

MATH H333 (Algebra I) Lecture Notes

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This is Haverford College's undergraduate MATH H333, instructed by Tarik Aougab. All errors are my responsibility.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

This class is being taught remotely via Zoom.

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1 Binary Operations (September 09, 2020)

1.1 Why Algebra?

Algebra is the study of symmetry. An object has a symmetry when we can do something to it (transform it in some way) and without changing its appearance.

Example 1.1

A circle has a rotational symmetry: if we rotate the circle about its center, we get the same circle.

Example 1.2

The algebraic equation $x^2 + y^2 + z^2 - 3xyz = 0$ has a symmetry: for example, we can change the roles of x and z , which gives us the same equation.

Symmetry appears all over Mathematics, so Algebra is a prevalent topic abroad Mathematics.

1.2 Places where Algebra arises in Mathematics

Number Theory. The following theorem will be proven in this course.

Theorem 1.1 (Fermat's Little Theorem)

Let p be a prime integer number. Let a be a positive integer number. Then, $a^p - a$ is a multiple of p .

Topology.

Theorem 1.2

There is no continuous bijection $f : S \rightarrow T$.



Sketch. Associate a “group” to S and another to T . A continuous bijection would send the S -group perfectly to the T -group. But the two groups are different.

1.3 Binary Operations

Definition 1.3

If S is a set, then a *binary operation* on S is a function $f : S \times S \rightarrow S$. Here, $S \times S = \{(a, b) \mid a, b \in S\}$.

Example 1.3

If $S = \mathbb{R}$, then $f(a, b) = a + b$ and $g(a, b) = a \cdot b$ are binary operations.

Example 1.4

If $S = \mathbb{N}$, then $h(a, b) = a - b$ is not a binary operation.

Definition 1.4

A binary operation $f : S \times S \rightarrow S$ is *associative* if, for all $a, b, c \in S$,

$$f(f(a, b), c) = f(a, f(b, c)).$$

Example 1.5

If $S = \mathcal{M}_n(\mathbb{R})$, then $f(A, B) = AB$ is an associative binary operation.

Example 1.6

If $S = \mathbb{R}$, then $f(a, b) = a - b$ is a non-associative binary operation.

A key concept in Algebra is *transformation*.

Example 1.7

Let S be a non-empty set. Define $g(S) = \{T : S \rightarrow S\}$. Then, composition is an associative binary operation on $g(S)$, i.e., $f(T_1, T_2) = T_1 \circ T_2$ is an associative binary operation on $g(S)$.

2 Groups (September 11, 2020)

In the last class, we focused on binary (associative) operations.

2.1 Defining Groups

Definition 2.1 (Notation)

If $a, b \in S$, then ab or $a \cdot b$ will commonly be used to denote $f(a, b)$. We will also commonly call this operation a *product*.

Associativity allows us to be less careful when writing down long products.

Example 2.1

In general, $a_1 a_2 a_3 a_4 a_5 a_6 a_7$ has no meaning. However, if the binary operation is associative, no matter in which order we do the product, there will be no ambiguity about what value the expression have.

Definition 2.2

A binary operation on S is called *commutative* if for all $a, b \in S$, $ab = ba$ holds.

Example 2.2

- (i) $(\mathbb{R}, +)$, (\mathbb{C}, \cdot) have commutative binary operations.
- (ii) $(\mathcal{M}_n(\mathbb{R}), \text{matrix multiplication})$ has a non-commutative operation.
- (iii) $(\mathbb{R}, \text{distance})$, i.e. $f(a, b) = |a - b|$, has a commutative, but non-associative operation.

Definition 2.3

Given S equipped with a binary operation, we say (S, \cdot) , has an identity element if there exists $e \in S$ such that, for all $a \in S$, $a \cdot e = e \cdot a = a$ holds.

Example 2.3

- (i) $(\mathbb{R}, +)$ has 0 as an identity.
- (ii) (\mathbb{R}, \cdot) has 1 as an identity.
- (iii) $(\mathcal{M}_n(\mathbb{R}), \text{matrix multiplication})$ has I_n as an identity.

Definition 2.4

An element a of (S, \cdot) , that has an identity element (which we are going to call e), is called invertible if there exists $b \in S$ so that $ab = ba = e$.

Example 2.4

- (i) Every element of $(\mathbb{R}, +)$ is invertible.
- (ii) Every element, except 0, of (\mathbb{R}, \cdot) is invertible.
- (iii) Some elements, but not all, of $\mathcal{M}_n(\mathbb{R})$, equipped with matrix multiplication, are invertible.

Definition 2.5

A *group* is a set (G, \cdot) with a binary operation so that:

- (i) The binary operation is associative.
- (ii) There exists an identity element in G .
- (iii) Every element in G is invertible.

If \cdot is commutative, G is called an *abelian group*.

Example 2.5

- (i) $(\mathbb{R}, +)$ is a group.
- (ii) $(\mathbb{C}, +)$ is a group.
- (iii) $(\mathbb{Z}, +)$ is a group.
- (iv) $(\mathbb{R} \setminus \{0\}, \cdot)$ is a group.
- (v) $(\mathbb{C} \setminus \{0\}, \cdot)$ is a group.
- (vi) $(\mathbb{Z} \setminus \{0\}, \cdot)$ is not a group, because 2 does not have an inverse element.
 - However, $(\mathbb{Q} \setminus \{0\}, \cdot)$ is a group.
- (vii) $\mathcal{M}_n(\mathbb{R})$, equipped with matrix multiplication is not a group, because the zero matrix does not have an inverse element.
 - However, if we define $GL_n(\mathbb{R}) = \{A \in \mathcal{M}_n(\mathbb{R}) : A \text{ is invertible}\}$, then $GL_n(\mathbb{R})$, equipped with matrix multiplication is a group.^a
- (viii) Define $D_8 = \{\text{affine bijections } T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \text{ such that } T(\mathcal{S}) = \mathcal{S}\}$, where $\mathcal{S} = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, also known as the standard unit square.

^aIt is important to prove that matrix multiplication is closed under $GL_n(\mathbb{R})$. In addition, this is the first example of a non-abelian group.

3 Subgroups (September 14, 2020)

Let's look more closely to $D_8 = \{\text{affine bijections } T : \mathbb{R} \rightarrow \mathbb{R} \text{ such that } T(\mathcal{S}) = \mathcal{S}\}$.

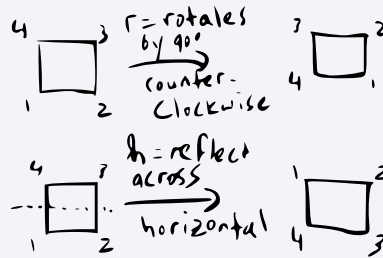
Proposition 3.1

D_8 is a group. The order of the group D_8 is 8.

Proposition 3.2

Let $r, h \in D_8$ be described as follows:

- (i) r denotes the rotation of \mathcal{S} by 90° , counter-clockwise.
- (ii) h denotes the reflect across the horizontal perpendicular bisector.



