Combinatorics I, Exam 1

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Problem 1

Show that every graph G gas a bipartite subgraph with at least $e(G)/2$ edges.

Solution for Problem 1. Let A be a random subset of $V(G)$, with the distribution given by $\mathbb{P}(v \in A) = \frac{1}{2}$, for all $v \in V(G)$, chosen independently. Let $B = V(G) \setminus A$. Let $E(A, B)$ be the set of edges of $E(G)$ with endpoints on A and B. Let $e(A, B) = |E(A, B)|$. Then,

$$\begin{aligned} \mathbb{E}\left[e(A, B)\right] &= \sum_{e \in E(G)} \mathbb{P}(e \in E(A, B)) \\ &= \sum_{e \in E(G)} \frac{1}{2} \\ &= \frac{e(G)}{2}. \end{aligned}$$

Therefore, there exists a choice of A such that $e(A, B) \geqslant \frac{e(G)}{2}$, i.e., the biparitite subgraph of G given by the edges of G from A to B has at least $e(G)/2$ edges.

Problem 2

Show that, for every graph H, there exists $\delta > 0$ such that the following holds for all sufficiently large n:

If G is a graph on n vertices with $e(G) > (1-\delta)\binom{n}{2}$, then in every r-colouring of $E(G)$ there are at least $\delta n^{v(H)}$ monochromatic copies of H.

Remark. For the next solution, I consulted my own notes to Problems 1 and 2 from List 2.

Solution for Problem 2. Let $r = R_r(H)$. For any $\delta < \frac{1}{r-1}$, then

$$e(G) > (1-\delta)\binom{n}{2} > \left(1 - \frac{1}{r-1}\right)\binom{n}{2} + o(n^2).$$

Thus, by Lemma 1, there exists ε (note that ε does not depend on δ) such that there are at least εn^r copies of K_r in G. By definition of r, there exists a monochromatic copy of H inside each K_r.

Bu double-counting (monochormatic copy of H, copy of K_r) with $H \subset K_r$, we have

$$\varepsilon n^r \underset{\substack{\uparrow \\ \text{counting per } K_r}}{\leq} \#(H, K_r) \underset{\substack{\uparrow \\ \text{per } H}}{\leq} n^{r-v(H)} \#(H)$$
$$\epsilon n^{v(H)} \leqslant \#(H).$$

Finally, pick $\delta < \frac{1}{r-1}$ and $\delta < \varepsilon$, then for any graph with $v(G) = n$ sufficiently large and $e(G) > (1-\delta)\binom{n}{2}$, there are at least $\delta n^{v(H)}$ monochromatic copies of H.

Lemma 1

A graph with $o(n^r)$ of K_r has at most $\mathrm{ex}(n, K_r) + o(n^2)$ edges.

Proof of Lemma 1. Let $t = 2(r-1)$. Let $\mathcal{A} = \{T \subset V(G) : |T| = t \text{ and } K_r \subset G[T]\}$.

Since there are $o(n^r)$ copies of K_r, and each copy of K_r is in at most n^{t-r} sets in $\mathcal{A}$, by double-counting (copy of K_r, T) with $K_r \subset G[T]$, we get

$$|\mathcal{A}| \leqslant o(n^r) \cdot n^{t-r} = o(n^t).$$

By double-counting (T, e) with $T \subset V(G)$, $|T| = t$ and $e \in E(G[T])$, we get

$$e(G)\binom{n-2}{t-2} \underset{\substack{\uparrow \\ \text{counting per } e}}{=} \#(T, e) \underset{\substack{\uparrow \\ \text{per } T}}{\leq} \underbrace{o(n^t)\binom{t}{2}}_{T \in \mathcal{A}} + \underbrace{\binom{n}{t}\mathrm{ex}(t, K_r)}_{T \notin \mathcal{A}}.$$

