Combinatorics I Lecture Notes

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This is IMPA's master class Combinatorics 1, instructed by Robert Morris. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Gooogle Meet.

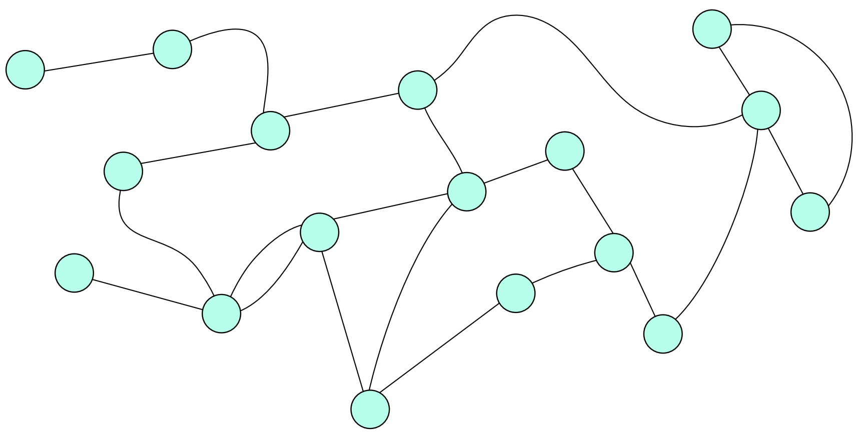

Figure 1: This is a graph.

Contents

1 Which problems we'll study?

In this summer course, we'll study extremal, counting and probabilistic problems. Here are some examples:

Problem 1.1

Let A be a subset of $\{1, 2, \dots, 2n\}$ such that $a \nmid b$, for all $a \neq b \in A$.

How large can $|A|$ be?

Solution. $A = \{n+1, \dots, 2n\}$ is a good example. This yields $|A| = n$.

Consider the partition of $\{1, 2, \dots, 2n\}$ given by the following sets:

- $\{2^t\}$
- $\{3 \cdot 2^t\}$
- $\{5 \cdot 2^t\}$

$\vdots$

- $\{(2n-1) \cdot 2^t\}$

There can't be two elements in the same set of the partition, so $|A| \leq n$.

Problem 1.2

Let A be a subset of $\{1, 2, \dots, 2n\}$ such that $a+b \neq c$, for all $a, b, c \in A$. We'll call such set *sum-free*.

How large can $|A|$ be?

Solution. $A = \{n+1, \dots, 2n\}$ is a good example. Another good example are the odd numbers. Both yield $|A| = n$.

Suppose $|A| \geq n+1$. Let $a = \max A$.

Consider the following partition with $\left\lfloor \frac{a}{2} \right\rfloor$ sets:

- $\{1, a-1\}$
- $\{2, a-2\}$

$\vdots$

- $\{\left\lfloor \frac{a}{2} \right\rfloor, \left\lceil \frac{a}{2} \right\rceil\}$

There can't be two elements in the same set of the partition.

If $a \leq 2n-1$, then there are at most $n-1$ sets listed above, which implies $|A| \leq n$.

If $a = 2n$, then $n \notin A$, and then the $n-1$ first sets listed above cover A, thus $|A| \leq n$.

Theorem 1.1 (Schur, 1916)

Given $c \colon \mathbb{Z}_{>0} \to \{1, \dots, r\}$, the there are x, y, z such that:

- $x + y = z$
- $c(x) = c(y) = c(z)$

Problem 1.3

How many sum-free sets are in $[n]$?

Conjecture 1.2 (Cameron and Erdős)

The number of sum-free sets in $[n]$ is $\leq C \cdot 2^{n/2}$.

2 Ramsey's Theory

Theorem 2.1 (Ramsey's Theorem)

If $c : \binom{\mathbb{N}}{2} \to \{1, \ldots, r\}$, then there exists $A \subset \mathbb{N}$ infinite and monochromactic, i.e, such that $c(ab) = c$, for all $a, b \in A$.

Proof of Theorem 2.1. Let $S_0 = \mathbb{N}$.

For each i, do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1}. Since S_{i-1} is infinite and there are finitely many colors, there is some color that appears infinitely many times; we'll call this color c_i, and define $S_i = \{u \in S_{i-1} \colon c(\{v_i, u\}) = c_i\}$.

Now, we have an infinite sequence $v_1, v_2, \ldots$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are finitely many colors, there is some color that appears in infinitely many c_i's; call this color c, and define $A = \{v_i \colon c_i = c\}$.

The set A satisfies our condition.

Proof of Theorem 1.1. Given a coloring $c : \mathbb{N} \to \{1, \ldots, r\}$, we define $c' : \binom{\mathbb{N}}{2} \to \{1, \ldots, r\}$ by $c'(\{a, b\}) = c(b - a)$, for $b > a$.

By Theorem 2.1, there is A infinite and monochromactic. Pick $x < y < z \in A$, then we have $c(y - x) = c(z - y) = c(z - x)$, and $(y - x) + (z - y) = z - y$, so we're done!

