MATH H333 (Algebra I) Lecture Notes

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This is Haverford College's undergraduate MATH H333, instructed by Tarik Aougab. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Zoom.

Contents

1 Binary Operations (September 09, 2020)

1.1 Why Algebra?

Algebra is the study of symmetry. An object has a symmetry when we can do something to it (transform it in some way) and without changing its appearance.

Example 1.1. A circle has a rotational symmetry: if we rotate the circle about its center, we get the same circle.

Example 1.2. The algebraic equation $x^2 + y^2 + z^2 - 3xyz = 0$ has a symmetry: for example, we can change the roles of x and z, which gives us the same equation.

Symmetry appears all over Mathematics, so Algebra is a prevalent topic abroad Mathematics.

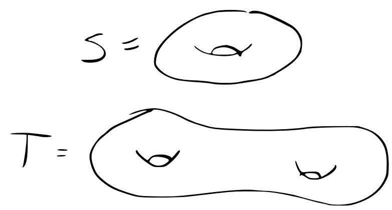
1.2 Places where Algebra arises in Mathematics

Number Theory. The following theorem will be proven in this course.

Theorem 1.1 (Fermat's Little Theorem). Let p be a prime integer number. Let a be a positive integer number. Then, $a^p - a$ is a multiple of p.

Topology.

Theorem 1.2. There is no continous bijection $f : S \to T$.



Sketch. Associate a "group" to S and another to T. A continous bujection would send the S-group perfectly to the T-group. But the two groups are different.

1.3 Binary Operations

Definition 1.1. If S is a set, then a *binary operation* on S is a function $f : S \times S \to S$. Here, $S \times S = \{(a, b) \mid a, b \in S\}$.

Example 1.3. If $S = \mathbb{R}$, then $f(a, b) = a + b$ and $g(a, b) = a \cdot b$ are binary operations.

Example 1.4. If $S = \mathbb{N}$, then $h(a, b) = a - b$ is not a binary operation.

Definition 1.2. A binary operation $f : S \times S \to S$ is *associative* if, for all $a, b, c \in S$,

$$f(f(a, b), c) = f(a, f(b, c)).$$

Example 1.5. If $S = \mathcal{M}_n(\mathbb{R})$, then $f(A, B) = AB$ is an associative binary operation.

Example 1.6. If $S = \mathbb{R}$, then $f(a, b) = a - b$ is a non-associative binary operation.

A key concept in Algebra is *transformation.*

Example 1.7. Let S be a non-empty section. Define $g(S) = \{T : S \to S\}$. Then, composition is an associative binary operation on $g(S)$, i.e., $f(T_1, T_2) = T_1 \circ T_2$ is an associative binary operation on $g(S)$.

2 Groups (September 11, 2020)

In the last class, we focused on binary (associative) operations.

2.1 Defining Groups

Definition 2.1 (Notation). If $a, b \in S$, then ab or $a \cdot b$ will commonly be used to denote $f(a, b)$. We will also commonly call this operation a *product*.

Associativity allows us to be less careful when writing down long products.

Example 2.1. In general, $a_1a_2a_3a_4a_5a_6a_7$ has no meaning. However, if the binary opperation is associative, no matter in which order we do the product, there will be no ambiguity about what value the expression have.

Definition 2.2. A binary operation on S is called *commutative* if for all $a, b \in S$, $ab = ba$ holds.

Example 2.2.
- (i) $(\mathbb{R}, +)$, $(\mathbb{C}, \cdot)$ have commutative binary operations.
- (ii) $(\mathcal{M}_n(\mathbb{R}),$ matrix multiplication$)$ has a non-commutative operation.
- (iii) $(\mathbb{R},$ distance$)$, i.e, $f(a, b) = |a - b|$, has a commuative, but non-associative operation.

Definition 2.3. Given S equipped with a binary opperation, we say $(S, \cdot)$, has an identity element if there exists $e \in S$ such that, for all $a \in S$, $a \cdot e = e \cdot a = a$ holds.

Example 2.3.
- (i) $(\mathbb{R}, +)$ has 0 as an identity.
- (ii) $(\mathbb{R}, \cdot)$ has 1 as an identity.
- (iii) $(\mathcal{M}_n(\mathbb{R}),$ matrix multiplication$)$ has I_n as an identity.

