Problem Solving Group
Meeting Notes

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1 Meeting Notes from 2021-09-08

Problem 1.1 (Putnam 2018, B3)

Find all positive integers $n < 10^{100}$ for which simultaneously n divides 2^n, $n-1$ divides 2^n-1, and $n-2$ divides 2^n-2.

Solution A: Let's enumerate the conditions:

(1) $n \mid 2^n$.

(2) $n-1 \mid 2^n-1$.

(3) $n-2 \mid 2^n-2$.

Condition (1) is equivalent to n being a power of 2. Let's write $n = 2^k$. Then, conditions (2) and (3) are equivalent to:

(2) $2^k - 1 \mid 2^{2^k} - 1$.

(3) $2^{k-1} - 1 \mid 2^{2^k-1} - 1$.

Lemma 1

Let m, i be positive integers. Then,

$$m \mid i \iff 2^m - 1 \mid 2^i - 1.$$

Proof. Since $2^m \equiv 1 \pmod{2^m-1}$, we conclude that if $i \equiv j \pmod{m}$, then $2^i - 1 \equiv 2^j - 1 \pmod{2^m-1}$. Futhermore, the integers $2^0-1, 2^1-1, \ldots, 2^{m-1}-1$ are distinct integers between 0 and 2^m-2, so they are in distict residue classes modulo 2^m-1. Therefore,

$$i \equiv j \pmod{m} \iff 2^i - 1 \equiv 2^j - 1 \pmod{2^m - 1},$$

and in particular, the result follows from applying $j = 0$. $\square$

Applying the Lemma, conditions (2) and (3) are equivalent to:

(2) $k \mid 2^k$.

(3) $k-1 \mid 2^k - 1$.

These are the same conditions as (1) and (2) for n! (2) implies that $k = 2^p$, and (3) implies that

(3) $p \mid 2^p$,

thus p is a power of 2.

Now, we just need to use the "size" condition. $2^{2^p} = 2^k = n < 10^{100} < 2^{334} < 2^{2^9}$, thus $p < 9$, i.e., $p = 1, 2, 4, 8$ are the possible values of p. The possible values of n are $2^2, 2^{2^2}, 2^{2^4}, 2^{2^8}$.

Equivalently as proving the Lemma as stated, we could have argued that the order of 2 modulo $2^m - 1$ is m. In Algebra 1, one learns the definition of order for an element of any group, which is the smallest number of times you have to repeat the operation (in this case, multiplication) in a given element (in this case, 2) to get the identity. It is an important concept in Elementary Number Theory, and later in Algebra. You can find more about order in Number Theory in Orders Modulo a Prime, by Evan Chen.

Problem 1.2 (Putnam 2010, A1)

Given a positive integer n, what is the largest k such that the numbers $1, 2, \dots, n$ can be put into k boxes so that the sum of the numbers in each box is the same?

Solution A: We claim that the answer is $n/2$, if n is even, and $(n+1)/2$, if n is odd.

Note that there cannot be two boxes with only one number, since all the numbers are distincts. Therefore, if a proper configuration uses k boxes, $k-1$ of those boxes have at least 2 elements. Thus,

$$n \leq 1 + 2(k-1),$$

or equivalently,

$$k \leq (n+1)/2.$$

If n is even, a proper configuration with $n/2$ boxes is

$$\{\{n, 1\}, \{n-1, 2\}, \dots, \{n/2+1, n/2\}\}$$

If n is odd, a proper configuration with $(n+1)/2$ boxes is

$$\{\{n\}, \{n-1, 1\}, \{n-2, 2\}, \dots, \{(n+1)/2, (n-1)/2\}\}$$

Problem 1.3 (Putnam 2013, A1)

Recall that a regular icosahedron is a convex polyhedron having 12 vertices and 20 faces; the faces are congruent equilateral triangles. On each face of a regular icosahedron is written a nonnegative integer such that the sum of all 20 integers is 39. Show that there are two faces that share a vertex and have the same integer written on them.

Solution A: Suppose, by contradiction, that no two faces that share a vertex have the same number.

Suppose 5 faces have the same number. Then, there are 15 pairs of face and vertex such that the vertex is in such face and the face has such number. Therefore, by the pidgeonhole principle, since there are only 12 vertices, there exists a vertex with two faces with such number, a contradiction. Therefore, each number appears at most 4 times.

Therefore, the four smallest numbers are ≤ 0; without those, the four new smallest numbers are ≤ 1; without those, the four new smallest numbers are ≤ 2; without those, the four new smallest numbers are ≤ 3; and the four remaining numbers are ≤ 4. Thus,

$$39 = \text{total sum} \geq 4(0+1+2+3+4) = 40,$$

a contradiction.

Problem 1.4 (Putnam 2013, B1)

For positive integers n, let the numbers $c(n)$ be determined by the rules $c(1) = 1, c(2n) = c(n)$, and $c(2n+1) = (-1)^n c(n)$. Find the value of

$$\sum_{n=1}^{2013} c(n)c(n+2).$$

Solution A:

$$\begin{aligned}
\sum_{n=1}^{2013} c(n)c(n+2) &= \sum_{k=1}^{1006} c(2k)c(2k+2) + \sum_{k=0}^{1006} c(2k+1)c(2k+3) \\
&= \sum_{k=1}^{1006} c(k)c(k+1) + \left(c(1)c(3) + \sum_{k=1}^{1006} (-1)^k c(k)(-1)^{k+1} c(k+1) \right) \\
&= \sum_{k=1}^{1006} c(k)c(k+1) + c(1)c(3) - \sum_{k=1}^{1006} c(k)c(k+1) \\
&= -c(1)c(3) \\
&= 1.
\end{aligned}$$

2 Meeting Notes from 2021-09-15

Problem 2.1 (Putnam 1999, A1)

Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x+2 & \text{if } -1 \le x \le 0 \\ -2x+2 & \text{if } x > 0. \end{cases}$$

Solution A: The polynomials

$$\begin{aligned} f(x) &= \frac{1}{2}((3x+2) - (-1)) = \frac{3}{2}x + \frac{3}{2}, \\ g(x) &= \frac{1}{2}((-2x+2) - (3x+2)) = \frac{5}{2}x, \\ h(x) &= -x + \frac{3}{2} \end{aligned}$$

satisfy the requirement.

