Analysis I
Lecture Notes

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This is Haverford College's undergraduate MATH H317, instructed by Robert Manning. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

1 What are the real numbers?

Lecture 1

The main idea is to derive $\mathbb{R}$ from $\mathbb{Q}$. We will layout some properties that $\mathbb{Q}$ has that we also want $\mathbb{R}$ to have; and then add an additional property that will distinguish $\mathbb{Q}$ from $\mathbb{R}$.

1.1 Algebraic Structure

First, $\mathbb{Q}$ is a field, and we also want $\mathbb{R}$ to be a field.

defn:field

Definition 1.1 (Field Axioms)

A set F is a *field* if there exist two operations — addition and multiplication — that satisfy the following list of conditions:

i. (Commutativity) $x + y = y + x$ and $xy = yx$ for all $x, y \in F$.

ii. (Associativity) $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$ for all $x, y, z \in F$.

iii. (Identities) There exist two special elements, denoted by 0 and 1, such that $x + 0 = x$ and $x1 = x$ for all $x \in F$.

iv. (Inverses) Given $x \in F$, there exists an element $-x \in F$ such that $x + (-x) = (-x) + x = 0$. If $x \neq 0$, there exits an element x^{-1} such that $xx^{-1} = x^{-1}x = 1$.

v. (Distributivity) $x(y + z) = xy + xz$ for all $x, y, z \in F$.

Being a field is not restrictive enough, since it allows for finite fields, such as $\mathbb{Z}/p\mathbb{Z}$, or complex numbers $\mathbb{C}$.

1.2 Order Structure

1.2.1 Ordering

Another feature of $\mathbb{Q}$ (and a desired feature of $\mathbb{R}$) is order.

defn:ordering

Definition 1.2 (Ordering)

An *ordering* on a set F is a relation, represented by $\leq$, with the following properties:

i. $x \leq y$ or $y \leq x$, for all $x, y \in F$.

ii. If $x \leq y$ and $y \leq x$, then $x = y$.
iii. If $x \leq y$ and $y \leq z$, then $x \leq z$.

We define $x < y$ as equivalent to $x \leq y$ and $x \neq y$. We define $y \geq x$ as equivalent to $x \leq y$. We define $y > x$ as equivalent to $x < y$.

Additionally, a field F is called an *ordered field* if F is endowed with an ordering $\leq$ that satisfies

iv. If $y \leq z$, then $x + y \leq x + z$.
v. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$.

Lecture 2

1.2.2 Bounds

Now, we need to add a feature that distinguishes $\mathbb{Q}$ and our desired $\mathbb{R}$. Intuitively, "$\mathbb{Q}$ has holes", meaning that one can build a sequence in $\mathbb{Q}$ that approaches a limit that is not in $\mathbb{Q}$; on the other hand, "$\mathbb{R}$ has no holes", meaning that any sequence in $\mathbb{R}$ that converges can only converge to a limit that is in $\mathbb{R}$.

defn:upperbound

Definition 1.3 (Upper bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is an *upper bound of A* if $a \leq b$ for all $a \in A$.

If a set A has an upper bound, we say that A is *bounded above*.

defn:supremum

Definition 1.4 (Supremum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *least upper bound of A*, or *supremum of A*, denoted by $\sup(A)$, if:
i. s is an upper bound of A, and
ii. if b is any upper bound of A, then $s \leq b$.

prop:supisunique

Proposition 1.5 (The supremum, if it exists, is unique)

If s and s' are both supremum of A, then $s = s'$.

Example

Let $F = \mathbb{Q}$ and $A = \{0, -1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \dots\}$. Then, $\sup(A) = 0$.

Analogously, we can define lower bounds and least upper bounds.

defn:lowerbound

Definition 1.6 (Lower bound)

If F is an ordered field, and $A \subset F$, then we say that some $b \in F$ is a *lower bound of A* if $a \geq b$ for all $a \in A$.

If a set A has a lower bound, we say that A is *bounded below*.

defn:infimum

Definition 1.7 (Infimum)

If F is an ordered field, and $A \subset F$, we say $s \in F$ is the *greatest lower bound of A*, or *infimum of A*, denoted by $\inf(A)$, if:

i. s is a lower bound of A, and
ii. if b is any lower bound of A, then $s \geq b$.

1.2.3 Completeness

defn:completeness

Definition 1.8 (Completeness)

Given F an ordered field, we say F is *complete* if, for any subset $A \subset F$ bounded above and nonempty, the supremum of A exists[a].

[a] and is an element of F, as the definition requires.

thm:uniqueorderedfield

Theorem 1.9 (Unique complete ordered field)

There exists a unique complete ordered field, up to isomorphism.

The proof of this theorem is beyond the scope of this course. One can show the existence of such a field by creating a field of Dedekind cuts. A Dedekind cut is a subset $C \subset \mathbb{Q}$ such that, if $c \in C$, then all rational numbers $x < c$ also are in C; and that C has no supremum in $\mathbb{Q}$. Addition can be defined using set addition. Multiplication is harder to define, since it is needed a separation between "non-negative" and "negative" numbers. Ordering can be defined using subsets. Finally, one has to prove all the axioms (field axioms, ordering axioms, and the axiom of completeness).

defn:realnumbers

Definition 1.10 (Real numbers)

The set of real numbers, denoted by $\mathbb{R}$, is the complete ordered field.

Question. If $\mathbb{R}$ is defined in such a axiomatic way, how can we say that $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$?

Recall that the statement $\mathbb{Z} \subset \mathbb{Q}$ is also a strange statement. A rational number is actually a pair of integers; how can a single integer be also a pair of integers? To be fair, in a set-theoretical sense, it is in fact untrue that $\mathbb{Z} \subset \mathbb{Q}$. However, the set of rational numbers of the form $\frac{n}{1}$ has the same structure (with respect to multiplication and addition) as the set of integers. Therefore, when we say "$\mathbb{Z} \subset \mathbb{Q}$", we actually mean that there exists a subset of $\mathbb{Q}$ that is isomorphic to $\mathbb{Z}$. This difference is usually not interesting for us, when studying Analysis. In the rare cases where this kind of difference is relevant, we say sentences like "there is a copy of $\mathbb{Z}$ in $\mathbb{Q}$."

In a similar fashion, the additive group generated by the identity of $\mathbb{R}$[*] is isomorphic to $\mathbb{Z}$; as well as the field generated by the identity of $\mathbb{R}$[†] is isomorphic to $\mathbb{Q}$. Therefore, there are copies of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{R}$, or, more informally, $\mathbb{Z} \subset \mathbb{R}$ and $\mathbb{Q} \subset \mathbb{R}$.

1.3 Appendix: Absolute value

Before we dig more deeply into the idea of a supremum, consider this definition that comes just from the structure of an ordered field.

defn:absolutevalue

Definition 1.11 (Absolute value)

If F is an ordered field, and $x \in F$, let

$$|x| = \begin{cases} x \text{ if } x \geq 0, \\ -x \text{ if } x < 0. \end{cases}$$

Theorem 1.12

If F is an ordered field, and $x \in F$, then $|x| \geq 0$.

Proof. If $x \geq 0$, then $|x| = x \geq 0$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x = |x|$. ∎

thm:|-x|=|x|

Theorem 1.13

If F is an ordered field, and $x \in F$, then $|-x| = |x|$.

[*] In other words, the smallest additive group that contains the identity of $\mathbb{R}$.

[†] In other words, the smallest field that contains the identity of $\mathbb{R}$.

