Analysis II
Lecture Notes

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Haverford College — Fall 2021
Last updated: January 24, 2022

This is Haverford College's undergraduate MATH H318, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 Limits of Functions **4**
1.1 Interaction with Boundness 6

1 Limits of Functions

Lecture 1

Definition 1.1 (Pointwise Convergence)

Given a sequence (f_n) of functions with $f_n\colon A \subset \mathbb{R} \to \mathbb{R}$, f_n *converges pointwise on* A to $f\colon A \to \mathbb{R}$ if, for each $x \in A$,

$$\lim_{n\to\infty} f_n(x) = f(x).$$

This notion of convergence for functions has undesirable properties. For example, even if all f_n are continuous, it may be the case that their limit function is not a continuous function. Let's define a stronger convergence condition for functions.

Definition 1.2 (Uniform Convergence)

Given a sequence (f_n) of functions with $f_n\colon A \subset \mathbb{R} \to \mathbb{R}$, f_n *converges uniformly on* A to $f\colon A \to \mathbb{R}$ if, for all $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in A$ and $n \geq N$.

Note that we can rewrite the definition of pointwise convergence as: for all $x \in A$ and all $\epsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. The key difference between pointwise convergence and uniform convergence is that N depends on both x and ϵ in pointwise convergence, while N only depends on ϵ in uniform convergence.

Proposition 1.3

Given a sequence (f_n) of functions with $f_n\colon A \subset \mathbb{R} \to \mathbb{R}$ such that f_n converges uniformly on A to $f\colon A \to \mathbb{R}$, then f_n converes poiwise on A to f.

Proof. Since f_n converges uniformly on A to f, for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in A$ and $n \geq N$.

Therefore, for all $x \in A$ and $\epsilon > 0$, the integer $N(\epsilon)$ has the property that

$$|f_n(x) - f(x)| < \epsilon,$$

i.e., for all $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$. Therefore, f_n converges pointwise on A to f. ■

Theorem 1.4

Given a sequence (f_n) of functions with $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$, f_n converfes uniformly on A to f if, and only if,

$$\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0.$$

Proof. Suppose f_n converges uniformly on A to f. Then, for all $\epsilon > 0$, there exists an integer N such that $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in A$ and $n \geq N$. This implies that, for all $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$. Finally, using the definition of limit, we conclude that $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$.

Now, suppose that $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$. This implies that, for all $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon/2$. Then, for all $\epsilon > 0$, there exists an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$ and $n \geq N$. Therefore, f_n converges uniformly on A to f. ■

Lecture 2

Uniform convergence of f_n to f says that, for all $\epsilon > 0$, there exists N large enough so that the graph of f_n, for all $n \geq N$, is entirely in the "ϵ-tube" of the graph of f.

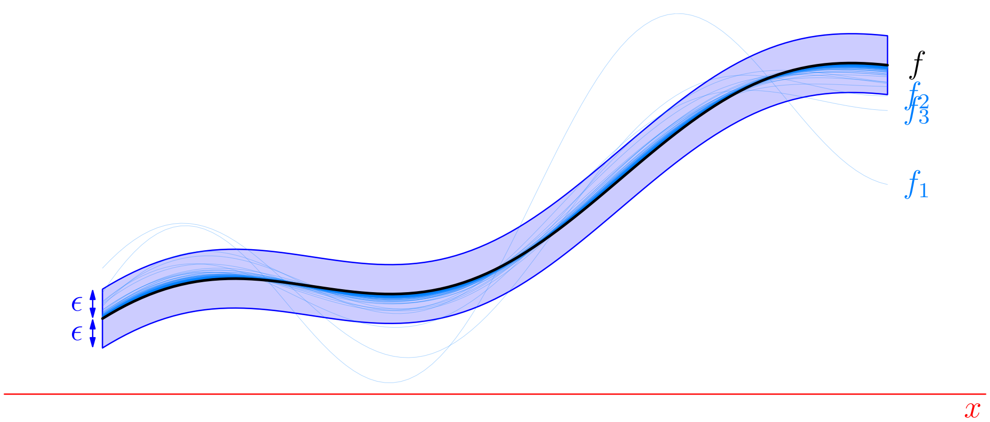

Figure 1.1: Graph of the "ϵ-tube." In this example, all f_n, for $n \geq 4$, are in the ϵ-tube.

Example

Let $f_n(x) = \frac{1}{1+nx^2}$ and let f be its poitwise limit, i.e., $f(x) = 0$. Then, f_n uniformly converges to f on $(\epsilon, 1)$, for all $\epsilon > 0$; however, f_n does not converge uniformly to f on $(0, 1)$.

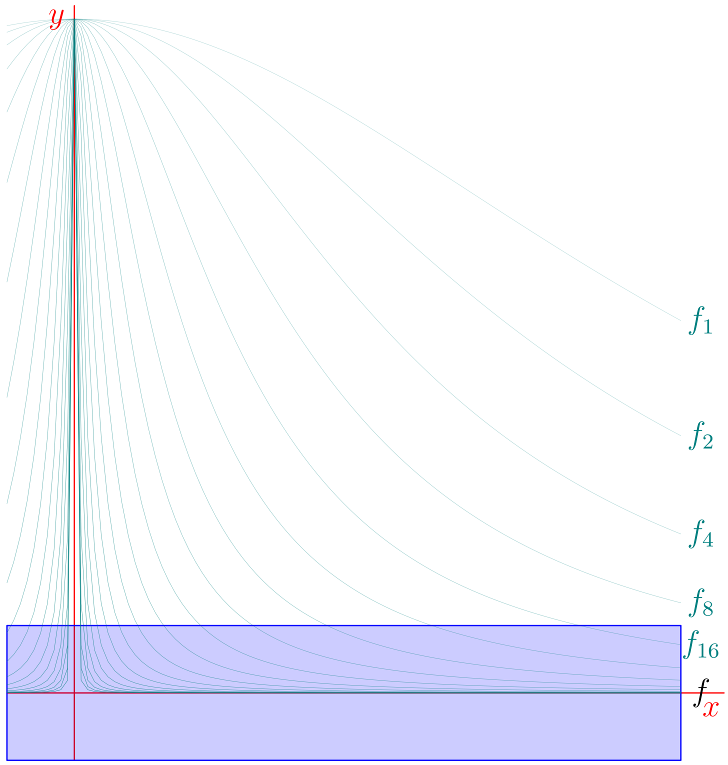

Figure 1.2: Graph of some functions $f_n(x) = \frac{1}{1+nx^2}$.

Lecture 3

1.1 Interaction with Boundness

Proposition 1.5

There exists a sequence of functions $f_n\colon A \subset \mathbb{R} \to \mathbb{R}$, all of them bounded on A, and $f_n \to f$ pointwise on A for a unbounded function f on A.

Proof. Consider $f_n\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} x & |x| < n \\ 0 & \text{otherwise,} \end{cases}$$

which converges pointwise to $f(x) = x$ on $\mathbb{R}$. ■

Proposition 1.6

If $f_n\colon A \subset \mathbb{R} \to \mathbb{R}$ is bounded for each n, and if $f_n \to f$ uniformly on A, then f is bounded on A.

Proof. Plug $\epsilon \mapsto 1$ on the definition of uniformly convergence. Then, there exists N such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and all $x \in A$.

Since f_N is bounded, there exists M such that $|f_N(x)| < M$ for all $x \in A$. Finally, by triangular inequality, we conclude that

$$|f(x)| < M + 1,$$

for all $x \in A$; therefore, f is bounded. ■