Combinatorics I Lecture Notes

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This is IMPA's master class Combinatorics 1, instructed by Robert Morris. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Gooogle Meet and YouTube videos. The recommended material can be found here.

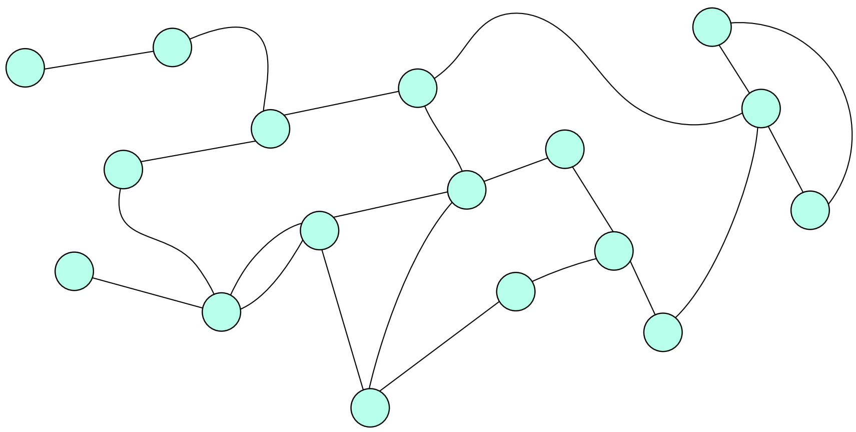

Figure 1: This is a graph.

Contents

1 Which problems we'll study?

Lec. 1

In this summer course, we'll study extremal, counting and probabilistic problems. Here are some examples:

Problem 1.1

Let A be a subset of $\{1, 2, \ldots, 2n\}$ such that $a \nmid b$, for all $a \neq b \in A$.

How large can $|A|$ be?

Solution. $A = \{n+1, \ldots, 2n\}$ is a good example. This yields $|A| = n$.

Consider the partition of $\{1, 2, \ldots, 2n\}$ given by the following sets:

- $\{2^t\}$
- $\{3 \cdot 2^t\}$
- $\{5 \cdot 2^t\}$

$\vdots$

- $\{(2n-1) \cdot 2^t\}$

There can't be two elements in the same set of the partition, so $|A| \leq n$.

Problem 1.2

Let A be a subset of $\{1, 2, \ldots, 2n\}$ such that $a + b \neq c$, for all $a, b, c \in A$. We'll call such set *sum-free*.

How large can $|A|$ be?

Solution. $A = \{n+1, \ldots, 2n\}$ is a good example. Another good example are the odd numbers. Both yield $|A| = n$.

Suppose $|A| \geq n + 1$. Let $a = \max A$.

Consider the following partition with $\left\lfloor \frac{a}{2} \right\rfloor$ sets:

- $\{1, a-1\}$
- $\{2, a-2\}$

$\vdots$

- $\{\left\lfloor \frac{a}{2} \right\rfloor, \left\lceil \frac{a}{2} \right\rceil\}$

There can't be two elements in the same set of the partition.

If $a \leq 2n - 1$, then there are at most $n - 1$ sets listed above, which implies $|A| \leq n$.

If $a = 2n$, then $n \notin A$, and then the $n - 1$ first sets listed above cover A, thus $|A| \leq n$.

Theorem 1.1 (Schur, 1916)

Given $c \colon \mathbb{Z}_{>0} \to \{1, \ldots, r\}$, the there are x, y, z such that:

- $x + y = z$
- $c(x) = c(y) = c(z)$

Problem 1.3

How many sum-free sets are in $[n]$?

Conjecture 1.2 (Cameron and Erdős)

The number of sum-free sets in $[n]$ is $\leq C \cdot 2^{n/2}$.

2 Ramsey's Theory

Theorem 2.1 (Ramsey's Theorem)

If $c : \binom{\mathbb{N}}{2} \to \{1, \dots, r\}$, then there exists $A \subset \mathbb{N}$ infinite and monochromactic, i.e, such that $c(ab) = c$, for all $a, b \in A$.

