Analysis I
Lecture Notes

Guilherme Zeus Dantas e Moura
gdantasemo@haverford.edu

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This is Haverford College's undergraduate MATH H317, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 What are the real numbers?

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1.1 Defining the real numbers: an axiomatic aproach

The main idea is to derive $\mathbb{R}$ from $\mathbb{Q}$. We will layout some properties that $\mathbb{Q}$ has that we also want $\mathbb{R}$ to have; and then add an additional property that will distinguish $\mathbb{Q}$ from $\mathbb{R}$.

First, $\mathbb{Q}$ is a field, and we also want $\mathbb{R}$ to be a field.

Definition 1.1 (Field Axioms)

A set F is a *field* if there exist two operations — addition and multiplication — that satisfy the following list of conditions:

i. (Commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.

ii. (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.

iii. (Identities) There exist two special elements, denoted by 0 and 1, such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.

iv. (Inverses) Given $x \in F$, there exists an element $-x \in F$ such that $x+(-x) = (-x)+x = 0$. If $x \neq 0$, there exits an element x^{-1} such that $xx^{-1} = x^{-1}x = 1$.

v. (Distributivity) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Being a field is not restrictive enough, since it allows for finite fields, such as $\mathbb{Z}/p\mathbb{Z}$, or complex numbers $\mathbb{C}$. Another feature of $\mathbb{Q}$ (and a desired feature of $\mathbb{R}$) is order.

Definition 1.2 (Ordering)

An *ordering* on a set F is a relation, represented by $\leq$, with the following properties:

i. $x \leq y$ or $y \leq x$, for all $x, y \in F$.

ii. If $x \leq y$ and $y \leq x$, then $x = y$.

iii. If $x \leq y$ and $y \leq z$, then $x \leq z$.

We define $x < y$ as equivalent to $x \leq y$ and $x \neq y$. We define $y \geq x$ as equivalent to $x \leq y$. We define $y > x$ as equivalent to $x < y$.
Additionally, a field F is called an *ordered field* if F is endowed with an ordering $\leq$ that satisfies

iv. If $y \leq z$, then $x + y \leq x + z$.

v. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

Now, we need to add a feature that distinguishes $\mathbb{Q}$ and our desired $\mathbb{R}$. Intuitively, "$\mathbb{Q}$ has holes", meaning that one can build a sequence in $\mathbb{Q}$ that approaches a limit that is not in $\mathbb{Q}$; on the other hand, "$\mathbb{R}$ has no holes", meaning that any sequence in $\mathbb{R}$ that converges can only converge to a limit that is in $\mathbb{R}$.