Analysis II
Lecture Notes

Guilherme Zeus Dantas e Moura
gdantasemo@haverford.edu

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This is Haverford College's undergraduate MATH H318, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 Limits of Functions

Definition 1.1 (Pointwise Convergence)

Given a sequence (f_n) of functions with $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$, f_n *converges pointwise on* A to $f \colon A \to \mathbb{R}$ if, for each $x \in A$,

$$\lim_{n \to \infty} f_n(x) = f(x).$$

This notion of convergence for functions has undesirable properties. For example, even if all f_n are continuous, it may be the case that their limit function is not a continuous function. Let's define a stronger convergence condition for functions.

Definition 1.2 (Uniform Convergence)

Given a sequence (f_n) of functions with $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$, f_n *converges uniformly on* A to $f \colon A \to \mathbb{R}$ if, for all $\epsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in A$ and $n \geq N$.

Note that we can rewrite the definition of pointwise convergence as: for all $x \in A$ and all $\epsilon > 0$, there exists N such that $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$. The key difference between pointwise convergence and uniform convergence is that N depends on both x and ϵ in pointwise convergence, while N only depends on ϵ in uniform convergence.

Proposition 1.3 (Uniform convergence implies pointwise convergence)

Given a sequence (f_n) of functions with $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$ such that f_n converges uniformly on A to $f \colon A \to \mathbb{R}$, then f_n converes pointwise on A to f.

Proof. Since f_n converges uniformly on A to f, for all $\epsilon > 0$, there exists an integer $N(\epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon$$

for all $x \in A$ and $n \geq N$.

Therefore, for all $x \in A$ and $\epsilon > 0$, the integer $N(\epsilon)$ has the property that

$$|f_n(x) - f(x)| < \epsilon,$$

i.e., for all $x \in A$, $\lim_{n\to\infty} f_n(x) = f(x)$. Therefore, f_n converges pointwise on A to f. ■

Theorem 1.4

Given a sequence (f_n) of functions with $f_n\colon A \subset \mathbb{R} \to \mathbb{R}$, f_n converfes uniformly on A to f if, and only if,

$$\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0.$$

Proof. Suppose f_n converges uniformly on A to f. Then, for all $\epsilon > 0$, there exists an integer N such that $|f_n(x) - f(x)| < \epsilon/2$ for all $x \in A$ and $n \geq N$. This implies that, for all $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon$. Finally, using the definition of limit, we conclude that $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$.

Now, suppose that $\lim_{n\to\infty} \sup_{x\in A} |f_n(x) - f(x)| = 0$. This implies that, for all $\epsilon > 0$, there exists an integer N such that, for all $n \geq N$, $\sup_{x\in A} |f_n(x) - f(x)| < \epsilon/2$. Then, for all $\epsilon > 0$, there exists an integer N such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in A$ and $n \geq N$. Therefore, f_n converges uniformly on A to f. ■

Uniform convergence of f_n to f says that, for all $\epsilon > 0$, there exists N large enough so that the graph of f_n, for all $n \geq N$, is entirely in the "ϵ-tube" of the graph of f.

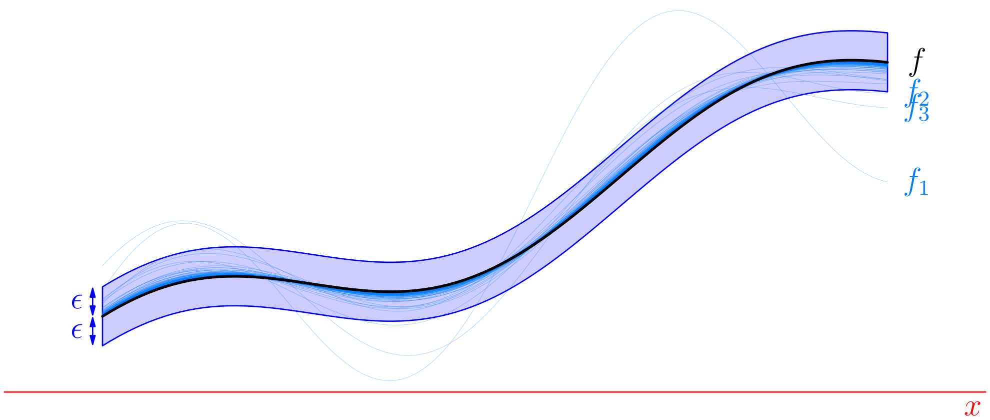

Figure 1.1: Graph of the "ϵ-tube." In this example, all f_n, for $n \geq 4$, are in the ϵ-tube.

