Algebra II
Lecture Notes

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This is Haverford College's undergraduate MATH H334, instructed by Elizabeth Milićević. All errors are my responsability.

Use these notes only as a guide. There is a non-trivial chance that some things here are wrong or incomplete (especially proofs).

Contents

Intro Text about Algebra and Representation Theory, by Liz

We encounter symmetry in nearly every aspect of our daily lives: looking at our faces in the mirror, watching snowflakes fall from the sky, and driving across bridges. Symmetric organisms persist through evolution in nature, symmetric protagonists are perceived as especially beautiful in art, and symmetric components are critical to engineering structures that can withstand powerful forces. The set of symmetries of a particular physical object enjoys a rich algebraic structure, since symmetries are operations that can be composed together.

This group of all symmetries can then be conveniently studied by encoding each symmetry as a rectangular array of numbers called a matrix. This process of passing from a symmetric object in the natural world to a related collection of matrices is the hallmark of the mathematical field of representation theory. Representation theory thus reduces the wild and complex study of symmetry in nature to questions in the well understood area of mathematics called linear algebra.

As such, the proposed projects have broad potential to substantially impact our understanding of many symmetric structures occurring throughout the mathematical and natural sciences.

1 Group Actions

Example

Let's look into S_6, the group of permutations on $[6] = \{1, 2, 3, 4, 5, 6\}$. Consider

$$\pi = (2\,5\,6)(1\,3) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 4 & 6 & 2 \end{bmatrix} \in S_6.$$

The idea is that a group action is a way to mix up things in a set. In this example, π mixes up $\{1, 2, 3, 4, 5, 6\}$ in the way the second row says.

Groups *act* on sets. Group actions capture the symmetries of the set. Let's formalize that concept.

defn:groupactions

Definition 1.1 (Group Actions)

An *action* of a group G on a set S is a function $*\colon G \times S \to S$, denoted by $(g, s) \mapsto g*s$, such that

i. $e * s = s$; and

ii. $g * (h * s) = (gh) * s$.

Example

The symmetric group S_n acts on $[n]$ by the action $\pi * i := \pi(i)$.

Example

Let G be the set of translations $t_{\vec{v}}$ in the plane by any vector $\vec{v} \in \mathbb{R}^2$. Endow G with the operation of composition.

Consider the rule $t_{\vec{v}} * \vec{x} = \vec{v} + \vec{x}$. This is rule describes an action of G on $\mathbb{R}^2$, since

i. $t_{\vec{0}}$ is the identity of G and, for all $\vec{x} \in \mathbb{R}^2$

$$t_{\vec{0}} * \vec{x} = \vec{0} + \vec{x} = \vec{x},$$

ii. for all $t_{\vec{v}}, t_{\vec{u}} \in G$ and $\vec{x} \in \mathbb{R}^2$,

$$\begin{aligned} t_{\vec{v}} * (t_{\vec{u}} * \vec{x}) &= t_{\vec{v}} * (\vec{u} + \vec{x}) \\ &= \vec{v} + (\vec{u} + \vec{x}) \\ &= (\vec{v} + \vec{u}) + \vec{x} \\ &= t_{\vec{v}+\vec{u}} * \vec{x} \\ &= (t_{\vec{v}} \circ t_{\vec{u}}) * \vec{x}. \end{aligned}$$

Example (Example of Non-Action)

If G is a non-commutative group, then *right multiplication* of G on itself defined by $g * s = sg$ is not a group action.

Example

If G is a group, then *left multiplication* of G on itself defined by $g * s = gs$ is a group action.

1.1 Orbits

defn:orbits

Definition 1.2 (Orbits)

Suppose G acts on S. Fix $s \in S$. The *orbit of s* is

$$\mathcal{O}_s = \{g * s : g \in G\}.$$

Example

Let $S = \mathbb{R}^2$ and G be the group of rotations of $\mathbb{R}$ around the origin. We denote by $\rho_\theta \in G$ the rotation around the origin by θ. Define $\rho_\theta * \vec{x} = \rho_\theta(\vec{x})$.

If we fix $\vec{x} \in \mathbb{R}^2$, the orbit $\mathcal{O}_{\vec{x}}$ is the circle centered at the origin with radius $|\vec{x}|$.

Example

Let G be a group. Define the group action *left multiplication* of G on itself defined by $g * s = gs$.

If we fix $x \in G$, note that $(yx^{-1}) * x = yx^{-1}x = y$; therefore the orbit $\mathcal{O}_x$ is G.