Definition 2.2 (Ramsey Number)

Let $R(k)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n, i.e., for every $c : E(K_n) \to \{R, B\}$, there exists a monochromatic copy of K_k.

Let $R(s, t)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n, i.e., for every $c : E(K_n) \to \{R, B\}$, there exists a red copy of K_s or a blue copy of K_t.

Clearly, $R(k) = R(k, k)$.

Theorem 2.3

$$R(k) \lesssim 2^{2k}.$$

Sketch. Let $n = 2^{2k}$, and pick any coloring c of K_n. Let $S_0 = [n]$.

For each $i < 2k$ do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1}. There is some color that appears more times; we'll call this color c_i, and define $S_i = \{u \in S_{i-1} \colon c(\{v_i, u\}) = c_i\}$.

~~Note that, the size of S_m is at least $\frac{n}{2^m} \geq 1$.~~ This is not quite correct. At each step, we're taking one vertice away, and then dividing by two. We'll properly prove a better bound later.

Now, we have an sequence $v_1, v_2, \ldots, v_{2k-1}$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are two colors, there is some color that appears at least k times; call this color c, and define $A = \{v_i \colon c_i = c\}$. The size of A is at least k. Pick any subset B of A that has exactly k elements.

The subgraph of K given by deleting all vertices but those in B is a monochromatic copy of K_k.

Lemma 2.4

$$R(s, t) \leq R(s - t, t) + R(s, t - 1).$$

Proof. Let $n = R(s,t) - 1$. By definition, there exists a coloring $c\colon E(K_n) \to \{R, B\}$ without a red K_s or a blue K_t.

Pick any vertex v. v it connected to some of the vertices through a red edge, which we'll put in the set S_R; the others are connected to v through a blue edge, those we'll put in the set S_B.

Since there are no red K_s or blue K_t, there can't be any red K_{s-1} or blue K_t in S_R; thus, $|S_R| \leq R(s-1,t)$. Analougously, $|S_B| \leq R(s,t-1)$.

Therefore,

$$\begin{aligned} R(s,t) - 1 &\leq R(s-1,t) - 1 + R(s,t-1) - 1 + 1 \\ R(s,t) &\leq R(s-1,t) + R(s,t-1). \end{aligned}$$

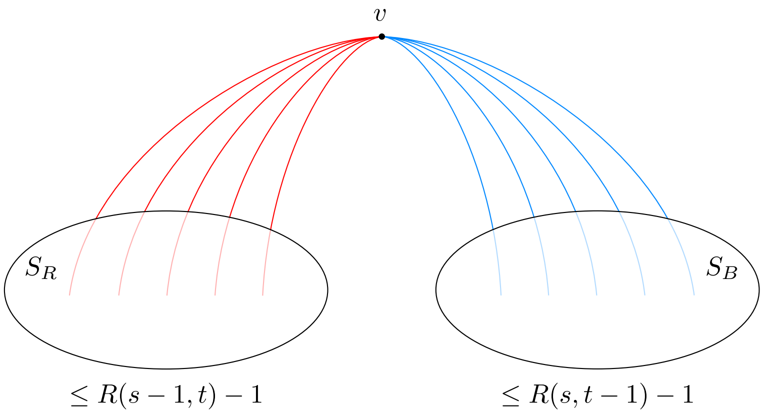

Figure 2: S_R and S_B.

Theorem 2.5

$$R(s,t) \leq \binom{s+t}{s}.$$

Proof. Follows from Lemma 2.4.

Theorem 2.6 (Erdős-Szekeres, 1935)

$$R(k) \leq \binom{2k}{k} \approx \frac{1}{\sqrt{k}} 4^k.$$

Let's now try to find a lower bound. It is very difficult to show a good construction. Luckly, we are not going to do that.

Theorem 2.7 (Erdős, 1947)

$$\sqrt{2}^k \leq R(k)$$

Proof. Let $n \leq \sqrt{2}^k$. Let's choose colors randomly. Let

$$\mathbb{P}(c(e) = R) = \frac{1}{2},$$

for every edge e in K_n, independently.

We want to show that

$$\mathbb{P}(\text{there is not a monochromatic copy of } K_k) > 0.$$

Let X be the number of monochromactic copies of K_k in c. Then,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}\left[\sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{1}[S \text{ is monochromatic}]\right] \\
&= \sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{E}\left[\mathbb{1}[S \text{ is monochromatic}]\right] \\
&= \sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{P}(S \text{ is monochromatic})) \\
&= \sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&= \binom{k}{n}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&\leq 2\left(\frac{en}{k}\right)^k\left(\frac{1}{2}\right)^{\frac{k(k-1)}{2}} \\
&\leq 2\left(\frac{e\sqrt{2}}{k}\right)^k \\
&< 1, \text{ for } k \geq 5.
\end{aligned}$$

Therefore, since $\mathbb{E}[X] < 1$, we have $\mathbb{P}(X = 0) > 0$.

The bounds have not improved much since then

Theorem 2.8 (Conlon, 2009)

$$R(k) \leq \frac{4^k}{k^{\sqrt{\log k}}}$$

3 Extremal Graph Theory

Definition 3.1

Let $ex(n, H)$ be the maximum number of edges a graph $G \subset K_n$ can have such that there are no copies of H in G.

