Combinatorics I, List 1

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Problem 1

By defining, and calculating the expectation of, a suitable random variable, show that every graph G has a bipartite subgraph with at least $e(G)/2$ edges.

Sketch. Let's pick a random bipartition (A, B) of $V(G)$, with probability $1/2$ of $v \in A$, chosen independently.

Define
$$X = \#\left(a \sim b : a \in A \text{ and } b \in B\right).$$

Problem 2

Show that $R(3,4) \leqslant 9$, $R(4,4) \leqslant 18$ and $R(3,3,3) \leqslant 17$.

Solution.

Problem 3

Show that every graph of average degree d contains a subgraph of minimum degree at least $d/2$. Deduce that $\mathrm{ex}(n,T) \leqslant (k-1)n$ for every tree T with k vertices.

Solution.

Problem 4

Show that if T is a tree with k vertices and G is a graph with minimum degree $k-1$, then $T \subset G$. Deduce that $r(K_3, T) = 2k-1$.

Sketch. Lower bound: Two blobs with $k-1$. Blue edge iff two vertices are in the same blob.

Upper bound: Show that $d_R(v) \leq k-1$. Therefore, $d_B(v) \geq k-1$.

Problem 5

Let $T_1, \ldots, T_k$ be subtrees of a tree T, any two of which have at least one vertex in common. Prove that there is a vertex in all the T_i.

Sketch. Induction on k. Merge two trees $T_1, T_2 \subset T$ into one tree $T' \subset T$ that contains all vertices that were in T_1 and T_2.

Problem 6

Let $R_r(3)$ denote the r-colour Ramsey number of a triangle. Show that

$$2^r \leqslant R_r(3) \leqslant 3 \cdot r!.$$

Show moreover that $R_r(3) \leqslant 5^{r/2}$.

Sketch. Recursion. Pick a vertex and look to the blobs for each edge color.

Problem 7

Let $g(n)$ be the largest integer such that there exists a graph with the following properties: $|V(G)| = n$, $e(G) = m$, and it is possible to red-blue colour the edges of G without creating a monochromatic triangle.

Show that $g(n)/\binom{n}{2}$ converges, and find c such that $g(n)/\binom{n}{2} \to c$ as $n \to \infty$.

Sketch. Lower bound: Turan's graph with 5 blobs.

Upper bound: If there are more edges than Turan with 5 blobs, then there is a K_6, which implies there is a monochromatic triangle.

Problem 8

Recall that $\alpha(G)$ denotes the size of the largest independent set in G. Show that, for every graph G,

$$\alpha(G) \geqslant \sum_{v \in v(G)} \frac{1}{d(v)+1}.$$

Sketch. Let $V(G) = [n]$. Pick a random permutation $\pi\colon [n] \to [n]$. For each $v \in [n]$, define

$$A_v := \{u \in V(G) : u \sim v \text{ and } \pi(v) < \pi(u)\}.$$

Problem 9

Let $C(s)$ be the smallest n such that every connected graph on n vertives has, as an *induced* subgraph, either a complete K_s, a star $K_{1,s}$ or a path P_s of lenfth s.

Show that $C(s) \leqslant R(s)^s$, where $R(S)$ is the Ramsey number of s.

Solution.

Problem 10

Prove that $R(3, k) \geqslant k^{1+c}$ for some $c > 0$.

Sketch.

$$X_k = \#(K_k \subset G(n,p))$$
$$X_3 = \#(K_k \subset \overline{G(n,p)})$$

Problem 11

Show that there is an infinite set S of positive integers such that the sum of any two distinct elements of S has an even number of distinct prime factors.

Sketch. Let $S = \{n : n \equiv 1 \pmod 3\}$.

Let $x \sim y$ be blue iff $x + y$ has a even number of distinct prime factors. Ramsey's Theorem implies that there is an infinite. If it's blue, we're done! If it's red, then multiply every number by 3. (a new prime factor!)

Problem 12

Suppose we are given n points and n lines in the plane. Show that there are at most $O(n^{3/2})$ point-line incidences.

Solution.