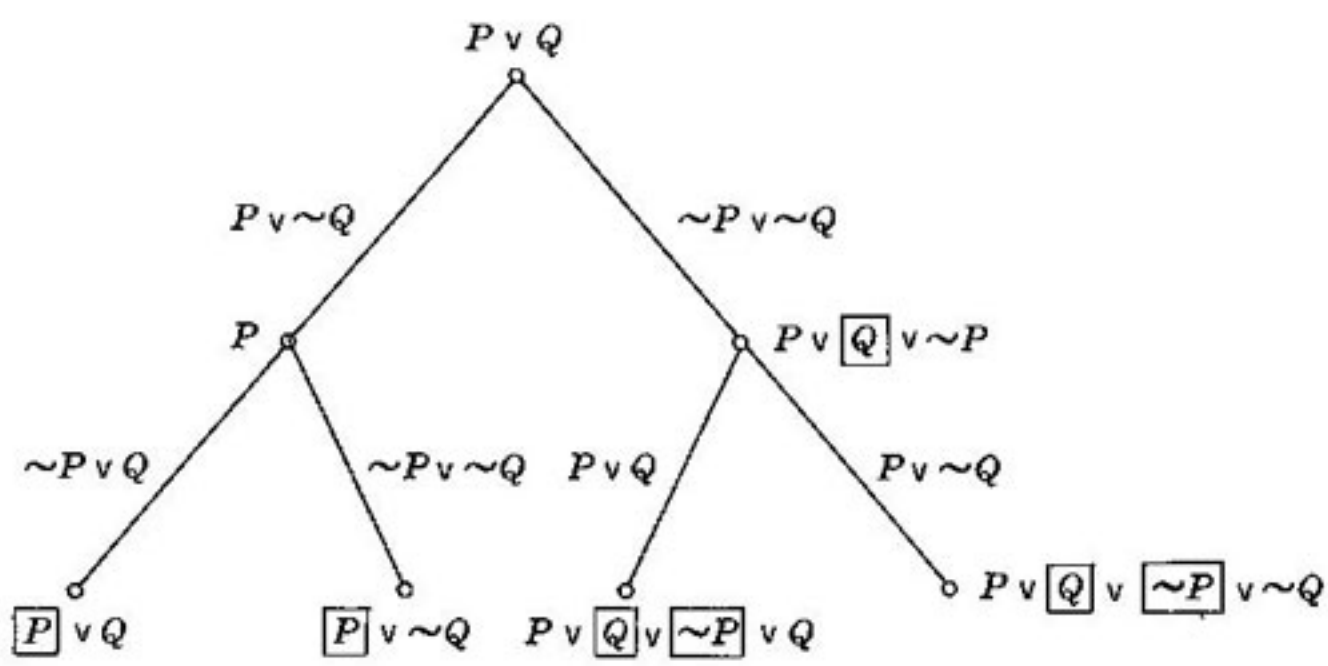

Figure 7.14

When an ordered resolvent is generated, we always check whether it is reducible. If it is, we always reduce it to the reduced ordered clause. In Fig. 7.14, $P \vee \boxed{Q} \vee \boxed{\sim P} \vee \sim Q$ is reducible. Its reduced ordered clause is P.

Therefore, the tree in Fig. 7.14 can be simplified to the tree shown in Fig. 7.15.

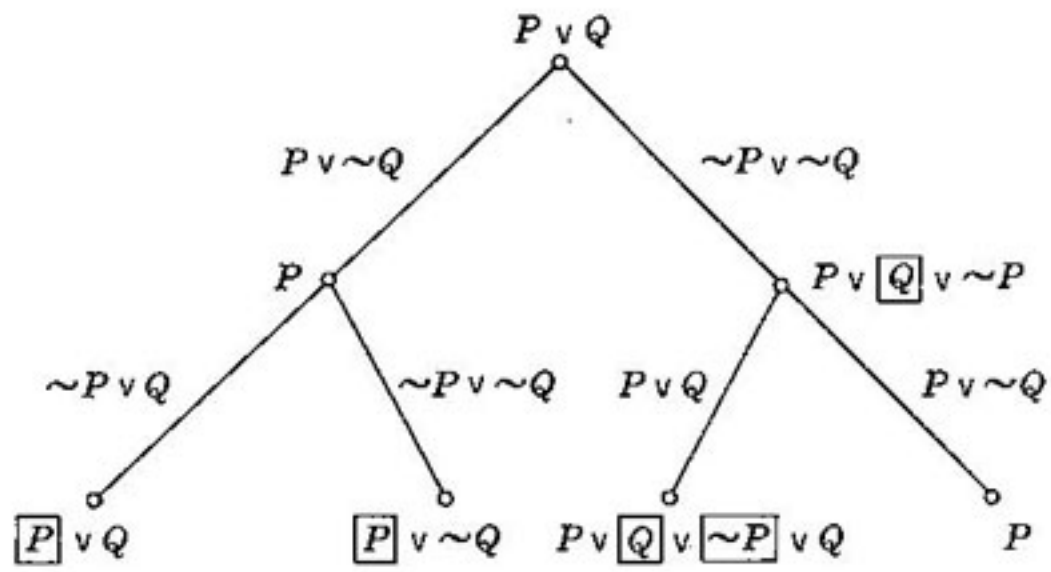

Figure 7.15

Now there are four possible center clauses. Again, for every center clause, we can single out all possible clauses to resolve. Thus, we obtain the tree shown in Fig. 7.16. The ordered resolvents $\boxed{P} \vee \boxed{Q} \vee \sim P$ and

