

Combinatorics I Lecture Notes

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This is IMPA's master class Combinatorics 1, instructed by Robert Morris. All errors are my responsibility.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Google Meet and [YouTube videos](#). The recommended material can be found [here](#).

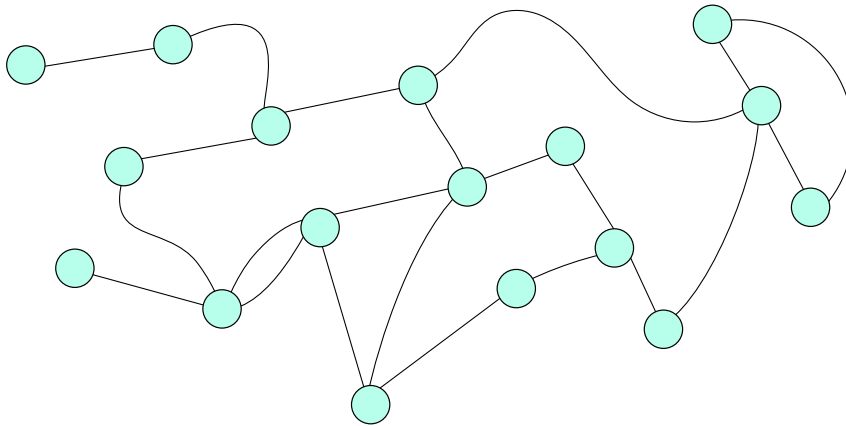


Figure 1: This is a graph.

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1 Which problems we'll study?

In this summer course, we'll study extremal, counting and probabilistic problems. Here are some examples:

Problem 1.1

Let A be a subset of $\{1, 2, \dots, 2n\}$ such that $a \nmid b$, for all $a \neq b \in A$.

How large can $|A|$ be?

Solution. $A = \{n+1, \dots, 2n\}$ is a good example. This yields $|A| = n$.

Consider the partition of $\{1, 2, \dots, 2n\}$ given by the following sets:

- $\{2^t\}$
- $\{3 \cdot 2^t\}$
- $\{5 \cdot 2^t\}$
- \vdots
- $\{(2n-1) \cdot 2^t\}$

There can't be two elements in the same set of the partition, so $|A| \leq n$.

Problem 1.2

Let A be a subset of $\{1, 2, \dots, 2n\}$ such that $a+b \neq c$, for all $a, b, c \in A$. We'll call such set *sum-free*.

How large can $|A|$ be?

Solution. $A = \{n+1, \dots, 2n\}$ is a good example. Another good example are the odd numbers. Both yield $|A| = n$.

Suppose $|A| \geq n+1$. Let $a = \max A$.

Consider the following partition with $\lfloor \frac{a}{2} \rfloor$ sets:

- $\{1, a-1\}$
- $\{2, a-2\}$
- \vdots
- $\{\lfloor \frac{a}{2} \rfloor, \lceil \frac{a}{2} \rceil\}$

There can't be two elements in the same set of the partition.

If $a \leq 2n-1$, then there are at most $n-1$ sets listed above, which implies $|A| \leq n$.

If $a = 2n$, then $n \notin A$, and then the $n-1$ first sets listed above cover A , thus $|A| \leq n$.

Theorem 1.1 (Schur, 1916)

Given $c: \mathbb{Z}_{>0} \rightarrow \{1, \dots, r\}$, there are x, y, z such that:

- $x + y = z$
- $c(x) = c(y) = c(z)$

Problem 1.3

How many sum-free sets are in $[n]$?

Conjecture 1.2 (Cameron and Erdős)

The number of sum-free sets in $[n]$ is $\leq C \cdot 2^{n/2}$.

2 Ramsey's Theory

Theorem 2.1 (Ramsey's Theorem)

If $c : \binom{\mathbb{N}}{2} \rightarrow \{1, \dots, r\}$, then there exists $A \subset \mathbb{N}$ infinite and monochromatic, i.e., such that $c(ab) = c$, for all $a, b \in A$.

Proof of Theorem 2.1. Let $S_0 = \mathbb{N}$.

For each i , do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1} . Since S_{i-1} is infinite and there are finitely many colors, there is some color that appears infinitely many times; we'll call this color c_i , and define $S_i = \{u \in S_{i-1} : c(\{v_i, u\}) = c_i\}$.

Now, we have an infinite sequence v_1, v_2, \dots , such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are finitely many colors, there is some color that appears in infinitely many c_i 's; call this color c , and define $A = \{v_i : c_i = c\}$.

The set A satisfies our condition.

Proof of Theorem 1.1. Given a coloring $c : \mathbb{N} \rightarrow \{1, \dots, r\}$, we define $c' : \binom{\mathbb{N}}{2} \rightarrow \{1, \dots, r\}$ by $c'(\{a, b\}) = c(b - a)$, for $b > a$.

