Combinatorics I Lecture Notes

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This is IMPA's master class Combinatorics 1, instructed by Robert Morris. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Gooogle Meet and YouTube videos. The recommended material is can be found here.

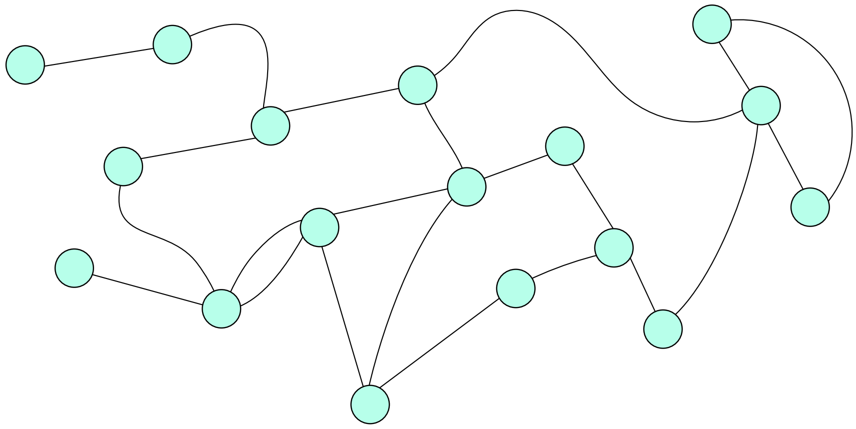

Figure 1: This is a graph.

Contents

1 Which problems we'll study?

In this summer course, we'll study extremal, counting and probabilistic problems. Here are some examples:

Problem 1.1

Let A be a subset of $\{1, 2, \ldots, 2n\}$ such that $a \nmid b$, for all $a \neq b \in A$.

How large can $|A|$ be?

Solution. $A = \{n+1, \ldots, 2n\}$ is a good example. This yields $|A| = n$.

Consider the partition of $\{1, 2, \ldots, 2n\}$ given by the following sets:

- $\{2^t\}$
- $\{3 \cdot 2^t\}$
- $\{5 \cdot 2^t\}$

$\vdots$

- $\{(2n-1) \cdot 2^t\}$

There can't be two elements in the same set of the partition, so $|A| \leq n$.

Problem 1.2

Let A be a subset of $\{1, 2, \ldots, 2n\}$ such that $a+b \neq c$, for all $a, b, c \in A$. We'll call such set *sum-free*.

How large can $|A|$ be?

Solution. $A = \{n+1, \ldots, 2n\}$ is a good example. Another good example are the odd numbers. Both yield $|A| = n$.

Suppose $|A| \geq n+1$. Let $a = \max A$.

Consider the following partition with $\left\lfloor \frac{a}{2} \right\rfloor$ sets:

- $\{1, a-1\}$
- $\{2, a-2\}$

$\vdots$

- $\{\left\lfloor \frac{a}{2} \right\rfloor, \left\lceil \frac{a}{2} \right\rceil\}$

There can't be two elements in the same set of the partition.

If $a \leq 2n-1$, then there are at most $n-1$ sets listed above, which implies $|A| \leq n$.

If $a = 2n$, then $n \notin A$, and then the $n-1$ first sets listed above cover A, thus $|A| \leq n$.

Theorem 1.1 (Schur, 1916)

Given $c\colon \mathbb{Z}_{>0} \to \{1, \ldots, r\}$, the there are x, y, z such that:

- $x + y = z$
- $c(x) = c(y) = c(z)$

Problem 1.3

How many sum-free sets are in $[n]$?

Conjecture 1.2 (Cameron and Erdős)

The number of sum-free sets in $[n]$ is $\leq C \cdot 2^{n/2}$.

2 Ramsey's Theory

Theorem 2.1 (Ramsey's Theorem)

If $c : \binom{\mathbb{N}}{2} \to \{1, \ldots, r\}$, then there exists $A \subset \mathbb{N}$ infinite and monochromactic, i.e, such that $c(ab) = c$, for all $a, b \in A$.