Problem 2.2 (Putnam 1999, B2)

Let $P(x)$ be a polynomial of degree n such that $P(x) = Q(x)P''(x)$, where $Q(x)$ is a quadratic polynomial and $P''(x)$ is the second derivative of $P(x)$. Show that if $P(x)$ has at least two distinct roots then it must have n distinct roots.

Solution A: If $n \le 2$, then it always holds that, if $P(x)$ has at least to distinct roots, then it has at least n distinct roots. Suppose that $n > 3$.

We'll equivalently prove that, if $P(x)$ has a root of multiplicity at least 2, then it has a root with multiplicity n.

In other words, suppose $(x-\alpha)^2 \mid P(x)$. We will show that $(x-\alpha)^n \mid P(x)$.

Throughout the solution, we'll use the following theorems.

Lemma 1

If $(x-\alpha)^k \mid P(x)$, then $(x-\alpha)^{k-1} \mid P'(x)$.

Lemma 2

If $(x-\alpha)$ divides $P(x)$, $P'(x)$, ..., $P^{(k-1)}(x)$, then $(x-\alpha)^k \mid P(x)$.

First, if we compare the leading coefficient in the expression

$$P(x) = Q(x)P''(x),$$

then we conclude the leading coefficient of $Q(x)$ is $\frac{1}{n(n-1)}$.

Suppose $Q(x) \neq \frac{1}{n(n-1)}(x-\alpha)^2$. Then, $(x-\alpha)^2 \mid P(x) = Q(x)P''(x) \implies (x-\alpha) \mid P''(x)$, since the two factors $x-\alpha$ cannot be both in $Q(x)$. By the first lemma, $(x-\alpha) \mid P'(x)$. By the second lemma, $(x-\alpha)^3 \mid P(x)$.

We'll prove, using induction, that $(x-\alpha)^k \mid P(x)$ for any positive integer k.

Suppose $(x-\alpha)^k \mid P(x) = Q(x)P''(x) \implies (x-\alpha)^{k-1} \mid P''(x)$, since the two factors $x-\alpha$ cannot be both in $Q(x)$. By the first lemma, $(x-\alpha)^{k-1} \mid P''(x) \implies (x-\alpha)^{k-2} \mid P^{(3)}(x) \implies \cdots \implies (x-\alpha) \mid P^{(k)}(x)$. By the second lemma, $(x-\alpha)^{k+1} \mid P(x)$; which finishes the induction.

This implies that $P(x)$ has a root with multiplicity $n+1$, which contradicts the fact that the degree of $P(x)$ is n. Therefore, $Q(x) = \frac{1}{n(n-1)}(x-\alpha)^2$.

Let's differentiate the original equation twice:

$$P'(x) = Q(x)P^{(3)}(x) + Q'(x)P''(x)$$
$$P''(x) = Q(x)P^{(4)}(x) + 2Q'(x)P^{(3)}(x) + Q''(x)P''(x)$$

Notice that $(x-\alpha)$ divides $Q(x)P^{(4)}(x) + 2Q'(x)P^{(3)}(x)$, therefore, it also must divide $P''(x)(1-Q''(x)) = P''(x)\left(1 - \frac{1}{\binom{n}{2}}\right)$. Since $1 - \frac{1}{\binom{n}{2}} \neq 0$, we conclude $(x-\alpha)$ divides $P''(x)$.

In general, using that $Q^{(3)}(x) = 0$, we have

$$P^{(k)} = Q(x)P^{(k+2)}(x) + kQ'(x)P^{(k+1)}(x) + \binom{k}{2}Q''(x)P^{(k)}(x).$$

So we similarly conclude $(x-\alpha)$ divides $\left(1 - \frac{\binom{k}{2}}{\binom{n}{2}}\right)P^{(k)}$, and, as long as $k \neq n$, we conclude that $(x-\alpha)$ divides $P^{(k)}(x)$. Thus, by the second lemma, $(x-\alpha)^n \mid P(x)$, as desired.

Problem 2.3 (Putnam 2014, A1)

Prove that every nonzero coefficient of the Taylor series of

$$(1-x+x^2)e^x$$

about $x=0$ is a rational number whose numerator (in lowest terms) is either 1 or a prime number.