By Theorem 2.1, there is A infinite and monochromatic. Pick $x < y < z \in A$, then we have $c(y - x) = c(z - y) = c(z - x)$, and $(y - x) + (z - y) = z - x$, so we're done!

Definition 2.2 (Ramsey Number)

Let $R(k)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n , i.e., for every $c : E(K_n) \rightarrow \{R, B\}$, there exists a monochromatic copy of K_k .

Let $R(s, t)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n , i.e., for every $c : E(K_n) \rightarrow \{R, B\}$, there exists a red copy of K_s or a blue copy of K_t .

Clearly, $R(k) = R(k, k)$.

Theorem 2.3

$$R(k) \lesssim 2^{2^k}.$$

Sketch. Let $n = 2^{2^k}$, and pick any coloring c of K_n . Let $S_0 = [n]$.

For each $i < 2k$ do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1} . There is some color that appears more times; we'll call this color c_i , and define $S_i = \{u \in S_{i-1} : c(\{v_i, u\}) = c_i\}$.

~~Note that, the size of S_m is at least $\frac{n}{2^m} \geq 1$.~~ This is not quite correct. At each step, we're taking one vertex away, and then dividing by two. We'll properly prove a better bound later.

Now, we have an sequence $v_1, v_2, \dots, v_{2k-1}$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are two colors, there is some color that appears at least k times; call this color c , and define $A = \{v_i : c_i = c\}$. The size of A is at least k . Pick any subset B of A that has exactly k elements.

The subgraph of K given by deleting all vertices but those in B is a monochromatic copy of K_k .

Lemma 2.4

$$R(s, t) \leq R(s - t, t) + R(s, t - 1).$$

Proof. Let $n = R(s, t) - 1$. By definition, there exists a coloring $c: E(K_n) \rightarrow \{R, B\}$ without a red K_s or a blue K_t .

Pick any vertex v . v is connected to some of the vertices through a red edge, which we'll put in the set S_R ; the others are connected to v through a blue edge, those we'll put in the set S_B .

Since there are no red K_s or blue K_t , there can't be any red K_{s-1} or blue K_t in S_R ; thus, $|S_R| \leq R(s-1, t)$. Analogously, $|S_B| \leq R(s, t-1)$.

Therefore,

$$\begin{aligned} R(s, t) - 1 &\leq R(s-1, t) - 1 + R(s, t-1) - 1 + 1 \\ R(s, t) &\leq R(s-1, t) + R(s, t-1). \end{aligned}$$

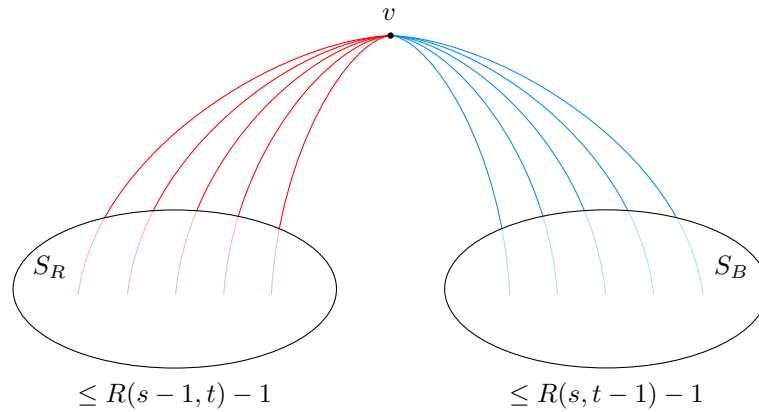


Figure 2: S_R and S_B .

Theorem 2.5

$$R(s, t) \leq \binom{s+t}{s}.$$

Proof. Follows from Lemma 2.4.

Theorem 2.6 (Erdős-Szekeres, 1935)

$$R(k) \leq \binom{2k}{k} \approx \frac{1}{\sqrt{k}} 4^k.$$

Lec. 2

Let's now try to find a lower bound. It is very difficult to show a good construction. Luckily, we are not going to do that.

Theorem 2.7 (Erdős, 1947)

$$\sqrt{2}^k \leq R(k)$$

Proof. Let $n \leq \sqrt{2}^k$. Let's choose colors randomly. Let

$$\mathbb{P}(c(e) = R) = \frac{1}{2},$$

for every edge e in K_n , independently.

We want to show that

$$\mathbb{P}(\text{there is not a monochromatic copy of } K_k) > 0.$$

Let X be the number of monochromatic copies of K_k in c . Then,

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E} \left[\sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{1}[S \text{ is monochromatic}] \right] \\ &= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{E}[\mathbb{1}[S \text{ is monochromatic}]] \\ &= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{P}(S \text{ is monochromatic}) \\ &= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\ &= \binom{k}{n} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\ &\leq 2 \left(\frac{en}{k}\right)^k \left(\frac{1}{2}\right)^{\frac{k(k-1)}{2}} \\ &\leq 2 \left(\frac{e\sqrt{2}}{k}\right)^k \\ &< 1, \text{ for } k \geq 5. \end{aligned}$$

Therefore, since $\mathbb{E}[X] < 1$, we have $\mathbb{P}(X = 0) > 0$.

