Problem Solving Group
Meeting Notes

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1 Meeting Notes from 2021-09-08

Problem 1.1 (Putnam 2018, B3)**.** Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides $2^n - 1$, and $n-2$ divides $2^n - 2$.

Solution. Let's enumerate the conditions:

(1) $n \mid 2^n$.

(2) $n-1 \mid 2^n - 1$.

(3) $n-2 \mid 2^n - 2$.

Condition (1) is equivalent to n being a power of 2. Let's write $n = 2^k$. Then, conditions (2) and (3) are equivalent to:

(2) $2^k - 1 \mid 2^{2^k} - 1$.

(3) $2^{k-1} - 1 \mid 2^{2^k - 1} - 1$.

Lemma. Let m, i be positive integers. Then,

$$m \mid i \iff 2^m - 1 \mid 2^i - 1.$$

Proof. Since $2^m \equiv 1 \pmod{2^m - 1}$, we conclude that if $i \equiv j \pmod{m}$, then $2^i - 1 \equiv 2^j - 1 \pmod{2^m - 1}$. Futhermore, the integers $2^0 - 1, 2^1 - 1, \ldots, 2^{m-1} - 1$ are distinct integers between 0 and $2^m - 2$, so they are in distict residue classes modulo $2^m - 1$. Therefore,

$$i \equiv j \pmod{m} \iff 2^i - 1 \equiv 2^j - 1 \pmod{2^m - 1},$$

and in particular, the result follows from applying $j = 0$. □

Applying the Lemma, conditions (2) and (3) are equivalent to:

(2) $k \mid 2^k$.

(3) $k-1 \mid 2^k - 1$.

These are the same conditions as (1) and (2) for n! (2) implies that $k = 2^p$, and (3) implies that

(3) $p \mid 2^p$,

thus p is a power of 2.

Now, we just need to use the "size" condition. $2^{2^p} = 2^k = n < 10^{100} < 2^{334} < 2^{2^9}$, thus $p < 9$, i.e., $p = 1, 2, 4, 8$ are the possible values of p. The possible values of n are $2^2, 2^{2^2}, 2^{2^4}, 2^{2^8}$.

Comments. Equivalently as proving the Lemma as stated, we could have argued that the order of 2 modulo $2^m - 1$ is m. In Algebra 1, one learns the definition of order for an element of any group, which is the smallest number of times you have to repeat the operation (in this case, multiplication) in a given element (in this case, 2) to get the identity. It is an important concept in Elementary Number Theory, and later in Algebra. You can find more about order in Number Theory in Orders Modulo a Prime, by Evan Chen.

Problem 1.2 (Putnam 2010, A1)**.** Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same?

Solution. We claim that the answer is $n/2$, if n is even, and $(n+1)/2$, if n is odd.

Note that there cannot be two boxes with only one number, since all the numbers are distincts. Therefore, if a proper configuration uses k boxes, $k-1$ of those boxes have at least 2 elements. Thus,

$$n \leq 1 + 2(k-1),$$

or equivalently,

$$k \leq (n+1)/2.$$

If n is even, a proper configuration with $n/2$ boxes is

$$\{\{n, 1\}, \{n-1, 2\}, \dots, \{n/2+1, n/2\}\}$$

If n is odd, a proper configuration with $(n+1)/2$ boxes is

$$\{\{n\}, \{n-1, 1\}, \{n-2, 2\}, \dots, \{(n+1)/2, (n-1)/2\}\}$$

Problem 1.3 (Putnam 2013, A1)**.** Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Solution. Suppose, by contradiction, that no two faces that share a vertex have the same number.

Suppose 5 faces have the same number. Then, there are 15 pairs of face and vertex such that the vertex is in such face and the face has such number. Therefore, by the pidgeonhole principle, since there are only 12 vertices, there exists a vertex with two faces with such number, a contradiction. Therefore, each number appears at most 4 times.

Therefore, the four smallest numbers are ≤ 0; without those, the four new smallest numbers are ≤ 1; without those, the four new smallest numbers are ≤ 2; without those, the four new smallest numbers are ≤ 3; and the four remaining numbers are ≤ 4. Thus,

$$39 = \text{total sum} \geq 4(0 + 1 + 2 + 3 + 4) = 40,$$

a contradiction.

Problem 1.4 (Putnam 2013, B1)**.** For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Solution.

$$\begin{aligned}
\sum_{n=1}^{2013} c(n)c(n+2) &= \sum_{k=1}^{1006} c(2k)c(2k+2) + \sum_{k=0}^{1006} c(2k+1)c(2k+3) \\
&= \sum_{k=1}^{1006} c(k)c(k+1) + \left(c(1)c(3) + \sum_{k=1}^{1006} (-1)^k c(k)(-1)^{k+1} c(k+1) \right) \\
&= \sum_{k=1}^{1006} c(k)c(k+1) + c(1)c(3) - \sum_{k=1}^{1006} c(k)c(k+1) \\
&= -c(1)c(3) \\
&= 1.
\end{aligned}$$

2 Meeting Notes from 2021-09-15

Problem 2.1 (Putnam 1999, A1)**.** Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x+2 & \text{if } -1 \leq x \leq 0 \\ -2x+2 & \text{if } x > 0. \end{cases}$$

Solution. The polynomials

$$\begin{aligned} f(x) &= \frac{1}{2}((3x+2) - (-1)) = \frac{3}{2}x + \frac{3}{2}, \\ g(x) &= \frac{1}{2}((-2x+2) - (3x+2)) = \frac{5}{2}x, \\ h(x) &= -x + \frac{3}{2} \end{aligned}$$

satisfy the requirement.