By (Turán, 1941), $\mathrm{ex}(n, K_r) = t_{r-1}(n) \approx (1 - \frac{1}{r-1})\binom{n}{2}$. Since t is a multiple of $(r-1)$, we also have $\mathrm{ex}(t, K_r) = t_{r-1}(t) = (1 - \frac{1}{r-1})\binom{t}{2}$. Thus,

$$\mathrm{ex}(t, K_r) \approx \frac{\binom{t}{2}}{\binom{n}{2}} \mathrm{ex}(n, K_r).$$

Back to the inequality,

$$
\begin{aligned}
e(G)\binom{n-2}{t-2}\binom{n}{2} &\leqslant o(n^t)\binom{t}{2}\binom{n}{2} + \binom{n}{t}\binom{t}{2}\operatorname{ex}(n,K_r) \\
e(G)\binom{n}{t}\binom{t}{2} &\leqslant o(n^t)\binom{t}{2}\binom{n}{2} + \binom{n}{t}\binom{t}{2}\operatorname{ex}(n,K_r) \\
e(G)\binom{n}{t} &\leqslant o(n^t)\binom{n}{2} + \binom{n}{t}\operatorname{ex}(n,K_r) \\
e(G) &\leqslant o(n^t)\frac{\binom{n}{2}}{\binom{n}{t}} + \operatorname{ex}(n,K_r) \\
e(G) &\leqslant o(n^2) + \operatorname{ex}(n,K_r).
\end{aligned}
$$

Problem 3

For every $\varepsilon > 0$, prove that

$$k^{2-\varepsilon} \leqslant R(4,k) \leqslant k^3,$$

for all sufficiently large k.

Solution for the upper bound. For any k, $R(2,k) = k$, since, on K_k there must be a blue edge or all other edges are red, forming a red K_k.

I will show by induction that, for $k \geqslant 3$, $R(3,k) \leqslant \frac{2}{3}k^2$.

Proof. For $k = 3$, $R(3,3) = 6 \leqslant \frac{2}{3}3^2$.

For $k > 3$,

$$\begin{aligned} R(3,k) &\leqslant R(3,k-1) + R(2,k) \\ &\leqslant \frac{2}{3}(k-1)^2 + k = \frac{2k^2 - k + 2}{3} \\ &\leqslant \frac{2}{3}k^2. \end{aligned}$$

Now, I will show by induction that, for $k \geqslant 3$, $R(4,k) \leqslant \frac{1}{3}k^3$.

Proof. For $k = 3$, $R(4,3) = 9 \leqslant \frac{1}{3}3^3$.

For $k > 3$,

$$\begin{aligned} R(4,k) &\leqslant R(4,k-1) + R(3,k) \\ &\leqslant \frac{1}{3}(k-1)^3 + \frac{2}{3}k^2 = \frac{1}{3}(k^3 - k^2 + 3k - 1) \\ &\leqslant \frac{1}{3}k^3. \end{aligned}$$

Remark. For the next solution, I consulted my own solution to Problem 10 from List 1.

Solution for the lower bound. I shall prove that $R(4,k) \geqslant O(k^\alpha)$, for any $\alpha < 2$.

Let $n = k^\alpha, p = k^{-1+\varepsilon}$, with $0 < \varepsilon < 1 - \frac{1}{2}\alpha$. Define the random variables

$$X = \#(K_4 \subset G(n,p))$$

$$Y = \#(K_k \subset \overline{G(n,p)})$$

The expected value of X is

$$\begin{aligned} \mathbb{E}[X] &= \binom{n}{3}p^3 \\ &= O(n^3p^3) \\ &= O(k^{4\alpha+6\varepsilon-6}) \\ &= o(k^\alpha), \end{aligned}$$

thus, $X = o(k^\alpha)$ with high probability.

The expected value of Y is

$$\begin{aligned}
\mathbb{E}[Y] &= \binom{n}{k}(1-p)^{\binom{k}{2}} \\
&\leqslant \left(\frac{en}{k}(1-p)^{\frac{k-1}{2}}\right)^k \\
&\leqslant \left(ek^{\alpha-1}\exp\left(-p\frac{k-1}{2}\right)\right)^k \\
&\approx \left(\frac{ek^{\alpha-1}}{\exp(k^{\varepsilon})}\right)^k \to 0,
\end{aligned}$$

as $k \to \infty$; thus $Y = 0$ with high probability.