Definition 2.4. An element a of $(S, \cdot)$, that has an identity element (which we are going to call e), is called invertible if there exists $b \in S$ so that $ab = ba = e$.

Example 2.4.
- (i) Every element of $(\mathbb{R}, +)$ is invertible.
- (ii) Every element, except 0, of $(\mathbb{R}, \cdot)$ is invertible.
- (iii) Some elements, but not all, of $\mathcal{M}_n(\mathbb{R})$, equipped with matrix multiplication, are invertible.

Definition 2.5. A *group* is a set $(G, \cdot)$ with a binary opperation so that:
- (i) The binary opperation is associative.
- (ii) There exists an identity element in G.
- (iii) Every element in G is invertible.

If $\cdot$ is commutative, G is called an *abelian group.*

Example 2.5.
- (i) $(\mathbb{R}, +)$ is a group.
- (ii) $(\mathbb{C}, +)$ is a group.
- (iii) $(\mathbb{Z}, +)$ is a group.
- (iv) $(\mathbb{R}\backslash\{0\}, \cdot)$ is a group.
- (v) $(\mathbb{C}\backslash\{0\}, \cdot)$ is a group.
- (vi) $(\mathbb{Z}\backslash\{0\}, \cdot)$ is not a group, because 2 does not have an inverse element.
  - However, $(\mathbb{Q}\backslash\{0\}, \cdot)$ is a group.
- (vii) $\mathcal{M}_n(\mathbb{R})$, equipped with matrix multiplication is not a group, because the zero matrix does not have an inverse element.
  - However, if we define $GL_n(\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) \ : \ A \text{ is invertible}\}$, then $GL_n(\mathbb{R})$, equipped with matrix multiplication is a group.[1]
- ($viii$) Define $D_8 = \{\text{affine bijections } T : \mathbb{R}^2 \to \mathbb{R}^2 \text{ such that } T(\mathcal{S}) = \mathcal{S}\}$, where $\mathcal{S} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, also known as the standard unit square.

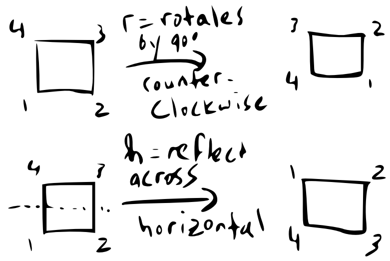
[1] It is important to prove that matrix multiplication is closed under $GL_n(\mathbb{R})$. In addition, this is the first example of a non-abelian group.

3 Subgroups (September 14, 2020)

Let's look more closely to $D_8 = \{\text{affine bijections } T : \mathbb{R} \to \mathbb{R} \text{ such that } T(\mathcal{S}) = \mathcal{S}\}$.

Proposition 3.1. D_8 is a group. The order of the group D_8 is 8.

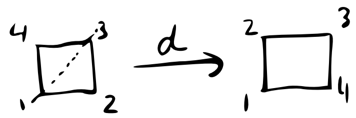
Proposition 3.2. Let $r, h \in D_8$ be described as follows:
(i) r denotes the rotation of $\mathcal{S}$ by 90°, counter-clockwise.
(ii) h denotes the reflect across the horizontal perpendicular bisector.



If $\phi \in D_8$, then ϕ can be expressed as $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \phi_1$, where $\phi_i = h$ or $\phi_i = r$, for all i.

The proposition above should resemble the concept of basis in Linear Algebra. In some sense, h and r gererate the group D_8.

Example 3.1. Let d be the reflection through the diagonal line through $(0, 0)$ and $(1, 1)$. We have $d = h \circ r \circ r \circ r = hr^3$.



Example 3.2. Let v be the reflection though the vertical perpendicular bisector. We have $v = hr^2$.

Note that $h^2 = r^4 = e$, and $2 \cdot 4 = 8$, which is the number of elements in D_8. What a coincidence, isn't it?

Definition 3.1. A *subgroup* H of a group $(G, \cdot)$ is a subset of G that is a group itself, with respect to the same operation $\cdot$.