Proof. If $x \geq 0$, then $0 = x + (-x) \geq 0 + (-x) = -x$, therefore $|-x| = -(-x) = x = |x|$. If $x \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$, therefore $|-x| = -x = |x|$. ∎

Theorem 1.14

If F is an ordered field, and $x, y \in F$, then $|xy| = |x||y|$.

Proof. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$ and $|xy| = xy = |x||y|$.

If $x \geq 0$ and $y \leq 0$, then $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the previous case with x and $-y$ and also Theorem 1.13 to obtain $|xy| = |-xy| = |x(-y)| = |x||-y| = |x||y|$.

If $x \leq 0$ and $y \geq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$. So we apply the first case with $-x$ and y and also Theorem 1.13 to obtain $|xy| = |-xy| = |(-x)y| = |-x||y| = |x||y|$.

If $x \leq 0$ and $y \leq 0$, then $0 = x + (-x) \leq 0 + (-x) = -x$ and $0 = y + (-y) \leq 0 + (-y) = -y$. So we apply the first case with $-x$ and $-y$ and also Theorem 1.13 to obtain $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$. ∎

thm:triangleinequality

Theorem 1.15 (Triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x + y| \leq |x| + |y|.$$

Proof. If $x \geq 0$, then $|x| = x$. If $x \leq 0$, then $x \leq 0 = x + (-x) \leq 0 + (-x) = x$, so $|x| = -x \geq x$. In either case, $|x| \geq x$.

Thus, $|x| + |y| \geq x + y$ and $|x| + |y| = |-x| + |-y| \geq -x - y = -(x + y)$. Since $|x + y| = x + y$ or $|x + y| = -(x + y)$, in either case, $|x| + |y| \geq |x + y|$. ∎

thm:reversetriangleinequality

Theorem 1.16 (Reverse triangle inequality)

If F is an ordered field, and $x, y \in F$, then

$$|x - y| \geq ||x| - |y||.$$

Proof. Triangle inequality implies that $|x| = |(x - y) + y| \leq |x - y| + |y|$ and $|y| = |(y - x) + x| \leq |y - x| + |x|$. Equivalently, we have $|x - y| \geq |x| - |y|$ and $|x - y| \geq |y| - |x|$; consequently, $|x - y| \geq ||x| - |y||$. ∎

2 Getting to know the Real Numbers

Lecture 3

thm:e-sup

Theorem 2.1 (ϵ-sup Theorem)

Given $A \subset \mathbb{R}$ nonempty and bounded above, and given s an upper bound of A, then $s = \sup(A)$ if, and only if, for all $\epsilon > 0$, there exists $a \in A$ such that $a > s - \epsilon$.

Proof. Supppose $s = \sup(A)$. Then, $s - \epsilon$ is not an upper bound of A. Therefore, there exists $a \in A$ such that $a > s - \epsilon$.

Suppose $s \neq \sup(A)$. Then, there exists an upper bound of A that is smaller than s, say $s - \delta$. Then, it follows that, for $\epsilon = \delta/2$, there is no $a \in A$ such that $a > s - \delta/2$, because $a \leq s - \delta < s - \delta/2$ for all $a \in A$. ∎

defn:sumofsets

Definition 2.2 (Sum of Sets)

Given $A, B \subset \mathbb{R}$, we define their sum as

$$A + B = \{a + b : a \in A, b \in B\}$$

Theorem 2.3 (Supremum of Sum of Sets)

If $A, B \subset \mathbb{R}$ are both nonempty and bounded above, then

$$\sup(A + B) = \sup(A) + \sup(B).$$

Proof. Since $\sup(A)$ is an upper bound of A, it holds that $a \leq \sup(A)$ for all $a \in A$. Since $\sup(B)$ is an upper bound of B, it holds that $b \leq \sup(B)$ for all $b \in B$. Therefore, $a+b \leq \sup(A)+\sup(B)$ for all $a \in A$ and $b \in B$, i.e., $x \leq \sup(A)+\sup(B)$ for all $x \in A + B$; thus, $\sup(A) + \sup(B)$ is an upper bound of $A + B$.

Let $\epsilon > 0$ be any positive real number. thm:e-sup ϵ-sup Theorem implies that there exists $a \in A$ such that $a > \sup(A) - \epsilon/2$. thm:e-sup ϵ-sup Theorem also implies that there exists $b \in B$ such that $b > \sup(B) - \epsilon/2$. Therefore, there exist $a \in A$ and $b \in B$ such that $a + b > \sup(A) + \sup(B) - \epsilon$; thus, there exists $x \in X$ such that $x > \sup(A) + \sup(B) - \epsilon$.

thm:e-sup
Finally, by ϵ-sup Theorem, $\sup(A+B) = \sup(A) + \sup(B)$. ∎

Lecture 4

2.1 Archimedean Properties

thm:archimedeanproperties

Theorem 2.4 (Archimedean Properties)

i. Given any $x \in \mathbb{R}$, there exists some $n \in \mathbb{Z}_{>0}$ with $n > x$.
ii. Given any $y > \mathbb{R}_{>0}$, there exists some $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < y$.

Proof. The first statement is equivalent to $\mathbb{Z}_{>0}$ is not bounded above.

Suppose $\mathbb{Z}_{>0}$ is bounded above. Then, there exists $s = \sup(\mathbb{Z}_{>0})$. Therefore, $s-1$ is not an upper bound of $\mathbb{Z}_{>0}$, i.e., there exists $n \in \mathbb{Z}_{>0}$ such that $s-1 < n$. However, this implies $s < n+1 \in \mathbb{Z}_{>0}$, implies s is not an upper bound of $\mathbb{Z}_{>0}$, which is a contradiction.

The second statement follows from the first one by setting $x = \frac{1}{n}$. ∎

thm:QisdenseinR

Theorem 2.5 (Density of $\mathbb{Q}$ in $\mathbb{R}$)

For all $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ with $a < q < b$.

thm:archimedeanproperties
Proof. By Archimedean Properties with $y = b - a > 0$, there exists $n \in \mathbb{Z}_{>0}$ with $\frac{1}{n} < b - a$.

Let m be the smallest natural number greater than na. Then,

$$\begin{aligned} m - 1 \leq na < m \\ \frac{m}{n} - \frac{1}{n} \leq a < \frac{m}{n}. \end{aligned}$$

The first inequality implies that $\frac{m}{n} \leq a + \frac{1}{n} < b$, so finally, we conclude that

$$a < \frac{m}{n} < b.$$

∎

Corolary 2.6

For all $a, b \in \mathbb{R}$, with $a < b$, there exist infinitely many $q \in \mathbb{Q}$ with $a < q < b$.

Lecture 5

2.2 Nested Interval Property

thm:nestedintervalproperty

Theorem 2.7 (Nested Interval Property)

Suppose we have a sequence of closed intervals $I_n = [a_n, b_n]$, with $a_n \leq b_n$, that are nested decreasing, i.e.,

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots .$$

Then $\bigcap_{n=1}^{\infty} I_n \neq \varnothing$.

Proof. Let $A = \{a_1, a_2, \dots\}$ and $B = \{b_1, b_2, \dots\}$.

The "nested" condition implies that, if $i < j$, then $[a_i, b_i] \supset [a_j, b_j]$. Therefore, $a_j, b_j \in [a_i, b_i]$, which implies that $a_i \leq a_j \leq b_j \leq b_i$ for all $i < j$. Note that this implies that

$$a_i \leq b_j \text{ and } a_j \leq b_i, \text{ for all } i < j.$$

We can rewrite it as

$$a_i \leq b_j, \text{ for all } i \text{ and } j.$$

This implies that a_i is a lower bound of B for any i, and also implies that b_j is an upper bound of A for any j. Since A is bounded above, we can define $x = \sup(A)$. Clearly, x is an upper bound of A.