Proof of Theorem 2.1. Let $S_0 = \mathbb{N}$.

For each i, do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1}. Since S_{i-1} is infinite and there are finitely many colors, there is some color that appears infinitely many times; we'll call this color c_i, and define $S_i = \{u \in S_{i-1} : c(\{v_i, u\}) = c_i\}$.

Now, we have an infinite sequence $v_1, v_2, \ldots$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are finitely many colors, there is some color that appears in infinitely many c_i's; call this color c, and define $A = \{v_i : c_i = c\}$.

The set A satisfies our condition.

Proof of Theorem 1.1. Given a coloring $c : \mathbb{N} \to \{1, \ldots, r\}$, we define $c' : \binom{\mathbb{N}}{2} \to \{1, \ldots, r\}$ by $c'(\{a, b\}) = c(b - a)$, for $b > a$.

By Theorem 2.1, there is A infinite and monochromactic. Pick $x < y < z \in A$, then we have $c(y - x) = c(z - y) = c(z - x)$, and $(y - x) + (z - y) = z - y$, so we're done!

Definition 2.2 (Ramsey Number)

Let $R(k)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n, i.e., for every $c : E(K_n) \to \{R, B\}$, there exists a monochromatic copy of K_k.

Let $R(s, t)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n, i.e., for every $c : E(K_n) \to \{R, B\}$, there exists a red copy of K_s or a blue copy of K_t.

Clearly, $R(k) = R(k, k)$.

Theorem 2.3

$$R(k) \lesssim 2^{2k}.$$

Sketch. Let $n = 2^{2k}$, and pick any coloring c of K_n. Let $S_0 = [n]$.

For each $i < 2k$ do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1}. There is some color that appears more times; we'll call this color c_i, and define $S_i = \{u \in S_{i-1} : c(\{v_i, u\}) = c_i\}$.

~~Note that, the size of S_m is at least $\frac{n}{2^m} \geq 1$.~~ This is not quite correct. At each step, we're taking one vertice away, and then dividing by two. We'll properly prove a better bound later.

Now, we have an sequence $v_1, v_2, \ldots, v_{2k-1}$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are two colors, there is some color that appears at least k times; call this color c, and define $A = \{v_i : c_i = c\}$. The size of A is at least k. Pick any subset B of A that has exactly k elements.

The subgraph of K given by deleting all vertices but those in B is a monochromatic copy of K_k.

Lemma 2.4

$$R(s, t) \leq R(s - t, t) + R(s, t - 1).$$

Proof. Let $n = R(s,t) - 1$. By definition, there exists a coloring $c\colon E(K_n) \to \{R, B\}$ without a red K_s or a blue K_t.

Pick any vertex v. v it connected to some of the vertices through a red edge, which we'll put in the set S_R; the others are connected to v through a blue edge, those we'll put in the set S_B.

Since there are no red K_s or blue K_t, there can't be any red K_{s-1} or blue K_t in S_R; thus, $|S_R| \leq R(s-1,t)$. Analougously, $|S_B| \leq R(s,t-1)$.

Therefore,

$$
\begin{aligned}
R(s,t) - 1 &\leq R(s-1,t) - 1 + R(s,t-1) - 1 + 1 \\
R(s,t) &\leq R(s-1,t) + R(s,t-1).
\end{aligned}
$$