Proof of Theorem 2.1. Let $S_0 = \mathbb{N}$.

For each i, do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1}. Since S_{i-1} is infinite and there are finitely many colors, there is some color that appears infinitely many times; we'll call this color c_i, and define $S_i = \{u \in S_{i-1} \colon c(\{v_i, u\}) = c_i\}$.

Now, we have an infinite sequence $v_1, v_2, \dots$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are finitely many colors, there is some color that appears in infinitely many c_i's; call this color c, and define $A = \{v_i \colon c_i = c\}$.

The set A satisfies our condition.

Proof of Theorem 1.1. Given a coloring $c : \mathbb{N} \to \{1, \dots, r\}$, we define $c' : \binom{\mathbb{N}}{2} \to \{1, \dots, r\}$ by $c'(\{a, b\}) = c(b - a)$, for $b > a$.

By Theorem 2.1, there is A infinite and monochromactic. Pick $x < y < z \in A$, then we have $c(y - x) = c(z - y) = c(z - x)$, and $(y - x) + (z - y) = z - y$, so we're done!

Definition 2.2 (Ramsey Number)

Let $R(k)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n, i.e., for every $c : E(K_n) \to \{R, B\}$, there exists a monochromatic copy of K_k.

Let $R(s, t)$ denote the smallest n such that, for every coloring with two colors of the edges of the complete graph K_n, i.e., for every $c : E(K_n) \to \{R, B\}$, there exists a red copy of K_s or a blue copy of K_t.

Clearly, $R(k) = R(k, k)$.

Theorem 2.3

$$R(k) \lesssim 2^{2k}.$$

Sketch. Let $n = 2^{2k}$, and pick any coloring c of K_n. Let $S_0 = [n]$.

For each $i < 2k$ do the following: Pick $v_i \in S_{i-1}$. Look at the colors of $\{v_i, u\}$, for u in S_{i-1}. There is some color that appears more times; we'll call this color c_i, and define $S_i = \{u \in S_{i-1} \colon c(\{v_i, u\}) = c_i\}$.

~~Note that, the size of S_m is at least $\frac{n}{2^m} \geq 1$.~~ This is not quite correct. At each step, we're taking one vertice away, and then dividing by two. We'll properly prove a better bound later.

Now, we have an sequence $v_1, v_2, \dots, v_{2k-1}$, such that $c(\{v_i, v_j\}) = c_i$, for $i < j$. Since there are two colors, there is some color that appears at least k times; call this color c, and define $A = \{v_i \colon c_i = c\}$. The size of A is at least k. Pick any subset B of A that has exactly k elements.

The subgraph of K given by deleting all vertices but those in B is a monochromatic copy of K_k.

Lemma 2.4

$$R(s, t) \leq R(s - t, t) + R(s, t - 1).$$

Proof. Let $n = R(s,t) - 1$. By definition, there exists a coloring $c\colon E(K_n) \to \{R, B\}$ without a red K_s or a blue K_t.

Pick any vertex v. v it connected to some of the vertices through a red edge, which we'll put in the set S_R; the others are connected to v through a blue edge, those we'll put in the set S_B.

Since there are no red K_s or blue K_t, there can't be any red K_{s-1} or blue K_t in S_R; thus, $|S_R| \leq R(s-1,t)$. Analougously, $|S_B| \leq R(s,t-1)$.