Suppose x is not a lower bound of B. Then, there exists n such that $x > b_n$. The thm:e-sup ϵ-sup Theorem, with $\epsilon = x - b_n > 0$, implies that there exists m such that $a_m > x-(x-b_n) = b_n$, which contradicts the previous displayed equation. Therefore, x is a lower bound of B.

Finally, x is both an upper bound of A and a lower bound of B, thus, for all n, $a_n \leq x \leq b_n$, i.e., $x \in [a_n, b_n]$. Therefore, x is in such intersection. ∎

2.3 Cardinality

Question. Are all sets with an infinite number of elements the same size?

Definition 2.8 (Cardinality)

Given two sets A and B, we say that A *and* B *have the same cardinality* if there exists a bijection $f\colon A \to B$. We will write $A \sim B$ to say that A and B have the same cardinality.

Example

The sets $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ and $\{2, 4, 6, 8, \dots\}$ have the same cardinality.

defn:countability

Definition 2.9 (Countability)

We say a set S is *countable* if it has the same cardinality as $\mathbb{N}$. If a set is not a finite set and not countable, then we say it is *uncountable*.

prop:N2iscountable

Proposition 2.10 ($\mathbb{N}^2$ is countable)

$\mathbb{N}^2$ is countable.

Proof. The function $f\colon \mathbb{N}^2 \to \mathbb{N}$, defined by

$$f(i, j) = \frac{(i + j - 1)(i + j - 2)}{2} + i$$

is a bijection. ∎

Lecture 6

Theorem 2.11

If A is countable and B is countable, then $A \times B$ is countable.

Similarly, if $A_1, A_2, \dots, A_n$ are each countable, then $A_1 \times \dots \times A_n$ is countable.

Similarly, if $A_1, A_2, \dots, A_n$ are each countable or finite, then $A_1 \times \dots \times A_n$ is countable or finite.

thm:countableunion

Theorem 2.12

If $S_1, S_2, \dots$ are each countable, then their union is countable.

Similarly, if $\{S_i\}_{i \in I}$ is a countable or finite collection of sets, which are each countable or finite; then their union is countable.

Example

Let $\mathcal{T}$ be the collection of finite subsets of $\mathbb{N}$. For each $i \in \mathbb{N}$, let A_i be the collection of subsets of $\{1, 2, \ldots, i\}$. Note that $|A_i| = 2^i$, thus A_i is finite. Then, note that $\varnothing \in A_1$, and, if $S \in \mathcal{T}$ is non-empty, it holds that $S \in A_{\max(S)}$; so $\mathcal{T} = \bigcup_{i=1}^{\infty} A_i$.

thm:countableunion

Therefore, by Theorem 2.12, we conclude that $\mathcal{T}$ is countable or finite. Since $\mathcal{T}$ is not finite, then it is countable.

thm:countabletransitivityviafunctions

Theorem 2.13

If A is countable, and $f : A \to B$ is surjective, then B is countable or finite.

Similarly, if A is countable, and $f : B \to A$ is injective, then B is countable or finite.

In particular, if A is countable, and $A \supseteq B$, then B is countable or finite.

prop:qiscountable

Proposition 2.14 ($\mathbb{Q}$ is countable)

$\mathbb{Q}$ is countable.

Proof. Consider the function $f : \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) \to \mathbb{Q}$ defined by $f(a, b) = \frac{a}{b}$. Clearly, $f(p, q) = \frac{p}{q}$ for any $\frac{p}{q} \in \mathbb{Q}$. ∎

Proposition 2.15

$\mathbb{R}$ is not countable.

Proof (using nested intervals). Assume $\mathbb{R}$ is countable. So, there exists a bijection $f : \mathbb{N} \to \mathbb{R}$.

Let $I_1 = [f(1) + 1, f(2) + 2]$. Note that $f(1) \notin I_1$. We will define I_{n+1} recursively. Suppose $I_n = [a, b]$, then, define I_{n+1} as either $[a, \frac{2a+b}{3}]$ or $[\frac{a+2b}{3}, b]$ such that $f(i + 1) \notin I_{n+1}$; that is possible since $f(i + 1)$ cannot be in both sets.

By the Nested Interval Property, there exists a real number $r \in \bigcap_{i=1}^{\infty} I_n$. However, since f is a bijection, there exists $m \in \mathbb{N}$ such that $f(m) = r$. Therefore, $r \notin I_m$, a contradiction. ∎

Lecture 7

Proof (using Cantor's diagonalization). We'll prove $(0,1)$ is uncountable, which implies $\mathbb{R}$ is uncountable.

Assume $(0,1)$ is countable, therefore, there exists a bijective function $f : \mathbb{N} \to (0,1)$.

Let's write out decimal expansions[a] of $f(1), f(2), \ldots$. If there's doubt between a recurrent 9 or a recurrent 0 in the end, we choose the latter form. We write

$$f(i) = 0.a_{i1}a_{i2}a_{i3}\ldots,$$

with $a_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$. Let $b_k = 1$, if a_{kk} is odd, and $b_k = 2$, if a_{kk} is even. Note that $c_k \neq b_{kk}$ and $c_k \notin \{0,9\}$ for all k. Therefore, $x = 0.b_1b_2b_3\ldots$ cannot be on the image of f; a contradiction. ■

[a] What are decimal expansions? We only need to know that decimal expansions are unique except for some duplication, like $0.09999 = 0.1$.

Another perspective on the Cantor's proof arises by using the binary base, instead of the decimal base. For each real number $x = 0.x_1x_2x_3\ldots$, we can define a $f(x) = \{n \in \mathbb{N} : a_n = 1\}$. This is almost[*] a bijection because, but nevertheless, we can conclude that, in some sense,

$$|\mathbb{R}| = 2^{|\mathbb{N}|}.$$

[*] The same number with two expansions yields a problem.

3 Normed Vector Spaces and Metric Spaces

3.1 Complex Numbers

defn:complexnumber

Definition 3.1 (Complex number)

The set of complex numbers, denoted by $\mathbb{C}$, is the set of pairs (a, b) of real numbers. On top of that, we define addition and multiplication of complex numbers by

- $(a, b) + (c, d) = (a + c, b + d)$.
- $(a, b) \cdot (c, d) = (ac - bd, ad + bc)$.

prop:cisafield

Proposition 3.2 ($\mathbb{C}$ is a field)

$(\mathbb{C}, +, \cdot)$ is a field.

Consider the complex numbers of the form $(a, 0)$. Note that $(a, 0) + (a', 0) = (a + a', 0)$ and $(a, 0) \cdot (a', 0) = (aa', 0)$. Therefore, this subset of the complex numbers is isomorphic to $\mathbb{R}$. Similarly as we've seen in previous chapters, we can say that $\mathbb{R} \subset \mathbb{C}$, referring to this natural homomorphism; i.e., if a is a real number, then we'll also use a to talk about the complex number $(a, 0)$.

defn:imaginaryunit

Definition 3.3 (Imaginary unit)

Let $i = (0, 1) \in \mathbb{C}$.

Proposition 3.4

Let $(a, b) \in \mathbb{C}$. Then,

$$(a, b) = a + bi.$$

Proof.