Solution A: Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

we conclude that

$$\begin{aligned}
(1 - x - x^2)e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!} \\
&= \sum_{n=0}^{\infty} \frac{x^n}{n!} - \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} + \sum_{n=2}^{\infty} \frac{x^n}{(n-2)!} \\
&= 1 + \sum_{n=2}^{\infty} x^n \left(\frac{1}{n!} - \frac{1}{(n-1)!} + \frac{1}{(n-2)!} \right) \\
&= 1 + \sum_{n=2}^{\infty} x^n \frac{1 - n + n(n-1)}{n!} \\
&= 1 + \sum_{n=2}^{\infty} x^n \frac{(n-1)^2}{n!} \\
&= 1 + \sum_{n=2}^{\infty} x^n \frac{(n-1)}{(n-2)! \cdot n}
\end{aligned}$$

If $n-1$ is prime, we're good. If $n-1=4$, then $\frac{4}{3! \cdot 5} = \frac{2}{15}$, so we're good. If $n-1=p^2$, with $p>2$, then $n-1=p^2 \mid p \cdot (2p) \mid (n-2)!$, so the numerator of the fraction is 1. Otherwise, we can find $n-1 > a > b > 1$ such that $n-1=ab$, therefore $n-1=ab \mid (n-2)!$ so the numeratior is 1.

3 Meeting Notes from 2021-09-22

Problem 3.1

Can three points with integer coornidates in the plane be vertices of an equilateral triangle? What about in three dimentions?

Solution A: Let ABC be an equilateral triangle in the plane so that $A = (a_x, a_y) \in \mathbb{Z}^2$, $B = (b_x, b_y) \in \mathbb{Z}^2$, and $C = (c_x, c_y)$, with A, B, and C being distinct. Without loss of generality, suppose A, B, C are in counterclockwise order. Then,

$$\begin{aligned}
(C - A) &= \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{pmatrix} (B - A) \\
&= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}2 & 1/2 \end{pmatrix} \begin{pmatrix} b_x - a_x \\ b_y - a_y \end{pmatrix} \\
&= \begin{pmatrix} 1/2(b_x - a_x) - \sqrt{3}/2(b_y - a_y) \\ \sqrt{3}/2(b_x - a_x) + 1/2(b_y - a_y) \end{pmatrix}.
\end{aligned}$$

Finally, since $A \neq B$, $b_x - a_x \neq 0$ or $b_y - b_x \neq 0$, which implies $C - A \notin \mathbb{Q}^2 \implies C \notin \mathbb{Q}^2 \implies C \notin \mathbb{Z}^2$. Therefore, there is no equilateral triangle in the plane with vertices with integer coordinates.

In three dimensions, the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ form an equilateral triangle.

Solution B (using areas)**:** Let's use the same notation as above, and suppose $A, B, C \in \mathbb{Z}^2$. On one hand, using the same notation as above, the area of the triangle ABC is the absolute value of the determinant of

$$\frac{1}{2} \begin{pmatrix} a_x & a_y & 1 \\ b_x & b_y & 1 \\ c_x & c_y & 1 \end{pmatrix},$$

which is a rational number. On the other hand, the area of an equilateral triangle is

$$\frac{\ell^2 \sqrt{3}}{2},$$

where ℓ is the length of the side. Pythagoras' theorem implies that $\ell^2 = |AB|^2$ is an integer, so the area of ABC is an irrational number; a contradiction. Thus, no such triangle exists.

4 Meeting Notes from 2021-09-29

Problem 4.1 (IMO 1975, 4)

When 4444^{4444} is written in decimal notation, the sum of its digits is A. Let B be the sum of the digits of A. Find the sum of the digits of B.

Solution A: Let C be the sum of the digits of B.

First, we will investigate the size of the numbers. Since $0 < 4444^{4444} < 10000^{4444} = 10^{4 \cdot 4444}$, we conclude $0 < A \leq 9 \cdot 4 \cdot 4444$. Since $0 < A < 10^6$, $0 < B \leq 9 \cdot 6 = 36$. Finally, this implies $0 < C \leq 2 + 9 = 11$.

If we write any number n in its decimal representation, i.e., $n = a_0 + a_1 10 + a_2 10^2 + \cdots + a_k 10^k$, then we conlude

$$n = a_0 + a_1 10 + a_2 10^2 + \cdots + a_k 10^k \equiv a_0 + a_1 + a_2 + \cdots + a_k \pmod 9.$$

Therefore, $7 \equiv 4444^{4444} \equiv A \equiv B \equiv C \pmod 9$. Since $0 < C \leq 11$, C must be 7.

Problem 4.2 (Putnam 2003, B3)

Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple, and $\lfloor x \rfloor$ denotes the greatest integer $\leq x$.)

Sketch A (UNFINISHED SOLUTION)**:** Let $\nu_p(n)$ be the largest α so that $p^\alpha \mid n$.[a] In order to show that LHS equals RHS, it suffices to show that $\nu_p(\text{LHS}) = \nu_p(\text{RHS})$ for all primes p.

We know that[b]

$$\begin{aligned} \nu_p\left(n!\right) &= \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor \\ &= \sum_{i=1}^{n} \left\lfloor \frac{n}{p^i} \right\rfloor. \end{aligned}$$

We also know that

$$\nu_p\left(\prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}\right) = \sum_{i=1}^{n} \nu_p\left(\operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}\right).$$

Finally, recall that $\nu_p(\operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\})$ is the largest number α so that

$$p^\alpha \mid \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\},$$

which means that, if we set $\alpha = \nu_p(\operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\})$, it holds that

$$p^\alpha \mid \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}$$

and

$$p^{\alpha+1} \nmid \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

By definition of least common multiple, we conlcude $p^\alpha \mid k$ for some $1 \le k \le \lfloor n/i \rfloor$ and $p^{\alpha+1} \nmid k$ for all $1 \le k \le \lfloor n/i \rfloor$.

By definition of floor, the statement above is equivalent to $p^\alpha \mid k$ for some $1 \le k \le n/i$ and $p^{\alpha+1} \nmid k$ for all $1 \le k \le n/i$.