The bounds have not improved much since then

Theorem 2.8 (Conlon, 2009)

$$R(k) \leq \frac{4^k}{k^{\sqrt{\log k}}}$$

3 Extremal Graph Theory

3.1 Complete Graphs

Definition 3.1

Let $\text{ex}(n, H)$ be the maximum number of edges a graph $G \subset K_n$ can have such that there are no copies of H in G .

Theorem 3.2 (Mantel, 1907)

$$\text{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. The example is the bipartite graph with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices.

Let's prove by induction on n .

Now, suppose G does not have a triangle. Pick an edge uv . Let G' be the graph G deleting u and v . The subgraph G' also does not contain triangles, so $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Notice that cannot exist $w \in G'$ such that uw and vw are edges of G , because G does not have triangles. Therefore, there can be at most $n - 2$ edges from u or v to vertices on G' . Including the edge uv , we conclude that

$$\begin{aligned} e(G) &\leq e(G') + n - 1 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

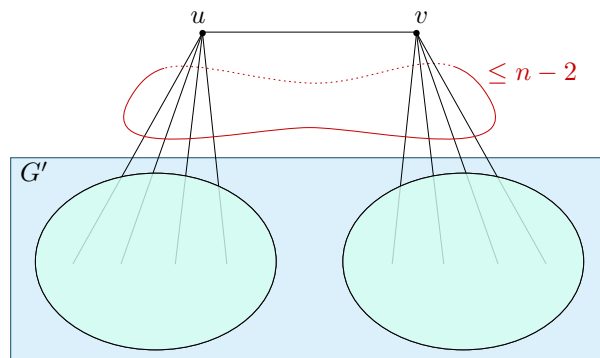


Figure 3: Edge uv on a triangle-free graph.

Definition 3.3 (Turán's Graph)

The graph $T_r(n)$ consists of r sets with roughly n/r elements each (some rounded up, some rounded down).; we create an edge uv if, and only if, u and v are on different sets.

We'll denote by $t_r(n)$ the number of edges in $T_r(n)$.

Theorem 3.4 (Turán, 1941)

$$\text{ex}(n, K_{r+1}) = t_r(n) \approx \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

Proof. We'll use induction on n . For $n \leq r$, we're good.

Pick a maximal graph G that doesn't have a copy of K_{r+1} . Pick a copy of K_r , let's call it H . Define $G' = G - H$. Of course, G' has no copies of K_r ; thus $e(G') \leq t_r(n - r)$, by induction.

Futhermore, if $v \in G'$, there can be at most $r - 1$ edges connecting v to some vertex in H .

Wrapping everything up, we have

$$\begin{aligned} e(G) &\leq e(G') + (n - r)(r - 1) + \binom{r}{2} \\ &\leq t_r(n - r) + (n - r)(r - 1) + \binom{r}{2} \\ &\leq t_r(n). \end{aligned}$$

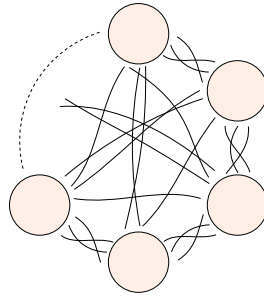


Figure 4: Turán's Graph

3.2 Bipartite Graphs

Theorem 3.5 (Erdős, 1935)

$$\text{ex}(n, C_4) \leq \frac{n^{3/2}}{2}.$$

Proof. Let's count cherries! A *cherry* is a pair $(v, \{u, w\})$, in which vu and vw are edges of the graph.

Since there is no C_4 , there is at most one cherry for each pair $\{u, w\}$. This implies that:

$$\begin{aligned} \binom{n}{2} &\geq \#(\text{cherries}) = \sum_{v \in G} \binom{d(v)}{2} \\ &\geq n \binom{\frac{2e(G)}{n}}{2}. \end{aligned}$$

Solving this quadratic inequation on $e(G)$ yields to

$$e(g) \geq \frac{n^{3/2}}{2}.$$

Question 3.1

For which graphs we have

$$\text{ex}(n, H) = \Theta(n^2)?$$

Proposition 3.6

For every non-bipartite graph H , we have

$$\text{ex}(n, H) \geq \frac{n^2}{4}.$$

Proof. Take G as the complete bipartite graph with n vertices. It has roughly $\frac{n^2}{4}$ edges and it cannot contain a non-bipartite graph.