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (0, b) \\ &= (a, b). \end{aligned}$$

defn:conjugate

Definition 3.5 (Conjugate)

Given real numbers a, b and a complex number $z = a + bi$, we define *the conjugate of* z as $a - bi$, denoted by $\overline{z}$.

defn:realimaginarypart

Definition 3.6 (Real and imaginary part)

Given real numbers a, b and a complex number $z = a + bi$, the numbers a and b are called *the real part* and *the imaginary part of* z, respectively. Symbolically, we can write

$$\mathrm{Re}(z) = a, \qquad \mathrm{Im}(z) = b.$$

Proposition 3.7

Given $z, w \in \mathbb{C}$, we have

- $\overline{z + w} = \overline{z} + \overline{w}$.
- $\overline{zw} = \overline{z} \cdot \overline{w}$.
- $z + \overline{z} = 2\mathrm{Re}(z)$ and $z - \overline{z} = 2i\mathrm{Im}(z)$.
- $z\overline{z}$ is a non-negative real number.

defn:absolutevaluecomplex

Definition 3.8 (Absolute value of a complex number)

If $z = a + bi$ is a complex number, then we define $|z| = (z\overline{z})^{1/2} = (a^2 + b^2)^{1/2}$.

3.2 Euclidean Spaces

defn:euclideanspace

Definition 3.9 (Euclidean Space)

Given an positive integer n, let $\mathbb{R}^n$ be the set of all n-uples

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)$$

of real numbers. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ are in $\mathbb{R}^n$, and $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} \mathbf{x} + \mathbf{y} &= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\ \alpha\mathbf{x} &= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n). \end{aligned}$$

These definitions make the set $\mathbb{R}^n$ into a vector space over the real field.

We further define the *inner product* by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}.$$

This structure ($\mathbb{R}^n$, equipped with the inner product and norm) is called the Euclidean n-dimensional space.

Proposition 3.10 ($\mathbb{R}^n$ is a metric space)

Define $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$. $(\mathbb{R}^n, d)$ is a metric space.

3.3 Normed Vector Space

defn:normedvectorspace

Definition 3.11 (Normed Vector Space)

Let F be either $\mathbb{R}$ or $\mathbb{C}$.

A *normed vector space* is a *vector space* W over F, equipped with a norm $||Ĺ||\colon W \to \mathbb{R}$, satistying the following conditions:

i. $||v|| \geq 0$ for all $v \in W$.
ii. $||v|| = 0$ if, and only if, $v = 0$.
iii. $||cv|| = |c| \cdot ||v||$ for all $v \in W$ and for all $c \in F$.
iv. $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in W$.

Example

$\mathbb{R}$ is a normed vector space, considering the norm $|x|$. $\mathbb{C}$ is a normed vector space, considering the norm $|z|$.

3.3.1 Euclidean Spaces

defn:euclideanspace

Definition 3.12 (Euclidean Space)

Given an positive integer n, consider the vector space $\mathbb{R}^n$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$, we define the *inner product* by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

and the *norm* of $\mathbf{x}$ by

$$||\mathbf{x}|| = (\mathbf{x} \cdot \mathbf{x})^{1/2}.$$

This structure is called the Euclidean n-dimensional space.

Proposition 3.13

The Euclidean n-dimensional space is a normed vector space.

3.4 Metric Space

defn:metricspace

Definition 3.14 (Metric Space)

A *metric space* is a pair (X, d), where X is a set and $d\colon X \times X \to \mathbb{R}$ is a function, called *metric*, that satisfies:

- $d(x, y) \geq 0$ for all $x, y \in X$; with equality if, and only if, $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example

Consider $d(x, y) = |x - y|$, in all following examples (with the appropriate domain).

- $(\mathbb{Z}, d)$ is a metric space.
- $(\mathbb{Q}, d)$ is a metric space.
- $(\mathbb{R}, d)$ is a metric space.
- $(\mathbb{C}, d)$ is a metric space.

Every normed vector space W is naturally also a metric space, by considering the metric $d : W \times W \to \mathbb{R}$ defined by $d(v, u) = ||v - u||$.

Example ($\mathbb{R}^n$ is a metric space)

Define $d\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$. $(\mathbb{R}^n, d)$ is a metric space.

There are metric spaces which are not normed vector spaces, but they are out of the scope of this course.

From now on in these notes, whenever you read "X is a metric space" or "X is a normed vector space," it is useful to think about the prototypical examples of $X = \mathbb{R}$, $X = \mathbb{C}$, or $X = \mathbb{R}^2$.

4 Limits

4.1 Sequences

defn:limitsequence

Definition 4.1 (Limit of a sequence)

Let X be a metric space. We say a sequence $(a_n) = a_1, a_2, a_3, \ldots$, where $a_i \in X$, *converges to* $a \in X$ if, for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N$.

If this definition holds for some a, we write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$.

If this definition does not hold for any a, we say $\lim_{n\to\infty} a_n$ does not exist, or that the sequence diverges.

Proposition 4.2 (The limit, if it exists, is unique)

If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} a_n = a'$, then $a = a'$.

Proof. For all $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $d(a_n, a) < \epsilon$ for all $n \geq N_\epsilon$. For all $\epsilon > 0$, there exists $M_\epsilon \in \mathbb{N}$ such that $d(a_n - a') < \epsilon$ for all $n \geq M_\epsilon$.

Therefore, for all $\epsilon > 0$, there exists $L_\epsilon \in \mathbb{N}$, namely $\max\{N_\epsilon, M_\epsilon\}$, such that $d(a_n, a) < \epsilon$ and $d(a_n, a') < 0$ for all $n \geq L_\epsilon$. Triangle inequality implies that $d(a, a') < 2\epsilon$ for all $\epsilon > 0$; thus $d(a, a') = 0$, and consequently $a = a'$. ∎

Example

We claim that $\lim_{n\to\infty} \frac{1}{n^2} = 0$.

This is true because, given $\epsilon > 0$, we can choose N be a natural number larger than $\sqrt{\frac{1}{\epsilon}}$. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N^2} > \frac{1}{n^2} = \left|\frac{1}{n^2} - 0\right|.$$

Example (The limit does not exist)

We claim that $\lim_{n\to\infty}(-1)^n$ does not exist.

Suppose it does exist, namely a. Then, consider $\epsilon = \frac{1}{2}\max\{|a-1|, |a+1|\}$. Not both $|a-1|$ and $|a+1|$ can be zero, so $\epsilon > 0$. However, since $\lim_{n\to\infty} = a$, for that ϵ, it must hold that there exists $N \in \mathbb{N}$ so that for all $n \geq N$, $|a - (-1)^n| < \epsilon$.

In particular, note that plugging in $n \mapsto N$ and $n \mapsto N+1$ imply that $|a-1| < \epsilon$ and $|a+1| < \epsilon$; which is a contradiction given our choice of ϵ.

Example

We claim that $\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$. Note that we can rewrite $\frac{2n+1}{n+3} = 2 - \frac{5}{n+3}$. For any ϵ, there exists $N \in \mathbb{N}$ such that $N > \frac{5}{\epsilon}$. Therefore, for all $n \geq N$, it holds that

$$\left|\left(2 - \frac{5}{n+3}\right) - 2\right| = \frac{5}{n+3} < \frac{5}{N} < \epsilon,$$

and our claim follows.

Lecture 8

Example

We claim that $\lim_{n\to\infty} \frac{2n^2}{5n^3-7} = 0$.

This is true because, given $\epsilon > 0$, we can choose N to be a natural number larger than $\frac{1}{\epsilon}$ and larger than 2. Then, for all $n \geq N$, we have

$$\epsilon > \frac{1}{N} > \frac{1}{n} > \frac{2n^2}{4n^3} > \frac{2n^2}{4n^3 + (n^3 - 7)} = \left|\frac{2n^2}{5n^3-7} - 0\right|$$

thm:manipulationlimits

Theorem 4.3 (Algebraic Manipulation of Limits)

Let W be a normed vector space over F.