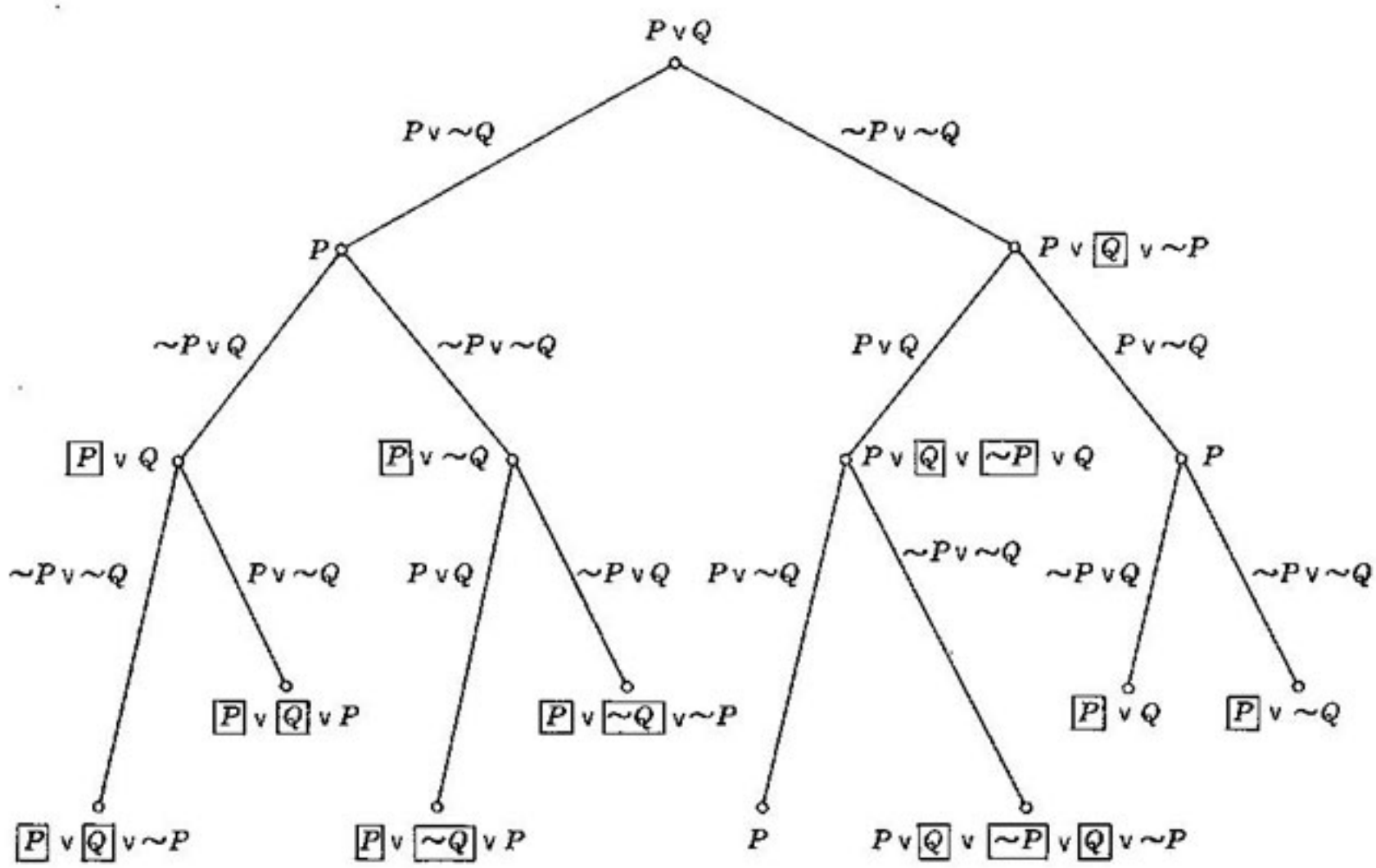

Figure 7.16

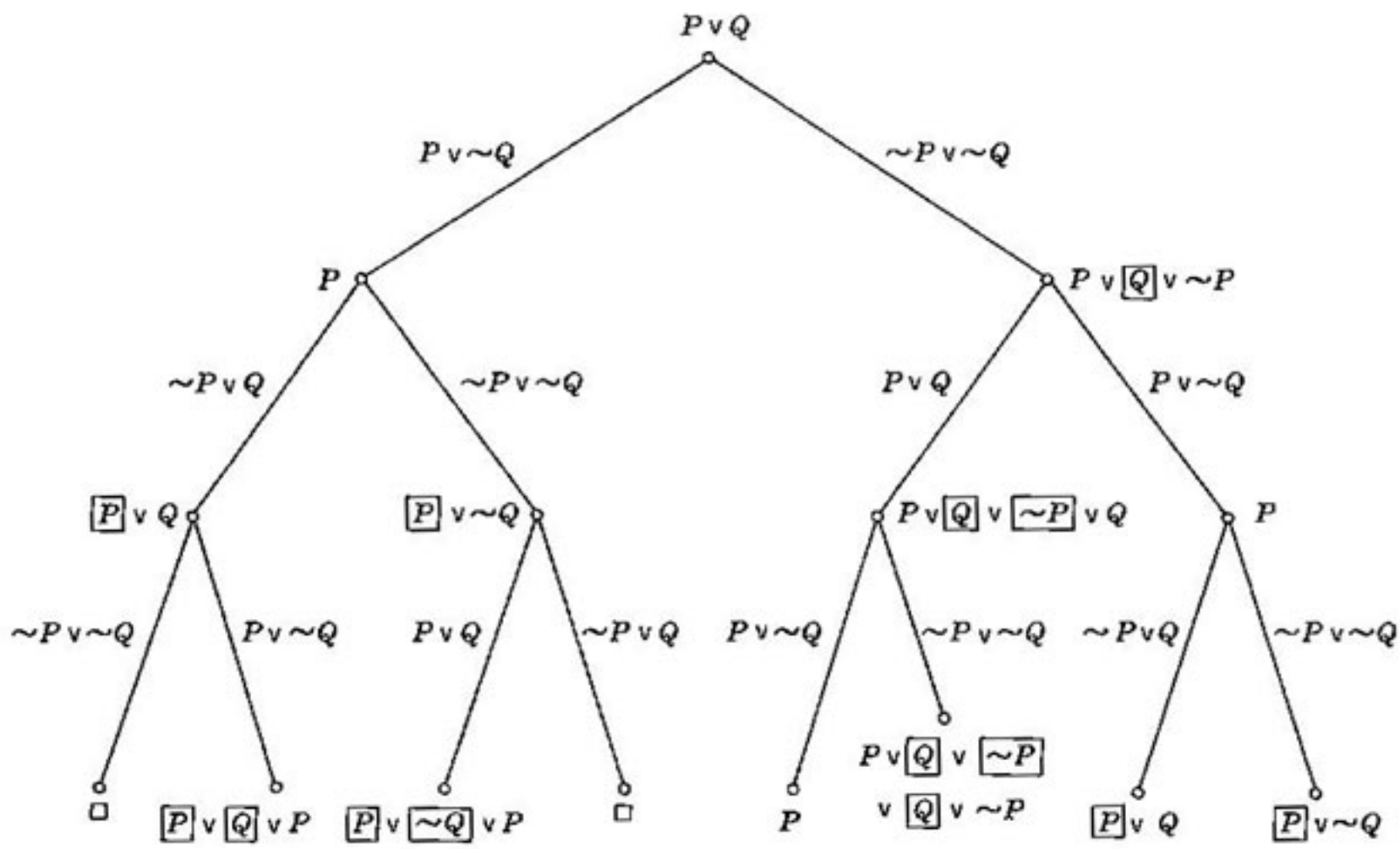

Figure 7.17

$[P] \vee [\sim Q] \vee \sim P$ are reducible. Reducing them, we obtain the tree shown in Fig. 7.17. Since the empty clause has been generated, the process is terminated. The leftmost branch of the tree in Fig. 7.17 corresponds to the OL-refutation shown in Fig. 7.18.

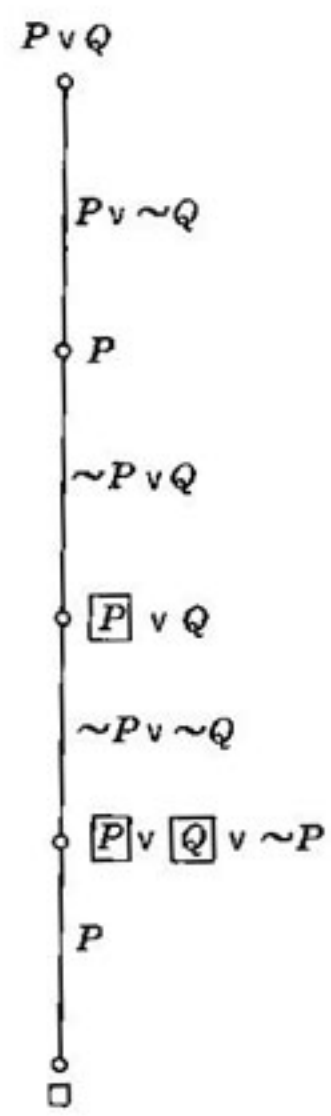

Figure 7.18

The reader can now see that the problem of finding an OL-refutation can be viewed as a tree-searching problem. Many techniques developed for tree searching can be applied to OL-deduction. Thanks to the persistent effort of researchers in tree searching, we have seen great progress made in this field. For a comprehensive study of general tree searching techniques, see [Slagle, 1971] and [Nilsson, 1971].

In the above example, the method of developing the tree in Fig. 7.17 is called the *breadth-first* method. In tree-searching terminology, C_0 is the *top node* of the tree. The side clauses may be considered as *operators*, which can be used to develop the successor nodes of a node. Let C be an ordered clause that denotes a node. Single out all the possible side clauses and resolve them with C to obtain ordered resolvents $R_1, \ldots, R_m$. $R_1, \ldots, R_m$ will be called the *immediate successor nodes* of C. When all the possible immediate successor nodes of a node C are generated, we say that node C has been *expanded*. That is, expanding a node means generating all its immediate successor nodes.

Suppose that S is a set of ordered clauses to be proved. Suppose that C_0 is an ordered clause in S that is selected to be the top clause. Then, the breadth-first method is described as follows.

The Breadth-First Method

Step 1 Let CLIST $= (C_0)$.

Step 2 If CLIST is empty, terminate without a proof. Otherwise, continue.

Step 3 Let C be the first ordered clause in CLIST. Delete C from CLIST.

Step 4 Find all the ordered clauses in S that can be side clauses of C. If no such side clauses exist, go to Step 2. Otherwise, resolve C with all these side clauses. Let $R_1, \ldots, R_m$ denote these ordered resolvents. Let R_i^* be the reduced ordered clause of R_i if R_i is reducible. If R_i is not reducible, let $R_i^* = R_i$.