Theorem 3.7 (Kővári–Sós–Turán, 1954)

Let H be a bipartite graph. Then,

$$\text{ex}(n, H) = o(n^2).$$

Proof. Since H is bipartite, there is some $K_{s,t}$ such that $H \subset K_{s,t}$. Then,

$$\text{ex}(n, H) \leq \text{ex}(n, K_{s,t}).$$

Let's bound $\text{ex}(n, K_{s,t})$.

We'll count s -cherries: (v, S) , in which S has size s and $vx \in E(G)$ for all $x \in S$.

There are at most $t - 1$ s -cherries for each subset S with size s . This implies that:

$$\begin{aligned} (t-1) \binom{n}{s} &\geq \#(\text{cherries}) = \sum_{v \in G} \binom{d(v)}{s} \\ &\geq n \binom{\frac{2e(G)}{n}}{s} \geq \frac{e(G)^s}{s^s \cdot n^{s-1}}. \end{aligned}$$

This implies that, for some constant C ,

$$e(G) \leq C \cdot n^{2-\frac{1}{s}}$$

Question 3.2

For which H it holds that

$$\text{ex}(n, H) = O(n)?$$

3.3 Trees

Lec. 3

Definition 3.8 (Tree)

A tree is a connected graph that has no cycles.

Proposition 3.9

Given a graph G , the following are equivalent:

- (i) G is a tree;
- (ii) G is a maximal graph without cycles, i.e., G does not have cycles and there is no graph $H \supset G$ such that H does not have cycles;
- (iii) G is a minimal connected graph, i.e., G is connected and there is no graph $H \subset G$ such that H is connected.

Theorem 3.10

Let T be a graph with k vertices. Then,

$$\frac{(k-2)}{2}n \leq \text{ex}(n, T) \leq (k-1) \cdot n.$$

Proof of the lower bound. Pick $\frac{n}{k-1}$ disjoint $k-1$ -cliques. There cannot be a copy of a connected graph with k vertices inside this graph. This graph has roughly

$$\binom{k-1}{2} \frac{n}{k-1} = \frac{k-2}{2}n$$

edges.

Proof of the upper bound. Let's start with a lemma.

Lemma 3.11

Let G be a graph with mean degree d , then, there exists a subgraph $G' \subset G$ with minimum degree at least $\frac{d}{2}$.

Proof. While there are vertices with degree smaller than $\frac{d}{2}$, throw them away.

If we stopped before throwing away all vertices, we're done. Suppose we threw away all vertices. At each step, we threw away at most $\frac{d}{2}$ edges. Since we threw away all edges, this means $n \cdot \frac{d}{2} < e(G) = n \frac{d}{2}$; a contradiction.

Lemma 3.12

Let G be a graph with $\delta(G) \geq k-1$. Then, there is a copy of T in G for every tree T with k vertices.

Proof. We'll use induction on k . If $k=1$, we're done!

Pick a leaf v of T . Its unique edge connects it to u . Let T' be the tree without v . By induction, there is a copy $C_{T'}$ of T' in G . Let c_u be the copy of u in $C_{T'}$. Since $\deg(c_u) \leq k-2$ in $C_{T'}$, but $\deg(c_u) \geq k-1$ in G , there is some vertex that is connected to u outside of $C_{T'}$, say c_v . Thus, let C_T be $C_{T'}$, adding c_v . C_T is a copy of T inside G .

Finally, $e(G) = (k-1)n \implies \bar{d}(G) = 2(k-1) \implies$ there exists a subgraph $G' \subset G$ such that $\delta(G') \geq k-1 \implies T \subset G'$.

Conjecture 3.13 (Erdős-Sós, 1960's)

$$\text{ex}(n, T) \leq \frac{(k-2)n}{2}$$

Definition 3.14 (Random graph of Erdős-Rényi)

We define $G(n, p)$ as a random distribution of graphs with n vertices, with

$$\mathbb{P}(e \in E(G(n, p))) = p,$$

chosen independently.

Lemma 3.15 (Markov's inequality)

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

Proof. Left to the reader. Use the definition of $\mathbb{E}[X]$.

Theorem 3.16

$$\text{ex}(n, C_t) \geq O\left(n^{1+\frac{1}{2t-1}}\right) \gg n.$$

Proof. Let $t = 2k$. We want to choose $p = p(n)$ such that:

- $e(G(n, p)) \gg n$;
- $C_{2k} \not\subset G(n, p)$.

$$\mathbb{E}[e(G(n, p))] = p \binom{n}{2}.$$

Moreover, $e(G(n, p))$ is a binomial distribution, therefore, $e(G(n, p)) \approx np^2$ with high probability. Thus, we should pick $p \gg 1/n$, i.e., $pn \rightarrow \infty$.