Example

Let $f_n(x) = \frac{1}{1+nx^2}$ and let f be its poitwise limit, i.e., $f(x) = 0$. Then, f_n uniformly converges to f on $(\epsilon, 1)$, for all $\epsilon > 0$; however, f_n does not converge uniformly to f on $(0, 1)$.

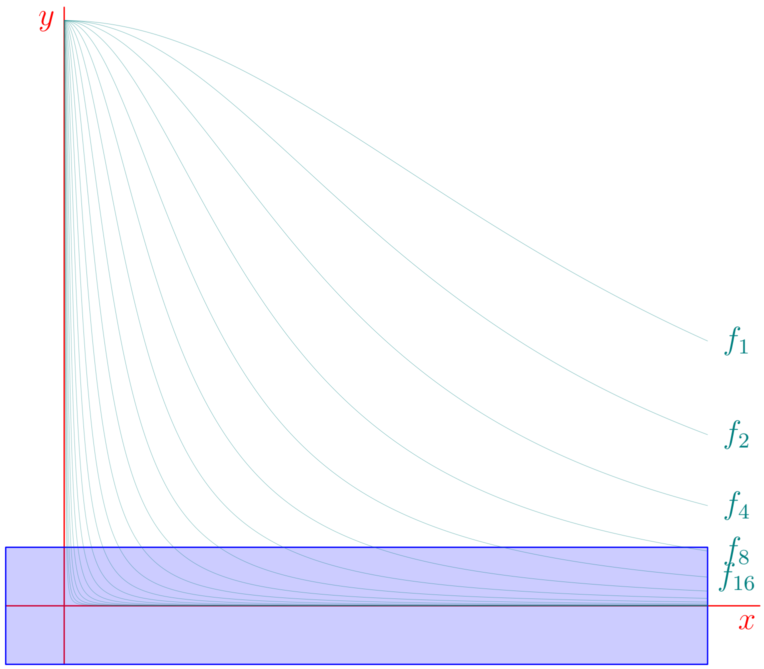

Figure 1.2: Graph of some functions $f_n(x) = \frac{1}{1+nx^2}$.

1.1 Interaction with Boundness

Proposition 1.5 (Pointwise convergence does not preserve boundness)

There exists a sequence of functions $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$, all of them bounded on A, and $f_n \to f$ pointwise on A for a unbounded function f on A.

Proof. Consider $f_n \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} x & |x| < n \\ 0 & \text{otherwise,} \end{cases}$$

which converges pointwise to $f(x) = x$ on $\mathbb{R}$. ■

Proposition 1.6 (Uniform convergence preserves boundness)

If $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$ is bounded for each n, and if $f_n \to f$ uniformly on A, then f is bounded on A.

Proof. Plug $\epsilon \mapsto 1$ on the definition of uniformly convergence. Then, there exists N such that

$$|f_n(x) - f(x)| < 1$$

for all $n \geq N$ and all $x \in A$.

Since f_N is bounded, there exists M such that $|f_N(x)| < M$ for all $x \in A$. Finally, by triangular inequality, we conclude that

$$|f(x)| < M + 1,$$

for all $x \in A$; therefore, f is bounded. ■

1.2 Interaction with Continuity

Proposition 1.7 (Uniform convergence preserves continuity)

If $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$ is continuous for each n, and if $f_n \to f$ uniformly on A, then f is continuous on A.

Proof. Let $c \in A$ be arbitrary. Let $\epsilon > 0$ be arbitrary.

Since $f_n \to f$, there exists N such that

$$|f_n(x) - f(x)| < \epsilon/3$$

for all $n \geq N$ and $x \in A$.

Since f_N is continous at c, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(c)| < \epsilon/3$$

for all $x \in A$ satisfying $|x - c| < \delta$.

Therefore, by triangle inequality,

$$\begin{aligned}|f(c) - f(x)| &= |f(c) - f_n(c) + f_n(c) - f_n(x) + f_n(x) - f(x)| \\ &\leq |f(c) - f_n(c)| + |f_n(c) - f_n(x)| + |f_n(x) - f(x)| \\ &< \epsilon\end{aligned}$$

for all $x \in A$ satisfying $|x - c| < \delta$. Since ϵ was arbitrary, this implies f is continuous at c. Since c was arbitrary, this implies f is continuous on A. ∎

1.3 Interaction with Differentiability

Proposition 1.8 (Uniform convergence does not preserve differentiability)

There exists a sequence of functions $f_n \colon A \subset \mathbb{R} \to \mathbb{R}$, all of them differentiable on A, and $f_n \to f$ uniformly on A for a non-differentiable function f on A.