If $\phi \in D_8$, then ϕ can be expressed as $\phi = \phi_n \circ \phi_{n-1} \circ \cdots \circ \phi_1$, where $\phi_i = h$ or $\phi_i = r$, for all i .

The proposition above should resemble the concept of basis in Linear Algebra. In some sense, h and r generate the group D_8 .

Example 3.1

Let d be the reflection through the diagonal line through $(0,0)$ and $(1,1)$. We have $d = h \circ r \circ r \circ r = hr^3$.



Example 3.2

Let v be the reflection through the vertical perpendicular bisector. We have $v = hr^2$.

Note that $h^2 = r^4 = e$, and $2 \cdot 4 = 8$, which is the number of elements in D_8 . What a coincidence, isn't it?

Definition 3.3

A *subgroup* H of a group (G, \cdot) is a subset of G that is a group itself, with respect to the same operation \cdot .

Example 3.3

- (i) If G is a group, it has an identity, say e . Then $\{e\}$ is a subgroup of G .
- (ii) G is always a subgroup of G .

Lemma 3.4

Given a group G , a non-empty subset $H \subset G$ is a subgroup of G if, and only if, both following conditions are met:

- (i) $ab \in H$, for all $a, b \in H$.
- (ii) $a^{-1} \in H$, for all $a \in H$.

Example 3.4

$2\mathbb{Z} = \{\text{even integers}\}$ is a subgroup of $(\mathbb{Z}, +)$.

Definition 3.5 (Symmetric group on n elements)

Given $n \in \mathbb{N}$, define $S_n = \{\text{bijections } \tau : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}\}$, equipped with composition.

Example 3.5

Let $n = 5$, then consider $\tau : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 2 & 5 \end{pmatrix}$. Then $\tau \in S_5$.

Alternatively, we can use the following notation for $\tau = (13)(24)(5)$, which is called *cycle notation*.

Example 3.6

Consider $\tau' : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$. We can write $\tau' = (124)(35)$, using cycle notation.

Example 3.7

Consider $\tau'' : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$. We can write $\tau'' = (12345)$, using cycle notation.

Remark. Cycle notation is “not unique”, e.g., $(12345) = (34512)$.

4 Integers (September 16, 2020)

Proposition 4.1

S_n is a finite group, and $|S_n| = n! = n \cdot (n-1) \cdot \dots \cdot 2 \cdot 1$.

Proof. An arbitrary element $\tau \in S_n$ is described by determining $\tau(1), \tau(2), \dots, \tau(n)$. We have n choices for $\tau(1)$; after that, we have $n-1$ choices for $\tau(2)$; \dots ; after that, we have 1 choice for $\tau(n)$. \square

Example 4.1

Suppose $q, p \in S_5$, $q = (14325)$ and $p = (15)(34)$. Determine qp in cycle notation.

Answer (Cheat). $qp = (14325)(15)(34)$.

Answer (More useful). $qp = (425)$.

Definition 4.2

Given $\tau \in S_n$, define M_τ as a $n \times n$ matrix obtained by permuting the rows of I_n in accordance with τ .

Example 4.2

If $\tau \in S_4$, $\tau = (134)$, then

$$M_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

$$\text{Given } \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \text{ we have } M_\tau \vec{x} = \begin{pmatrix} x_4 \\ x_2 \\ x_1 \\ x_3 \end{pmatrix}.$$

Theorem 4.3

$$\text{Given } \tau \in S_n, \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ then } M_\tau \vec{x} = \begin{pmatrix} x_{\tau^{-1}(1)} \\ x_{\tau^{-1}(2)} \\ \vdots \\ x_{\tau^{-1}(n)} \end{pmatrix}.$$

Theorem 4.4

$$\det(M_\tau) = \pm 1.$$

Theorem 4.5

Given $p, q \in S_n$, then $M_{pq} = M_p M_q$.

Definition 4.6

The *sign* of $\tau \in S_n$ is either ± 1 , and it is just $\det(M_\tau)$.

Problem 4.1. If $G = (\mathbb{Z}, +)$, what are all subgroups of G ?

Solution. Let H be a subgroup of G . $0 \in H$, because 0 is the identity element.

If $H = \{0\}$, we have a group – note that $H = 0\mathbb{Z}$. Otherwise, H has an element distinct from 0. Since $a \in H \iff -a \in H$, then there is a positive integer in H .

Let h be the smallest positive integer in H . Since addition is a binary operation in H , we have $h\mathbb{Z} \subset H$.

Suppose $H \neq h\mathbb{Z}$. Therefore, there is an element $x \in H$, such that $x \notin h\mathbb{Z}$. Therefore, by Euclid's Algorithm, there is an integer q such that $nh < x < (n+1)h$; namely, q the quotient of x when evenly divided by h . Therefore, $0 < x - qh < h$.

However, $qh, x \in S$ implies that $x - qh \in H$. This is a contradiction, because we have found a positive integer smaller than h (the smallest positive element of H), which is also an element of H .

Therefore, $H = h\mathbb{Z}$, with $h \in \mathbb{Z}_{\geq 0}$, are all the subgroups of G .

Let us see some applications of Problem 4.1.

Given $a, b \in \mathbb{Z}$, consider $S = a\mathbb{Z} + b\mathbb{Z} = \{n \in \mathbb{Z} : n = ra + sb, r, s \in \mathbb{Z}\}$. Verify that S is a subgroup of \mathbb{Z} . Using Problem 4.1, we have that $S = d\mathbb{Z}$, for some integer d .

5 Cyclic Groups (September 18, 2020)

Recall that every subgroup S of $(\mathbb{Z}, +)$ is of the form $d\mathbb{Z}$, for some integer d .

Also, if a, b are integers, we can consider $S = a\mathbb{Z} + b\mathbb{Z}$, which is a subgroup of \mathbb{Z} . Therefore, $S\mathbb{Z} = d\mathbb{Z}$ for some integer d .

Since $a, b \in S = a\mathbb{Z} + b\mathbb{Z}$, then $a, b \in d\mathbb{Z}$, which means that d is a divisor of both a, b .

Now, let $n \in \mathbb{Z}$ such that n divides both a and b . Thus, n divides any number of the form $sa + rb$. But, $d\mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}$, which means $d = ra + bs$, for right choices of r and s . Therefore, n divides d .

Definition 5.1

For $a, b \in \mathbb{Z}$, we define d as above as the *greatest common divisor* of a and b , which we denote by $\gcd(a, b)$.

We have shown not only that d is the greatest common divisor of a and b , but also that any other common divisor of a and b divides d .

