Analysis I
Lecture Notes

Guilherme Zeus Dantas e Moura
gdantasemo@haverford.edu

Haverford College — Fall 2021
Last updated: September 30, 2021

This is Haverford College's undergraduate MATH H317, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 What are the real numbers? **4**

1.1 Defining the real numbers: an axiomatic aproach 4

1.2 Bounds 5

1.3 Absolute value 5

1.4 Completeness: the key to define the real numbers 7

2 Getting to know the Real Numbers **8**

2.1 Archimedean Properties 9

2.2 Nested Interval Property 9

2.3 Cardinality 10

3 Limits **13**

3.1 Sequences 13

3.2 Subsequences 17

3.3 Series 20

1 What are the real numbers?

Lecture 1

1.1 Defining the real numbers: an axiomatic aproach

The main idea is to derive $\mathbb{R}$ from $\mathbb{Q}$. We will layout some properties that $\mathbb{Q}$ has that we also want $\mathbb{R}$ to have; and then add an additional property that will distinguish $\mathbb{Q}$ from $\mathbb{R}$.

First, $\mathbb{Q}$ is a field, and we also want $\mathbb{R}$ to be a field.

Definition 1.1 (Field Axioms)

A set F is a *field* if there exist two operations — addition and multiplication — that satisfy the following list of conditions:

i. (Commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.
ii. (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.
iii. (Identities) There exist two special elements, denoted by 0 and 1, such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.
iv. (Inverses) Given $x \in F$, there exists an element $-x \in F$ such that $x+(-x) = (-x)+x = 0$. If $x \neq 0$, there exits an element x^{-1} such that $xx^{-1} = x^{-1}x = 1$.
v. (Distributivity) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Being a field is not restrictive enough, since it allows for finite fields, such as $\mathbb{Z}/p\mathbb{Z}$, or complex numbers $\mathbb{C}$. Another feature of $\mathbb{Q}$ (and a desired feature of $\mathbb{R}$) is order.

Definition 1.2 (Ordering)

An *ordering* on a set F is a relation, represented by $\leq$, with the following properties:

i. $x \leq y$ or $y \leq x$, for all $x, y \in F$.
ii. If $x \leq y$ and $y \leq x$, then $x = y$.
iii. If $x \leq y$ and $y \leq z$, then $x \leq z$.

We define $x < y$ as equivalent to $x \leq y$ and $x \neq y$. We define $y \geq x$ as equivalent to $x \leq y$. We define $y > x$ as equivalent to $x < y$.

Additionally, a field F is called an *ordered field* if F is endowed with an ordering $\leq$ that satisfies

iv. If $y \leq z$, then $x + y \leq x + z$.
v. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

Now, we need to add a feature that distinguishes $\mathbb{Q}$ and our desired $\mathbb{R}$. Intuitively, "$\mathbb{Q}$ has holes", meaning that one can build a sequence in $\mathbb{Q}$ that approaches a limit that is not in $\mathbb{Q}$; on the other hand, "$\mathbb{R}$ has no holes", meaning that any sequence in $\mathbb{R}$ that converges can only converge to a limit that is in $\mathbb{R}$.

That's the main idea we will formalize next.

Lecture 2

1.2 Bounds

Definition 1.3 (Upper bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is an *upper bound of A* if $a \leq b$ for all $a \in A$.

If a set A has an upper bound, we say that A is *bounded above*.

Definition 1.4 (Supremum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *least upper bound of A*, or *supremum of A*, denoted by $\sup(A)$, if:

i. s is an upper bound of A, and
ii. if b is any upper bound of A, then $s \leq b$.

Example

Let $F = \mathbb{Q}$ and $A = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\}$. Then, $\sup(A) = 0$.

Definition 1.5 (Lower bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is a *lower bound of A* if $a \geq b$ for all $a \in A$.

If a set A has a lower bound, we say that A is *bounded below*.

Definition 1.6 (Infimum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *greatest lower bound of A*, or *infimum of A*, denoted by $\inf(A)$, if:

i. s is a lower bound of A, and
ii. if b is any lower bound of A, then $s \geq b$.

1.3 Absolute value

Before we dig more deeply into the idea of a supremum, consider this definition that comes just from the structure of an ordered field.

Definition 1.7 (Absolute value)

If F is an ordered field, and $x \in F$, let

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

Theorem 1.8

If F is an ordered field, and $x \in F$, then $|x| \geq 0$.