1.2 Stabilizers

Definition 1.3 (Stabilizer)

Suppose G acts on S. Fix $s \in S$. The *stabilizer of s* is

$$G_s = \{g \in G : g * s = s\}.$$

Example

Let $S = \mathbb{R}^2$ and G be the group of rotations of $\mathbb{R}$ around the origin. We denote by $\rho_\theta \in G$ the rotation around the origin by θ. Define $\rho_\theta * \vec{x} = \rho_\theta(\vec{x})$.

If we fix $\vec{x} \neq \vec{0} \in \mathbb{R}^2$, the stabilizer $S_{\vec{x}}$ is $\{\rho_0\}$. The stabilizer $S_{\vec{0}}$ is G.

Example

Let G be a group. Define the group action *left multiplication* of G on itself defined by $g * s = gs$.

If we fix $x \in G$, note that $(gx = x \iff g = e$; therefore the orbit $\mathcal{O}_x$ is $\{e\}$.

1.3 Global Properties

defn:transitiveaction

Definition 1.4 (Transitive Action)

Let G act on S. The action is *transitive* if, for all $s, s' \in S$, there exists $g \in G$ such that $g * s = s'$.

Equivalenty, the action is transitive if, for all $s \in S$, the orbit of s is S.

defn:faithfulaction

Definition 1.5 (Faithful Action)

Let G act on S. The action is *faithful* if the only group element that fixes every set element is e, i.e., whenever $g * s = s$ for all s implies $g = e$.

Equivalently, the action is faithful if $\bigcap_{s \in S} G_s = \{e\}$.

Example

Let G be the group of rotations of $\mathbb{R}$ around the origin. Let G act on $\mathbb{R}$ naturally. This action is not transitive, but it it faithful.

Example

Let G be a group. Let G act on itself by multiplication. This action is transitive and faithful.

1.4 Propositions on Orbits and Stabilizers

prop:orbitspartition

Proposition 1.6

Suppose G acts on S. The set of orbits

$$\{\mathcal{O}_s : s \in S\}$$

is a partition of S.

Equivalently, if $s, s' \in S$, then either $\mathcal{O}_s = \mathcal{O}_{s'}$ or $\mathcal{O}_s \cap \mathcal{O}_{s'} = \varnothing$.

Proof. Define a relation $\sim$ by $s_1 \sim s_2$ if, and only if, there exists $g \in G$ such that $g * s_1 = s_2$. This is an equivalence relation, since

- **i.** $s \sim s$ follows from $e * s = s$;
- **ii.** $s_1 \sim s_2 \iff s_2 \sim s_1$ follows from $g * s_1 = s_2 \iff g^{-1} * s_2 = s_1$; and
- **iii.** $s_1 \sim s_2$ and $s_2 \sim s_3 \implies s_1 \sim s_3$ follows from $g * s_1 = s_2$ and $h * s_2 = s_3 \implies (hg) * s_1 = s_3$.

Therefore, since $\mathcal{O}_s$ is the equivalence class of s with respect to $\sim$, it follows that the set of orbits partitions S. ∎

prop:stabilizersaresubgroups

Proposition 1.7

Suppose G acts on S. Fix $s \in S$. The stabilizer G_s is a subgroup of G.

Proof. Note that

- **i.** $e * s = s$, therefore $e \in G_s$.
- **ii.** if $g, h \in G_s$, then $(gh) * s = g * (h * s) = g * (s) = s$, therefore $gh \in G_s$.
- **iii.** if $g \in G_s$, then $g^{-1} * s = g^{-1} * (g * s) = (g^{-1}g) * s = e * s = s$, therefore $g^{-1} \in G_s$.

Therefore, G_s is a subgroup of G. ∎

thm:orbit-stabilizer

Theorem 1.8 (Orbit–Stabilizer Theorem)

Let G act on S. Let $s \in S$. Then

$$|\mathcal{O}_s| = |G/G_s| = [G : G_s].$$

If G is finite, then,

$$|\mathcal{O}_s| = \frac{|G|}{|G_s|}.$$

Proof. Note that, for all $x, y \in G$,

$$\begin{aligned} xG_s = yG_s &\iff G_s = x^{-1}yG_s \\ &\iff x^{-1}y \in G_s \\ &\iff x^{-1}ys = s \\ &\iff x * s = y * s. \end{aligned}$$

Consider the map $\varphi\colon G/G_s \to O_s$ by $\varphi(xG_s) = x * s$.

By our first observation, φ is well-defined, as well as injective.

Note that, if $s' \in O_s$, then there exists $g \in G$ such that $s' = g * s$. Thus, $\phi(gG_s) = g * s = s'$. Therefore, φ is onto.

Finally, φ is a bijection. Therefore, $|G/G_s| = |O_s|$, as desired. ∎

Application

Question. What are all rotational symmetries of a cube?