Therefore,

$$\begin{aligned} R(s,t) - 1 &\leq R(s-1,t) - 1 + R(s,t-1) - 1 + 1 \\ R(s,t) &\leq R(s-1,t) + R(s,t-1). \end{aligned}$$

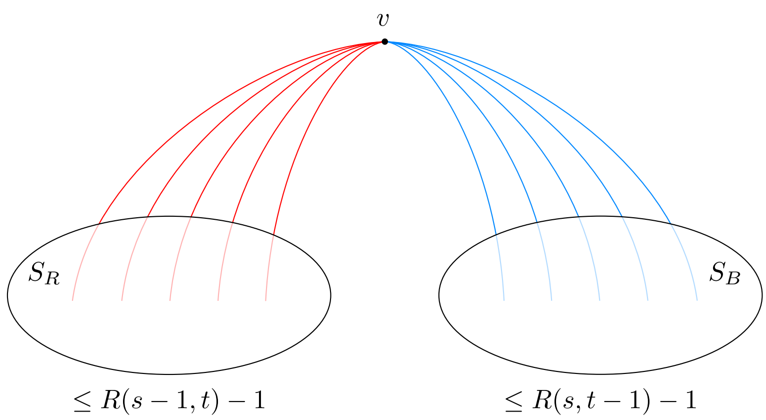

Figure 2: S_R and S_B.

Theorem 2.5

$$R(s,t) \leq \binom{s+t}{s}.$$

Proof. Follows from Lemma 2.4.

Theorem 2.6 (Erdős-Szekeres, 1935)

$$R(k) \leq \binom{2k}{k} \approx \frac{1}{\sqrt{k}} 4^k.$$

Lec. 2

Let's now try to find a lower bound. It is very difficult to show a good construction. Luckly, we are not going to do that.

Theorem 2.7 (Erdős, 1947)

$$\sqrt{2}^k \leq R(k)$$

Proof. Let $n \leq \sqrt{2}^k$. Let's choose colors randomly. Let

$$\mathbb{P}(c(e) = R) = \frac{1}{2},$$

for every edge e in K_n, independently.

We want to show that

$$\mathbb{P}(\text{there is not a monochromatic copy of } K_k) > 0.$$

Let X be the number of monochromactic copies of K_k in c. Then,

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}\left[\sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{1}[S \text{ is monochromatic}]\right] \\
&= \sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{E}\left[\mathbb{1}[S \text{ is monochromatic}]\right] \\
&= \sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \mathbb{P}(S \text{ is monochromatic})) \\
&= \sum_{\substack{S\subset K_n \\ S \text{ is a copy of } K_k}} \left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&= \binom{k}{n}\left(\frac{1}{2}\right)^{\binom{k}{2}-1} \\
&\leq 2\left(\frac{en}{k}\right)^k\left(\frac{1}{2}\right)^{\frac{k(k-1)}{2}} \\
&\leq 2\left(\frac{e\sqrt{2}}{k}\right)^k \\
&< 1, \text{ for } k \geq 5.
\end{aligned}$$

Therefore, since $\mathbb{E}[X] < 1$, we have $\mathbb{P}(X = 0) > 0$.

The bounds have not improved much since then

Theorem 2.8 (Conlon, 2009)

$$R(k) \leq \frac{4^k}{k^{\sqrt{\log k}}}$$

3 Extremal Graph Theory

3.1 Complete Graphs

Definition 3.1

Let $\operatorname{ex}(n, H)$ be the maximum number of edges a graph $G \subset K_n$ can have such that there are no copies of H in G.

Theorem 3.2 (Mantel, 1907)

$$\operatorname{ex}(n, K_3) = \left\lfloor \frac{n^2}{4} \right\rfloor.$$

Proof. The example is the bipartite graph with $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$ vertices.

Let's prove by indution on n.

Now, suppose G does not have a triangle. Pick an edge uv. Let G' be the graph G deleting u and v. The subgraph G' also does not contain triangles, so $e(G') \geq \left\lfloor \frac{n^2}{4} \right\rfloor$.