Proof. If $x \geq 0$, then $|x| = x \geq 0$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x = |x|$. ∎

Theorem 1.9

If F is an ordered field, and $x \in F$, then $|-x| = |x|$.

Proof. If $x \geq 0$, then $0 = x + (-x) \geq 0 + (-x) = -x$, therefore $|-x| = -(-x) = x = |x|$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$, therefore $|-x| = -x = |x|$. ∎

Theorem 1.10

If F is an ordered field, and $x, y \in F$, then $|xy| = |x||y|$.

Proof. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.

If $x \geq 0$ and $y \leq 0$, then $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the previous case with x and $-y$ and also Theorem 1.9 to obtain $|xy| = |-xy| = |x(-y)| = |x||-y| = |x||y|$.

If $x \leq 0$ and $y \geq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$. So we apply the first case with $-x$ and y and also Theorem 1.9 to obtain $|xy| = |-xy| = |(-x)y| = |-x||y| = |x||y|$.

If $x \leq 0$ and $y \leq 0$, then $0 = x+(-x) \leq 0+(-x) = -x$ and $0 = y+(-y) \leq 0+(-y) = -y$. So we apply the first case with $-x$ and $-y$ and also Theorem 1.9 to obtain $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$. ∎

Theorem 1.11 (Triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x + y| \leq |x| + |y|.$$

Proof. If $x \geq 0$, then $|x| = x$. If $x \leq 0$, then $x \leq 0 = x + (-x) \leq 0 + (-x) = x$, so $|x| = -x \geq x$. In either case, $|x| \geq x$.

Thus, $|x|+|y| \geq x+y$ and $|x|+|y| = |-x|+|-y| \geq -x-y = -(x+y)$. Since $|x+y| = x+y$ or $|x+y| = -(x+y)$, in either case, $|x|+|y| \geq |x+y|$. ∎

Theorem 1.12 (Reverse triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x - y| \geq ||x| - |y||.$$

Proof. Triangle inequality implies that $|x| = |(x-y)+y| \leq |x-y|+|y|$ and $|y| = |(y-x)+x| \leq |y - x| + |x|$. Equivalently, we have $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$; consequently, $|x - y| \geq ||x| - |y||$. ∎

1.4 Completeness: the key to define the real numbers

Definition 1.13 (Completeness)

Given F an ordered field, we say F is *complete* if, for any subset $A \subset F$ bounded above and nonempty, the supremum of A exists[a].

[a]and is an element of F, as the definition requires.

Definition 1.14 (Real numbers)

The set of real numbers is a complete ordered field. In other words, we define $\mathbb{R}$ to be any set that obeys the field axioms, the order axioms and the Axiom of Completeness.

Subtely, this leaves open the possibility that there is more than one set that is "the real numbers", or no such set. However, there is a theorem that states that there is a unique complete ordered field[*].

[*]up to isomorphism.

2 Getting to know the Real Numbers

Lecture 3

Theorem 2.1 (ϵ-sup Theorem)

Given $A \subset \mathbb{R}$ nonempty and bounded above, and given s an upper bound of A, then $s = \sup(A)$ if, and only if, for all $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.

Proof. Supppose $s = \sup(A)$. Then, $s - \epsilon$ is not an upper bound of A. Therefore, there exists $a \in A$ such that $a > s - \epsilon$.

Suppose $s \neq \sup(A)$. Then, there exists an upper bound of A that is smaller than s, say $s - \delta$. Then, it follows that, for $\epsilon = \delta/2$, there is no $a \in A$ such that $a > s - \delta/2$, because $a \leq s - \delta < s - \delta/2$ for all $a \in A$. ∎

Definition 2.2 (Sum of Sets)

Given $A, B \subset \mathbb{R}$, we define their sum as

$$A + B = \{a + b : a \in A, b \in B\}$$

Theorem 2.3 (Supremum of Sum of Sets)

If $A, B \subset \mathbb{R}$ are both nonempty and bounded above, then

$$\sup(A + B) = \sup(A) + \sup(B).$$

Proof. Since $\sup(A)$ is an upper bound of A, it holds that $a \leq \sup(A)$ for all $a \in A$. Since $\sup(B)$ is an upper bound of B, it holds that $b \leq \sup(B)$ for all $b \in B$. Therefore, $a + b \leq \sup(A) + \sup(B)$ for all $a \in A$ and $b \in B$, i.e., $x \leq \sup(A) + \sup(B)$ for all $x \in A + B$; thus, $\sup(A) + \sup(B)$ is an upper bound of $A + B$.

