

Combinatorics I Lecture Notes

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This is IMPA's master class Combinatorics 1, instructed by Robert Morris. All errors are my responsibility.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Google Meet and [YouTube videos](#). The recommended material can be found [here](#).

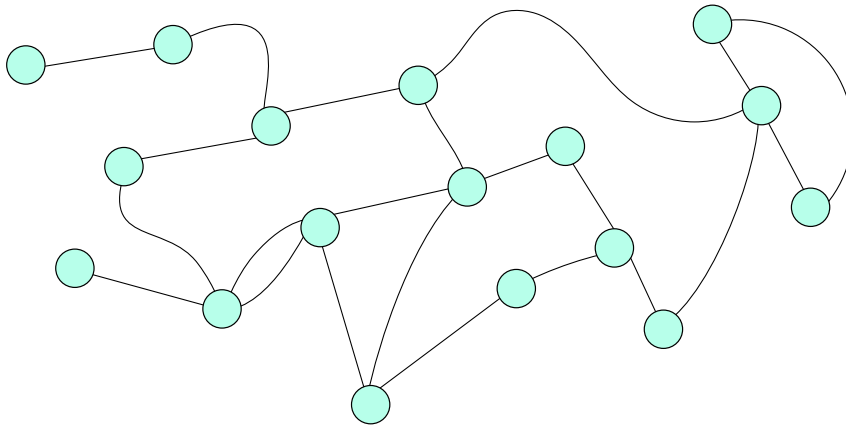


Figure 1: This is a graph.

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1 Which problems we'll study?

In this summer course, we'll study extremal, counting and probabilistic problems. Here are some examples:

Problem 1.1

Let A be a subset of $\{1, 2, \dots, 2n\}$ such that $a \nmid b$, for all $a \neq b \in A$.

How large can $|A|$ be?

Solution. $A = \{n+1, \dots, 2n\}$ is a good example. This yields $|A| = n$.

Consider the partition of $\{1, 2, \dots, 2n\}$ given by the following sets:

- $\{2^t\}$
- $\{3 \cdot 2^t\}$
- $\{5 \cdot 2^t\}$
- \vdots
- $\{(2n-1) \cdot 2^t\}$

There can't be two elements in the same set of the partition, so $|A| \leq n$.

Problem 1.2

Let A be a subset of $\{1, 2, \dots, 2n\}$ such that $a+b \neq c$, for all $a, b, c \in A$. We'll call such set *sum-free*.

How large can $|A|$ be?

Solution. $A = \{n+1, \dots, 2n\}$ is a good example. Another good example are the odd numbers. Both yield $|A| = n$.

Suppose $|A| \geq n+1$. Let $a = \max A$.

Consider the following partition with $\lfloor \frac{a}{2} \rfloor$ sets:

- $\{1, a-1\}$
- $\{2, a-2\}$
- \vdots
- $\{\lfloor \frac{a}{2} \rfloor, \lceil \frac{a}{2} \rceil\}$

There can't be two elements in the same set of the partition.

If $a \leq 2n-1$, then there are at most $n-1$ sets listed above, which implies $|A| \leq n$.

If $a = 2n$, then $n \notin A$, and then the $n-1$ first sets listed above cover A , thus $|A| \leq n$.

Theorem 1.1 (Schur, 1916)

Given $c: \mathbb{Z}_{>0} \rightarrow \{1, \dots, r\}$, there are x, y, z such that:

- $x + y = z$
- $c(x) = c(y) = c(z)$

Problem 1.3

How many sum-free sets are in $[n]$?

Conjecture 1.2 (Cameron and Erdős)

The number of sum-free sets in $[n]$ is $\leq C \cdot 2^{n/2}$.

2 Ramsey's Theory

Theorem 2.1 (Ramsey's Theorem)

If $c : \binom{\mathbb{N}}{2} \rightarrow \{1, \dots, r\}$, then there exists $A \subset \mathbb{N}$ infinite and monochromatic, i.e., such that $c(ab) = c$, for all $a, b \in A$.

Proof of Theorem 2.1. Let $S_0 = \mathbb{N}$.

For each i , do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1} . Since S_{i-1} is infinite and there are finitely many colors, there is some color that appears infinitely many times; we'll call this color c_i , and define $S_i = \{u \in S_{i-1} : c(\{v_i, u\}) = c_i\}$.