Example 3.3.
(i) If G is a group, it has an identity, say e. Then $\{e\}$ is a subgroup of G.
(ii) G is always a subgroup of G.

Lemma 3.1. Given a a group G, a non-empty subset $H \subset G$ is a subgroup of G if, and only if, both following conditions are met:
(i) $ab \in H$, for all $a, b \in H$.
(ii) $a^{-1} \in H$, for all $a \in H$.

Example 3.4. $2\mathbb{Z} = \{\text{even integers}\}$ is a subgroup of $(\mathbb{Z}, +)$.

Definition 3.2 (Symmetric group on n elements). Given $n \in N$, define $S_n = \{\text{bijections } \tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}\}$, equipped with composition.

Example 3.5. Let $n = 5$, then consider $\tau : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$. Then $\tau \in S_5$.

Alternatively, we can use the following notation for $\tau = (13)(24)(5)$, which is called *cycle notation.*

Example 3.6. Consider $\tau' : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$. We can write $\tau' = (124)(35)$, using cycle notation.

Example 3.7. Consider $\tau'' : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$. We can write $\tau'' = (12345)$, using cycle notation.

Remark. Cycle notation is "not unique", e.g., $(12345) = (34512)$.

4 Integers (September 16, 2020)

Proposition 4.1. S_n us a finite group, and $|S_n| = n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$.

Proof. An arbitrary element $\tau \in S_n$ is described by determining $\tau(1), \tau(2), \dots, \tau(n)$. We have n choices for $\tau(1)$; after that, we have $n-1$ choices for $\tau(2)$; $\dots$; after that, we have 1 choice for $\tau(n)$. $\square$

Example 4.1. Suppose $q, p \in S_5$, $q = (14325)$ and $p = (15)(34)$. Determine qp in cycle notation.

Answer (Cheat). $qp = (14325)(15)(34)$.

Answer (More useful). $qp = (425)$.

Definition 4.1. Given $\tau \in S_n$, define M_τ as a $n \times n$ matrix obtained by permuting the rows of I_n in accordance with τ.

Example 4.2. If $\tau \in S_4$, $\tau = (134)$, then

$$M_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Given $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, we have $M_\tau \vec{v} = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \\ x_3 \end{pmatrix}$.

Theorem 4.1. Given $\tau \in S_n, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then $M_\tau \vec{x} = \begin{pmatrix} x_{\tau^{-1}(1)} \\ x_{\tau^{-1}(2)} \\ \vdots \\ x_{\tau^{-1}(n)} \end{pmatrix}$.

Theorem 4.2. $\det(M_\tau) = \pm 1$.

Theorem 4.3. Given $p, q \in S_n$, then $M_{pq} = M_p M_q$.

Definition 4.2. The *sign of* $\tau \in S_n$ is either ± 1, and it is just $\det(M_\tau)$.

Problem 4.1. If $G = (\mathbb{Z}, +)$, what are all subgroups of G?

Solution. Let H be a subgroup of G. $0 \in H$, because 0 is the identity element.

If $H = \{0\}$, we have a group – note that $H = 0\mathbb{Z}$. Otherwise, H has an element distinct from 0. Since $a \in H \iff -a \in H$, then there is a positive integer in H.

Let h be the smallest positive integer in H. Since addition is a binary opperation in H, we have $h\mathbb{Z} \subset H$.

Supposse $H \neq h\mathbb{Z}$. Therefore, there is an element $x \in H$, such that $x \notin h\mathbb{Z}$. Therefore, by Euclid's Algorithm, there is an integer q such that $nh < x < (n+1)h$; namely, q the quotient of x when evenly divided by h. Therefore, $0 < x - qh < h$.

However, $qh, x \in S$ implies that $x - qh \in H$. This is a contradition, because we have found a positive ineger smaller than h (the smallest positive element of H), which is also an element of H.

Therefore, $H = h\mathbb{Z}$, with $H \in \mathbb{Z}_{\geq 0}$, are all the subgroups of G.

Let us see some applications of Problem 4.1.