Let $\epsilon > 0$ be any positive real number. ϵ-sup Theorem implies that there exists $a \in A$ such that $a > \sup(A) - \epsilon/2$. ϵ-sup Theorem also implies that there exists $b \in B$ such that $b > \sup(B) - \epsilon/2$. Therefore, there exist $a \in A$ and $b \in B$ such that $a + b > \sup(A) + \sup(B) - \epsilon$; thus, there exists $x \in X$ such that $x > \sup(A) + \sup(B) - \epsilon$. Finally, by ϵ-sup Theorem, $\sup(A + B) = \sup(A) + \sup(B)$. ∎

Lecture 4

2.1 Archimedean Properties

Theorem 2.4 (Archimedean Properties)

i. Given any $x \in \mathbb{R}$, there exists some $n \in \mathbb{Z}_{>0}$ with $n > x$.
ii. Given any $y > \mathbb{R}_{>0}$, there exists some $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < y$.

Proof. The first statement is equivalent to $\mathbb{Z}_{>0}$ is not bounded above.

Suppose $\mathbb{Z}_{>0}$ is bounded above. Then, there exists $s = \sup(\mathbb{Z}_{>0})$. Therefore, $s - 1$ is not an upper bound of $\mathbb{Z}_{>0}$, i.e., there exists $n \in \mathbb{Z}_{>0}$ such that $s - 1 < n$. However, this implies $s < n + 1 \in \mathbb{Z}_{>0}$, implies s is not an upper bound of $\mathbb{Z}_{>0}$, which is a contradiction.

The second statement follows from the first one by setting $x = \frac{1}{n}$. ∎

Theorem 2.5 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For all $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ with $a < q < b$.

Proof. By Archimedean Properties with $y = b - a > 0$, there exists $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < b - a$.

Let m be the smallest natural number greater than na. Then,

$$\begin{gathered} m - 1 \leq na < m \\ \frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n}. \end{gathered}$$

The first inequality implies that $\frac{m}{n} \leq a + \frac{1}{n} < b$, so finally, we conclude that

$$a < \frac{m}{n} < b.$$

∎

Corolary 2.6

For all $a, b \in \mathbb{R}$, with $a < b$, there exist infinitely many $q \in \mathbb{Q}$ with $a < q < b$.

Lecture 5

2.2 Nested Interval Property

Theorem 2.7 (Nested Interval Property)

Suppose we have a sequence of closed intervals $I_n = [a_n, b_n]$, with $a_n \leq b_n$, that are nested decreasing, i.e.,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots .$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

Proof. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$.

The "nested" condition implies that, if $i < j$, then $[a_i, b_i] \supset [a_j, b_j]$. Therefore, $a_j, b_j \in [a_i, b_i]$, which implies that $a_i \leq a_j \leq b_j \leq b_i$ for all $i < j$. Note that this implies that

$$a_i \leq b_j \text{ and } a_j \leq b_i, \text{ for all } i < j.$$

We can rewrite it as

$$a_i \leq b_j, \text{ for all } i \text{ and } j.$$

This implies that a_i is a lower bound of B for any i, and also implies that b_j is an upper bound of A for any j. Since A is bounded above, we can define $x = \sup(A)$. Clearly, x is an upper bound of A.

Suppose x is not a lower bound of B. Then, there exists n such that $x > b_n$. The ϵ-sup Theorem, with $\epsilon = x - b_n > 0$, implies that there exists m such that $a_m > x - (x - b_n) = b_n$, which contradicts the previous displayed equation. Therefore, x is a lower bound of B.

Finally, x is both an upper bound of A and a lower bound of B, thus, for all n, $a_n \leq x \leq b_n$, i.e., $x \in [a_n, b_n]$. Therefore, x is in such intersection. ■

2.3 Cardinality

Question. Are all sets with an infinite number of elements the same size?

Definition 2.8 (Cardinality)

Given two sets A and B, we say that *A and B have the same cardinality* if there exists a bijection $f\colon A \to B$. We will write $A \sim B$ to say that A and B have the same cardinality.

Example

The sets $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\{2, 4, 6, 8, \dots\}$ have the same cardinality.

Definition 2.9 (Countability)

We say a set S is *countable* if it has the same cardinality as $\mathbb{N}$. If a set is not a finite set and not countable, then we say it is *uncountable*.