Therefore, $X = o(k^\alpha)$ and $Y = 0$ with high probability; which implies there is a graph G, with k^α vertices, with $t = o(k^\alpha)$ copies of K_4 and with no copy of $K_k \subset \overline{G}$.

Let's create G', with $k^\alpha - t = O(k^\alpha)$ vertices, by removing one vertex from each of the t copies of K_4 from G. The graph G' has no copies of K_4, no copy of $K_k \subset \overline{G}$, and has $O(k^\alpha)$ vertices. Thus,

$$R(4,k) \geqslant O(k^\alpha).$$

Problem 4

Prove that for any constant $0 < c < 1/3$, the threshold for the event that $G(n,p)$ contains a collection of cn vertex-disjoint triangles is $n^{-2/3}$.

Proposition 2

If $p \ll n^{-2/3}$, i.e., $p^3n^2 \to 0$, then $\mathbb{P}(\exists\ cn$ vertex-disjoint triangles in $G(n,p)) \to 0$.

Proof of Proposition 2. Define X as the number of copies of K_3 in $G(n,p)$. Then,

$$\mathbb{E}[X] = \binom{n}{3}p^3 \leq (p^3n^2)n = o(n).$$

Therefore, with high probability, there are $o(n)$ triangles in $G(n,p)$, which implies there is no collection of cn vertex-disjoint triangles in $G(n,p)$.

Remark. I consulted this problem set and its solutions. The consulted problem was for $c = 1/6$.

Proposition 3

If $p \gg n^{-2/3}$, i.e., $p^3n^2 \to \infty$, then $\mathbb{P}(\exists\ cn$ vertex-disjoint triangles in $G(n,p)) \to 1$.

Proof of Proposition 3. Note that, if $p(n) > \log(n)n^{-2/3}$, then $G(n, \log(n)n^{-2/3}) \subset G(n,p)$. Thus, finding a collection of triangles in $G(n, \log(n)n^{-2/3})$ with high probability implies finding a collection of triangles in $G(n, \log(n)n^{-2/3})$ with high probability. Thus, suppose $p(n) \leq \log(n)n^{-2/3}$.

Let $\epsilon = 1-3c$, let S be any ϵn-subset of $[n]$, and A_j $(1 \leq j \leq \binom{\epsilon n}{3})$ be all the triangles with vertices on S. Let B_j be the event $\{A_j \subset G(n,p)\}$. Let $X = \sum_{j=1}^{\binom{\epsilon n}{3}} \mathbb{1}[B_j]$, $\mu = \mathbb{E}[X]$, and $\Delta = \sum_{i \sim j} \mathbb{P}(B_i \cap B_j)$. Then,

$$\begin{aligned}
\mu &= \binom{\epsilon n}{3}p^3 \\
&\approx \frac{1}{3}\epsilon^3 n^3 p^3 \\
&\approx \frac{1}{3}\epsilon^3 (pn^{3/2})^3 n.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Delta &= 6\binom{\epsilon n}{4}p^5 \\
&\approx \frac{1}{4}\epsilon^4 n^4 p^5 \\
&\approx \frac{1}{4}\epsilon^4 (pn^{3/2})^5 n^{2/3}.
\end{aligned}$$

By Janson, 1987,

$$\begin{aligned}
\mathbb{P}(S \text{ has no triangles}) &= \mathbb{P}\left(\bigcap_{j=1}^{\binom{\epsilon n}{3}} \overline{B_j}\right) \\
&\leq \exp(-\mu + \Delta/2).
\end{aligned}$$

Thus,

$$\begin{aligned}\mathbb{E}[\#(S : |S| = \epsilon n \text{ and } S \text{ has no triangles})] &\le \binom{n}{\epsilon n} \exp(-\mu + \Delta/2) \\ &\le \left(\frac{en}{\epsilon n}\right)^{\epsilon n} \exp(-\mu + \Delta/2) \\ &\le \exp((1 - \log \epsilon)\epsilon n - \mu + \Delta/2).\end{aligned}$$