Theorem 1.9

If

- **i.** f_n is differentiable on $[a, b]$, for all integers n,
- **ii.** f_n' converges uniformly on $[a, b]$ to g, and
- **iii.** f_n converges pointwise on $[a, b]$ to f,

then f is differentiable on $[a, b]$, and $f' = g$.

Aside: Series solutions for differential equations

Question. Derive functions $y(x)$ that obey

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0.$$

Proof (Sketch). Plug $y(x) = \sum_{n=0}^{\infty} a_n x^n$ in the equation above yields

$$a_1 + \sum_{n=1}^{\infty} \left((n+1)^2 a_{n+1} + a_{n-1}\right) x^n = 0,$$

thus we can infer $a_{2k+1} = 0$ and $(2k)^2 a_{2k} + a_{2k-2} = 0$ for all $k \in \{1, 2, \ldots\}$. ∎

1.4 Uniformly Cauchy

Definition 1.10 (Uniformly Cauchy)

The sequence of functions $f_n\colon A \to \mathbb{R}$ is uniformly Cauchy on A if, for all $\epsilon > 0$, there exists a positive integer N such that, for all $x \in A$ and all $m, n \geq N$,

$$|f_m(x) - f_n(x)| < \epsilon.$$

Theorem 1.11

The sequence of functions $f_n\colon A \to \mathbb{R}$ converges uniformly on A if, and only if, it is uniformly Cauchy on A.

Proof (Uniformly Convergence implies Uniformly Cauchy). If $f_n\colon A \to \mathbb{R}$ converges uniformly on A, then for all $\epsilon > 0$, there exists a positive integer N such that

$$|f_n(x) - f(x)| < \epsilon/2$$

for all $x \in A$ and all $n \geq N$.

Therefore, it follows that, for all ϵ, there exists a positive integer N such that

$$|f_m(x) - f_n(x)| = |f_m(x) - f(x)| + |f(x) - f_n(x)| < \epsilon.$$

for all $x \in A$ and all $m, n \geq N$. ∎

Proof (Uniformly Cauchy implies Uniformly Convergence). To be done. ∎

1.5 Weierstrass M-test

Theorem 1.12 (Weierstrass M-test)

If $g_n\colon A \to \mathbb{R}$ is a sequence of functions and if there exists constants $M_n \geq 0$ so that

$$|g_n(x)| \leq M_n$$

for all $x \in A$ and $\sum_n M_n$ converges, then $\sum_n g_n(x)$ converges uniformly on A.

Proof. Since $\sum M_n$ converges, by the Cauchy criterion, for all $\epsilon > 0$, there exists a positive integer N such that

$$M_{m+1} + \cdots + M_n = |M_{m+1} + \cdots + M_n| < \epsilon$$

for all $n > m \geq N$.

Thus, for all ϵ, for the N above, implies that

$$|g_{m+1}(x) + \cdots + g_n(x)| < \epsilon.$$

for all $m > n \geq N$ and all $x \in A$. ∎

2 Function Spaces

2.1 Our first function spaces

Definition 2.1

Given $A \subset \mathbb{R}$, let $C(A)$ be the set of all functions that are continuous on A, and let $B(A)$ be the set of all functions that are bounded on A.

Proposition 2.2

$C([a, b]) \subseteq B([a, b])$.

Definition 2.3 (Infinity Norm)

Given a bounded function $f : [a, b] \to \mathbb{R}$, let

$$||f||_\infty = \sup_{x \in [a,b]} |f(x)|.$$

The infinity norm is a valid norm for both $B([a, b])$ and $C([a, b])$.

2.2 Topology in function spaces

Last semester, we defined ϵ-neighborhoods, open sets, sequence convergence, closed sets, etc. on normed vector spaces. Therefore, we have those definitions for $C([a, b])$ and $B([a, b])$

Example

The set $\{f \in C[(0, 1)] : f(1/3) + f(2/3) < 1\}$ is open, but not closed.

The set $\{f \in C[(0, 1)] : f(0) = 0\}$ is closed, but not open.

The set $\{f \in C[(0, 1)] : f \text{ is a constant function}\}$ is closed, but not open.

The set $\{f \in C[(0, 1)] : f \text{ is a polynomial}\}$ is neither open nor closed.

Proposition 2.4

Convergence with respect to the infinity norm is equal to uniform convergence.