Suppose that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$ are elements of W and $c, d \in F$. Then,

i. $\lim_{n\to\infty}(ca_n + db_n) = ca + db$

If W is a field, then,

ii. $\lim_{n\to\infty} a_n b_n = ab$
iii. $\lim_{n\to\infty}(1/a_n) = 1/a$ if the $a_n \neq 0$ for all n and $a \neq 0$.
iv. $\lim_{n\to\infty}(a_n/b_n) = a/b$ if the $b_n \neq 0$ for all n and $b \neq 0$.

Proof.

i. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < \frac{\epsilon}{2|c|}$$

for all $n \geq N$. Similarly, there exists M such that

$$|b_n - b| < \frac{\epsilon}{2|d|}$$

for all $n \geq M$. Therefore, for all $n \geq \max\{N, M\}$, it holds that

$$\begin{aligned} |(ca_n + db_n) - (ca + db)| &= |(ca_n - ca) + (db_n - db)| \\ &\leq |ca_n - ca| + |db_n - db| \\ &\leq |c||a_n - a| + |d||b_n - b| \\ &< \epsilon, \end{aligned}$$

thus, $\lim_{n\to\infty} ca_n + db_n$.

ii. Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$|a_n - a| < 1$$

for all $n \geq N$; therefore, $|a_n| < |a| + 1$ for all $n \geq N$.
Since $\lim_{n\to\infty} a_n = a$, there exist M such that

$$|a_n - a| < \frac{\epsilon}{|b|}$$

for all $n \geq M$. Similarly, there exist O such that

$$|b_n - b| < \frac{\epsilon}{2(|a| + 1)}$$

for all $n \geq O$. Therefore, for all $n \geq \max\{N, M, O\}$, it holds that

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\ &= |a_n(b_n - b) + b(a_n - a)| \\ &\leq |a_n(b_n - b)| + |b(a_n - a)| \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n b_n = ab$.

iii. Without loss of generality, suppose $a > 0$. Since $\lim_{n\to\infty} a_n = a$, there exist N such that

$$a_n > \frac{a}{2} > 0$$

for all $n \geq N$. Therefore, $0 < \frac{1}{a_n} < \frac{2}{a}$ for all $n \geq N$.
Let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = a$, there exists M so that

$$|a_n - a| < \frac{\epsilon a^2}{2}.$$

Then, for all $n \geq \max\{N, M\}$, it holds that

$$\begin{aligned} \left|\frac{1}{a_n} - \frac{1}{a}\right| &= |a - a_n| \cdot \frac{1}{a} \cdot \left|\frac{1}{a_n}\right| \\ &< \frac{\epsilon a^2}{2} \cdot \frac{1}{a} \cdot \frac{2}{a} \\ &< \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$.

iv. Using **ii** and **iii**, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \left(a_n \frac{1}{b_n}\right) \\ &= \left(\lim_{n\to\infty} a_n\right) \left(\lim_{n\to\infty} \frac{1}{b_n}\right) \\ &= a \cdot \frac{1}{b} = \frac{a}{b}. \end{aligned}$$

■

Example

Since $\lim_{n\to\infty}(1 + 1/n) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n) = 1 + 0 = 1$ and $\lim_{n\to\infty}(1 + 1/n^2) = \lim_{n\to\infty} 1 + \lim_{n\to\infty}(1/n^2) = 1 + 0 = 1$, we can conlude that

$$\begin{aligned} \lim_{n\to\infty} \frac{n^2 + n}{n^2 + 1} &= \lim_{n\to\infty} \frac{1 + 1/n}{1 + 1/n^2} \\ &= \frac{\lim_{n\to\infty} 1 + 1/n}{\lim_{n\to\infty} 1 + 1/n^2} \\ &= \frac{1}{1} = 1. \end{aligned}$$

Lecture 9

defn:seqbounded

Definition 4.4 (Boundness)

Let W be a normed vector space. A sequence $(a_n)_{n\in\mathbb{N}}$, where $a_i \in W$, is bounded if there exists $M \in \mathbb{R}$ so that $|a_n| \le M$ for all $n \in \mathbb{N}$.

Theorem 4.5 (A convergent sequence is bounded)

If $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence, then (a_n) is bounded.

Proof. Let L be the limit of such sequence. Let $\epsilon = 1$. Then, there exists $N \in \mathbb{N}$ so that $|a_n - L| < 1$ for all $n \ge N$. Triangle inequality implies that $|a_n| < |L| + 1$ for all $n \ge N$. Define

$$M = \max\{|a_1| + 1, |a_2| + 1, \dots, |a_{N-1}| + 1, |L| + 1\}.$$

Then, for this choice of M, it holds that $|a_n| < M$ for all $n \in \mathbb{N}$. Therefore, (a_n) is bounded. ∎

Lecture 10

defn:monotone

Definition 4.6 (Monotone sequences)

Let (a_n) be a sequence of elements of an ordered set (for example, the real numbers).

We say (a_n) is *monotone increasing* if $a_{n+1} \ge a_n$ for all n.

We say (a_n) is *strictly monotone increasing* if $a_{n+1} < a_n$ for all n.

We say (a_n) is *monotone decreasing* if $a_{n+1} \le a_n$ for all n.

We say (a_n) is *strictly monotone decreasing* if $a_{n+1} < a_n$ for all n.

thm:monotoneconvergence

Theorem 4.7 (Monotone Convergence Theorem)

Let (a_n) be a sequence of real numbers. If (a_n) is monotone increasing and bounded above, then it converges.

Similarly, if (a_n) is monotone decreasing and bounded below, then it converges.

Proof. We will only prove the first statement. Let $\epsilon > 0$. Let $a = \sup\{a_1, a_2, a_3, \dots\}$. thm:e-sup ϵ-sup Theorem implies that there exists N so that $a - a_N < \epsilon$. Since the sequence is monotone increasing, for all $n \ge N$, we have that

$$|a - a_n| = a - a_n < \epsilon;$$

thus, $\lim_{n\to\infty} a_n = n$. ∎

Example

What in the world is $\sqrt{6+\sqrt{6+\sqrt{6+\cdots}}}$? If it exists, it would be plausible to be the limit of the sequence

$$\sqrt{6}, \quad \sqrt{6+\sqrt{6}}, \quad \sqrt{6+\sqrt{6+\sqrt{6}}}, \dots$$

The easier way to make sense of this sequence is using recursion. We will define it as

$$a_1 = \sqrt{6}, \quad \text{and} \quad a_n = \sqrt{6+a_{n-1}} \text{ for } n \geq 2.$$

We know that $a_1 = \sqrt{6} < \sqrt{6+\sqrt{6}} = a_2$. Suppose that $a_{n-1} < a_n$. Then, $a_n = \sqrt{6+a_{n-1}} < \sqrt{6+a_n} = a_{n+1}$. Therefore, by induction, $a_{n+1} > a_n$ for all $n \geq 1$, i.e., the sequence a_n is monotone increasing.