Algorithm 5.2 (Euclidean Algorithm) —

Example 5.1

Let $a = 314$ and $b = 136$. We divide 314 by 136 and get $314 = 2 \cdot 136 + 42$. Thus,

$$\begin{aligned} n \in 314\mathbb{Z} + 136\mathbb{Z} &\iff n = r \cdot 314 + s \cdot 136 \\ &\iff n = r \cdot (2 \cdot 136 + 402) + s \cdot 136 \\ &\iff n = r \cdot (2r + s) \cdot 136 + r \cdot 42 \\ &\iff n \in 136\mathbb{Z} + 42\mathbb{Z}. \end{aligned}$$

Therefore, $\gcd(314, 136) = \gcd(136, 42)$. We can further use

Definition 5.3

Given $a, b \in \mathbb{Z}$, $a, b \neq 0$, then a and b are relatively prime if, and only if, $\gcd(a, b) = 1$.

Proposition 5.4

The $\gcd(a, b)$ is the product of the prime powers common to prime factorizations of a and b .

Example 5.2

Let $a = 52 = 2^2 \cdot 13$, and $b = 2^3 \cdot 3$. Therefore, $\gcd(52, 24) = 2^2$.

Corollary 5.5

If a and b are relatively prime if, and only if, there are integers r and s such that $ra + sb = 1$.

Corollary 5.6

Suppose p is a prime. Then, given $a, b \in \mathbb{Z}$, if p divides ab , therefore p divides a or p divides b .

Proof. If p divides a , we are done.

Suppose that p does not divide a . Thus, $\gcd(p, a) = 1$. It implies that

$$1 = rp + sa,$$

for some integers r and s . If we multiply both sides by b , we have

$$b = rbp + sab.$$

Notice that p divides both rbp and sab , therefore, p divides their sum, which is b . □

Theorem 5.7

Let $G = (G, \cdot)$ be a group, let I be a set, and let $\{H_i\}_{i \in I}$ be a family of subgroups of G indexed by I . Then, the set

$$\bigcap_{i \in I} H_i$$

is a group.

Proof. We want to show:

- (i) $\bigcap_{i \in I} H_i \neq \emptyset$.

For this item, $e \in \bigcap_{i \in I} H_i$.

- (ii) $a, b \in \bigcap_{i \in I} H_i \implies ab \in \bigcap_{i \in I} H_i$.

For this item, $a, b \in H_i$, for all $i \in I$, which implies $ab \in H_i$ for all i

□

Back to $(\mathbb{Z}, +)$. Given $a, b \in \mathbb{Z}$, let $S = a\mathbb{Z} \cap b\mathbb{Z}$. By the last theorem, S is a subgroup. By Wednesday's theorem, $S = a\mathbb{Z} + b\mathbb{Z} = m\mathbb{Z}$, for some $m \in \mathbb{Z}$. Since $m \in m\mathbb{Z} = a\mathbb{Z} \cap b\mathbb{Z}$, m is a multiple of a and b .

Now, for any number n that is multiple of both a and $b \implies n \in a\mathbb{Z} \cap b\mathbb{Z} = m\mathbb{Z} \implies n$ is a multiple of m .

Definition 5.8

The m described above is called the *lowest common multiple* of a and b , denoted by $\text{lcm}(a, b)$.

We have proved above only that m is the lowest common multiple, but also that m divides every common multiple of a and b .

Definition 5.9

Let (G, \cdot) be a group and $x \in G$. Then the cyclic subgroup generated by x , denoted by $\langle x \rangle$, is all powers of x , i.e.,

$$\langle x \rangle = \{\dots, x^{-1}, e, x^1, x^2, \dots\}.$$

Theorem 5.10

In G , let $\Gamma(x) = \{H \subseteq G : H \text{ is a subgroup of } G \text{ and } x \in H\}$. Then

$$\bigcap_{H \in \Gamma(x)} H = \langle x \rangle.$$

6 Isomorphisms (September 21, 2020)

This class happened during IMO. The lecture notes are to do.

7 Cosets (September 23, 2020)

This class happened during IMO. The lecture notes are to do.

8 Coset Properties (September 25, 2020)

Definition 8.1 (Equivalence Relation)

An *equivalence relation* is a relation on a set S , i.e., a way to say that certain pairs of elements can be in relationship to one another; so long as the pair satisfies whatever rules we choose for that relationship, AND our rules need to satisfy these properties.

- (i) $x \sim x$;
- (ii) if $x \sim y$, then $y \sim x$;
- (iii) if $x \sim y$ and $y \sim z$, then $x \sim z$.

Remark. If a pair (x, y) satisfy our rules, we write $x \sim y$, “ x is equivalent to y ”.

Definition 8.2 (Equivalence Class)

Given a set S , $s \in S$, and an equivalence relation \sim , the *equivalence class of x* , denoted $[x]$, is $[x] = \{y \in S : x \sim y\}$.

Example 8.1

Let $S = \mathbb{Z} \times (\mathbb{Z} - \{0\})$, and we will say that $(a, b) \sim (c, d) \iff ad = bc$. Let us check if the three properties are ensured:

- (i) $(a, b) \sim (a, b)$, because $ab = ba$;
- (ii) $(a, b) \sim (c, d) \iff ad = bc \iff cb = da \iff (c, d) \sim (a, b)$;
- (iii) If $(a, b) \sim (c, d)$ and $(c, d) \sim (r, s)$. Then, $ad = bc$ and $cs = dr$. Therefore, $adcs = bcdr$, which means that $as = br$ (since $c \neq 0 \neq d$). In other words, $(a, b) \sim (r, s)$.
In this case, $[(a, b)] = \{(c, d) \in S : ad = bc\}$.

Theorem 8.3

If S is a set, with an equivalence relation \sim , then the equivalence classes of \sim *disjointly partition* S , i.e., every element of S is contained in **exactly** one equivalence class.

Given S , equipped with an equivalence class \sim on S , we define $\bar{S} = \{[x] : x \in S\}$, i.e., the set of equivalence classes.

In this situation, there exists a map $\pi : S \rightarrow \bar{S}$, defined by $x \mapsto [x]$.

Example 8.2

Let $S = \mathbb{Z}$, and $a \sim b \iff a - b$ is a multiple of 5. (You should verify that this is an equivalence relation.)

Then $\bar{S} = \{[0], [1], [2], [3], [4]\}$. E.g., $\pi(7) = [2]$.

Definition 8.4

Let $H \leq G$ be groups, and $a \in G$. Then, the *right coset of H with respect to a* is

$$Ha = \{g \in G : \exists h \in H \text{ such that } ha = g\} = \{ha : h \in H\}.$$

Lemma 8.5

$$Ha = Hb \iff ab^{-1} \in H$$

Lemma 8.6

Given $H \leq G$ groups, the relation defined by $a \sim b \iff ab^{-1} \in H$ is an equivalence relation.

So, what are the equivalence classes of this equivalence relation? They are exactly the right cosets of H , i.e, $[a] = Ha$.

Therefore, right cosets, if distinct, share no elements in common.

On Monday, we'll prove the following theorem.

Theorem 8.7 (Lagrange's Theorem)

If G is a finite group and H is a subgroup of G , then $|H|$ divides $|G|$.