Define X as the number of copies of C_{2k} in $G(n, p)$.

$$\begin{aligned} \mathbb{E}[X] &= \sum_{\substack{\text{copies } S \text{ of} \\ C_{2k} \text{ in } K_n}} \mathbb{P}(S \subset G(n, p)) \\ &\approx n^{2k} p^{2k} = (pn)^{2k}. \end{aligned}$$

Let $0 < \varepsilon < \frac{1}{2k-1}$, and define $p = p(n) = n^{-1+\varepsilon}$. Then, we have $pn \gg n^{-1}$ and $(pn)^{2k} \ll pn^2$. Therefore, each of the following happen with high probability:

- $e(G(n, p)) \approx pn^2$;
- The number of copies of C_{2k} in $G(n, p) \approx (pn)^{2k}$.

Therefore, the intersection also occurs with high probability. Pick a graph G in the intersection.

For each of the $(pn)^{2k}$ cycles in G delete an edge in it; call this new graph G' . Thus $e(G') \approx pn^2 - (pn)^{2k} \approx n^{1+\varepsilon}$, and G' has no C_{2k} .

Theorem 3.17

$$\text{ex}(n, H) = O(n) \iff H \text{ does not have cycles.}$$

Proof. All the work has been done. The proof, which is simply a jigsaw puzzle, is left to the reader.

4 Planar graphs

Definition 4.1 (Planar Graph)

A planar graph is a graph that can be drawn on the plane without having crossing edges. Edges may not be straight.

Lemma 4.2 ($V + F - E = 2$)

Let G be a planar connected graph, and $v(G) \geq 1$. For any planar drawing of G , we have

$$v(G) + f(G) - e(G) = 2.$$

Sketch. Induction on $e(G)$.

(i) **If there is a leaf**, then we can take it away.

$$\begin{aligned} v(G') &= v(G) - 1, \\ e(G') &= e(G) - 1, \\ f(G') &= f(G). \end{aligned}$$

(ii) **If there is no leaf**, there is a cycle, take away an edge from the cycle.

$$\begin{aligned} v(G') &= v(G), \\ e(G') &= e(G) - 1, \\ f(G') &= f(G) - 1. \end{aligned}$$

Watch an animated version of this classic demonstration at [3Blue1Brown](#).

Theorem 4.3

Let G be a planar graph with $n \geq 3$ vertices. Then,

$$e(G) \leq 3n - 6$$

Proof. Without loss of generality G is maximal.

Maximal and $n \geq 3$ implies all regions are triangles. Double counting implies

$$3f(G) = 2e(G).$$

Also,

$$v(G) + f(G) - e(G) = 2.$$

It follows that

$$e(G) = 3n - 6.$$

Theorem 4.4

K_5 is not planar.

Proof.

$$e(K_5) = 10 > 3 \cdot 5 - 6 = 3v(K_5) - 6.$$

Theorem 4.5

Let G be a triangle-free planar graph with $n \geq 4$ vertices. Then,

$$e(G) \leq 2n - 2$$

Proof. Without loss of generality G is maximal.

Maximal and $n \geq 4$ implies all regions have at least 4 sides. Double counting implies

$$4f(G) \leq 2e(G).$$

Also,

$$v(G) + f(G) - e(G) = 2.$$

It follows that

$$e(G) = 2n - 4.$$

Theorem 4.6

$K_{3,3}$ is not planar.

Proof. $K_{3,3}$ is triangle-free.

$$e(K_{3,3}) = 9 > 2 \cdot 6 - 4 = 2v(K_{3,3}) - 4$$

Theorem 4.7

G is planar if, and only if, G does not have a topological copy of K_5 or $K_{3,3}$ if, and only if, G does not have a K_5 -minor or a $K_{3,3}$ -minor.

5 More colors

Definition 5.1 (Chromatic Number of a Graph)

The chromatic number of G , denoted by $\chi(G)$, is the smallest r such that there is a coloring $c: V(G) \rightarrow [r]$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$.

Lec. 4

Definition 5.2

Let $r(G, H)$ denote the minimum n such that, for every coloration $c: E(K_n) \rightarrow \{R, B\}$, there must exist a red G or a blue H .

Proposition 5.3

$$\chi(G) \leq \Delta(G) + 1.$$

Sketch. Greedy algorithm.

Theorem 5.4 (4-color Theorem, 1970's)

If G is planar, then $\chi(G) \leq 4$.

Proposition 5.5

If G is planar, then $\chi(G) \leq 6$.

Proof. Induction on n .

Since G is planar, $e(G) \leq 3n - 6$, thus $\delta(G) \leq 5$. Pick v with degree at most 5. Define G' as G without v , then G' has a proper coloring. Now, v has at most five neighbors, thus we can pick one color for v out of six such that no neighbor of v has this color.

