Algebraic Geometry
Lecture Notes

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This is Haverford College's undergraduate MATH H334, officially named Algebra II, instructed by Tarik Aougab. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Zoom.

Contents

1 Introduction: It's all connected

The Markov equation is

$$x^2 + y^2 + z^2 = 3xyz.$$

Let's understand the integer solutions for the Markov equation.

Definition 1.1 (Markov number)
A Markov number $n \in \mathbb{N}$ is any number such that there exists y_0, z_0 such that (n, y_0, z_0) is a solution to the Markov equation. Let m_n be the n-th positive integer Markov number.

Example 1.1 (Markov number)
$(1, 2, 5)$ is a solution to the Markov equation. Thus, $1, 2, 5$ are Markov numbers.

Theorem 1.2 (Caracterization of irrational numbers)
Let $\alpha \in \mathbb{R}$. Then, α is irrational $\iff$ there are infinitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}q^2}.$$

Theorem 1.3 ($\sqrt{5}$ is the best constant)
If $\alpha = \phi = \frac{1+\sqrt{5}}{2}$ and $\beta > \sqrt{5}$, there are only finitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\beta q^2}.$$

If we disregard ϕ and its derivatives, then we can change $\sqrt{5}$ to $2\sqrt{2}$.

Theorem 1.4 (Caracterization of irrational numbers not related to ϕ)
Let $\alpha \in \mathbb{R}$. Then, $\alpha \notin \mathbb{Q}[\phi] \iff$ there are infinitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{2\sqrt{2}q^2}.$$

Theorem 1.5 ($2\sqrt{2}$ is the best constant)
If $\alpha = \sqrt{2}$ and $\beta > 2\sqrt{2}$, there are only finitely many coprime (p, q) such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\beta q^2}.$$

We can disregard $\sqrt{2}$ and its derivatives, and change $2\sqrt{2}$ to $\frac{\sqrt{221}}{5}$; and so on.

This naturally creates a sequence of real numbers, called *Lagrange numbers*, which starts as $\sqrt{5}$, $2\sqrt{2}$, $\frac{\sqrt{221}}{5}$, $\ldots$, L_n, $\ldots$.

Surprisingly, there is a conection between the Markov and Lagrange numbers.

Theorem 1.6 (Markov)

$$L_n = \sqrt{9 - \frac{4}{m_n^2}}$$

Theorem 1.7
Let $\rho_1, \rho_2 \in \operatorname{Hom}(F_2, SL(2,\mathbb{R}))$. If $\operatorname{Tr}(\rho_1(a)) = \operatorname{Tr}(\rho_2(a))$, $\operatorname{Tr}(\rho_1(b)) = \operatorname{Tr}(\rho_2(b))$ and $\operatorname{Tr}(\rho_1(ab^{-1})) = \operatorname{Tr}(\rho_2(ab^{-1}))$, then there exists $A \in SL(2,\mathbb{R})$ such that $\rho_1(w) = A\rho_2 A^{-1}$ for all $w \in F_2$.

The upshot of this theorem is that $\operatorname{Hom}(F_2, SL(2,\mathbb{R}))/\text{conjugation}$ is, in some sense, a subset inside $\mathbb{R}^3$.

For certain homomorphisms $\rho : F_2 \to SL(2,\mathbb{R})$, there exists a magical machine, which we will call *hyperbolic geometry machine*, that sends ρ to the following figure.

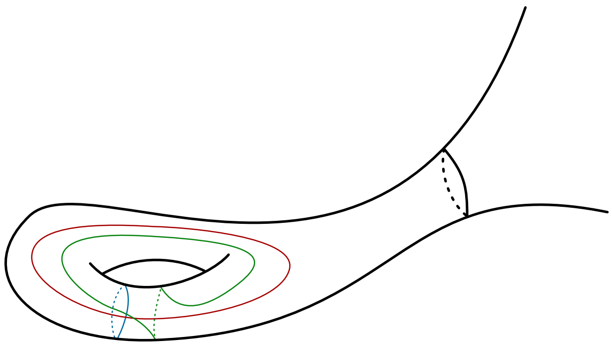


Figure 1: Result of the hyperbolic geometry machine on certain homomorphisms

The length of the blue, green and red loops are replated to $\operatorname{Tr}(\rho(a))$, $\operatorname{Tr}(\rho(b))$ and $\operatorname{Tr}(\rho(ab^{-1}))$.