Again, the statement above implies that $1 \le p^\alpha \le n/i$, while $n/i < p^{\alpha+1}$, i.e., α is the unique number satisfying

$$\frac{n}{p} < ip^\alpha \le n$$

[a]See p-adic order on Wikipedia.

[b]This is known as Legendre's formula.

5 Meeting Notes from 2021-10-06

Problem 5.1

Prove that

$$\sin\left(\frac{\pi}{11}\right)\sin\left(\frac{2\pi}{11}\right)\cdots\sin\left(\frac{10\pi}{11}\right)=\frac{11}{2^{10}},$$

or more generally, prove that

$$\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\cdots\sin\left(\frac{(n-1)\pi}{n}\right)=\frac{n}{2^{n-1}}.$$

Solution A: Recall the formula

$$\sin(\theta)=\frac{1}{2i}(e^{i\theta}-e^{-i\theta}).$$

Therefore,

$$\begin{aligned}\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)&=\prod_{k=1}^{n-1}\left(\frac{1}{2i}\left(e^{ki\pi/n}-e^{-ki\pi/n}\right)\right)\\&=\frac{1}{(2i)^{n-1}}\left(\prod_{k=1}^{n-1}e^{ki\pi/n}\right)\left(\prod_{k=1}^{n-1}(1-e^{-2ki\pi/n})\right).\end{aligned}$$

The first product evaluates to $\prod_{k=1}^{n-1}e^{ki\pi/n}=e^{\sum_{k=1}^{n-1}ki\pi/n}=e^{(n-1)i\pi/2}=i^{n-1}$.

To calculate the second product, notice each of the $n-1$ distinct terms $e^{-2ki\pi/n}$ is a root of the polynomial $1+z+z^2+\cdots+z^{n-1}=\frac{z^n-1}{z-1}$. Therefore, we have that

$$\prod_{n=1}^{n-1}(z-e^{-2ki\pi/n})=1+z+z^2+\cdots+z^{n-1}.$$

By plugging $z\to 1$ above, we have that $\prod_{n=1}^{n-1}(1-e^{-2ki\pi/n})=n$.

Finally, we conclude that

$$\begin{aligned}\prod_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)&=\frac{1}{(2i)^{n-1}}\left(\prod_{k=1}^{n-1}e^{ki\pi/n}\right)\left(\prod_{k=1}^{n-1}(1-e^{-2ki\pi/n})\right)\\&=\frac{1}{(2i)^{n-1}}i^{n-1}n\\&=\frac{n}{2^{n-1}}.\end{aligned}$$

Problem 5.2 (Putnam 2004, A3)

Define a sequence $(u_n)_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$ and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \geq 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

Sketch A: One can prove, using induction, that

$$u_{2k} = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2k-1) = \frac{2k!}{2^k k!}$$

and

$$u_{2k+1} = 2 \cdot 4 \cdot 6 \cdot \dots \cdot 2k = 2^k k!$$

which solves the problem.

Problem 5.3 (Putnam 2005, A1)

Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, $23 = 9+8+6$.)

Solution A: If $N = 0$, then we can write 0 as the empty sum.

Supoose $N \geq 1$, and for all $0 \leq n < N$, n can be written as asked.

If N is even, then $0 \leq \frac{N}{2} < N$, so we can write $\frac{N}{2}$ as

$$\frac{N}{2} = 2^{s_1} 3^{r_1} + \dots + 2^{s_k} r_k$$

where no summand divides another, and consequently,

$$N = 2^{s_1+1} s^{r_1} + \dots + 2^{s_k+1} 3^{r_k}$$

where no summand divides another.

If N is odd, let α be the largest integer so that $3^\alpha \leq N$. Therefore, $N < 3^{\alpha+1}$, and consequently, $0 \leq \frac{N-3^\alpha}{2} < 3^\alpha \leq N$. Thus, we can write $\frac{N-3^\alpha}{2}$ as

$$\frac{N - 3^\alpha}{2} = 2^{s_1} 3^{r_1} + \dots + 2^{s_k} 3^{r_k}$$

where no summand divides another. Since the number above is smaller than 3^α, all r_i are smaller than α. Consequently, we can write

$$N = 3^\alpha + 2^{s_1+1} 3^{r_1} + \dots 2^{s_k+1} r_k$$

where no summand divides another (3^α does not divide $2^{s_i+1} 3^{r_i}$ since $r_i < \alpha$; and $2^{s_i+1} 3^{r_i}$ does not divide 3^α since 2 does not divide 3^α).

Therefore, by induction all numbers can be written in such form.

6 Meeting Notes from 2021-10-20

Problem 6.1

Can you show how to express any positive fraction as a sum of distinct positive reciprocal whole numbers? For example, $7/3 = 1/1 + 1/2 + 1/3 + 1/4 + 1/5 + 1/20$.

Solution A: Let's divide the solution into two lemmas.

Lemma

If $\frac{p}{q} < 1$, then there exists a finite subset $I \subset \mathbb{Z}_{>0}$ so that

$$\frac{p}{q} = \sum_{i \in I} \frac{1}{i}.$$

Proof. Induction on p. If $p = 0$, set $I = \varnothing$. If $p = 1$, set $I = \{q\}$.

Supoose that $p > 1$, and that, for all rational numbers $\frac{p'}{q'}$ with $p' < p$, the statement is true.

Let n be the smallest integer so that

$$np - q \geq 0.$$

By minimality of n, $(n-1)p - q < 0$, thus $np - q < p$.