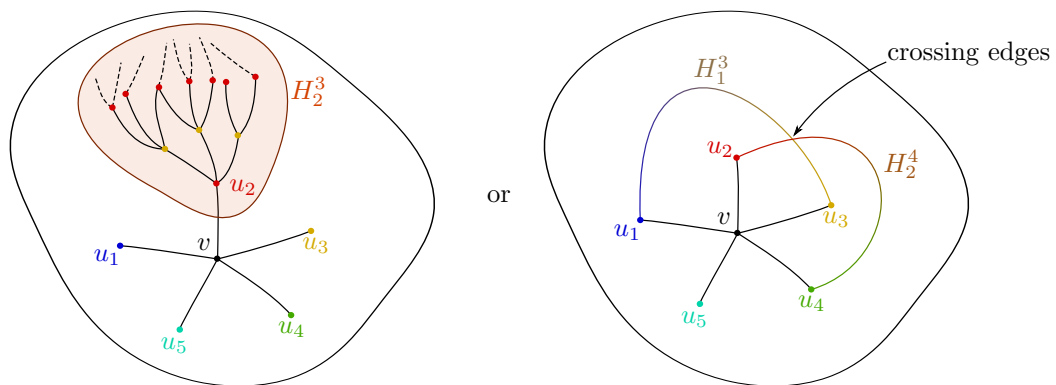


Figure 5: Five color theorem

Theorem 5.6

If G is planar, then $\chi(G) \leq 5$.

Proof. Induction on n .

Since G is planar, $e(G) \leq 3n - 6$, thus $\delta(G) \leq 5$. Pick v with degree at most 5. Define G' as G without v , then G' has a proper coloring. Now, v has at most five neighbors. If there at most four colors are used in the neighbors of v , we can paint v with a distinct color.

Suppose all neighbors of v have different colors. Let's call the neighbors u_1, u_2, u_3, u_4, u_5 , in clockwise order, with colors 1, 2, 3, 4, 5.

Define $G'_a{}^b$ as the subgraph of G' that only contains vertices with colors a and b . Let H_a^b be the connected component of $G'_a{}^b$ that contains u_a .

- **If there exists a, b such that $u_b \notin H_a^b$** , then we flip the colors a and b inside H_a^b and define $c(v) := a$.
- **If, for all a, b , $u_b \in H_a^b$** , $H_{1,3}$ and $H_{2,4}$ are vertex disjoint, but have to go through each other; a contradiction. See fig. 5.

Lemma 5.7

Se T é uma árvore, então $\chi(T) \leq 2$.

Sketch 1. Induction on number of vertices. Paint a leaf by the oposite color of its neighbor.

Theorem 5.8 (Erdős-Stone, 1946)

$$\text{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right) \binom{n}{2}.$$

Sketch. The example is the Turán's Graph $T_{\chi(H)-1}(n)$.

Let $r = \chi(H)$. We'll show it by induction on r .

If $r \leq 2$, then the theorem says $\text{ex}(n, H) = o(n^2)$, which is true by [Kővári-Sós-Turán, 1954](#).

Lemma 5.9

Let $\varepsilon > 0$ such that $\epsilon \binom{n}{2} > \binom{m_0}{2}$ and G be a graph with density β . Then, there exists $G^* \subset G$ with $m \geq m_0$ vertices and

$$\delta(G^*) \geq (\beta - \varepsilon)m.$$

Sketch. Throw away vertices with small degree. The first one we threw away had degree at most $< (\beta - \varepsilon)n$, the second one had degree at most $< (\beta - \varepsilon)(n - 1)$, and so on.

If we threw $n - m_0$ vertices away, then

$$\begin{aligned} e(G) &< (\beta - \varepsilon)(n + (n - 1) + \dots + m_0) + \binom{m_0}{2} \\ &< (\beta - \varepsilon) \binom{n}{2} + \binom{m_0}{2} \\ &< \beta \binom{n}{2}. \end{aligned}$$

The graph H is contained in $K_r(t)$, the complete r -partite with t vertices on each part, with $t = t(H)$.

Suppose $e(G) \geq \left(1 - \frac{1}{r-1} + \alpha\right) \binom{n}{2}$. Applying Lemma 5.9 with $\varepsilon = \frac{\alpha}{2}$, $m_0 = \frac{\alpha n}{2}$, and $\beta = 1 - \frac{1}{r-1} + \alpha$, we conclude that there exists $G^* \subset G$, with $m \geq \frac{\alpha n}{2}$ vertices, and $\delta(G^*) \geq \left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right) m$.

Induction hypothesis implies that, for large n , G^* has a copy F of $K_{r-1}(q)$, the complete $(r - 1)$ -partite graph with q vertices on each part, for $q > \frac{2(t-1)}{(r-1)\alpha}$.

Let $X = V(G^*) \setminus V(F)$. Let Y be the set of vertices in X that have at least $(r - 2)q + t$ neighbors in $V(F)$.

Let's call F_1, F_2, \dots, F_{r-1} the parts of F , a complete $(r - 1)$ -partite graph. Let's count the number of *hyper-cherries* $(v, S_1, S_2, \dots, S_{r-1})$, in which $v \in X$, $S_1 \subset F_1, \dots, S_{r-1} \subset F_{r-1}$, and $v \sim u$, for all u in some S_i . See fig. 6.