Step 5 If some R_q^* is an empty clause, $1 \leqslant q \leqslant m$, terminate with a proof; otherwise, continue.

Step 6 Put $R_1^*, \ldots, R_m^*$ (in an arbitrary order) at the end of CLIST and go to Step 2.

A minimal proof from S with top clause C_0 is an OL-refutation (from S with top clause C_0) that involves the smallest number of resolutions. The breadth-first method will always find such a minimal proof if one exists. However, obviously, many irrelevant and redundant clauses may be generated by this breadth-first method before the empty clause is found. This is evident in the above example. Another way to search a tree is the *depth-first*

method. The idea is that, unlike the breadth-first method which expands nodes from top to bottom, the depth-first method expands nodes from left to right. The depth-first method is almost the same as the breadth-first method except that in Step 5 we put $R_1^*, \ldots, R_m^*$ at the beginning of CLIST, instead of at the end of CLIST. In addition, we must provide a *depth bound* for the depth-first method in order to prevent us from expanding nodes along a wrong path (branch), that is, a path that cannot lead to the empty clause. The depth of a clause in an OL-deduction is defined as follows.

Definition In an OL-deduction with top ordered clause C_0, the *depth of* C_0 is 0. If the depth of a certain ordered clause C is k and R is an ordered resolvent of C and some side clause, then the *depth of* R is $(k+1)$. The *length* of a proof (refutation) with top clause C_0 is the depth of the empty clause.

Let d^* be a threshold number specified beforehand. The depth-first method is now described as follows:

The Depth-First Method

Step 1 Let CLIST $= (C_0)$.

Step 2 If CLIST is empty, terminate without a proof. Otherwise, continue.

Step 3 Let C be the first ordered clause in CLIST. Delete C from CLIST. If the depth of C is greater than d^*, go to Step 2. Otherwise, continue.

Step 4 Find all the ordered clauses in S that can be side clauses of C. If no such side clauses exist, go to Step 2. Otherwise, resolve C with all of its side clauses. Let $R_1, \ldots, R_m$ denote these ordered resolvents. Let R_i^* be the reduced ordered clause of R_i if R_i is reducible. Otherwise, let $R_i^* = R_i$.

