Combinatorics I, List 2

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Problem 1
Prove the following supersaturaion theorem for cliques:

A graph with $o(n^r)$ copies of K_r has at most $\mathrm{ex}(n, K_r) + o(n^2)$ edges.

Sketch. Seja $B = \{T \subset V(G) : |T| = t, K_r \subset G[T]\}$.

$$|B| \leq \epsilon n^k \cdot n^{t-k}.$$

Vamos contar os pares de (T, e) onde $T \subset V(G), |T| = t, e \in e(G(T))$.

$$e(G)\binom{n-2}{t-2} = \#(T, e) \leq (\epsilon n^t)\binom{t}{2} + \binom{n}{t} t_{r-1}(t).$$

Problem 2

Show that, for every graph H and $\varepsilon > 0$, there exists $\delta > 0$ such that the following golds for all sufficiently large $n \in \mathbb{N}$.

If G is a graph on n vertices with

$$e(G) > (1-\delta)\binom{n}{2},$$

then in every r-colouring of $E(G)$ there are at least $\varepsilon n^{v(H)}$ monochromatic copies of H.

Sketch. Pegar k grande suitable.

Usar o item anterior pra achar muitos K_k.

Fazer contagem dupla de (C_{K_k}, C_H), $C_H \subset C_{K_k}$.

Problem 3

Shay that a k-uniform hypergraph G is said to be 2-*colourable* if there exists a partition $V(G) = A \cup B$ with no edges entirely contained in either A or B. Let $b(k)$ denote the minimum number of edges k-uniform hypergraph that is not 2-colourable.

(a) By considering a random coloring, show that $b(k) \geq 2^{k-1}$.

(b) By considering a random hypergraph, probe an upper bound for $b(k)$.

Sketch for (a). It suffices to show that k-uniform hypergraph with $2^{k-1}-1$ edges is 2-colourable.

Pick a random bipartition, $p = 1/2$.

$$\mathbb{P}[e \text{ is monochromatic}] = \left(\frac{1}{2}\right)^{k-1}.$$

Then,

$$\mathbb{P}[\exists e \text{ monochromatic}] = \frac{v(\mathcal{H})}{2^{k-1}} < 1.$$

Sketch for an upper bound. The k-uniform complete hypergraph with $2k-1$ edges works.

Sketch for (b). Escolha um hypergrafo k-uniforme com m arestas aleatório. Para facilitar, escolha as m arestas independentemente, e se calhar de ser a mesma, o seu grafo vai ter menos arestas.

Let (A, B) be a bipartition. Say $|A| = a$.

$$\mathbb{P}(e \text{ is monochromatic}) = \frac{\binom{r}{k}}{\binom{n}{k}} + \frac{\binom{n-r}{k}}{\binom{n}{k}} \geq \frac{2\binom{n/2}{k}}{\binom{n}{k}}.$$

Logo,

$$\mathbb{P}(\text{todas arestas não são monocromáticas}) = \left(1 - \frac{2\binom{n/2}{k}}{\binom{n}{k}}\right)^m.$$

Note que, vale

$$2^n \left(1 - \frac{\binom{n/2}{k}}{\binom{n}{k}}\right)^m > 1$$

se $m >?$.

Problem 4

Show that any finite set A of integers contains a sum-free subset of size at least $|A|/3$.

Solution. Mergulha em $\mathbb{Z}/p\mathbb{Z}$, para p primo suficientemente grande.

Escolhe $x \in \mathbb{Z}/p\mathbb{Z}$ aleatório. Calcula o esperado de elementos de xA que são $\{p/3, \dots, 2p/3\}$.

Problem 5

Problem 6

Let $(e_1, \ldots, e_m)$ be an arbitrary ordering of the edges of a graph G on n vertices. Show that there exists an incresing walk (in this ordering) of length at least $d = 2m/n$.

Sketch. Coloca uma pessoa em cada vértice.

Visite as arestas da menor para a maior, cada pessoa em um endpoint dessa aresta anda pro outro endpoint.

No total, tiveram $2m$ passos.

Por P.C.P., alguma pessoa andou pelo menos $2m/n$ passos.

Problem 7

Sketch. Troca cada intersecção por um vértice. Agora o grafo é planar.