For certain super special homomorphisms $\rho : F_2 \to SL(2,\mathbb{R})$, this machine sends ρ to this other figure.

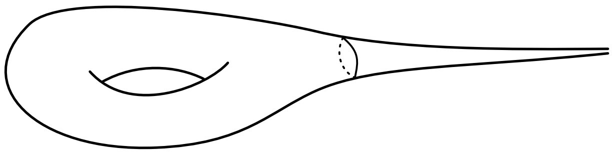


Figure 2: Result of the hyperbolic geometry machine on super special homomorphisms

Theorem 1.8
ρ is super special if, and only if,

$$\operatorname{Tr}(\rho(a))^2 + \operatorname{Tr}(\rho(b))^2 + \operatorname{Tr}(\rho(ab^{-1}) = \operatorname{Tr}(\rho(a)) \operatorname{Tr}(\rho(b)) \operatorname{Tr}(\rho(ab^{-1})),$$

i.e., $\left(\frac{\operatorname{Tr}(\rho(a))}{3}, \frac{\operatorname{Tr}(\rho(b))}{3}, \frac{\operatorname{Tr}(\rho(ab^{-1}))}{3}\right)$ is a solution to the Markov equation.

2 Introducing Algebraic Varieties

February 15, 2021

2.1 Definition

Definition 2.1 (Affine hypersurface)
Let K be a field. $K[x_1, \ldots, x_n]$ is the ring of polynomials with coefficients in K, and Suppose $p \in K[x_1, \ldots x_n]$ and p is not constant. Then,

$$V(p) := \{(k_1, \ldots, k_n) \in K^n \mid p(k_1, \ldots, k_n) = 0\}$$

Example 2.1
Let $K = \mathbb{R}$ and $n = 2$. Consider $p(x_1, x_2) = x_1^2 + x_2^2 - 1$. In this case,

$$V(p) = \{(r_1, r_2) \in \mathbb{R}^2 \mid r_1^2 + r_2^2 - 1 = 0\}.$$

In this case, $V(p)$ represents a circle.

More generally, ellipses, hyperbolas, parabolas are all $V(p)$, for the right choice of p.

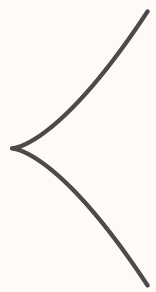
Definition 2.2 (Algebraic variety)
More generally, if $\mathcal{P}$ is a collection of polynomials in $K[X]$, not constants. Define

$$V(\mathcal{P}) = \{(k_1, \ldots, k_n) \in K^n \mid p(k_1, \ldots, k_n) = 0, \forall p \in \mathcal{P}\}.$$

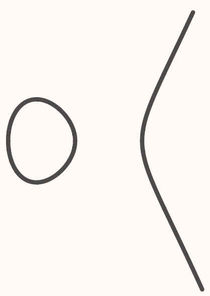
2.2 Examples with $K = \mathbb{R}$

Question 2.1
What sorts of geometric properties can algebraic varieties have?

Example 2.2
Consider $p(x, y) = y^2 - x^3$. Then, $V(p)$ looks like:

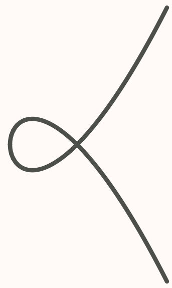


Example 2.3
Consider $q(x, y) = y^2 - x(x^2 - 1)$. Then, $V(q)$ looks like:



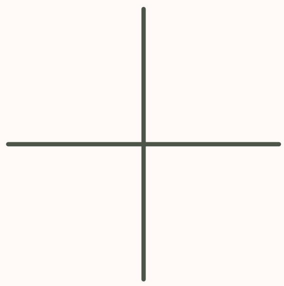
Example 2.4

Consider $r(x, y) = y^2 - x^2(x + 1)$. Then, $V(r)$ looks like:



Example 2.5

Consider $s(x, y) = xy$. Then, $V(s)$ looks like:



Definition 2.3

We say a variety has dimension d if a subset of it "looks like $\mathbb{R}^d$" and if it is the disjoint union of finitely many pieces that each "look like $\mathbb{R}^i$" with $0 \leq i \leq d$

Example 2.6

The dimension of $V(x^2 + y^2 - 1)$ is 1.