Step 5 If some R_q^* is an empty clause, $1 \leqslant q \leqslant m$, terminate with a proof. Otherwise, continue.

Step 6 Put $R_1^*, \ldots, R_m^*$ (in an arbitrary order) at the beginning of CLIST and go to Step 2.

It is obvious that applying the depth-first method with $d^* = 2$ on the tree in Fig. 7.17 will generate the tree shown in Fig. 7.19. Clearly, the tree in Fig. 7.19 is smaller than the one in Fig. 7.17.

Although, in general, the depth-first method searches a smaller tree than the breadth-first method does, we would like to improve it still futher. We notice that in the depth-first method, when a node is chosen for expansion, it is completely expanded. That is, all the possible successor nodes of the node are generated. However, we can modify this and generate only one successor node at a time. Let us consider Fig. 7.19. The top node of the

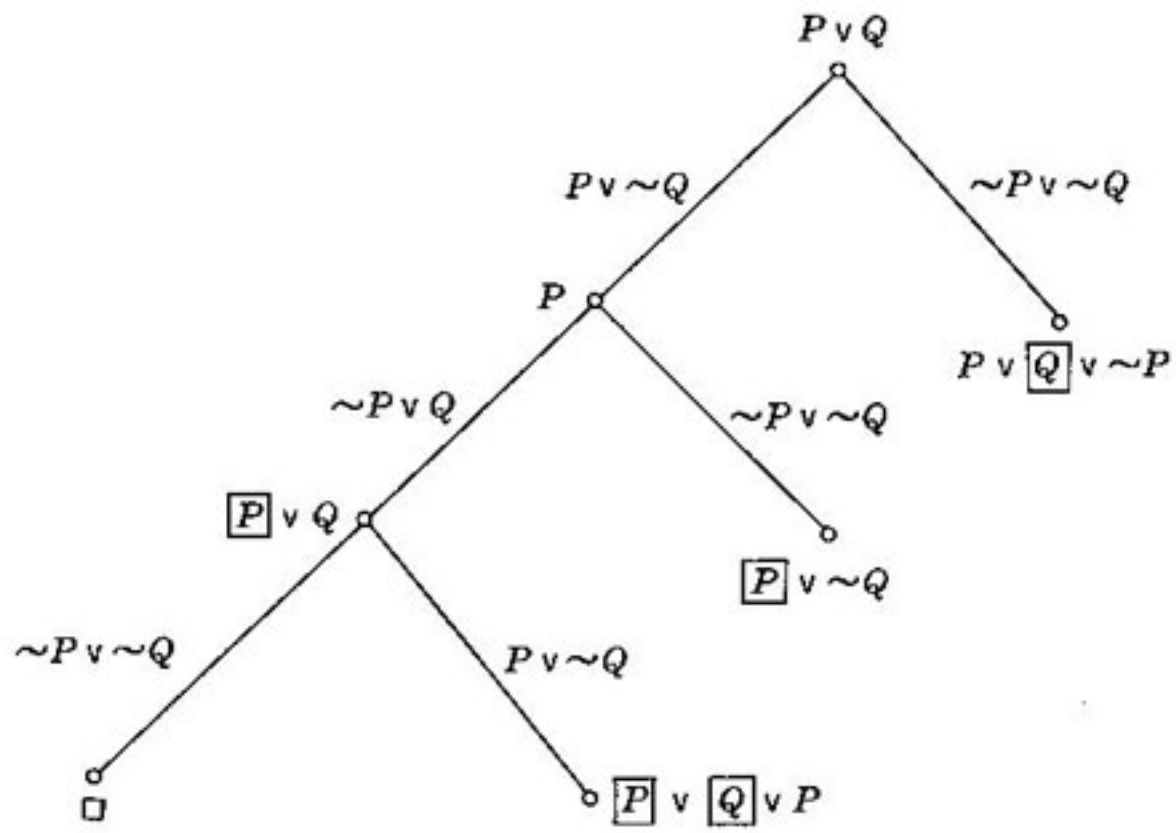

Figure 7.19

tree is $P \vee Q$. There are two successor nodes, namely, P and $P \vee \boxed{Q} \vee \sim P$, of this top node. Suppose only P is generated. Now, P is a center clause. There are two successor nodes of node P. Suppose only $\boxed{P} \vee Q$ is generated. Again, $\boxed{P} \vee Q$ has two successor nodes, namely $\square$ and $\boxed{P} \vee \boxed{Q} \vee P$. If only $\square$ is generated, we obtain an empty clause. The entire process is depicted by the tree shown in Fig. 7.20, which is even smaller than the tree in Fig. 7.19. Because of the above consideration, we give a modified depth-first method as follows, where d^* is a prespecified threshold number.

A Modified Depth-First Method

Step 1 Find all ordered clauses in S that can be side clauses of C_0. If no such side clauses exist, terminate without a proof. Otherwise, let $B_0{}^1, \ldots, B_0{}^r$

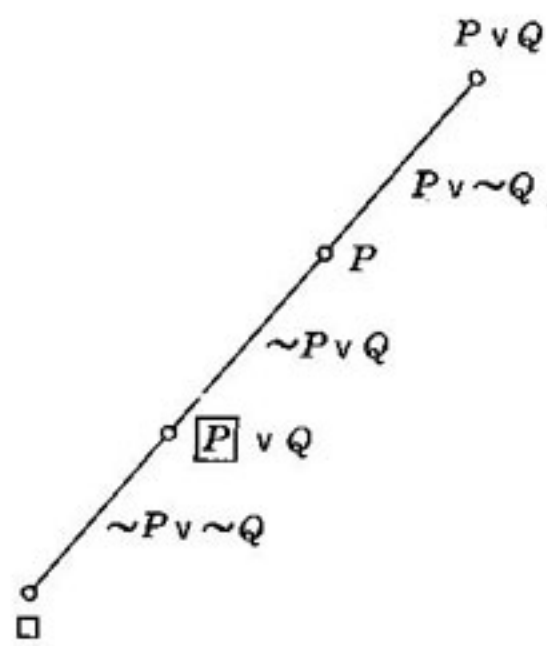

Figure 7.20

be all these side clauses. Construct pairs $(C_0, B_0^1), \ldots, (C_0, B_0^r)$. Let CLIST be a list of all these pairs arranged in an arbitrary order.