Notice that cannot exist $w \in G'$ such that uw and vw are edges of G, because G does not have triangles. Therefore, there can be at most $n - 2$ edges from u or v to vertices on G'. Including the edge uv, we conclude that

$$\begin{aligned} e(G) &\leq e(G') + n - 1 \\ &\leq \left\lfloor \frac{(n-2)^2}{4} \right\rfloor + n - 1 \\ &\leq \left\lfloor \frac{n^2}{4} \right\rfloor. \end{aligned}$$

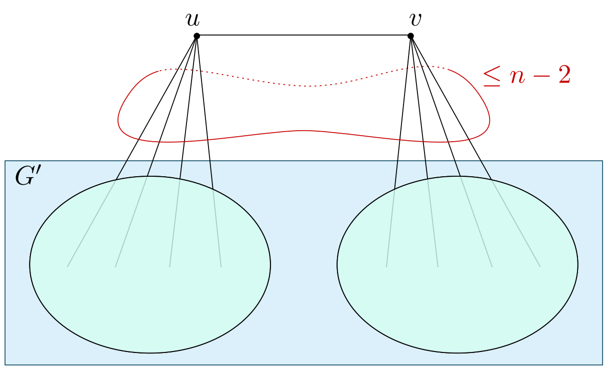

Figure 3: Edge uv on a triangle-free graph.

Definition 3.3 (Turán's Graph)

The graph $T_r(n)$ consists of r sets with roughly n/r elements each (some rounded up, some rounded down).; we create an edge uv if, and only if, u and v are on different sets.

We'll denote by $t_r(n)$ the number of edges in $T_r(n)$.

Theorem 3.4 (Turán, 1941)

$$\operatorname{ex}(n, K_{r+1}) = t_r(n) \approx \left(1 - \frac{1}{r}\right)\binom{n}{2}.$$

Proof. We'll use induction on n. For $n \leq r$, we're good.

Pick a maximal graph G that doesn't have a copy of K_{r+1}. Pick a copy of K_r, let's call it H. Define $G' = G - H$. Of course, G' has no copies of K_r; thus $e(G') \leq t_r(n-r)$, by induction.

Futhermore, if $v \in G'$, there can be at most $r-1$ edges connecting v to some vertex in H.

Wrapping everything up, we have

$$\begin{aligned} e(G) &\leq e(G') + (n-r)(r-1) + \binom{r}{2} \\ &\leq t_r(n-r) + (n-r)(r-1) + \binom{r}{2} \\ &\leq t_r(n). \end{aligned}$$

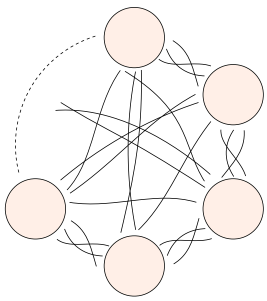

Figure 4: Turán's Graph

3.2 Bipartite Graphs

Theorem 3.5 (Erdős, 1935)

$$\mathrm{ex}(n, C_4) \leq \frac{n^{3/2}}{2}.$$

Proof. Let's count cherries! A *cherry* is a pair $(v, \{u, w\})$, in which vu and vw are edges of the graph.

Since there is no C_4, there is at most one cherry for each pair $\{u, w\}$. This implies that:

$$\begin{aligned} \binom{n}{2} \geq \#(\text{cherries}) &= \sum_{v \in G} \binom{d(v)}{2} \\ &\geq n \binom{\frac{2e(G)}{n}}{2}. \end{aligned}$$

Solving this quadractic inequation on $e(G)$ yields to

$$e(g) \geq \frac{n^{3/2}}{2}.$$

Question 3.1

For which graphs we have

$$\mathrm{ex}(n, H) = \Theta(n^2)?$$

Proposition 3.6

For every non-bipartite graph H, we have

$$\mathrm{ex}(n, H) \geq \frac{n^2}{4}.$$

Proof. Take G as the complete bipartite graph with n vertices. It has roughly $\frac{n^2}{4}$ vertices and it cannot contain a non-bipartite graph.