We also know that $a_1 < 10$. Suppose that $a_{n-1} < 10$. Then, $a_n = \sqrt{6+a_n} < \sqrt{16} < 10$. Therefore, by induction, $a_n < 10$ for all $n \geq 1$, i.e., 10 is an upper bound of a_n.

thm:monotoneconvergence

By the Monotone Convergence Theorem, we conclude that a_n has a limit. Finally,

$$\begin{aligned}\left(\lim_{n\to\infty} a_n\right)^2 &= \lim_{n\to\infty} a_n^2 \\ &= \lim_{n\to\infty} (6+a_{n-1}) \\ &= 6 + \lim_{n\to\infty} a_n;\end{aligned}$$

therefore, $\lim_{n\to\infty} a_n = 3$ or $\lim_{n\to\infty} a_n = -2$. Since a_n evaluates to positive real numbers, the latter proposition yields a contradiction when plugging $\epsilon \mapsto 1$. Therefore, the former proposition must be true, i.e.,

$$\lim_{n\to\infty a_n} = 3.$$

thm:limitpreserveleq

Theorem 4.8 (Limits preserve $\leq$)

Let (a_n) and (b_n) be sequences of real numbers. Suppose $a_n \leq b_n$ for all n, and $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then, $a \leq b$.

4.2 Subsequences

Lecture 11

defn:subsequence

Definition 4.9 (Subsequence)

Given a sequence (a_n) and a strictly monotone increasing sequence of natural numbers (n_i), the sequence (a_{n_i}) is called a *subsequence* of (a_n).

In other words, we can say that (b_k) is a subsequence of (a_n) if there exists a strictly monotone increasing $f : \mathbb{N} \to \mathbb{N}$ so that $b_k = a_{f(k)}$ for all k.

thm:seqconvsubseqconv

Theorem 4.10

Let X be a metric space. A sequence of elements in X converges to $L \in X$ if, and only if, every of its subsequences converges to $L \in X$.

Proof. The inverse implication is straightforward, since the sequence is a subsequence of itself. Let's prove the direct implication. Let (a_n) be a sequence so that $a_n \to L$. Let (a_{n_i}) be a subsequence of (a_n). Let $\epsilon > 0$. Since $a_n \to L$, there exists N so that

$$d(L, a_n) < \epsilon,$$

for all $n \geq N$. Note that $n_i \geq i$. Therefore, for the same choice of N, it holds that

$$d(L, a_{n_i}) < \epsilon$$

for all $i \geq N$. Therefore, $a_{n_i} \to L$. ∎

Lecture 12

thm:squeeze

Theorem 4.11 (Squeeze Theorem)

Let (x_n), (y_n), and (z_n) be sequences of real numbers. If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$, then $\lim_{n\to\infty} y_n = L$.

Proof. For all $n \in \mathbb{N}$, since $x_n \leq y_n \leq z_n$, $|z_n - x_n| = |z_n - y_n| + |y_n - x_n|$, which implies

eqn:zn-xn>zn-yn

$$|z_n - x_n| \geq |y_n - x_n|. \tag{4.1}$$

Theorem 4.3 implies that $\lim_{n\to\infty}(z_n - x_n) = \lim_{n\to\infty} z_n - \lim_{n\to\infty} x_n = 0$.

Let $\epsilon > 0$. Therefore, since $(z_n - x_n) \to 0$, there exists N such that $|z_n - x_n| < \epsilon$ for all $n \geq N$. Equation (4.1) implies that, for the same choice of N, it holds that $|y_n - x_n| < \epsilon$ for all $n \geq N$. Therefore, $(y_n - x_n) \to 0$. Since $(x_n) \to L$ and $(y_n - x_n) \to 0$, theorem 4.3 implies $(y_n) \to L$. ∎

Example

We claim that $\lim_{n\to\infty} \sqrt{n^2+4n} - n = 2$.

A good intuition for that to be true is that $\sqrt{n^2+4n} - n \approx \sqrt{n^2+4n+4} - n = 2$.

Formally,

$$\begin{aligned}\sqrt{n^2+4n} - n &= \frac{(n^2+4n)-n^2}{\sqrt{n^2+4n}+n^2} \\ &= \frac{4}{\sqrt{1+4/n}+1} \to 2.\end{aligned}$$

thm:bw

Theorem 4.12 (Bolzano-Weierstrass Theorem)

Every bounded sequence of real numbers has a convergent subsequence.

Proof. Since (a_n) is bounded, there exists M such that $a_n \leq M$ for all n. Let $I_1 = [-M, M]$. Note that infinitely many terms of (a_n) are in I_1.

Suppose $I_k = [a_k, b_k]$ contains infinitely many terms of (a_n). Define I_{k+1} as either $[a_k, \frac{a_k+b_k}{2}]$ or $[\frac{a_k+b_k}{2}, b_k]$ such that I_{k+1} contais infinitely many terms of (a_n).

thm:nestedintervalproperty
Nested Interval Property implies that there exists $x \in I_j$ for all j.

Let $n_1 = 1$, so that $a_{n_1} \in I_1$. Define $n_{i+1} > n_i$, so that $a_{n_{i+1}} \in I_{i+1}$; which is possible since I_{n+1} has infinitely many terms.

For each j, both a_{n_j} and x are in I_j. Since the width of I_j is $2M/2^{j-1}$, we conclude

$$-\frac{2M}{2^{j-1}} + x \leq a_{n_j} \leq \frac{2M}{2^{j-1}} + x,$$

thm:squeeze
thus the Squeeze Theorem implies $(a_{n_j}) \to x$. ■

Definition 4.13 (Cauchy sequence)

Let X be a metric space. A sequence of elements in X is *Cauchy* if, for all $\epsilon > 0$, there exists N so that $d(a_m, a_n) < \epsilon$ for all $m, n \geq N$.

Example

We claim that the sequence $a_n = \frac{(-1)^n}{n}$ is Cauchy.

Let $\epsilon > 0$. Choose N larger than $\frac{1}{2\epsilon}$.

Then, for all $n, m \geq N$, it holds that

$$\begin{aligned}\left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| &= \left|\frac{1}{n} \pm \frac{1}{m}\right| \\ &\leq \frac{1}{n} + \frac{1}{m} \\ &\leq \frac{2}{N} \\ &< \epsilon.\end{aligned}$$

Proposition 4.14

Every convergent sequence is Cauchy.

Proof. Let $\epsilon > 0$. Since $(a_n) \to L$, there exists N so that

$$d(a_n, L) < \frac{\epsilon}{2}$$

for all $n \geq N$. Therefore, using the triangle inequality,

$$d(a_n, a_m) \leq d(a_n, L) + d(L, a_m) < \epsilon$$

for all $n, m \geq N$; thus the sequence is Cauchy. ∎

Lecture 13

prop:cauchybounded

Proposition 4.15

Let W be a normed vector space. Every Cauchy sequence of elements in W is a bounded sequence.

Proof. Let $\epsilon = 1$. There exist N so that $|a_m - a_n| < 1$ for all $m, n \geq N$. This implies that $|a_m - a_N| < 1$ for all $m \geq N$, and consequently, by triangle inequality, $|a_m| = |a_m - 0| \leq |a_m - a_N| + |a_N - 0| < 1 + |a_N|$ for all $m \geq N$.

Therefore, if we set

$$M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\},$$

we conclude $|a_m| < M$ for all m. ∎

prop:subsequencecauchyconverges

Proposition 4.16

Let X be a metric space. Let (a_n) be a sequence of elements of X. If (a_n) is Cauchy, and if some subsequence of (a_n) converges to some limit $a \in X$, then the whole sequence (a_n) converges to $a \in X$.

Proof. Let $\epsilon > 0$. Let (a_{k_i}) be such sequence that converges to a. Thus, there exists N so that

$$d(a_{k_n}, a) < \epsilon/2$$

for all $n > N$.