Let G be the group of rotations of the cube. Let S be the set of (the six) faces of the cube. Let G act on S naturally. Note that this action is transitive, i.e., you can send any face to any other face via rotations. Also, if we fix $s \in S$ to be a face, then there are only 4 rotations of the cube that fix s.

Therefore, using the thm:orbit-stabilizer Orbit–Stabilizer Theorem, we conclude that

$$|G| = |\mathcal{O}_s||G_s| = 6 \cdot 4 = 24.$$

Now, if we want to describe all rotational symmetries of a cube and be sure there is no sneaky rotation that we didn't account for, it suffices to describe 24 distinct rotations. Here they are:

- 1 identity.
- 3 rotations by 90° through an axis through midpoints of opposite faces.
- 3 rotations by 180° through an axis through midpoints of opposite faces.
- 3 rotations by 270° through an axis through midpoints of opposite faces.
- 6 rotations by 180° through an axis through midpoints of opposite edges.
- 4 rotations by 120° through an axis through opposite vertices.
- 4 rotations by 240° through an axis through opposite vertices.

1.5 Burnside's Lemma

defn:fixedset

Definition 1.9 (Fixed set)

Suppose G acts on S. Let $g \in G$. The fixed set of g is

$$S^g = \{s \in S : g * s = s\}.$$

thm:burnside

Theorem 1.10 (Burnside's Lemma)

Suppose a finite group G acts on S. Then,

$$\#(\text{orbits}) = \frac{1}{|G|} \sum_{g \in G} |S^g|,$$

i.e., the number of orbits is the average size of the fixed sets.

Proof. Note that

$$\begin{aligned}
\#(\text{orbits}) &= \sum_{s \in S} \frac{1}{|\mathcal{O}_s|} \\
&= \sum_{s \in S} \frac{|G_s|}{|G|} \\
&= \frac{1}{|G|} \sum_{s \in S} |G_s| \\
&= \frac{1}{|G|} \sum_{g \in G} |S^g| && \text{(by double counting)}
\end{aligned}$$

∎

Application

> *Question.* A board game piece is a cube, in which each face is colored in one of three colors. Thus, each piece is associated with a 3-coloration of the 6 faces of the cube.
>
> We say that two board game pieces are indistinguishable when one can be rotated into the other. What is the maximum number of distinguishable game pieces?

Let G be the group of rotations of the cube. Let S be the set of 3-colorations of the faces of the cube. Let G act on S naturally.

The answer to the question is the number of orbits of this action; since distinguishable colorations must be in distinct orbits.

There are some types of rotations:

- 1 identity.
- 3 rotations by 90° through an axis through midpoints of opposite faces.
- 3 rotations by 180° through an axis through midpoints of opposite faces.
- 3 rotations by 270° through an axis through midpoints of opposite faces.
- 6 rotations by 180° through an axis through midpoints of opposite edges.
- 4 rotations by 120° through an axis through opposite vertices.
- 4 rotations by 240° through an axis through opposite vertices.

Respectively, their fixed sets have 3^6, 3^3, 3^4, 3^3, 3^3, 3^2, 3^2, elements. Finally, by thm:burnside Burnside's Lemma, we conclude that the number of orbits is

$$\#(\text{orbits}) = \frac{1}{24}\left(1 \cdot 3^6 + 3 \cdot 3^3 + 3 \cdot 3^4 + 3 \cdot 3^3 + 6 \cdot 3^3 + 4 \cdot 3^2 + 4 \cdot 3^2\right) = \frac{1368}{24} = 57.$$

Application

> *Question* (Benzene Compounds). Let S be the set of 3-colorings of the vertices of an hexagon. Let G_6 act naturally on S. How many orbits there are?

2 Representation Theory

2.1 Our first group representation

Example

Let $G = GL_2(\mathbb{F}_2)$, and let

$$S = \mathbb{F}_2^2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}.$$

Let G act on S by $A * \mathbf{v}_i = A\mathbf{v}_i$.

prop:pigbijection

Proposition 2.1

Let G act on S. Fix $\pi_g : S \to S$ be defined by $s \mapsto g * s$. Then, π_g is a bijection.

Proof. I claim that π_g is injective. Indeed, if $\pi_g(s) = \pi_g(r)$, then $g * s = g * r$, then $s = g^{-1} * (g * s) = g^{-1} * (g * r) = r$.

I claim that π_g is surjective. Indeed, for all $s \in S$, $\pi_g(g^{-1} * s) = s$. ■

Definition 2.2 (Permutations of S)

Let $\mathrm{Perm}(S)$ be the group of bijections from S to S endowed with composition.

