# Analysis I
# Lecture Notes

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This is Haverford College's undergraduate MATH H317, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

# Contents

# 1 What are the real numbers?

2021-08-30

## 1.1 Defining the real numbers: an axiomatic aproach

The main idea is to derive $\mathbb{R}$ from $\mathbb{Q}$. We will layout some properties that $\mathbb{Q}$ has that we also want $\mathbb{R}$ to have; and then add an additional property that will distinguish $\mathbb{Q}$ from $\mathbb{R}$.

First, $\mathbb{Q}$ is a field, and we also want $\mathbb{R}$ to be a field.

**Definition 1.1** (Field Axioms)

A set $F$ is a *field* if there exist two operations — addition and multiplication — that satisfy the following list of conditions:

- **i.** (Commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.
- **ii.** (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.
- **iii.** (Identities) There exist two special elements, denoted by 0 and 1, such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.
- **iv.** (Inverses) Given $x \in F$, there exists an element $-x \in F$ such that $x+(-x) = (-x)+x = 0$. If $x \neq 0$, there exits an element $x^{-1}$ such that $xx^{-1} = x^{-1}x = 1$.
- **v.** (Distributivity) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Being a field is not restrictive enough, since it allows for finite fields, such as $\mathbb{Z}/p\mathbb{Z}$, or complex numbers $\mathbb{C}$. Another feature of $\mathbb{Q}$ (and a desired feature of $\mathbb{R}$) is order.

**Definition 1.2** (Ordering)

An *ordering* on a set $F$ is a relation, represented by $\leq$, with the following properties:

- **i.** $x \leq y$ or $y \leq x$, for all $x, y \in F$.
- **ii.** If $x \leq y$ and $y \leq x$, then $x = y$.
- **iii.** If $x \leq y$ and $y \leq z$, then $x \leq z$.

We define $x < y$ as equivalent to $x \leq y$ and $x \neq y$. We define $y \geq x$ as equivalent to $x \leq y$. We define $y > x$ as equivalent to $x < y$.

Additionally, a field $F$ is called an *ordered field* if $F$ is endowed with an ordering $\leq$ that satisfies

- **iv.** If $y \leq z$, then $x + y \leq x + z$.
- **v.** If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

Now, we need to add a feature that distinguishes $\mathbb{Q}$ and our desired $\mathbb{R}$. Intuitively, "$\mathbb{Q}$ has holes", meaning that one can build a sequence in $\mathbb{Q}$ that approaches a limit that is not in $\mathbb{Q}$; on the other hand, "$\mathbb{R}$ has no holes", meaning that any sequence in $\mathbb{R}$ that converges can only converge to a limit that is in $\mathbb{R}$.

That's the main idea we will formalize next.

2021-09-01

## 1.2 Bounds

**Definition 1.3** (Upper bound)

If $F$ is an ordered field, and $A \subset F$, then we say that some $b \in F$ is an *upper bound of $A$* if $a \leq b$ for all $a \in A$.

If a set $A$ has an upper bound, we say that $A$ is *bounded above*.

**Definition 1.4** (Supremum)

If $F$ is an ordered field, and $A \subset F$, we say $s \in F$ is the *least upper bound of $A$*, or *supremum of $A$*, denoted by $\sup(A)$, if:

**i.** $s$ is an upper bound of $A$, and
**ii.** if $b$ is any upper bound of $A$, then $s \leq b$.

**Example**

Let $F = \mathbb{Q}$ and $A = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\}$. Then, $\sup(A) = 0$.

**Definition 1.5** (Lower bound)

If $F$ is an ordered field, and $A \subset F$, then we say that some $b \in F$ is a *lower bound of $A$* if $a \geq b$ for all $a \in A$.

If a set $A$ has a lower bound, we say that $A$ is *bounded below*.

**Definition 1.6** (Infimum)

If $F$ is an ordered field, and $A \subset F$, we say $s \in F$ is the *greatest lower bound of $A$*, or *infimum of $A$*, denoted by $\inf(A)$, if:

**i.** $s$ is a lower bound of $A$, and
**ii.** if $b$ is any lower bound of $A$, then $s \geq b$.

## 1.3 Absolute value

Before we dig more deeply into the idea of a supremum, consider this definition that comes just from the structure of an ordered field.

