Combinatorics I, Exam 1

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Problem 1

Let G be a planar graph with no cycles of length 3, 4 or 5. Show that $\chi(G) \leq 3$.

Solution. We will prove the result by induction on $v(G)$. If $v(G) \leq 3$, we are good!

Let G^* be a graph obtained by taking G and adding edges between distinct connected components until it becomes connected. Since G^* is planar and connected, Euler's formula for planar graphs implies $v(G^*) - e(G^*) + f(G^*) = 2$.

Adding edges between distinct connected components does not create new faces, so $f(G) = f(G^*)$. Also, $e(G) \leq e(G^*)$ and $v(G) = v(G^*)$; thus

$$v(G) - e(G) + f(G) \geq 2.$$

By double conting edges, and using that each face has at least 6 edges, we conclude

$$2e(G) \geq 6f(G).$$

The last two equations imply

$$\sum_{v \in V(G)} \deg(v) = 2e(G) \geq 3(v(G) - 2),$$

therefore there exists $u \in V(G)$ such that $\deg(u) \leq 2$.

Induction hypothesis implies that there exists a proper 3-colouring of the vertices of $G - \{u\}$. Since u has at most two neighbors, there exists a colour that does not appear on its neighbors. Define this as the colour of u. Now, we have a proper 3-colouring of G; thus $\chi(G) \leq 3$.

Theorem 1 (Euler's formula for planar graphs)

If G is a planar and connected graph, $v(G) - e(G) + f(G) = 2$.

Proof. Induction on $e(G)$. If $e(G) = 0$, then $v(G) = 1$ and $f(G) = 1$, so we are good! If $e(G) > 0$:

1. **If there is a leaf v,** then we define G' by taking v away.

$$v(G') = v(G) - 1, \qquad e(G') = e(G) - 1, \qquad f(G') = f(G).$$
$$v(G') - e(G') + f(G') = 2 \implies v(G) - e(G) + f(G) = 2.$$

2. **If there is no leaf,** there is a cycle, then we define G' by taking away an edge from the cycle.

$$v(G') = v(G), \qquad e(G') = e(G) - 1, \qquad f(G') = f(G) - 1.$$
$$v(G') - e(G') + f(G') = 2 \implies v(G) - e(G) + f(G) = 2.$$

Problem 2

Prove that

$$e(G) \geq \binom{\chi(G)}{2}.$$

Solution. We will prove, by induction on $v(G)$, that if $e(G) < \binom{r}{2}$, then there exists a proper $(r-1)$-colouring of the vertices of G.

If $v(G) \leq r-1$, we can assign each vertex a colour; so we are good!

Suppose $v(G) \geq r$. Since

$$\sum_{v \in V(G)} \deg(v) = 2e(G) < r(r-1),$$

there exists a vertex u such that $\deg(u) < r-1$.

Induction hypothesis implies that there exists a proper $(r-1)$-colouring of the vertices of $G - \{u\}$. Since u has at most $r-2$ neighbors, there exists a colour that does not appear on its neighbors. Define this as the colour of u. Now, we have a proper $(r-1)$-colouring of G.

Wrapping everything up, by definition, there is no proper $(\chi(G)-1)$-colouring of the vertices G, so

$$e(G) \geq \binom{\chi(G)}{2}.$$

Problem 3

Let H be a k-uniform hypergraph with m edges. Show that if $m < \frac{4^{k-1}}{3^k}$, then there exists a 4-colouring of the vertices of H such that every edge contains all four colours.

Solution. Pick a random 4-colouring $c\colon V(H) \to \{R, G, B, Y\}$, with $\mathbb{P}(c(v) = R) = \mathbb{P}(c(v) = G) = \mathbb{P}(c(v) = B) = \mathbb{P}(c(v) = Y) = \frac{1}{4}$.

For any $e \in E(H)$,

$$\mathbb{P}(e \text{ does not have four colours}) \underset{\substack{\uparrow \\ \text{union bound}}}{\leq} \binom{4}{3}\frac{3^k}{4^k} = \frac{3^k}{4^{k-1}}.$$

Thus, if $m < \frac{4^{k-1}}{3^k}$,

$$\mathbb{P}(\exists e \in E(H) : e \text{ does not have four colours}) \underset{\substack{\uparrow \\ \text{union bound}}}{\leq} m\frac{3^k}{4^{k-1}} < 1.$$

Finally, this implies that, with positive probability, all edges $e \in E(H)$ have four colours. Thus, there exists a 4-colouring of the vertices of H such that every edge contain all four colours.