Step 2 If CLIST is empty, terminate without a proof. Otherwise, continue.

Step 3 Let (C, B) be the first pair in CLIST. Delete (C, B) from CLIST. If the depth of C is greater than d^*, go to Step 2. Otherwise, continue.

Step 4 Resolve C with B. Let $R_1, \ldots, R_m$ denote all the ordered resolvents of C against B. Let R_i^* be the reduced ordered clause of R_i if R_i is reducible. Otherwise, let $R_i^* = R_i$.

Step 5 If some R_q^* is the empty clause, $1 \leqslant q \leqslant m$, terminate with a proof. Otherwise, continue.

Step 6 For each $i = 1, 2, \ldots, m$, find ordered clauses in S that can be side clauses of R_i^*. If no such side clauses exist, delete R_i^*. Otherwise, let $B_{i1}, B_{i2}, \ldots, B_{is_i}$ be all these side clauses. Construct pairs $(R_i^*, B_{i1}), \ldots, (R_i^*, B_{is_i})$. Put all these pairs (in an arbitrary order) at the beginning of CLIST and go to Step 2.

In most situations, both the ordinary and the modified depth-first methods are better than the breadth-first method. If the set S of ordered clauses has a proof with a length less than d^*, both versions of the depth-first method guarantee finding a proof. Although the modified depth-first method is more difficult to implement than the ordinary depth-first method, it is clearly superior. In the next section, we assume that the modified depth-first method is used.

7.7 HEURISTICS IN TREE SEARCHING

In this section, we shall discuss "heuristics" that may be used with the modified depth-first method to speed up the searching. The use of a heuristic may not guarantee finding a proof even though there is one. Nevertheless, it is often a necessary tool for proving a large or difficult theorem, that is, a theorem with a long proof. There are many heuristics in the field of theorem proving and we shall now discuss some of them. We note that, although the following heuristics are introduced to be used with the modified depth-first method, they can also be applied to other kinds of resolution, not necessarily only to linear resolution.

A Deletion Strategy

An ordered clause C_1 is said to *subsume* another ordered clause C_2 if and only if the clause consisting of the unframed literals of C_1 subsumes the

clause consisting of the unframed literals of C_2. An ordered clause is said to be a *tautology* if and only if it contains a complementary pair of unframed literals. Since tautologies and subsumed clauses are irrelevant and redundant, we should delete them whenever possible. In the modified depth-first method, at the end of Step 4 we should check whether R_i^* is a tautology, $i = 1, \ldots, m$. If R_i^* is a tautology, it should be deleted. In addition, if there is a pair (C, B) in CLIST such that R_i^* is subsumed by C, then R_i^* should also be deleted.

B The Fewest-Literal Preference Strategy

In the modified depth-first method, pairs of CLIST are arranged in an arbitrary order. Actually, we should arrange the pairs of CLIST in such a way that the potentially "good" pairs appear before the "bad" ones. Thus, in Step 3 of the method, when we take the first pair from CLIST, we can be sure that we have a good one that can lead to an empty clause promptly.

Let the length of an ordered clause E, denoted as length(E), be defined as the number of unframed literals in E. Suppose (C, B) is a pair in CLIST. Let R denote an ordered resolvent of C against B. Since length(R) indicates the minimum number of resolutions necessary to lead R to an empty clause, we may use length(R) as an indicator of the goodness of the pair (C, B). That is, the smaller the value of length(R), the better the pair (C, B). Actually, in practice, we do not have to calculate R in order to calculate length(R). A mere estimation of length(R) will be sufficient. Clearly,

$$\text{length}(R) \leqslant \text{length}(C) + \text{length}(B) - 2. \tag{7.1}$$

Moreover, since we only need a relative number, we may simply use (length(C) + length(B)) as a measurement of goodness. Let

$$f(C, B) = \text{length}(C) + \text{length}(B). \tag{7.2}$$

Then $f(C, B)$ will be used to evaluate the goodness of the pair (C, B). In the modified depth-first method, just before Step 3 we may arrange pairs of CLIST by putting pairs with small values of f before those with large values of f. This ordering scheme was proposed by Slagle [1965], and is called the *fewest-literal preference strategy*.

C Use of Heuristic Evaluation Functions

In the above discussion, the only measurement of the goodness of a clause is the number of literals it contains. This is, of course, far too simple. For a pair (C, B) in CLIST of the modified depth-first method, let us define $h^*(C, B)$ to be the number of resolutions in a minimal proof (refutation) with top ordered clause C and with first side clause B. If $h^*(C, B)$ is known, then we could use $h^*(C, B)$ to order pairs of CLIST. However, $h^*(C, B)$ is generally not known. Therefore, $h^*(C, B)$ has to be estimated. Let $h(C, B)$ be an estimate

of $h^*(C, B)$. Usually, $h(C, B)$ is obtained from a set of known data. In the sequel, we shall consider how to obtain an $h(C, B)$. Suppose $h(C, B)$ can be expressed as

$$h(C, B) = w_0 + w_1 f_1(C, B) + \cdots + w_n f_n(C, B) \tag{7.3}$$

where each $f_i, i = 1, \ldots, n$, is a real-valued function of C and B, and w_i is a weight associated with f_i. Each f_i is called a *feature* of the pair (C, B). Usually, features are selected ad hoc by the experimenter who believes that they are relevant to $h^*(C, B)$. For example, we may use some of the features suggested in Table 7.1. Actually, $h(C, B)$ can be a function, linear or non-

TABLE 7.1

1. number of unframed literals in C
2. number of framed literals in C
3. number of side clauses of C
4. number of constants in the last literal of C
5. number of function symbols in the last literal of C
6. number of framed literals of C, each of which subsumes the last literal of C
7. number of distinct variables in C and B
8. length(C) + length(B) − 2
9. number of constants in C/(1 + number of variables of C)
10. number of constants in C/(1 + number of distinct variables of C)
11. depth of C
12. number of unframed literals that are in both C and B
13. number of literals in B, each of which has a framed complement in C
14. number of distinct predicate symbols in C and B

linear, of $f_1(C, B), \ldots, f_n(C, B)$. For illustrative purposes, we shall only consider linear functions in the sequel. The function $h(C, B)$ may be used to order pairs of CLIST, that is, to put pairs with small values of h before those with large values of h. We shall call this $h(C, B)$ a *heuristic evaluation function*, or *evaluation function* for short. In the next section, we shall give a method for obtaining an $h(C, B)$.