Also, since (a_n) is Cauchy, there exists M so that

$$d(a_m, a_n) < \epsilon/2$$

for all $m, n \geq M$. In particular, by setting $m = k_n \geq n$, we conclude

$$d(a_{k_n}, a_n) < \epsilon/2$$

for all $n \geq M$.

Therefore, for all $n \geq \max\{N, M\}$, it holds that

$$d(a_n, a) \leq d(a_n, a_{k_n}) + d(a_{k_n}, a) < \epsilon;$$

in other words, (a_n) converges to a. ∎

thm:cauchyconvergentreal

Theorem 4.17

Every Cauchy sequence of real numbers is convergent.

Proof. Let (a_n) be a Cauchy sequence o

To be finished. ∎

4.3 Series

Lecture 14

defn:series

Definition 4.18 (Series)

Given a sequence (a_n), we associate it with a sequence (s_n), defined by

$$s_n = \sum_{k=1}^{n} a_k.$$

As an abuse of notation[a], we denote (s_n) using the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or

$$\sum_{n=1}^{\infty} a_n.$$

We call those expressions *(infinite) series*. Each s_n is called a *partial sum* of this series. If (s_n) converges to s, we say that the series *converges*, which we denote symbolically[b] by

$$\sum_{n=1}^{\infty} a_n = s,$$

which we call the sum of the series; though it is actually the limit of a sequence of partial sums.

If (s_n) diverges, we say that the series diverges.

[a] In my honest opinion, this is a really bad notation.

[b] Using the same symbolic arrangement as before! Who did this?

Note that theorems about sequences can be stated in terms of series and vice versa, by defining $a_1 = s_1$ and $a_n = s_n - s_{n-1}$.

Example

Suppose $a_n = (-1)^n$. Consider the infinite series $-1+1-1+1-1+1-\cdots$. Then, a formula for the partial sums is $s_n = \begin{cases} -1, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even.} \end{cases}$ Therefore, the sum of the infinite series does not converge, since $\lim_{n\to\infty} s_n$ does not exist.

Example

Suppose $a_n = \frac{1}{2^n}$. Consider the infinite series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Then, a formula for the partial sums is $s_n = 1 - \frac{1}{2^n}$. Therefore, the sum of the infinite series is 1, since

$\lim_{n\to\infty} s_n = 1$.

prop:geometricseries

Proposition 4.19 (Geometric Series)

$$\sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r}, & \text{if } -1 < r < 1 \\ \text{does not converge}, & \text{otherwise.} \end{cases}$$

Proof. Note that

$$s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}.$$

If $-1 < r < 1$, then $(r_{n+1}) \to 0$, which implies $(s_n) \to \frac{1}{1-r}$. Otherwise, then (r_{n+1}) does not converge, which implies (s_n) does not converge. ∎

prop:monotoneconvergenceforseries

Proposition 4.20

Suppose (a_n) is a sequence and $a_n \geq 0$ for all n. Then, $\sum_{n=1}^{\infty} a_n$ converges if, and only if, the partial sums $\sum_{k=1}^{n} a_n$ are bounded.

prop:monotoneconvergenceforseries thm:monotoneconvergence

This proposition 4.20 is a direct corollary of Monotone Convergence Theorem.

thm:condensationtest

Theorem 4.21 (Condensation Test)

Suppose (a_n) is monotone decreasing and $a_n \geq 0$ for all n. Then, $\sum_{n=1}^{\infty} a_n$ converges if, and only if, $\sum_{n=1}^{\infty} 2^n a_{2^n}$.

prop:monotoneconvergenceforseries

Proof. Proposition 4.20 implies that it suffices to show that

eqn:condensationtest:anbounded

$$\left(\sum_{k=1}^{n} a_k\right)_{n\in\mathbb{N}} \quad \text{is bounded} \tag{4.2}$$

if, and only if,

eqn:condensationtest:2na2nbouned

$$\left(\sum_{k=1}^{m} 2^k a_{2^k}\right)_{m\in\mathbb{N}} \quad \text{is bounded.} \tag{4.3}$$

eqn:condensationtest:anbounded

Suppose (4.2) is true. Therefore, there exists a constant N so that $\sum_{k=1}^{n} a_k < N$ for all n. Given any $m \in \mathbb{N}$, we will plug $n = 2^m - 1$ in the previous statement. This implies that

$$\sum_{k=1}^{2^m} a_k < N,$$

which implies,

$$\sum$$

■

thm:pseriesconverges

Theorem 4.22 (p-series converges)

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if, and only if, $p > 1$.

Proof. thm:condensationtest Condensation Test implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if, and only if,

$$\sum_{n=1}^{\infty} \frac{2^n}{2^{np}} = \sum_{n=1}^{\infty} \left(2^{1-p}\right)^n$$

converges. prop:geometricseries Geometric Series implies that the series above converges if, and only if, $-1 < 2^{1-p} < 1$, which is equivalent to $p < 1$. ■

Lecture 15

thm:manipulationseries

Theorem 4.23 (Algebraic Manipulation of Series)

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge. Then, for any $c, d \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} (ca_n + db_n)$$

converges to

$$c \cdot \sum_{n=1}^{\infty} a_n + d \cdot \sum_{n=1}^{\infty} b_n.$$

This theorem is a corollary of thm:manipulationlimits Algebraic Manipulation of Limits.

Theorem 4.24 (Comparison Test)

Suppose $0 \leq a_n \leq b_n$ for all n. Then,

i. if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii. if $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. If $\sum_{n=1}^{\infty} b_n$ converges, then, by Proposition 4.20, the partial sums $\sum_{k=1}^{n} b_k$ are bounded. Since $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$, we conclude the partial sums $\sum_{k=1}^{n} a_k$ are also bounded. Therefore, by Proposition 4.20, $\sum_{n=1}^{\infty} a_n$ converges. Therefore, **i.** is true.

ii. follows from **i.** by contraposition. ∎

Theorem 4.25 (Cauchy Criterion for Series)

A series $\sum_{n=1}^{\infty} a_n$ converges if, and only if, for all $\epsilon > 0$, there exists N so that

$$\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$$

for all $n > m \geq N$.

This theorem is a corollary of Theorem 4.17.

With this theorem, we can provide another proof for **i.** of Comparison Test.

Proof (of **i.** of Comparison Test). If $\sum_{n=1}^{\infty} b_n$ converges, then, by the Cauchy Criterion for Series, for all $\epsilon > 0$, there exists N, so that

$$\left| \sum_{k=m+1}^{n} b_k \right| < \epsilon$$

for all $n > m \geq N$.

For any $\epsilon > 0$, with the choice of N given above, we have that

$$\left| \sum_{k=m+1}^{n} a_k \right| \leq \left| \sum_{k=m+1}^{n} b_k \right| < \epsilon$$

for all $n > m \geq N$. Therefore, by the Cauchy Criterion for Series, $\sum_{n=1}^{\infty} a_n$ converges. ∎

Lecture 16

thm:ratiotest

Theorem 4.26 (Ratio Test)

Given $\sum_{n=1}^{\infty} a_n$, suppose that the limit

$$R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

exists.

If $R < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges[a]. If $R > 1$, the series $\sum_{n=1}^{\infty} a_n$ diverges.

[a] In fact, it converges absolutely.

thm:divergencetest

Theorem 4.27 (Divergence Test)

If a series $\sum n = 1^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

4.3.1 Mixed-sign series

Now, we seek to determine if $\sum_{n=1}^{\infty} a_n$ converges when the a_n are a mix of non-negative and negative terms.