Proposition 2.3

$\mathrm{Perm}(S)$ is a group.

thm:GtoPermS

Theorem 2.4

Let G act on S. Define a map $\phi : G \to \mathrm{Perm}(S)$ defined which sends g to π_g, as defined in Proposition 2.1 prop:pigbijection.

Then, ϕ is a group homomorphism.

Proof. Let $g, h \in G$ be arbitrary.

Note that, for all $s \in S$,

$$\begin{aligned}(\pi_h \circ \pi_g)(s) &= \pi_h(\pi_g(s)) \\ &= \pi_h(g * s) \\ &= h * (g * s) \\ &= (hg) * s \\ &= \pi_{hg}(s).\end{aligned}$$

Therefore, it follows that $\phi(h) \circ \phi(g) = \phi(hg)$, for all $g, h \in G$; thus, ϕ is a group homomorphism. ∎

thm:GfaithfultoPermSinjective

Theorem 2.5

Suppose G acts faithfully on S. Then, ϕ, as defined in Theorem 2.4 (thm:GtoPermS), is injective.

Proof. If $\phi(g) = \phi(h)$, then $\phi(gh^{-1}) = \phi(h)$, then $(gh^{-1}) * s = s$ for all s; since the action is faithful, then $gh^{-1} = e$, then $g = h$.

Therefore, ϕ is injective. ∎

2.2 Representation

Definition 2.6 (Representation)

A *representation* of a group G is a homomorphism $\rho\colon G \to GL(V)$ for some (finite-dimensional) vector space V. The *degree/dimension* of ρ is the dimension of V.

Proposition 2.7

Define $\psi\colon S_n \to GL_n(\mathbb{R})$ by mapping π to a matrix $M = [a_{ij}]_{n\times n}$ with

$$a_{ij} = \begin{cases} 1 & \text{if } j = \pi(i) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the map

$$G \to \mathrm{Perm}(S) \to S_n \to GL_n(\mathbb{R}),$$

that sends

$$g \mapsto \pi_g \to \pi \mapsto M,$$

is a representation.

Definition 2.8 (Character of a Representation)

Given a representation $\rho\colon G \to GL_n(\mathbb{C})$, let $\chi_\rho\colon G \to \mathbb{C}$ be defined by $\chi_\rho = \operatorname{Tr} \circ \rho$.

If a representation is indexed, e.g. ρ_T, then we usually denote its character by the same index, e.g. $\chi_T = \chi_{\rho_T}$.

Example (Trivial Representation)

Example (Permutation Representation)

Example ((Left) Regular Representation)

Example (Sign Representation of S_n)

Example (Liz's Standard Representation)

Suppose G act on $\mathbb{C}^n$, and $\pi_g\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $z \mapsto g{*}z$ is a linear trasformations. The *standard representation* $\rho_A\colon G \to GL_n(\mathbb{C})$ is defined by

$$g \mapsto \begin{bmatrix} g * \mathbf{e}_1 & \cdots & g * \mathbf{e}_n \end{bmatrix}$$

Example (Observation)

If G is generated by S, and $\rho\colon G \to GL_n(\mathbb{C})$ is an representation; then we can just calculate $\rho(s)$ for $s \in S$, and let a computer calculate the others by doing matrix multiplications.

Example

Consider $\rho_{\text{3D}}, \rho_{\text{2D}}, \rho_{\text{sign}}\colon D_3 \to GL_n(\mathbb{C})$. Then,

$$\rho_{\text{3D}} = \rho_{\text{2D}} \oplus \rho_{\text{sign}}.$$

Magically, we also have

$$\chi_{\text{3D}} = \chi_{\text{2D}} + \chi_{\text{sign}}.$$

Definition 2.9 (G-invariant Subspace)

Suppose G acts on $\mathbb{C}^n$ by linear transformations, then a vector subspace $V \subset \mathbb{C}^n$ is *G-invariant* if

$$\rho(g)\mathbf{v} \in V$$

for all $g \in G$ and all $\mathbf{v} \in V$.

Definition 2.10 (Irreducible Representation)

A representation of G is called *irreducible* if it has no proper G-invariant subspace.

2.2.1 Closer look into Characters

Definition 2.11 (Isomorphic Representations)

Two representations are isomorphic if they differ only by a change of basis.

Theorem 2.12 (Characters and Isomorphisms)

Let G be a finite group. Then:

- **i.** Two representations ρ_1, ρ_2 of G are isomorphic if, and only if, $\chi_1 = \chi_2$.
- **ii.** The number of non-isomorphic irreducibe representations of G equals the number of conjugacy classes of G.