Therefore, by the induction hypothesis, there exists a finite subset $I' \subset \mathbb{Z}_{>0}$ so that

$$\frac{np - q}{qn} = \sum_{i \in I'} \frac{1}{i}.$$

Notice that, since $np - q < p < q$, $\frac{np-q}{qn} < \frac{1}{n}$, so $n \notin I'$. Therefore, define $I = I' \cup \{n\}$, and we have

$$\begin{aligned} \frac{p}{q} &= \frac{pn - q}{qn} + \frac{1}{n} \\ &= \sum_{i \in I'} \frac{1}{i} + \frac{1}{n} \\ &= \sum_{i \in I} \frac{1}{i}, \end{aligned}$$

as desired. $\square$

Lemma

Any fraction $\frac{p}{q}$ can be written as a sum of distinct positive reciprocal of integers.

Proof. Let k be the largest integer so that

$$x \leq \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k}.$$

(Since the harmonic series tends to infinity, k is well-defined.)

Let $x' = x - \frac{1}{1} - \frac{1}{2} - \cdots - \frac{1}{k}$. By maximality of k, $x' < \frac{1}{k+1}$.

From the previous lemma, since $x' < \frac{1}{k+1} < 1$, we know that there exists a finite set $I' \subset \mathbb{Z}_{>0}$ so that

$$x' = \sum_{i \in I'} \frac{1}{i}.$$

Since $x' < \frac{1}{k+1}$, we conclude that $1, 2, \ldots, k$ are not elements of I'. Define $I = \{1, 2, \ldots, k\} \cup I'$. Then, we have

$$\begin{aligned} x &= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{k} + x' \\ &= \sum_{i \in \{1,2,\ldots,k\}} \frac{1}{i} + \sum_{i \in I'} \frac{1}{i} \\ &= \sum_{i \in I} \frac{1}{i}, \end{aligned}$$

as desired. □

7 Meeting Notes from 2021-10-27

Problem 7.1 (Putnam 2015, A2)

Given a list of the positive integers $1, 2, 3, 4, \ldots,$ take the first three numbers $1, 2, 3$ and their sum 6 and cross all four numbers off the list. Repeat with the three smallest remaining numbers $4, 5, 7$ and their sum 16. Continue in this way, crossing off the three smallest remaining numbers and their sum and consider the sequence of sums produced: $6, 16, 27, 36, \ldots$. Prove or disprove that there is some number in this sequence whose base 10 representation ends with 2015.

Sketch A: The full solution is to be done. The answer is yes. You can prove, using induction, that the $(k+1)$-th term of the sequence will be $10k + s_k$, with $s_k \in \{5, 6, 7\}$. You will be able to recursevely evaluate in terms of s_q, you write $k = 3q + r$, $0 \leq r < 3$.

With that, you will be able to find k such that $k \equiv 201 \pmod 1)000$ and $s_k = 5$.

8 Meeting Notes from 2021-11-03

Problem 8.1

Find the 2000^{th} digit in the square root of $N = 11\ldots1$, where N contains 1998 digits, all of them 1's.

Solution A: The answer is $\boxed{6.}$ We can estimate

$$\sqrt{\frac{10^{1998}-1}{9}} \approx \sqrt{\frac{10^{1998}}{9}} = \frac{10^{999}}{3}.$$

How good is this estimate?

$$\begin{aligned}\left(\frac{10^{999}}{3} - \sqrt{\frac{10^{1998}-1}{9}}\right)\left(\frac{10^{999}}{3} + \sqrt{\frac{10^{1998}-1}{9}}\right) &= \frac{10^{1998}}{9} - \frac{10^{1998}-1}{9} \\ &= \frac{1}{9},\end{aligned}$$

thus, by using the estimative $\sqrt{\frac{10^{1998}-1}{9}} \approx \frac{10^{999}}{3}$,

$$\frac{10^{999}}{3} - \sqrt{\frac{10^{1998}-1}{9}} \approx \frac{1}{6 \cdot 10^{999}},$$

which sadly implies that our estimative is not good enough to evaluate the 2000^{th}. However, now we have a new estimative:

$$\sqrt{\frac{10^{1998}-1}{9}} \approx \frac{10^{999}}{3} - \frac{1}{6 \cdot 10^{999}}.$$

Again, how good is this estimative?

$$\begin{aligned}\left(\sqrt{\tfrac{10^{1998}-1}{9}} - \left(\tfrac{10^{999}}{3} - \tfrac{1}{6\cdot 10^{999}}\right)\right)\left(\sqrt{\tfrac{10^{1998}-1}{9}} + \left(\tfrac{10^{999}}{3} - \tfrac{1}{6\cdot 10^{999}}\right)\right) &= \tfrac{10^{1998}-1}{9} - \left(\tfrac{10^{999}}{3} - \tfrac{1}{6\cdot 10^{999}}\right)^2 \\ &= \tfrac{1}{36\cdot 10^{1998}},\end{aligned}$$

so we know that

$$\sqrt{\tfrac{10^{1998}-1}{9}} - \left(\tfrac{10^{999}}{3} - \tfrac{1}{6\cdot 10^{999}}\right) < \frac{1}{10^{1010}},$$

which is good enough to imply that the 2000^{th} digit of $\sqrt{\frac{10^{1998}-1}{9}}$ is equal to the 2000^{th} digit of $\frac{10^{999}}{3} - \frac{1}{6\cdot 10^{999}}$, which is 6.

Problem 8.2 (Putnam 2014, A3)

Let $a_0 = 5/2$ and $a_k = a_{k-1}^2 - 2$ for $k \geq 1$. Compute

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$

in closed form.