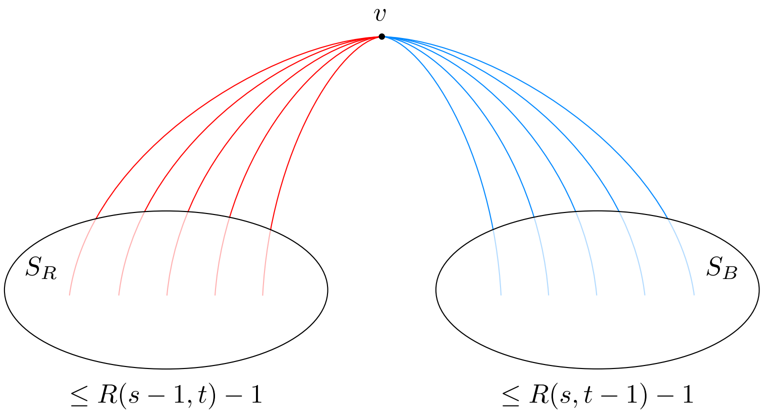

Figure 2: S_R and S_B.

Theorem 2.5

$$R(s,t) \leq \binom{s+t}{s}.$$

Proof. Follows from Lemma 2.4.

Theorem 2.6 (Erdős-Szekeres, 1935)

$$R(k) \leq \binom{2k}{k} \approx \frac{1}{\sqrt{k}} 4^k.$$

Let's now try to find a lower bound. It is very difficult to show a good construction. Luckly, we are not going to do that.

Theorem 2.7 (Erdős, 1947)

$$\sqrt{2}^k \leq R(k)$$

Proof. Let $n \leq \sqrt{2}^k$. Let's choose colors randomly. Let

$$\mathbb{P}(c(e) = R) = \frac{1}{2},$$

for every edge e in K_n, independently.

We want to show that

$$\mathbb{P}(\text{there is not a monochromatic copy of } K_k) > 0.$$

Let X be the number of monochromactic copies of K_k in c. Then,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}\left[\sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{1}[S \text{ is monochromatic}]\right] \\
&= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{E}\left[\mathbb{1}[S \text{ is monochromatic}]\right] \\
&= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{P}(S \text{ is monochromatic})) \\
&= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&= \binom{k}{n}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&\leq 2\left(\frac{en}{k}\right)^k\left(\frac{1}{2}\right)^{\frac{k(k-1)}{2}} \\
&\leq 2\left(\frac{e\sqrt{2}}{k}\right)^k \\
&< 1, \text{ for } k \geq 5.
\end{aligned}$$

Therefore, since $\mathbb{E}[X] < 1$, we have $\mathbb{P}(X = 0) > 0$.

The bounds have not improved much since then

Theorem 2.8 (Conlon, 2009)

$$R(k) \leq \frac{4^k}{k^{\sqrt{\log k}}}$$

3 Extremal Graph Theory

Definition 3.1

Let $ex(n, H)$ be the maximum number of edges a graph $G \subset K_n$ can have such that there are no copies of H in G.

Theorem 3.2 (Mantel, 1907)

$$ex(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. The example is the bipartite graph with $\left\lfloor \frac{n}{2} \right\rfloor$ and $\left\lceil \frac{n}{2} \right\rceil$ vertices.

Let's prove by indution on n.

Now, suppose G does not have a triangle. Pick an edge uv. Let G' be the graph G deleting u and v. The subgraph G' also does not contain triangles, so $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Notice that cannot exist $w \in G'$ such that uw and vw are edges of G, because G does not have triangles. Therefore, there can be at most $n-2$ edges from u or v to vertices on G'. Including the edge uv, we conclude that

$$\begin{aligned} e(G) &\leq e(G') + n - 1 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

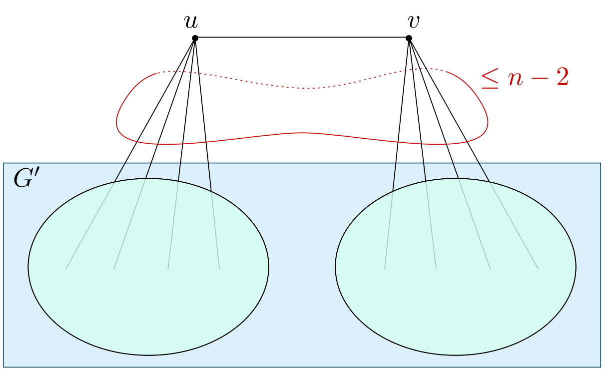

Figure 3: Edge uv on a triangle-less graph.