Note that, for all sufficienly large n,

$$\begin{aligned}-\mu + \Delta/2 &\le -\mu/2 \\ &\le -\frac{\epsilon^3}{6}(pn^{3/2})^3 n.\end{aligned}$$

Since $pn^{3/2} \to \infty$, for all sufficiently large n,

$$-\mu + \Delta/2 \le -(2 - \log \epsilon)n.$$

Thus,

$$\mathbb{E}[\#(S : |S| = \epsilon n \text{ and } S \text{ has no triangles})] \le \exp(-n),$$

which tends to 0.

Thus, almost every set of size at least $(1 - 3c)n$ contains a triangle. We can greedly construct our collection of vertex-disjoint triangles by choosing an $(1 - 3c)n$-subset, taking its triangle, choosing another $(1 - 3c)n$-subset that avoids the previous triangle, taking its triangle, and so on. With high probability, we will not be able to get a new triangle if there are less than $(1 - 3c)n$ vertices not chosen, i.e., at least $3cn$ vertices chosen, i.e., cn vertex-disjoint triangles.

Remark. The proof below was obtained from my own class notes.

Theorem 4 (Janson, 1987)

Let $A_1, \ldots, A_m \subset [N]$. Define the events $B_j := \{A_j \subset R\}$, in which $\mathbb{P}(i \in R) = p$, $\forall i \in [N]$ independently.

Let $X = \sum_{j=1}^m \mathbb{1}[B_j]$, $\mu = \mathbb{E}[X]$, $\Delta = \sum_{i \sim j} \mathbb{P}(B_i \cap B_j)$, in which $i \sim j$ means $A_i \cap A_j \neq \varnothing$.

Then,

$$\mathbb{P}\left(\bigcap_{j=1}^m \overline{B_j}\right) \le \exp\left(-\mu + \frac{\Delta}{2}\right).$$

Proof of Janson, 1987.

$$\begin{aligned}\mathbb{P}\left(\bigcap_{j=1}^m \overline{B_j}\right) &= \prod_{j=1}^m \left(1 - \mathbb{P}\left(B_j \mid \overline{B_1} \cap \cdots \cap \overline{B_{i-1}}\right)\right) \\ &\le \exp\left(-\sum_{j=1}^m \mathbb{P}(B_j \mid \overline{B_1} \cap \cdots \cap \overline{B_{i-1}})\right) \\ &\underset{\substack{\uparrow \\ \text{Proposition 5}}}{\le} \exp\left(-\mu + \frac{\Delta}{2}\right).\end{aligned}$$

It suffices to prove the following claim:

Proposition 5

$$\mathbb{P}\left(B_j \mid \overline{B_1} \cap \cdots \cap \overline{B_{i-1}}\right) \geq \mathbb{P}(B_j) - \sum_{\substack{i\sim j\\ i<j}} \mathbb{P}(B_i \cap B_j)$$

Proof. Let $E = \bigcap_{\substack{i\sim j\\ i<j}} \overline{B_j}$, and $F = \bigcap_{\substack{i\not\sim j\\ i<j}} \overline{B_i}$. Then,

$$\begin{aligned}
\mathbb{P}\left(B_j \mid E \cap F\right) &\geq \mathbb{P}(B_j \cap E \mid F) \\
&= \mathbb{P}(B_j \mid F)\mathbb{P}(E \mid B_j \cap F) \\
&= \mathbb{P}(B_j)\mathbb{P}(E \mid B_j \cap F) \\
&\underset{\substack{\uparrow\\ \text{Harris}}}{\geq} \mathbb{P}(B_j)\mathbb{P}(E \mid B_j) \\
&\underset{\substack{\uparrow\\ \text{union bound}}}{\geq} \mathbb{P}(B_j)\left(1 - \sum_{\substack{i\sim j\\ i<j}} \mathbb{P}(B_i \mid B_j)\right) \\
&\geq \mathbb{P}(B_j) - \sum_{\substack{i\sim j\\ i<j}} \mathbb{P}(B_i \cap B_j).
\end{aligned}$$

Problem 5

Let k be a sufficiently large constant, let G be a cycle with kn vertices, and let $c : V(G) \to [n]$ be a colouring of the vertices of G with exactly k vertices of each colour. Show that there exists an independent set of size n with one vertex of each colour.