Problem 2.2 (Putnam 1999, B2)**.** Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

Solution. If $n \leq 2$, then it always holds that, if $P(x)$ has at least to distinct roots, then it has at least n distinct roots. Suppose that $n > 3$.

We'll equivalently prove that, if $P(x)$ has a root of multiplicity at least 2, then it has a root with multiplicity n.

In other words, suppose $(x-\alpha)^2 \mid P(x)$. We will show that $(x-\alpha)^n \mid P(x)$.

Throughout the solution, we'll use the following theorems.

Lemma. If $(x-\alpha)^k \mid P(x)$, then $(x-\alpha)^{k-1} \mid P'(x)$.

Lemma. If $(x-\alpha)$ divides $P(x)$, $P'(x)$, ..., $P^{(k-1)}(x)$, then $(x-\alpha)^k \mid P(x)$.

First, if we compare the leading coefficient in the expression

$$P(x) = Q(x)P''(x),$$

then we conclude the leading coefficient of $Q(x)$ is $\frac{1}{n(n-1)}$.

Suppose $Q(x) \neq \frac{1}{n(n-1)}(x-\alpha)^2$. Then, $(x-\alpha)^2 \mid P(x) = Q(x)P''(x) \implies (x-\alpha) \mid P''(x)$, since the two factors $x-\alpha$ cannot be both in $Q(x)$. By the first lemma, $(x-\alpha) \mid P'(x)$. By the second lemma, $(x-\alpha)^3 \mid P(x)$.

We'll prove, using induction, that $(x-\alpha)^k \mid P(x)$ for any positive integer k.

Suppose $(x-\alpha)^k \mid P(x) = Q(x)P''(x) \implies (x-\alpha)^{k-1} \mid P''(x)$, since the two factors $x-\alpha$ cannot be both in $Q(x)$. By the first lemma, $(x-\alpha)^{k-1} \mid P''(x) \implies (x-\alpha)^{k-2} \mid P^{(3)}(x) \implies \cdots \implies (x-\alpha) \mid P^{(k)}(x)$. By the second lemma, $(x-\alpha)^{k+1} \mid P(x)$; which finishes the induction.

This implies that $P(x)$ has a root with multiplicity $n+1$, which contradicts the fact that the degree of $P(x)$ is n. Therefore, $Q(x) = \frac{1}{n(n-1)}(x-\alpha)^2$.

Let's differentiate the original equation twice:

$$P'(x) = Q(x)P^{(3)}(x) + Q'(x)P''(x)$$
$$P''(x) = Q(x)P^{(4)}(x) + 2Q'(x)P^{(3)}(x) + Q''(x)P''(x)$$

Notice that $(x-\alpha)$ divides $Q(x)P^{(4)}(x) + 2Q'(x)P^{(3)}(x)$, therefore, it also must divide $P''(x)(1 - Q''(x)) = P''(x)\left(1 - \frac{1}{\binom{n}{2}}\right)$. Since $1 - \frac{1}{\binom{n}{2}} \neq 0$, we conclude $(x-\alpha)$ divides $P''(x)$.

In general, using that $Q^{(3)}(x) = 0$, we have

$$P^{(k)} = Q(x)P^{(k+2)}(x) + kQ'(x)P^{(k+1)}(x) + \binom{k}{2}Q''(x)P^{(k)}(x).$$

So we similarly conclude $(x-\alpha)$ divides $\left(1 - \frac{\binom{k}{2}}{\binom{n}{2}}\right)P^{(k)}$, and, as long as $k \neq n$, we conclude that $(x-\alpha)$ divides $P^{(k)}(x)$. Thus, by the second lemma, $(x-\alpha)^n \mid P(x)$, as desired.

Problem 2.3 (Putnam 2014, A1)**.** Prove that every nonzero coefficient of the Taylor series of

$$(1 - x + x^2)e^x$$

about $x = 0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Solution. Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

we conclude that

$$\begin{aligned}
(1 - x - x^2)e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} + \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} \\
&= 1 + \sum_{n=2}^{\infty} x^n \left(\frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} \right) \\
&= 1 + \sum_{n=2}^{\infty} x^n \frac{1 - n + n(n-1)}{n!} \\
&= 1 + \sum_{n=2}^{\infty} x^n \frac{(n-1)^2}{n!} \\
&= 1 + \sum_{n=2}^{\infty} x^n \frac{(n-1)}{(n-2)! \cdot n}
\end{aligned}$$

If $n - 1$ is prime, we're good. If $n - 1 = 4$, then $\frac{4}{3! \cdot 5} = \frac{2}{15}$, so we're good. If $n - 1 = p^2$, with $p > 2$, then $n - 1 = p^2 \mid p \cdot (2p) \mid (n-2)!$, so the numerator of the fraction is 1. Otherwise, we can find $n - 1 > a > b > 1$ such that $n - 1 = ab$, therefore $n - 1 = ab \mid (n-2)!$ so the numeratior is 1.