Analysis I
Lecture Notes

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This is Haverford College's undergraduate MATH H317, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 What are the real numbers?

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1.1 Defining the real numbers: an axiomatic aproach

The main idea is to derive $\mathbb{R}$ from $\mathbb{Q}$. We will layout some properties that $\mathbb{Q}$ has that we also want $\mathbb{R}$ to have; and then add an additional property that will distinguish $\mathbb{Q}$ from $\mathbb{R}$.

First, $\mathbb{Q}$ is a field, and we also want $\mathbb{R}$ to be a field.

Definition 1.1 (Field Axioms)

A set F is a *field* if there exist two operations — addition and multiplication — that satisfy the following list of conditions:

i. (Commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.
ii. (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.
iii. (Identities) There exist two special elements, denoted by 0 and 1, such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.
iv. (Inverses) Given $x \in F$, there exists an element $-x \in F$ such that $x+(-x) = (-x)+x = 0$. If $x \neq 0$, there exits an element x^{-1} such that $xx^{-1} = x^{-1}x = 1$.
v. (Distributivity) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Being a field is not restrictive enough, since it allows for finite fields, such as $\mathbb{Z}/p\mathbb{Z}$, or complex numbers $\mathbb{C}$. Another feature of $\mathbb{Q}$ (and a desired feature of $\mathbb{R}$) is order.

Definition 1.2 (Ordering)

An *ordering* on a set F is a relation, represented by $\leq$, with the following properties:

i. $x \leq y$ or $y \leq x$, for all $x, y \in F$.
ii. If $x \leq y$ and $y \leq x$, then $x = y$.
iii. If $x \leq y$ and $y \leq z$, then $x \leq z$.

We define $x < y$ as equivalent to $x \leq y$ and $x \neq y$. We define $y \geq x$ as equivalent to $x \leq y$. We define $y > x$ as equivalent to $x < y$.

Additionally, a field F is called an *ordered field* if F is endowed with an ordering $\leq$ that satisfies

iv. If $y \leq z$, then $x + y \leq x + z$.
v. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

Now, we need to add a feature that distinguishes $\mathbb{Q}$ and our desired $\mathbb{R}$. Intuitively, "$\mathbb{Q}$ has holes", meaning that one can build a sequence in $\mathbb{Q}$ that approaches a limit that is not in $\mathbb{Q}$; on the other hand, "$\mathbb{R}$ has no holes", meaning that any sequence in $\mathbb{R}$ that converges can only converge to a limit that is in $\mathbb{R}$.

That's the main idea we will formalize next.

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1.2 Bounds

Definition 1.3 (Upper bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is an *upper bound of* A if $a \leq b$ for all $a \in A$.

If a set A has an upper bound, we say that A is *bounded above*.

Definition 1.4 (Supremum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *least upper bound of* A, or *supremum of* A, denoted by $\sup(A)$, if:

i. s is an upper bound of A, and

ii. if b is any upper bound of A, then $s \leq b$.

Example

Let $F = \mathbb{Q}$ and $A = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\}$. Then, $\sup(A) = 0$.

Definition 1.5 (Lower bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is a *lower bound of* A if $a \geq b$ for all $a \in A$.

If a set A has a lower bound, we say that A is *bounded below*.

Definition 1.6 (Infimum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *greatest lower bound of* A, or *infimum of* A, denoted by $\inf(A)$, if:

i. s is a lower bound of A, and

ii. if b is any lower bound of A, then $s \geq b$.

1.3 Absolute value

Before we dig more deeply into the idea of a supremum, consider this definition that comes just from the structure of an ordered field.

Definition 1.7 (Absolute value)

If F is an ordered field, and $x \in F$, let

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

Theorem 1.8

If F is an ordered field, and $x \in F$, then $|x| \geq 0$.

Proof. If $x \geq 0$, then $|x| = x \geq 0$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x = |x|$. ∎

Theorem 1.9

If F is an ordered field, and $x \in F$, then $|-x| = |x|$.

Proof. If $x \geq 0$, then $0 = x + (-x) \geq 0 + (-x) = -x$, therefore $|-x| = -(-x) = x = |x|$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$, therefore $|-x| = -x = |x|$. ∎

Theorem 1.10

If F is an ordered field, and $x, y \in F$, then $|xy| = |x||y|$.

Proof. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.