Sketch A: The full solution is to be done. The answer is $\boxed{3/7.}$ Prove by induction that

$$a_n = 2^{2^n} + 2^{-2^n}.$$

Then, the product we want to evaluate is

$$\prod_{k=0}^{\infty} \frac{(1/4)^{2^k} - (1/2)^{2^k} + 1}{(1/4)^{2^k} + 1} = \prod_{k=0}^{\infty} \frac{(1/8)^{2^k} + 1}{((1/4)^{2^k} + 1)((1/2)^{2^k} + 1)}.$$

It suffices to show that

$$\prod_{k=0}^{\infty} x^{2^k} + 1 = \frac{1}{1-x},$$

and plug the results for $x = 1/2$, $x = 1/4$, and $x = 1/8$.

Finally, the answer is

$$\frac{8/7}{(2/1)(4/3)} = \frac{3}{7}.$$

9 Meeting Notes from 2021-11-10

Problem 9.1 (Putnam 2014, B2)

Suppose that f is a function on the interval $[1, 3]$ such that $-1 \le f(x) \le 1$ for all x and $\int_1^3 f(x)\,dx = 0$. How large can $\int_1^3 \frac{f(x)}{x}\,dx$ be?

Solution A: The answer is $\boxed{\log \frac{4}{3}.}$

Let $f^\star \colon [1, 3] \to [-1, 1]$ be defined as follows

$$f^\star(x) = \begin{cases} 1 & 1 \le x \le 2 \\ -1 & 2 < x \le 3. \end{cases}$$

Let $f \colon [1, 3] \to [-1, 1]$ be any function such that $\int_1^3 f(x)\,dx = 0$. Then,

$$\begin{aligned} \int_1^3 \frac{f(x)}{x}\,dx - \int_1^3 \frac{f^\star(x)}{x}\,dx &= \int_1^3 \frac{f(x) - f^\star(x)}{x}\,dx \\ &= \int_1^2 \frac{f(x) - 1}{x}\,dx + \int_2^3 \frac{f(x) + 1}{x}\,dx \\ &\le \int_1^2 \frac{f(x) - 1}{2}\,dx + \int_2^3 \frac{f(x) + 1}{2}\,dx \\ &= \int_1^3 \frac{f(x)}{2}\,dx = 0. \end{aligned}$$

In other words,

$$\int_1^3 \frac{f(x)}{x}\,dx \le \int_1^3 \frac{f^\star(x)}{x}\,dx.$$

Thus, the answer to the problem is

$$\int_1^3 \frac{f^\star(x)}{x}\,dx = \int_1^2 \frac{1}{x} - \int_2^3 \frac{1}{x} = \log \frac{4}{3}.$$

Problem 9.2 (IMO 1995, 6)

Let p be an odd prime number. Determine how many p-element subsets A of $\{1, 2, \ldots, 2p\}$ are such that the sum of elements of A is divisible by p.

Solution A: The answer is

$$\frac{\binom{2p}{p}-2}{p}+2.$$

Let $\mathcal{S}$ denote the collection of all subsets of $\{1,\ldots,2p\}$ with p elements. Let's interpret this sets of this collection by assigning 0's and 1's in a $2\times p$ matrix. For example, if $p=5$ and $S=\{1,2,4,6,10\}\in\mathcal{S}$, we will visualize S as the matrix

$$\begin{pmatrix}1&1&0&1&0\\1&0&0&0&1\end{pmatrix}.$$

Now, for each matrix with the first row consisting of not all zeros or not all ones, we will group it with the cyclical permutations of the first row.

For example, the matrix above will form the following group of matrices:

$$\begin{pmatrix}1&1&0&1&0\\1&0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&1&0&1\\1&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&1&0\\1&0&0&0&1\end{pmatrix},\begin{pmatrix}0&1&0&1&1\\1&0&0&0&1\end{pmatrix},\begin{pmatrix}1&0&1&0&1\\1&0&0&0&1\end{pmatrix}.$$

Note that, in this example, the sums of the corresponding sets are, in modulo 5, respectively,

$$3,1,4,2,0.$$

Unless the first row is entirely made up of zeros or entirely made up of ones, since p is prime, we have that this will generate a group with p matrices. Furthermore, when going from one to another, the sum of the corresponding sets increases by the amount of ones in the first line. This implies that, in each group, there will be exactly one set with the desired property.

Accounting for the matrices

$$\begin{pmatrix}1&1&\ldots&1\\0&0&\ldots&0\end{pmatrix},\text{ and }\begin{pmatrix}0&0&\ldots&0\\1&1&\ldots&1\end{pmatrix},$$

that correspond to the sets $\{1,\ldots,p\}$ and $\{p+1,\ldots,2p\}$, and have the desired property; we imply that the final answer is

$$\frac{\binom{2p}{p}-2}{p}+2.$$

10 Meeting Notes from 2021-11-27

11 Meeting Notes from 2021-11-23

Problem 11.1 (Putnam 2011, A3)

Find a real number c and a positive number L for which

$$\lim_{r\to\infty} \frac{r^c \int_0^{\pi/2} x^r \sin x\, dx}{\int_0^{\pi/2} x^r \cos x\, dx} = L.$$

Solution A: The answer is $c = -1$ and $L = 2/\pi$. Taylor series implies that

$$\begin{aligned}
\sin x &= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} x^{2i+1} \\
\cos x &= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i)!} x^{2i}
\end{aligned}$$