Problem 4

Use Van der Waerden's theorem to show that, for every r and k, there exists a constant $\delta = \delta(r, k) > 0$ such that the following holds for all sufficiently large n:

In every colouring of $\{1, \ldots, n\}$ with r colours, there are at least δn^2 monochromatic k-term arithmetic progressions.

Solution. Define $AP_x(a, d) := \{a, a + d, \ldots, a + (x - 1)d\}$.

Let T be the number of pairs (S_1, S_2) such that:

- S_1 is monochromatic;
- $S_1 \subset S_2 \subset [n]$;
- $S_1 = AP_k(a, d)$, for some a, d;
- $S_2 = AP_{W(r,k)}(a', d')$, for some a', d'.

Since each $S_1 = AP_{W(r,k)}(a', d')$ has size $W(r, k)$, it has a monochromatic k-term arithmetic progression. Thus

$$T \geq \#\big(AP_{W(r,k)}(a', d')\big) = \#\big((a', d') : a + (W(r, k) - 1)d \leq n\big) \geq \frac{n^2}{2W(r, k)}.$$

On the other hand, for each $S_1 = AP_k(a, d)$, let's count the number of $S_2 = AP_{W(r,k)}(a', d')$ such that $S_1 \subset S_2$. Since $S_1 \subset S_2$, d' divides d. Define $x := \frac{d}{d'} \leq \frac{W(r,k)}{k}$. For a fixed positive integer $x \leq \frac{W(r,k)}{k}$, there are at most $W(r, k)$ arithmetic progressions S_2 that work.

Thus, for a fixed S_1, there are at most $\frac{W(r,k)^2}{k}$ arithmetic progressions S_2 such that $S_1 \subset S_2$.

Finally,

$$\frac{n^2}{2W(r, k)} \leq T \leq \frac{W(r, k)^2}{k} \cdot \#\big(\text{monochromatic } AP_k(a, d)\big),$$

which implies that, for $\delta := \frac{k}{2W(r,k)^3}$,

$$\#\big(\text{monochromatic } AP_k(a, d)\big) \geq \delta n^2.$$

Problem 5

Prove that $r(K_r, T) = (r-1)(k-1)+1$ for every tree T with k vertices.

Solution. We will prove the statement using induction on r.

Consider the graph G with $r-1$ disjoint red copies of K_{k-1}, and blue edges between any two vertices in distinct K_{k-1}. The graph G contains no blue K_r and no red T. Thus,

$$r(K_r, T) > (r-1)(k-1).$$

Let $n \geq (r-1)(k-1)+1$. We will show that any colouring of K_n contains either a blue K_r or a red T.

Suppose there is a colouring of K_n that does not have a blue K_r or a red T. Let $N_B(v)$ and $N_R(v)$ denote the blue neighborhood and red neighborhood of v, respectively. There is no blue K_{r-1} nor a red T in $N_B(v)$. Thus, using the induction hypothesis, $|N_B(v)| \leq (r-2)(k-1)$, which implies $|N_R(v)| \geq k-1$, for all v. By Lemma 2, there exists a red copy of T in K_n; a contradiction.

Therefore,

$$r(K_r, T) \leq (r-1)(k-1)+1,$$

which implies

$$r(K_r, T) = (r-1)(k-1)+1.$$

Lemma 2 (Problem 3 from List 1)

If T is a tree with k vertices and G is a graph with minimum degree $k-1$, then $T \subset G$.

Proof. We'll use induction on k. If $k=1$, we're done!

Pick a leaf v of T. Its unique edge connects it to u. Let T' be the tree without v. By induction, there is a copy $C_{T'}$ of T' in G. Let c_u be the copy of u in $C_{T'}$. Since $\deg(c_u) \leqslant k-2$ in $C_{T'}$, but $\deg(c_u) \geqslant k-1$ in G, there is some vertex that is connected to u outside of $C_{T'}$, say c_v. Thus, let C_T be $C_{T'}$, adding c_v. C_T is a copy of T inside G.

Problem 6

Show that, for every $p = p(n)$,

$$\chi(G(n,p)) \geq \frac{pn}{2\log n},$$

with high probability as $n \to \infty$.