**Definition 1.7** (Absolute value)

If $F$ is an ordered field, and $x \in F$, let

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

**Theorem 1.8**

If $F$ is an ordered field, and $x \in F$, then $|x| \geq 0$.

*Proof.* If $x \geq 0$, then $|x| = x \geq 0$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x = |x|$. ∎

**Theorem 1.9**

If $F$ is an ordered field, and $x \in F$, then $|-x| = |x|$.

*Proof.* If $x \geq 0$, then $0 = x + (-x) \geq 0 + (-x) = -x$, therefore $|-x| = -(-x) = x = |x|$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$, therefore $|-x| = -x = |x|$. ∎

**Theorem 1.10**

If $F$ is an ordered field, and $x, y \in F$, then $|xy| = |x||y|$.

*Proof.* If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.

If $x \geq 0$ and $y \leq 0$, then $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the previous case with $x$ and $-y$ and also Theorem 1.9 to obtain $|xy| = |-xy| = |x(-y)| = |x||-y| = |x||y|$.

If $x \leq 0$ and $y \geq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$. So we apply the first case with $-x$ and $y$ and also Theorem 1.9 to obtain $|xy| = |-xy| = |(-x)y| = |-x||y| = |x||y|$.

If $x \leq 0$ and $y \leq 0$, then $0 = x+(-x) \leq 0+(-x) = -x$ and $0 = y+(-y) \leq 0+(-y) = -y$. So we apply the first case with $-x$ and $-y$ and also Theorem 1.9 to obtain $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$. ∎

**Theorem 1.11** (Triangle inequality)

If $F$ is an ordered field, and $x, y \in F$, then

$$|x + y| \leq |x| + |y|.$$

*Proof.* If $x \geq 0$, then $|x| = x$. If $x \leq 0$, then $x \leq 0 = x + (-x) \leq 0 + (-x) = x$, so $|x| = -x \geq x$. In either case, $|x| \geq x$.

Thus, $|x|+|y| \geq x+y$ and $|x|+|y| = |-x|+|-y| \geq -x-y = -(x+y)$. Since $|x+y| = x+y$ or $|x+y| = -(x+y)$, in either case, $|x| + |y| \geq |x + y|$. ∎

**Theorem 1.12** (Reverse triangle inequality)

If $F$ is an ordered field, and $x, y \in F$, then

$$|x - y| \geq ||x| - |y||.$$

*Proof.* Triangle inequality implies that $|x| = |(x-y)+y| \leq |x-y|+|y|$ and $|y| = |(y-x)+x| \leq |y - x| + |x|$. Equivalently, we have $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$; consequently, $|x - y| \geq ||x| - |y||$. ∎

## 1.4 Completeness: the key to define the real numbers

**Definition 1.13** (Completeness)

Given $F$ an ordered field, we say $F$ is *complete* if, for any subset $A \subset F$ bounded above and nonempty, the supremum of $A$ exists[a].

[a]and is an element of $F$, as the definition requires.

**Definition 1.14** (Real numbers)

The set of real numbers is a complete ordered field. In other words, we define $\mathbb{R}$ to be any set that obeys the field axioms, the order axioms and the Axiom of Completeness.

Subtely, this leaves open the possibility that there is more than one set that is “the real numbers”, or no such set. However, there is a theorem that states that there is a unique complete ordered field[*].

[*]up to isomorphism.

# 2 Getting to know the Real Numbers

2021-09-03

**Theorem 2.1** ($\epsilon$-sup Theorem)

Given $A \subset \mathbb{R}$ nonempty and bounded above, and given $s$ an upper bound of $A$, then $s = \sup(A)$ if, and only if, for all $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.

*Proof.* Supppose $s = \sup(A)$. Then, $s - \epsilon$ is not an upper bound of $A$. Therefore, there exists $a \in A$ such that $a > s - \epsilon$.