For each vertex v in Y (of $|Y|$), there are $\prod_i \binom{\deg_i(v)}{t} \geq \binom{q}{t}^{r-2}$ hyper-cherries. On the other hand,

for each possible subsets S_1, \dots, S_{r-1} (of $\binom{q}{t}^{r-1}$), there are at most $t-1$ hyper-cherries. This implies

$$|Y| \leq (t-1) \binom{q}{t}.$$

Thereore, the number of edges between X and $V(F)$ is at most

$$\left(m - (r-1)q - \binom{q}{t}(t-1) \right) ((r-2)q + t - 1) + \binom{q}{t}(t-1)(r-1)q,$$

which simplifies to

$$m((r-2)q + t - 1) + \text{constant}.$$

On the other hand, since every vertex of $V(F)$ has degree at least $\left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right)m$ the number of vertices between X and $V(F)$ is at least

$$\left(\left(1 - \frac{1}{r-1} + \frac{\alpha}{2}\right)m - (r-2)q \right) (r-1)q,$$

which simplifies to

$$m \left((r-2)q + \frac{(r-1)q\alpha}{2} \right) + \text{constant},$$

which yeilds to a contradiction to large n (i.e. large m).

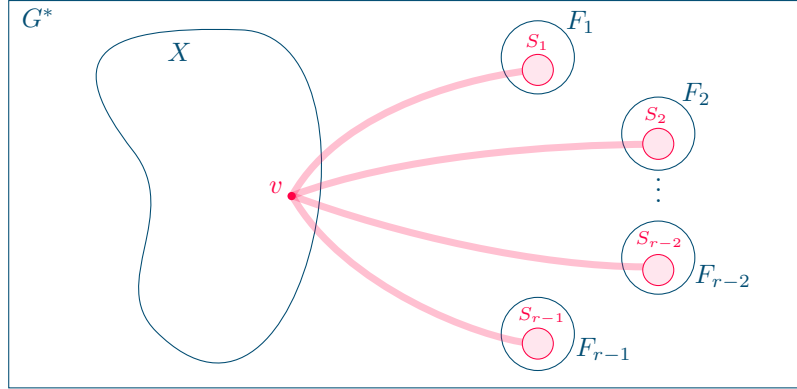


Figure 6: Hyper-cherry

6 Ramsey's Theory again

Lec. 5

Definition 6.1

Let $R_r^{(k)}(m)$ is the minimal n such that, for all colorings $c: \binom{[n]}{k} \rightarrow [r]$, there exists a monochromatic copy of $K_m^{(k)}$.

We'll consider $r = 2$ and $k = 2$, if not otherwise stated.

Remark. $K_m^{(k)}$ is the k -uniform complete hypergraph with n vertices. $E(K_m^{(k)}) = \binom{V(K_m^{(k)})}{k}$. See [Wikipedia](#).

Theorem 6.2 (Ramsey, 1930)

$$R_r^{(k)}(m) < \infty.$$

Sketch. Induction on k .

Pick $v_1 \in G$. Given $c: \binom{V(G)}{k} \rightarrow [r]$, define $c_1: \binom{V(G) \setminus \{v_1\}}{k-1} \rightarrow [r]$. Induction hypothesis implies that there exists a monochromatic copy of $K_{m_1}^{(k-1)}$, for $n \geq R_r^{(k-1)}(m_1)$.

Repeat the process inside this copy of $K_{m_1}^{(k-1)}$.

Similarly to the proof of Theorem 2.1, we'll have a sequence v_1, v_2, \dots, v_ℓ (that gets larger as n gets larger), for which $c(\{v_{a_1}, v_{a_2}, \dots, v_{a_k}\}) = f(a_1)$, if $a_1 < a_2 < \dots < a_r$.

Pick large n such that $\ell \geq (r-1)m + 1$, for which there exists a subsequence a_{b_1}, \dots, a_{b_r} such that $f(a_{b_i})$ is the same for all i .

Theorem 6.3 (Erdős–Hajnal)

$$R^{(k)}(m) \leq 2^{\binom{R^{(k-1)}(m)}{k-1}}$$

Sketch for $k = 3$. Suppose $e(G) \gtrsim 2^{\binom{R(m)}{2}}$

Pick a edge $v_1 v_2 \in E(G)$. Given $c: \binom{V(G)}{3} \rightarrow \{1, 2\}$, define $c': \binom{V(G) \setminus \{v_1, v_2\}}{2} \rightarrow \{1, 2\}$ by $c'(v) := c(v_1 v_2 v)$. The coloring c' naturally partitions $V(G) \setminus \{v_1, v_2\}$ into two parts, one for each color — denote the largest part by A_3 , this has $\gtrsim n/2$ vertices. This implies that $c(v_1 v_2 v)$ is constant for all $v \in A_3$ — denote this constant by $f(v_1 v_2)$.