Some previous test/tools can be applied to the mixed-sign case:

- Definition of infinite series convergence;
- Cauchy Criterion;
- Geometric Series Test;
- Ratio Test;
- Divergence Test;

but one key test cannot (at least not immediately):

- Comparison Test.

defn:absoluteconvergence

Definition 4.28 (Absolute Convergence)

If $\sum_{n=1}^{\infty} |a_n|$ converges, we say $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

defn:conditionalconvergence

Definition 4.29 (Conditional Convergence)

If $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges, we say $\sum_{n=1}^{\infty}$ *converges conditionally.*

thm:absoluteconvergencetest

Theorem 4.30 (Absolute Convergence Test)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

thm:alternatingseriestest

Theorem 4.31 (Alternating Series Test)

Consider (a_n) monotone decreasing with $a_n \geq 0$ for all n and $\lim_{n\to\infty} a_n = 0$. Then, the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_1 = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Proof. Consider the partial sums $s_n = \sum_{i=1}^{n} a_i$.

Define $I_{2k} = [s_{2k}, s_{2k-1}]$ and $I_{2k+1} = [s_{2k}, s_{2k+1}]$.

Note that $s_{2k+2} - s_{2k} = a_{2k+1} - a_{2k+2} \geq 0$ and $s_{2k+2} - s_{2k+1} = -a_{2k+2} \leq 0$. Therefore

$$s_{2k+2} \in [s_{2k}, s_{2k+1}].$$

and consequently,

$$I_{2k+2} \subset I_{2k+1}.$$

Similarly, since $s_{2k+1} - s_{2k-1} = -a_{2k} + a_{2k+1} \leq 0$ and $s_{2k+1} - s_{2k} = a_{2k+1} \geq 0$. Therefore

$$s_{2k+1} \in [s_{2k}, s_{2k-1}].$$

and consequently,

$$I_{2k+1} \subset I_{2k}.$$

Given any $\epsilon > 0$, since $(a_n) \to 0$, there exists ∎

Lecture 17

thm:limitcomparisontest

Theorem 4.32 (Limit Comparison Test)

Suppose $a_n \geq 0$ and $b_n > 0$ for all n, $\sum_{n=1}^{\infty} b_n$ converges and $L = \lim_{n\to\infty} \frac{a_n}{b_n}$ exists. Then, $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Since $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, there exists N so that

$$\frac{a_n}{b_n} < L + 1$$

for all $n \geq N$. Therefore,

$$\begin{aligned}\sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \\ &< \sum_{n=1}^{N-1} a_n + (L+1)\sum_{n=1}^{\infty} b_n.\end{aligned}$$

Since the first sum is finite and the second sum converges, we conclude that $\sum_{n=1}^{\infty} a_n$ converges. ■

4.3.2 Reordering a series

Theorem 4.33

If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then any reordering of $\sum_{n=1}^{\infty}$ will converge to the same value as $\sum_{n=1}^{\infty} a_n$.

Theorem 4.34

If $\sum_{n=1}^{\infty} a_n$ converges conditionally and α is any real number, then there exists an reordering of $\sum_{n=1}^{\alpha}$ that converges to α.

Proof. Define

$$\begin{aligned}a_n^+ &= \max\{a_n, 0\} \\ a_n^- &= \max\{-a_n, 0\}.\end{aligned}$$

Notice that

$$\begin{aligned}a_n &= a_n^+ - a_n^- \\ |a_n| &= a_n^+ + a_n^-.\end{aligned}$$

Suppose, by contradiction, one of the sequences $\sum a_n^+$ or $\sum a_n^-$ converges. Without loss of generality, $\sum a_n^+$ converges. Since $\sum a_n$ converges, we conclude $\sum a_n^- = \sum (a_n^+ - a_n)$ converges. Therefore, $\sum |a_n| = \sum (a_n^+ + a_n^-)$ converges; a contradiction of the conditional convergence.

Therefore, both series $\sum a_n^+$ and $\sum a_n^-$ diverge.

To be finished. ■

Lecture 18

5 Basic Topology

defn:neighborhood

Definition 5.1 (Neighborhood)

Let X be a metric space. Given any $a \in X$ and $\epsilon > 0$, we define the ϵ-*neighborhood centered at* a as

$$V_\epsilon(a) = \{x \in \mathbb{R} : d(a, x) < \epsilon\}.$$

One can see that, if $X = \mathbb{R}$,

$$V_\epsilon(a) = (a - \epsilon, a + \epsilon).$$

defn:open

Definition 5.2 (Open Set)

Let X be a metric space. We say $O \subset X$ is open with respect to X if, for all $a \in O$, there exists $\epsilon > 0$ so that

$$V_\epsilon(a) \subset O.$$

We'll usually omit "with respect to X" when the metric space is clear by context. Usually, in the examples, we'll consider $X = \mathbb{R}$.

Example

With respect to $\mathbb{R}$, $(1, 4)$ is an open set; $[1, 4)$ is not open; $(0, \infty)$ is open; $\mathbb{Q}$ is not open; $(1, 3) \cup (4, 6)$ is open; the empty set is open; $\mathbb{R}$ is open.

With respect to $\mathbb{R}^2$,

$$\{(x, y) \in \mathbb{R}^2 : 0 < x < 1 \text{ and } 0 < y < 1\}$$

is open;

$$\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$$

is open;

$$\{(x, y) \in \mathbb{R}^2 : y = 0 \text{ and } 1 < x < 4\}$$

is not open.

prop:unionopensets

Proposition 5.3 (Union of open sets)

Let $\mathcal{C}$ be a collection of open sets. Then,

$$\bigcup_{O \in \mathcal{C}} O$$

is an open set.

Proof. Let $x \in \bigcup_{O \in \mathcal{C}} O$. By definition of union, there exists a set $O_x \in \mathcal{C}$ so that $x \in O_x$. Since O_x is open, there exists $\epsilon > 0$ so that $V_\epsilon(x) \subset O_x$. Since $O_x \subset \bigcup O \in \mathcal{C} O$, we conclude $V_\epsilon(x) \subset \bigcup_{O \in \mathcal{C}} O$.

Since this argument was done for arbitrary x, we conclude $\bigcup_{O \in \mathcal{C}} O$ is open. ■

prop:intersectionopensets

Proposition 5.4 (Finite intersection of open sets)

Let $\mathcal{C}$ be a finite collection of open sets. Then,

$$\bigcap_{O \in \mathcal{C}} O$$

is an open set.

Proposition 5.5

There exists a collection $\mathcal{C}$ of open sets such that $\bigcap_{O \in \mathcal{C}} O$ is not open.

Proof. Let $\mathcal{C} = \left\{(-\frac{1}{n}, \frac{1}{n}) : n \in \mathbb{Z}_{>0}\right\}$. Then,

$$\bigcap_{O \in \mathcal{C}} O = \{0\},$$

which is not open. ■

prop:open

Proposition 5.6 (ϵ-neighborhoods are open)

Given any $a \in X$, and any $\epsilon > 0$, the set $V_r(a)$ is open.

Proof. Let $b \in V_\epsilon(a)$. Therefore, $d(a, b) < \epsilon$. Let $\delta = \epsilon - d(a, b) > 0$.

Let $c \in V_\delta(b)$. Therefore, $d(b, c) < \delta = \epsilon - d(a, b)$. Therefore, by the triangle

inequality,
$$d(a,c) \leq d(a,b) + d(b,c) < \epsilon,$$
i.e., $c \in V_\epsilon(a)$. Since this was done for arbitrary c, we conclude $V_\delta(b) \subset V_\epsilon(a)$.

Since this was done for arbitrary b, we conclude $V_\epsilon(a)$ is open. ■