Suppose $s \neq \sup(A)$. Then, there exists an upper bound of $A$ that is smaller than $s$, say $s - \delta$. Then, it follows that, for $\epsilon = \delta/2$, there is no $a \in A$ such that $a > s - \delta/2$, because $a \leq s - \delta < s - \delta/2$ for all $a \in A$. ∎

**Definition 2.2** (Sum of Sets)

Given $A, B \subset \mathbb{R}$, we define their sum as

$$A + B = \{a + b : a \in A, b \in B\}$$

**Theorem 2.3** (Supremum of Sum of Sets)

If $A, B \subset \mathbb{R}$ are both nonempty and bounded above, then

$$\sup(A + B) = \sup(A) + \sup(B).$$

*Proof.* Since $\sup(A)$ is an upper bound of $A$, it holds that $a \leq \sup(A)$ for all $a \in A$. Since $\sup(B)$ is an upper bound of $B$, it holds that $b \leq \sup(B)$ for all $b \in B$. Therefore, $a + b \leq \sup(A) + \sup(B)$ for all $a \in A$ and $b \in B$, i.e., $x \leq \sup(A) + \sup(B)$ for all $x \in A + B$; thus, $\sup(A) + \sup(B)$ is an upper bound of $A + B$.

Let $\epsilon > 0$ be any positive real number. $\epsilon$-sup Theorem implies that there exists $a \in A$ such that $a > \sup(A) - \epsilon/2$. $\epsilon$-sup Theorem also implies that there exists $b \in B$ such that $b > \sup(B) - \epsilon/2$. Therefore, there exist $a \in A$ and $b \in B$ such that $a + b > \sup(A) + \sup(B) - \epsilon$; thus, there exists $x \in X$ such that $x > \sup(A) + \sup(B) - \epsilon$. Finally, by $\epsilon$-sup Theorem, $\sup(A + B) = \sup(A) + \sup(B)$. ∎

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## 2.1 Archimedean Properties

**Theorem 2.4** (Archimedean Properties)

**i.** Given any $x \in \mathbb{R}$, there exists some $n \in \mathbb{Z}_{>0}$ with $n > x$.
**ii.** Given any $y > \mathbb{R}_{>0}$, there exists some $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < y$.

*Proof.* The first statement is equivalent to $\mathbb{Z}_{>0}$ is not bounded above.

Suppose $\mathbb{Z}_{>0}$ is bounded above. Then, there exists $s = \sup(\mathbb{Z}_{>0})$. Therefore, $s - 1$ is not an upper bound of $\mathbb{Z}_{>0}$, i.e., there exists $n \in \mathbb{Z}_{>0}$ such that $s - 1 < n$. However, this implies $s < n + 1 \in \mathbb{Z}_{>0}$, implies $s$ is not an upper bound of $\mathbb{Z}_{>0}$, which is a contradiction.

The second statement follows from the first one by setting $x = \frac{1}{n}$. ∎

**Theorem 2.5** (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For all $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ with $a < q < b$.

*Proof.* By Archimedean Properties with $y = b - a > 0$, there exists $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < b - a$.

Let $m$ be the smallest natural number greater than $na$. Then,

$$\begin{aligned} m - 1 &\leq na < m \\ \frac{m}{n} - \frac{1}{n} &\leq a < \frac{m}{n}. \end{aligned}$$

The first inequality implies that $\frac{m}{n} \leq a + \frac{1}{n} < b$, so finally, we conclude that

$$a < \frac{m}{n} < b.$$

∎

**Corolary 2.6**

For all $a, b \in \mathbb{R}$, with $a < b$, there exist infinitely many $q \in \mathbb{Q}$ with $a < q < b$.

2021-09-10

## 2.2 Nested Interval Property

**Theorem 2.7** (Nested Interval Property)

Suppose we have a sequence of closed intervals $I_n = [a_n, b_n]$, with $a_n \leq b_n$, that are nested decreasing, i.e.,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots .$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

*Proof.* Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$.

The "nested" condition implies that, if $i < j$, then $[a_i, b_i] \supset [a_j, b_j]$. Therefore, $a_j, b_j \in [a_i, b_i]$, which implies that $a_i \leq a_j \leq b_j \leq b_i$ for all $i < j$. Note that this implies that

$$a_i \leq b_j \text{ and } a_j \leq b_i, \text{ for all } i < j.$$

We can rewrite it as

$$a_i \leq b_j, \text{ for all } i \text{ and } j.$$

This implies that $a_i$ is a lower bound of $B$ for any $i$, and also implies that $b_j$ is an upper bound of $A$ for any $j$. Since $A$ is bounded above, we can define $x = \sup(A)$. Clearly, $x$ is an upper bound of $A$.