Example 2.7

The dimension of $V(\{x, y\}) = \{(0, 0)\}$ is 0.

In Linear Algebra, the number of linearly independent always equals the codimension of the solution set.

Question 2.2

Does this hold for varieties?

Answer. No. $V(x^2 + y^2) = \{(0, 0)\}$, which has dimension 0 (as opposed to the expected $2 - 1 = 1$). Another example is $V(y, y - 1) = \varnothing$, which has dimension -1 (as opposed to the expected $2 - 2 - 2$).

So, linear algebraic dimension count fail for varieties for at least two reasons:

- non-existence of solutions to certain types of algebraic equations (e.g., $x^2 = -1$).
- non-extistence of intersections between parallel lines. February 17, 2021

To solve the first problem, we'll use the complex numbers instead of the real numbers. To solve the second problem, we'll need to develop *projective spaces*.

3 Introducing Projective Spaces

Basic idea Start with a initial space and add new point to it which keep track of the different "ways" of goign off to infinity in a straight line.

Notation $\mathbb{P}^{\text{dimension}}(\text{field})$.

Example 3.1 (Real projective line)

Consider the projective space $\mathbb{P}^1(\mathbb{R})$: this is just $\mathbb{R}$ plus one additional point "at infinity".

Example 3.2 (Real projective plane)

Consider the projective space $\mathbb{P}^2(\mathbb{R})$: this is $\mathbb{R}^2$ plus an additional point for each line in $\mathbb{R}^2$ through the origin.

Any two parallel lines in $\mathbb{R}^2$ intersect in the $\mathbb{P}^2(\mathbb{R})$.

So, any two lines in $\mathbb{P}^2(\mathbb{R})$ intersect at a point in $\mathbb{P}^2(\mathbb{R})$

Example 3.3 (Complex projective line)

Consider the projective space $\mathbb{P}(\mathbb{C})$: this is just $\mathbb{C}$ plus one additional point "at infinity".

Definition 3.1

In $\mathbb{C}^2$, a *complex line through the origin* is a subvector space of $\mathbb{C}^2$ over $\mathbb{C}$ with dimension 1.

February 19, 2021

Example 3.4

Consider the projective space $\mathbb{P}^2(\mathbb{C})$: this is $\mathbb{C}^2$ plus an additional point "at infinity" for each complex line through the origin.

"How many" new points are there? There is one for each complex line. In fact, if we look to the "slope" of each complex line, it lives inside $\mathbb{P}(\mathbb{C})$ and uniquely identifies each complex line.

4 Exploring some varieties

February 26, 2021

Example 4.1

$$y^2 = \dots$$

yields a sphere on $\mathbb{P}^2(\mathbb{C})$.

Example 4.2

$$y^2 = x(x^2 - 1)$$

yields a torus on $\mathbb{P}^2(\mathbb{C})$.

Theorem 4.1 (WRONG! Dream Theorem)

Example 4.3

The polynomial $p(x, y) = xy$ yeilds to two spheres that touch at one point in $\mathbb{P}^(\mathbb{C}^2)$; which is not on the list.

Theorem 4.2 (Correct Theorem)

If $p \in \mathbb{C}[x, y]$ ...

5 Projective Spaces

March 01, 2021

Let K be a field (in practice, for this class, $K = \mathbb{R}$ or $\mathbb{C}$).

Theorem 5.1 (n-dimension Projective Space)
Given $n \in \mathbb{Z}_{\geq 0}$, the projective n-dimension space over K, denoted by $\mathbb{P}^n(K)$, is defined as the set

$$\mathbb{P}^n(K) = \{A \subset K^{n+1} : A \text{ is a subspace of } K^{n+1} \text{ with dimension 1over } K\}$$

Small, informal aside: $\mathbb{P}^n(K)$ is more than just a set. It is a topological space — more on this soon.

Example 5.1
$\mathbb{P}^1(\mathbb{R})$ is the set of lines in $\mathbb{R}^2$ that go through the origin.

We can try to use the blue circle to "keep track" of the lines.

