Math 317 Fall 2021: In-Class Activity for Mon. Aug. 30 (Completeness)

Our first goal is to define the set $\mathbb{R}$ of real numbers. We will *assume* we know definitions[1] of:

- the set of natural numbers $= \mathbb{N} = \{1, 2, 3, \cdots\}$
- the set of integers $= \mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$
- the set of rational numbers = the set of all ratios m/n where m, n are integers and $n \neq 0$

To make the jump from $\mathbb{Q}$ to $\mathbb{R}$ turns out to include a subtle new idea called "completeness". To explore this idea (before we define it later this week), try the following. For each of the five situations described below, provide an example as requested, or say "NO SUCH EXAMPLE EXISTS".[2]

Give an example of:

(1) a sequence of rational numbers that has a finite limit, with that limit being a rational number.

(2) a sequence of rational numbers that has a finite limit, with that limit being an irrational number.

(3) a sequence of irrational numbers that has a finite limit, with that limit being a rational number.

(4) a sequence of irrational numbers that has a finite limit, with that limit being an irrational number.

(5) a sequence of real numbers that has a finite limit, with that limit **not** being a real number.

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[1] A set theory course would dig carefully into these three definitions, but, intuitively at least, we could think that we "know" $\mathbb{N}$ by being able to count things; then $\mathbb{Z}$ comes from $\mathbb{N}$ by including the idea of additive inverse (and zero), and $\mathbb{Q}$ comes from $\mathbb{Z}$ by including the idea of division.

[2] Each example involves the "limit of a sequence", which we have not defined (but will). For now, proceed using whatever *intuition* you have about limits.