Given $a, b \in \mathbb{Z}$, consider $S = a\mathbb{Z} + b\mathbb{Z} = \{n \in \mathbb{Z} : n = ra + sb, r, s \in \mathbb{Z}\}$. Verify that S is a subgroup of $\mathbb{Z}$. Using Problem 4.1, we have that $S = d\mathbb{Z}$, for some integer d.

5 Cyclic Groups (September 18, 2020)

Recall that every subgroup S of $(\mathbb{Z}, +)$ is of the form $d\mathbb{Z}$, for some integer d.

Also, if a, b are integers, we can consider $S = a\mathbb{Z} + b\mathbb{Z}$, which is a subgroup of $\mathbb{Z}$. Therefore, $S\mathbb{Z} = d\mathbb{Z}$. for some integer d.

Since $a, b \in S = a\mathbb{Z} + b\mathbb{Z}$, then $a, b \in d\mathbb{Z}$, which means that d is a divisor of both a, b.

Now, let $n \in \mathbb{Z}$ such that n divides both a and b. Thus, n divides any number of the form $sa + rb$. But, $d\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$, which means $d = ra + bs$, for right choices of r and s. Therefore, n divides d.

Definition 5.1. For $a, b \in \mathbb{Z}$, we define d as above as the *greatest common divisor* of a and b, which we denote by $\gcd(a, b)$.

We have shown not only that d is the greatest common divisor of a and b, but also that any other common divisor of a and b divides d.

Algorithm 5.1 (Euclidean Algorithm).

Example 5.1. Let $a = 314$ and $b = 136$. We divide 314 by 136 and get $314 = 2 \cdot 136 + 42$. Thus,

$$\begin{aligned} n \in 314\mathbb{Z} + 136\mathbb{Z} &\iff n = r \cdot 314 + s \cdot 136 \\ &\iff n = r \cdot (2 \cdot 136 + 402) + s \cdot 136 \\ &\iff n = r \cdot (2r + s) \cdot 136 + r \cdot 42 \\ &\iff n \in 136\mathbb{Z} + 42\mathbb{Z}. \end{aligned}$$

Therefore, $\gcd(314, 136) = \gcd(136, 42)$. We can further use

Definition 5.2. Given $a, b \in \mathbb{Z}$, $a, b \neq 0$, then a and b are relatively prime if, and only if, $\gcd(a, b) = 1$.

Proposition 5.1. The $\gcd(a, b)$ is the product of the prime powers common to prime factorizations of a and b.

Example 5.2. Let $a = 52 = 2^2 \cdot 13$, and $b = 2^3 \cdot 3$. Therefore, $\gcd(52, 24) = 2^2$.

Corollary 5.1. If a and b are relatively prime if, and only if, there are integers r and s such that $ra + sb = 1$.

Corollary 5.2. Suppose p is a prime. Then, given $a, b \in \mathbb{Z}$, if p divides ab, therefore p divides a or p divides b.

Proof. If p divides a, we are done.

Suppose that p does not divide a. Thus, $\gcd(p, a) = 1$. It implies that

$$1 = rp + sa,$$

for some integers r and s. If we multiply both sides by b, we have

$$b = rbp + sab.$$

Notice that p divides both rbp and sab, therefore, p divides their sum, which is b. $\square$

Theorem 5.1. Let $G = (G, \cdot)$ be a group, let I be a set, and let $\{H_i\}_{i \in I}$ be a family of subgroups of G indexed by I. Then, the set

$$\bigcap_{i \in I} H_i$$

is a group.

Proof. We want to show:

(i) $\bigcap_{i \in I} H_i \neq \varnothing$.
For this item, $e \in \bigcap_{i = I} H_i$.

(ii) $a, b \in \bigcap_{i \in I} H_i \implies ab \in \bigcap_{i \in I}$.
For this item, $a, b \in H_i$, for all $i \in I$, which implies $ab \in H_i$ for all i

$\square$

Back to $(\mathbb{Z}, +)$. Given $a, b \in \mathbb{Z}$, let $S = a\mathbb{Z} \cap b\mathbb{Z}$. By the last theorem, S is a subgroup. By Wednesday's theorem, $S = a\mathbb{Z} + b\mathbb{Z} = m\mathbb{Z}$, for some $m \in \mathbb{Z}$. Since $m \in m\mathbb{Z} = a\mathbb{Z} \cap b\mathbb{Z}$, m is a multiple of a and b.