Theorem 3.2 (Mantel, 1907)

$$ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. The example is the bipartite graph with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices.

Let's prove by indution on n.

Now, suppose G does not have a triangle. Pick an edge uv. Let G' be the graph G deleting u and v. The subgraph G' also does not contain triangles, so $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Notice that cannot exist $w \in G'$ such that uw and vw are edges of G, because G does not have triangles. Therefore, there can be at most $n - 2$ edges from u or v to vertices on G'. Including the edge uv, we conclude that

$$\begin{aligned} e(G) &\leq e(G') + n - 1 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

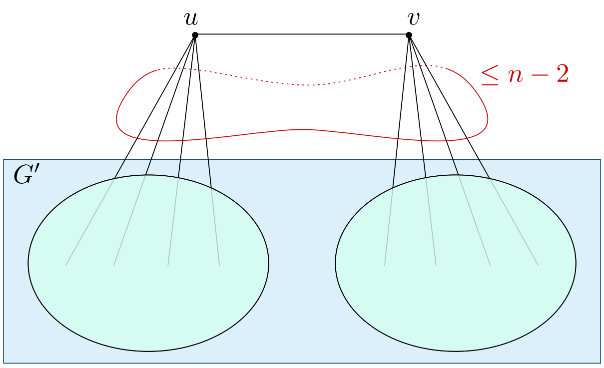

Figure 3: Edge uv on a triangle-less graph.

Definition 3.3 (Turán's Graph)

The graph $T_r(n)$ consists of r sets with roughly n/r elements each (some rounded up, some rounded down).; we create an edge uv if, and only if, u and v are on different sets.

We'll denote by $t_r(n)$ the number of edges in $T_r(n)$.

Theorem 3.4 (Turán, 1941)

$$ex(n, K_{r+1}) = t_r(n) \approx \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

Proof. We'll use induction on n. For $n \leq r$, we're good.

Pick a maximal graph G that doesn't have a copy of K_{r+1}. Pick a copy of K_r, let's call it H. Define $G' = G - H$. Of course, G' has no copies of K_r; thus $e(G') \leq t_r(n-r)$, by induction.

Futhermore, if $v \in G'$, there can be at most $r-1$ edges connecting v to some vertex in H.

Wrapping everything up, we have

$$\begin{aligned} e(G) &\leq e(G') + (n-r)(r-1) + \binom{r}{2} \\ &\leq t_r(n-r) + (n-r)(r-1) + \binom{r}{2} \\ &\leq t_r(n). \end{aligned}$$

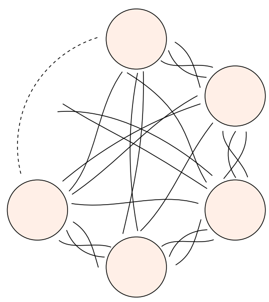

Figure 4: Turán's Graph

Theorem 3.5 (Erdős, 1935)

$$ex(n, C_4) \leq \frac{n^{3/2}}{2}.$$

Proof. Let's count cherries! A *cherry* is a pair $(v, \{u, w\})$, in which vu and vw are edges of the graph.

Since there is no C_4, there is at most one cherry for each pair $\{u, w\}$. This implies that:

$$\begin{aligned} \binom{n}{2} \geq \#(\text{cherries}) &= \sum_{v \in G} \binom{d(v)}{2} \\ &\geq n \binom{\frac{2e(G)}{n}}{2}. \end{aligned}$$

Solving this quadractic inequation on $e(G)$ yields to

$$e(g) \geq \frac{n^{3/2}}{2}.$$

Question 3.1

For which graphs we have

$$ex(n, H) = \Theta(n^2)?$$

Proposition 3.6

For every non-bipartite graph H, we have

$$ex(n, H) \geq \frac{n^2}{4}.$$

Proof. Take G as the complete bipartite graph with n vertices. It has roughly $\frac{n^2}{4}$ vertices and it cannot contain a non-bipartite graph.

Theorem 3.7 (Kővári–Sós–Turán, 1954)

Let H be a bipartite graph. Then,

$$ex(n, H) = o(n^2).$$

Proof. Since H is bipartite, there is some $K_{s,t}$ such that $H \subset K_{s,t}$. Then,

$$ex(n, H) \leq ex(n, K_{s,t}).$$

Let's bound $ex(n, K_{s,t})$.

We'll count s-cherries: (v, S), in which S has size s and $vx \in E(G)$ for all $x \in S$.

There are at most $t - 1$ s-cherries for each subset S with size s. This implies that:

$$\begin{aligned}(t-1)\binom{n}{s} \geq \#(\text{cherries}) &= \sum_{v \in G} \binom{d(v)}{s} \\ &\geq n \binom{\frac{2e(G)}{n}}{s} \geq \frac{e(G)^s}{s^s \cdot n^{s-1}}.\end{aligned}$$

This implies that, for some constant C,

$$e(G) \leq C \cdot n^{2-\frac{1}{s}}$$

Question 3.2

For which H it holds that

$$ex(n, H) = O(n)?$$