Suppose $x$ is not a lower bound of $B$. Then, there exists $n$ such that $x > b_n$. The $\epsilon$-sup Theorem, with $\epsilon = x - b_n > 0$, implies that there exists $m$ such that $a_m > x - (x - b_n) = b_n$, which contradicts the previous displayed equation. Therefore, $x$ is a lower bound of $B$.

Finally, $x$ is both an upper bound of $A$ and a lower bound of $B$, thus, for all $n$, $a_n \leq x \leq b_n$, i.e., $x \in [a_n, b_n]$. Therefore, $x$ is in such intersection. ∎

## 2.3 Cardinality

*Question.* Are all sets with an infinite number of elements the same size?

**Definition 2.8** (Cardinality)

Given two sets $A$ and $B$, we say that *$A$ and $B$ have the same cardinality* if there exists a bijection $f \colon A \to B$. We will write $A \sim B$ to say that $A$ and $B$ have the same cardinality.

**Example**

The sets $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\{2, 4, 6, 8, \dots\}$ have the same cardinality.

**Definition 2.9** (Countability)

We say a set $S$ is *countable* if it has the same cardinality as $\mathbb{N}$. If a set is not a finite set and not countable, then we say it is *uncountable*.

**Proposition 2.10** ($\mathbb{N}^2$ is countable)

$\mathbb{N}^2$ is countable.

*Proof.* The function $f\colon \mathbb{N}^2 \to \mathbb{N}$, defined by

$$f(i,j) = \frac{(i+j-1)(i+j-2)}{2} + i$$

is a bijection. ∎

2021-09-13

**Theorem 2.11**

If $A$ is countable and $B$ is countable, then $A \times B$ is countable.

Similarly, if $A_1, A_2, \dots, A_n$ are each countable, then $A_1 \times \cdots \times A_n$ is countable.

Similarly, if $A_1, A_2, \dots, A_n$ are each countable or finite, then $A_1 \times \cdots \times A_n$ is countable or finite.

**Theorem 2.12**

If $S_1, S_2, \dots$ are each countable, then their union is countable.

Similarly, if $\{S_i\}_{i\in I}$ is a countable or finite collection of sets, which are each countable or finite; then their union is countable.

**Example**

Let $\mathcal{T}$ be the collection of finite subsets of $\mathbb{N}$. For each $i \in \mathbb{N}$, let $A_i$ be the collection of subsets of $\{1,2,\dots,i\}$. Note that $|A_i| = 2^i$, thus $A_i$ is finite. Then, note that $\varnothing \in A_1$, and, if $S \in \mathcal{T}$ is non-empty, it holds that $S \in A_{\max(S)}$; so $\mathcal{T} = \bigcup_{i=1}^{\infty} A_i$.

Therefore, by Theorem 2.12, we conclude that $\mathcal{T}$ is countable or finite. Since $\mathcal{T}$ is not finite, then it is countable.

**Theorem 2.13**

If $A$ is countable, and $f : A \to B$ is surjective, then $B$ is countable or finite.

Similarly, if $A$ is countable, and $f : B \to A$ is injective, then $B$ is countable or finite.

In particular, if $A$ is countable, and $A \supseteq B$, then $B$ is countable or finite.

**Proposition 2.14** ($\mathbb{Q}$ is countable)

$\mathbb{Q}$ is countable.

*Proof.* Consider the function $f : \mathbb{Z}\times(\mathbb{Z}\setminus\{0\}) \to \mathbb{Q}$ defined by $f(a,b) = \frac{a}{b}$. Clearly, $f(p,q) = \frac{p}{q}$ for any $\frac{p}{q} \in \mathbb{Q}$. ∎

**Proposition 2.15**

$\mathbb{R}$ is not countable.

*Proof* (using nested intervals). Assume $\mathbb{R}$ is countable. So, there exists a bijection $f : \mathbb{N} \to \mathbb{R}$.

Let $I_1 = [f(1) + 1, f(2) + 2]$. Note that $f(1) \notin I_1$. We will define $I_{n+1}$ recursively. Suppose $I_n = [a, b]$, then, define $I_{n+1}$ as either $[a, \frac{2a+b}{3}]$ or $[\frac{a+2b}{3}, b]$ such that $f(i + 1) \notin I_{n+1}$; that is possible since $f(i + 1)$ cannot be in both sets.