Now, we have an infinite sequence v_1, v_2, \dots , such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are finitely many colors, there is some color that appears in infinitely many c_i 's; call this color c , and define $A = \{v_i : c_i = c\}$.

The set A satisfies our condition.

Proof of Theorem 1.1. Given a coloring $c : \mathbb{N} \rightarrow \{1, \dots, r\}$, we define $c' : \binom{\mathbb{N}}{2} \rightarrow \{1, \dots, r\}$ by $c'(\{a, b\}) = c(b - a)$, for $b > a$.

By Theorem 2.1, there is A infinite and monochromatic. Pick $x < y < z \in A$, then we have $c(y - x) = c(z - y) = c(z - x)$, and $(y - x) + (z - y) = z - x$, so we're done!

Definition 2.2 (Ramsey Number)

Let $R(k)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n , i.e., for every $c : E(K_n) \rightarrow \{R, B\}$, there exists a monochromatic copy of K_k .

Let $R(s, t)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n , i.e., for every $c : E(K_n) \rightarrow \{R, B\}$, there exists a red copy of K_s or a blue copy of K_t .

Clearly, $R(k) = R(k, k)$.

Theorem 2.3

$$R(k) \lesssim 2^{2^k}.$$

Sketch. Let $n = 2^{2^k}$, and pick any coloring c of K_n . Let $S_0 = [n]$.

For each $i < 2k$ do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1} . There is some color that appears more times; we'll call this color c_i , and define $S_i = \{u \in S_{i-1} : c(\{v_i, u\}) = c_i\}$.

~~Note that, the size of S_m is at least $\frac{n}{2^m} \geq 1$.~~ This is not quite correct. At each step, we're taking one vertex away, and then dividing by two. We'll properly prove a better bound later.

Now, we have an sequence $v_1, v_2, \dots, v_{2k-1}$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are two colors, there is some color that appears at least k times; call this color c , and define $A = \{v_i : c_i = c\}$. The size of A is at least k . Pick any subset B of A that has exactly k elements.

The subgraph of K given by deleting all vertices but those in B is a monochromatic copy of K_k .

Lemma 2.4

$$R(s, t) \leq R(s - t, t) + R(s, t - 1).$$

Proof. Let $n = R(s, t) - 1$. By definition, there exists a coloring $c: E(K_n) \rightarrow \{R, B\}$ without a red K_s or a blue K_t .

Pick any vertex v . v is connected to some of the vertices through a red edge, which we'll put in the set S_R ; the others are connected to v through a blue edge, those we'll put in the set S_B .

Since there are no red K_s or blue K_t , there can't be any red K_{s-1} or blue K_t in S_R ; thus, $|S_R| \leq R(s-1, t)$. Analogously, $|S_B| \leq R(s, t-1)$.

Therefore,

$$\begin{aligned} R(s, t) - 1 &\leq R(s-1, t) - 1 + R(s, t-1) - 1 + 1 \\ R(s, t) &\leq R(s-1, t) + R(s, t-1). \end{aligned}$$

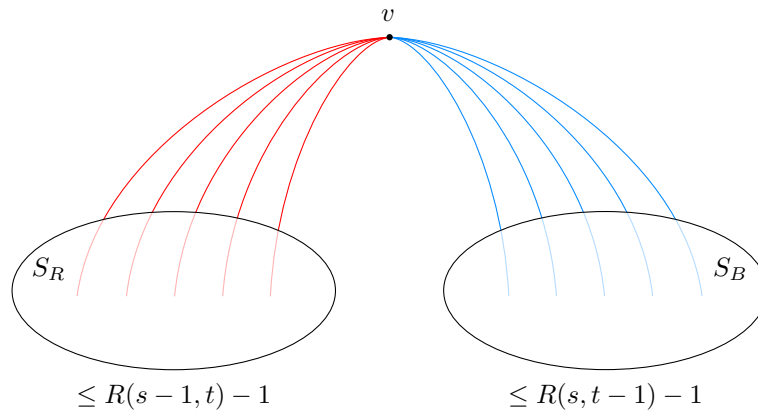


Figure 2: S_R and S_B .