Now, for any number n that is multiple of both a and $b \implies n \in a\mathbb{Z} \cap b\mathbb{Z} = m\mathbb{Z} \implies n$ is a multuple of m.

Definition 5.3. The m described above is called the *lowest common multiple* of a and b, denoted by $\operatorname{lcm}(a, b)$.

We have proved above only only that m is the lowest common multiple, but also that m divides every common multiple of a and b.

Definition 5.4. Let $(G, \cdot)$ be a group and $x \in G$. Then the cyclic subgroup generated by x, denoted by $< x >$, is all powers of x, i.e.,

$$\langle x \rangle = \{\dots, x^{-1}, e, x^1, x^2, \dots\}.$$

Theorem 5.2. In G, let $\Gamma(x) = \{H \subseteq G : H \text{ is a subgroup of } G \text{ and } x \in H\}$. Then

$$\bigcap_{H \in \Gamma(x)} H = \langle x \rangle .$$

6 Isomorphisms (September 21, 2020)

This class happened during IMO. The lecture notes are to do.

7 Cosets (September 23, 2020)

This class happened during IMO. The lecture notes are to do.

8 Coset Properties (September 25, 2020)

Definition 8.1 (Equivalence Relation). An *equivalence relation* is a relation on a set S, i.e., a way to say that certain pairs of elements can be in relationship to one another; so long as the pair satisfies whatever rules we choose for that relationship, AND our rules need to satisfy these properties.

(i) $x \sim x$;
(ii) if $x \sim y$, then $y \sim x$;
(iii) if $x \sim y$ and $y \sim z$, then $x \sim z$.

Remark. If a pair (x, y) satisfy our rules, we write $x \sim y$, "x is equivalent to y".

Definition 8.2 (Equivalence Class). Given a set S, $s \in S$, and an equivalence relation $\sim$, the *equivalence class of* x, denoted $[x]$, is $[x] = \{y \in S : x \sim y\}$.

Example 8.1. Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$, and we will say that $(a, b) \sim (c, d) \iff ad = bc$. Let us check if the three properties are ensured:

(i) $(a, b) \sim (a, b)$, because $ab = ba$;
(ii) $(a, b) \sim (c, d) \iff ad = bc \iff cb = da \iff (c, d) \sim (a, b)$;
(iii) If $(a, b) \sim (c, d)$ and $(c, d) \sim (r, s)$. Then, $ad = bc$ and $cs = dr$. Therefore, $adcs = bcdr$, which means that $as = br$ (since $c \neq 0 \neq d$). In other words, $(a, b) \sim (r, r)$.

In this case, $[(a, b)] = \{(c, d) \in S : ad = bc\}$.

Theorem 8.1. If S is a set, with an equivalence relation $\sim$, then the equivalence classes of $\sim$ *disjointly partition* S, i.e., every element of S is contained in **exactly** one equivalence class.

Given S, equipped with an equivalence class $\sim$ on S, we define $\bar{S} = \{[x] : x \in S\}$, i.e., the set of equivalence classes.

In this situation, there exists a map $\pi : S \to \bar{S}$, defined by $x \mapsto [x]$.

Example 8.2. Let $S = \mathbb{Z}$, and $a \sim b \iff a - b$ is a multiple of 5. (You should verify that this is an equivalence relation.)

Then $\bar{S} = \{[0], [1], [2], [3], [4]\}$. E.g., $\pi(7) = [2]$.

Definition 8.3. Let $H \leq G$ be groups, and $a \in G$. Then, *the right coset of H with respect to a is*

$$Ha = \{g \in G : \exists h \in H \text{ such that } ha = g\} = \{ha : h \in H\}.$$

Lemma 8.1.

$$Ha = Hb \iff ab^{-1} \in H$$

Lemma 8.2. Given $H \leq G$ groups, the relation defined by $a \sim b \iff ab^{-1} \in H$ is an equivalence relation.

So, what are the equivalence classes of this equivalence relation? They are exactly the right cossets of H, i.e, $[a] = Ha$.

Therefore, right cosets, if distinct, share no elements in common.

