Algebra II
Lecture Notes

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Haverford College — Spring 2022
Last updated: January 24, 2022

This is Haverford College's undergraduate MATH H334, instructed by Elizabeth Milićević. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

Intro Text about Algebra and Representation Theory, by Liz

Lecture 1

We encounter symmetry in nearly every aspect of our daily lives: looking at our faces in the mirror, watching snowflakes fall from the sky, and driving across bridges. Symmetric organisms persist through evolution in nature, symmetric protagonists are perceived as especially beautiful in art, and symmetric components are critical to engineering structures that can withstand powerful forces. The set of symmetries of a particular physical object enjoys a rich algebraic structure, since symmetries are operations that can be composed together.

This group of all symmetries can then be conveniently studied by encoding each symmetry as a rectangular array of numbers called a matrix. This process of passing from a symmetric object in the natural world to a related collection of matrices is the hallmark of the mathematical field of representation theory. Representation theory thus reduces the wild and complex study of symmetry in nature to questions in the well understood area of mathematics called linear algebra.

As such, the proposed projects have broad potential to substantially impact our understanding of many symmetric structures occurring throughout the mathematical and natural sciences.

1 Group Actions

Example

Let's look into S_6, the group of permutations on $[6] = \{1,2,3,4,5,6\}$. Consider

$$\pi = (2\,5\,6)(1\,3) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{bmatrix} \in S_6.$$

The idea is that a group action is a way to mix up things in a set. In this example, π mixes up $\{1,2,3,4,5,6\}$ in the way the second row says.

Groups *act* on sets. Group actions capture the symmetries of the set. Let's formalize that concept.

Definition 1.1 (Group Actions)

An *action* of a group G on a set S is a function $*\colon G \times S \to S$, denoted by $(g, s) \mapsto g*s$, such that

i. $e * s = s$; and

ii. $g * (h * s) = (gh) * s$.

Example

The symmetric group S_n acts on $[n]$ by the action $\pi * i := \pi(i)$.

Lecture 2

Example

Let G be the set of translations $t_{\vec{v}}$ in the plane by any vector $\vec{v} \in \mathbb{R}^2$. Endow G with the operation of composition.

Consider the rule $t_{\vec{v}} * \vec{x} = \vec{v} + \vec{x}$. Note that

i. $t_{\vec{0}}$ is the identity of G and, for all $\vec{x} \in \mathbb{R}^2$

$$t_{\vec{0}} * \vec{x} = \vec{0} + \vec{x} = \vec{x},$$

ii. for all $t_{\vec{v}}, t_{\vec{u}} \in G$ and $\vec{x} \in \mathbb{R}^2$,

$$\begin{aligned} t_{\vec{v}} * (t_{\vec{u}} * \vec{x}) &= t_{\vec{v}} * (\vec{u} + \vec{x}) \\ &= \vec{v} + (\vec{u} + \vec{x}) \\ &= (\vec{v} + \vec{u}) + \vec{x} \\ &= t_{\vec{v}+\vec{u}} * \vec{x} \\ &= (t_{\vec{v}} \circ t_{\vec{u}}) * \vec{x}. \end{aligned}$$

Example (Example of Non-Action)

If G is a non-commutative group, then *right multiplication* of G on itself defined by $g * s = sg$ is not a group action.

Example

If G is a group, then *left multiplication* of G on itself defined by $g * s = gs$ is a group action.

1.1 Orbits

Lecture 3

Definition 1.2 (Orbits)

Suppose G acts on S. Fix $s \in S$. The *orbit of* s is

$$O_s = \{g * s : g \in G\}.$$

Example

Let $S = \mathbb{R}^2$ and G be the group of rotations of $\mathbb{R}$ around the origin. We denote by $\rho_\theta \in G$ the rotation around the origin by θ. Define $\rho_\theta * \vec{x} = \rho_\theta(\vec{x})$.

If we fix $\vec{x} \in \mathbb{R}^2$, the orbit $O_{\vec{x}}$ is the circle centered at the origin with radius $|\vec{x}|$.

Example

Let G be a group. Define the group action *left multiplication* of G on itself defined by $g * s = gs$.

If we fix $x \in G$, note that $(yx^{-1}) * x = yx^{-1}x = y$; therefore the orbit O_x is G.

Definition 1.3 (Stabilizer)

Suppose G acts on S. Fix $s \in S$. The *stabilizer of* s is

$$G_s = \{g \in G : g * s = s\}.$$

Example

Let $S = \mathbb{R}^2$ and G be the group of rotations of $\mathbb{R}$ around the origin. We denote by $\rho_\theta \in G$ the rotation around the origin by θ. Define $\rho_\theta * \vec{x} = \rho_\theta(\vec{x})$.

If we fix $\vec{x} \neq \vec{0} \in \mathbb{R}^2$, the stabilizer $S_{\vec{x}}$ is $\{\rho_0\}$. The stabilizer $S_{\vec{0}}$ is G.

Example

Let G be a group. Define the group action *left multiplication* of G on itself defined by $g * s = gs$.

If we fix $x \in G$, note that $(gx = x \iff g = e$; therefore the orbit O_x is $\{e\}$.

Proposition 1.4

Suppose G acts on S. Fix $s \in S$. The orbit O_s is a group.

Proof (Sketch). Just check the axioms. ∎