Proposition 2.10 ($\mathbb{N}^2$ is countable)

$\mathbb{N}^2$ is countable.

Proof. The function $f\colon \mathbb{N}^2 \to \mathbb{N}$, defined by

$$f(i,j) = \frac{(i+j-1)(i+j-2)}{2} + i$$

is a bijection. ■

Lecture 6

Theorem 2.11

If A is countable and B is countable, then $A \times B$ is countable.

Similarly, if $A_1, A_2, \ldots, A_n$ are each countable, then $A_1 \times \cdots \times A_n$ is countable.

Similarly, if $A_1, A_2, \ldots, A_n$ are each countable or finite, then $A_1 \times \cdots \times A_n$ is countable or finite.

Theorem 2.12

If $S_1, S_2, \ldots$ are each countable, then their union is countable.

Similarly, if $\{S_i\}_{i \in I}$ is a countable or finite collection of sets, which are each countable or finite; then their union is countable.

Example

Let $\mathcal{T}$ be the collection of finite subsets of $\mathbb{N}$. For each $i \in \mathbb{N}$, let A_i be the collection of subsets of $\{1, 2, \ldots, i\}$. Note that $|A_i| = 2^i$, thus A_i is finite. Then, note that $\varnothing \in A_1$, and, if $S \in \mathcal{T}$ is non-empty, it holds that $S \in A_{\max(S)}$; so $\mathcal{T} = \bigcup_{i=1}^{\infty} A_i$.

Therefore, by Theorem 2.12, we conclude that $\mathcal{T}$ is countable or finite. Since $\mathcal{T}$ is not finite, then it is countable.

Theorem 2.13

If A is countable, and $f : A \to B$ is surjective, then B is countable or finite.

Similarly, if A is countable, and $f : B \to A$ is injective, then B is countable or finite.

In particular, if A is countable, and $A \supseteq B$, then B is countable or finite.

Proposition 2.14 ($\mathbb{Q}$ is countable)

$\mathbb{Q}$ is countable.

Proof. Consider the function $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ defined by $f(a,b) = \frac{a}{b}$. Clearly, $f(p,q) = \frac{p}{q}$ for any $\frac{p}{q} \in \mathbb{Q}$. ■

Proposition 2.15

$\mathbb{R}$ is not countable.

Proof (using nested intervals). Assume $\mathbb{R}$ is countable. So, there exists a bijection $f : \mathbb{N} \to \mathbb{R}$.

Let $I_1 = [f(1) + 1, f(2) + 2]$. Note that $f(1) \notin I_1$. We will define I_{n+1} recursively. Suppose $I_n = [a, b]$, then, define I_{n+1} as either $[a, \frac{2a+b}{3}]$ or $[\frac{a+2b}{3}, b]$ such that $f(i + 1) \notin I_{n+1}$; that is possible since $f(i + 1)$ cannot be in both sets.

By the Nested Interval Property, there exists a real number $r \in \bigcap_{i=1}^{\infty} I_n$. However, since f is a bijection, there exists $m \in \mathbb{N}$ such that $f(m) = r$. Therefore, $r \notin I_m$, a contradiction. ■

Lecture 7

Proof (using Cantor's diagonalization). We'll prove $(0, 1)$ is uncountable, which implies $\mathbb{R}$ is uncountable.

Assume $(0, 1)$ is countable, therefore, there exists a bijective function $f : \mathbb{N} \to (0, 1)$.

Let's write out decimal expansions[a] of $f(1), f(2), \ldots$. If there's doubt between a recurrent 9 or a recurrent 0 in the end, we choose the latter form. We write

$$f(i) = 0.a_{i1}a_{i2}a_{i3}\ldots,$$

with $a_{ij} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Let $b_k = 1$, if a_{kk} is odd, and $b_k = 2$, if a_{kk} is even. Note that $c_k \neq b_{kk}$ and $c_k \notin \{0, 9\}$ for all k. Therefore, $x = 0.b_1b_2b_3\ldots$ cannot be on the image of f; a contradiction. ■

[a] What are decimal expansions? We only need to know that decimal expansions are unique except for some duplication, like $0.09999 = 0.1$.

Another perspective on the Cantor's proof arises by using the binary base, instead of the decimal base. For each real number $x = 0.x_1x_2x_3\ldots$, we can define a $f(x) = \{n \in \mathbb{N} : a_n = 1\}$. This is almost[*] a bijection because, but nevertheless, we can conclude that, in some sense,

$$|\mathbb{R}| = 2^{|\mathbb{N}|}.$$

[*] The same number with two expansions yields a problem.