Remark. I consulted a Math StackExchange post, "prove or supply counter example about-graph". The solution was written after I understood the original solution, without looking to the sources.

Solution. Proposition 6 shows the result for $k = 4$.

For any larger k, we can use what we'll show for $k = 4$ in the subgraph induced by any set with 4 vertices of each color — *note that this induced subgraph has $4n$ vertices, n colors with 4 vertices each, and it is a subgraph of C_{4n} (since each connected component is a path).*

Proposition 6

Let $c : V(C_{4n}) \to [n]$ be a colouring of the vertices of C_{4n} with exactly 4 vertices of each colour. Show that there exists an independent set of size n with one vertex of each colour.

Proof of Proposition 6. Paint the edges of the cycle with alternating colors, so that each vertex is in exactly one red edge and one blue edge.

Let $V_1, V_2, \ldots, V_n$ be the sets of the vertices with the colors $1, 2, \ldots, n$, respectively. Create a partition of V_i into W_{2n-1} and W_{2n}, with 2 vertices on each part.

We shall show that there are vertices $w_1, w_2, \ldots, w_{2n}$, with $w_i \in W_i$, such that $w_i w_j$ is not a red edge, for all i, j.

Proof. Let $i_1 = 1$.

Define w_{i_1} as a point in W_{i_1}. w_{i_1} sends a unique red edge to a vertex in some of the W's. If this set is not W_{i_1}, say W_{i_2}, define w_{i_2} as the vertex in W_{i_2} that does not recieve an red edge from w_{i_1}. Again, w_{i_2} sends a unique red edge to a vertex in some of the W's. If this set is not W_{i_1} or W_{i_2}, say W_{i_3}, define w_{i_3} as the vertex in W_{i_3} that does not recieve an edge from w_{i_2}; and so on. If, at some point, $w_{i_{k-1}}$ sends its unique red edge to a vertex in W_{i_1}, W_{i_2}, $\ldots$ or $W_{i_{k-1}}$, pick any set W not chosen before, call it W_{i_k}, call any of its vertices w_{i_k}.

After $2n$ iterations, each set W_j will have assigned a vertex w_j, and there cannot be any red edges between w_i and w_j.

Let $V'_i = \{w_{2i-1}, w_{2i}\}$; *note that $V'_i \subset V_i$.*

We shall show that there are vertices $v_1, v_2, \ldots, v_n$, with $v_i \in V'_i$, such that $v_i v_j$ is not a blue edge, for all i, j.

Proof. Let $i_1 = 1$.

Define v_{i_1} as a point in V'_{i_1}. v_{i_1} sends a unique blue edge to a vertex in some of the W's. If this set is not V'_{i_1}, say V'_{i_2}, define v_{i_2} as the vertex in V'_{i_2} that does not recieve an blue edge from v_{i_1}. Again, v_{i_2} sends a unique blue edge to a vertex in some of the W's. If this set is not V'_{i_1} or V'_{i_2}, say V'_{i_3}, define v_{i_3} as the vertex in V'_{i_3} that does not recieve an edge from v_{i_2}; and so on. If, at some point, $v_{i_{k-1}}$ sends its unique blue edge to a vertex in V'_{i_1}, V'_{i_2}, $\ldots$ or $V'_{i_{k-1}}$, pick any set W not chosen before, call it V'_{i_k}, call any of its vertices v_{i_k}.

After n iterations, each set V'_j will have assigned a vertex v_j, and there cannot be any blue edges between v_i and v_j.

The set $S = \{v_1, \ldots, v_n\}$ has no red edge and no blue edge, thus it is independent; and has one vertice of each colour.