9 Normal Subgroups (September 28, 2020)

Lemma 9.1

Given $H \leq G$, if $|G| < \infty$, then given $a, b \in G$, it holds $\#(Ha) = \#(Hb)$.

Proof. Note that H is a right coset ($H = He$). So, suffices to show that for all $a \in G$, $\#(Ha) = |H|$. Define a function $\varphi : H \rightarrow Ha$, defined by $h \mapsto ha$. We shall prove that φ is a bijection.

Let's show that φ is onto. Given $g \in Ha$, then $g = ha$ for some $h \in H$. But $\varphi(h) = ha = g$, which means that $g \in \text{Im}(\varphi)$.

Let's show that φ is one-to-one. If $\varphi(h_1) = \varphi(h_2) \implies h_1a = h_2a \implies h_1aa^{-1} = h_2aa^{-1} \implies a =$
Therefore φ is a bijection, which implies that $\#(Ha) = |H|$, and we're done! \square

Theorem 9.2 (Lagrange's Theorem)

If $H \leq G$ are finite groups, then $|H|$ divides $|G|$.

Proof. The right cosets of H partitionate G , i.e., they are disjoint and their union is G ; and they all have the same number of elements. Let $[G : H]$ denote the number of right cosets of H sitting inside G , which is called index of H in G . Therefore,

$$G = [G : H] \cdot |H|.$$

\square

Corollary 9.3

Given a group G and $a \in G$, if $|G| < \infty$, then $\text{order}(a)$ divides $|G|$.

Proof. Consider $\langle a \rangle \leq G$, then, by Lagrange's Theorem, $|\langle a \rangle| = \text{order}(a)$ divides $|G|$ \square

Definition 9.4

A subgroup H of G is called *normal*, denoted by $H \triangleleft G$ if, for all $g \in G$, the image of H under the g -conjugation isomorphism (the g -conjugation isomorphism is the map $\phi_g : G \rightarrow G$ defined by $a \mapsto gag^{-1}$) is contained in H , i.e, $\phi_g(H) \subset H$, for all $g \in G$.

Lemma 9.5

If G and G' are subgroups, $\phi : G \rightarrow G'$ a homomorphism, then $\text{Ker}\phi \triangleleft G$.

Example 9.1

$SL_n(\mathbb{R}) \triangleleft GL_n(\mathbb{R})$.

Use Lemma 9.5 with $\det : GL_n(\mathbb{R}) \rightarrow \mathbb{R}^\times$.

Example 9.2

$\langle (1\ 2) \rangle \not\triangleleft G$, because $\phi_{(2\ 3)}(\langle (1\ 2) \rangle) = \{e, (1\ 3)\} \not\subset H$

Theorem 9.6

The following are equivalent:

- (i) $H \triangleleft G$;
- (ii) $gHg^{-1} = H$, for any $g \in G$;
- (iii) $gH = Hg$, for any $g \in G$;
- (iv) Every left coset of H is a right coset of H .

10 Example of Quotients (September 30, 2020)

From last time, we discussed the following theorem:

Theorem 10.1

The following are equivalent:

- (i) $H \triangleleft G$;
- (ii) $gHg^{-1} = H$, for any $g \in G$;
- (iii) $gH = Hg$, for any $g \in G$;
- (iv) Every left coset of H is a right coset of H .

Proof (i \implies ii). $H \triangleleft G \implies \phi_g(H) \in H \implies gHg^{-1} \subset H$. Analogously, $g^{-1}Hg \subset H$. This last one implies that $H \subset gHg^{-1}$.

Therefore, $gHg^{-1} = H$. □

Proof (ii \iff iii). $gHg^{-1} = H \iff gH = Hg$. □

Proof (iii \implies iv). If $gH = Hg$, then gH is a right coset. □

Proof (iv \implies iii). Assume that, given aH , then there is b such that $aH = Hb$. Note that gH shares an element (namely, g) with Hg . Since gH is a left coset, then $gH = Hb$ for some b .

Since $g \in gH = Hb$, then Hb intersects with Hg , then $Hb = Hg$ (because, if two left cosets share an element, then they are equal). □

Proof (ii \implies i). $gHg^{-1} = \phi_g(H) = H$, then $\phi_g \subset H$, which implies $H \triangleleft G$. □

Recall from Linear Algebra:

Theorem 10.2

Let $T : V \rightarrow W$ a linear map, then

$$\dim V = \dim \ker T + \dim \operatorname{Im} T.$$

If T is onto, then

$$\dim V = \dim \ker T + \dim W.$$

The goal is to reproduce this idea with groups and homomorphisms, i.e., given G, G' groups, and an onto homomorphism $\phi : G \rightarrow G'$, then understand G as being a "stacking" of cosets of $\ker(\phi)$ and when we collapse each coset to a point, we get G' .

Our goal will be related to the following theorem:

Theorem 10.3

Given G and a subgroup H , then $H \triangleleft G$ if, and only if, there is a group G' and a homomorphism $\phi : G \rightarrow G'$ such that $\ker \phi = H$.

Definition 10.4 (Notation)

Let G/H (" $G \bmod H$ ") be the set of all right cosets of H sitting inside G .

Theorem 10.5

When $H \triangleleft G$, there exists a binary operation on G/H and an homomorphism $\phi : G \rightarrow G/H$ such that $\ker \phi = H$.

Spoiler: The operation $*$ will be, for $A, B \in G/H$, $A * B = AB = \{g \in G : \exists a_1 \in A, b_1 \in B, g = a_1 b_1\}$.

11 Quotient Groups (October 02, 2020)

Here's a rephrasing of our motivation from Linear Algebra:

Suppose V is a vector space, and $S \subset V$ a subspace. Let $\vec{x} \in V$. In Linear Algebra, we learned we write \vec{x} as a sum of a vector in S with an vector \vec{z} orthogonal to S , a.k.a., $\vec{z} \in S^\perp$.

V can be decomposed into parallel copies of S , and there exists a vector space W and a linear map $T : V \rightarrow W$ so that T has the effect of collapsing each parallel copy of S to a point. And, $\ker T = S$.

To summarize: Given S a subspace of V , there exists a decomposition of V into parallel copies of S and there exists a vector space W and a linear map $T : V \rightarrow W$ so that T collapses the parallel copies to points and $\ker T = S$.

Our goal in the Group Theory setting: Given $H \triangleleft G$, there exists a decomposition of G into right cosets of H in G and there exists a group G' and a homomorphism $\phi : G \rightarrow G'$ so that ϕ collapses a right cosets of H to a point and $\ker \phi = H$.

On Wednesday, we defined $G/H = \{\text{right cosets of } H \text{ in } G\}$ as our candidate for G' .

Given $Ha, Hb \in G/H$, we defined $Ha * Hb = (Ha)(Hb) = \{h_1 a h_2 b : h_1, h_2 \in H\}$.

We know that $aH = Ha$, then $HaHb = HHab$. Since H is closed under operation and $He = H$, we have that $HH = H$. Thus,

$$Ha * Hb = (Ha)(Hb) = H(ab),$$

which means that $*$ is a binary operation.