3 Limits

3.1 Sequences

Definition 3.1 (Limit of a sequence)

We say a sequence $(a_n) = a_1, a_2, a_3, \ldots$ *converges to a real number* a if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N$.

If this definition holds for some a, we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$.

If this definition does not hold for any a, we say $\lim_{n\to\infty} a_n$ does not exist, or that the sequence diverges.

Proposition 3.2 (The limit, if it exists, is unique)

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = a'$, then $a = a'$.

Proof. For all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $|a_n - a| < \epsilon$ for all $n \geq N_\epsilon$. For all $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that $|a_n - a'| < \epsilon$ for all $n \geq M_\epsilon$.

Therefore, for all $\epsilon > 0$, there exists $L_\epsilon \in \mathbb{N}$, namely $\max\{N_\epsilon, M_\epsilon\}$, such that $|a_n - a| < \epsilon$ and $|a_n - a'| < 0$ for all $n \geq L_\epsilon$. Triangle inequality implies that $|a - a'| < 2\epsilon$ for all $\epsilon > 0$; thus $a = a'$. ∎

Example

We claim that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

This is true because, given $\epsilon > 0$, we can choose N be a natural number larger than $\sqrt{\frac{1}{\epsilon}}$. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N^2} > \frac{1}{n^2} = \left|\frac{1}{n^2} - 0\right|.$$

Example (The limit does not exist)

We claim that $\lim_{n\to\infty}(-1)^n$ does not exist.

Suppose it does exist, namely a. Then, consider $\epsilon = \frac{1}{2}\max\{|a-1|, |a+1|\}$. Not both $|a-1|$ and $|a+1|$ can be zero, so $\epsilon > 0$. However, since $\lim_{n\to\infty} = a$, for that ϵ, it must hold that there exists $N \in \mathbb{N}$ so that for all $n \geq N$, $|a - (-1)^n| < \epsilon$.

In particular, note that plugging in $n \mapsto N$ and $n \mapsto N+1$ imply that $|a-1| < \epsilon$ and

$|a+1| < \epsilon$; which is a contradiction given our choice of ϵ.

Example

We claim that $\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$. Note that we can rewrite $\frac{2n+1}{n+3} = 2 - \frac{5}{n+3}$. For any ϵ, there exists $N \in \mathbb{N}$ such that $N > \frac{5}{\epsilon}$. Therefore, for all $n \geq N$, it holds that

$$\left|\left(2 - \frac{5}{n+3}\right) - 2\right| = \frac{5}{n+3} < \frac{5}{N} < \epsilon,$$

and our claim follows.

Lecture 8

Example

We claim that $\lim_{n\to\infty} \frac{2n^2}{5n^3-7} = 0$.

This is true because, given $\epsilon > 0$, we can choose N to be a natural number larger than $\frac{1}{\epsilon}$ and larger than 2. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N} > \frac{1}{n} > \frac{2n^2}{4n^3} > \frac{2n^2}{4n^3 + (n^3 - 7)} = \left|\frac{2n^2}{5n^3-7} - 0\right|$$

Theorem 3.3 (Algebraic Manipulation of Limits)

Suppose that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ and c, d are constant real numbers. Then,

- **i.** $\lim_{n\to\infty}(ca_n + db_n) = ca + db$
- **ii.** $\lim_{n\to\infty} a_n b_n = ab$
- **iii.** $\lim_{n\to\infty}(1/a_n) = 1/a$ if the $a_n \neq 0$ for all n and $a \neq 0$.
- **iv.** $\lim_{n\to\infty}(a_n/b_n) = a/b$ if the $b_n \neq 0$ for all n and $b \neq 0$.

Proof.

i. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < \frac{\epsilon}{2|c|}$$

for all $n \geq N$. Similarly, there exists M such that

$$|b_n - b| < \frac{\epsilon}{2|d|}$$

for all $n \geq M$. Therefore, for all $n \geq \max\{N, M\}$, it holds that

$$\begin{aligned} |(ca_n + db_n) - (ca + db)| &= |(ca_n - ca) + (db_n - db)| \\ &\leq |ca_n - ca| + |db_n - db| \\ &\leq |c||a_n - a| + |d||b_n - b| \\ &< \epsilon, \end{aligned}$$

thus, $\lim_{n\to\infty} ca_n + db_n$.