Then, we have

$$\begin{aligned}
&\lim_{r\to\infty} r(\pi/2)^{-r-1} \int_0^{\pi/2} x^r \sin x\, dx \\
&= \lim_{r\to\infty} \int_0^{\pi/2} \sum_{i=0}^{\infty} r(\pi/2)^{-r-1} \frac{(-1)^i}{(2i+1)!} x^{2i+1+r}\, dx \\
&= \lim_{r\to\infty} \sum_{i=0}^{\infty} \int_0^{\pi/2} r(\pi/2)^{-r-1} \frac{(-1)^i}{(2i+1)!} x^{2i+1+r}\, dx \\
&= \lim_{r\to\infty} \sum_{i=0}^{\infty} r(\pi/2)^{-r-1} \frac{(-1)^i}{(2i+2+r)(2i+1)!} (\pi/2)^{2i+2+r} \\
&= \lim_{r\to\infty} \sum_{i=0}^{\infty} \frac{r}{2i+2+r} \frac{(-1)^i}{(2i+1)!} (\pi/2)^{2i+1} \\
&= \sum_{i=0}^{\infty} \lim_{r\to\infty} \frac{r}{2i+2+r} \frac{(-1)^i}{(2i+1)!} (\pi/2)^{2i+1} \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} (\pi/2)^{2i+1} \\
&= \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)!} (\pi/2)^{2i+1} \\
&= \sin(\pi/2) = 1
\end{aligned}$$

And we also have

$$
\begin{aligned}
&\lim_{r\to\infty} r^2(\pi/2)^{-r-1}\int_0^{\pi/2} x^r \cos x\,dx \\
=&\lim_{r\to\infty}\int_0^{\pi/2}\sum_{i=0}^{\infty} r^2(\pi/2)^{-r-1}\frac{(-1)^i}{(2i)!}x^{2i}\,dx \\
=&\lim_{r\to\infty}\sum_{i=0}^{\infty}\int_0^{\pi/2} r^2(\pi/2)^{-r-1}\frac{(-1)^i}{(2i)!}x^{2i}\,dx \\
=&\lim_{r\to\infty}\sum_{i=0}^{\infty}\frac{r^2}{2i+1+r}\frac{(-1)^i}{(2i)!}(\pi/2)^{2i} \\
=&\lim_{r\to\infty}\left(\frac{r^2}{r+1}+\sum_{i=1}^{\infty}\frac{r^2}{2i+1+r}\frac{(-1)^i}{(2i)!}(\pi/2)^{2i}\right) \\
=&\lim_{r\to\infty}\left(\frac{r^2}{r+1}+\sum_{i=1}^{\infty}\frac{r^2}{(2i+1+r)(2i)}\frac{(-1)^i}{(2i-1)!}(\pi/2)^{2i}\right) \\
=&\lim_{r\to\infty}\left(\frac{r^2}{r+1}+\sum_{i=1}^{\infty}\frac{r}{r+1}\left(\frac{r}{2i}-\frac{r}{2i+1+r}\right)\frac{(-1)^i}{(2i-1)!}(\pi/2)^{2i}\right) \\
=&\lim_{r\to\infty}\left(\left(\sum_{i=0}^{\infty}\frac{r^2}{r+1}\frac{(-1)^i}{(2i)!}(\pi/2)^{2i}\right)-\left(\sum_{i=1}^{\infty}\frac{r^2}{(r+1)(2i+1+r)}\frac{(-1)^i}{(2i-1)!}(\pi/2)^{2i}\right)\right) \\
=&\lim_{r\to\infty}\left(\frac{r^2}{r+1}\left(\sum_{i=0}^{\infty}\frac{(-1)^i}{(2i)!}(\pi/2)^{2i}\right)-\left(\sum_{i=1}^{\infty}\frac{r^2}{(r+1)(2i+1+r)}\frac{(-1)^i}{(2i-1)!}(\pi/2)^{2i}\right)\right) \\
=&\lim_{r\to\infty}\left(\frac{r^2}{r+1}\cos(\pi/2)-\left(\sum_{i=1}^{\infty}\frac{r^2}{(r+1)(2i+1+r)}\frac{(-1)^i}{(2i-1)!}(\pi/2)^{2i}\right)\right) \\
=&\lim_{r\to\infty}\sum_{i=1}^{\infty}\frac{r^2}{(r+1)(2i+1+r)}\frac{(-1)^{i+1}}{(2i-1)!}(\pi/2)^{2i} \\
=&\sum_{i=1}^{\infty}\lim_{r\to\infty}\frac{r^2}{(r+1)(2i+1+r)}\frac{(-1)^{i+1}}{(2i-1)!}(\pi/2)^{2i} \\
=&\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{(2i-1)!}(\pi/2)^{2i} \\
=&\sum_{i=0}^{\infty}\frac{(-1)^{i}}{(2i+1)!}(\pi/2)^{2i+2} \\
=&(\pi/2)\sum_{i=0}^{\infty}\frac{(-1)^{i}}{(2i+1)!}(\pi/2)^{2i+1} \\
=&(\pi/2)\sin(\pi/2)=\pi/2.
\end{aligned}
$$

Finally, this implies that

$$\begin{aligned}\lim_{r\to\infty} \frac{r^{-1}\int_0^{\pi/2} x^r \sin x\,dx}{\int_0^{\pi/2} x^r \cos x\,dx} &= \lim_{r\to\infty} \frac{r(\pi/2)^{-r-1}\int_0^{\pi/2} x^r \sin x\,dx}{r^2(\pi/2)^{-r-1}\int_0^{\pi/2} x^r \cos x\,dx} \\ &= \frac{\lim_{r\to\infty} r(\pi/2)^{-r-1}\int_0^{\pi/2} x^r \sin x\,dx}{\lim_{r\to\infty} r^2(\pi/2)^{-r-1}\int_0^{\pi/2} x^r \cos x\,dx} = \frac{2}{\pi}\end{aligned}$$

12 Meeting Notes from 2021-12-01

Problem 12.1 (Putnam 2010, A2)

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n.