Theorem 3.7 (Kővári–Sós–Turán, 1954)

Let H be a bipartite graph. Then,

$$\mathrm{ex}(n, H) = o(n^2).$$

Proof. Since H is bipartite, there is some $K_{s,t}$ such that $H \subset K_{s,t}$. Then,

$$\mathrm{ex}(n, H) \leq \mathrm{ex}(n, K_{s,t}).$$

Let's bound $\mathrm{ex}(n, K_{s,t})$.

We'll count s-cherries: (v, S), in which S has size s and $vx \in E(G)$ for all $x \in S$.

There are at most $t-1$ s-cherries for each subset S with size s. This implies that:

$$\begin{aligned}(t-1)\binom{n}{s} \geq \#(\text{cherries}) &= \sum_{v \in G} \binom{d(v)}{s} \\ &\geq n \binom{\frac{2e(G)}{n}}{s} \geq \frac{e(G)^s}{s^s \cdot n^{s-1}}.\end{aligned}$$

This implies that, for some constant C,

$$e(G) \leq C \cdot n^{2-\frac{1}{s}}$$

Question 3.2

For which H it holds that

$$\mathrm{ex}(n, H) = O(n)?$$

3.3 Trees

Lec. 3

Definition 3.8 (Tree)

A tree is a connected graph that has no cycles.

Proposition 3.9

Given a graph G, the following are equivalent:

(i) G is a tree;

(ii) G is a maximal graph without cycles, i.e., G does not have cycles and there is no graph $H \supset G$ such that H does not have cycles;

(iii) G is a minimal connected graph, i.e., G is connected and there is no graph $H \subset G$ such that H is connected.

Theorem 3.10

Let T be a graph with k vertices. Then,

$$\frac{(k-2)}{2}n \leq \mathrm{ex}(n,T) \leq (k-1)\cdot n.$$

Proof of the lower bound. Pick $\frac{n}{k-1}$ disjoint $k-1$-cliques. There cannot be a copy of a connected graph with k vertices inside this graph. This graph has roughly

$$\binom{k-1}{2}\frac{n}{k-1} = \frac{k-2}{2}n$$

edges.

Proof of the upper bound. Let's start with a lemma.

Lemma 3.11

Let G be a graph with mean degree d, then, there exists a subgraph $G' \subset G$ with minimum degree at least $\frac{d}{2}$.

Proof. While there are vertices with degree smaller than $\frac{d}{2}$, throw them away.

If we stopped before throwing away all vertices, we're done. Suppose we threw away all vertices. At each step, we threw away at most $\frac{d}{2}$ edges. Since we threw away all edges, this means $n \cdot \frac{d}{2} < e(G) = n\frac{d}{2}$; a contradiction.

Lemma 3.12

Let G be a graph with $\delta(G) \geq k-1$. Then, there is a copy of T in G for every tree T with k vertices.

Proof. We'll use induction on k. If $k=1$, we're done!

Pick a leaf v of T. Its unique edge connects it to u. Let T' be the tree without v. By induction, there is a copy $C_{T'}$ of T' in G. Let c_u be the copy of u in $C_{T'}$. Since $\deg(c_u) \leq k-2$ in $C_{T'}$, but $\deg(c_u) \geq k-1$ in G, there is some vertex that is connected to u outside of $C_{T'}$, say c_v. Thus, let C_T be $C_{T'}$, adding c_v. C_T is a copy of T inside G.

Finally, $e(G) = (k-1)n \implies \bar{d}(G) = 2(k-1) \implies$ there exists a subgraph $G' \subset G$ such that $\delta(G') \geq k-1 \implies T \subset G'$.

Conjecture 3.13 (Erdős-Sós, 1960's)

$$\mathrm{ex}(n,T) \leq \frac{(k-2)n}{2}$$

Definition 3.14 (Random graph of Erdős-Rónyi)

We define $G(n,p)$ as a random distribution of graphs with n vertices, with

$$\mathbb{P}(e \in E(G(n,p))) = p,$$

chosen independently.