So far, we have that G/H has a binary operation. We also have a candidate for ϕ ! Define $\phi : G \rightarrow G/H$, with $g \rightarrow Hg$. Given, $a, b \in G$,

$$\phi(ab) = H(ab) = (Ha)(Hb) = \phi(a)\phi(b),$$

thus ϕ has the homomorphism property. Note also that ϕ is onto.

Lemma 11.1

If G is a group, Y is a set with a binary operation, $\phi : G \rightarrow Y$ such that ϕ has the homomorphism property, and ϕ is onto. Then Y is a group and ϕ is a homomorphism.

Proof. We need to show the following items:

- (i) Associativity. Given $a, b, c \in Y$, since ϕ is onto, we have $a = \phi(a'), b = \phi(b'), c = \phi(c')$, for some $a', b', c' \in G$. So

$$\begin{aligned} (ab)c &= (\phi(a')\phi(b'))\phi(c') \\ &= \phi(a'b')\phi(c') \\ &= \phi((a'b')c') \\ &= \phi(a'(b'c')) \\ &= \phi(a')\phi(b'c') \\ &= \phi(a')(\phi(b')\phi(c')) \\ &= a(bc). \end{aligned}$$

- (ii) Identity. The same strategy as above.

- (iii) Inverses. The same strategy as above.

Thus, Y is a group. □

12 First Isomorphism Theorem (October 05, 2020)

Summary of our work from last week:

Start with a group G and a normal subgroup H in G . Then, we showed that there exists a way to turn G/H into a group, and, the “natural” map $\phi : G \rightarrow G/H$, defined by $g \rightarrow Hg$ is an onto homomorphism. The identity of G/H is H . Also, $\ker(\phi) = H$. AND, ϕ has the effect of collapsing each coset of H to a single element.

So, starting with $H \triangleleft G$, we constructed an onto homomorphism $\phi : G \rightarrow G/H$.

Question. If we start with an onto homomorphism $\phi : G \rightarrow G'$, when is it the case that $\bar{\phi}$ arose via what we did last week?

Theorem 12.1 (1st Isomorphism Theorem)

If $\phi : G \rightarrow G'$ is an onto homomorphism, and N denotes $\ker(\phi)$, then G' is isomorphic to G/N .

And, there exists a unique isomorphism $\bar{\phi} : G/N \rightarrow G'$ so that $\bar{\phi}$ commutes with the natural map $\pi : G \rightarrow G/N$, defined by $g \mapsto gN$, i.e.,

$$\bar{\phi} \circ \pi = \phi.$$

Proof. We want to find a map $\bar{\phi} : G/N \rightarrow G'$. Let's define

$$\bar{\phi}(Ng) = \phi(g).$$

There is a subtle problem: what if $Ng = Nh$? As we defined $\bar{\phi}$, it sends Ng to $\phi(g)$, and Nh to $\phi(h)$. This is a problem unless $\phi(g) = \phi(h)$.

This definition is coherent, because

$$\begin{aligned} Ng = Nh &\implies gh^{-1} \in N = \ker \phi \\ &\implies \phi(gh^{-1}) = e' && (e' \text{ is the identity element of } G') \\ &\implies \phi(g)\phi(h)^{-1} = e' \\ &\implies \phi(g) = \phi(h). \end{aligned}$$

Let's show that $\bar{\phi}$ is a homomorphism:

$$\begin{aligned} \bar{\phi}(NaNb) &= \bar{\phi}(Nab) && (N \text{ is normal}) \\ &= \phi(ab) \\ &= \phi(a)\phi(b) && (\phi \text{ is a homomorphism}) \\ &= \bar{\phi}(Na)\bar{\phi}(Nb). \end{aligned}$$

Let's show that $\bar{\phi}$ is onto: Given $g' \in G'$, there exists $y \in G$ such that $\phi(y) = g'$, since ϕ is onto. So, consider $\pi(y) = Ny \in G/N$. Then, $\bar{\phi}(Ny) = \phi(y) = g'$.

Let's show that $\bar{\phi}$ is one-to-one:

$$\begin{aligned} \bar{\phi}(Na) = \bar{\phi}(Nb) &\implies \phi(a) = \phi(b) \\ &\implies \phi(a)\phi(b)^{-1} = e' \\ &\implies \phi(ab^{-1}) = e' \\ ab^{-1} &\in N \\ Na &= Nb. \end{aligned}$$

Thus, $\bar{\phi}$ satisfies the commuting property! □

13 Example of First Isomorphism Theorem (October 07, 2020)

General idea of today's lecture: 1st iso thm and the conversation about quotient groups from last week are useful for studying a group G from the point of view of onto hom's $\phi : G \rightarrow$ something else.

Example 13.1

$G' = \mathbb{Z}^2 = \{(a, b) : a, b \in \mathbb{Z}\}, (a, b) + (c, d) = (a + c, b + d)$. Note: \mathbb{Z}^2 is an abelian group (vector addition is commutative).

$G = F_2 =$ "free group on 2 generators" = {finite strings using symbols a, b, a^{-1}, b^{-1}, e }, with the operation of concatenation.

Define $\phi : F_2 \rightarrow \mathbb{Z}^2$ by $\phi(a) = (1, 0), \phi(b) = (0, 1)$ and send anything else to where you would have to sent it to make ϕ a homomorphism, e.g., $g \in F_2, g = a^3b^{-2}a^5$; then

$$\begin{aligned}\phi(g) &= 3\phi(a) - 2\phi(b) + 5\phi(a) \\ &= (8, -2).\end{aligned}$$

The function ϕ is onto, since $\phi(a^c b^d) = (c, d)$, for every $(c, d) \in \mathbb{Z}^2$.

Then, by the 1st iso thm, $\mathbb{Z}^2 \simeq F_2 / \ker \phi$. (Which implies that, e.g., $F_2 / \ker \phi$ is abelian.)

So far, we have that $\mathbb{Z}^2 \simeq F_2 / \ker \phi$. So, the idea is: learn something about F_2 by understanding what $\ker \phi$ is. Thus, we want to understand what $\ker \phi$ is.

For example, $aba^{-1}b^{-1} \in \ker \phi$. More generally, if $g_1, g_2 \in F_2$, then $g_1 g_2 g_1^{-1} g_2^{-1} \in \ker \phi$.

So, $\ker \phi \supset$ group generated by all expressions of the form $g_1 g_2 g_1^{-1} g_2^{-1}$. It is left to the reader to show that the equality holds. (Think about what it really means for something to be in the kernel.)

Remember, F_2 is not abelian. And a group G is abelian $\iff gh = hg \iff ghg^{-1}h^{-1}$.

The 1st isomorphism theorem says that the elements of \mathbb{Z}^2 are representing cosets of $\ker \phi$. This means that $g \in \ker \phi \iff \pi(g) = \text{identity in } F_2 / \ker \phi$.