Now, pick a vertex in $v_3 \in A_3$. Create similar colorings for the edges $v_1 v_3$ and $v_2 v_3$. There is a subset $A_4 \subset A_3$, with $\gtrsim n/8$ vertices, such that $c(v_1 v_3 v)$ and $c(v_2 v_3 v)$ are constant for all $v \in A_4$ — denote those constants by $f(v_1 v_3)$ and $f(v_2 v_3)$.

Repeat this process $R(m)$ times, which we can because $n \geq 2^{\binom{R(m)}{2}}$. Now, we have vertices $v_1, \dots, v_{\binom{R(m)}{2}}$, with a coloring f of each 2-edge, in which $f(v_{a_1} v_{a_2}) = c(v_{a_1} v_{a_2} v_{a_3})$, for all $a_1 < a_2 < a_3$. By definition, there is a monochromatic K_m over the coloring f , which implies that there exists a monochromatic $K_m^{(3)}$ over the coloring c .

6.1 Happy Ending Problem

Problem 6.1

Given 5 points on the plane, prove that there are 4 of them that form a convex polygon.

Solution. If the convex hull has size 5 or 4, we're ok. If it has size 3, then draw a line through the 2 points inside the convex hull, it meets two of the three sides of the convex hull. The two points inside and the two points in the side not crossed form a convex polygon.

Definition 6.4

Let $f(k)$ be the minimal n such that, for any set of n points in \mathbb{R}^2 in general position, there are k points that form a convex polygon.

Theorem 6.5 (Erdős-Szekeres, 1935)

$$f(k) \leq R^{(4)}(k) \leq 2^{2^{c_k}}.$$

Proof. Suppose $n > R^{(4)}(k)$.

Define $c: \binom{[n]}{4} \rightarrow R, B$ by $c(\{A, B, C, D\}) = R$ if, and only if, $\{A, B, C, D\}$ does form a convex polygon.

By definition, there exists a monochromatic $K_k^{(4)}$. For $k \geq 5$, it cannot be blue. Therefore, it's red, which would not be possible if those k vertices didn't form a convex polygon.

6.2 Monochromatic Arithmetic Progression

Definition 6.6

Let $W(r, k)$ be the minimal n such that for all $c: [n] \rightarrow [r]$, there exists a monochromatic arithmetic progression of size k .

Theorem 6.7 (Van der Waerden, 1927)

Let $c: \mathbb{N} \rightarrow [r]$. There is a monochromatic arithmetic progression of size k , for all positive integers k .

Equivalently,

$$W(r, k) < \infty.$$

Definition 6.8

Denote $\{a, a + d, a + 2d, \dots, a + (k - 1)d\}$ by $PA_k(a, d)$.

The arithmetic progressions $PA_k(a_1, d_1), PA_k(a_2, d_2), \dots, PA_k(a_s, d_s)$ are color-focused if:

- (i) They are monochromatic with different colors.
- (ii) They have the same "focus" f , i.e.,

$$a_1 + kd_1 = \dots = a_s + kd_s = f$$

Proof of Van der Waerden, 1927. We will use induction on k . Note that $W(r, 1) = 1$.

We shall find r color-focused $(k - 1)$ -arithmetic progressions.

Lemma 6.9

There exists $n = n(s, r)$ such that, for every coloring $c: [n] \rightarrow [r]$, there exists a monochromatic k -arithmetic progression or s color-focused $(k - 1)$ -arithmetic progressions.

Proof. Induction on s . $n(1, r) = W(r, k - 1) < \infty$.

Let $N = 2n(s - 1, r)$. Consider $W(r^N, k - 1) < \infty$ blocks of size N . There is an arithmetic progression of equally-colored blocks of size $k - 1$, let D be the distance of consecutive blocks in the arithmetic progression of blocks. Since the first half of the block has $n(s - 1, r)$ elements, there exists a monochromatic k -arithmetic progression (which means we're done), or $s - 1$ color-focused $(k - 1)$ -arithmetic progressions – their focus f surely lies inside the block of size N .

Let the $s - 1$ color-focused $(k - 1)$ -arithmetic progressions in the first block be $PA_{k-1}(a_1, d_1), \dots, PA_{k-1}(a_{s-1}, d_{s-1})$, with focus f_1 . The proposed s color-focused $(k - 1)$ -arithmetic progressions are $PA_{k-1}(a_1, d_1 + d), \dots, PA_{k-1}(a_{s-1}, d_{s-1} + d), PA_{k-1}(f_1, d)$.

Therefore,

$$n(s, r) \leq 2 \cdot W(r^{2n(s-1, r)}, k - 1) \cdot 2n(s - 1, r).$$

Therefore, for suitable large n , there must exist a large k -arithmetic progression.