Lemma 3.15 (Markov's inequality)

$$\mathbb{P}(X \geq t) \geq \frac{\mathbb{E}[X]}{t}.$$

Proof. Left to the reader. Use the definition of $\mathbb{E}[X]$.

Theorem 3.16

$$\text{ex}(n, C_t) \geq O\left(n^{1+\frac{1}{2k-1}}\right) \gg n.$$

Proof. Let $t = 2k$. We want to choose $p = p(n)$ such that:

- $e(G(n,p)) \gg n$;
- $C_{2k} \not\subset G(n,p)$.

$$\mathbb{E}[e(G(n,p))] = p\binom{n}{2}.$$

Moreover, $e(G(n,p))$ is a binomial distribution, therefore, $e(G(n,p)) \approx np^2$ with high probability. Thus, we should pick $p \gg 1/n$, i.e., $pn \to \infty$.

Define X as the number of copies of C_{2k} in $G(n,p)$.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{\substack{\text{copies } S \text{ of} \\ C_{2k} \text{ in } K_n}} \mathbb{P}(S \subset G(n,p)) \\ &\approx n^{2k}p^{2k} = (pn)^{2k}.\end{aligned}$$

Let $0 < \varepsilon < \frac{1}{2k-1}$, and define $p = p(n) = n^{-1+\varepsilon}$. Then, we have $pn \gg n^{-1}$ and $(pn)^{2k} \ll pn^2$. Therefore, each of the following happen with high probability:

- $e(G(n,p)) \approx pn^2$;
- The number of copies of C_{2k} in $G(n,p) \approx (pn)^{2k}$.

Therefore, the intersection also occours with high probability. Pick a graph G in the intersection.

For each of the $(pn)^{2k}$ cycles in G delete an edge in it; call this new graph G'. Thus $e(G') \approx pn^2 - (pn)^{2k} \approx n^{1+\epsilon}$, and G' has no C_{2k}.

Theorem 3.17

$$\text{ex}(n, H) = O(n) \iff H \text{ does not have cycles.}$$

Proof. All the work has been done. The proof, which is simply a jigsaw puzzle, is left to the reader.

4 Planar graphs

Definition 4.1 (Planar Graph)
A planar graph is a graph that can be drawn on the plane without having crossing edges. Edges may not be straight.

Lemma 4.2 ($V + F - E = 2$)
Let G be a planar connected graph, and $v(G) \geq 1$. For any planar drawing of G, we have

$$v(G) + f(G) - e(G) = 2.$$

Sketch. Induction on $e(G)$.

(i) **If there is a leaf,** then we can take it away.

$$\begin{aligned} v(G') &= v(G) - 1, \\ e(G') &= e(G) - 1, \\ f(G') &= f(G). \end{aligned}$$

(ii) **If there is no leaf,** there is a cycle, take away an edge from the cycle.

$$\begin{aligned} v(G') &= v(G), \\ e(G') &= e(G) - 1, \\ f(G') &= f(G) - 1. \end{aligned}$$

Watch an animated version of this classic demonstration at 3Blue1Brown.

Theorem 4.3
Let G be a planar graph with $n \geq 3$ vertices. Then,

$$e(G) \leq 3n - 6$$

Proof. Without loss of generalitym G is maximal.

Maximal and $n \geq 3$ implies all regions are triangles. Double counting implies

$$3f(G) = 2e(G).$$

Also,

$$v(G) + f(G) - e(G) = 2.$$

It follows that

$$e(G) = 3n - 6.$$

Theorem 4.4
K_5 is not planar.

Proof.

$$e(K_5) = 10 > 3 \cdot 5 - 6 = 3v(K_5) - 6.$$

Theorem 4.5
Let G be a triangle-free planar graph with $n \geq 4$ vertices. Then,

$$e(G) \leq 3n - 6$$

Proof. Without loss of generalitym G is maximal.