Theorem 2.5

$$R(s, t) \leq \binom{s+t}{s}.$$

Proof. Follows from Lemma 2.4.

Theorem 2.6 (Erdős-Szekeres, 1935)

$$R(k) \leq \binom{2k}{k} \approx \frac{1}{\sqrt{k}} 4^k.$$

Lec. 2

Let's now try to find a lower bound. It is very difficult to show a good construction. Luckily, we are not going to do that.

Theorem 2.7 (Erdős, 1947)

$$\sqrt{2}^k \leq R(k)$$

Proof. Let $n \leq \sqrt{2}^k$. Let's choose colors randomly. Let

$$\mathbb{P}(c(e) = R) = \frac{1}{2},$$

for every edge e in K_n , independently.

We want to show that

$$\mathbb{P}(\text{there is not a monochromatic copy of } K_k) > 0.$$

Let X be the number of monochromatic copies of K_k in c . Then,

$$\begin{aligned} \mathbb{E}[X] &= \mathbb{E} \left[\sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{1}[S \text{ is monochromatic}] \right] \\ &= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{E}[\mathbb{1}[S \text{ is monochromatic}]] \\ &= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{P}(S \text{ is monochromatic}) \\ &= \sum_{\substack{S \subset K_n \\ S \text{ is a copy of } K_k}} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\ &= \binom{k}{n} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\ &\leq 2 \left(\frac{en}{k}\right)^k \left(\frac{1}{2}\right)^{\frac{k(k-1)}{2}} \\ &\leq 2 \left(\frac{e\sqrt{2}}{k}\right)^k \\ &< 1, \text{ for } k \geq 5. \end{aligned}$$

Therefore, since $\mathbb{E}[X] < 1$, we have $\mathbb{P}(X = 0) > 0$.

The bounds have not improved much since then

Theorem 2.8 (Conlon, 2009)

$$R(k) \leq \frac{4^k}{k^{\sqrt{\log k}}}$$

3 Extremal Graph Theory

3.1 Complete Graphs

Definition 3.1

Let $\text{ex}(n, H)$ be the maximum number of edges a graph $G \subset K_n$ can have such that there are no copies of H in G .

Theorem 3.2 (Mantel, 1907)

$$\text{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. The example is the bipartite graph with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices.

Let's prove by induction on n .

Now, suppose G does not have a triangle. Pick an edge uv . Let G' be the graph G deleting u and v . The subgraph G' also does not contain triangles, so $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Notice that cannot exist $w \in G'$ such that uw and vw are edges of G , because G does not have triangles. Therefore, there can be at most $n - 2$ edges from u or v to vertices on G' . Including the edge uv , we conclude that

$$\begin{aligned} e(G) &\leq e(G') + n - 1 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

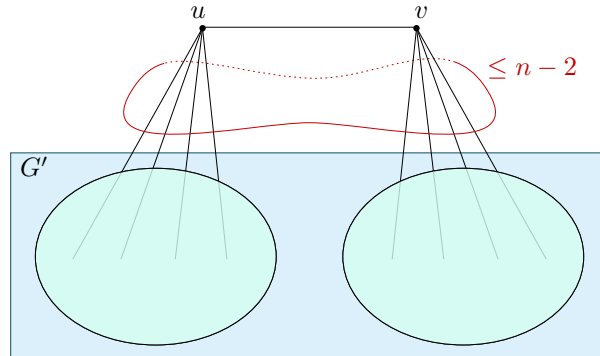


Figure 3: Edge uv on a triangle-less graph.

Definition 3.3 (Turán's Graph)

The graph $T_r(n)$ consists of r sets with roughly n/r elements each (some rounded up, some rounded down).; we create an edge uv if, and only if, u and v are on different sets.

We'll denote by $t_r(n)$ the number of edges in $T_r(n)$.

Theorem 3.4 (Turán, 1941)

$$\text{ex}(n, K_{r+1}) = t_r(n) \approx \left(1 - \frac{1}{r}\right) \binom{n}{2}.$$

Proof. We'll use induction on n . For $n \leq r$, we're good.

Pick a maximal graph G that doesn't have a copy of K_{r+1} . Pick a copy of K_r , let's call it H . Define $G' = G - H$. Of course, G' has no copies of K_r ; thus $e(G') \leq t_r(n - r)$, by induction.