On Monday, we'll prove the following theorem.

Theorem 8.2 (Lagrange's Theorem). If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$.

9 Normal Subgroups (September 28, 2020)

Lemma 9.1. Given $H \leq G$, if $|G| < \infty$, then given $a, b \in G$, it holds $\#(Ha) = \#(Hb)$.

Proof. Note that H is a right coset ($H = He$). So, suffices to show that for all $a \in G$, $\#(Ha) = |H|$. Define a function $\varphi : H \to Ha$, defined by $h \mapsto ha$. We shall prove that φ is a bijection.

Let's show that φ is onto. Given $g \in Ha$, then $g = ha$ for some $h \in H$. But $\phi(h) = ha = g$, which means that $g \in \text{Im}(\varphi)$.

Let's show that φ is one-to-one. If $\varphi(h_1) = \varphi(h_2) \implies h_1 a = h_2 a = \implies h_1 aa^{-1} = h_2 aa^{-1} \implies a =$

Therefore φ is a bijection, which implies that $\#(Ha) = |H|$, and we're done! □

Theorem 9.1 (Lagrange's Theorem). If $H \leq G$ are finite groups, then $|H|$ divides $|G|$.

Proof. The right cosets of H partitionate G, i.e., they are disjoint and their union is G; and they all have the same number of elements. Let $[G : H]$ denote the number of right cosets of H sitting inside G, which is called index of H in G. Therefore,

$$G = [G : H] \cdot |H|.$$

□

Corollary 9.1. Given a group G and $a \in G$, if $|G| < \infty$, then order(a) divides $|G|$.

Proof. Consider $\langle a \rangle \leq G$, then, by Lagrange's Theorem, $|\langle a \rangle| = \text{order}(a)$ divides $|G|$ □

Definition 9.1. A subgroup H of G is called *normal*, denoted by $H \lhd G$ if, for all $g \in G$, the image of H under the g-conjugation isomorphism (the g-conjugation isomorphism is the map $\phi_g : G \to G$ defined by $a \mapsto gag^{-1}$) is cointained in H, i.e, $\phi_g(H) \subset H$, for all $g \in G$.

Lemma 9.2. If G and G' are subgroups, $\phi : G \to G'$ a homomorphism, then $\text{Ker}\phi \lhd G$.

Example 9.1. $SL_n(\mathbb{R}) \lhd GL_n(\mathbb{R})$.

Use Lemma 9.2 with $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$.

Example 9.2. $\langle (1\ 2) \rangle \ntriangleleft G$, because $\phi_{(2\ 3)}(\langle (1\ 2) \rangle) = \{e, (1\ 3)\} \not\subset H$

Theorem 9.2. The following are equivalent:

(i) $H \lhd G$;
(ii) $gHg^{-1} = H$, for any $g \in G$;
(iii) $gH = Hg$, for any $g \in G$;
(iv) Every left coset of H is a right coset of H.

10 Example of Quotients (September 30, 2020)

From last time, we discussed the following theorem:

Theorem 10.1. The following are equivalent:
(i) $H \lhd G$;
(ii) $gHg^{-1} = H$, for any $g \in G$;
(iii) $gH = Hg$, for any $g \in G$;
(iv) Every left coset of H is a right coset of H.

Proof ($i \implies ii$). $H \lhd G \implies \phi_g(H) \in H \implies gHg^{-1} \subset H$. Analougously, $g^{-1}Hg \subset H$. This last one implies that $H \subset gHg^{-1}$.

Therefore, $gHg^{-1} = H$. □

Proof ($ii \iff iii$). $gHg^{-1} = H \iff gH = Hg$. □

Proof ($iii \implies iv$). If $gH = Hg$, then gH is a right coset. □

Proof ($iv \implies iii$). Assume that, given aH, then there is b such that $aH = Hb$. Note that gH sahres an element shares an element (namely, g) with Hg. Since gH is a left coset, then $gH = Hb$ for some b.