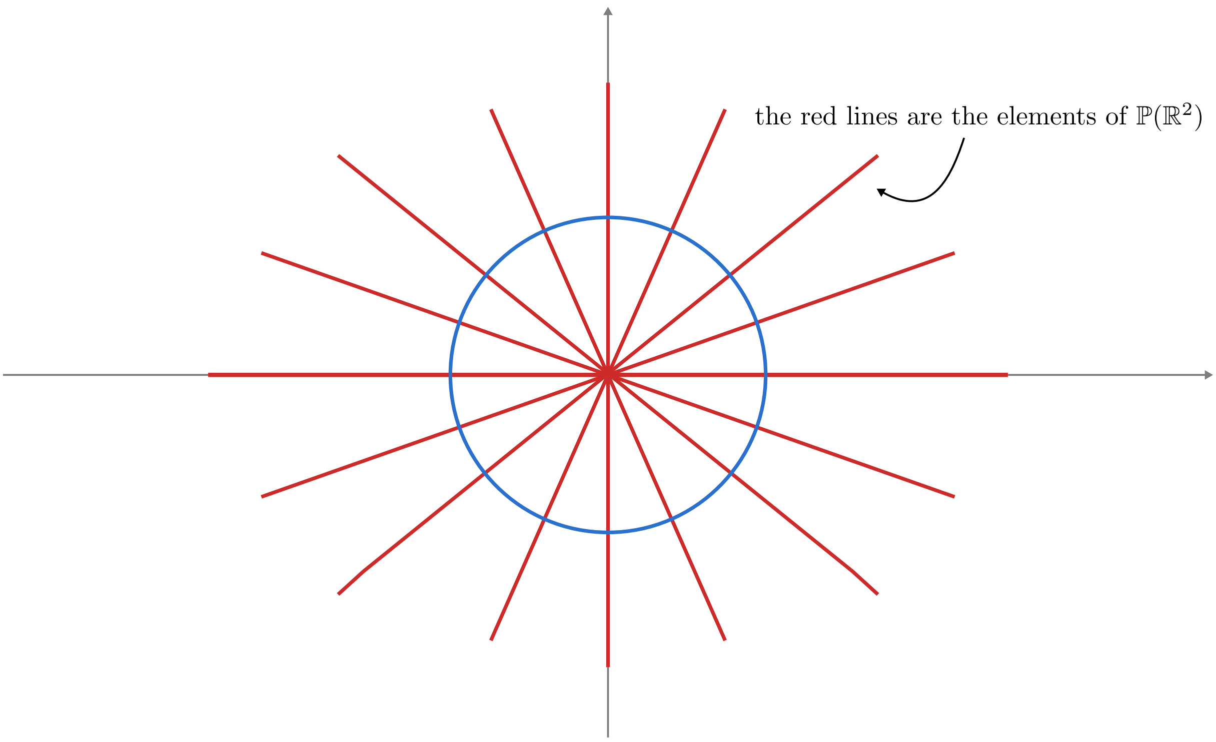


Figure 3: Projective real line

Example 5.2
$\mathbb{P}^2(\mathbb{R})$ is the set of lines in $\mathbb{R}^3$ that go through the origin.

We can try to use the unit sphere to "keep track" of the lines.

Definition 5.2
If U is a $(r+1)$-dimensional subspace of K^{n+1}, then the 1-subspaces of U yield a subset of $\mathbb{P}^n(K)$, and called a r-dimension projective subspace.

Proposition 5.3
Any r-dimension projective subspace is naturally a copy of $\mathbb{P}^r(K)$ inside of $\mathbb{P}^n(k)$.

Proof. Any two r-dimension subspaces of K^{n+1} are related by an isomorphism of $K^{n+1} \to K^{n+1}$ (change of basis). Notice also that $K^{r+1} \subset K^{n+1}$, corr. to zero-ing out the last $n-r$ coordinates is an $(r+1)$-subspace of K^{n+1}. And its 1-subspaces are the elements of $\mathbb{P}^r(K)$, by definition.