Proof. Note that $f_n \to f$ with respect to $||\bullet||_\infty$ is equivalent to $||f_n - f||_\infty \to 0$. In turn, this is equivalent to $\lim_{n\to\infty} \sup_{x\in[a,b]} |f_n(x) - f(x)|$. Finally, this is equivalent to $f_n \to f$ uniformly. ■

2.3 Measure

We are going to cheat a little.

Definition 2.5 (Measure Zero)

A set $A \subset \mathbb{R}$ *has measure zero* if, for all $\epsilon > 0$, there exists a finite or countable collection of open intervals (a_n, b_n) such that

$$A \subset \bigcup_n (a_n, b_n) \qquad \text{and} \qquad \sum_n (b_n - a_n) \leq \epsilon.$$

Proposition 2.6

If A has measure zero, and B has measure zero, then $A \cup B$ has measure zero.

Proposition 2.7

If $\mathcal{C}$ is a countable collection of sets with measure zero, then their union also has measure zero.

Example

The sets $\{4, 8\}$, $\mathbb{N}$, $\mathbb{Q}$, and the cantor set C have measure zero.

Proposition 2.8

If $a < b$, then $[a, b]$ does not have measure zero.

Proof (Sketch). Suppose we could cover $[a, b]$ with an open cover with total length at most $\frac{b-a}{4}$. Since $[a, b]$ is compact, this cover has a finite sub-cover; which still has length at most $\frac{b-a}{4}$. Suppose that that subcover contains (a_1, b_1), $\ldots$, (a_n, b_n), with

$a_1 \leq a_2 \leq \cdots \leq a_n$. To be finished. ■

Corolary 2.9

If $a < b$, then $(a, b]$, $[a, b)$, (a, b) do not have measure zero.

2.3.1 Step Functions

A step function is a function that is non-zero on a finite set of disjoint bounded intervals, constant on each of those intervals, and zero everywhere else.

Definition 2.10 (Characteristic Function)

Given $S \subset \mathbb{R}$, the corresponding characteristic function is

$$\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

Definition 2.11 (Step Function)

A step function is a function of the form

$$f(x) = c_1\chi_{I_1}(x) + c_2\chi_{I_2}(x) + \cdots + c_n\chi_{I_n}(x),$$

where c_j are real numbers and I_j is a collection of disjoint bounded intervals. Equivalently, I_j can be a collection of (not necessarily disjoint) bounded intervals.

Definition 2.12 (Lebesgue integral of a step function)

Given a step function

$$f(x) = c_1\chi_{I_1}(x) + c_2\chi_{I_2}(x) + \cdots + c_n\chi_{I_n}(x),$$

we define its Lebesgue integral to be

$$\int_{-\infty}^{\infty} f(x) = c_1 m(I_1) + c_2 m(I_2) + \cdots + c_n m(I_n),$$

where $m(I_i)$ is the absolute value of the difference of I_i's endpoints.

Definition 2.13 (Lebesgue integral of some other functions)

Given a function $f\colon \mathbb{R} \to \mathbb{R}$, if we can find a sequence ϕ_n of step functions such that

i. $\phi_1(x), \phi_2(x), \phi_3(x), \ldots$ is monotone increasing for each $x \in \mathbb{R}$, and

ii. $\phi_n \to f$ pointwise except possibly on a set of measure zero,

then we define

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)dx,$$

if that limit exists.

Example

Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ and } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\phi_n(x) := 0$ converges pointwise to f except on $\mathbb{Q}\cap[0,1]$, which has measure zero. Therefore,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)dx = 0.$$

Example

Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = 1.$$

Define $\phi_n\colon \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) := \begin{cases} 1 & \text{if } x \in [-n, n] \\ 0 & \text{otherwise.} \end{cases}$$

Since ϕ_n converges pointwise to f,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)dx = \lim_{n\to\infty} 2n = +\infty.$$

Example

Consider $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/2^k & \text{if } k-1 < x \leq k \text{ for } k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Define $\phi_n \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) := \begin{cases} 1/2^k & \text{if } k-1 < x \le k \text{ for } k \in \{1, 2, \ldots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

Since ϕ_n converges pointwise to f,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)dx \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^n} = 1. \end{aligned}$$

Example

Consider $f \colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/\sqrt{x} & \text{for } 0 < x \le 1 \\ 0 & \text{anywhere.} \end{cases}$$

Define $\phi_n \colon \mathbb{R} \to \mathbb{R}$ by

$$\phi_n(x) := f\left(\frac{k^2}{4^n}\right) = \frac{2^n}{k},$$

if $\frac{(k-1)^2}{4^n} < x \le \frac{k^2}{4^n}$. Since ϕ_n converges pointwise to f,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= \lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)dx \\ &= \lim_{n\to\infty} \sum_{j=1}^{2^n} \frac{2^n}{j} \frac{j^2-(j-1)^2}{4^n} \\ &= \lim_{n\to\infty} \frac{1}{2^n} \sum_{j=1}^{2^n} \frac{j^2-(j-1)^2}{j} \end{aligned}$$

To be continued.