ii. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < 1$$

for all $n \geq N$; therefore, $|a_n| < |a| + 1$ for all $n \geq N$.
Since $\lim_{n\to\infty} a_n = a$, there exist M such that

$$|a_n - a| < \frac{\epsilon}{|b|}$$

for all $n \geq M$. Similarly, there exist O such that

$$|b_n - b| < \frac{\epsilon}{2(|a| + 1)}$$

for all $n \geq O$. Therefore, for all $n \geq \max\{N, M, O\}$, it holds that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n b_n = ab$.

iii. Without loss of generality, suppose $a > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$a_n > \frac{a}{2} > 0$$

for all $n \geq N$. Therefore, $0 < \frac{1}{a_n} < \frac{2}{a}$ for all $n \geq N$.
Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exists M so that

$$|a_n - a| < \frac{\epsilon a^2}{2}.$$

Then, for all $n \geq \max\{N, M\}$, it holds that

$$\begin{aligned} \left|\frac{1}{a_n} - \frac{1}{a}\right| &= |a - a_n| \cdot \frac{1}{a} \cdot \left|\frac{1}{a_n}\right| \\ &< \frac{\epsilon a^2}{2} \cdot \frac{1}{a} \cdot \frac{2}{a} \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

iv. Using **ii** and **iii**, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(a_n \frac{1}{b_n}\right) \\ &= \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} \frac{1}{b_n}\right) \\ &= a \cdot \frac{1}{b} = \frac{a}{b}. \end{aligned}$$

■

Example

Since $\lim_{n\to\infty}(1+1/n) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n) = 1+0 = 1$ and $\lim_{n\to\infty}(1+1/n^2) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n^2) = 1+0 = 1$, we can conlude that

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2+n}{n^2+1} &= \lim_{n\to\infty} \frac{1+1/n}{1+1/n^2} \\ &= \frac{\lim_{n\to\infty} 1+1/n}{\lim_{n\to\infty} 1+1/n^2} \\ &= \frac{1}{1} = 1.\end{aligned}$$

Lecture 9

Definition 3.4 (Boundness)

A sequence $(a_n)_{n\in\mathbb{N}}$ is bounded if there exists $M \in \mathbb{R}$ so that $|a_n| \le M$ for all $n \in \mathbb{N}$.

Theorem 3.5 (A convergent sequence is bounded)

If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then (a_n) is bounded.

Proof. Let L be the limit of such sequence. Let $\epsilon = 1$. Then, there exists $N \in \mathbb{N}$ so that $|a_n - L| < 1$ for all $n \ge N$. Triangle inequality implies that $|a_n| < |L| + 1$ for all $n \ge N$. Define

$$M = \max\{|a_1|+1, |a_2|+1, \ldots, |a_{N-1}|+1, |L|+1\}.$$

Then, for this choice of M, it holds that $|a_n| < M$ for all $n \in \mathbb{N}$. Therefore, (a_n) is bounded. ∎

Lecture 10

Definition 3.6 (Monotone sequences)

We say (a_n) is *monotone increasing* if $a_{n+1} \ge a_n$ for all n.

We say (a_n) is *strictly monotone increasing* if $a_{n+1} < a_n$ for all n.

We say (a_n) is *monotone decreasing* if $a_{n+1} \le a_n$ for all n.

We say (a_n) is *strictly monotone decreasing* if $a_{n+1} < a_n$ for all n.

Theorem 3.7 (Monotone Convergence Theorem)

If (a_n) is monotone increasing and bounded above, then it converges.

Similarly, if (a_n) is monotone decreasing and bounded below, then it converges.

Proof. We will only prove the first statement. Let $\epsilon > 0$. Let $a = \sup\{a_1, a_2, a_3, \ldots\}$. ϵ-sup Theorem implies that there exists N so that $a - a_N < \epsilon$. Since the sequence is monotone increasing, for all $n \ge N$, we have that

$$|a - a_n| = a - a_n < \epsilon;$$

thus, $\lim_{n\to\infty} a_n = n$. ■

Example

What in the world is $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$? If it exists, it would be plausible to be the limit of the sequence

$$\sqrt{6}, \quad \sqrt{6+\sqrt{6}}, \quad \sqrt{6+\sqrt{6+\sqrt{6}}}, \ldots$$

The easier way to make sense of this sequence is using recursion. We will define it as

$$a_1 = \sqrt{6}, \quad \text{and} \quad a_n = \sqrt{6+a_{n-1}} \text{ for } n \geq 2.$$

We know that $a_1 = \sqrt{6} < \sqrt{6+\sqrt{6}} = a_2$. Suppose that $a_{n-1} < a_n$. Then, $a_n = \sqrt{6+a_{n-1}} < \sqrt{6+a_n} = a_{n+1}$. Therefore, by induction, $a_{n+1} > a_n$ for all $n \geq 1$, i.e., the sequence a_n is monotone increasing.