Futhermore, if $v \in G'$, there can be at most $r - 1$ edges connecting v to some vertex in H .

Wrapping everything up, we have

$$\begin{aligned} e(G) &\leq e(G') + (n - r)(r - 1) + \binom{r}{2} \\ &\leq t_r(n - r) + (n - r)(r - 1) + \binom{r}{2} \\ &\leq t_r(n). \end{aligned}$$

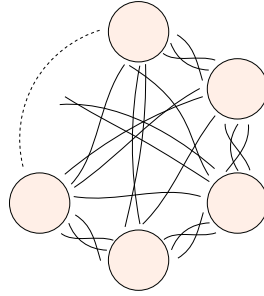


Figure 4: Turán's Graph

3.2 Bipartite Graphs

Theorem 3.5 (Erdős, 1935)

$$\text{ex}(n, C_4) \leq \frac{n^{3/2}}{2}.$$

Proof. Let's count cherries! A *cherry* is a pair $(v, \{u, w\})$, in which vu and vw are edges of the graph.

Since there is no C_4 , there is at most one cherry for each pair $\{u, w\}$. This implies that:

$$\begin{aligned} \binom{n}{2} &\geq \#(\text{cherries}) = \sum_{v \in G} \binom{d(v)}{2} \\ &\geq n \binom{\frac{2e(G)}{n}}{2}. \end{aligned}$$

Solving this quadratic inequation on $e(G)$ yields to

$$e(g) \geq \frac{n^{3/2}}{2}.$$

Question 3.1

For which graphs we have

$$\text{ex}(n, H) = \Theta(n^2)?$$

Proposition 3.6

For every non-bipartite graph H , we have

$$\text{ex}(n, H) \geq \frac{n^2}{4}.$$

Proof. Take G as the complete bipartite graph with n vertices. It has roughly $\frac{n^2}{4}$ edges and it cannot contain a non-bipartite graph.

Theorem 3.7 (Kővári–Sós–Turán, 1954)

Let H be a bipartite graph. Then,

$$\text{ex}(n, H) = o(n^2).$$

Proof. Since H is bipartite, there is some $K_{s,t}$ such that $H \subset K_{s,t}$. Then,

$$\text{ex}(n, H) \leq \text{ex}(n, K_{s,t}).$$

Let's bound $\text{ex}(n, K_{s,t})$.

We'll count s -cherries: (v, S) , in which S has size s and $vx \in E(G)$ for all $x \in S$.

There are at most $t - 1$ s -cherries for each subset S with size s . This implies that:

$$\begin{aligned} (t-1) \binom{n}{s} &\geq \#(\text{cherries}) = \sum_{v \in G} \binom{d(v)}{s} \\ &\geq n \binom{\frac{2e(G)}{n}}{s} \geq \frac{e(G)^s}{s^s \cdot n^{s-1}}. \end{aligned}$$

This implies that, for some constant C ,

$$e(G) \leq C \cdot n^{2-\frac{1}{s}}$$

Question 3.2

For which H it holds that

$$\text{ex}(n, H) = O(n)?$$

3.3 Trees

Lec. 3

Definition 3.8 (Tree)

A tree is a connected graph that has no cycles.

Proposition 3.9

Given a graph G , the following are equivalent:

- (i) G is a tree;
- (ii) G is a maximal graph without cycles, i.e., G does not have cycles and there is no graph $H \supset G$ such that H does not have cycles;
- (iii) G is a minimal connected graph, i.e., G is connected and there is no graph $H \subset G$ such that H is connected.

Theorem 3.10

Let T be a graph with k vertices. Then,

$$\frac{(k-2)}{2}n \leq \text{ex}(n, T) \leq (k-1) \cdot n.$$

Proof of the lower bound. Pick $\frac{n}{k-1}$ disjoint $k-1$ -cliques. There cannot be a copy of a connected graph with k vertices inside this graph. This graph has roughly

$$\binom{k-1}{2} \frac{n}{k-1} = \frac{k-2}{2}n$$

edges.

Proof of the upper bound. Let's start with a lemma.

Lemma 3.11

Let G be a graph with mean degree d , then, there exists a subgraph $G' \subset G$ with minimum degree at least $\frac{d}{2}$.