If $x \geq 0$ and $y \leq 0$, then $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the previous case with x and $-y$ and also Theorem 1.9 to obtain $|xy| = |-xy| = |x(-y)| = |x||-y| = |x||y|$.

If $x \leq 0$ and $y \geq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$. So we apply the first case with $-x$ and y and also Theorem 1.9 to obtain $|xy| = |-xy| = |(-x)y| = |-x||y| = |x||y|$.

If $x \leq 0$ and $y \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$ and $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the first case with $-x$ and $-y$ and also Theorem 1.9 to obtain $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$. ∎

Theorem 1.11 (Triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x + y| \leq |x| + |y|.$$

Proof. If $x \geq 0$, then $|x| = x$. If $x \leq 0$, then $x \leq 0 = x + (-x) \leq 0 + (-x) = x$, so $|x| = -x \geq x$. In either case, $|x| \geq x$.

Thus, $|x| + |y| \geq x + y$ and $|x| + |y| = |-x| + |-y| \geq -x - y = -(x + y)$. Since $|x + y| = x + y$ or $|x + y| = -(x + y)$, in either case, $|x| + |y| \geq |x + y|$. ∎

Theorem 1.12 (Reverse triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x - y| \geq ||x| - |y||.$$

Proof. Triangle inequality implies that $|x| = |(x-y)+y| \leq |x-y|+|y|$ and $|y| = |(y-x)+x| \leq |y - x| + |x|$. Equivalently, we have $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$; consequently, $|x - y| \geq ||x| - |y||$. ■

1.4 Completeness: the key to define the real numbers

Definition 1.13 (Completeness)

Given F an ordered field, we say F is *complete* if, for any subset $A \subset F$ bounded above and nonempty, the supremum of A exists[a].

[a]and is an element of F, as the definition requires.

Definition 1.14 (Real numbers)

The set of real numbers is a complete ordered field. In other words, we define $\mathbb{R}$ to be any set that obeys the field axioms, the order axioms and the Axiom of Completeness.

Subtely, this leaves open the possibility that there is more than one set that is "the real numbers", or no such set. However, there is a theorem that states that there is a unique complete ordered field[*]. 2021-09-03

Theorem 1.15 (ϵ-sup Theorem)

Given $A \subset \mathbb{R}$ nonempty and bounded above, and given s an upper bound of A, then $s = \sup(A)$ if, and only if, for all $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.

Proof. Supppose $s = \sup(A)$. Then, $s - \epsilon$ is not an upper bound of A. Therefore, there exists $a \in A$ such that $a > s - \epsilon$.

Suppose $s \neq \sup(A)$. Then, there exists an upper bound of A that is smaller than s, say $s - \delta$. Then, it follows that, for $\epsilon = \delta/2$, there is no $a \in A$ such that $a > s - \delta/2$, because $a \leq s - \delta < s - \delta/2$ for all $a \in A$. ■

Definition 1.16 (Sum of Sets)

Given $A, B \subset \mathbb{R}$, we define their sum as

$$A + B = \{a + b : a \in A, b \in B\}$$

[*]up to isomorphism.

Theorem 1.17 (Supremum of Sum of Sets)

If $A, B \subset \mathbb{R}$ are both nonempty and bounded above, then

$$\sup(A + B) = \sup(A) + \sup(B).$$

Proof. Since $\sup(A)$ is an upper bound of A, it holds that $a \leq \sup(A)$ for all $a \in A$. Since $\sup(B)$ is an upper bound of B, it holds that $b \leq \sup(B)$ for all $b \in B$. Therefore, $a + b \leq \sup(A) + \sup(B)$ for all $a \in A$ and $b \in B$, i.e., $x \leq \sup(A) + \sup(B)$ for all $x \in A + B$; thus, $\sup(A) + \sup(B)$ is an upper bound of $A + B$.

Let $\epsilon > 0$ be any positive real number. ϵ-sup Theorem implies that there exists $a \in A$ such that $a > \sup(A) - \epsilon/2$. ϵ-sup Theorem also implies that there exists $b \in B$ such that $b > \sup(B) - \epsilon/2$. Therefore, there exist $a \in A$ and $b \in B$ such that $a + b > \sup(A) + \sup(B) - \epsilon$; thus, there exists $x \in X$ such that $x > \sup(A) + \sup(B) - \epsilon$. Finally, by ϵ-sup Theorem, $\sup(A + B) = \sup(A) + \sup(B)$. ∎