Definition 3.3 (Turán's Graph)

The graph $T_r(n)$ consists of r sets with roughly n/r elements each (some rounded up, some rounded down).; we create an edge uv if, and only if, u and v are on different sets.

We'll denote by $t_r(n)$ the number of edges in $T_r(n)$.

Theorem 3.4 (Turán, 1941)

$$ex(n, K_{r+1}) = t_r(n) \approx \left(1 - \frac{1}{r}\right)\binom{n}{2}.$$

Proof. We'll use induction on n. For $n \leq r$, we're good.

Pick a maximal graph G that doesn't have a copy of K_{r+1}. Pick a copy of K_r, let's call it H. Define $G' = G - H$. Of course, G' has no copies of K_r; thus $e(G') \leq t_r(n-r)$, by induction.

Futhermore, if $v \in G'$, there can be at most $r-1$ edges connecting v to some vertex in H.

Wrapping everything up, we have

$$\begin{aligned} e(G) &\leq e(G') + (n-r)(r-1) + \binom{r}{2} \\ &\leq t_r(n-r) + (n-r)(r-1) + \binom{r}{2} \\ &\leq t_r(n). \end{aligned}$$

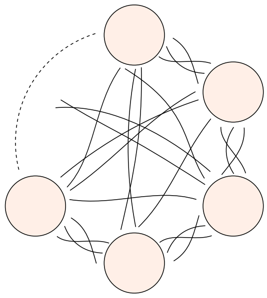

Figure 4: Turán's Graph

Theorem 3.5 (Erdős, 1935)

$$ex(n, C_4) \leq \frac{n^{3/2}}{2}.$$

Proof. Let's count cherries! A *cherry* is a pair $(v, \{u, w\})$, in which vu and vw are edges of the graph.

Since there is no C_4, there is at most one cherry for each pair $\{u, w\}$. This implies that:

$$\begin{aligned} \binom{n}{2} \geq \#(\text{cherries}) &= \sum_{v \in G} \binom{d(v)}{2} \\ &\geq n \binom{\frac{2e(G)}{n}}{2}. \end{aligned}$$

Solving this quadractic inequation on $e(G)$ yields to

$$e(g) \geq \frac{n^{3/2}}{2}.$$

Question 3.1

For which graphs we have

$$ex(n, H) = \Theta(n^2)?$$

Proposition 3.6

For every non-bipartite graph H, we have

$$ex(n,H) \geq \frac{n^2}{4}.$$

Proof. Take G as the complete bipartite graph with n vertices. It has roughly $\frac{n^2}{4}$ vertices and it cannot contain a non-bipartite graph.

Theorem 3.7 (Kővári–Sós–Turán, 1954)

Let H be a bipartite graph. Then,

$$ex(n,H) = o(n^2).$$

Proof. Since H is bipartite, there is some $K_{s,t}$ such that $H \subset K_{s,t}$. Then,

$$ex(n,H) \leq ex(n,K_{s,t}).$$

Let's bound $ex(n,K_{s,t})$.

We'll count s-cherries: (v,S), in which S has size s and $vx \in E(G)$ for all $x \in S$.

There are at most $t-1$ s-cherries for each subset S with size s. This implies that:

$$\begin{aligned}(t-1)\binom{n}{s} \geq \#(\text{cherries}) &= \sum_{v \in G} \binom{d(v)}{s} \\ &\geq n\binom{\frac{2e(G)}{n}}{s} \geq \frac{e(G)^s}{s^s \cdot n^{s-1}}.\end{aligned}$$

This implies that, for some constant C,

$$e(G) \leq C \cdot n^{2-\frac{1}{s}}$$

Question 3.2

For which H it holds that

$$ex(n,H) = O(n)?$$