7.8 ESTIMATIONS OF EVALUATION FUNCTIONS

As discussed in Section 7.7C, we assume that

$$h(C, B) = w_0 + w_1 f_1(C, B) + \cdots + w_n f_n(C, B).$$

In this section, we shall discuss the following two important questions:

1. Given a set of features, how can we determine the appropriate values of $w_0, \ldots, w_n$?

2. How can we know that a set of features is good (relevant to $h^*(C, B)$) or not?

We shall now answer the first question. Let us assume that we have collected a set of pairs, $(C_1, B_1), \ldots, (C_q, B_q)$. Also assume that we know the values of $h^*(C_1, B_1), \ldots, h^*(C_q, B_q)$. Then the problem is to determine $w_0, w_1, \ldots, w_n$ such that

$$\sum_{i=1}^{q} [h^*(C_i, B_i) - (w_0 + w_1 f_1(C_i, B_i) + \cdots + w_n f_n(C_i, B_i))]^2 \qquad (7.4)$$

is minimum. This is the so called *least-square estimation* [Draper and Smith, 1966]. Let the matrices F, H, and W be defined as

$$F = \begin{bmatrix} 1 & f_1(C_1, B_1) & \cdots & f_n(C_1, B_1) \\ 1 & f_1(C_2, B_2) & \cdots & f_n(C_2, B_2) \\ \vdots & & & \\ 1 & f_1(C_q, B_q) & \cdots & f_n(C_q, B_q) \end{bmatrix}$$

$$H = \begin{bmatrix} h^*(C_1, B_1) \\ h^*(C_2, B_2) \\ \vdots \\ h^*(C_q, B_q) \end{bmatrix}, \qquad W = \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{bmatrix}.$$

Then the W that minimizes the expression (7.4) is given by

$$W = (F' F)^{-1} F' H \qquad (7.5)$$

where F' is the transpose of F and $(F' F)^{-1}$ is the inverse of $(F' F)$. Substituting W into (7.3), an evaluation function can be obtained.

Example 7.7

Consider the tree shown in Fig. 7.17. Let us redraw the tree in Fig. 7.21, where each ordered clause is numbered. For this example, suppose we use the features defined as:

$f_1(C, B) =$ length(C) + length$(B) - 2$

$f_2(C, B) =$ number of framed literals in C

$f_3(C, B) =$ number of unframed literals that are in both C and B

$f_4(C, B) =$ number of literals in B, each of which has a framed complement in C

$f_5(C, B) =$ number of framed literals in C, each of which subsumes the last literal of C.

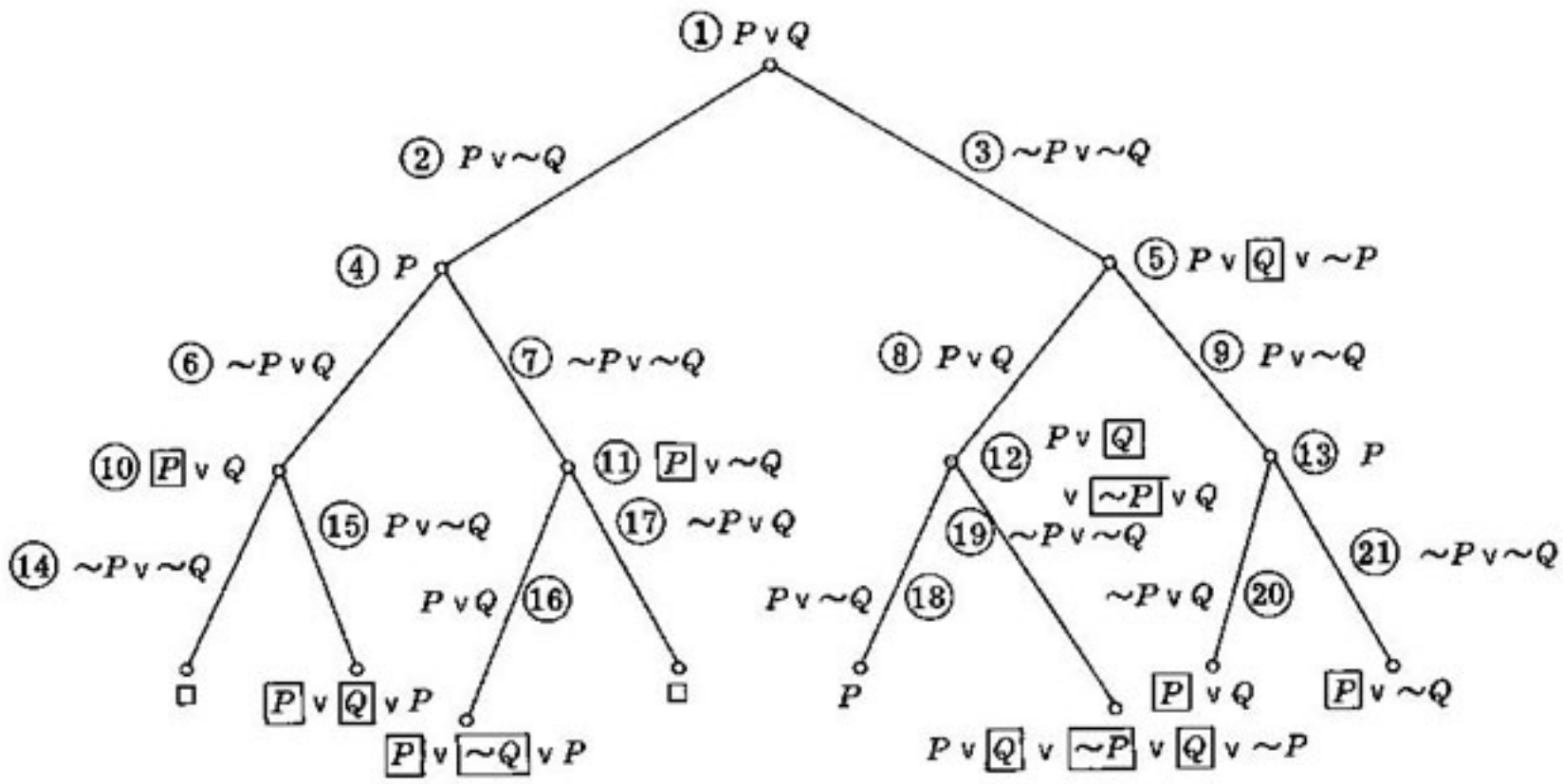

Figure 7.21

Calculating the values of the features of each pair in the tree of Fig. 7.21, we obtain Table 7.2.