Maximal and $n \geq 4$ implies all regions have at least 4 sides. Double counting implies

$$4f(G) \leq 2e(G).$$

Also,

$$v(G) + f(G) - e(G) = 2.$$

It follows that

$$e(G) = 2n - 4.$$

Theorem 4.6

$K_{3,3}$ is not planar.

Proof. $K_{3,3}$ is triangle-free.

$$e(K_{3,3}) = 9 > 2 \cdot 6 - 4 = 2v(K_{3,3}) - 4$$

Theorem 4.7

G is planar if, and only if, G does not have a topological copy of K_5 or $K_{3,3}$ if, and only if, G does not have a K_5-minor or a $K_{3,3}$-minor.

5 More colors

Definition 5.1 (Chromatic Number of a Graph)

The chromatic number of G, denoted by $\chi(G)$, is the smallest r such that there is a coloring $c\colon V(G) \to [r]$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$.

Lec. 4

Definition 5.2

Let $r(G, H)$ denote the minimum n such that, for every coloration $c\colon E(K_n) \to \{R, B\}$, there must exist a red G or a blue H.

Proposition 5.3

$$\chi(G) \leq \Delta(G) + 1.$$

Sketch. Greedy algorithm.

Theorem 5.4 (4-color Theorem, 1970's)

If G is planar, then $\chi(G) \leq 4$.

Proposition 5.5

If G is planar, then $\chi(G) \leq 6$.

Proof. Induction on n.

Since G is planar, $e(G) \leq 3n - 6$, thus $\delta(G) \leq 5$. Pick v with degree at most 5. Define G' as G without v, then G' has a proper coloring. Now, v has at most five neighbors, thus we can pick one color for v out of six such that no neighbor of v has this color.

Theorem 5.6

If G is planar, then $\chi(G) \leq 6$.

Proof. Induction on n.

Since G is planar, $e(G) \leq 3n - 6$, thus $\delta(G) \leq 5$. Pick v with degree at most 5. Define G' as G without v, then G' has a proper coloring. Now, v has at most five neighbors. If there at most four colors are used in the neighbors of v, we can paint v with a distinct color.

Suppose all neighbors of v have different colors. Let's call the neightbors u_1, u_2, u_3, u_4, u_5, in clockwise order, with colors $1, 2, 3, 4, 5$.

Define ${G'}_a^b$ as the subgraph of G' that only contains vertices with colors a and b. Let H_a^b be the connected component of ${G'}_a^b$ that contains u_a.

- **If there exists $\boldsymbol{a, b}$ such that $\boldsymbol{u_b \notin H_a^b}$,** then we flip the colors a and b inside H_a^b and define $c(v) := a$.
- **If, for all $\boldsymbol{a, b}$, $\boldsymbol{u_b \in H_a^b}$,** $H_{1,3}$ and $H_{2,4}$ are vertex disjoint, but have to go through each other; a contradiction. See fig. 5.

Theorem 5.7 (Erdős-Stone, 1946)

$$\operatorname{ex}(n, H) = \left(1 - \frac{1}{\chi(H) - 1} + o(1)\right)\binom{n}{2}.$$

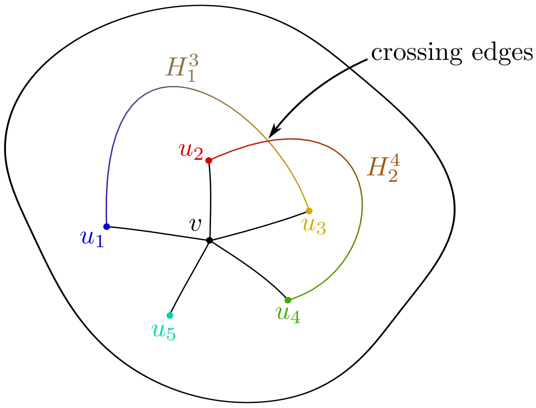

Figure 5: Second case of five color theorem

Sketch. The example is the Turán's Graph $T_{\chi(H)-1}(n)$.