Since $g \in gH = Hb$, then Hb intersects with Hg, then $Hb = Hg$ (because, if two left cosets share an element, then they are equal). □

Proof ($ii \implies i$). $gHg^{-1} = \phi_g(H) = H$, then $\phi_g \subset H$, which implies $H \lhd G$. □

Recall from Linear Algebra:

Theorem 10.2. Let $T : V \to W$ a linear map, then

$$\dim V = \dim \ker T + \dim \operatorname{Im} T.$$

If T is onto, then

$$\dim V = \dim \ker T + \dim W.$$

The goal is to reproduce this idea with groups and homomorphisms, i.e., given G, G' groups, and an onto homomorphism $\phi : G \to G'$, then understand G as being a "stacking" of cosets of $\ker(\phi)$ and when we collapse each coset to a point, we get G'.

Our goal will be realted to the following theorem:

Theorem 10.3. Given G and a subgroup H, then $H \lhd G$ if, and only if, there is a group G' and a homomorphism $\phi : G \to G'$ such that $\ker \phi = H$.

Definition 10.1 (Notation). Let G/H ("G mod H") be the set of all right cosets of H sitting inside G.

Theorem 10.4. When $H \lhd G$, there exists a binary operation on G/H and an homomorphism $\phi : G \to G/H$ such that $\ker \phi = H$.

Spoiler: The operation $*$ will be, for $A, B \in G/H$, $A * B = AB = \{g \in G : \exists\ a_1 \in A, b_1 \in B, g = a_1 b_1\}$.

11 Quotient Groups (October 02, 2020)

Here's a rephrasing of our motivation from Linear Alegbra:

Suppose V is a vector space, and $S \subset V$ a subspace. Let $\vec{x} \in V$. In Linear Algebra, we learned we write $\vec{x}$ as a sum of a vector in S with an vector $\vec{z}$ ortogonal to S, a.k.a., $\vec{z} \in S^{\perp}$.

V can be decomposed into parallel copies of S, and there exists a vector space W and a linear map $T : V \to W$ so that T has the effect of collapsing each parallel copy of S to a point. And, $\ker T = S$.

To summarize: Given S a subspace of V, there exists a decomposition of V into parallel copies of S and there exists a vector space W and a linear map $T : V \to W$ so that T collapses the parallel copies to points and $\ker T = S$.

Our goal in the Group Theory setting: Given $H \lhd G$, there exists a decomposition of G into right cosets of H in G and there exists a group G' and a homomorphism $\phi : G \to G'$ so that ϕ collapses a right cosets of H to a point and $\ker\phi = H$.

On Wednesday, we defined $G/H = \{\text{right cosets of } H \text{ in } G\}$ as our candidate for G'.

Given $Ha, Hb \in G/H$, we defined $Ha * Hb = (Ha)(Hb) = \{h_1 a h_2 b : h_1, h_2 \in H\}$.

We know that $aH = Ha$, then $HaHb = HHab$. Since H is closed under operation and $He = H$, we have that $HH = H$. Thus,

$$Ha * Hb = (Ha)(Hb) = H(ab),$$

which means that $*$ is a binary operation.

So far, we have that G/H has a binary operation. We also have a candidate for ϕ! Define $\phi : G \to G/H$, with $g \to Hg$. Given, $a, b \in G$,

$$\phi(ab) = H(ab) = (Ha)(Hb) = \phi(a)\phi(b),$$

thus ϕ has the homomorphism property. Note also that ϕ is onto.

Lemma 11.1. If G is a group, Y is a set with a binary operation, $\phi : G \to Y$ such that ϕ has the homomorphism property, and ϕ is onto. Then Y is a group and ϕ is a homomorphism.

Proof. We need to show the following items:

(i) Assoiativity. Given $a, b, c \in Y$, since ϕ is onto, we have $a = \phi(a'), b = \phi(b'), c = \phi(c')$, for some $a', b', c' \in G$. So

$$\begin{aligned}
(ab)c &= (\phi(a')\phi(b'))\phi(c') \\
&= \phi(a'b')\phi(c') \\
&= \phi((a'b')c') \\
&= \phi(a'(b'c')) \\
&= \phi(a')\phi(b'c') \\
&= \phi(a')(\phi(b')\phi(c')) \\
&= a(bc).
\end{aligned}$$

(ii) Identity. The same strategy as above.

(iii) Inverses. The same strategy as above.

Thus, Y is a group. ∎