Solution A: I claim that the only functions f that satisfy the requirements are $f(x) = ax + b$, with real parameters a, b. It is easy to check that those functions indeed satisfy the requirements; both RHS and LHS are always a.

Now, suppose f is a function that satisfies the requirements. Let $g(x) = f(x) - f(0) - xf'(0)$. Then, $g'(x) = f'(x) - f'(0)$.

Lemma 1

$$g'(x) = \frac{g(x+n) - g(x)}{n}$$

for all real numbers x and all positive integers n.

Proof. Note that

$$\begin{aligned}
\frac{g(x+n) - g(n)}{n} &= \frac{f(x+n) - f(x) - nf'(0)}{n} \\
&= \frac{f(x+n) - f(x)}{n} - f'(0) \\
&= f'(x) - f'(0) \\
&= g'(x),
\end{aligned}$$

for all real numbers x and all positive integers n. □

Lemma 2

$$g'(x) = g'(x+1)$$

for all real numbers x.

Proof. Note that

$$g'(x) = \frac{g(x+2) - g(x)}{2}$$

and

$$g'(x) = \frac{g(x+1) - g(x)}{1},$$

therefore we have

$$g'(x) = \frac{(g(x+2) - g(x)) - (g(x+1) - g(x))}{2-1} = \frac{g(x+2) - g(x+1)}{1} = g'(x+1).$$

□

Since $g'(0) = 0$, Lemma 2 implies that $g'(n) = 0$ for all integers n. Then, with Lemma 1, we conclude that $g(n) = g(0) = 0$ for all integers n.

Lemma 3

$$g(x) = g(x+1)$$

for all real numbers x.

Proof.

$$\begin{aligned}
g(x+1) - g(x) &= \int_x^{x+1} g(t)\,dt \\
&= \int_x^{\lfloor x \rfloor} g(t)\,dt + \int_{\lfloor x \rfloor}^{x+1} g(t)\,dt \\
&= \int_x^{\lfloor x \rfloor} g(t)\,dt + \int_{\lfloor x \rfloor - 1}^{x} g(t)\,dt && \text{by Lemma 2} \\
&= \int_{\lfloor x \rfloor - 1}^{\lfloor x \rfloor} g(t)\,dt \\
&= g(\lfloor x \rfloor) - g(\lfloor x \rfloor - 1) = 0.
\end{aligned}$$

□

Finally, Lemma 1 with $n = 1$, together with Lemma 3, imply that

$$g'(x) = 0$$

for all real numbers x; since $g(0) = 0$, we have that

$$g(x) = 0$$

for all real numbers x, and consequently

$$f(x) = f'(0)x + f(0),$$

which is in the format we desired.

Problem 12.2 (Putnam 2011, B1)

Let h and k be positive integers. Prove that for every $\varepsilon > 0$, there are positive integers m and n such that

$$\varepsilon < \left|h\sqrt{m} - k\sqrt{n}\right| < 2\varepsilon.$$

Solution A:

Lemma 1

Let $\epsilon > 0$. There are positive integers m_0 and n_0 such that

$$0 < \sqrt{m_0} - \sqrt{n_0} < \epsilon.$$

Proof. Let $m_0 = n_0 + 1$. Let n_0 be large enough so that $n_0 > \epsilon^{-2}$. Then,

$$0 < \sqrt{n_0 + 1} - \sqrt{n_0} = \frac{1}{\sqrt{n_0 + 1} + \sqrt{n_0}} < n_0^{-1/2} < \epsilon.$$

$\square$

Lemma 2

Let $\epsilon > 0$. There are positive integers m_1 and n_1 such that

$$\epsilon < \sqrt{m_1} - \sqrt{n_1} < 2\epsilon.$$

Proof. Consider m_0 and n_0 from Lemma 1. Let $\delta = \sqrt{m_0} - \sqrt{n_0}$. We know that $0 < \delta < \epsilon$. Let $\ell = \lfloor \frac{\epsilon}{\delta} \rfloor + 1$. Then, we know that

$$\frac{\epsilon}{\delta} < \ell \leq \frac{\epsilon}{\delta} + 1,$$

i.e.

$$\epsilon < \ell\delta \leq \epsilon + \delta < 2\epsilon.$$

Define $m_1 = \ell^2 m_0$ and $n_1 = \ell^2 n_0$. Then, $\sqrt{m_1} - \sqrt{n_1} = \ell\delta$, which is between ϵ and 2ϵ, as desired. $\square$

Lemma 3 (a.k.a., the problem)

Let h and k be positive integers. Let $\varepsilon > 0$. There are positive integers m and n such that

$$\varepsilon < h\sqrt{m} - k\sqrt{n} < 2\varepsilon.$$

Proof. Lemma 2 with $\epsilon \mapsto \frac{\varepsilon}{hk}$ implies that there exist m_1 and n_1 such that

$$\frac{\varepsilon}{hk} < \sqrt{m_1} - \sqrt{n_1} < \frac{2\varepsilon}{hk}.$$

Define $m = k^2 m_1$ and $n = h^2 n_1$. Then, $\varepsilon < h\sqrt{m} - k\sqrt{n} < 2\varepsilon$, as desired. $\square$