TABLE 7.2

i	pair (C_i, B_i)	$f_1(C_i, B_i)$	$f_2(C_i, B_i)$	$f_3(C_i, B_i)$	$f_4(C_i, B_i)$	$f_5(C_i, B_i)$	$h^*(C_i, B_i)$
1	(1, 2)	2	0	1	0	0	3
2	(1, 3)	2	0	0	0	0	4
3	(4, 6)	1	0	0	0	0	2
4	(4, 7)	1	0	0	0	0	2
5	(5, 8)	2	1	1	0	0	4
6	(5, 9)	2	1	1	1	0	3
7	(10, 14)	1	1	0	1	0	1
8	(10, 15)	1	1	0	0	0	3
9	(11, 16)	1	1	0	0	0	3
10	(11, 17)	1	1	0	1	0	1
11	(12, 18)	2	2	1	2	1	3
12	(12, 19)	2	2	0	1	1	4
13	(13, 20)	1	0	0	0	0	2
14	(13, 21)	1	0	0	0	0	2

Computing formula (7.5) on a computer, we obtain

$$\begin{aligned} w_0 &= 0.30 & w_1 &= 1.68 \\ w_2 &= 0.76 & w_3 &= -0.24 \\ w_4 &= -1.44 & w_5 &= 0.60. \end{aligned} \tag{7.6}$$

Thus we have

$$h(C,B) = 0.30 + 1.68f_1(C,B) + 0.76f_2(C,B) - 0.24f_3(C,B) - 1.44f_4(C,B) + 0.60f_5(C,B). \tag{7.7}$$

In the modified depth-first method, just before Step 3, if we use the above function $h(C,B)$ to order pairs of CLIST, then the tree that is searched by the modified depth-first method will be the one shown in either Fig. 7.22a or b.

In the above example, we showed how to obtain an evaluation function. The evaluation function obtained in Example 7.7 is only for sets of *ground* clauses. It was based upon one example. Obviously, the more examples we use to estimate an evaluation function, the better it will be. Nevertheless, we shall now apply the evaluation function obtained in Example 7.7 to a new example to see how good the estimation is.

Example 7.8

Let us consider the following set of ordered clauses:

(1) $\sim P_1 \vee \sim P_2 \vee P_3$

(2) P_1

(3) P_2

(4) $\sim P_1 \vee \sim P_3$

(5) $\sim P_3 \vee P_4$

(6) $\sim P_3 \vee P_5$

(7) $\sim P_4 \vee P_5$.

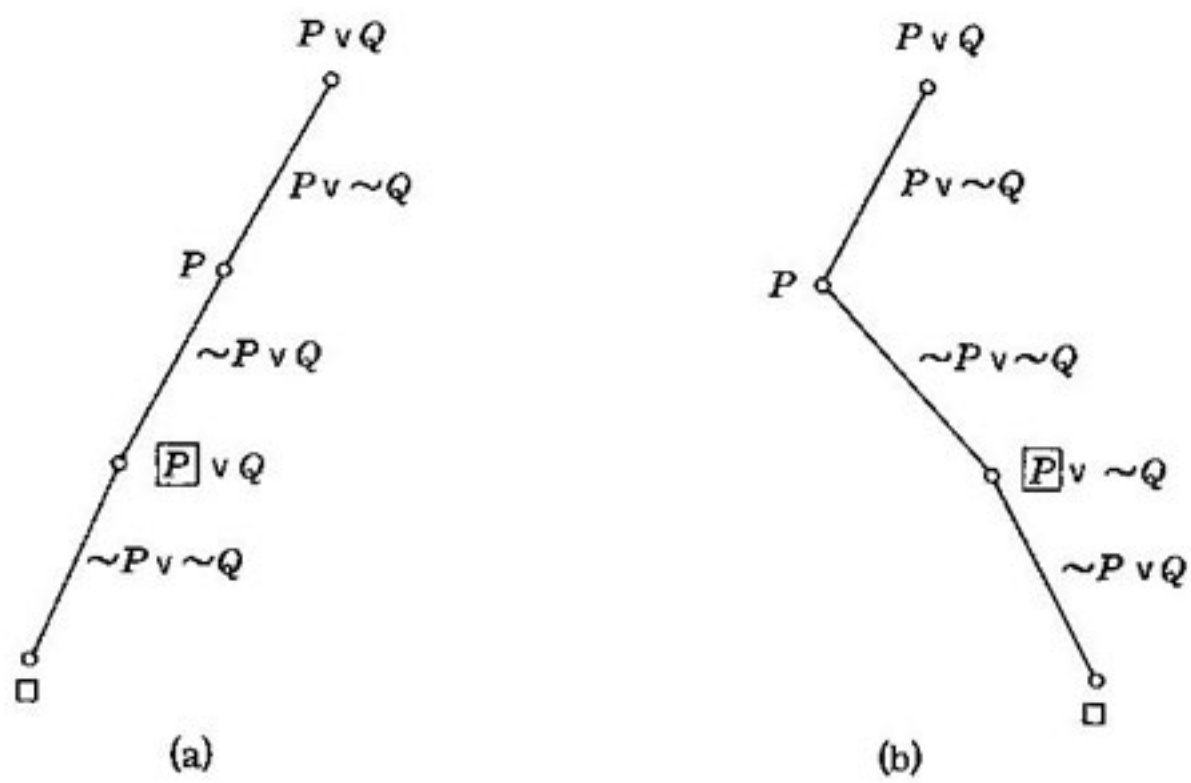

Figure 7.22

Let us use clause (1) as the top clause. There are three ordered clauses that can be side clauses of clause (1). They are clauses (4), (5), and (6). We now use $h(C, B)$ defined by Eq. (7.7) to calculate the following:

$$h(1,4) = 0.30 + 1.68 \times (3+2-2) + 0.76 \times 0 - 0.24 \times 1$$
$$-1.44 \times 0 - 0.60 \times 0 = 3.42$$
$$h(1,5) = 0.30 + 1.68 \times (3+2-2) + 0.76 \times 0 - 0.24 \times 0$$
$$-1.44 \times 0 - 0.60 \times 0 = 3.66$$
$$h(1,6) = 0.30 + 1.68 \times (3+2-2) + 0.76 \times 0 - 0.24 \times 0$$
$$-1.44 \times 0 - 0.60 \times 0 = 3.66.$$

Since $h(1,4)$ is the smallest, we choose clause (4) as the side clause of clause (1).