All this to say: $\phi F_2 \rightarrow \mathbb{Z}^2$ collapses $\ker \phi$ to a point and collapses each coset of $\ker \phi$ to a different point.

Remember that $\ker \phi = \langle \text{commutators} \rangle$. The commutators are exactly the stuff that, if they're not identity, the group is not abelian.

So, F_2 is not abelian because $\langle \text{commutators} \rangle \neq \{\text{identity element}\}$ and we get an abelian group when we quotient out by this subgroup.

In summary:

- We characterized $\ker \phi$ as $\langle \text{commutators} = g_1 g_2 g_1^{-1} g_2^{-1} \rangle$ and commutators prevent abelian-ness. So, $F_2 / \ker \phi$ should be abelian. And, 1st iso thm implies $F_2 / \ker \phi \simeq \mathbb{Z}^2$, which is abelian.

14 More Examples (October 09, 2020)

Remark. This class has a lot of drawings, which are on Moodle's video of the lecture. The lecture notes for this class are not really useful without the drawings.

Let's show another example of the first isomorphism theorem:

In spirit, the first isomorphism theorem says that if $\phi : G \rightarrow G'$ onto, then G is comprised of copies of $\ker \phi$, organized into the pattern of G' .

Example 14.1

Let $G = (\mathbb{Z}, +)$ and $G' = \{0, 1, 2, 3, 4\}$, equipped with addition modulo 5. Let $\phi : \mathbb{Z} \rightarrow G'$ be the map defined by $k \mapsto k \pmod{5}$.

Then, the first isomorphism theorem implies that:

- (i) $G' \simeq \mathbb{Z} / \ker \phi$.
- (ii) If $\pi : \mathbb{Z} \rightarrow \mathbb{Z} / \ker \phi$ is the natural map defined by $n \mapsto (\ker \phi)n$, then there is an isomorphism $\bar{\phi} : \mathbb{Z} / \ker \phi \rightarrow G'$ such that $\phi = \bar{\phi} \circ \pi$.

This implies that, for any $k_1, k_2 \in \mathbb{Z}$, it holds that $\phi(k_1) = \phi(k_2) \iff \pi(k_1) = \pi(k_2)$, i.e., $(\ker \phi) + k_1 = (\ker \phi) + k_2$. (Here, we are using $+$ because that is the group operation.)

SEE THE PICTURES IN THE LECTURE VIDEO.

Example 14.2

Let's apply the same reasoning for $\phi : F_2 \rightarrow \mathbb{Z}^2$, from last class.

SEE THE PICTURES IN THE LECTURE.

15 Rings (October 12, 2020)

For groups, our prototype was the *symmetries* of a geometric object.

For rings, our prototypes will be $(\mathbb{Z}, +, \cdot)$, and $(\{f : \mathbb{R} \rightarrow \mathbb{R}\}, +, \cdot)$.

Definition 15.1

A ring is a set R , equipped with two binary operations, $+$ and \cdot , satisfying:

- (i) $(R, +)$ is an abelian group.
- (ii) The operation \cdot is commutative, associative, and has an identity.
- (iii) For all $a, b, c \in R$, it holds that $(a + b) \cdot c = a \cdot c + b \cdot c$.

Definition 15.2 (Notations)

We denote $(R, +)$ as R^+ . We denote the additive element as 0. We denote the multiplicative element as 1.

Definition 15.3

A subring of a ring R is a subset $S \subset R$, equipped with the same operations as R .

Subrings of \mathbb{C} were studied as examples of rings in the lecture. The notes for this segment of the class are to do.

16 Units, Fields (October 14, 2020)

Definition 16.1

An element $r \in R$ is called a unit if there exists $s \in R$ such that $rs = 1$.

Lemma 16.2

0 is never a unit, unless R is the trivial group, $\{0\}$.

Definition 16.3

Any non-trivial ring R in which every element except 0 is a unit is called a field.

In other words, a field is a ring in which all non-zero elements have multiplicative inverses.

Example 16.1

\mathbb{C} , \mathbb{R} , \mathbb{Q} , $\mathbb{Z}/p\mathbb{Z}$ (p is prime) are examples of fields.

17 — (October 16, 2020)

In last class, we learned about the substitution principle. If $\phi : R \rightarrow R'$ is a ring homomorphism, and $a \in R'$, then there exists a unique ring homomorphism $\Phi : R[x] \rightarrow R'$ satisfying

- (i) $\Phi|_R = \phi$. (In general, if $f : X \rightarrow Y$ and $A \subset X$, we can consider $f|_A : A \rightarrow Y$ given by restricting the domain of f to A)
- (ii) $\Phi(x) = a$.

On Wednesday, we said that “in spirit”, this is saying that any ring homomorphism “looks like” a restriction of an evaluation map.

Example 17.1 (Example of Evaluation Map)

Given R a ring and $a \in R$, let $\phi_a : R[x] \rightarrow R$ be defined by $p(x) = a_n x^n + \cdots + a_1 x + a_0 \mapsto a_n a^n + \cdots + a_1 a + a_0$. This is an evaluation map, and it is a ring homomorphism

Definition 17.1 (Generalized Evaluation Map)

Assume $\phi : R \rightarrow R'$ is a ring homomorphism and fix $a \in R'$. Let $\phi_a : R[x] \rightarrow R'$ be defined by

$$p(x) = a_n x^n + \cdots + a_1 x + a_0 \mapsto \phi(a_n) a^n + \cdots + \phi(a_1) a + \phi(a_0).$$

Every ring homomorphism $\phi : R \rightarrow R'$ is the restriction of a generalized evaluation map from $R[X]$ to R' . And, if we specify $a \in R'$, there is a unique generalized evaluation $\phi_a : R[x] \rightarrow R'$ such that ϕ is a restriction of ϕ_a .

Therefore, we can interpret the substitution principle as follows:

Theorem 17.2

Given $\phi : R \rightarrow R'$, and $a \in R'$, the generalized evaluation map ϕ_a is a ring homomorphism and it is the only ring homomorphism from $R[x]$ to R' agreeing with ϕ on R and sending x to a .

Let's prove the theorem above (and, consequently, prove the substitution principle).

Proof. We are given $\phi : R \rightarrow R'$ and $a \in R'$. We want to show that:

- (i) The map $\Phi : R[x] \rightarrow R'$ defined by $p(x) = a_n x^n + \cdots + a_1 x + a_0 \mapsto \phi(a_n) a^n + \cdots + \phi(a_1) a + \phi(a_0)$ is a ring homomorphism.
- (ii) It holds that $\Phi|_R = \phi$, and $\Phi(x) = a$.
- (iii) The map Φ is the only ring homomorphism with these properties.