Definition 2.14

Let $L^0(\mathbb{R})$ be the set of functions $f \colon \mathbb{R} \to \mathbb{R}$ such that there exists a sequence ϕ_n of step functions satisfying

i. $\phi_n(x) \le \phi_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$;

ii. $\phi_n \to f$ pointwise, expect on a set of measure zero;
iii. $\lim_{n\to\infty} \int_{-\infty}^{\infty} \phi_n(x)$ is a real number.

This definition only makes sense given the following theorem:

Theorem 2.15

If two sequence of step functions converge pointwise to f, then the limit of their integrals are equal (or both don't exist).

Definition 2.16

Let $L^1(\mathbb{R})$ be the space of functions f such that there are $g, h \in L^0(\mathbb{R})$ satisfying $f = g - h$. In that case, we define

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx - \int_{-\infty}^{\infty} h(x)\,dx.$$

Theorem 2.17 ($L^1(\mathbb{R})$ is a vector space)

$L^1(\mathbb{R})$ is a vector space.

Theorem 2.18 (Order Integral Theorem)

If $f_1, f_2 \in L^1(\mathbb{R})$ and $f_1(x) \geq f_2(x)$ for all $x \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} f_1(x)\,dx \geq \int_{-\infty}^{\infty} f_2(x)\,dx.$$

Theorem 2.19

If $f \in L^1(\mathbb{R})$, then $|f| \in L^1(\mathbb{R})$, and

$$\left|\int_{-\infty}^{\infty} f(x)\,dx\right| \leq \int_{-\infty}^{\infty} |f(x)|\,dx.$$

Definition 2.20 (1-"norm")

Given $f \in L^1(\mathbb{R})$, let $\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|$.

The 1-"norm" is not actally an norm. Let's "fix" $L^1(\mathbb{R})$ so that 1-"norm" is actually

Definition 2.21

Let $L^1_{nvs}(\mathbb{R})$ be the equivalent classes of functions, in which f and g are equivalent if, and only if, f and g agree except on a set of measure zero.

Notationally, nobody calls this $L^1_{nvs}(\mathbb{R})$; they just call it $L^1(\mathbb{R})$. And most of the time, they describe an object in $L^1(\mathbb{R})$ as if it were a function, even though in fact is a "collection of functions that all equal each other except on a set of measure zero".

Definition 2.22

If $f\colon A \supset [a,b] \to \mathbb{R}$ and the restriction of f to $[a,b]$ is in $L^1([a,b])$, then

$$\int_a^b f(x)\,dx := \int_{-\infty}^{\infty} g(x)\,dx,$$

where $g\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x) \begin{cases} f(x) & x \in [a,b] \\ 0 & \text{otherwise.} \end{cases}$$

Since we are lazy, we will say that $f \in L^1([a,b])$.

Proposition 2.23

If $f \in L^1([a,b])$ and $f \in L^1([b,c])$, then $f \in L^1([a,c]$ and

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Theorem 2.24 (Lebegue's Fundamental Theorem of Calculus)

If $f \in L^1([a,b])$ define $F(x) = \int_a^b f(t)\,dt$ for any $x \in [a,b]$. If f is continuous at $c \in (a,b)$, then $F'(c)$ exists and equals $f(c)$.

Proof. Let $\epsilon > 0$ be arbitrary.

Since f is continous at c, there exists $\delta > 0$ so that for every x δ-close to c we have $|f(x) - f(c)| < \delta$.

Let h be arbitrary such that $0 < |h| < \delta$. Without loss of generality, suppose

$0 < h < \delta$.

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| = \left|\frac{\int_a^{c+h} f(t)\,dt - \int_a^c f(t)\,dt}{h} - f(c)\right|$$
$$= \left|\frac{\int_c^{c+h} f(t)\,dt|}{h} - \frac{\int_c^{c+h} f(c)\,dt}{h}\right|$$
$$= \left|\frac{\int_c^{c+h} (f(t) - f(c))\,dt|}{h}\right| < \epsilon.$$

Therefore, the result follows. ■