We also know that $a_1 < 10$. Suppose that $a_{n-1} < 10$. Then, $a_n = \sqrt{6+a_n} < \sqrt{16} < 10$. Therefore, by induction, $a_n < 10$ for all $n \geq 1$, i.e., 10 is an upper bound of a_n.

By the Monotone Convergence Theorem, we conclude that a_n has a limit. Finally,

$$\begin{aligned} \left(\lim_{n\to\infty} a_n\right)^2 &= \lim_{n\to\infty} a_n^2 \\ &= \lim_{n\to\infty} (6 + a_{n-1}) \\ &= 6 + \lim_{n\to\infty} a_n; \end{aligned}$$

therefore, $\lim_{n\to\infty} a_n = 3$ or $\lim_{n\to\infty} a_n = -2$. Since a_n evaluates to positive real numbers, the latter proposition yields a contradiction when plugging $\epsilon \mapsto 1$. Therefore, the former proposition must be true, i.e.,

$$\lim_{n\to\infty a_n} = 3.$$

Theorem 3.8 (Limits preserve $\leq$)

Let $N \in \mathbb{N}$. Suppose $a_n \leq b_n$ for all $n \geq N$, and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then, $a \leq b$.

3.2 Subsequences

Lecture 11

Definition 3.9 (Subsequence)

Given a sequence (a_n) and a strictly monotone increasing sequence of natural numbers (n_i), the sequence (a_{n_i}) is called a *subsequence* of (a_n).

In other words, we can say that (b_k) is a subsequence of (a_n) if there exists a strictly monotone increasing $f : \mathbb{N} \to \mathbb{N}$ so that $b_k = a_{f(k)}$ for all k.

Theorem 3.10

A sequence converges to L if, and only if, every of its subsequences converges to L.

Proof. The inverse implication is straightforward, since the sequence is a subsequence of itself. Let's prove the direct implication. Let (a_n) be a sequence so that $a_n \to L$. Let (a_{n_i}) be a subsequence of (a_n). Let $\epsilon > 0$. Since $a_n \to L$, there exists N so that

$$|L - a_n| < \epsilon,$$

for all $n \geq N$. Note that $n_i \geq i$. Therefore, for the same choice of N, it holds that

$$|L - a_{n_i}| < \epsilon$$

for all $i \geq N$. Therefore, $a_{n_i} \to L$. ∎

Lecture 12

Theorem 3.11 (Squeeze Theorem)

Show that if $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then $\lim_{n\to\infty} y_n = L$.

Proof. For all $n \in \mathbb{N}$, since $x_n \leq y_n \leq z_n$, $|z_n - x_n| = |z_n - y_n| + |y_n - x_n|$, which implies

$$|z_n - x_n| \geq |y_n - x_n|. \tag{3.1}$$

Theorem 3.3 implies that $\lim_{n\to\infty}(z_n - x_n) = \lim_{n\to\infty} z_n - \lim_{n\to\infty} x_n = 0$.

Let $\epsilon > 0$. Therefore, since $(z_n - x_n) \to 0$, there exists N such that $|z_n - x_n| < \epsilon$ for all $n \geq N$. Equation (3.1) implies that, for the same choice of N, it holds that $|y_n - x_n| < \epsilon$ for all $n \geq N$. Therefore, $(y_n - x_n) \to 0$. Since $(x_n) \to L$ and $(y_n - x_n) \to 0$, theorem 3.3 implies $(y_n) \to L$. ∎

Example

We claim that $\lim_{n\to\infty} \sqrt{n^2 + 4n} - n = 2$.

A good intuition for that to be true is that $\sqrt{n^2 + 4n} - n \approx \sqrt{n^2 + 4n + 4} - n = 2$.

Formally,

$$\begin{aligned} \sqrt{n^2 + 4n} - n &= \frac{(n^2 + 4n) - n^2}{\sqrt{n^2 + 4n} + n^2} \\ &= \frac{4}{\sqrt{1 + 4/n} + 1} \to 2. \end{aligned}$$

Theorem 3.12 (Bolzano-Weierstrass Theorem)

Every bounded sequence has a convergent subsequence.