Let's prove each item:

- (i) We need to show the following items
 - (a) $\Phi(1_R) = 1_{R'}$, because $\Phi(1_R) = \phi(1_R) = 1_{R'}$.
 - (b) $\Phi(f + g) = \Phi(f) + \Phi(g)$. This item is left to the reader.*
 - (c) $\Phi(fg) = \Phi(f)\Phi(g)$. Let $f = \sum_i a_i x^i$ and $g = \sum_j b_j x^j$. Then,

$$\begin{aligned} \Phi(fg) &= \Phi\left(\sum_i \sum_j a_i b_j x^{i+j}\right) \\ &= \sum_i \sum_j \phi(a_i b_j) a^{i+j} \\ &= \sum_i \sum_j \phi(a_i) \phi(b_j) a^{i+j} \\ &= \left(\sum_i \phi(a_i) a^i\right) \left(\sum_j \phi(b_j) a^j\right) \\ &= \Phi(f) \Phi(g) \end{aligned}$$

- (ii) Exercise left to the reader.
- (iii) Try it!

□

*Use the definition of Φ and use that ϕ is a ring homomorphism.

Theorem 17.3 (Slightly more general version of subprinciple)

If $\phi : R \rightarrow R'$ is a ring homomorphism and $\alpha_1, \alpha_2, \dots, \alpha_n \in R'$, then there exists a unique ring homomorphism $\Phi : R[x_1, x_2, \dots, x_n] \rightarrow R'$ satisfying $\Phi|_R = \phi$ and $\Phi(x_i) = \alpha_i$.

Theorem 17.4

For any ring R , it holds that $R[x, y] \simeq (R[x])[y]$.

Proof. R is a subring of $R[x]$ and $R[x]$ is a subring of $(R[x])[y]$. Thus, R is a subring of $(R[x])[y]$. Therefore, we have a natural obvious ring homomorphism $\phi : R \rightarrow (R[x])[y]$.

The substitution principle for $n = 2$, there exists a ring homomorphism $\Phi : R[x, y] \rightarrow (R[x])[y]$, extending ϕ .

$R[x]$ is also a subring of $R[x, y]$. Therefore, we have $\gamma : R[x] \rightarrow R[x, y]$. The substitution principle, there exists $\Gamma : (R[x])[y] \rightarrow R[x, y]$.

Check that Γ is the inverse of Φ . □

18 Ideals (October 19, 2020)

Our plan is to build up the analog of normal subgroups and quotient groups (kernels of homomorphisms), but for rings.

Definition 18.1

An ideal in a ring $(R, +, \cdot)$ is a non-empty subset $I \subseteq R$, such that

- (i) I is closed under $+$.
- (ii) For all $r \in R$ and $s \in I$, then $rs \in I$.

Lemma 18.2

If $\phi : R \rightarrow R'$ is a ring homomorphism, then $\ker \phi = \{r \in R \mid \phi(r) = 0_{R'}\}$ is an ideal of R .

Proof. Left to the reader. (It is straightfoward). □

Lemma 18.3

I is an ideal if, and only if $I \neq \emptyset$ and any linear combination $r_1 s_1 + \cdots + r_k s_k$ is in I .

Prove this on your own.

Example 18.1 (i) Kernels of ring homomorphisms.

- (ii) (Principal ideal generated by a) If $a \in R$, then $\{ra \mid r \in R\}$ forms an ideal. We write (a) to refer to this ideal.
- (iii) More generally, given a_1, \dots, a_n , the ideal denoted by (a_1, \dots, a_n) is $\{a_1 r_1 + \cdots + a_n r_n\}$ is an ideal.

Definition 18.4

A proper ideal of R is an ideal of R that is not $\{0_R\}$ nor R .

Lemma 18.5

A proper ideal is never a subring.

Example 18.2

Let $\phi : \mathbb{R}[x] \rightarrow \mathbb{R}$ given by $\phi(p(x)) = p(17)$. Then, by the lemma, $\ker \phi$ is an ideal in $\mathbb{R}[x]$.

$$\begin{aligned}\ker \phi &= \{p \in \mathbb{R}[x] \mid p(17) = 0\} \\ &= \{f(x)(x - 17) \mid f \in \mathbb{R}[x]\} \\ &= (x - 17).\end{aligned}$$

Proposition 18.6

If F is a field, then any ideal in $F[x]$ is principal.

Algorithm 18.7 (Polynomial division) — Given $g \in R[x]$ and $f \in R[x]$ monic, there are unique polynomials $p, r \in R[x]$ such that $g = fp + r$, and $\deg(r) < \deg(f)$.

Definition 18.8

If R is a ring and every ideal of R is principal, then it is called a principal ideal domain.

Proposition 18.9 (Rewrite of Proposition 18.6)

If F is a field, then $F[x]$ is a principal ideal domain.

Proof (of Proposition 18.6). Assume that $I \neq \{0\}$. Pick $f(x) \in I$ such that $f(x) \neq 0$ and $\deg(f)$ is minimal. Say that $f(x) = a_n x^n + \cdots + a_0$.

Since F is a field, there is a multiplicative inverse of a_n in F . Thus, let $\tilde{f} = a_n^{-1} \cdot f \in I$, and is monic; also with minimal degree.

Since $\tilde{f} \in I$, we have $(\tilde{f}) \subseteq I$.

Fix $g \in I$. Then, by polynomial division, there are unique $p, r \in F[x]$, with $\deg(r) < \deg(\tilde{f})$ such that $g = p\tilde{f} + r$. Then, $r \in I$, which implies $r = 0$. Thus, $g = p\tilde{f}$. This means that $I \subseteq (\tilde{f})$.

Therefore, $I = (\tilde{f})$. □

Lemma 18.10

If R is a ring, then there exists only one ring homomorphism $\phi : \mathbb{Z} \rightarrow R$.

Proof. Define, $\phi(0) = 0_R$ for $n > 0$, $\phi(n) = 1_R + 1_R + \cdots + 1_R$, where 1_R appears n times, and $\phi(-n) = -\phi(n)$. This is the only ring homomorphism that exists. □

Note that \mathbb{Z} is also a principal ideal domain.

So, let $\phi : \mathbb{Z} \rightarrow R$ be the unique ring homomorphism. Then $\ker \phi$ is an ideal in \mathbb{Z} , so it's principal.

Definition 18.11

The characteristic of R is the positive integer generating $\ker \phi$. If no such integer exists, then $\ker \phi = \{0\}$; in this case, we'll say that the characteristic is 0.

19 — (October 21, 2020)

20 Quotient out by an ideal (October 23, 2020)

The point of today's lecture is to understand that “quotienting out by an ideal” is “imposing relations on the original ring of the form (some elements) = 0.”

Lemma 20.1

$$R/(a, b) \approx (R/(a))/\pi((b))$$

21 Making rings bigger (October 26, 2020)

Example 21.1

Let $R = (\mathbb{R}, +, \cdot)$. We can define a new symbol, i , defined by the equation $i^2 = -1$. Then we get a new ring

$$\mathbb{R}[i] = \{r_0 + r_1i + r_2i^2 + \cdots + r_ni^n \mid r_i \in \mathbb{R}\}.$$

But $i^2 = -1$, thus,

$$\mathbb{R}[i] = \{a + bi \mid a, b \in \mathbb{R}\}.$$

Two observations are:

- $\mathbb{R}[i]$ is “a lot smaller” than expected.
- $\mathbb{R}[i] \approx \mathbb{R}[x]/\langle x^2 + 1 \rangle$.