(7) $\sim P_1 \vee \sim P_2$ from (1) and (4).

The rest of the proof is trivial. The reader can easily see that the following steps will complete the proof:

(8) $\sim P_1$ from (7) and (3)

(9) $\square$ from (8) and (2).

For this example, no irrelevant clauses were generated. We note that should the evaluation function make us choose clause (5) or (6) as a side clause of clause (1), irrelevant clauses would be generated. Thus, we have shown that the evaluation function is useful for this example.

We can now answer the second question: How can we know that a set of features is good? Evidently, if a set of features is good, then after using a large number of examples to estimate an evaluation function, the function should generalize to new examples. That is, the evaluation function should give good performance on new examples. If it does not generalize to new problems, then the experimenter should analyze the data and consider new features. For an excellent discussion of estimations of evaluations in theorem proving, the reader is encouraged to read [Slagle and Farrell, 1971].

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EXERCISES

Section 7.2

1. Use linear resolution to prove that the following sets of clauses are unsatisfiable:

   a. $S = \{\sim P \vee Q, P \vee R, \sim Q, \sim R\}$

   b. $S = \{P \vee Q \vee R, \sim P, \sim Q, \sim R \vee W, \sim W\}$

   c. $S = \{P \vee Q, Q \vee R, R \vee W, \sim R \vee \sim P, \sim W \vee \sim Q, \sim Q \vee \sim R\}$.

2. Use linear resolution to prove the unsatisfiability of the following set of clauses:

   a. $\sim E(x) \vee V(x) \vee S(x, f(x))$ b. $\sim E(x) \vee V(x) \vee C(f(x))$

   c. $P(a)$ d. $E(a)$

   e. $\sim S(a, y) \vee P(y)$ f. $\sim P(x) \vee \sim V(x)$

   g. $\sim P(x) \vee \sim C(x)$.

   Use clause (g) as the top clause.

3. Prove the unsatisfiability of the following clauses by linear resolution. Use the last clause as the top clause.

   a. $P(x, I(x), e)$

   b. $\sim S(x) \vee \sim S(y) \vee \sim P(x, I(y), z) \vee S(z)$

   c. $S(a)$

   d. $\sim S(e)$.

Section 7.3

4. Prove the unsatisfiability of the following sets of clauses by input resolution.

   a.

$$S = \begin{cases} P(I(x), x, e), \\ P(e, x, x), \\ \sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(u, z, w) \vee P(x, v, w), \\ \sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(x, v, w) \vee P(u, z, w), \\ \sim P(a, x, e). \end{cases}$$

   b.

$$S = \begin{cases} D(x, x), \\ \sim D(x, y) \vee \sim D(y, z) \vee D(x, z), \\ P(x) \vee D(g(x), x), \\ P(x) \vee L(l, g(x)), \\ P(x) \vee L(g(x), x), \\ L(l, a), \\ \sim P(x) \vee \sim D(x, a), \\ \sim L(l, x) \vee \sim L(x, a) \vee P(f(x)), \\ \sim L(l, x) \vee \sim L(x, a) \vee D(f(x), x). \end{cases}$$

5. Consider Exercise 4 again. This time use unit resolution.
6. Let S be an unsatisfiable set of ground clauses. Let L be a unit clause in S. Let S' be that set obtained from S by deleting those clauses containing L and by deleting $\sim L$ from the remaining clauses. Prove that if there is a unit refutation from S, there must be a unit refutation from S'.

Section 7.4

7. Prove the unsatisfiability of the following set of ordered clauses by OL-deduction:

$P(e, x, x)$

$P(x, e, x)$

$P(x, I(x), e)$

$P(I(x), x, e)$

$S(b)$

$\sim S(x) \vee \sim S(y) \vee \sim P(x, I(y), z) \vee S(z)$

$\sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(x, v, w) \vee P(u, z, w)$

$\sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(u, z, w) \vee P(x, v, w)$

$\sim S(I(b))$.

8. Prove the unsatisfiability of the following set of clauses by OL-deduction:

$P(I(x), x, e)$

$P(e, x, x)$

$\sim P(x, y, u) \vee \sim P(y, z, u) \vee \sim P(u, z, w) \vee P(x, v, w)$

$\sim P(x, y, u) \vee \sim P(y, z, v) \vee \sim P(x, v, w) \vee P(u, z, w)$

$\sim P(a, e, a)$.

9. Prove Lemma 7.2.
10. Prove that Theorem 7.2 implies Theorem 6.2.

Section 7.6

11. Consider the following set of ordered clauses:

$$S = \{P \vee Q, P \vee \sim Q \vee R, P \vee \sim Q \vee \sim R, \sim P \vee R, \sim P \vee \sim R\}.$$

Let $P \vee Q$ be the top ordered clause. Let $d^* = 6$. Prove that S is unsatisfiable by (a) the breadth-first method, (b) the depth-first method, and (c) the modified depth-first method.

Section 7.8

12. Let S be the set of following ordered clauses:

$P \vee Q \vee R$

$P \vee Q \vee \sim R$

$P \vee \sim Q \vee R$

$P \vee \sim Q \vee \sim R$

$\sim P \vee Q \vee R$

$\sim P \vee Q \vee \sim R$

$\sim P \vee \sim Q \vee R$

$\sim P \vee \sim Q \vee \sim R.$

Let $P \vee Q \vee R$ be the top ordered clause. Using the evaluation function defined by Eq. (7.7) to order pairs of CLIST, find the tree that will be searched by the modified depth-first method.