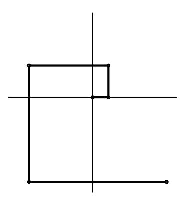
Haverford Problem Solving Group
December 1, 2021

In order to prepare ourselves to work on fresh new problems this weekend (Putnam, Dec 4^{th}), here is a totally new set of problems. If you are interested in thinking about some problem in previous lists, you can access them in our Discord server. Scan this QR code to join.

Problem 20. Define a *growing spiral* in the plane to be a sequence of points with integer coordinates $P_0 = (0,0), P_1, \ldots, P_n$ such that $n \geq 2$ and:

- The directed line segments P_0P_1, P_1P_2, $\ldots$, $P_{n-1}P_n$ are in successive coordinate directions east (for P_0P_1), north, west, south, east, etc.
- The lengths of these line segments are positive and strictly increasing.



How many of the points (x, y) with integer coordinates $0 \leq x \leq 2011, 0 \leq y \leq 2011$ *cannot* be the last point, P_n, of any growing spiral?

Problem 21. Let $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ be sequences of positive real numbers such that $a_1 = b_1 = 1$ and $b_n = b_{n-1}a_n - 2$ for $n = 2, 3, \ldots$. Assume that the sequence (b_j) is bounded. Prove that

$$S = \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}$$

converges, and evaluate S.

Problem 22. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Problem 23. Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < \left| h\sqrt{m} - k\sqrt{n} \right| < 2\varepsilon.$$

Problem 24. Let S be a class of functions from $[0, \infty)$ to $[0, \infty)$ that satisfies:

- The functions $f_1(x) = e^x - 1$ and $f_2(x) = \ln(x+1)$ are in S;
- If $f(x)$ and $g(x)$ are in S, the functions $f(x) + g(x)$ and $f(g(x))$ are in S;
- If $f(x)$ and $g(x)$ are in S and $f(x) \geq g(x)$ for all $x \geq 0$, then the function $f(x) - g(x)$ is in S.

Prove that if $f(x)$ and $g(x)$ are in S, then the function $f(x)g(x)$ is also in S.

Problem 25. Let P be a given (non-degenerate) polyhedron. Prove that there is a constant $c(P) > 0$ with the following property:

> If a collection of n balls whose volumes sum to V contains the entire surface of P, then $n > c(P)/V^2$.