Definition 5.4

If a vector space V has dimension n, and $U \subset V$ is a subspace, the *co-dimension of* U, denoted $\text{cod}(U)$, is
$$\text{cod}(U) = n - \dim(U).$$

Lemma 5.5

Let S_1, S_2 be any two projective subspaces of $\mathbb{P}^n(K)$. Then,
$$\text{cod}(S_1 \cap S_2) \leq \text{cod}(S_1) + \text{cod}(S_2).$$

Equivalentely,
$$\dim(S_1 \cap S_2) \geq \dim(S_1) + \dim(S_2) - n.$$

March 03, 2021

Sketch. Using tools from Linear Algebra, we can conclude that given two subspaces $V_1, V_2 \subset V$,
$$\dim(V_1 \cap V_2) \geq \dim(V_1) + \dim(V_2) - \dim(V).$$

An $(r+1)$-dimensional subspace $\tilde{S}_1$ of K^{n+1} has codimension $(n+1)-(r+1) = n-r$. And, the associated projective subspace S_1 of $\mathbb{P}^n(K)$ has same codimension. So, the inequality for vector spaces implies the inequality for projective spaces.

We will use a lot the connection between vector spaces and projective spaces.

Example 5.3

Any two projective 2-spaces in $\mathbb{P}^3(\mathbb{R})$ intersect in at least a (projective) line.

Definition 5.6

Let $p \in \mathbb{P}^n(K)$ — we may call p a *projective point*, or simply a *point* — and let L_p be the corresponding line throught the origin in K^{n+1}. (*Technically, those are the same, but it is useful to separate them.*)

Then, if $\vec{a} \in K^{n+1}$, $\vec{a} \in L_p$, $\vec{a} \neq \vec{0}$, we call $\vec{a}$ a *coordinate set for* p — or simply *coordinates for* p.

An unforunate fact is that a single projective point p doesn't have a unique coordinate set.

Proposition 5.7

Given two non-zero $\vec{a}, \vec{b} \in K^{n+1}$, they are coordinate for the same point in $\mathbb{P}^n(K)$ if, and only if, there exists $\lambda \in K$ such that
$$\vec{a} = \lambda \cdot \vec{b},$$
i.e., if 0, $\vec{a}$ and $\vec{b}$ are collinear.

Example 5.4

Let's think about $\mathbb{P}^2(\mathbb{R})$.

For any point $(x, y, z) \in \mathbb{R}^3$ such that $z \neq 0$, we can divide by z and get $\left(\frac{x}{z}, \frac{y}{z}, 1\right)$ — *which represents the same projective point in* $\mathbb{P}^2(\mathbb{R})$ *as* (x, y, z).

Therefore, except for the projective points (lines) in the xy-plane, we can handle the problem of non-unique representation of projective points by referring to a projective point by the *unique* point in $\mathbb{R}^3$ with a 1 in the last coordinate. See fig. 4.

So, the plane $z = 1$ (a copy of $\mathbb{R}^2$) can be naturally identified with the subset of $\mathbb{P}^2(\mathbb{R})$ consisting of projective points that represent lines *not* in the xy-plane. The remaining projective points can be identified with a copy of $\mathbb{P}^1(\mathbb{R})$ — which we usually call *the line at infinity*.

In general, one can always imagine $\mathbb{P}^n(K)$ as a copy of K^n together with a copy of $\mathbb{P}^{n-1}(K)$ *"at infinity"* — the latter we call *the hyperplane at infinity*.

Remark. There is no preferred hyperplane at infinity. In our example, the choice of the plane $z = 1$ was completely arbitrary.

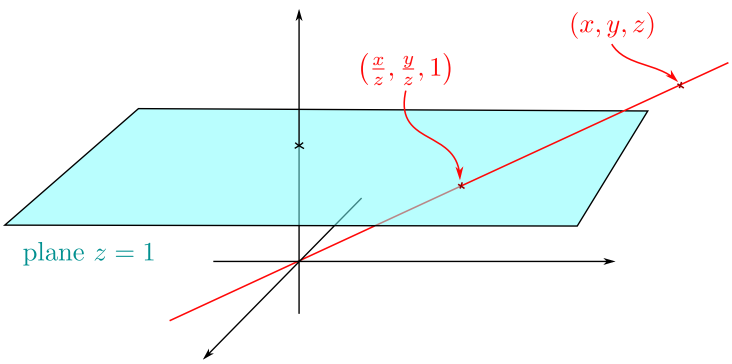


Figure 4: Real Projective Plane

Lemma 5.8

Any $(n-1)$-dimensional projective subspace W in $\mathbb{P}^n(K)$ can be chosen as the hyperplane at infinity.