Proof. Since (a_n) is bounded, there exists M such that $a_n \leq M$ for all n. Let $I_1 = [-M, M]$. Note that infinitely many terms of (a_n) are in I_1.

Suppose $I_k = [a_k, b_k]$ contains infinitely many terms of (a_n). Define I_{k+1} as either $[a_k, \frac{a_k+b_k}{2}]$ or $[\frac{a_k+b_k}{2}, b_k]$ such that I_{k+1} contais infinitely many terms of (a_n).

Nested Interval Property implies that there exists $x \in I_j$ for all j.

Let $n_1 = 1$, so that $a_{n_1} \in I_1$. Define $n_{i+1} > n_i$, so that $a_{n_{i+1}} \in I_{i+1}$; which is possible since I_{n+1} has infinitely many terms.

For each j, both a_{n_j} and x are in I_j. Since the width of I_j is $2M/2^{j-1}$, we conclude

$$-\frac{2M}{2^{j-1}} + x \leq a_{n_j} \leq \frac{2M}{2^{j-1}} + x,$$

thus the Squeeze Theorem implies $(a_{n_j}) \to x$. ∎

Definition 3.13 (Cauchy sequence)

A sequence is *Cauchy* if, for all $\epsilon > 0$, there exists N so that $|a_m - a_n| < \epsilon$ for all $m, n \geq N$.

Example

We claim that the sequence $a_n = \frac{(-1)^n}{n}$ is Cauchy.

Let $\epsilon > 0$. Choose N larger than $\frac{1}{2\epsilon}$.

Then, for all $n, m \geq N$, it holds that

$$\begin{aligned}\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| &= \left|\frac{1}{n} \pm \frac{1}{m}\right| \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{2}{N} \\ &< \epsilon.\end{aligned}$$

Proposition 3.14

Every convergent sequence is Cauchy.

Proof. Let $\epsilon > 0$. Since $(a_n) \to L$, there exists N so that

$$|a_n - L| < \frac{\epsilon}{2}$$

for all $n \geq N$. Therefore, using the triangle inequality,

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \epsilon$$

for all $n, m \geq N$; thus the sequence is Cauchy. ∎

Lecture 13

Proposition 3.15

Every Cauchy sequence of real numbers is bounded.

Proof. Let $\epsilon = 1$. There exist N so that $|a_m - a_n| < 1$ for all $m, n \geq N$. This implies that $|a_m - a_N| < 1$ for all $m \geq N$, and consequently, by triangle inequality, $|a_m| = |a_m - 0| \leq |a_m - a_N| + |a_N - 0| < 1 + |a_N|$ for all $m \geq N$.

Therefore, if we set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\},$$

we conclude $|a_m| < M$ for all m. ∎

Proposition 3.16

If (a_n) is Cauchy, and if some subsequence of (a_n) converges to some limit a, then the whole sequence (a_n) converges to a.

Proof. Let $\epsilon > 0$. Let (a_{k_i}) be such sequence that converges to a.

Also, since (a_n) is Cauchy, there exists N so that

$$|a_m - a_n| < \epsilon$$

for all $m, n \geq N$.

In particular, by setting $m = k_n \geq n$, we conclude

$$|a_{k_n} - a_n| < \epsilon$$

for all $n \geq N$. Therefore, $(a_{k_n} - a_n) \to 0$. Since $(a_{k_n}) \to 0$, 3.3 ∎

Theorem 3.17

Every Cauchy sequence of real numbers is convergent.

Proof. Let (a_n) be a Cauchy sequence o ∎

3.3 Series

Lecture 14

Definition 3.18 (Series)

Given a sequence (a_n), we associate it with a sequence (s_n), defined by

$$s_n = \sum_{k=1}^{n} a_k.$$

As an abuse of notation, we denote (s_n) using the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or

$$\sum_{n=1}^{\infty} a_n.$$

We call those expressions *(infinite) series*. Each s_n is called a *partial sum* of this series. If (s_n) converges to s, we say that the series *converges*, which we denote symbolically by

$$\sum_{n=1}^{\infty} a_n = s,$$

which we call the sum of the series; though it is actually the limit of a sequence of partial sums.

If (s_n) diverges, we say that the series diverges.

Note that theorems about sequences can be stated in terms of series and vice versa, by defining $a_1 = s_1$ and $a_n = s_n - s_{n-1}$.