Solution. Let $S \in \binom{[n]}{k}$, for $k \approx \frac{2\log n}{p}$. Then,

$$\mathbb{P}(S \text{ is independent}) = (1-p)^{\binom{k}{2}}.$$

Therefore,

$$\begin{aligned}
\mathbb{P}\left(\exists S \in \binom{[n]}{k} : S \text{ is independent}\right) &\underset{\substack{\uparrow \\ \text{union bound}}}{\leq} \binom{n}{k}(1-p)^{\binom{k}{2}} \\
&\leq \left(\frac{en}{k}\exp\left(-\frac{p(k-1)}{2}\right)\right)^k \\
&\leq \left(\frac{enp}{\log n}\exp\left(-\log n + \frac{p}{2}\right)\right)^k \\
&\leq \left(\frac{e^{3/2}}{\log n}\right)^k \to 0, \text{ as } n \to \infty.
\end{aligned}$$

Thus, with high probability, there is no independent set of size $k \approx \frac{2\log n}{p}$, i.e., with high probability,

$$\alpha(G(n,p)) \leq \frac{2\log n}{p}.$$

Since, for any graph G, $\chi(G)\alpha(G) \geq n$, we conclude that, with high probability,

$$\chi(G(n,p)) \geq \frac{np}{2\log n}.$$

Lemma 3

For any graph G with n vertices,

$$\chi(G)\alpha(G) \geq n.$$

Proof. By definition, there exists a partition of $V(G)$ into $\chi(G)$ independent sets. Each of those indepentent sets has size at most $\alpha(G)$. Therefore,

$$\chi(G)\alpha(G) \geq n.$$

Problem 7

Let $r_k(H)$ be the smallest n such that every colouring of $E(K_n)$ with k colours contains a monochromatic copy of H. Show that

$$k^{1+c} \leq r_k(C_4) \leq C \cdot k^2$$

for some constants $C > c > 0$ and all sufficiently large k.

Solution for the upper bound. Suppose $c\colon E(K_n) \to [k]$ produces no monochromatic C_4. By counting monochromatic cherries, we conclude

$$\frac{n(n-1)(n-k-1)}{2k} = \sum_v \left(k \binom{\frac{n-1}{k}}{2} \right) \leq \sum_v \sum_{\text{colour } i} \binom{d_i(v)}{2} \underset{\substack{\uparrow \\ \text{counting per vertex}}}{\leq} \#\text{monochromatic cherries} \underset{\substack{\uparrow \\ \text{counting per pair}}}{\leq} \binom{n}{2} k.$$

This implies $n \leq k^2 + k + 1$. Therefore, for any fixed $C > 1$, for all sufficiently large k,

$$r_k(C_4) \lesssim Ck^2.$$

Idea for the lower bound. Let $n = k^{1+\epsilon}$.

Let's colour the edges of K_n randomly and independently, with the probability of an edge being of a given colour is $\frac{1}{k}$. For a fixed colour, the graph of this colour is $G(n, \frac{1}{k})$.

The expected number of C_4 of a given colour is

$$\mathbb{E}\left[\# \left(C_4 \text{ in } G\left(k^{1+\epsilon}, \frac{1}{k} \right) \right) \right] \leq \frac{1}{2} k^{4\epsilon}.$$

Thus, the expected number of monochromatic C_4 is $\frac{1}{2}k^{1+4\epsilon}$.

We cannot remove a vertex from each monochromatic C_4, because $\frac{1}{2}k^{1+4\epsilon} \gg k^{1+\epsilon}$.

Problem 8

Define
$$\hat{r}(H) := \min\{e(G) : G \to H\},$$
where $G \to H$ means that every two-colouring of the edges of G contains a monochromatic copy of H. Prove that
$$\hat{r}(K_t) = \binom{R(t)}{2},$$
for every $t \in \mathbb{N}$.

Sketch. Note that, by the definition of $R(t)$, $K_{R(t)} \to K_t$. Therefore,
$$\hat{r}(K_t) \leq \binom{R(t)}{2},$$
for every $t \in \mathbb{N}$.

Conversely, let G be a graph with $e(G) < \binom{R(t)}{2}$. We want to show that there exists a 2-colouring of $E(G)$ that avoids a monochromatic copy of K_t.

> *Idea 1.* Let's try induction on $v(G)$.
>
> If $v(G) < R(t)$, then $G \subset K_{R(t)-1}$. By definition, there exists a 2-colouring of $E(K_{R(t)-1})$ that avoids monochromatic K_t; the restriction to $E(G)$ of this 2-colouring also avoids monochromatic K_t.
>
> If $v(G) \geq R(t)$, then $\sum_v \deg(v) = 2e(G) < R(t)(R(t)-1)$. Thus, there exists u such that $\deg(u) \leq R(t) - 1$. Apply the induction hypothesis on $G - \{u\}$. *Is there a smart way to colour the edges from u?*

> *Idea 2.* Let's pick a random 2-colouring of $E(G)$. *Maybe we can show*
> $$\mathbb{P}(\textit{monochromatic copy of } H \textit{ in } G) < 1?$$