Proof. While there are vertices with degree smaller than $\frac{d}{2}$, throw them away.

If we stopped before throwing away all vertices, we're done. Suppose we threw away all vertices. At each step, we threw away at most $\frac{d}{2}$ edges. Since we threw away all edges, this means $n \cdot \frac{d}{2} < e(G) = n \frac{d}{2}$; a contradiction.

Lemma 3.12

Let G be a graph with $\delta(G) \geq k-1$. Then, there is a copy of T in G for every tree T with k vertices.

Proof. We'll use induction on k . If $k=1$, we're done!

Pick a leaf v of T . Its unique edge connects it to u . Let T' be the tree without v . By induction, there is a copy $C_{T'}$ of T' in G . Let c_u be the copy of u in $C_{T'}$. Since $\deg(c_u) \leq k-2$ in $C_{T'}$, but $\deg(c_u) \geq k-1$ in G , there is some vertex that is connected to u outside of $C_{T'}$, say c_v . Thus, let C_T be $C_{T'}$, adding c_v . C_T is a copy of T inside G .

Finally, $e(G) = (k-1)n \implies \bar{d}(G) = 2(k-1) \implies$ there exists a subgraph $G' \subset G$ such that $\delta(G') \geq k-1 \implies T \subset G'$.

Conjecture 3.13 (Erdős-Sós, 1960's)

$$\text{ex}(n, T) \leq \frac{(k-2)n}{2}$$

Definition 3.14 (Random graph of Erdős-Rényi)

We define $G(n, p)$ as a random distribution of graphs with n vertices, with

$$\mathbb{P}(e \in E(G(n, p))) = p,$$

chosen independently.

Lemma 3.15 (Markov's inequality)

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

Proof. Left to the reader. Use the definition of $\mathbb{E}[X]$.

Theorem 3.16

$$\text{ex}(n, C_t) \geq O\left(n^{1+\frac{1}{2k-1}}\right) \gg n.$$

Proof. Let $t = 2k$. We want to choose $p = p(n)$ such that:

- $e(G(n, p)) \gg n$;
- $C_{2k} \not\subset G(n, p)$.

$$\mathbb{E}[e(G(n, p))] = p \binom{n}{2}.$$

Moreover, $e(G(n, p))$ is a binomial distribution, therefore, $e(G(n, p)) \approx np^2$ with high probability. Thus, we should pick $p \gg 1/n$, i.e., $pn \rightarrow \infty$.

Define X as the number of copies of C_{2k} in $G(n, p)$.

$$\begin{aligned} \mathbb{E}[X] &= \sum_{\substack{\text{copies } S \text{ of} \\ C_{2k} \text{ in } K_n}} \mathbb{P}(S \subset G(n, p)) \\ &\approx n^{2k} p^{2k} = (pn)^{2k}. \end{aligned}$$

Let $0 < \varepsilon < \frac{1}{2k-1}$, and define $p = p(n) = n^{-1+\varepsilon}$. Then, we have $pn \gg n^{-1}$ and $(pn)^{2k} \ll pn^2$. Therefore, each of the following happen with high probability:

- $e(G(n, p)) \approx pn^2$;
- The number of copies of C_{2k} in $G(n, p) \approx (pn)^{2k}$.

Therefore, the intersection also occurs with high probability. Pick a graph G in the intersection.

For each of the $(pn)^{2k}$ cycles in G delete an edge in it; call this new graph G' . Thus $e(G') \approx pn^2 - (pn)^{2k} \approx n^{1+\varepsilon}$, and G' has no C_{2k} .

Theorem 3.17

$$\text{ex}(n, H) = O(n) \iff H \text{ does not have cycles.}$$

Proof. All the work has been done. The proof, which is simply a jigsaw puzzle, is left to the reader.

4 Planar graphs

Definition 4.1 (Planar Graph)

A planar graph is a graph that can be drawn on the plane without having crossing edges. Edges may not be straight.

Lemma 4.2 ($V + F - E = 2$)

Let G be a planar connected graph. For any planar drawing of G , we have

$$v(G) + f(G) - e(G) = 2.$$

Watch an animated version of this classic demonstration at [3Blue1Brown](#).

Theorem 4.3

Let G be a planar graph with n vertices. Then

$$n$$