Proposition 21.1

Let R be a ring, let $f \in R[x]$ be a monic polynomial, and let n be $\deg f$. Then the quotient ring $R[x]/\langle f \rangle$ satisfies:

- (i) If we define α , a new element, so that $f(\alpha) = 0$, then $R[x]/\langle f \rangle$ can be identified with $R[\alpha]$, and, any element $\lambda \in R[\alpha]$ can be uniquely expressed as $\lambda = r_0 + r_1\alpha + \cdots + r_{n-1}\alpha^{n-1}$ for some (r_0, \dots, r_{n-1}) , i.e., $(1_R, \alpha, \alpha^2, \dots, \alpha^{n-1})$ acts like a basis for $R[\alpha]$.
- (ii) elements in $R[\alpha]$ are added just like vector addition.
- (iii) If $\beta_1, \beta_2 \in R[\alpha]$, let $g_1, g_2 \in R[x]$ such that $g_1(\alpha) = \beta_1$ and $g_2(\alpha) = \beta_2$. Then polynomial division implies that there are polynomials $q(x), r(x)$, with $\deg(r) < n$, such that $g_1g_2(x) = f(x)q(x) + r(x)$. Then $\beta_1\beta_2 = r(\alpha)$.

How else can we make a ring bigger?

Example 21.2

Let $R = (\mathbb{Z}, +, \cdot)$. Then \mathbb{Q} is called the “field of fractions of \mathbb{Z} ”, which is a field you get by starting with \mathbb{Z} and adding in the multiplicative inverses of everything except 0.

Definition 21.2 (Field of Fractions)

Let R be an integral domain (a ring with no zero-divisors, i.e., $xy = 0$ implies $x = 0$ or $y = 0$). Given $R \times (R - \{0_R\})$, let \sim be an equivalence relation defined by $(a, b) \sim (c, d) \iff ad = bc$.

Then the field of fractions of R is the set of equivalence classes of this relation. It has the following notions of addition and multiplication:

- (i) $(a, b) + (c, d) = (ad + bc, bd)$;
- (ii) $(a, b)(c, d) = (ac, bd)$.

As always, whenever an object has different representations, it is important to show that the operations are consistent (if I change (a, b) to an equivalent (a', b') , then the result is the same).

22 Creating fields from rings (October 28, 2020)

Given a ring R , how can we use it to create a field F ?

Philosophically, there are two ways to do this:

- (i) add elements to R to create some field F in which R is a subring.
- (ii) kill elements of R , e.g., create R/I .

For (i), we can only do that if R does not contain zero-divisors, i.e., if R is an integral domain.

Definition 22.1 (Field of Fractions)

Let R be an integral domain (a ring with no zero-divisors, i.e., $xy = 0$ implies $x = 0$ or $y = 0$). Given $R \times (R - \{0_R\})$, let \sim be an equivalence relation defined by $(a, b) \sim (c, d) \iff ad = bc$.

Then the field of fractions of R is the set of equivalence classes of this relation. It has the following notions of addition and multiplication:

- (i) $(a, b) + (c, d) = (ad + bc, bd)$;
- (ii) $(a, b)(c, d) = (ac, bd)$.

As always, whenever an object has different representations, it is important to show that the operations are consistent (if I change (a, b) to an equivalent (a', b') , then the result is the same).

Problem 22.1. Prove that the set $R \times (R - \{0\})$, equipped with the operations defined above, is a ring.

Example 22.1

For $(\mathbb{Z}, +, \cdot)$, $F(\mathbb{Z}) = \{[(a, b)] : a, b \in \mathbb{Z}\}$. In this case, we have $F(\mathbb{Z}) \approx \mathbb{Q}$.

Note also that in $F(R)$, the additive identity is $[(0, 1)]$, and $(0, 1) \sim (0, a)$ for $a \in R - \{0\}$; and the multiplicative identity is $[(1, 1)]$.

So, to show that $F(R)$ is a field, it suffices to show that $[(a, b)]$, where $a \neq 0$, has a multiplicative inverse. If $a \neq 0$, then $[(b, a)] \in F(R)$ and $[(a, b)] \times [(b, a)] = [(ab, ba)] = [(1, 1)]$.

Notationally, we will often denote $[(a, b)]$ as $\frac{a}{b}$.

Proposition 22.2

$F(R)$ is the “smallest” field containing R .

Theorem 22.3 (Mapping principle)

If F_1 is a field, R is an integral domain and $R \subset F_1$, then there is a ring homomorphism $\phi : F(R) \rightarrow F_1$. Namely, ϕ maps $\frac{a}{b}$ to ab^{-1} .

Proposition 22.4

The map $\psi : R \rightarrow F(R)$, defined by $r \mapsto [(r, 1)]$ is one-to-one.

Before moving to (ii), let's introduce some terminology.

- R is a ring (not necessarily an integral domain).
- $u \in R$ is a *unit* if it has a multiplicative inverse.
- a *divides* b , written as $a \mid b$, if there is $q \in R$ such that $b = aq$.
- a is a *proper divisor* of b if a is not a unit and there is $q \in R$, q is not a unit and $b = aq$.
- a and b are *associates* if each divides each other.
- a is *irreducible* if a does not have proper divisors.
- a is *prime* if it holds that $a \mid bc \implies a \mid b$ or $a \mid c$.

Lemma 22.5

- u is a unit if, and only if, $(u) = R$.
- $a \mid b \iff (b) \subset (a)$.
- a is a proper divisor of b if, and only if, $(b) \subsetneq (a) \subsetneq R$.
- a and b are associates if, and only if, $(a) = (b)$.
- a is irreducible if, and only if, $(a) \subsetneq R$ and there is no $c \in R$ such that $(a) \subsetneq (c) \subsetneq R$.
- a is prime if, and only if, $bc \in (a) \implies b \in (a) \text{ or } c \in (a)$.

Definition 22.6 (Prime ideal)

A prime ideal I is an ideal such that, if $bc \in I$, then $b \in I$ or $c \in I$.

Definition 22.7

An ideal $I \neq R$ of a ring R is called *maximal* if, whenever J is an ideal and $J \supset I$, then $J = I$ or $J = R$.

In the next class, we will discuss the following theorem:

Theorem 22.8

R/I is a field if, and only if, I is maximal.

Lemma 22.9

A ring R is a field if, and only if, R contains exactly two ideals, namely $\{0\}$ and R .

Proof. If R is a field, then if J is an ideal, suppose $J \neq \{0_R\}$. Then there is some $a \in J$. However, since R is a field, a is a unit. Therefore, $(a) = R$, which implies that $J = R$.

Conversely, for all $a \neq 0$, (a) must be R , thus a is a unit. □