By the Nested Interval Property, there exists a real number $r \in \bigcap_{i=1}^{\infty} I_n$. However, since $f$ is a bijection, there exists $m \in \mathbb{N}$ such that $f(m) = r$. Therefore, $r \notin I_m$, a contradiction. ∎

2021-09-15

*Proof* (using Cantor's diagonalization). We'll prove $(0, 1)$ is uncountable, which implies $\mathbb{R}$ is uncountable.

Assume $(0, 1)$ is countable, therefore, there exists a bijective function $f : \mathbb{N} \to (0, 1)$.

Let's write out decimal expansions[a] of $f(1), f(2), \ldots$. If there's doubt between a recurrent 9 or a recurrent 0 in the end, we choose the latter form. We write

$$f(i) = 0.a_{i1}a_{i2}a_{i3}\ldots,$$

with $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $b_k = 1$, if $a_{kk}$ is odd, and $b_k = 2$, if $a_{kk}$ is even. Note that $c_k \neq b_{kk}$ and $c_k \notin \{0, 9\}$ for all $k$. Therefore, $x = 0.b_1b_2b_3\ldots$ cannot be on the image of $f$; a contradiction. ∎

[a] What are decimal expansions? We only need to know that decimal expansions are unique except for some duplication, like $0.09999 = 0.1$.

Another perspective on the Cantor's proof arises by using the binary base, instead of the decimal base. For each real number $x = 0.x_1x_2x_3\ldots$, we can define a $f(x) = \{n \in \mathbb{N} : a_n = 1$. This is almost[*] a bijection because, but nevertheless, we can conclude that, in some sense,

$$|\mathbb{R}| = 2^{|\mathbb{N}|}.$$

**Definition 2.16** (Limit of a sequence)

We say a sequence $(a_n) = a_1, a_2, a_3, \ldots$ *converges to a real number* $a$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$.

If this definition holds for some $a$, we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$.

If this definition does not hold for any $a$, we say $\lim_{n\to\infty} a_n$ does not exist, or that the sequence diverges.

**Proposition 2.17** (The limit, if it exists, is unique)

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = a'$, then $a = a'$.

[*] The same number with two expansions yields a problem.

*Proof.* For all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N_\epsilon$. For all $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that $|a_n - a'| < \epsilon$ for all $n \geq M_\epsilon$.

Therefore, for all $\epsilon > 0$, there exists $L_\epsilon \in \mathbb{N}$, namely $\max\{N_\epsilon, M_\epsilon\}$, such that $|a_n - a| < \epsilon$ and $|a_n - a'| < 0$ for all $n \geq L_\epsilon$. Triangle inequality implies that $|a - a'| < 2\epsilon$ for all $\epsilon > 0$; thus $a = a'$. ∎

**Example**

We claim that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

This is true because, given $\epsilon > 0$, we can choose $N$ be a natural number larger than $\sqrt{\frac{1}{\epsilon}}$. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N^2} > \frac{1}{n^2} = \left|\frac{1}{n^2} - 0\right|.$$

**Example** (The limit does not exist)

We claim that $\lim_{n\to\infty}(-1)^n$ does not exist.

Suppose it does exist, namely $a$. Then, consider $\epsilon = \frac{1}{2}\max\{|a-1|, |a+1|\}$. Not both $|a-1|$ and $|a+1|$ can be zero, so $\epsilon > 0$. However, since $\lim_{n\to\infty} = a$, for that $\epsilon$, it must hold that there exists $N \in \mathbb{N}$ so that for all $n \geq N$, $|a - (-1)^n| < \epsilon$.

In particular, note that plugging in $n \mapsto N$ and $n \mapsto N+1$ imply that $|a-1| < \epsilon$ and $|a+1| < \epsilon$; which is a contradiction given our choice of $\epsilon$.

**Example**

We claim that $\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$. Note that we can rewrite $\frac{2n+1}{n+3} = 2 - \frac{5}{n+3}$. For any $\epsilon$, Archimedean Properties imply that there exists $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{5}$. Therefore, for all $n \geq N$, it holds that

$$\left|\left(2 - \frac{5}{n+3}\right) - 2\right| = \frac{5}{n+3} < \frac{5}{N} < \epsilon,$$

and our claim follows.